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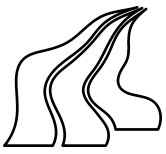

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# Total domination in partitioned graphs

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**Abstract**

Let $G = (V, E)$ be a graph of order $n$. For sets $S, T \subseteq V$, $S$ totally dominates $T$ if every vertex in $T$ is adjacent to some vertex of $S$. The minimum number of vertices needed to totally dominate $V$ is the total domination number $\gamma_t(G)$. For a partition $V_1, V_2$ of $V$, we define $f_t(G; V_1, V_2) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2)$ and $f_t(G) = \max\{f_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}$. For a graph $H$, we denote by $H \circ P_2$ the graph obtained from $H$ by attaching a path of length 2 to each vertex of $H$ so that the resulting paths are vertex-disjoint. We show that if $G$ is a tree of order $n \geq 4$ and $G \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then $f_t(G) \leq 14n/9$ with equality if and only if $G \in \{P_9, P_{18}\}$ or $G = (T \circ P_2) \circ P_2$ for some tree $T$. If $G$ is a graph with minimum degree at least two, we establish that $f_t(G) \leq 3n/2$ with equality if and only if $G$ is a cycle of order congruent to zero modulo 4.

**Keywords:** partitioned graphs, total domination
**AMS subject classification: 05C69**

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# 1 Introduction

In this paper, we continue the study of concept of partitions and domination in graphs introduced by Hartnell and Vestergaard [4], and studied, for example, in [6, 7, 8]. Here we study partitions and total domination in graphs.

Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$, and with no isolated vertex. For sets $S, T \subseteq V$, $S$ *totally dominates* $T$ if every vertex in $T$ is adjacent to some vertex of $S$. If $S$ totally dominates $V$, then $S$ is called a *total dominating set*, denoted TDS, of $G$. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. For $U \subseteq V$, we let $\gamma_t(G, U)$ denote the minimum cardinality of a set of vertices in $G$ that totally dominates $U$. Hence, $\gamma_t(G, V) = \gamma_t(G)$. If $U = \emptyset$, we define $\gamma_t(G, U) = 0$. A set of cardinality $\gamma_t(G, U)$ that totally dominates $U$ in $G$ we call a $\gamma_t(G, U)$-set. If $U = V$, we also call a $\gamma_t(G, U)$-set a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [1] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [2, 3].

By a *partition* of the vertices of a graph $G = (V, E)$, we shall mean two disjoint subsets $V_1, V_2$ of $V$ with $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$; $\{V_1, V_2\} = \{\emptyset, V\}$ is permitted. Given a partition $\mathcal{P} = (V_1, V_2)$ of $V$, we define the *label* of a vertex $v$ in $\mathcal{P}$, denoted $\ell_{\mathcal{P}}(v)$, as the number $i \in \{1, 2\}$ such that $v \in V_i$. For a graph $G$, and a partition $V_1, V_2$ of $V$, we define $f_t(G; V_1, V_2)$ and $g_t(G; V_1, V_2)$ by

$$\begin{aligned} f_t(G; V_1, V_2) &= \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2), \\ g_t(G; V_1, V_2) &= \gamma_t(G, V_1) + \gamma_t(G, V_2), \end{aligned}$$

and $f_t(G)$ and $g_t(G)$ by

$$\begin{aligned} f_t(G) &= \max\{f_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}, \\ g_t(G) &= \max\{g_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}. \end{aligned}$$

For notation and graph theory terminology we in general follow [2]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Let $v \in V$ and let $S \subseteq V$. The *open neighborhood* of $v$ in $G$ is $N(v) = \{u \in V \mid uv \in E\}$, while the *open neighborhood* of $S$ is the set $N(S) = \cup_{v \in S} N(v)$. Hence for a set $U \subseteq V$, the set $S$ *totally dominates* $U$ if $U \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. A vertex of degree $k$ we call a *degree-$k$ vertex*. A degree-1 vertex we call a *leaf* (or an end-vertex), and a vertex adjacent to a leaf we call a *support vertex*. The minimum (resp., maximum) degree among the vertices of $G$ is denoted by $\delta(G)$ (resp., $\Delta(G)$). For disjoint subsets $S$ and $T$ of vertices, we denote by $[S, T]$ the set of edges of $G$ with one end in $S$ and the other in $T$.

A subset $S$ of vertices in a graph $G$ is an *open packing* if the open neighborhoods of vertices in $S$ are pairwise disjoint, i.e. no two vertices from $S$ have a common neighbour, but they may be adjacent. The *open packing number* $\rho^o(G)$ is the maximum cardinality of an open packing in $G$.

A set $M$ of edges of $G$ is a *matching* if no two edges in $M$ are incident to the same vertex. A *perfect matching* in $G$ is a matching with the property that every vertex is incident with an edge of the matching.

A cycle on $n$ vertices is denoted by $C_n$ and a path on $n$ vertices by $P_n$. For $r \geq 3$ and $s \geq 1$, we denote by $L_{r,s}$ the graph obtained by joining with an edge a vertex in $C_r$ to an end-vertex of $P_s$. We call the graph $L_{r,s}$ a *key*.

For a graph $H$, we denote by $H \circ P_2$ the graph of order $3|V(H)|$ obtained from $H$ by attaching a path of length 2 to each vertex of $H$ so that the resulting paths are vertex-disjoint. The graph $H \circ P_2$ is also called the 2-*corona* of $H$.

## 2 Main Results

We shall prove:

**Theorem 1** *If $T$ is a tree of order $n \geq 4$ and $T \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$ or $T = (T' \circ P_2) \circ P_2$ for some tree $T'$.*

The tree $(K_1 \circ P_2) \circ P_2$, for example, is shown in Figure 1.

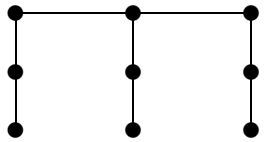

Figure 1: The tree $(K_1 \circ P_2) \circ P_2$.

**Theorem 2** *If $G$ is a connected graph of order $n$ with $\delta(G) \geq 2$, then $f_t(G) \leq 3n/2$ with equality if and only if $G \cong C_n$ where $n \equiv 0 \pmod 4$.*

## 3 Proof of Theorem 1

### 3.1 Preliminary Results

The total domination number of a cycle $C_n$ or a path $P_n$ on $n \geq 3$ vertices is easy to compute.

**Lemma 1** ([5]) *For $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Thus for $G \in \{P_n, C_n\}$, if $n \geq 3$ is odd, then $\gamma_t(G) = (n+1)/2$ and if $n$ is congruent to zero modulo 4, then $\gamma_t(G) = n/2$. Finally if $n$ is congruent to two modulo 4, then $\gamma_t(G) = (n+2)/2$.

The total domination number of a key $L_{r,s}$ of order (and size) $r+s$ was determined in [5]. As a consequence of this result, we have the following upper bound on $\gamma_t(L_{r,s})$.

**Lemma 2** ([5]) *For $r \geq 3$ and $s \geq 1$, if $G$ is a key $L_{r,s}$ of order $n = r + s$, then $\gamma_t(G) \leq (n+2)/2$ with equality if and only if $r \equiv 2 \pmod 4$ and $s \equiv 0 \pmod 4$.*

The following lemmas follow immediately from the definitions of $f_t(G)$ and $g_t(G)$.

**Lemma 3** *If $G'$ is a spanning subgraph of a graph $G$ with $\delta(G') \geq 1$, then $g_t(G) \leq g_t(G')$.*

**Lemma 4** *If $G$ is a graph with no isolated vertex, then $f_t(G) = \gamma_t(G) + g_t(G)$.*

We shall use the obvious observation that for a graph $G$ with induced subgraphs $G_1, G_2$ having no isolated vertices and satisfying $V(G) = V(G_1) \cup V(G_2)$, we have that

$$\gamma_t(G) \leq \gamma_t(G_1) \ + \gamma_t(G_2),$$

$$g_t(G) \leq g_t(G_1) + g_t(G_2),$$

$$f_t(G) \leq f_t(G_1) + f_t(G_2).$$

The following lemma follows readily from the definition of an open packing.

**Lemma 5** *Let $G = (V, E)$ be a path $v_1, v_2, \ldots, v_n$ of order $n$, and let $V_1$, $V_2$ be a partition of $V$. If both $V_1$ and $V_2$ are open packings in $G$, then the labels of $V(P_n)$ come in alternating pairs but the beginning and the end may be a pair or a single label. More precisely, renaming the sets $V_1$ and $V_2$ if necessary, we have*

$$V_1 = \left( \bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left( \bigcup_{i=0}^{\lfloor (n-2)/4 \rfloor} \{v_{4i+2}\} \right).$$

*or*

$$V_1 = \left( \bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left( \bigcup_{i=0}^{\lfloor (n-4)/4 \rfloor} \{v_{4(i+1)}\} \right),$$

*with the remaining vertices in $V_2$.*

**Definition 1** *For a graph $G = (V, E)$, we define a partition $V_1$, $V_2$ of $V$ to be a* good partition *if both $V_1$ and $V_2$ are open packings in $G$.*

The following lemmas will prove to be useful when proving our main results.

**Lemma 6** *Let $G = (V, E)$ be a graph of order $n \geq 2$,with no isolated vertex and let $V_1$, $V_2$ be a partition of $V$. Then, $V_1$, $V_2$ is a good partition of $V$ if and only if $\gamma_t(G, V_1)+\gamma_t(G, V_2) = n$.*

**Proof.** Suppose that $V_1$, $V_2$ is a good partition of $V$. Then for $i \in \{1, 2\}$, no two vertices from $V_i$ can be dominated by a common vertex, and so $\gamma_t(G, V_1)+\gamma_t(G, V_2) = |V_1|+|V_2| = n$. This establishes the necessity. To prove the sufficiency, suppose that $V_1$, $V_2$ is not a good partition of $V$. We may assume that $V_1$ is not an open packing in $G$. Thus there exist two vertices in $V_1$ that have a common neighbor, implying that $\gamma_t(G, V_1) \leq |V_1| - 1$. Hence since $\gamma_t(G, V_2) \leq |V_2|$, we have that $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. □

**Lemma 7** *For $n \geq 2$, $g_t(P_n) = n$ and $f_t(P_n) = \lfloor 3n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

**Proof.** Since every path has a good partition of its vertex set, we have by Lemma 6 that $g_t(P_n) = n$. The desired result now follows from Lemmas 1 and 4. □

Thus by Lemma 7, if $n \geq 3$ is odd, then $f_t(P_n) = (3n + 1)/2$; if $n \equiv 0 \pmod 4$, then $f_t(P_n) = 3n/2$; if $n \equiv 2 \pmod 4$, then $f_t(P_n) = (3n + 2)/2$.

**Lemma 8** *If $G = (V, E)$ is a path of order $n \geq 2$, and $V_1$, $V_2$ is not a good partition of $V$, then $f_t(G; V_1, V_2) \leq 3n/2$ with strict inequality if $n \not\equiv 2 \pmod 4$.*

**Proof.** By Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. By Lemma 1, $\gamma_t(G) \leq (n + 2)/2$ with strict inequality if $n \not\equiv 2 \pmod 4$. Hence, $f_t(G; V_1, V_2) \leq 3n/2$ with strict inequality if $n \not\equiv 2 \pmod 4$. □

The following lemma is an immediate consequence of Lemma 8.

**Lemma 9** *If $G = (V, E)$ is a path of order $n \geq 2$, and $V_1$, $V_2$ is a partition of $V$ for which $f_t(G; V_1, V_2) > 3n/2$, then $V_1$, $V_2$ is a good partition of $V$.*

**Lemma 10** *If $G$ is a graph of order $n$ without isolated vertices and $S \subseteq V(G)$, then $g_t(G) \leq n + 2|S| - |N(S)|$.*

**Proof.** Let $G = (V, E)$ and let $V_1$, $V_2$ be a partition of $V$. Let $i \in \{1, 2\}$. For each vertex $v \in V_i \setminus N(S)$, we choose an adjacent vertex and call the resulting set of such vertices $S'_i$. Then, $S \cup S'_i$ totally dominates $V_i$ in $G$, and so $\gamma_t(G, V_i) \leq |S| + |S'_i|$. Thus, $g_t(G; V_1, V_2) \leq 2|S| + |S'_1| + |S'_2| \leq 2|S| + |V \setminus N(S)| = n + 2|S| - |N(S)|$. Thus for every partition $V_1$, $V_2$ of $V$, $g_t(G; V_1, V_2) \leq n + 2|S| - |N(S)|$. Therefore, $g_t(G) \leq n + 2|S| - |N(S)|$. □

As a special case of Lemma 10, we have the following result.

**Lemma 11** *If $G$ is a graph of order $n$ with no isolated vertex and maximum degree at least 3, then $g_t(G) \leq n - 1$.*

**Proof.** Let $v$ be a vertex of maximum degree at least 3 and let $S = \{v\}$. Then, $|S| = 1$ and $|N(S)| \geq 3$, and so the desired result follows from Lemma 10. □

**Lemma 12** *If $T$ is a graph of order $n$ that can be obtained from a star on at least four vertices by subdividing some (including the possibility of none) of the edges exactly once, then $f_t(T) < 3n/2$.*

**Proof.** For integers $r \geq k \geq 0$ with $r \geq 3$, let $T = (V, E)$ be obtained from a star $K_{1,r}$ by subdividing $k$ edges exactly once. If $k = 0$, then $n = r + 1 \geq 4$ and $f_t(T) \leq 5 < 3n/2$. Hence we may assume that $k \geq 1$. Then, $\gamma_t(T) = k + 1$. Let $V_1$, $V_2$ be a partition of $V$. Then, $\gamma_t(T, V_1) + \gamma_t(T, V_2) \leq k + 3$, and so $f_t(T; V_1, V_2) \leq 2k + 4$. Since $r \geq k$ and $r \geq 3$, we have $3n/2 = 3(k + r + 1)/2 = (3k + r)/2 + r + 3/2 \geq 2k + 9/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(T; V_1, V_2) < 3n/2$. Therefore, $f_t(T) < 3n/2$. □

Next we define a special set $\mathcal{S}$ of small paths.

**Definition 2** *Let $\mathcal{S} = \{P_1, P_2, P_3, P_5, P_6, P_7, P_{10}, P_{14}\}$.*

As a consequence of the remark after Lemma 7 we have the following result.

**Lemma 13** *If $T \in \mathcal{S}$ has order $n \geq 2$, then $f_t(T) = (3n + 1)/2$ if $n$ is odd; otherwise, $f_t(T) = (3n + 2)/2$.*

A proof of the following lemma is a simple exercise and is omitted.

**Lemma 14** *Let $T = (V, E)$ be a path in $\mathcal{S}$. If $|V| \geq 2$ and $v \in V$ is neither a leaf of a $P_5$ nor a center of a $P_7$, then there exists a $\gamma_t(T)$-set containing $v$.*

**Definition 3** *Let $\mathcal{T} = \{T \mid T = (T' \circ P_2) \circ P_2$ for some tree $T'\}$.*

### 3.2 Proof of Theorem 1

Recall Theorem 1.

**Theorem 1** *If $T \notin \mathcal{S}$ is a tree of order $n \geq 4$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$ or $T \in \mathcal{T}$.*

**Proof.** We proceed by induction on $n$. When $n = 4$, either $T = K_{1,3}$, in which case $f_t(T) = 5$, or $T = P_4$, in which case $f_t(T) = 6$. In both cases, $f_t(T) < 14n/9$. This

establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for all trees $T' \notin \mathcal{S}$ of order $n'$, where $4 \leq n' < n$, that $f_t(T') \leq 14n'/9$ with equality if and only if $T' \in \{P_9, P_{18}\}$ or $T' \in \mathcal{T}$.

So let $T = (V, E)$ be a tree of order $n$ with $T \notin \mathcal{S}$. The following observation follows from Lemma 1.

**Observation 1** *If $T = P_n$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$.*

By Observation 1, we may assume that $T$ is not a path, for otherwise the desired result follows. With this assumption, we have the following observation by Lemma 11.

**Observation 2** $g_t(T) \leq n - 1$.

**Observation 3** *If $T$ contains a path on five vertices with one end a leaf in $T$ and with each internal vertex a degree-2 vertex in $T$, then $f_t(T) < 14n/9$.*

**Proof.** Let $P\colon v, v_1, v_2, v_3, v_4$ be a path in $T$ where $\deg_T(v_4) = 1$ and $\deg_T(v_i) = 2$ for $i = 1, 2, 3$. Let $T_1$ and $T_2$ be the components of $T - vv_1$ containing $v$ and $v_1$, respectively. Then, $T_1$ is a tree of order $n_1 = n-4$, while $T_2 = P_4$, and so $g_t(T_2) = n_2 = 4$ and $f_t(T_2) = 6$. Since $T$ is not a path, $n_1 \geq 3$.

Suppose $T_1$ is a path. Then, $g_t(T_1) = n_1$ and, by Lemma 1, $f_t(T_1) \leq (3n_1 + 2)/2$. Thus, $g_t(T_1) + g_t(T_2) = n$. By Observation 2, $g_t(T) \leq n - 1$, and so $g_t(T) \leq g_t(T_1) + g_t(T_2) - 1$. Thus, by Lemmas 3 and 4, $f_t(T) = \gamma_t(T) + g_t(T) \leq \gamma_t(T_1) + \gamma_t(T_2) + g_t(T_1) + g_t(T_2) - 1 = f_t(T_1) + f_t(T_2) - 1 \leq (3n_1 + 2)/2 + 6 - 1 = 3n/2 < 14n/9$. Hence we may assume that $T_1$ is not a path. In particular, $T_1 \notin \mathcal{S}$ and $n_1 \geq 4$. Thus, by the inductive hypothesis, $f_t(T) \leq f_t(T_1) + f_t(T_2) \leq 14n_1/9 + 6 < 14n/9$. □

By Observation 3, we may assume that $T$ contains no path on five vertices with one end a leaf in $T$ and with each internal vertex a degree-2 vertex in $T$.

Let $V_1$, $V_2$ be a partition of $V$. For each edge $uv \in E$, let $T_u$ and $T_v$ denote the components of $T - uv$ containing $u$ and $v$, respectively. If $T_u \in \mathcal{S}$, then we orient the edge from $u$ to $v$, while if $T_v \in \mathcal{S}$, then we orient the edge from $v$ to $u$. (Possibly an edge may be oriented in both directions.)

**Observation 4** *If an edge of $T$ has no orientation, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \mathcal{T}$.*

**Proof.** Suppose that an edge $uv \in E$ has no orientation. Applying the inductive hypothesis to $T_u$ and $T_v$, we have that for $x \in \{u, v\}$, $f_t(T_x) \leq 14|V(T_x)|/9$ with equality if and only if $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$. Hence, $f_t(T) \leq f_t(T_u) + f_t(T_v) \leq 14|V(T_u)|/9 + 14|V(T_v)|/9 =$

$14n/9$. Thus if $f_t(T_x) < 14|V(T_x)|/9$ for some $x \in \{u, v\}$, then $f_t(T) < 14n/9$. Suppose then that for $x \in \{u, v\}$, $f_t(T_x) = 14|V(T_x)|/9$, and so $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$.

Suppose that one of $T_u$ and $T_v$, say $T_u$, is a path. Then, $T_u \in \{P_9, P_{18}\}$ and at least one leaf in $T_u$ is a leaf in $T$ that is the end of a path on five vertices every internal vertex of which has degree 2 in $T$, contrary to assumption.

Hence both $T_u$ and $T_v$ are in the family $\mathcal{T}$. Let $G \cong (P_1 \circ P_2) \circ P_2$. Then both $T_u$ and $T_v$ have disjoint copies of $G$ as a spanning subgraph. Thus, $T$ has as a spanning subgraph the graph $H = kG$, consisting of $k$ disjoint copies of $G$, for some integer $k \geq 2$, where $u$ and $v$ belong to different copies of $G$ in $H$. Hence, $n = 9k$. Let $G_u$ and $G_v$ be the copies of $G$ in $H$ that contain $u$ and $v$, respectively. Let $T_{uv} = G_u \cup G_v \cup \{uv\}$.

We proceed further with two observations about the graph $G$. We observe first that $\gamma_t(G) = 6$, while $g_t(G) = |V(G)| - 1 = 8$, and so $f_t(G) = 14 = 14|V(G)|/9$. We observe secondly that for every vertex of $G$ there exists a $\gamma_t(G)$-set containing it and if $w$ is a leaf in $G$ or a support vertex in $G$, then $\gamma_t(G, V(G) \setminus \{w\}) = \gamma_t(G) - 1$.

Suppose that $u$ is a leaf or a support vertex in $G_u$. Then it follows from our two earlier observations about the graph $G$ that $\gamma_t(T_{uv}) \leq \gamma_t(G_u) + \gamma_t(G_v) - 1$, implying that $\gamma_t(T) \leq k\gamma_t(G) - 1 = 6k - 1$. Thus since $g_t(T) \leq kg_t(G) = 8k$, we have that $f_t(T) \leq 14k - 1 = 14n/9 - 1$. Hence we may assume that $u$ is neither a leaf nor a support vertex in $G_u$. Similarly, $v$ is neither a leaf nor a support vertex in $G_v$.

Suppose that $u$ or $v$ is the vertex of degree-3 in $G_u$ or $G_v$, respectively. Then applying Lemma 10 to the tree $T_{uv}$ with $S = \{u, v\}$ we have that $g_t(T_{uv}) \leq |V(G_u)| + |V(G_v)| + 2|S| - |N(S)| \leq 18 + 4 - 7 = 15$. Thus, $g_t(T) \leq g_t(T_{uv}) + (k-2)g_t(G) \leq 8k - 1$ while $\gamma_t(T) \leq k\gamma_t(G) = 6k$, and so $f_t(T) \leq 14k - 1 = 14n/9 - 1$. Hence we may assume that neither $u$ nor $v$ is the vertex of degree 3 in $G_u$ or $G_v$, respectively.

If $k = 2$, then $T = (T' \circ P_2) \circ P_2$ where $T' = P_2$ consists of the vertices $u$ and $v$, whence $T \in \mathcal{T}$. Hence we may assume that $k \geq 3$.

Assume that $F \cup (k-3)G$ is a spanning subgraph of $T$ where $F = P_9 \circ P_2$. Let $v_1, v_2, \ldots, v_9$ be the vertices from the path $P_9$ in $F$. Then applying Lemma 10 to the graph $F$ with $S = \{v_2, v_3, v_6, v_7\}$ we obtain $g_t(F) \leq 27+8-12 = 23$. Thus, $g_t(T) \leq g_t(F)+(k-3)g_t(G) \leq 8k-1$ while $\gamma_t(T) \leq k\gamma_t(G) = 6k$, and so $f_t(T) \leq 14k - 1 = 14n/9 - 1$. Hence we may assume that $(P_9 \circ P_2) \cup (k-3)G$ is not a spanning subgraph of $T$. It follows that the degree of every vertex in $G_u \cup G_v$, different from $u$ and $v$, is unchanged in $T$. Thus for $x \in \{u, v\}$, if $T_x = (T'_x \circ P_2) \circ P_2$ for some tree $T'_x$, then we have that $u \in V(T'_u)$ and $v \in V(T'_v)$. This implies that $T = (T' \circ P_2) \circ P_2$ where $T'$ is the tree $T'_u \cup T'_v \cup \{uv\}$. Thus, $T \in \mathcal{T}$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T \in \mathcal{T}$. □

**Observation 5** *If an edge of $T$ is oriented in both directions, then $f_t(T) \leq 14n/9$ with equality if and only if $T = (P_1 \circ P_2) \circ P_2$.*

**Proof.** Suppose that an edge $uv \in E$ is oriented in both directions. Hence both components $T_u$ and $T_v$ of $T - uv$ are contained in $\mathcal{S}$. Since both $T_u$ and $T_v$ are paths, $g_t(T_u) + g_t(T_v) = n$.

By Observation 2, $g_t(T) \leq n-1$, and so $g_t(T) \leq g_t(T_u) + g_t(T_v) - 1$.

Since $T$ is not a path, $\deg_T(u) \geq 3$ or $\deg_T(v) \geq 3$. If both $\deg_T(u) \geq 3$ and $\deg_T(v) \geq 3$, then applying Lemma 10 to the tree $T$ with $S = \{u, v\}$, we have $g_t(T) \leq n-2 = g_t(T_u) + g_t(T_v) - 2$. Thus since $\gamma_t(T) \leq \gamma_t(T_u) + \gamma_t(T_v)$, we have by Lemma 13 that $f_t(T) \leq f_t(T_u) + f_t(T_v) - 2 \leq (3|V(T_u)| + 2)/2 + (3|V(T_u)| + 2)/2 - 2 = 3n/2 < 14n/9$.

Hence we may assume that either $\deg_T(u) \geq 3$ or $\deg_T(v) \geq 3$, but not both. We may assume that $\deg_T(u) \geq 3$, and so $\deg_T(v) \leq 2$. By our assumption following Observation 3, we have that $T_v \in \{P_1, P_2, P_3\}$.

Suppose $T_v = P_1$, and so $|V(T_u)| = n-1$. If there is a $\gamma_t(T_u)$-set containing $u$, then $\gamma_t(T) \leq \gamma_t(T_u)$, implying that $f_t(T) \leq \gamma_t(T_u) + g_t(T) \leq (|V(T_u)|+2)/2 + n - 1 = (3n-1)/2 < 14n/9$. On the other hand, if there is no $\gamma_t(T_u)$-set containing $u$, then, by Lemma 14, $T_u = P_7$ and $u$ is the central vertex of this $P_7$. But then $n = 8$, $\gamma_t(T) = 5$ and $g_t(T) \leq n - 1 = 7$, implying that $f_t(T) \leq 12 = 3n/2 < 14n/9$. Hence we may assume that $T_v \in \{P_2, P_3\}$.

As observed earlier, $g_t(T) \leq g_t(T_u) + g_t(T_v) - 1$. Thus, $f_t(T) \leq f_t(T_u) + f_t(T_v) - 1$. Hence, by Lemma 13, $f_t(T) \leq (3n + \ell)/2$ where $\ell$ denotes the number of even components of $T - uv$. If $\ell = 0$, then $f_t(T) \leq 3n/2 < 14n/9$, as desired. Hence we may assume that $\ell \in \{1, 2\}$.

Suppose that $\ell = 1$, and so $f_t(T) \leq (3n+1)/2$. If $n > 9$, then $f_t(T) < 14n/9$. Hence we may assume that $n \leq 9$. Suppose firstly that $P_v = P_2$ and $T_u$ is of odd order. If $T_u \neq P_7$ or if $T_u = P_7$ but $u$ is not the central vertex of $P_u$, then there is a $\gamma_t(T_u)$-set containing $u$, and so $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)|+1)/2 + 1 + n - 1 < 3n/2 < 14n/9$. Hence we may assume that $T_u = P_7$ and that $u$ is the central vertex of $T_u$. But then $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$. Suppose secondly that $P_v = P_3$. Then, since $n \leq 9$, $P_u = P_6$. By our assumption following Observation 3, the vertex $u$ is a not a support vertex of $P_u$. But then again $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$.

Suppose finally that $\ell = 2$. Then, $T_v = P_2$ and $T_u \in \{P_2, P_6, P_{10}, P_{14}\}$. Since there is a $\gamma_t(T_u)$-set containing $u$, we have $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)| + 2)/2 + 1 + n - 1 = 3n/2 < 14n/9$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T = (P_1 \circ P_2) \circ P_2$.That proves Observation 5. □

By Observations 4 and 5, we may assume that every edge of $T$ is oriented in exactly one direction. Since $T$ is a tree, it follows that there exist a vertex $v$ with out-degree zero in this oriented tree. Thus for every edge $uv$ in $T$, $T_u \in \mathcal{S}$ and $T_v \notin \mathcal{S}$. If $v$ is a leaf and $u$ the support vertex adjacent with $v$, then $T_v = P_1 \in \mathcal{S}$ in $T - uv$, and so $v$ would have out-degree one in the oriented tree, a contradiction. Hence, $\deg_T(v) \geq 2$.

If every neighbor of $v$ in $T$ has degree at most two we define $I = 0$; otherwise, we define $I = 1$. Applying Lemma 10 to the tree $T$ with $S = \{v\}$, we have $g_t(T) \leq n + 2 - \deg_T(v)$. If $I = 1$, and $u$ is a neighbor of $v$ with $\deg_T(u) \geq 3$, then applying Lemma 10 to the tree $T$ with $S = \{u, v\}$, we have $g_t(T) \leq n + 4 - \deg_T(u) - \deg_T(v) \leq n + 1 - \deg_T(v)$. Hence we have the following observation.

**Observation 6** $g_t(T) \leq n + 2 - \deg_T(v) - I$.

If $v$ is adjacent only to vertices that are isolated in $T - v$ or leaves of a $P_5$ in $T - v$ or the central vertices of a $P_7$ in $T - v$, then we define $J = 1$; otherwise, we define $J = 0$. For a graph $G$, let $\mathrm{oc}(G)$ denote the number of odd components of $G$ and $\mathrm{ec}(G)$ the number of even components of $G$, and let $k_2(G)$ denotes the number of $P_2$-components in $G$. Then it follows from Lemmas 1 and 14 that

$$\gamma_t(T) \leq \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + J,$$

and if $k_2(T - v) \geq 1$, then

$$\gamma_t(T) \leq \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + 1 - k_2(T-v).$$

Hence, by Observation 6 and since $\deg_T(v) = \mathrm{ec}(T-v) + \mathrm{oc}(T-v)$, we have the following two upper bounds on $f_t(T)$.

**Observation 7** $f_t(T) \leq \frac{3n}{2} + \frac{3}{2} - \frac{\mathrm{oc}(T-v)}{2} - I + J$.

**Observation 8** *If* $k_2(T - v) \geq 1$, *then* $f_t(T) \leq \frac{3n}{2} + \frac{5}{2} - \frac{\mathrm{oc}(T-v)}{2} - I - k_2(T-v)$.

We proceed further with three observations.

**Observation 9** *If* $J = 1$, *then* $f_t(T) < 14n/9$.

**Proof.** Suppose $J = 1$. Then $\mathrm{oc}(T - v) = \deg_T(v) \geq 2$. By our assumption following Observation 3 there can be no $P_5$-component of $T - v$. Hence, $v$ is adjacent only to vertices that are isolated in $T - v$ or to the central vertices of a $P_7$ in $T - v$. If $T$ is a star, then the result follows from Lemma 12. Hence we may assume that $v$ is adjacent to the central vertex of a $P_7$ in $T - v$. But then $I = 1$. Thus, by Observation 7, we have that $f_t(T) \leq 3n/2 + (3 - \deg_T(v))/2$. If $\deg_T(v) \geq 3$, then $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\deg_T(v) = 2$, and so $f_t(T) \leq (3n + 1)/2$. If one component of $T - v$ is $P_1$ and the other one is $P_7$ with central vertex $u$, we have that $T_v = P_2 \in \mathcal{S}$, contradicting the fact that $v$ has out-degree zero in the oriented tree. Hence both components of $T - v$ are $P_7$-components, and so $n = 15$, whence $f_t(T) \leq (3n + 1)/2 < 14n/9$. □

**Observation 10** *If* $I = J = 0$, *then* $f_t(T) \leq 14n/9$ *with equality if and only if* $T = (P_1 \circ P_2) \circ P_2$.

**Proof.** Suppose $I = J = 0$. Then every neighbor of $v$ in $T$ has degree at most two. By our assumption following Observation 3 every component of $T - v$ is therefore isomorphic to $P_1, P_2$ and $P_3$ (and so, $\mathrm{ec}(T - v) = k_2(T - v)$). Since $T$ is not a path, $\deg_T(v) \geq 3$. If $T - v$ has no $P_3$-component, then by Lemma 12, $f_t(T) < 14n/9$. Hence we may assume that $T - v$ has a $P_3$-component. If $\mathrm{oc}(T - v) \geq 3$, then by Observation 7, $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\mathrm{oc}(T - v) \leq 2$. If $k_2(T - v) \geq 2$, then by Observation 8, $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $k_2(T - v) \leq 1$. Thus, since $\deg_T(v) \geq 3$, we have that $\mathrm{oc}(T - v) = 2$ and $k_2(T - v) = 1$. Since $v$ has out-degree zero in the oriented tree, there can be no $P_1$-component in $T - v$. Hence, $T - v$ consists of one $P_2$-component and two $P_3$-components and $v$ is adjacent to a leaf in each of these components. Thus, $T = (P_1 \circ P_2) \circ P_2$. □

**Observation 11** *If $I = 1$ and $J = 0$, then $f_t(T) < 14n/9$.*

**Proof.** Suppose $I = 1$ and $J = 0$. Then, by Observation 7, $f_t(T) \leq 3n/2 + (1 - \mathrm{oc}(T - v))/2$. If $\mathrm{oc}(T - v) \geq 1$, then $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\mathrm{oc}(T - v) = 0$, and so $f_t(T) \leq (3n + 1)/2$. If $n \leq 9$, then since $v$ by assumption is adjacent to a vertex $u$ of degree at least 3 in $T$, it follows that $T - v = P_2 \cup P_6$. But then if we consider the edge $uv$ we have that $T_v = P_3 \in \mathcal{S}$, contradicting the fact that $v$ has out-degree zero in the oriented tree. Hence, $n > 9$, whence $f_t(T) \leq (3n + 1)/2 < 14n/9$. □ The proof of Theorem 1 now follows from Observations 9, 10 and 11. □

# 4 Proof of Theorem 2

## 4.1 Preliminary Results

**Lemma 15** *If $T$ is a tree of order $n$ that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k + 1$ vertices, where $k \geq 0$, by attaching paths isomorphic to $P_1$ or $P_2$ to vertices in $\{v_1, v_3, \ldots, v_{2k+1}\}$ such that $\deg_T v_{2i+1} = 3$ for each $i \in \{0, \ldots, k\}$, then $f_t(T) < 3n/2$.*

**Proof.** We proceed by induction on $k$. If $k = 0$, then $T$ is a star or a subdivided star and the result follows from Lemma 12 and if $k = 1$, then $T$ is one of six small trees (of orders 7, 8, 9, 9, 10, 11) and the result is straightforward to check. This establishes the base cases. Hence we may assume that $k \geq 2$ and that the result of the theorem is true for all trees that can be obtained from a path on $2k' + 1$ vertices where $0 \leq k' < k$. Let $T$ be a tree of order $n$ that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k + 1$ vertices by the procedure described in the statement of the lemma.

We now consider the forest $F = T - v_3v_4$. Let $F_1$ and $F_2$ be the components of $F$ containing $v_3$ and $v_4$, respectively. Then, $F_1 \neq (P_1 \circ P_2) \circ P_2$ and $F_1$ is a tree with at least six and at most nine vertices and with three leaves, one vertex of degree 3, and with the remaining vertices of degree 2. Thus $F_1 \notin S$ and it follows from Theorem 1

that $f_t(F_1) \leq 3|V(F_1)|/2$. Applying the inductive hypothesis to the tree $F_2$, we have $f_t(F_2) < 3|V(F_2)|/2$. Hence, $f_t(T) \leq f_t(F_1) + f_t(F_2) < 3n/2$. □

**Lemma 16** *For $n \geq 3$, $f_t(C_n) \leq 3n/2$ with equality if and only if $n \equiv 0 \pmod 4$.*

**Proof.** Let $G = C_n$, and let $V_1$ and $V_2$ be a partition of $V(G)$ satisfying $f_t(G) = f_t(G; V_1, V_2)$. Suppose that both $V_1$ and $V_2$ are open packings in $G$. Let $i \in \{1, 2\}$. Since no two vertices of $V_i$ have a common neighbor, every vertex in $G[V_i]$ has degree one and the set of edges $[V_1, V_2]$ therefore induces a matching in $G$. Thus since $G$ is 2-regular, we must have that $|V_1| = |V_2|$, $[V_1, V_2]$ induces a perfect matching in $G$, and that $G[V_i]$ is $K_2$ or the disjoint union of copies of $K_2$. Hence, $n \equiv 0 \pmod 4$.

If $n$ is odd, then at least one of the sets $V_1$ and $V_2$ is not an open packing in $G$, and so, by Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. By Lemma 1, $\gamma_t(C_n) = (n + 1)/2$ for $n$ odd. Hence, $f_t(G) \leq (3n - 1)/2$. Therefore we may assume that $n$ is even.

Suppose $n \equiv 2 \pmod 4$. Then, by Lemma 1, $\gamma_t(C_n) = (n + 2)/2$. If $V_1$ or $V_2$ is empty, then $f_t(G) \leq 2\gamma_t(C_n) = n + 2 < 3n/2$ since $n \geq 6$. Suppose $|V_1| = 1$. Then, $G[V_2] = P_{n-1}$, and so $\gamma_t(G, V_2) \leq \gamma_t(G[V_2], V_2) \leq \gamma_t(G[V_2]) = \gamma_t(P_{n-1}) = n/2$, implying that $f_t(G) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2) \leq (n + 2)/2 + 1 + n/2 = n + 2 < 3n/2$. Hence we may assume that $|V_1| \geq 2$ and $|V_2| \geq 2$.

For $i \in \{1, 2\}$, if there are two adjacent vertices with the same label $i$, then $\gamma_t(G, V_{3-i}) \leq \gamma_t(P_{n-2}) = (n - 2)/2$. Hence if both sets $V_1$ and $V_2$ contain adjacent vertices, then $f_t(G) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2) \leq (n + 2)/2 + n - 2 = (3n - 2)/2$. Thus we may assume that at least one of $V_1$ and $V_2$, say $V_1$, is an independent set. This implies that $V_2$ is not an open packing, and so $\gamma_t(G, V_2) \leq |V_2| - 1$. If $V_1$ is not an open packing, then $\gamma_t(G, V_1) \leq |V_1| - 1$, implying that $f_t(G) \leq (n + 2)/2 + |V_1| + |V_2| - 2 = (3n - 2)/2$. Hence we may assume that $V_1$ is both an independent set and an open packing. Thus since the vertices in the set $V_1$ have disjoint neighborhoods in $G$, $N(V_1) \subseteq V_2$ and $|N(V_1)| = 2|V_1|$. For each vertex $v \in V_2 \setminus N(V_1)$, we choose an adjacent vertex and call the resulting set of such vertices $V_2'$. Then, $V_1 \cup V_2'$ totally dominates $V_2$, and so $\gamma_t(G, V_2) \leq |V_1| + |V_2'| \leq |V_1| + |V_2 \setminus N(V_1)| = |V_1| + |V_2| - |N(V_1)| = |V_2| - |V_1|$. Thus since $\gamma_t(G, V_1) = |V_1|$ and $\gamma_t(G) = (n + 2)/2$, we have that $f_t(G) \leq (n+2)/2+|V_2| \leq (n+2)/2+n-2 = (3n-2)/2$. Hence if $n \equiv 2 \pmod 4$, then $f_t(G) \leq (3n - 2)/2 < 3n/2$.

Suppose, finally, that $n \equiv 0 \pmod 4$. Then, by Lemma 1, $\gamma_t(C_n) = n/2$. Since there is a good partition of $V(G)$ in this case, $g_t(G) = n$, implying that $f_t(G) = 3n/2$. □

**Lemma 17** *For $n \geq 3$, let $G = C_n$ where $n \equiv 0 \pmod 4$, and let $V_1$, $V_2$ be a partition of $V(G)$. Then, $f_t(G; V_1, V_2) \leq 3n/2$ with equality if and only if $V_1$, $V_2$ is a good partition of $V(G)$.*

**Proof.** By Lemma 16, $f_t(G; V_1, V_2) \leq f_t(G) = 3n/2$. If $V_1$, $V_2$ is not a good partition of $V(G)$, then $V_1$ or $V_2$ is not an open packing in $G$, and so, by Lemma 6, $\gamma_t(G, V_1)+\gamma_t(G, V_2) \leq$

$n-1$. Together with Lemma 1, $\gamma_t(G) = n/2$, we obtain $f_t(G; V_1, V_2) \leq 3n/2-1$. Conversely, if $V_1$, $V_2$ is a good partition of $V(G)$, then both $V_1$ and $V_2$ are open packings in $G$, implying by Lemma 6 that $\gamma_t(G, V_1) + \gamma_t(G, V_2) = n$, whence $f_t(G; V_1, V_2) = 3n/2$. □

**Lemma 18** *If $G$ is a graph of order $n$ that can be obtained from a cycle $v_0, v_1, \ldots, v_{2k-1}, v_0$ on $2k$ vertices, where $k \geq 2$, by attaching for each $i \in \{0, 1, \ldots, k-1\}$ a path isomorphic to $P_1$ or $P_2$ to $v_{2i}$, then $f_t(G) < 3n/2$.*

**Proof.** Let $G = (V, E)$. If $k = 2$, then $G$ is one of three graphs (of orders 6, 7 and 8) and the result is straightforward to check. Hence we may assume that $k \geq 3$. Let $i \in \{0, 1, \ldots, k-1\}$ and let $F_i$ and $G_i$ be the components of $G - \{v_{2i-1}v_{2i}, v_{2i+2}v_{2i+3}\}$ containing $v_{2i}$ and $v_{2i-1}$, respectively (where addition is taken modulo $2k$). Then, $F_i$ is a path of order 5, 6 or 7, while $G_i$ is a tree that can be obtained from a path on $2(k-3)+1$ vertices by the procedure described in the statement of the Lemma 15. By Lemma 15, $f_t(G_i) < 3|V(G_i)|/2$.

Let $V_1$, $V_2$ be a partition of $V$ such that $f_t(G; V_1, V_2) = f_t(G)$. For $j = 1, 2$, let $V_{i,j} = V_j \cap V(F_i)$. Suppose that $V_{i,1}$, $V_{i,2}$ is not a good partition of $V(F_i)$. Then, by Lemma 9, $f_t(F_i; V_{i,1}, V_{i,2}) \leq 3|V(F_i)|/2$. Thus, $f_t(G) = f_t(G; V_1, V_2) \leq f_t(F_i; V_{i,1}, V_{i,2}) + f_t(G_i) < 3|V(F_i)|/2 + 3|V(G_i)|/2 = 3n/2$. Hence we may assume that $V_{i,1}$, $V_{i,2}$ is a good partition of $V(F_i)$ for each $i \in \{0, 1, \ldots, k-1\}$, for otherwise the desired result follows.

Suppose that for some $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are isomorphic (either to $P_1$ or $P_2$). For notational convenience, we may assume that the small component of $G - v_0$ and the small component of $G - v_2$ are isomorphic. Let $T_1$ and $T_2$ be the components of $G - \{v_0v_{2k-1}, v_4v_5\}$ containing $v_0$ and $v_{2k-1}$, respectively. Then, $T_1$ is a tree with three leaves, with one vertex of degree 3, and with the remaining vertices of degree 2. Since $T_1$ is one of four small trees, and since $V_{i,1}$, $V_{i,2}$ is a good partition of $V(F_i)$ for every $i \in \{0, 1, \ldots, k-1\}$, and in particular for $i = 0, 1$, it is straightforward to check that $f_t(T_1) \leq 3|V(T_1)|/2$. If $k = 3$, then $V(T_2) = \{v_5\}$ and since there exists a $\gamma_t(T_1)$-set containing $v_0$, it follows that $f_t(G) \leq f_t(T_1) + 1 \leq 3(n-1)/2 + 1 < 3n/2$. If $k \geq 4$, then by Lemma 15, $f_t(T_2) < 3|V(T_2)|/2$, implying that $f_t(G) \leq f_t(T_1) + f_2(T_2) < 3|V(T_1)|/2 + 3|V(T_2)|/2 = 3n/2$.

Hence we may assume that for every $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are not isomorphic. Thus, $k$ must be even. We may assume that for $i \equiv 0 \,(\mathrm{mod}\, 4)$, $G - v_i$ has a component isomorphic to $P_2$ (and therefore for $i \equiv 2 \,(\mathrm{mod}\, 4)$, $G - v_i$ has a component isomorphic to $P_1$). Let $C$ denote the cycle in $G$ (of order $2k$). Let $H$ be the spanning subgraph of $G$ obtained from $G$ by deleting all edges on $C$ incident with vertices $v_i$ where $i \equiv 0 \,(\mathrm{mod}\, 4)$. Then, $H$ is isomorphic to $k/2$ disjoint copies of $P_3 \cup K_{1,3}$. Hence since $f_t(P_3 \cup K_{1,3}) = 10$, it follows that $f_t(G) \leq f_t(H) \leq 10|V(H)|/7 = 10n/7 < 3n/2$. □

## 4.2 Notation

Before proceeding with a proof of Theorem 2, we introduce some additional notation. By a proper subgraph of a graph $G$ we mean a subgraph of $G$ different from $G$. We define a vertex as ***small*** if it has degree $\leq 2$, and ***large*** if it has degree more than 2. In a graph $G$ let $\mathcal{L}$ denote the set of all its large vertices. Suppose $|\mathcal{L}| \geq 1$ and let $C$ be any component of $G - \mathcal{L}$; it is a path. If $C$ has only one vertex and that is adjacent to two large vertices, or if $C$ has at least two vertices and the two ends of $C$ are adjacent in $G$ to different large vertices, then we say that $C$ is a **2-*path***. Otherwise, when the ends of $C$ are adjacent to the same large vertex, we say that $C$ is a **2-*handle***.

## 4.3 Proof of Theorem 2

Recall Theorem 2.

**Theorem 2** *If $G$ is a connected graph of order $n$ with $\delta(G) \geq 2$, then $f_t(G) \leq 2n/3$ with equality if and only if $G \cong C_n$ where $n \equiv 0 \pmod 4$.*

**Proof.** We proceed by induction on $\ell = n + m$, where $m$ denotes the size of $G$. Note that $n \geq 3$ and $m \geq 3$, and so $\ell \geq 6$. When $\ell = 6$, the graph $G$ is a 3-cycle and $f_t(G) = 4 < 3n/2$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 7$ and assume that for all connected graphs $G'$ of order $n'$ and size $m'$ with $n' + m' < \ell$ and with $\delta(G') \geq 2$ that $f_t(G') \leq 2n'/3$ with equality if and only if $G' \cong C_{n'}$ where $n' \equiv 0 \pmod 4$.

So let $G = (V, E)$ be a connected graph of order $n$ and size $m$ with $m + n = \ell$ and with $\delta(G) \geq 2$. Suppose that $G$ contains at least one large vertex. Let $\mathcal{L}$ be set of all large vertices of $G$.

**Observation 12** *If $\mathcal{L}$ contains two adjacent vertices, then $f_t(G) < 3n/2$.*

**Proof.** Suppose that two large vertices $u$ and $v$ are adjacent. Let $G' = G - uv$. Then, $G'$ is a graph of order $n' = n$ and size $m' = m - 1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to every component of $G'$, we have that $f_t(G') \leq 3n'/2 = 3n/2$ with equality if and only if every component of $G'$ is a cycle of order congruent to zero modulo 4. By Lemma 3, $f_t(G) \leq f_t(G') \leq 3n/2$. Thus if $f_t(G') < 3n/2$, then $f_t(G) < 3n/2$. If $f_t(G') = 3n/2$, then every component of $G'$ is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$ for every partition $V_1$, $V_2$ of $V(G)$. Thus, $f_t(G) \leq 3n/2 - 1$. □

By Observation 12, we may assume that $\mathcal{L}$ is an independent set (for otherwise, the desired result follows).

**Observation 13** *If $G$ contains a path on six vertices each internal vertex of which has degree 2 in $G$ and whose end-vertices are not adjacent, then $f_t(G) < 3n/2$.*

**Proof.** Let $u$ and $v$ be the two end-vertices of a path $P$ on six vertices each internal vertex of which has degree 2. Let $G'$ be the graph obtained from $G$ by removing the four internal vertices of this path and adding the edge $uv$. Then, $G'$ is a connected graph of order $n' = n - 4$ and size $m' = m - 4$ with $\delta(G') \geq 2$. Applying the inductive hypothesis to $G'$, we have that $f_t(G') \leq 3n'/2 = 3n/2 - 6$ with equality if and only if $G'$ is a cycle of order congruent to zero modulo 4. Since the degree of every large vertex of $G$ remains unchanged in $G'$, $\Delta(G') \geq 3$, implying that $f_t(G') < 3n/2 - 6$.

Let $V_1$, $V_2$ be a partition of $V$, and let $P$ be the path $u, u_1, u_2, u_3, u_4, v$. Thus, $G' = (G - \{u_1, u_2, u_3, u_4\}) \cup \{uv\}$. Let $i \in \{1, 2\}$ and let $V_i' = V(G') \cap V_i$. Let $U \subseteq V(G')$ and let $S'$ be a minimum set of vertices in $G'$ that totally dominates $U$ in $G'$, and so $|S'| = \gamma_t(G', U)$. If $\{u, v\} \subseteq S'$, let $S = S' \cup \{u_1, u_4\}$. If $\{u, v\} \cap S' = \emptyset$, let $S = S' \cup \{u_2, u_3\}$. If $u \in S'$ and $v \notin S'$, let $S = S' \cup \{u_3, u_4\}$. If $u \notin S'$ and $v \in S'$, let $S = S' \cup \{u_1, u_2\}$. In all cases, $|S| = |S'| + 2$ and $S$ totally dominates $U \cup V(P)$ in $G$. In particular, if $U = V(G')$, then $S'$ is a $\gamma_t(G')$-set and $S$ is a TDS of $G$, whence $\gamma_t(G) \leq |S| = |S'| + 2 = \gamma_t(G') + 2$. If $U = V_i'$, then $S$ totally dominates $V_i$ in $G$, and so $\gamma_t(G, V_i) \leq |S| = |S'| + 2 = \gamma_t(G', V_i') + 2$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 < 3n/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. □

By Observation 13, we may assume that $G$ contains no path on six vertices each internal vertex of which has degree 2 in $G$ and whose end-vertices are not adjacent. Hence since $\mathcal{L}$ is an independent set, we have the observation.

**Observation 14** *Every 2-path contains at most three vertices, while every 2-handle contains at most five vertices.*

**Observation 15** *If $G$ contains a degree-3 vertex that is adjacent to the ends of a 2-handle, then $f_t(G) < 3n/2$.*

**Proof.** Assume that there is a degree-3 vertex $v$ that is adjacent to the ends of a 2-handle $C$. By Observation 14, $2 \leq |C| \leq 5$. By connectivity there exists a 2-path $P$ with an end adjacent to $v$. Let $u$ be the other large vertex adjacent with an end of $P$. By Observation 14, $1 \leq |P| \leq 3$. Let $G'$ be the spanning subgraph of graph obtained from $G$ by removing the edge joining $u$ with an end of $P$. Let $G_u$ and $G_v$ be the components of $G'$ containing $u$ and $v$, respectively. Let $|V(G_u)| = n_u$ and $|V(G_v)| = n_v$, and so $n = n_u + n_v$. Now, $\delta(G_u) \geq 2$ while $G_v$ is a key $L_{r,s}$ where $r = |C| + 1$ and $s = |P|$. Hence, $3 \leq r \leq 6$ and $1 \leq s \leq 3$. Thus, by Lemma 2, $\gamma_t(G_v) \leq (n_v + 1)/2$. By Lemma 11, $\gamma_t(G_v, V_1) + \gamma_t(G_v, V_2) \leq n_v - 1$ for every partition $V_1$, $V_2$ of $V(G_v)$. Thus, $f_t(G_v) \leq (3n_v - 1)/2$. Applying the inductive hypothesis to the graph $G_u$, $f_t(G_u) \leq 3n_u/2$. Hence, $f_t(G') = f_t(G_u) + f_t(G_v) \leq (3n-1)/2$. Thus, by Lemma 3, $f_t(G) \leq f_t(G') < 3n/2$. □

By Observation 15, we may assume that every large vertex that is adjacent to the ends of a 2-handle has degree at least 4.

**Observation 16** *If $G$ contains a 2-handle of order 2, 4 or 5, then $f_t(G) < 3n/2$.*

**Proof.** Suppose there is a 2-handle $C$ where $|C| = k$ and $k \in \{2, 4, 5\}$. Say its ends have common neighbor $v \in \mathcal{L}$. By assumption, $\deg_G v \geq 4$. Let $G' = G - V(C)$. Then, $G'$ is a connected graph of order $n' = n - k$ and size $m' = m - k - 1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to $G'$, we have that $f_t(G') \leq 3n'/2 = 3(n-k)/2$ with equality if and only if $G'$ is a cycle of order congruent to zero modulo 4.

Let $V_1$, $V_2$ be a partition of $V$ and for $i \in \{1, 2\}$, let $V_i' = V(G') \cap V_i$. Let $U \subseteq V'(G)$ and let $S'$ be a minimum set of vertices in $G'$ that totally dominates $U$ in $G'$, and so $|S'| = \gamma_t(G', U)$.

Suppose $k = 2$. Then, $S \cup \{v\}$ totally dominates $U \cup V(C)$ in $G$. It follows that $\gamma_t(G) \leq \gamma_t(G') + 1$, and for $i \in \{1, 2\}$, $\gamma_t(G, V_i) \leq \gamma_t(G', V_i') + 1$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 3 \leq f_t(G') + 3 \leq 3n/2$. If $f_t(G') < 3(n-2)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-2)/2$, then $G'$ is a cycle (congruent to zero modulo 4). But then we can choose a $\gamma_t(G')$-set to contain $v$, implying that $\gamma_t(G) \leq \gamma_t(G')$ and $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 2 \leq f_t(G') + 2 \leq 3n/2 - 1$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 4$. Let $C$ be the path $v_1, v_2, v_3, v_4$. Then, $S \cup \{v_2, v_3\}$ totally dominates $U \cup V(C)$ in $G$. It follows that $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 \leq 3n/2$. If $f_t(G') < 3(n-4)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-4)/2$, then $G'$ is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n'/2 = (n-4)/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$, and so $f_t(G; V_1, V_2) \leq 3n/2 - 1$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 5$. Let $C$ be the path $v_1, v_2, v_3, v_4, v_5$. For $i = 1, 2$, let $W_i = V_i \cap V(C)$. If $W_1$, $W_2$ is not a good partition of $V(C)$, then by Lemma 8, $f_t(C; W_1, W_2) \leq 3(k-1)/2 = 7$. Thus, $f_t(G; V_1, V_2) \leq f_t(C; W_1, W_2) + f_t(G'; V_1', V_2') \leq 7 + f_t(G') \leq 7 + 3(n-5)/2 = (3n-1)/2$. On the other hand, suppose that $W_1$, $W_2$ is a good partition of $V(C)$. Thus, renaming the sets $V_1$ and $V_2$ if necessary, we may assume that $W_1 = \{v_1, v_2, v_5\}$ (that is, the labels of $v_1, v_2, v_3, v_4, v_5$ are given by $1, 1, 2, 2, 1$, respectively). But then $\{v, v_1\}$ totally dominates $W_1$ in $G$, $\{v_3, v_4\}$ totally dominates $W_2$ in $G$, and $\{v, v_3, v_4\}$ totally dominates $V(C)$ in $G$. Hence, $f_t(G; V_1, V_2) \leq 7 + f_t(G'; V_1', V_2') \leq 7 + 3(n-5)/2 = (3n-1)/2$. Thus for every partition $V_1$, $V_2$ of $V$, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. □

By Observations 14 and 16, we have the observation.

**Observation 17** *Every 2-handle contains three vertices.*

We now construct a spanning subgraph $H$ of $G$ as follows. First from every 2-handle (of order 3) and every 2-path that contains two or three vertices, we delete exactly one edge (both of whose ends necessarily have degree 2). Thus in the resulting graph, there is no 2-handle and every 2-path, if any, has order 1. We then successively delete an edge that joins the single vertex of a 2-path with a large vertex of degree at least 4 in the graph obtained at each stage until no such edge remains. (Thus if a large vertex in the graph constructed at this stage is adjacent with the vertex of a 2-path, then this large vertex has degree 3.)

Finally in the resulting graph, we successively delete two of the three edges incident with every large vertex all of whose neighbors are vertices of 2-paths (of order 1) in the resulting graph at each stage until no such large vertex remains. Let $H$ denote the resulting spanning subgraph of $G$.

By construction, $H$ has no 2-handle and every 2-path in $H$, if any, has order 1. Further, every large vertex of $H$ that is adjacent to the vertex of a 2-path has degree 3 and has at least one neighbor (of degree 1 or 2) that is not on any 2-path. (Thus no large vertex is adjacent to the ends of more than two 2-paths.) Each leaf in $H$ is either adjacent to a large vertex of $H$ or is adjacent to a degree-2 vertex that is adjacent to a large vertex of $H$. It follows that every component $H'$ of the spanning subgraph $H$ of $G$ is isomorphic to one of the graphs described in Lemmas 12, 15 or 18: If $H'$ contains only one large vertex, then $H'$ is one of the graphs described in Lemma 12 (stars with possible subdivisions). If the vertices of $H'$ that belong to 2-paths (of order 1) and their neighbors (the large vertices in $H'$) induce a path in $H'$, then $H'$ is one of the graphs described in Lemma 15 (paths with pendants). If the vertices of $H'$ that belong to 2-paths and their neighbors induce a cycle in $H'$, then $H'$ is one of the graphs described in Lemma 18 (cycles with pendants). Hence by Lemma 3, and by Lemmas 12, 15 or 18, it follows that $f_t(G) \leq f_t(H) < 3n/2$.

Hence we have shown that if $G$ contains at least one large vertex, then $f_t(G) < 3n/2$. If $G$ contains no large vertex, then $G$ is a cycle, and the desired result follows from Lemma 16. □

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