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Szmidt, Jan Kazimerz

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STRUCTURAL RELIABILITY THEORY
PAPER NO. 37

JAN KAZIMIERZ SZMIDT
DISCRETE ANALYSIS OF A PLANE INITIAL-VALUE PROBLEM FOR AN OFFSHORE STRUCTURE

JANUARY 1988

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0. INTRODUCTION

Offshore structures are generally subjected to hydrodynamic sea forces, wind forces and sea-bed forces. The quantities such as compression forces, velocities and displacements of a structure etc. are of random nature and they are more or less correlated to each other. The general problem of describing structure-fluid interaction is a difficult task, even in the case of deterministic formulation and harmonic vibration of the structure. The difficulties result from the complicated form of the governing differential equations of fluid flow which are non-linear and from boundary conditions that should be satisfied. In a general case all the quantities such as compression forces, velocities of the fluid particles, displacements of the structure are functions of three space independent variables and are dependent on time. Because of the difficulties, the usual way of analysing such a phenomenon is to construct appropriate simplified models which describe some important features of behaviour. The simplifications are clearly related to a specified problem and they are based on our knowledge and experience in the field mentioned. For example in the analysis of surface water waves it is usually assumed the fluid is incompressible and non-viscous. Moreover, for the gravity waves, the commonly used assumption is that the fluid flow is irrotational, so that the velocity potential exists from which the velocity field may be derived. Such approximations lead to accurate results for many practical problems concerning surface wave propagation. Of course, there are also regions of our interest where the compressibility and or viscosity of the fluid must be taken into account. Surface wave breaking, box-drop problems and flow in a boundary layer are examples of the latter cases.

With respect to certain approximations it is possible to find closed analytical solutions to the problems mentioned - for special cases only. For many complicated geometrics encountered in practice it is not possible to find closed analytical solutions and therefore, it is necessary to resort to numerical methods. The discrete methods such as finite difference and finite element methods are universal to some extent and they are very useful in practical applications. On the other hand, with these methods only a finite number of nodal points can be considered. Thus, the discrete methods are not directly applicable to infinite systems which occur in wave propagation problems. Nevertheless, it is possible to apply the methods to infinite systems by means of special boundary conditions - so-called radiation or transmitting conditions assumed on an artificial boundary. Such operation enables us to consider a finite domain only and to obtain solutions having properties of solutions for infinite systems. It should be stressed, however, that in general a discrete model may have properties different from those of the continuum. Such differences are important when the discretization leads to solutions which have no physical meaning but result from the discretization. To apply the discrete methods to solve a given problem it is reasonable to examine them first for the case of relatively simple problem for which analytical solutions are available.

In the following we will confine our attention to the two-dimensional problem of a generation of surface waves in water of constant depth. The problem clearly corresponds to a generation of the surface waves by a wave-maker in a hydraulic channel. The simpliest case is the harmonic generation of the waves. After sufficiently long time from the starting point and far enough from the source of disturbances (from the wave-maker) the free surface elevation is repeated for a constant period of time. Thus, we can say that surface elevation is a steady surface wave and we can treat the problem as being steady-state one. But even in the case of harmonic generation it is not easy to solve the initial problem of forming a surface wave for the system starting from rest. Most analytical solutions of such unsteady problems available in the literature of hydrodynamics [3,6] describes the velocity potentials and free surface elevations far enough from sources of disturbances (from the wave-maker). The reason for such descriptions is that solutions of linearized problems of such kind have singularity in the domain of disturbances.

The aim of the present work is to solve the initial moving wall problem for a layer of fluid with the help of the finite difference method. Special attention will be paid to the explanation of singularity in the velocity field which results from approximation to the mathematical description of the phenomenon rather than from physical background. We will also discuss some properties of algebraic equations resulting from the method of solving the problem. In the following we apply some necessary approximations and we consider some auxiliary problems which will hopefully finally result in the solution we are looking for.

1 FORMULATION OF THE PROBLEM

Let us consider the semi-infinite layer of fluid shown in Fig. 1. The flow of the fluid is induced by horizontal vibrations of a rigid wall OA. It is assumed all of the system fluid and of the wall are at rest for $t \leqslant 0$.

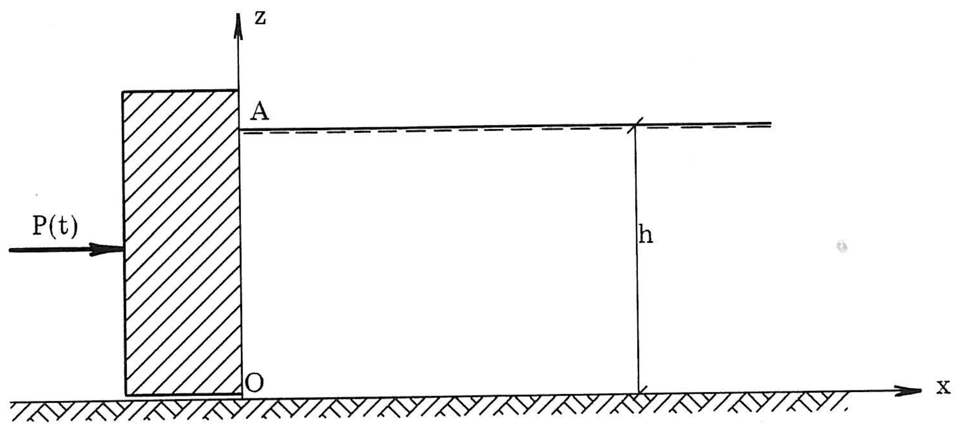


Fig. 1 Semi-infinite layer of fluid.

We will consider the following displacement of the wall:

$$\underset{\sim}{x} = d \cdot (1 - \cos \omega t), \tag{1.1}$$

where d is the amplitude and ω - is the circular frequency of vibrations, respectively.

In a more general case it is possible to substitute the amplitude "d" for a function, say f(z) which varies in the span OA. Without loss of generality it is possible to assume $d = 1$. Upon differentiating (1.1) with respect to time we obtain:

$$v_x = \dot{\underset{\sim}{x}} = d \cdot \omega \cdot \sin\omega t$$
$$a_x = a = \ddot{\underset{\sim}{x}} = d \cdot \omega^2 \cdot \cos\omega t \qquad (1.2)$$

In the last relations dots denote derivatives with respect to time. It is seen that the displacement $\underline{x}$ and the velocity v_x are both equal to zero at $t = 0$. At the same instant, the acceleration of the wall is equal to the finite value $a = d \cdot \omega^2$. Such situation corresponds to the case of a body which is suddenly loaded with a given force. Our aim is to describe the velocity field and the pressure distribution within the layer as well as the free surface elevation as functions of time. To do this we consider first the differential equations of the fluid motion together with appropriate boundary conditions. For the considered plane problem of an isotropic, compressible Newtonian fluid the momentum equations are [3]:

$$\frac{Du}{Dt} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} + \frac{1}{3}\nu \cdot \frac{\partial \Theta}{\partial x} + \nu \cdot \nabla^2 u$$
$$\frac{Dv}{Dt} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + \frac{1}{3}\nu \cdot \frac{\partial \Theta}{\partial z} + \nu \cdot \nabla^2 v - g \qquad (1.3)$$

where:

$\frac{D}{Dt}$ means material derivative, u and v are velocity components, p is the pressure, ρ is the fluid density, g is the gravitational acceleration, ν is the kinematic viscosity of the fluid and Θ is the rate of dilatation:

$$\Theta = \operatorname{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \qquad (1.4)$$

The momentum equations are supplemented by the equation of continuity:

$$\frac{D\rho}{Dt} + \rho \cdot \operatorname{div}\vec{v} = 0 \qquad (1.5)$$

In discussion of the generation of surface waves the small compressibility of the fluid and its viscosity are neglected and it is assumed that the flow is irrotational. For such case the equations (1.3) assume the form:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}\vec{v}^2\right) = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial z} \cdot \left(\frac{1}{2}\vec{v}^2\right) = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} - g \qquad (1.6)$$

and the equation (1.5) becomes:

$$\operatorname{div}\vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \qquad (1.7)$$

The assumption of irrotational flow leads to the velocity potential $\Phi(x,z,t)$ from which it follows that

$$\vec{v} = \operatorname{grad}\Phi, \rightarrow u = \frac{\partial \Phi}{\partial x} = \Phi,x \;, \quad v = \frac{\partial \Phi}{\partial z} = \Phi,z \qquad (1.8)$$

Substitution of the first (1.8) into (1.7) gives the Laplace equation:

$$\nabla^2\Phi = \Phi_{xx} + \Phi_{zz} = 0 \tag{1.9}$$

It is seen that the velocity potential Φ should satisfy the linear Laplace equation and Φ is thus a harmonic function. By using (1.8) it is readily verified that the equations of motion (1.6) can be written in a form of one vector equation which, after integration leads to well known Bernoulli's equation:

$$\dot{\Phi} + \frac{1}{2}\cdot(\vec{v})^2 + \frac{1}{\rho}\cdot p + g\cdot z = 0 \tag{1.10}$$

It is perhaps of importance to emphasize here that although the Laplace equation is linear, the momentum equations (1.6) and finally the Bernoulli equation are non-linear. Now, the boundary conditions can be written. For the free surface the kinematic condition [6] is:

$$\Phi_z = \dot{\eta} + \eta_x \cdot \Phi_x \tag{1.11}$$

and the dynamic condition:

$$[\dot{\Phi} + \frac{1}{2}(\Phi_x^2 + \Phi_z^2) + g\cdot\eta + \frac{1}{\rho}\cdot p_o]/_{z=\eta} = 0, \tag{1.12}$$

where $\eta(x,t)$ describes the free surface elevation and p_0 is the atmospheric pressure. The condition (1.11) means that a particle of the fluid being on the free surface at one instant remains on it. The second condition states that the fluid pressure on the freee surface equals the atmospheric pressure. In the following we assume $p_o = 0$ in the relation (1.12). At the bottom of the layer there is no flow through the boundary and thus we have:

$$\frac{\partial\Phi}{\partial z} = \Phi_z /_{z=0} = 0 \tag{1.13}$$

On the boundary $x = 0$, the fluid particles and the wall have the same horizontal velocity:

$$\frac{\partial\Phi}{\partial x}/_{x=0} = v_x \tag{1.14}$$

In fact, the last equation describes the linearized boundary condition, which in a general case should be written for an actual position of the wall:

$$\frac{\partial\Phi}{\partial x}/_{x=\underset{\sim}{x}(t)} = v_x \tag{1.14a}$$

To complete the boundary conditions, the Sommerfeld condition must be added so that the potential is bounded when going to infinity ($x \to \infty$)and no waves come from infinity.

Solution of the equation (1.9) satisfying prescribed boundary conditions will be performed in a numerical way for a discrete sequence of time steps $\Delta t, 2\cdot\Delta t, \dots$. For the procedure it is very important to have proper values of all quantities entering the basic equation (1.9) and the boundary conditions (1.11) ÷ (1.14) at the instant $t = \Delta t$. In other words it is necessary to calculate the surface elevation and the values of velocity potential at the first instant Δt, just after the starting point. To do this it is necessary to take into account the small compressibility of the fluid.

2. STEADY SOLUTIONS FOR COMPRESSIBLE FLUID

Let us consider now the problem of fluid flow which starts from a rest. At present, we are mainly interested in obtaining a solution for a short interval of time $0 \div \Delta t$ where Δt means the time passing from the beginning. At the instant $t = 0^+$ displacement of the wall and its velocity are both equal to zero. Hence, the fluid in the layer is at rest. At the fixed time 0^+ the only non-vanishing quantity is the acceleration of the wall which equals $a = d\omega^2$. We can assume that at the same instant of time the fluid is loaded with an unknown pressure distribution on the boundary $x = 0$, or in other words, the fluid is loaded with acceleration given on the boundary. During the time considered the fluid behaves as an elastic solid and the dilatational wave begins to propagate through the fluid. The speed of the wave - the sound wave is very high and, for example for water it equals $c_d = (1460 \div 1500)$ m/ sec. After a sufficiently long time from the starting point (as a matter of fact, not a very long time because of the high speed of the wave) the front of the wave will reach the lower and upper boundary of the layer (the bottom and the free surface which is flat at the moment). Then, the wave will reflect from the boundaries and the process will repeat several times. Hence, if Δt is long enough the reflections of dilatational waves will change the velocities of fluid particles (the velocities are much smaller than the sound velocity) in such a way that roughly speaking the state of the fluid flow approaches the steady state for a given period of time. Such explanation results from physical considerations. To obtain quantitative results for $t > 0^+$ it would be necessary to solve the initial -value problem for propagation of the dilatational wave. Fortunately, it is not necessary for our purposes. Of course the above explanation is plausible only for a small value of time interval Δt when the arising velocity and associated displacements of the fluid particles on the free surface are so small that they can be ignored. On the other hand, the interval Δt must be wide enough to ensure multiple reflections of the propagating waves from the boundaries. The reflections create standing dilatational waves and in the following we will assume that at a fixed time Δt we will deal with the standing wave only. A good criterion for choosing the proper Δt seems to be the condition that the propagating dilatational wave should cover the characteristic distance of a problem (in our case the layer depth) at least ten times, or better twenty times. Of course, such procedure cannot be applied for very large water depths. The procedure mentioned above is illustrated in Fig. 2 where the plots of acceleration, velocity and displacement of a fluid particle are given.

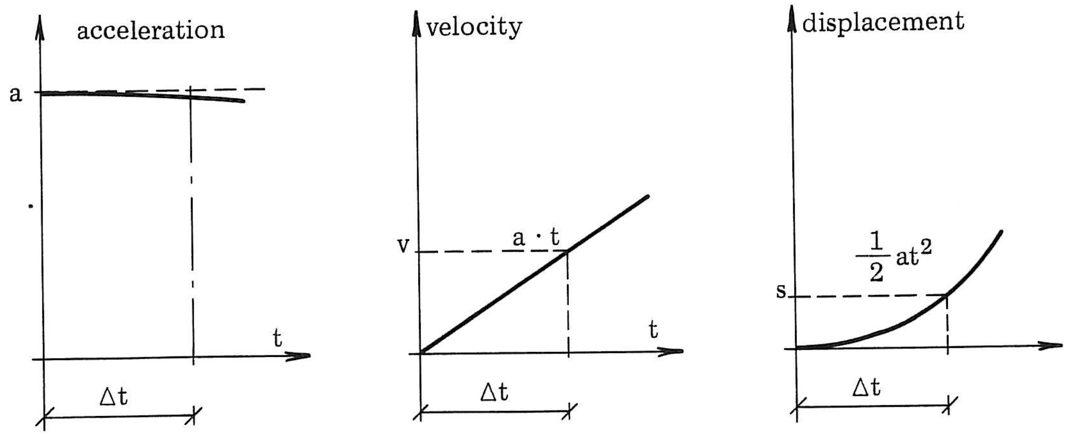


Fig. 2. Kinematic quantities for a fluid particle.

It is seen that the displacement of a fluid particle is a small quantity of the second order with respect to acceleration for small values of Δt. Thus, the assumption that the free surface of the layer remains unchanged in the interval Δt is justified. Following this the velocity field within the layer at instant Δt under condition of constant pressure on the surface z = h can be calculated (see Fig. 1). The last assumption will result in a number of simplifications in our further analysis. The approximation will be supported by analytical considerations in the next paragraph.

In accordance with the above discussion let us consider the case of small disturbances propagating in the layer of the fluid. It is assumed that the fluid is compressible and non-viscous. Neglecting the convective terms and the gravity force in (1.3) we have:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} \qquad (2.1)$$

For small changes of fluid density we may establish the relation [1]:

$$\rho = \rho_o \cdot (1 + S) = \rho_o \cdot (1 + \frac{\Delta p}{K}) \qquad (2.2)$$

where S is the condensation, ρ_o is the density for the undisturbed state of the fluid and K is the bulk modulus. The last relation is illustrated in Fig. 3.

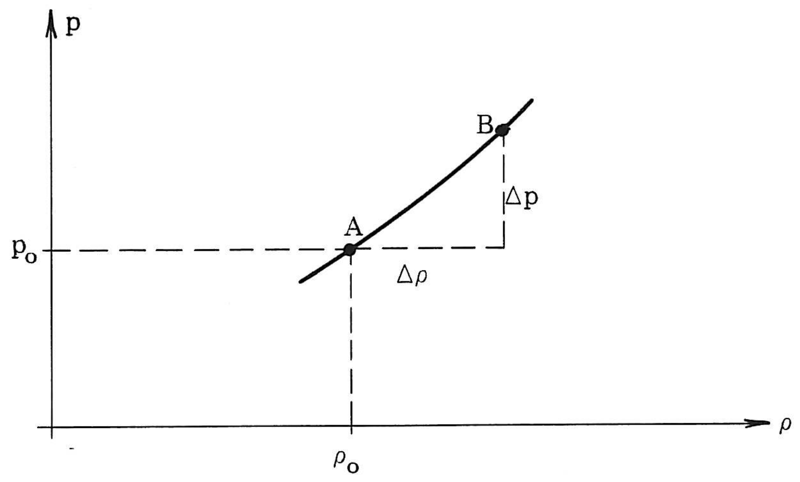


Fig. 3. Pressure-density relation.

Besides the following equation holds:

$$\left(\frac{\Delta p}{\Delta \rho}\right)_{\rho = \rho_o} \cong \left(\frac{dp}{d\rho}\right)_{\rho = \rho_o} = c^2 = \frac{K}{\rho_o} = \text{const.} \tag{2.3}$$

where c is the velocity of sound in the fluid.
From (2.2) and (2.3) the following can be seen:

$$p = p_o + \rho_o \cdot c^2 \cdot S \tag{2.4}$$

Using the latter relations we can write the approximated continuum equation [1,2]:

$$\frac{\partial S}{\partial t} + \Theta = 0, \tag{2.5}$$

where Θ is the rate of dilatation defined by (1.4). Upon differentiation of (2.5) with respect to time the formula

$$\frac{\partial^2 S}{\partial t^2} + \frac{\partial \Theta}{\partial t} = \frac{\partial^2 S}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial t}\right) = 0 \tag{2.6}$$

can be derived. Substitution of (2.1) and (2.4) into (2.6) and simple manipulation lead to the result:

$$\nabla^2 S - \frac{1}{c^2} \cdot \ddot{S} = 0 \tag{2.7}$$

Thus, we have obtained the wave equation for condensation. By means of (2.4) the latter equation may be transformed into the wave equation for pressure:

$$\nabla^2 p - \frac{1}{c^2} \cdot \ddot{p} = 0 \tag{2.8}$$

The equations (2.7) and (2.8) describe the propagation of small disturbances (small changes) of condensation (density) and pressure within isotropic fluid with the constant velocity c. In compliance with our previous considerations it is possible to introduce the velocity potential Φ and to transform (2.7) to the form:

$$\nabla^2\Phi - \frac{1}{c^2}\ddot{\Phi} = 0\,, \tag{2.9}$$

where:

$$\vec{v} = \text{grad}\ \Phi,\quad p = -\rho\cdot\dot{\Phi} \tag{2.10}$$

It is seen that if one takes the limit $c \to \infty$, the wave equation (2.9) will be reduced to the Laplace equation for incompressible fluid as it should be. A formal conclusion for the case $c \to \infty$ is that the interval Δt is going to zero and we are released from the approximations associated with proper choosing of $\Delta t > 0$.

However, the small compressibility of the fluid does not give rise to significant change of many results in many practical cases, but it enable us to learn more about the problem and finally to make proper approximations of the original task.

In accordance with the discussion let us consider now the problem of steady state harmonic vibrations of the wall in Fig. 1. Let the velocity of the wall be described by the first equation (1.2). For steady-state harmonic vibrations it is convenient to introduce the spatial velocity potential φ (x, z) according to the formula:

$$\Phi(x,z,t) = \varphi(x,z)\cdot e^{i\omega t}\,,\ \text{or}\ \Phi(x,z,t) = \varphi(x,z)\cdot \sin\omega t \tag{2.11}$$

Substitution of the last relation into (2.9) yields:

$$\nabla^2\varphi + k_c^2\cdot\varphi = 0,\quad k_c = \frac{\omega}{c} \tag{2.12}$$

It is seen that the problem has been reduced to the Helmholtz equation in space variables. The classical solution of (2.12) is obtained by separation of variables which leads to the result:

$$\varphi(x,z) = -\omega\cdot\sum_{j=1}^{\infty} A_j\cdot\frac{1}{r_j}\cdot e^{-r_j\cdot x}\cdot\cos k_j z \tag{2.13}$$

where A_j are constants and:

$$r_j = \sqrt{k_j^2 - k_c^2} = k_j\cdot\sqrt{1-\beta^2\cdot\left[\frac{2}{(2\cdot j-1)\,\pi}\right]^2}\,,\quad \beta = \frac{\omega\cdot h}{c} < \frac{\pi}{2}$$

$$k_j = \frac{2\cdot j-1}{2\cdot h}\cdot\pi,\quad j = 1,2,\ldots \tag{2.14}$$

The dimensionless parameter β corresponds to the Mach number for a fluid. The assumption $\beta < \pi/2$ is not a serious obstacle for problems discussed here. For $\beta > \pi/2$ (very high frequency of vibrations or very large fluid depth) one can obtain a propagating dilatational wave. Obviously, the solution (2.13) represents standing dilatational wave which decays exponetially with increasing values of $x > 0$. In the following it is assumed that for all cases considered $\beta < \pi/2$. The constants A_j (j = 1,2, . . .) of the solution are to be found from the boundary condition at x = 0. Simple procedure leads to the result:

$$A_j = \frac{2 \cdot d}{h} \cdot \int_0^h \cos k_j z \, dz = \frac{2 \cdot d}{k_j h} \cdot (-1)^{j+1}, \quad j = 1,2\ldots, \tag{2.15}$$

or, according to (2.14):

$$A_j = \frac{4 \cdot d}{\pi} \cdot \frac{(-1)^{j+1}}{(2 \cdot j - 1)}, \quad j = 1, 2, \ldots \tag{2.16}$$

For small values of β the second term under the square root can be disregarded and it can be assumed that

$$r_j \simeq k_j = \frac{2 \cdot j - 1}{2h} \pi, \; j = 1, 2, \ldots \tag{2.17}$$

Such approximation means that the small compressibility of the fluid has been neglected ($c \to \infty$, $k_c \to 0$). For the case, the velocity potential assumes the form:

$$\Phi = -\frac{8 \cdot d \cdot h \cdot \omega}{\pi^2} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)^2} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \sin \omega t \tag{2.18}$$

The solution obtained is the solution of the Laplace equation (1.9) for the condition of zero pressure on the line z = h (see Fig. 1). According to (2.10) the pressure is given by:

$$p = -\rho \cdot \dot{\Phi} = \frac{8 \cdot \rho \cdot d \cdot h \cdot \omega^2}{\pi^2} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)^2} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \cos \omega t, \tag{2.19}$$

where the constant hydrostatic pressure - p_o in (2.4) is omitted. The series (2.18) and (2.19) are absolutely convergent at all points of the semi-infinite layer of fluid. Hence, upon differentiation of the series (2.18) with respect to x and z we get:

$$u = \frac{4 \cdot d \cdot \omega}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \sin \omega t$$

$$v = \frac{4 \cdot d \cdot \omega}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)} \cdot e^{-k_j x} \cdot \sin k_j z \cdot \sin \omega t \tag{2.20}$$

where $d \cdot \omega$ means the amplitude of the velocity of the wall OA.

In a simiilar way formulae for the components of acceleration within the fluid can be obtained:

$$a_x = \ddot{x} = \frac{4 \cdot a}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \cos \omega t$$

$$a_z = \ddot{z} = \frac{4 \cdot a}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)} \cdot e^{-k_j x} \cdot \sin k_j z \cdot \cos \omega t \tag{2.21}$$

where $a = d \cdot \omega^2$ is the acceleration amplitude of the wall OA. The second series in (2.20) and (2.21) is divergent at the point (x = 0, z = h). To define the character of the singularity at the point the formulae given in reference [10] can be used:

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot (u)^n \cdot \cos \frac{n\pi x}{l} = -\frac{1}{2} \cdot \log \left(1 - 2 \cdot u \cdot \cos \frac{\pi \cdot x}{l} + u^2\right), \quad |u| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \cdot (u)^n \cdot \sin \frac{n\pi x}{l} = \arctan \frac{u \cdot \sin \frac{\pi \cdot x}{l}}{1 - u \cdot \cos \frac{\pi \cdot x}{l}}, \quad |u| < 1 \tag{2.22}$$

Using the latter relations, equations (2.20) may be rewritten in the closed analytical form:

$$u = \frac{2 \cdot d \cdot \omega}{\pi} \cdot [\tan^{-1} \frac{\cos\alpha z}{e^{\alpha x} + \sin\alpha z} + \tan^{-1} \frac{\cos\alpha z}{e^{\alpha x} - \sin\alpha z}] \cdot \sin\omega t$$

$$v = \frac{d \cdot \omega}{\pi} \cdot \log(\frac{\cosh\alpha x + \sin\alpha z}{\cosh\alpha x - \sin\alpha z}) \cdot \sin\omega t, \quad \alpha = \frac{\pi}{2h} \qquad (2.23)$$

where $\tan^{-1}$ means the inverse of tan.

It is seen that the vertical component v of the velocity field has logarithmic singularity at the point (x = 0, z = h). Although the solution obtained corresponds to the incompressible fluid ($c = \infty$) the main properties of it are valid also for compressible fluid. The singularity is the result of several approximations in the description of a real fluid where the infinite value of the considered quantity cannot be expected from physical considerations. The reason of the singularity in the velocity field seems to be followed by the assumption that the fluid is non-viscous. To examine the problem the subsidiary problem for incompressible viscous fluid will be considered.

3. HALF-PLANE OF A VISCOUS FLUID

Consider a half-plane of a viscous fluid with a prescribed velocity on its boundary (see Fig. 4.). The attention is confined to the linearized theory of Newtonian fluid and small disturbances within it.

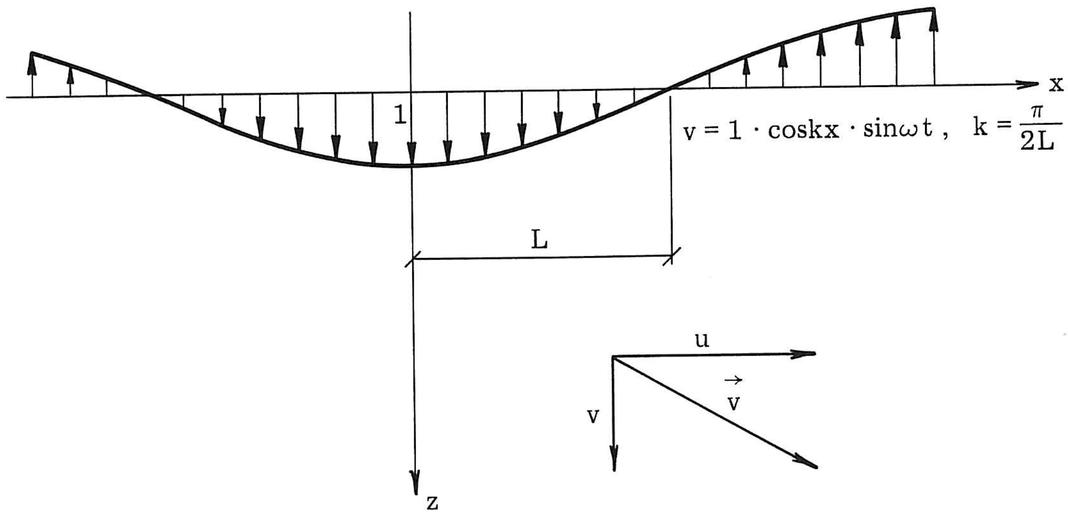


Fig. 4. Half-plane of a viscous fluid

It is assumed that the boundary condition for $z = 0, -\infty \leqslant x \leqslant \infty$, is:

$$u = 0$$
$$v = 1 \cdot \cos kx \cdot \sin\omega t \quad , \qquad k = \pi/2L \tag{3.1}$$

Neglecting the small compressibility of the fluid and convective terms in Narier-Stokes equations (1.3):

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial x} + \nu \cdot \nabla^2 u$$
$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \cdot \frac{\partial p}{\partial z} + \nu \cdot \nabla^2 v + g \tag{3.2}$$
$$\Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0$$

Following H. Lamb [3] the velocity components can be decomposed according to the formulae:

$$u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial z} , \quad v = \frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x} , \tag{3.3}$$

where Φ is the potential and Ψ is the rotational part of velocity field, respectively. Substitution of (3.3) into the third (3.2) gives:

$$\Theta = \nabla^2 \Phi = 0 \tag{3.4}$$

Thus, the function Φ should satisfy the Laplace equation. From the second equation in (3.2) upon integration with respect to z, it follows that:

$$\frac{1}{\rho} \cdot \int_0^z \frac{\partial p}{\partial z} dz = g \cdot z + \int_0^z [\nu \cdot \nabla^2 v - \frac{\partial v}{\partial t}] dz + C_1 \tag{3.5}$$

where C_1 is a constant of integration.

The integrand of the right hand side term in (3.5) may be expressed in the form:

$$\nu \cdot \nabla^2 v - \frac{\partial v}{\partial t} = \frac{\partial}{\partial x} [\dot{\Psi} - \nu \cdot \nabla^2 \Psi] + \frac{\partial}{\partial z} [\nabla^2 \Phi - \dot{\Phi}] =$$
$$= \frac{\partial}{\partial x} [\dot{\Psi} - \nu \cdot \nabla^2 \Psi] - \frac{\partial}{\partial z} (\dot{\Phi}) \tag{3.6}$$

Substitution of (3.6) into (3.5) yields:

$$\frac{1}{\rho} \cdot p /_0^z = g \cdot z - \dot{\Phi} /_0^z + \int_0^z \frac{\partial}{\partial x} [\dot{\Psi} - \nu \nabla^2 \Psi] dz + C_1 \tag{3.7}$$

According to Lamb [3] it is assumed that:

$$p = -\rho \cdot \dot{\Phi} + \rho \cdot g \cdot z + C_1 , \tag{3.8}$$

and:

$$\dot{\Psi} - \nu \cdot \nabla^2 \Psi = 0 \tag{3.9}$$

For our further purposes it is justified to neglect the hydrostatic pressure and to write:

$$\rho \cdot g \cdot z + C_1 = 0 \qquad (3.10)$$

Now, steady-state harmonic solutions of the equations (3.4) and (3.9) are sought. For the case it is convenient to make the following substitutions:

$$\Phi = \varphi_1 \cdot \cos\omega t + \varphi_2 \cdot \sin\omega t$$
$$\Psi = \Psi_1 \cdot \cos\omega t + \Psi_2 \cdot \sin\omega t \qquad (3.11)$$

Substitution of the first (3.11) into (3.4) leads to the conclusion that both the spatial functions $\varphi_1(x,z)$ and $\varphi_2(x,z)$ should satisfy the Laplace equation. Taking the boundary conditions (3.1) into account a solution of φ_1 (or φ_2) is sought in the form:

$$\varphi(x,z) = f_1(z) \cdot e^{ikx}, \quad k = \frac{\pi}{2L} \qquad (3.12)$$

Substituting (3.12) into (3.4) the ordinary differential equation is obtained:

$$\frac{d^2 f_1}{dz^2} - k^2 \cdot f_1 = 0 \qquad (3.13)$$

which has the solution:

$$f_1(z) = A \cdot e^{-kz}, \qquad (3.14)$$

where A is a constant.

With reference to the last result and substitutions (3.12) the solution of (3.4) is:

$$\Phi = e^{-kz}(A_1 \cos kx \cos\omega t + A_2 \cdot \cos kx \cdot \sin\omega t +$$
$$+ A_3 \cdot \sin kx \cdot \cos\omega t + A_4 \cdot \sin kx \cdot \sin\omega t), \qquad (3.15)$$

where $A_1, \ldots, A_4$ are constants of integration.

Substitution of the second (3.11) into (3.9) yields:

$$\nabla^2\Psi_1 - \frac{\omega}{\nu} \cdot \Psi_2 = 0$$
$$\nabla^2\Psi_2 + \frac{\omega}{\nu} \cdot \Psi_1 = 0 \qquad (3.16)$$

We can eliminate Ψ_1 (or Ψ_2) from the latter equations, but such operation will raise the order of differentiation in (3.16).

The effect of such a procedure is:

$$\nabla^2\nabla^2\Psi - \left(\frac{\omega}{\nu}\right)^2 \cdot \Psi = 0, \qquad (3.17)$$

where $\Psi(x,z)$ is a common description for Ψ_1 and Ψ_2. The latter equation is exactly the same as the one in Kinsman's book [2] in a discussion of a similar problem. Now, a solution of (3.17) is sought, which is to be of the form:

$$\Psi(x,z) = f(z) \cdot e^{ikx}, \quad k = \frac{\pi}{2L} \tag{3.18}$$

which is analogous to (3.12).

Substitution of (3.18) into (3.17) leads to the fourth-order ordinary differential equation:

$$f'''' - 2 \cdot k^2 \cdot f'' + (k^4 + \frac{\omega^2}{\nu^2}) \cdot f = 0 \tag{3.19}$$

where the primes denote the derivatives with respect to "z". Simple, but time-consuming manipulations give the solution of equation (3.19):

$$f(z) = B_1 \cdot e^{rz} + B_2 \cdot e^{-rz} + B_3 \cdot e^{r^{\star}z} + B_4 \cdot e^{-r^{\star}z} \quad , \tag{3.20}$$

where r and $r^{\star}$ are complex conjugate numbers:

$$r = a + ib \quad , \quad r^{\star} = a - ib \tag{3.21}$$

with real and imaginary parts:

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2\nu}} \cdot \sqrt{\sqrt{1+\beta^2}+\beta} \cong \sqrt{\frac{\omega}{2\nu}} \cdot (1 + \frac{1}{2}\beta) \\ b &= \sqrt{\frac{\omega}{2\nu}} \cdot \sqrt{\sqrt{1+\beta^2}-\beta} \cong \sqrt{\frac{\omega}{2\nu}} \cdot (1 - \frac{1}{2}\beta) \end{aligned} \tag{3.22}$$

where:

$$\beta = \frac{\nu}{\omega} \cdot k^2 \, , \quad k^2 = a^2 - b^2 \, , \quad 2ab = \frac{\omega}{\nu} \, , \quad k = \frac{2\pi}{L} \tag{3.23}$$

Substitution of (3.20) into (3.18) gives:

$$\Psi(x,z) = (B_1 \cdot e^{rz} + B_2 \cdot e^{-rz} + B_3 \cdot e^{r^{\star}z} + B_4 \cdot e^{-r^{\star}z}) \cdot e^{ikx} \tag{3.24}$$

Solutions for Ψ_1 and Ψ_2 (equations (3.16)) have the same form. Since the functions are not independent, they are coupled through the equations (3.16), we should substitute two independent solutions of the form (3.24) into the equations (3.16). Further, only the terms in (3.24) which decrease when going to infinity ($z \to \infty$) are chosen. Finally, after some rearrangements, the solutions assume the form:

$$\begin{aligned} \Psi_1(x,z) &= e^{-az}[(D_1 \cdot \cos bz + D_2 \cdot \sin bz) \cdot \cos kx + (D_3 \cdot \cos bz + D_4 \cdot \sin bz) \cdot \sin kx] \\ \Psi_2(x,z) &= e^{-az}[(D_1 \cdot \sin bz - D_2 \cdot \cos bz) \cdot \cos kx + (D_3 \cdot \sin bz - D_4 \cdot \cos bz) \cdot \sin kx] \end{aligned} \tag{3.25}$$

Substitution of (3.25) into (3.11) and then into boundary conditions (3.1) by means of (3.15) gives the final solution of the problem:

$$\Phi = e^{-kz} \cdot \cos kx \cdot [A \cdot \cos\omega t + B \cdot \sin\omega t]$$
$$\Psi = e^{-az} \cdot \sin kx [C \cdot \cos(bz - \omega t) + D \cdot \sin(bz - \omega t)] \quad , \tag{3.26}$$

where:

$$A = \frac{b}{2a} \cdot \frac{1}{a-k} , \quad B = -(\frac{1}{k} + \frac{1}{2a}),$$
$$C = -\frac{b}{2a} \cdot \frac{1}{a-k} = -A, \quad D = -\frac{1}{2a} \tag{3.27}$$

For a given amplitude of the velocity on the line z = 0, the constants (3.27) have to be multiplied by the excitation amplitude. It is important to note that the function $\Psi(x,z,t)$ describes the damped propagation (damped diffusion) of the rotational part of the velocity field from the boundary into the fluid. Since $\omega/2\nu$ is a relatively large number (for water $\nu \simeq 1.3 \cdot 10^{-2}\ cm^2 \cdot sec^{-1} \rightarrow$ [5]), the second solution (3.26) quickly decays with increasing values of z. In fact, the so–called boundary layer of thickness $\delta = \sqrt{2\nu/\omega}$ is obtained in which there is a very high reduction of the motion amplitude. To get a better insight into the problem discussed the vortex of the flow is considered.

$$\vec{\Omega} = \mathrm{curl}\, \vec{v} \tag{3.28}$$

For the plane problem there is only one non-zero component of $\vec{\Omega}$:

$$\Omega_1 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial z} = -\nabla^2 \Psi \tag{3.29}$$

It can be seen that the function Ψ represents the rotational part of the solution and it is the solution of the diffusion equation (3.9). Insertion of (3.9) into (3.29) gives:

$$\Omega_1 = -\frac{1}{\nu} \cdot \dot{\Psi} \tag{3.30}$$

Thus, the second equation (3.26) describes the diffusion of the vortex into the fluid. The vortex is damped exponentially with the depth z. The main reduction of the vortex takes place in the boundary layer of thickness $\delta = \sqrt{2\nu/\omega}$.

Substitution of (3.26) into (3.3) gives:

$$u = -\sin kx \ \{k \cdot e^{-kz} \cdot (A \cdot \cos\omega t + B \cdot \sin\omega t) +$$
$$+ e^{-az} [(a \cdot C - b \cdot D) \cdot \cos(bz - \omega t) + (b \cdot C + a \cdot D) \cdot \sin(bz - \omega t)] \} , \tag{3.31}$$
$$v = -k \cdot \cos kx \cdot \ \{e^{-kz} \cdot (A \cdot \cos\omega t + B \cdot \sin\omega t) +$$
$$+ e^{-az} \cdot [C \cdot \cos(bz - \omega t) + D \cdot \sin(bz - \omega t)] \}$$

For $z \to \infty$, both components of the velocity field are going to zero. The solution obtained is periodic with reference to the x-axis. It is convenient to rewrite the first of (3.31) in the form:

$$u = u_1 \cdot \cos\omega t + u_2 \cdot \sin\omega t \tag{3.32}$$

The components u_1 and u_2 may be expressed as follows:

$$\begin{aligned} u_1 &= -\sin kx \cdot E \cdot [e^{-kz} \cdot \cos\alpha - e^{-az} \cdot \cos(bz - \alpha)] \\ u_2 &= -\sin kx \cdot E \cdot [e^{-kz} \cdot \sin\alpha + e^{-az} \cdot \sin(bz - \alpha)] \;, \end{aligned} \tag{3.33}$$

where the following substitutions have been made:

$$\begin{aligned} E \cdot \cos\alpha &= \frac{b}{2a} \cdot \frac{k}{a-k} \\ E \cdot \sin\alpha &= 1 + \frac{k}{2a} \end{aligned} \tag{3.34}$$

The relevant amplitude of the horizontal component of velocity is given by the formula:

$$|u| = E \cdot \sqrt{e^{-2kz} + e^{-2az} - 2 \cdot e^{-(a+k)z} \cdot \cos bz} \tag{3.35}$$

where

$$E = \frac{k}{2a} \cdot \sqrt{\left(\frac{b}{a-k}\right)^2 + \left(1 + \frac{2a}{k}\right)^2} \tag{3.36}$$

The amplitude assumes extremum values at points where its first derivative with respect to "z" equals zero. The resulting condition may be written as follows:

$$(a + k)\cos bz + b \cdot \sin bz - [k \cdot e^{(a-k)z} + a \cdot e^{-(a-k)z}] = 0 \tag{3.37}$$

Roots of the last equation may be found in a numerical way. The results obtained above are used for a construction of a solution of a similar problem for a layer of viscous fluid.

4. SEMI-INFINITE LAYER OF VISCOUS FLUID

Let us consider the semi-infinite layer of viscous fluid $0 \leqslant x \leqslant h$, $0 \leqslant z$ and excitation of the fluid motion by horizontal harmonic vibration of the rigid wall OA - Fig. 5. The layer with the assumed excitation is extended by means of its symmetrical and antisymmetrical reflections with respect to the lines $x = 0$ and $x = L$ in such a way that the final result is a half-plane of the fluid $z \geqslant 0$.

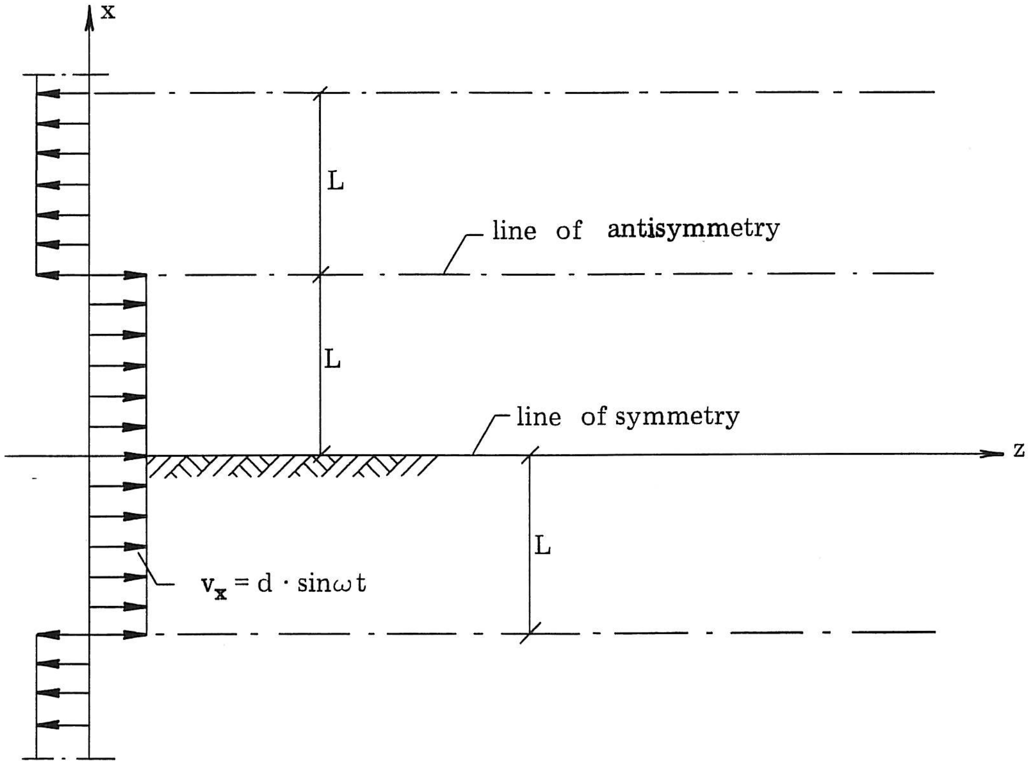


Fig. 5. Half-plane as an infinite set of semi-infinite layers.

The excitation of the fluid flow in the half-plane is similar to the one discussed previously, but now instead of the smooth curve coskx there is a periodical rectangular wave on the boundary z = 0. The latter is even function with respect to x = 0 and odd function with respect to x = L. The excitation function is the periodic function and it may be easily represented by the Fourier series:

$$d = \sum_{j=1}^{\infty} G_j \cdot \cos k_j x, \tag{4.1}$$

where:

$$G_j = \frac{4 \cdot d}{\pi} \cdot \frac{(-1)^{j+1}}{2 \cdot j - 1}, \quad k_j = \frac{2 \cdot j - 1}{2L} \cdot \pi, \qquad j = 1,2,\ldots \tag{4.2}$$

Substitution of (4.2) into (4.1) gives

$$d = \frac{4 \cdot d}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{2 \cdot j - 1} \cdot \cos k_j x \tag{4.3}$$

Finally, the velocity of excitement can be described by means of the formula:

$$v = \frac{4 \cdot d}{\pi} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)} \cdot \cos k_j x \cdot \sin \omega t \tag{4.4}$$

The problem is linear and we may apply the principle of superposition. Thus, the solution of equations (3.4) and (3.17) is expressed as a sum of solutions for each component of (4.4) which is treated independently from the others. Thus, for the j–th component of (4.4),

$$a_j = \sqrt{\frac{\omega}{2\nu}} \cdot \sqrt{\sqrt{1 + \beta_j^2} + \beta} \qquad b_j = \sqrt{\frac{\omega}{2\nu}} \cdot \sqrt{\sqrt{1 + \beta_j^2} - \beta}$$

$$\beta_j = \frac{\nu}{\omega} \cdot k_j^2 \,, \quad k_j = \frac{2 \cdot j - 1}{2L} \cdot \pi, \quad k_j^2 = a_j^2 - b_j^2 \,, \quad 2a_j b_j = \frac{\omega}{\nu} \tag{4.5}$$

and:

$$\Phi = \sum_{j=1}^{\infty} G_j \cdot e^{-k_j z} \cdot \cos k_j x \cdot (A_j \cdot \cos \omega t + B_j \cdot \sin \omega t)$$

$$\Psi = \sum_{j=1}^{\infty} G_j \cdot e^{-a_j z} \cdot \sin k_j x \cdot [C_j \cdot \cos(b_j z - \omega t) + D_j \cdot \sin(b_j z - \omega t)] \tag{4.6}$$

Instead of (3.27) the following relations hold:

$$A_j = \frac{b_j}{2a_j} \cdot \frac{1}{a_j - k_j} \,, \qquad B_j = -\left(\frac{1}{k_j} + \frac{1}{2a_j}\right)$$

$$C_j = -\frac{b_j}{2a_j} \cdot \frac{1}{a_j - k_j} = -A_j, \qquad D_j = -\frac{1}{2a_j} \tag{4.7}$$

Following (3.3) and (4.6) the components of fluid velocity are:

$$u = -\sum_{j=1}^{\infty} G_j \cdot \sin k_j x \cdot \{k_j \cdot e^{-k_j z} \cdot (A_j \cdot \cos \omega t + B_j \cdot \sin \omega t)$$
$$+ e^{-a_j z} \cdot [(a_j \cdot C_j - b_j \cdot D_j) \cdot \cos(b_j z - \omega t)$$
$$+ (b_j \cdot C_j + a_j \cdot D_j) \cdot \sin(b_j z - \omega t)]\} \tag{4.8}$$

$$v = -\sum_{j=1}^{\infty} G_j \cdot k_j \cdot \cos k_j x \, \{e^{-k_j z} \cdot (A_j \cdot \cos \omega t + B_j \cdot \sin \omega t)$$
$$+ e^{-a_j z} \cdot [C_j \cdot \cos(b_j z - \omega t) + D_j \cdot \sin(b_j z - \omega t)]\}$$

For $z = 0$, $u = 0$ and v is equal to (4.4). It is important for the further analysis to know if the series (4.8) are convergent within the region of consideration.. Let us investigate the problem for the vertical component of the velocity field. In compliance with the description (3.32) the first equation (4.8) leads to the components:

$$u_1 = \sum_{j=1}^{\infty} G_j \cdot \sin k_j x \cdot [(1 + \frac{k_j}{2a_j}) \cdot e^{-a_j z} \cdot \sin b_j z + $$
$$+ \frac{b_j k_j}{2a_j} \cdot \frac{1}{a_j - k_j} \cdot (e^{-a_j z} \cdot \cos b_j z - e^{-k_j z})] \,, \tag{4.9}$$

$$u_2 = \sum_{j=1}^{\infty} G_j \cdot \sin k_j x \cdot [\frac{b_j \cdot k_j}{2a_j} \cdot \frac{1}{a_j - k_j} \cdot e^{-a_j z} \cdot \sin b_j z$$
$$- (1 + \frac{k_j}{2a_j}) \cdot (e^{-a_j z} \cdot \cos b_j z - e^{-k_j z})]$$

To examine the convergence of the series they are divided into typical components. In view of (4.2) the components are:

$$S_1 = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{k_j} \cdot e^{-a_j z} \cdot \sin k_j x \cdot \begin{cases} \sin b_j z \\ \cos b_j z \end{cases}$$

$$S_2 = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{a_j} \cdot e^{-a_j z} \cdot \sin k_j x \cdot \begin{cases} \sin b_j z \\ \cos b_j z \end{cases} \tag{4.10}$$

$$S_3 = \sum_{j=1}^{\infty} (-1)^{j+1} \cdot \frac{b_j}{a_j} \cdot \frac{1}{a_j - k_j} \cdot e^{-a_j z} \cdot \sin k_j x \cdot \begin{cases} \sin b_j z \\ \cos b_j z \end{cases}$$

and,

$$S_4 = \sum_{j=1}^{\infty} (-1)^{j+1} \cdot e^{-k_j z} \cdot \sin k_j x \cdot \begin{cases} \frac{1}{k_j} \\ \frac{1}{a_j} \\ \frac{b_j}{a_j} \cdot \frac{1}{a_j - k_j} \end{cases}$$

In the following it will be shown that the series (4.9) are absolutely convergent. First of all, it is seen that $u_1 = u_2 = 0$ for $z = 0$. Thus the points $z = 0$ are excluded from further considerations. It is also seen that $j \to \infty$, $k_j \to \infty$, $\beta_j \to \infty$, $a_j \to \infty$ and $b_j \to 0$ in such a way that $a_j \cdot b_j = \omega/2\nu$. Now, consider the expression corresponding to the first of (4.10) series whose terms are real and non-negative:

$$T_1 = \sum_{j=1}^{\infty} |(-1)^{j+1} \cdot \sin k_j x \cdot \sin b_j z| \cdot \frac{1}{k_j} \cdot e^{-a_j z} \tag{4.11}$$

One can see that:

$$T_1 \leqslant \sum_{j=1}^{\infty} \frac{1}{k_j} \cdot e^{-a_j z} = \sum_{j=1}^{\infty} \frac{1}{k_j} \cdot e^{-k_j \cdot \epsilon_j z} = v_1 \,, \tag{4.12}$$

where:

$$a_j = k_j \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{\sqrt{1 + (1/\beta_j)^2} + 1} = k_j \cdot \epsilon_j \tag{4.13}$$

and

$\epsilon_j > 1$ for $j = 1, 2, \ldots$

Using the Taylor expansion of the exponential function entering (4.12) the following can be obtained:

$$v_1 = \sum_{j=1}^{\infty} \frac{1}{k_j} \cdot \frac{1}{k_j z} \cdot \frac{1}{[\epsilon_j + \frac{1}{k_j z} + \frac{1}{2!}(k_j z) \cdot \epsilon_j^2 + \frac{1}{3!}(k_j \cdot z)^2 \cdot \epsilon_j^3 + \ldots]} =$$

$$= \frac{1}{z} \cdot \sum_{j=1}^{\infty} \frac{1}{k_j^2} \cdot \frac{1}{R_j} < \frac{1}{z} \cdot \sum_{j=1}^{\infty} \frac{1}{k_j^2} = \tag{4.14}$$

$$= \frac{1}{z} \cdot (\frac{2L}{\pi})^2 \sum_{j=1}^{\infty} \frac{1}{(2_j - 1)^2} < \frac{1}{z} \cdot (\frac{2L}{\pi})^2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

where:

$$z > 0 \quad \text{and} \quad R_j > 1 \tag{4.15}$$

Thus, the series S_1(4.10) is absolutely convergent.

The similar procedure for the series S_2 (second equation 4.10) gives:

$$T_2 = \sum_{j=1}^{\infty} |(-1)^{j+1} \sin k_j x \cdot \sin b_j z| \cdot \frac{1}{a_j} \cdot e^{-a_j z} \leqslant \sum_{j=1}^{\infty} \frac{1}{a_j} \cdot e^{-a_j z} =$$

$$= \sum_{j=1}^{\infty} \frac{1}{\epsilon_j \cdot k_j} \cdot e^{-\epsilon_j k_j z} = v_2 \tag{4.16}$$

and:

$$v_2 = \sum_{j=1}^{\infty} \frac{1}{\epsilon_j} \cdot \frac{1}{k_j} \cdot \frac{1}{k_j z} \cdot \frac{1}{R_j} < \frac{1}{z} \cdot (\frac{2L}{\pi})^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{4.17}$$

where ϵ_j and R_j are defined by (4.13) and (4.14), respectively. Hence, the series S_2 is also absolutely convergent. The third series (4.10) may be transformed into the following:

$$S_4 = \frac{2 \cdot \nu}{\omega} \cdot \sum_{j=1}^{\infty} (-1)^{j+1} \sin k_j x \cdot \sin b_j z \cdot (a_j + k_j) \cdot e^{-a_j z} \tag{4.18}$$

The series corresponding to (4.18) with non-negative terms is:

$$T_3 = \sum_{j=1}^{\infty} a_j \cdot (1 + \frac{k_j}{a_j}) \cdot e^{-a_j z} < \sum_{j=1}^{\infty} 2a_j \cdot e^{-a_j z} = \frac{12}{z^3} \cdot \sum_{j=1}^{\infty} \frac{1}{a_j^2} \cdot R_j < \frac{12}{z^3} \cdot \sum_{j=1}^{\infty} \frac{1}{n^2} \tag{4.19}$$

where:

$$R_j = 1 + 3 \cdot \frac{1}{a_j z} + 6 \cdot \frac{1}{(a_j z)^2} + \frac{3!}{4!}(a_j z) + \ldots > 1 \; , \; z > 0 \tag{4.20}$$

The procedure described above leads to the conclusion that all the series (4.10) are absolutely convergent. Thus, the vertical component of velocity (first of (4.8)) is finite at points of the domain of consideration. In the same way it is possible to show that also the second component of (4.8) is finite at all points of the semi-infinite layer of fluid. The last statements are very important, because they explain that if the fluid viscosity, although even very small is taken into account, the resulting velocity field will be finite and no singularity will occur.

Therefore, the logarithmic singularity of the velocity field (2.23) results from neglecting the fluid viscosity which exists for every real fluid.

However, if the series (4.9) are absolutely convergent it is desirable to estimate a number of terms of the series which should be taken into account in practical calculations. The series are not monotone series, but it is possible to find such a term of the series that the remainder of the series after the term will lie within a monotone envelope. To investigate the problem, consider the components (4.9) for the case x = L.

Substitution of (4.2) into (4.9) gives:

$$u_1 = \frac{4 \cdot d}{\pi} \cdot \sum_{j=1}^{\infty} \frac{1}{2 \cdot j - 1} \cdot [(1 + \frac{k_j}{2a_j}) \cdot e^{-a_j z} \cdot \sin b_j z + \frac{\nu}{\omega} \cdot k_j \cdot (a_j + k_j) \cdot (e^{-a_j z} \cdot \cos b_j z - e^{-k_j z})]$$

$$u_2 = \frac{4 \cdot d}{\pi} \cdot \sum_{j=1}^{\infty} \frac{1}{2 \cdot j - 1} \cdot [\frac{\nu}{\omega} \cdot k_j \cdot (a_j + k_j) \cdot e^{-a_j z} \cdot \sin b_j z - (1 + \frac{k_j}{2a_j}) \cdot (e^{-a_j z} \cdot \cos b_j z - e^{-k_j z})] \qquad (4.21)$$

It is seen from (4.5) that $k_j < a_j$ and the pivotal series for the estimation is the series:

$$S = \sum_{j=1}^{\infty} (a_j + k_j) e^{-k_j z} = \sum_{j=1}^{\infty} H_j \qquad (4.22)$$

Such a number of j = N is sought that:

$$H_{N+1} < H_N \qquad (4.23)$$

Substitution of (4.22) into the last inequality gives:

$$(a_{N+1} + k_{N+1}) \cdot e^{-k_{N+1} \cdot z} < (a_N + k_N) e^{-k_N z} \qquad (4.24)$$

Taking into account the relations (4.13), the expression (4.24) may be written in the form:

$$k_{N+1} \cdot \epsilon \cdot e^{-k_{N+1} \cdot z} < k_N \cdot e^{-k_N \cdot z} , \qquad (4.25)$$

where:

$$\epsilon = \frac{1 + \frac{1}{\sqrt{2}} \cdot \sqrt{\sqrt{1 + (1/\beta_{N+1})^2} + 1}}{1 + \frac{1}{\sqrt{2}} \cdot \sqrt{\sqrt{1 + (1/\beta_N)^2} + 1}} < 1 \qquad (4.26)$$

Instead of (4.25), we may use the more intensive condition:

$$k_{N+1} \cdot e^{-k_{N+1} \cdot z} < k_N \cdot e^{-k_N \cdot z} , \qquad (4.27)$$

which results in:

$$N > \frac{1}{e^{\frac{\pi}{h} \cdot z} - 1} + \frac{1}{2} \qquad (4.28)$$

The last formula may serve as the initial measure of a number of terms of the series (4.21) that should be taken into account in computations. It is obvious that the number of terms of (4.9) in practical calculations should be greater than N in the last-mentioned relation.

Following (4.21) we may calculate the amplitude of the vertical component of velocity:

$$u = \sqrt{u_1^2 + u_2^2} \qquad (4.29)$$

and corresponding phase shift:

$$\alpha = \tan^{-1} \left(\frac{u_1}{u_2}\right) \qquad (4.30)$$

The last relations clearly correspond to the description (3.32).

To illustrate the considerations, numerical calculations have been performed. The results obtained are shown in Fig. 6, where the plots represent the distribution of the amplitudes of the vertical component of velocity at points z = h (see Fig. 1). The plots correspond to the solutions (4.29) and (4.21) for viscous fluid and to the solution (2.20) for non-viscous fluid, respectively. The results indicated in the figure correspond to the following input data:

$$\lambda = h = 40 \text{ cm}, \quad \omega^2 = g \cdot k_o \cdot \tanh k_o h, \quad k_o = 2\pi/\lambda \ , \qquad (4.31)$$

where k_o is the wave number, and λ is the wave length.

For the case of viscous fluid, because of its small viscosity (computations were performed for water viscosity), the component u_1 in (4.29) is negligibly small in comparison with u_2 and may be ignored in practical calculations for water. From the plots it is seen that the viscosity of the fluid influences the solution mainly in the neighbourhood of the wall OA. This confirms the well-known fact that influence of the viscosity of a fluid on a water wave phenomenon is important only in a very thin boundary layer. [5].

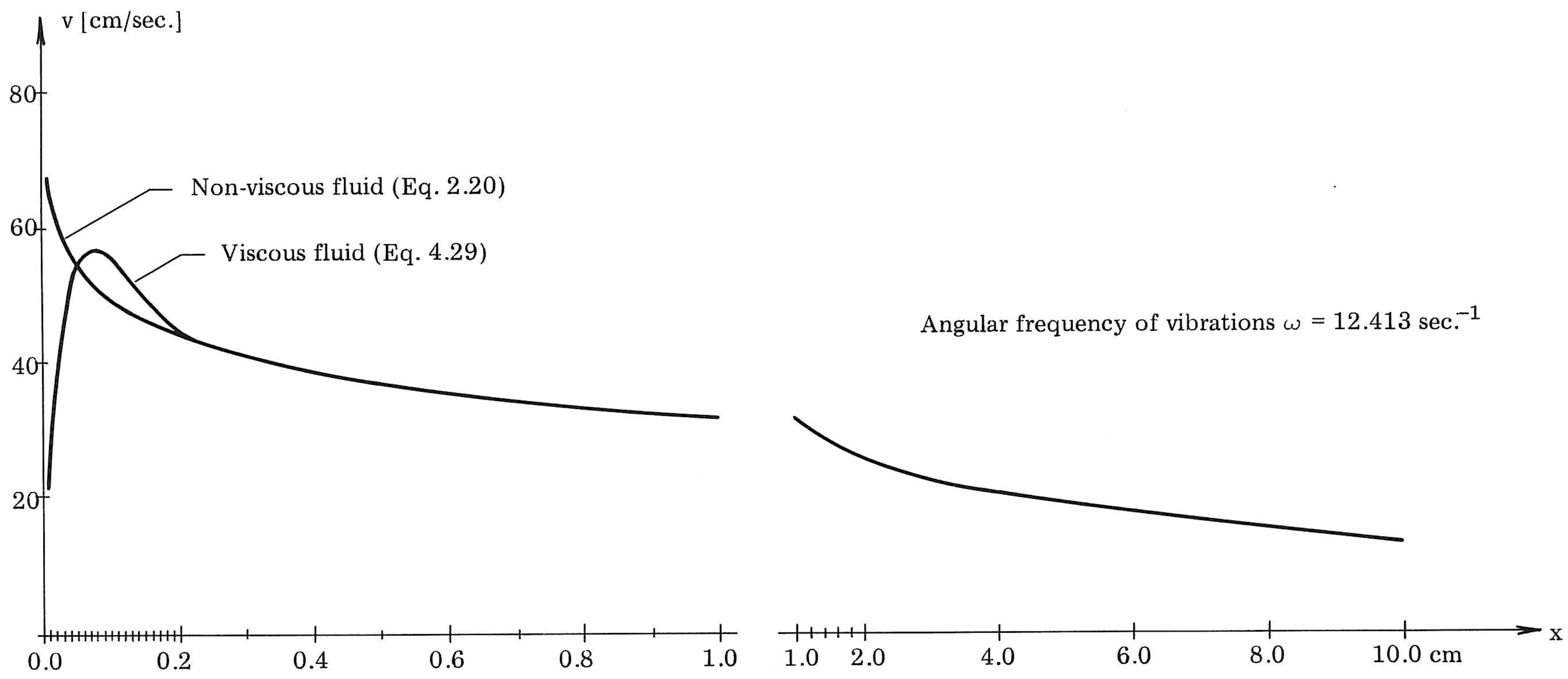


Fig. 6. Vertical component of velocity on the free surface.

5. INITIAL SOLUTION FOR INCOMPRESSIBLE FLUID

Some aspects of fluid motion starting from rest were considered in section two. Continuing the discussion, consider now the incompressible non-viscous fluid and assume that the velocity potential $\Phi(x,z,t)$ and the surface elevation $\eta(x,t)$ possess the power series representation with respect to time increment in the neighbourhood of the starting point. In other words it is assumed the quantities may be expand in Taylor series with respect to time. Thus, we have:

$$\begin{aligned} \Phi &= \phi_1 \cdot t + \phi_2 \cdot t^2 + \phi_3 \cdot t^3 + \ldots \\ \eta &= \eta_1 \cdot t + \eta_2 \cdot t^2 + \eta_3 \cdot t^3 + \ldots \quad , \end{aligned} \tag{5.1}$$

where t means small increment of time and the set of spatial functions $\phi_1, \phi_2, \ldots$ satisfy the Laplace equation and the boundary condition at the bottom of the layer. In the expressions constant terms are omitted, because without loss of generality we may assume $\Phi(x,z,t=0) = 0$ and, hence $\eta(x,t=0) = 0$. Substitution of the last-mentioned equations into the boundary conditions (1.11) and (1.12) gives:

$$\begin{aligned} &(\eta_1 + 2\cdot\eta_2\cdot t + 3\cdot\eta_3\cdot t^2 + \ldots) - (\phi_{1,z}\cdot t + \phi_{2,z}\cdot t^2 + \ldots) + \\ &+ (\eta_{1,x}\cdot t + \eta_{2,x}\cdot t^2 + \ldots)\cdot(\phi_{1,x}\cdot t + \phi_{2,x}\cdot t^2 + \ldots) = 0 \end{aligned} \tag{5.2}$$

and:

$$\begin{aligned} &(\phi_1 + 2\cdot\phi_2\cdot t + 3\cdot\phi_3\cdot t^2 + \ldots) + g\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots) + \\ &+ \frac{1}{2}\cdot(\phi_{1,x}\cdot t + \phi_{2,x}\cdot t^2 + \ldots) + \frac{1}{2}\cdot(\phi_{1,z}\cdot t + \phi_{2,z}\cdot t^2 + \ldots) = 0 , \end{aligned} \tag{5.3}$$

where $\eta_{i,x}$, $\phi_{j,x}$ and $\phi_{j,z}$ denote the derivatives of η_i and ϕ_j with respect to x and z. A remark is needed: all terms entering the last-mentioned expressions are to be calculated at points of the free surface, i.e. for $z = \eta(x,t)$. Assuming that the functions are continuous and differentiable to the required order, they may be expanded into Taylor's formula:

$$\phi_i/_{z=\eta} = \phi_i/_{z=h} + \phi_{i,z}/_{z=h}\cdot\eta + \frac{1}{2!}\cdot\phi_{i,zz}\cdot/_{z=h}\cdot\eta^2 + \ldots , \tag{5.4}$$

where $\eta(x,t)$ is measured from the line $z = h$.

Substitution of the last-mentioned equation into the conditions (5.2) and (5.3) leads to the result:

$$\begin{aligned} &\eta_1 + 2\cdot\eta_2\cdot t + 3\cdot\eta_3\cdot t^2 + \ldots \\ &-[\phi_{1,z} + \phi_{1,zz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots) + \frac{1}{2!}\cdot\phi_{1,zzz}(\eta_1\cdot t + \eta_2\cdot t^2 + \cdots)^2 + \ldots]\cdot t \\ &-[\phi_{2,z} + \phi_{2,zz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots) + \frac{1}{2!}\cdot\phi_{2,zzz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots)^2 + \ldots]\cdot t^2 \\ &+[\eta_{1,x}\cdot t + \eta_{2,x}\cdot t^2 + \cdots]\cdot\{[\phi_{1,x} + \phi_{1,xz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots) \\ &+ \frac{1}{2!}\phi_{1,xzz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots)^2 + \ldots]\cdot t + [\phi_{2,x} + \phi_{2,xz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots) \\ &+ \frac{1}{2!}\cdot\phi_{2,xzz}\cdot(\eta_1\cdot t + \eta_2\cdot t^2 + \ldots)^2 + \ldots]\cdot t^2 + \ldots\} = 0 \end{aligned} \tag{5.5}$$

and:

$$\phi_1 + \phi_{1,z} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \cdot \phi_{1,zz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots$$
$$+ 2 \, [\phi_2 + \phi_{2,z} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \, \phi_{2,zz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots] \cdot t$$
$$+ 3 \cdot [\phi_3 + \phi_{3,z} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \, \phi_{3,zz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots] \cdot t^2 + \ldots$$
$$+ g \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \qquad (5.6)$$
$$+ \frac{1}{2} \cdot \{[\phi_{1,x} + \phi_{1,xz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \, \phi_{1,xzz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots] \cdot t$$
$$+ [\phi_{2,x} + \phi_{2,xz} \cdot (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \, \phi_{2,xzz} \cdot (\eta_1 t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots] \cdot t^2 + \ldots\}^2 +$$
$$+ \frac{1}{2} \cdot \{[\phi_{1,z} + \phi_{1,zz} \cdot (\eta_1 t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \, \phi_{1,zzz} \cdot (\eta_1 \cdot t + \eta_2 t^2 + \ldots)^2 + \ldots] \cdot t$$
$$+ [\phi_{2,z} + \phi_{2,zz} \cdot (\eta_1 t + \eta_2 \cdot t^2 + \ldots) + \frac{1}{2!} \cdot \phi_{2,zzz} (\eta_1 \cdot t + \eta_2 \cdot t^2 + \ldots)^2 + \ldots] \cdot t^2 + \ldots\}^2 = 0,$$

where all components ϕ_j of the velocity potential together with their derivatives correspond to $z = h$.

Collecting terms with the same power in "t" the following relations are obtained:

$$\eta_1 = 0 \qquad (5.7)$$
$$2 \cdot \eta_2 - \phi_{1,z} = 0$$
$$3 \cdot \eta_3 - \eta_1 \cdot \phi_{1,zz} - \phi_{2,z} + \eta_{1,x} \cdot \phi_{1,x} = 0$$

.....

and:

$$\phi_1 = 0 \qquad (5.8)$$
$$\eta_1 \cdot \phi_{1,z} + 2 \cdot \phi_2 + g \cdot \eta_1 = 0$$
$$\eta_2 \cdot \phi_{1,z} + \frac{1}{2} \cdot (\eta_1)^2 \cdot \phi_{1,zz} + 2 \cdot \eta_1 \cdot \phi_{2,z} + 3 \cdot \phi_3 + g \cdot \eta_2 + \frac{1}{2} [(\phi_{1x})^2 + \frac{1}{2} (\phi_{1,z})^2] = 0$$

.....

The last-mentioned relations lead to the results:

$$\eta_1 = 0 \, , \quad \eta_2 = \frac{1}{2} \cdot \phi_{1,z/z=h} \quad , \quad \eta_3 = \frac{1}{3} \cdot \phi_{2,z/z=h} \qquad (5.9)$$
$$\phi_{1/z=h} = 0 \, , \quad \phi_{2/z=h} = 0 \quad , \quad \phi_{3/z=h} = -\frac{1}{6} \cdot [g \cdot \phi_{1,z} + (\phi_{1,x})^2 + 2 \, (\phi_{1,z})^2]_{/z=h}$$

It can be seen that the first-order approximation of the velocity potential with respect to the small quantity "t" leads to the second-order approximation in the description of the surface elevation with respect to the same quantity. Thus, the condition:

$$\phi_{1/z=h} = 0 \tag{5.10}$$

leads to the solution:

$$\eta(x,t) = \frac{1}{2} \cdot \phi_{1,z/z=h} \cdot t^2 \tag{5.11}$$

The last-mentioned conclusions are very important because they justify our earlier assumptions in the second paragraph and confirm conclusions derived there. Therefore, the condition (5.10) will be used for estimation of the surface elevation at the first time step $\Delta t > 0$ and for $x > 0$ just after beginning of the fluid motion.

6. DISCRETE FORMULATION OF THE PROBLEM – PRELIMINARY COMMENTS

Knowing the solutions (2.20), (2.23), and (3.31) for non-viscous and viscous fluid, respectively, we can calculate the velocity potential, velocity field and free surface elevation for the semi-infinite layer of fluid at the instant $\Delta t > 0$. Although the solutions correspond to steady-state vibrations we can use them also for unsteady problems. This may be justified as follows. For small values of Δt, the first equation (1.2) may be written in the form:

$$v_x = \dot{\underset{\sim}{x}} = d \cdot \omega^2 \cdot [\Delta t - \frac{1}{3!}\omega^2 \cdot (\Delta t)^3 + \frac{1}{5!} \cdot \omega^4 \cdot (\Delta t)^5 - \ldots] \tag{6.1}$$

and the velocity potential as:

$$\Phi = \phi_1 \cdot \Delta t + \phi_2 \cdot \Delta t^2 + \phi_3 \cdot (\Delta t)^3 + \ldots \tag{6.2}$$

The linearized boundary condition for $x = 0$ is:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \phi_1}{\partial x} \cdot \Delta t + \frac{\partial \phi_2}{\partial x} \cdot (\Delta t)^2 + \ldots = d \cdot \omega^2 [\Delta t - \frac{1}{3!}\omega^2 (\Delta t)^3 + \ldots] \tag{6.3}$$

From the last-mentioned relations it follows that:

$$\frac{\partial \phi_1}{\partial x}_{/x=0} = d\omega^2$$

or: (6.4)

$$\frac{\partial \phi_1}{\partial x}_{/x=0} \simeq \frac{d \cdot \omega}{\Delta t} \cdot \sin\omega\Delta t$$

To distinguish quantities associated with the instant Δt superscript 1 is used to denote them. Thus,

$$\Phi^1 = -\frac{8 \cdot d \cdot h \cdot \omega}{\pi^2} \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{(2 \cdot j - 1)^2} \cdot e^{-k_j x} \cdot \cos k_j z \cdot \sin\omega\Delta t$$

$$\frac{\partial \Phi^1}{\partial z}_{/z=h} = \frac{4 \cdot d \cdot \omega}{\pi} \cdot \sum_{j=1}^{\infty} \frac{1}{2 \cdot j - 1} \cdot e^{-k_j x} \cdot \sin\omega\Delta t \ , \qquad x \geqslant \epsilon$$

$$\frac{\partial \Phi^1}{\partial z}_{/z=h} = \frac{d \cdot \omega}{\pi} \cdot \log\frac{\cosh\alpha x + 1}{\cosh\alpha x - 1} \cdot \sin\omega\Delta t \ , \qquad \epsilon \geqslant x > 0 \tag{6.5}$$

$$\eta^1 = \frac{1}{2} \cdot \frac{\partial \Phi^1}{\partial z}_{/z=h} \cdot \Delta t$$

where ϵ should be chosen in a proper way.

In the neighbourhood of the point (x = 0, z = h) the following solution for viscous fluid may be used:

$$\Phi^1_{,z} = u_1 \cdot \cos\omega\Delta t + u_2 \cdot \sin\omega\Delta t \, , \tag{6.6}$$

where u_1 and u_2 are defined by (4.9).

In the following the attention is confined to the linearized boundary conditions. Instead of (1.11) and (1.12)

$$\dot{\eta} = \Phi_z \, , \quad (\dot{\Phi} + g \cdot \eta)\,||_{z=h} = 0 \, , \quad \frac{\partial \Phi}{\partial x} / _{x=0} = v_x \tag{6.7}$$

is applied. In discrete formulation the semi-infinite layer of fluid is replaced by a discrete system of chosen points. The number of the nodal points should be finite. The discrete system is shown schematically in figure 6. To solve the problem the finite difference method is used. Thus, the differential equation for the velocity potential is substituted by finite difference equations written for all nodal points of the assumed net. Besides, integration with respect to time is carried out for discrete steps of time Δt. In such a case, the problem will be reduced to algebraic equations and conditions written for nodal points.

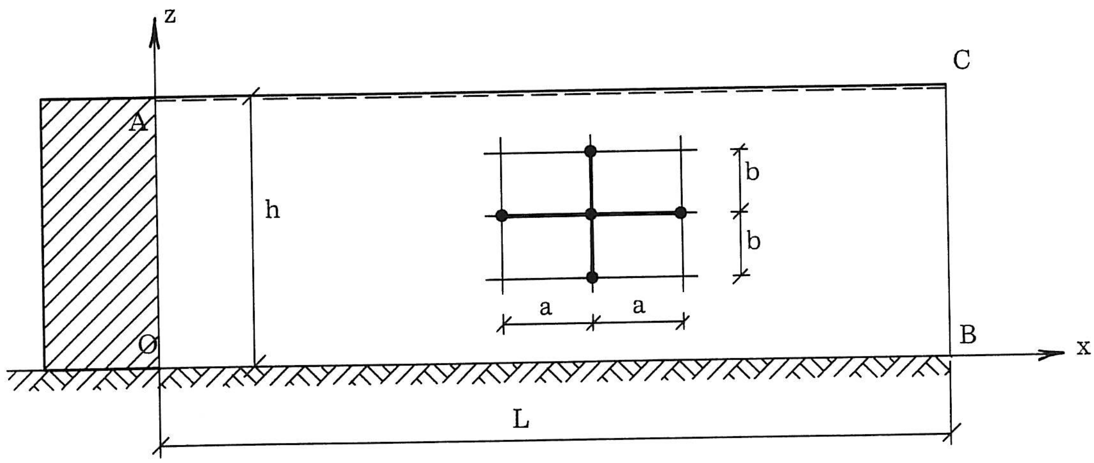


Fig. 7. Assumed discrete system.

There are some pivotal points of analysis of the problem in discrete formulation. The first one is a reply to the question how dense the assumed net should be and how to choose the ratio a/b (see Fig. 7). The second question arising is the extent of the finite domain considered (L = ?) and what kind of boundary conditions should be imposed on the boundary x = L (segment BC in Fig. 7). Besides, the dimensions "a" and "b" of the assumed net should be correlated with the time step integration $\Delta t > 0$. Additional task is associated with the evaluation of the vertical component of the velocity field at point A and the corresponding surface elevation at the point of fixed time $t = \Delta t$. The latter problem is a result of relatively high gradient of velocity field in the vicinity of the point which cannot be properly described by means of a mesh with big dimensions. It is not advisable to use a denser spacing of points in the neighbourhood of the wall OA, because such inhomogenity of an assumed net introduces inhomogenity into the velocity field. The result of changing dimensions an assumed grid is a spurious reflection of propagating waves which cannot be avoided. It is understood that replies to the questions cannot be unique because of the approximated character of the discrete analysis. Nevertheless, an investigation into some of the above-mentioned problems will be attempted which will hopefully result in a possibly proper discrete analysis of the original problem formulated in continuum.

In isotropic fluid there is no distinguished direction and because of that it is reasonable to choose a net with a = b. Of course this is no rigorous condition and for many practical problems it is possible to choose a ≠ b and to obtain accurate results. Knowing the circular frequency (see Eq. 4.31) it is possible to calculate the length of a surface wave and then to choose "a" in such a way that the discrete description of the wave is sufficiently good. For example, the dimension "a" should be smaller than 1/10 of the wave length.

To get a better insight into the problems consider the auxiliary problem of the one-dimensional wave equation:

$$\frac{\partial^2 U}{\partial x^2} - \frac{1}{c^2} \cdot \frac{\partial^2 U}{\partial t^2} = 0 , \qquad (6.8)$$

where c means the velocity of the wave.

The last-mentioned equation may describe for instance the propagation of a plane longitudinal wave in a bar of rectangular cross-section [9]. Such a problem of the longitudinal wave in the discrete description was investigated by Wilde in [11] and in the following the same method will be applied. Consider the case of steady-state harmonic solution of (6.8).

Thus:

$$U = u \cdot e^{\pm i\omega t} \qquad (6.9)$$

Substitution of (6.9) into (6.8) gives:

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\omega}{c}\right)^2 \cdot u = 0 \qquad (6.10)$$

Relevant to the last equation the finite difference equation is:

$$\frac{u_{n-1} - 2 \cdot u_n + u_{n+1}}{(\Delta x)^2} + \left(\frac{\omega}{c}\right) \cdot u_n = 0 \qquad (6.11)$$

For the considered case of a simple harmonic wave the solution:

$$u_n = u \cdot e^{ikn\Delta x} , \quad n \cdot \Delta x = x \qquad (6.12)$$

can be substituted and accordingly:

$$u_{n-1} = u \cdot e^{ik(n-1)\Delta x} , \quad u_{n+1} = u \cdot e^{ik \cdot (n+1)\Delta x} \qquad (6.13)$$

In the last-mentioned equations k is the wave number. Following (6.9) and (6.12), the phase velocity of the wave, or simply the wave velocity, is given by:

$$c_f = \frac{\omega}{k} = \frac{\lambda \cdot \omega}{2\pi}, \tag{6.14}$$

where λ is the wave length.

In a discrete description it may happen that c_f does not coincide with the velocity c in (6.8) Substituting (6.12) and (6.13) into (6.11) gives:

$$\frac{\cdot 1}{(\Delta x)^2} \cdot (2 \cdot \cos k\Delta x - 2) + (\frac{\omega}{c})^2 = 0 \tag{6.15}$$

or:

$$2 \cdot \sin^2 \frac{k\Delta x}{2} = \frac{1}{2} \cdot (\Delta x)^2 \cdot (\frac{\omega}{c})^2 \tag{6.16}$$

From the last-mentioned relations it follows that:

$$\sin \frac{k\Delta x}{2} = \frac{1}{2} \cdot \Delta x \cdot \frac{\omega}{c} = \frac{1}{2} \cdot k \cdot \Delta x \cdot (\frac{c_f}{c}) , \tag{6.17}$$

and finally:

$$\frac{c_f}{c} = \frac{\sin \pi/n}{\pi/n}, \quad \lambda = n \cdot \Delta x , \tag{6.18}$$

where n is a natural number.

According to the equation (6.18) numerical calculations were made. The results of the computations are indicated in table 1.

Table 1.

n =	1	2	3	4	6	8	10	12	14	16	20
c_f/c =	0	.6366	.8270	.9003	.9549	.9745	.9836	.9886	.9916	.9936	.9959

It is seen that the discrete system is the dispersive one. It means that the velocity of propagation of a wave depends on the wave length. The last feature of the system may be a serious obstacle in a discrete formulation of wave propagation phenomena.

Consider again the same problem of harmonic wave, but with a discrete description of both space and time:

$$x \rightarrow n \cdot \Delta x , \quad t \rightarrow r \cdot \Delta t \tag{6.19}$$

The finite difference equation for the equation (6.8) becomes:

$$\frac{1}{(\Delta x)^2} \cdot (u_{n-1,r} - 2 \cdot u_{n,r} + u_{n+1,r}) - \frac{1}{c^2} \cdot \frac{1}{(\Delta t)^2} \cdot (u_{n,r-1} - 2 \cdot u_{n,r} + u_{n,r+1}) = 0 \tag{6.20}$$

For the case of a simple harmonic wave the following expression is chosen:

$$u_{n,r} = \cos(k \cdot n \cdot \Delta x - \omega \cdot r \cdot \Delta t) \tag{6.21}$$

Let:

$$w = k \cdot n \cdot \Delta x - \omega \cdot r \cdot \Delta t, \tag{6.22}$$

so that the following relations hold:

$$u_{n-1,r} = w - k \cdot \Delta x = w - \frac{2\pi}{\lambda} \cdot \Delta x$$

$$u_{n+1,r} = w + k \cdot \Delta x = w + \frac{2\pi}{\lambda} \cdot \Delta x$$

and: (6.23)

$$u_{n,r-1} = w + \omega \cdot \Delta t$$

$$u_{n,r+1} = w - \omega \cdot \Delta t$$

Substituting (6.23) into (6.20) and making simple manipulations the following equation is derived:

$$\cos(2\pi \cdot \frac{\Delta x}{\lambda}) - 1 - \frac{1}{c^2} \cdot (\frac{\Delta x}{\Delta t})^2 \cdot (\cos\omega\Delta t - 1) = 0 \ , \tag{6.24}$$

from which it follows that:

$$\sin \frac{\pi \cdot \Delta x}{\lambda} = \frac{1}{c} \cdot \frac{\Delta x}{\Delta t} \cdot \sin \frac{\omega \Delta t}{2} \tag{6.25}$$

Knowing that $\lambda = m\Delta x$,

$$c_f = \frac{\omega \cdot m\Delta x}{2\pi} \ , \ \rightarrow \ \Delta x = \frac{2\pi}{\omega} \cdot \frac{c_f}{m} \tag{6.26}$$

and finally:

$$\frac{c_f}{c} = \frac{\omega \cdot m\Delta t}{2\pi} \cdot \frac{\sin \pi/m}{\sin \frac{\omega\Delta t}{2}} = \frac{\sin \pi/m}{\pi/m} \cdot \frac{\frac{\omega \cdot \Delta t}{2}}{\sin \frac{\omega\Delta t}{2}} \tag{6.27}$$

According to the formula (6.27) numerical calculations have been performed. The results obtained are shown in table 2.

Table 2.

c_f/c					
$\frac{\omega \cdot \Delta t}{2}$	m				
	2	4	8	16	20
0.2	0.6409	0.9063	0.9810	1.0002	1.0026
0.4	0.6539	0.9248	1.0010	1.0206	1.0230
0.6	0.6765	0.9567	1.0355	1.0558	1.0583
1	0.7566	1.0699	1.1581	1.1808	1.1835

From the table it is seen as expected that the discrete system is dispersive. It is also seen how important for a proper description of the propagating wave is to choose a proper time and space steps. Fortunately, if

$$\Delta x = c \cdot \Delta t \tag{6.28}$$

is assumed, then:

$$\frac{\pi}{m} = \frac{\pi}{m} \cdot \frac{\lambda}{\Delta x} \cdot \frac{\Delta x}{\lambda} = \frac{\pi \cdot \lambda}{m \cdot \Delta x} \cdot \frac{\Delta x}{\lambda} = \pi \cdot \frac{\Delta x}{\lambda} = \frac{\pi}{\lambda} \cdot c \cdot \Delta t \tag{6.29}$$

and because

$$\lambda = \frac{2\pi}{\omega} \cdot c, \tag{6.30}$$

the following is derived

$$\frac{\pi}{m} = \pi \cdot c \cdot \Delta t \cdot \frac{1}{\lambda} = \frac{\omega \cdot \Delta t}{2} \tag{6.31}$$

Therefore:

$$c_f = c\,, \qquad \Delta x = c \cdot \Delta t = c_f \cdot \Delta t \tag{6.32}$$

For the special case (6.28) the discrete system is not dispersive. Thus, in the following a net for which the condition (6.29) is satisfied will be used.

The solution of the problem with the magnitude of the dimension L(Fig. 6) depends on a character of a given problem. For the case discussed here (Eqs. 1.1 – 1.3) the plausible dimension seems to be four to six lengths of the surface wave. Of course, it is expected that at the beginning of the fluid motion (for a small elapse of time $t = m \cdot \Delta t$) changes of the velocity potential on the line $x = L$ are to be negligibly small. On the other hand it is possible to impose a transmitting condition on the boundary so as will be done in further analyses.

7. FINITE DIFFERENCE SOLUTIONS OF THE PROBLEM

Consider now the problem of a wave generation in the semi-infinite layer of fluid and its solutions by means of the finite difference method. The assumed net has the spacing of vertical lines equal to "a" and the spacing of horizontal lines equal to "b", respectively (Fig. 7). To simplify the considerations we will confine our discussion to the linearized boundary conditions at the upper surface of the layer. Thus, instead of (1.11) and (1.12)

$$\Phi_z - \dot{\eta} = 0$$

$$\dot{\Phi} + g \cdot \eta = 0 \qquad (7.1)$$

The remaining conditions (1.13) and (1.14) are still in force. Consider a typical nodal point (i, j) within the layer, where "i" denotes the horizontal line $z_i = (i - 1) \cdot b$ and "j" – the vertical line $x_j = (j - 1) \cdot a$ (Fig. 8). The finite difference equation for the Laplace equation at the point considered assumes the form:

$$\frac{1}{a^2} \cdot (\Phi_{i,j-1} - 2 \cdot \Phi_{i,j} + \Phi_{i,j+1}) + \frac{1}{b^2} \cdot (\Phi_{i-1,j} - 2 \cdot \Phi_{i,j} + \Phi_{i+1,j}) = 0 \qquad (7.2)$$

Equations of the type (7.2) should be written for all nodal points of the region: $0 \leqslant x \leqslant L$, $0 \leqslant z \leqslant h$. To write down the equations and corresponding boundary conditions at points on the boundaries of the region it is necessary to extend the net over the boundaries. The extension is indicated schematically in Fig. 8.

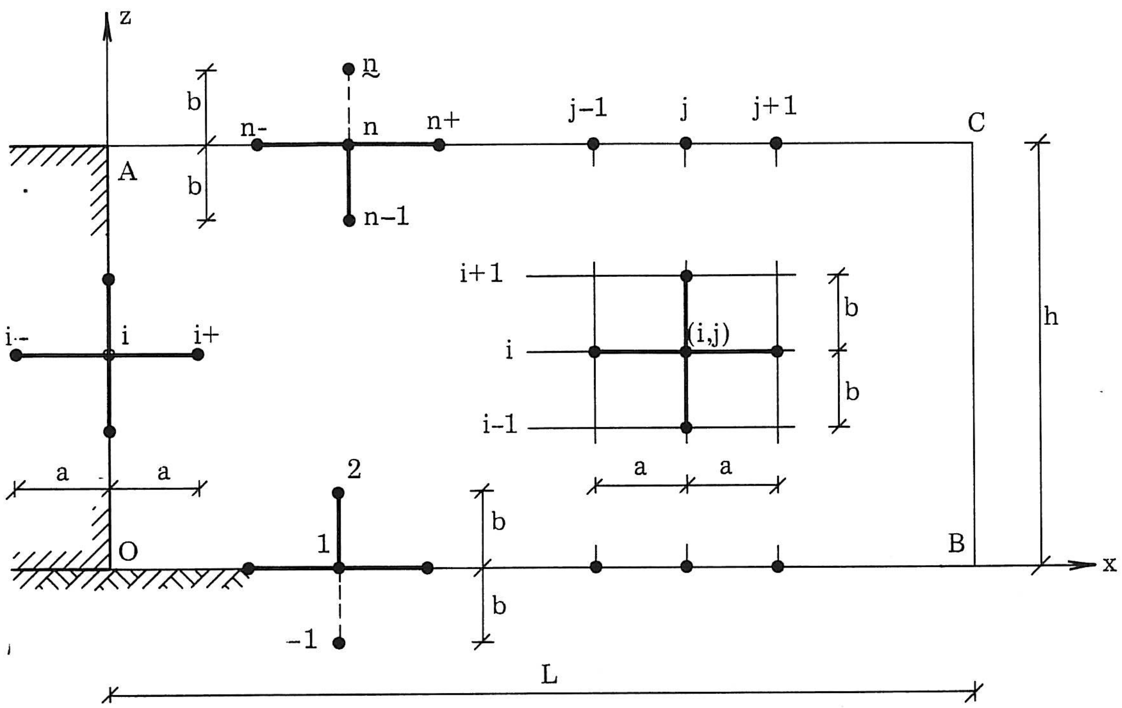


Fig. 8. Spacing of points in the finite domain.

For the points on the bottom and on the wall OA, finite difference formulation leads to the relations:

$$\left.\frac{\partial \Phi}{\partial x}\right|_{x=0} = v_x \,, \rightarrow \frac{1}{2a} \cdot (\Phi_{i+} - \Phi_{i-}) = v_x \,, \rightarrow \Phi_{i-} = \Phi_{i+} - 2 \cdot a \cdot v_x$$

$$\left.\frac{\partial \Phi}{\partial z}\right|_{z=0} = 0 \,, \rightarrow \frac{1}{2b} \cdot (\Phi_2 - \Phi_{-1}) = 0 \,, \rightarrow \Phi_{-1} = \Phi_2 \tag{7.3}$$

To write similar relations for nodal points on the upper surface of the layer the countable set of time increments $t_m = m \cdot \Delta t$ is introduced, where m is a natural number. With reference to the discrete sequence of the time steps, the time derivatives of the surface elevation and of the velocity potential in (7.1) may be expressed by forward, backward or central time differences (meaning relative to the time t at which space differences are expressed). The easiest way is to apply such a difference scheme which finally leads to the explicit system of equations in which the unknown values of η^{m+1} and Φ^{m+1} may be calculated directly from the known values η^m and Φ^m. (without solving a corresponding set of simultaneous system of equations). Unfortunately, such a procedure of solution is not stable for the problem considered. Therefore, we will use the Crank-Nicholson scheme for the first equation of (7.1) which yields:

$$\frac{\eta^{m+1} - \eta^m}{\Delta t} - \frac{1}{2} \cdot (\Phi^{m+1}_{,z} + \Phi^m_{,z}) = 0 \tag{7.4}$$

The Crank-Nicholson scheme is of the second order of approximation with respect to time increment and it is absolutely stable [4]. Besides, for such the scheme it is convenient to combine the equations (7.1) into the one condition for the velocity potential:

$$(\ddot{\Phi} + g \cdot \frac{\partial \Phi}{\partial z})/_{z=h} = 0 \quad (7.5)$$

Thus, the procedure of a solution will be performed in two steps: the first is to calculate the velocity potential and the second is the free surface elevation.

In finite differences, the last equation assumes the form:

$$\frac{\Phi_n^{m-1} - 2 \cdot \Phi_n^m + \Phi_n^{m+1}}{(\Delta t)^2} + g \cdot \frac{\partial \Phi^m}{\partial z}/_{z=h} = 0 \ , \quad (7.6)$$

where n - denotes the point $z_n = h$ (see Fig. 7) and (m− 1), m and (m+1) indicate the subsequent points of time. From equation (7.6) it follows that:

$$\frac{\partial \Phi^m}{\partial z}/_{z=h} = -\frac{1}{g(\Delta t)^2} \cdot [\Phi_n^{m-1} - 2 \cdot \Phi_n^m + \Phi_n^{m+1}] \quad (7.7)$$

On the other hand:

$$\frac{\partial \Phi^m}{\partial z}/_{z=h} \cong \frac{1}{2b} \cdot (\Phi_{\underset{\sim}{n}}^m - \Phi_{n-1}^m) \ , \rightarrow \Phi_{\underset{\sim}{n}}^m = \Phi_{n-1}^m + 2 \cdot b \cdot \frac{\partial \Phi^m}{\partial z}/_{z=h} \quad (7.8)$$

Substitution of (7.7) into (7.8) gives:

$$\Phi_{\underset{\sim}{n}}^m = \Phi_{n-1}^m - \frac{2b}{g \cdot (\Delta t)^2} \cdot [\Phi_n^{m-1} - 2 \cdot \Phi_n^m + \Phi_n^{m+1}] \quad (7.9)$$

In accordance with the equations (7.7) ÷ (7.9) the Laplace difference equation for a point on the free surface is:

$$-(\frac{b}{a})^2 \cdot (\Phi_{n-}^m + \Phi_{n+}^m) + 2 \cdot [1 + (\frac{b}{a})^2] \cdot \Phi_n^m - 2 \cdot \Phi_{n-1}^m +$$

$$+ \frac{2b}{g \cdot (\Delta t)^2} \cdot (\Phi_n^{m-1} - 2\Phi_n^m + \Phi_n^{m+1}) = 0 \ , \quad (7.10)$$

or:

$$-\epsilon \cdot \Phi_{n-}^m - 2 \cdot \Phi_{n-1}^m + 2 \cdot (1 + \epsilon - \beta)\Phi_n^m - \epsilon \cdot \Phi_{n+}^m = -\beta \cdot (\Phi_n^{m-1} + \Phi_n^{m+1}) \ , \quad (7.11)$$

where:

$$\epsilon = (\frac{b}{a})^2 \ , \quad \beta = \frac{2 \cdot b}{g \cdot (\Delta t)^2} \quad (7.12)$$

Denoting

$$K = 2 \cdot (1 + \epsilon)\,, \quad R = K - 2\beta\,, \tag{7.13}$$

and taking the boundary conditions (7.3) into account, the finite difference equations for the Laplace equation written for all nodal points on the vertical line x_j = const. assume the form:

$$\begin{aligned}
&-\epsilon \cdot \Phi^m_{1-} + K \cdot \Phi^m_1 - 2 \cdot \Phi^m_2 - \epsilon \cdot \Phi^m_{1+} = 0 \\
&-\epsilon \cdot \Phi^m_{2-} - \Phi^m_1 + K \cdot \Phi^m_2 - \Phi^m_3 - \epsilon \cdot \Phi^m_{2+} = 0 \\
&\qquad \vdots \\
&-\epsilon \cdot \Phi^m_{l-} - \Phi^m_{l-1} + K \cdot \Phi^m_l - \Phi^m_{l+1} - \epsilon \cdot \Phi^m_{l+} = 0 \\
&\qquad \vdots \\
&-\epsilon \cdot \Phi^m_{(n-1)-} - \Phi^m_{n-2} + K \cdot \Phi^m_{n-1} - \Phi^m_n - \epsilon \cdot \Phi^m_{(n-1)+} = 0 \\
&-\epsilon \cdot \Phi^m_{n-} - 2 \cdot \Phi^m_{n-1} + R \cdot \Phi^m_n - \epsilon \cdot \Phi^m_{n+} = -\beta \cdot (\Phi^{m-1}_n + \Phi^{m+1}_n)
\end{aligned} \tag{7.14}$$

The set of equations (7.14) is to be written for all vertical lines $x_j = 0, a \ldots L$. The result is a system of equations which is shown schematically in Fig. 9.

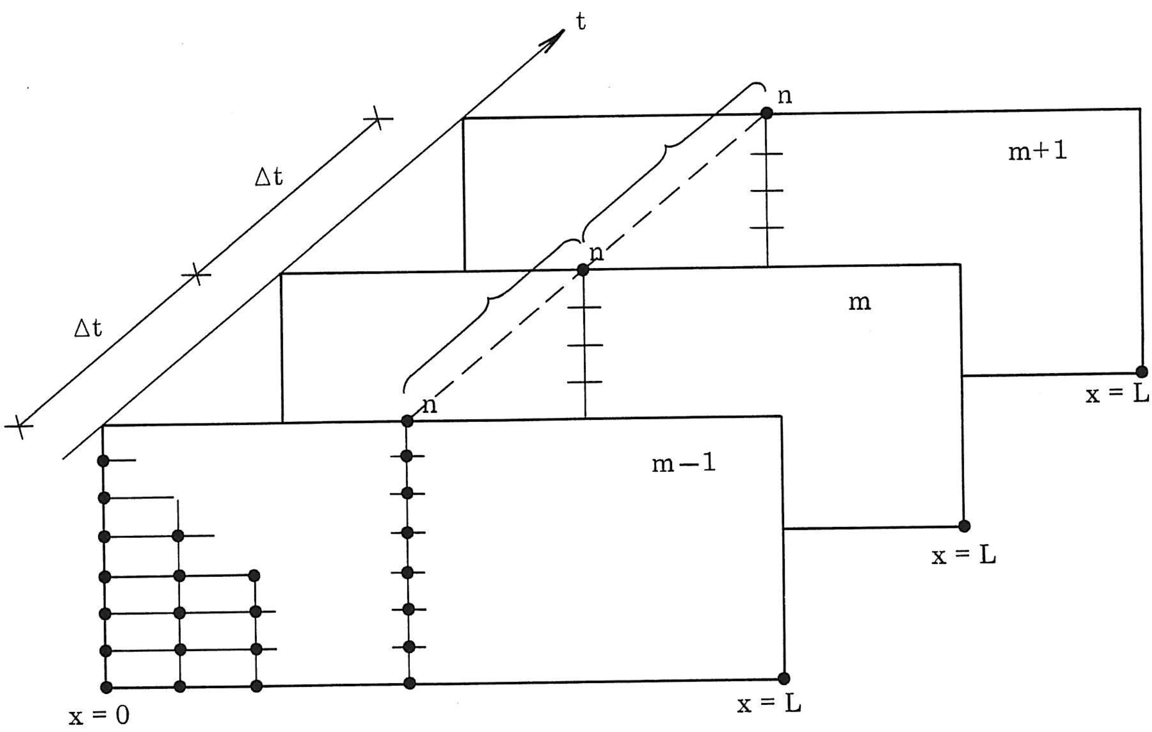


Fig. 9. Systems of equations of the problem.

It is seen that the equations of the problem considered are coupled through equations for nodal points of the free surface at subsequent points of time. A formal treatment of the problem leads to a very large system of equations. The form of the coupling suggests us to look for a formulation which allows us to consider the equations for a fixed time $t_m = m \cdot \Delta t$ only. Assuming that values of the velocity potential are known for $t_{m-1} = (m-1)\Delta t$ and for $t_m = (m \cdot \Delta t)$ it is possible to calculate the values of the velocity potential at points $z = h$ at the fixed time $t_{m+1} = (m+1)\Delta t$. Using the values it is easy to calculate the values of the velocity potential at all points of the assumed net. But, even in the last case, it is necessary to solve a relatively large number of equations in each time step. The aim of our further analysis is to find such a method of solving the problem which will allow us to obtain desirable results without the necessity of solving of the above-mentioned system of algebraic equations. To do this, let us divide the velocity potential into two parts:

$$\Phi = \phi + \Psi , \qquad (7.15)$$

where each of the components satisfies the Laplace equation and appropriate boundary conditions:

$$\frac{\partial \phi}{\partial x}\Big/_{x=0} = 0 , \qquad \Psi\Big/_{z=h} = 0 \qquad (7.16)$$

With reference to the latter conditions the boundary condition on the wall OA (x = 0) is solved by means of the second potential $\Psi(x,z,t)$. Substitution of (7.15) and (7.16) into (7.5) and (7.6) gives:

$$(\ddot{\phi} + g \cdot \frac{\partial \phi}{\partial z} + g \cdot \frac{\partial \Psi}{\partial z})\Big/_{z=h} = 0$$

and: (7.17)

$$\frac{\phi_n^{m-1} - 2\phi_n^m + \phi_n^{m+1}}{(\Delta t)^2} + g \cdot \frac{\partial \phi^m}{\partial z}\Big/_{z=h} + g \cdot \frac{\partial \Psi^m}{\partial z}\Big/_{z=h} = 0$$

In accordance with (7.16), the second potential $\Psi(x,z,t)$ satisfies the first $(n-1)$ equations (7.14). The same system of equations is valid for the potential $\phi(x,z,t)$, except the last equation (7.14) which has to be replaced by:

$$-\epsilon \cdot \phi_{n-}^m - 2 \cdot \phi_{n-1}^m + R \cdot \phi_n^m - \epsilon \cdot \phi_{n+}^m = -\beta \cdot (\phi_n^{m-1} + \phi_n^{m+1}) - 2 \cdot b \cdot \frac{\partial \Psi^m}{\partial z}\Big/_{z=h} \qquad (7.18)$$

It is important to note that the potential $\Psi(x,z,t)$ may be found independently of the second part $\phi(x,z,t)$ of the velocity potential. This may be easily done by means of (2.18) or by a similar formula that follows the discrete formulation of the problem. Hence, the term $\frac{\partial \Psi^m}{\partial z}$ which appears in (7.18) is assumed to be known in our further discussion. From the last equation it follows that:

$$\phi_n^{m+1} = -\phi_n^{m-1} - \frac{2b}{\beta} \cdot \frac{\partial \Psi^m}{\partial z}_{/z=h} + \frac{\epsilon}{\beta} \cdot (\phi_{n-}^m + \phi_{n+}^m) + \frac{2}{\beta} \cdot \phi_{n-1}^m - \frac{R}{\beta} \cdot \phi_n^m \tag{7.19}$$

Thus, having the values ϕ_n^{m-1}, ϕ_n^m and ϕ_{n-1}^{-m} of the velocity potential at points $z_n = h$ and $z_{n-1} = h - b$ at instants t_{m-1} and t_m it is possible to calculate the values ϕ_n^{m+1} at the subsequent point of time from the equation. To make such computations possible it is necessary to find additional relations which allow us to obtain the values ϕ_{n-1}^{m+1} of the potential at points $z_{n-1} = h - b$.

The free surface elevation at the instant t_{m+1} may be found from the relation (7.4):

$$\eta^{m+1} = \eta^m + \frac{1}{2} \cdot \Delta t \cdot (\Psi_{,z}^m + \Psi_{,z}^{m+1})_{/z=h} + \frac{1}{2} \Delta t \cdot (\phi_{,z}^m + \phi_{,z}^{m+1})_{/z=h} \tag{7.20}$$

To calculate the derivatives $\phi_{,z}^m$ and $\phi_{,z}^{m+1}$ for z = h the following relations may be used.

$$\phi_{,z\,/z=h} \simeq \frac{1}{2b} \cdot (\phi_{\tilde{n}} - \phi_{n-1})$$

$$\phi_{\tilde{n}} = -\epsilon \cdot (\phi_{n-} + \phi_{n+}) + 2 \cdot (1+\epsilon) \cdot \phi_n - \phi_{n-1} \tag{7.21}$$

where the second formula is in close accordance with the Laplace equation written for the point $z_n = h$. From the relations it follows that:

$$\phi_{,z}^{m+1}{}_{/z=h} \simeq \frac{1}{2b} \cdot [2 \cdot (1+\epsilon) \cdot \phi_n^{m+1} - 2 \cdot \phi_{n-1}^{m+1} - \epsilon \cdot (\phi_{n-}^{m+1} + \phi_{n+}^{m+1})]$$

$$\phi_{,z/z=h}^m \simeq \frac{1}{2b} \cdot [2 \cdot (1+\epsilon) \cdot \phi_n^m - 2 \cdot \phi_{n-1}^m - \epsilon \cdot (\phi_{n-}^m + \phi_{n+}^m)] \tag{7.22}$$

And, finally:

$$\eta^{m+1} = \eta^m + \frac{\Delta t}{4 \cdot b} \cdot [2 \cdot (1+\epsilon) \cdot (\phi_n^{m+1} + \phi_n^m) - 2 \cdot (\phi_{n-1}^{m+1} + \phi_{n-1}^m)$$

$$- \epsilon \cdot (\phi_{n-}^{m+1} + \phi_{n+}^{m+1} + \phi_{n-}^m + \phi_{n+}^m)] + \frac{\Delta t}{2} \cdot (\Psi_{,z}^{m+1} + \Psi_{,z}^m)_{/z=h} \tag{7.23}$$

At the moment, the main task is to find a way in which it would be possible to calculate ϕ_{n-1}^{m+1} assuming that the values ϕ_n^{m+1} are known. Knowing that the partial derivative of $\phi(x,z,t)$ with respect to x on the line x = 0 is equal to zero the function for negative vlaues of x may be extended in such a way that:

$$\phi(-x, z, t) = \phi(x, z, t) \tag{7.24}$$

Thus, the function is symmetric with respect to the line x = 0, and instead of the semi-infinite layer of fluid we will consider the infinite layer of fluid with prescribed values of ϕ at points of the free surface. The problem is linear and we may apply the principle of superposition.

Therefore, we may state that the value of the potential at any point within the layer is a linear function of independent values of the potential at points on the free surface. This suggests that a solution is found for the potential from a single value of ϕ at any given point on the line z = h. In other words, the values of ϕ at points z = h may be considered as a load on the layer. Hence, let us consider a layer of fluid with a point load at x_j = const. as it is illustrated in figure 10. Without loss of generality we may assume that:

$$\phi_{/z=h} = \begin{cases} 1 & \text{for } x = x_j \\ 0 & \text{for } x \neq x_j \end{cases} \tag{7.25}$$

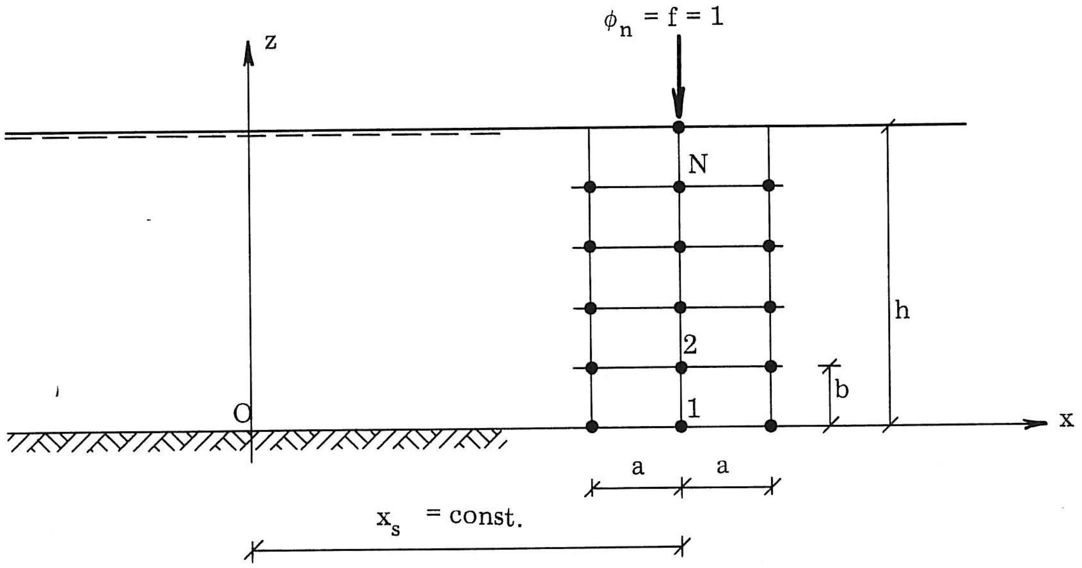


Fig. 10. Point-load of a layer of fluid.

Let $\varphi(x,z,t)$ denote the potential for the load (7.25). Equations (7.14) written for nodal points x_j = const. lead to the matrix equation:

$$-\epsilon \cdot (\varphi)^{\star} + [A] \cdot (\varphi) - \epsilon \cdot (\varphi^{\star\star}) = (RS) \, , \tag{7.26}$$

where $(RS)^T = (0, 0, \ldots, 0, f = 1)$, $(\varphi^{\star}) = (\varphi)_{/x = x_j - a}$ and $(\varphi^{\star\star}) = (\varphi)_{/x = x_j + a}$

The matrix [A] is indicated in table 3. It should be stressed that the number of equations (7.26) is equal to: N = n–1 where n is the total number of nodal points over the depth of the layer.

Table 3.

$$[A]=\begin{bmatrix} K & -2 & & & & \\ -1 & K & -1 & & & \\ & & \ddots & & & \\ & & & -1 & K & -1 \\ & & & & -1 & K \end{bmatrix}$$

To solve the set of equations (7.26) we apply a procedure given in [7]. Let us consider again the finite difference equation for the Laplace equation written for an arbitrary point ($x_j = (j-1)a$, $z_i = (i-1)b$) within the layer (see Fig. 8). Let us apply the well-known method of separation of variables for the discrete equation (7.2) and assume that:

$$\Phi_{i,j} = X(x_j) \cdot Z(z_i) = X_j \cdot Z_i \qquad (7.27)$$

where x_j and z_i correspond to the coordinates of the nodal point (i,j). Substitution of (7.27) into (7.2) yields:

$$\frac{1}{a^2} \cdot [X_{j-1} \cdot Z_i - 2 \cdot X_j \cdot Z_i + X_{j+1} \cdot Z_i] + \frac{1}{b^2} (X_j \cdot Z_{i-1} - 2 \cdot X_j \cdot Z_i + X_j \cdot Z_{i+1}) = 0 \qquad (7.28)$$

Upon division of the latter equation by $X_j \cdot Z_i$:

$$\frac{1}{a^2} \cdot \left(\frac{X_{j-1}}{X_j} - 2 + \frac{X_{j+1}}{X_j}\right) + \frac{1}{b^2} \cdot \left(\frac{Z_{i-1}}{Z_i} - 2 + \frac{Z_{i+1}}{Z_i}\right) = 0 \qquad (7.29)$$

The last equation should be satisfied for each value of "i" and "j". Hence the terms in parentheses should be equal to constants. Therefore:

$$\frac{X_{j-1} + X_{j+1}}{X_j} = 2 \cdot \lambda \ , \qquad (7.30)$$

where λ is a complex number in general.

From the equation it follows:

$$X_{j-1} - 2 \cdot \lambda \cdot X_j + X_{j+1} = 0 \qquad (7.31)$$

The latter expression describes the set of difference equations which leads to the solution:

$$X_j = \exp(-j \cdot r \cdot a) \ , \quad X_j = \exp(j \cdot r \cdot a) \ , \tag{7.32}$$

when the following notation is used:

$$\lambda = \cosh ra \tag{7.33}$$

According to the results, the following relation holds:

$$\Phi_{i,j+s} = \Phi_{i,j} \cdot X_s = \Phi_{i,j} \cdot e^{\pm rsa} \tag{7.34}$$

where r is an unknown parameter for the time being.

The last equation means that we express all the values at the points $x = (j + s - 1)a$ in terms of the values of the potential on the j – th vertical line. Now, return to equation (7.26) and consider the homogeneous case for which

$$\varphi_{/z=h} = 0 \tag{7.35}$$

Because of the Sommerfeld condition the first solution of (7.32) is chosen under the condition that a real part of r is greater than zero. Thus, for the nodal points $j-1, j$ and $j+1$:

$$\varphi_{i,j-1} = \varphi_i^{\star} = \varphi_{i,j} \cdot e^{ra} \ , \quad \varphi_i = \varphi_{i,j} \ , \quad \varphi_{i,j+1} = \varphi_i^{\star\star} = \varphi_{i,j} \cdot e^{-ra} \tag{7.36}$$

Substitution of the last expressions into i–th equation (7.26) leads to the result:

$$-\varphi_{i-1,j} + (K - \lambda^{\star}) \cdot \varphi_{i,j} - \varphi_{i+1,j} = 0 \ , \tag{7.37}$$

where:

$$\lambda^{\star} = 2 \cdot \epsilon \cdot \lambda = 2 \cdot \left(\frac{b}{a}\right)^2 \cdot \cosh ra \tag{7.38}$$

It is seen that equations (7.37) correspond directly to nodal points belonging to one vertical line. Thus, the index "j" may be omitted in the expression, and the set of equations (7.37) written for $i = 1, 2 \ldots, n-1 = N$ assumes the form:

$$[B] \cdot (\varphi) = 0 \tag{7.39}$$

where the matrix [B] is shown in table 4.

Table 4.

$$[B] = \begin{bmatrix} K-\lambda^{\star} & -2 & & & & \\ -1 & K-\lambda^{\star} & -1 & & & \\ & & \ddots & & & \\ & & & -1 & K-\lambda^{\star} & -1 \\ & & & & -1 & K-\lambda^{\star} \end{bmatrix}$$

Hence, the problem has been reduced to the eigenvalues and eigenvectors problem for the matrix [B]. Pre-multiplication of equation (7.39) by 1/2 leads to the equation:

$$[C] \cdot (\varphi) = 0 \tag{7.40}$$

where the symmetrical matrix [C] is shown in table 5.

Table 5.

$$[C] = \begin{bmatrix} \frac{1}{2}(K-\lambda^{\star}) & -1 & & & & \\ -1 & K-\lambda^{\star} & -1 & & & \\ & & \ddots & & & \\ & & & -1 & K-\lambda^{\star} & -1 \\ & & & & -1 & K-\lambda^{\star} \end{bmatrix}$$

The equation (7.40) has not a standard form for the eigenvalues problem. To bring the equation to standard form the following transformation is introduced:

$$(\varphi) = [D]\,(\varphi^{\star}) \,, \tag{7.41}$$

where the non-singular diagonal matrix [D] is given in table 6.

Table 6.

$$[D] = \begin{bmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

Substitution of (7.41) into (7.40) and left-multiplication of the result by the matrix [D] give:

$$[D] \cdot [C] \cdot [D] \cdot (\varphi^{\star}) = [G] \cdot (\varphi^{\star}) = 0 \tag{7.42}$$

The matrix [G] is indicated in table 7. The matrix is symmetrical and has the standard form of the eigenvalues problem.

Table 7.

$$[G] = \begin{bmatrix} K-\lambda^{\star} & -\sqrt{2} & & & \\ -\sqrt{2} & K-\lambda^{\star} & -1 & & \\ & & \ddots & & \\ & & -1 & K-\lambda^{\star} & -1 \\ & & & -1 & K-\lambda^{\star} \end{bmatrix}$$

All the elements of the matrix are real quantities. Therefore, it is possible to make the following substitutions [8]:

$$K - \lambda^{\star} = \begin{cases} 2 \cdot \cosh\kappa \cdot \text{ if } K - \lambda^{\star} \geqslant 2 & \text{a)} \\ 2 \cdot \cos\kappa \quad \text{if } -2 \leqslant K - \lambda^{\star} \leqslant 2 & \text{b)} \\ -\ 2 \cdot \cosh\kappa \quad \text{if } K - \lambda^{\star} \leqslant -2 & \text{c)} \end{cases} \tag{7.43}$$

Substitution of (7.43a) into (7.42) enables us to express $\varphi_2^{\star}$ in terms of $\varphi_1^{\star}$ from the first equation of (7.42) and then $\varphi_3^{\star}$ from the second equation of (7.42) and so on. The result of such the procedure is:

$$\varphi_r^{\star} = \sqrt{2} \cdot \varphi_1 \cdot \cosh\,(r-1)\kappa\ , \quad r = 2,3,\ldots,N \tag{7.44}$$

From substitution of the last expression into the last equation (7.42) it follows that:

$$\sqrt{2} \cdot \varphi_1^{\star} \cdot \cosh(n-1)\kappa = 0 \tag{7.45}$$

It can be seen that the above equation has no solution.

The similar procedure for the substitution (7.43b) leads to the formula:

$$\varphi_r^{\star} = \sqrt{2} \cdot \varphi_1^{\star} \cdot \cos(r-1)\kappa\ , \quad r = 2,3,\ldots,n-1 \tag{7.46}$$

For the case the last equation (7.42) gives:

$$\varphi_1^{\star} \cdot \sqrt{2} \cdot \cos(n-1)\kappa = 0 \tag{7.47}$$

The equation has an infinite number of roots:

$$\kappa_j = \frac{2 \cdot j - 1}{2(n-1)} \cdot \pi\ , \quad j = 1,2,\ldots \tag{7.48}$$

Although, the number of roots is infinite, it is sufficient to consider only the first (n - 1) values of (7.48) because the subsequent values for $j > n-1$ lead to eigenvectors existing for $j \leqslant n-1$. Assuming that $\varphi_1^{\star} = 1/\sqrt{2}$, the eigenvectors of the matrix [B] for the case (7.46) are expressed by the following formula:

$$(N_j)^T = \left(\frac{1}{\sqrt{2}}\ , \cos\kappa_j, \cos 2\kappa_j, \ldots, \cos(n-2)\kappa_j\right), \quad j = 1,2,\ldots,n-1 \tag{7.49}$$

The third substitution (7.43) leads to the solution:

$$\varphi_r^{\star} = \sqrt{2} \cdot (-1)^{r-1} \cdot \varphi_1^{\star} \cdot \cosh(r-1)\kappa\ , \quad r = 2,3,\ldots, n-1 \tag{7.50}$$

From substitution of the expression into the last equation of (7.42) it follows:

$$(-1)^{n-2} \cdot \sqrt{2} \cdot \varphi_1^{\star} \cdot \cosh(n-1)\kappa = 0 \tag{7.51}$$

The above equation has no solution. Thus as assumed (n – 1) eigenvalues and (n – 1) eigenvectors of the matrix [G] are obtained. The eigenvalues are distinct so, the eigenvectors are unique within to multiplication by any number. Because the matrix [G] is symmetric, the eigenvectors (7.49) are mutually orthogonal. The eigenvectors can be normalized. Their length may be found from the relation:

$$\|N_j\|^2 = (N_j)^T \cdot (N_j) = \frac{1}{2} + \cos^2\kappa_j + \cos^2 2\kappa_j + \ldots + \cos^2(n-2)\kappa_j \tag{7.52}$$

Knowing that:

$$\sum_{k=0}^{n} \cos^2 kx = \frac{n+2}{2} + \frac{\cos(n+1)x \cdot \sin nx}{2 \cdot \sin x} , \tag{7.53}$$

the expression (7.52) leads to the result:

$$\|N_j\|^2 = \frac{n-1}{2} = \frac{N}{2} , \tag{7.54}$$

where N means the dimension of the square matrix [G].

It is seen that the set of eigenvectors (7.49) has the same norm:

$$\|N_j\| = \sqrt{\frac{n-1}{2}} \quad , \quad j = 1,2,\ldots,n-1 \tag{7.55}$$

The eigenvalues of the matrix [A] (table 3) are exactly the same as those of the matrix [G] and they are found from the relations (7.43) and (7.48):

$$\lambda_j^{\star} = K - 2 \cdot \cos\kappa_j = 2 \cdot [1 + (\frac{b}{a})^2 - \cos\kappa_j] \tag{7.56}$$

The eigenvectors of the matrix [A] result from substitution of (7.49) into (7.41). Denoting them by (M_j):

$$(M_j)^T = (1, \cos\kappa_j, \cos 2\kappa_j, \ldots, \cos(n-2)\kappa_j) , \quad j = 1,2,\ldots, \quad n-1 \tag{7.57}$$

The set of eigenvectors (7.57) forms a linearly independent set of the vectors.

From substitution of (7.56) into (7.38) the following relation results:

$$\lambda_j = \cosh r_j a = 1 + (\frac{a}{b})^2 \cdot (1 - \cos\kappa_j), \qquad j = 1,2,\ldots,n-1 \tag{7.58}$$

It is seen that $\lambda_j > 1$ and r_j ($j = 1,2 \ldots , n-1$) are all real numbers. Thus, the substitution (7.33) and the solution (7.34) lead to the conclusion that for the boundary condition $\phi_{/z=h} = 0$, the discrete solution results in a standing wave only. This result corresponds directly to the analytical solution (2.18) for the case of zero pressure condition on the upper boundary of the layer (see Fig. 1).

Return to the set of equations (7.26). For the load on the layer of fluid as indicated in Fig. 9, the solution for velocity potential (φ) should be symmetric with respect to the line x_j = const.

Thus, $(\varphi^{\star}) = (\varphi^{\star\star})$ and (7.26) are replaced by the following set of equations:

$$[A] \cdot (\varphi) - 2 \cdot \epsilon \cdot (\varphi^{\star}) = (RS) \tag{7.59}$$

On the other hand, the matrix [A] is exactly the same as the matrix [B] and because of that it is reasonable to express the potentials (φ) and $(\varphi^{\star})$ in terms of eigenvectors of the matrices. In accordance with (7.36) and (7.39):

$$(\varphi) = \sum_{j=1}^{n-1} A_j \cdot (M_j) \ , \quad (\varphi^{\star}) = \sum_{j=1}^{n-1} A_j \cdot e^{-r_j a} \cdot (M_j) \ , \tag{7.60}$$

where A_j are unknown constants.

Substitution of the last results into (7.59) gives:

$$\sum_{j=1}^{n-1} A_j \cdot [A] \cdot (M_j) - 2 \cdot \epsilon \cdot \sum_{j=1}^{n-1} A_j \cdot e^{-r_j a} \cdot (M_j) = (RS) \tag{7.61}$$

But, from the above discussion it is known that:

$$[A] \cdot (M_j) = \lambda_j^{\star} \cdot (M_j) \ , \tag{7.62}$$

and finally:

$$\sum_{j=1}^{n-1} A_j \cdot [\lambda_j^{\star} - 2 \cdot \epsilon \cdot e^{-r_j a}] \cdot (M_j) = (RS) \tag{7.63}$$

Substituting:

$$(M_j) = [D] \cdot (N_j) \tag{7.64}$$

into (7.63) and making simple manipulations one can obtain:

$$A_j = \frac{1}{\lambda_j^{\star} - 2 \cdot \epsilon \cdot e^{-r_j a}} \cdot \frac{(N_j)^T \cdot [D]^{-1} \cdot (RS)}{\|N_j\|^2} \ , \quad j = 1,2 \ldots, n-1 \tag{7.65}$$

The last formula may be brought to a simpler form. Knowing that:

$$\lambda_j^{\star} - 2 \cdot \epsilon \cdot e^{-r_j a} = 2 \cdot \epsilon \cdot (\cosh r_j a - e^{-r_j a}) = 2 \cdot \epsilon \cdot \sinh r_j a \ , \tag{7.66}$$

and:

$$(N_j)^T \cdot [D]^{-1} (RS) = \cos(n-2)\kappa_j = \cos(N-1)\kappa_j \ , \tag{7.67}$$

we get:

$$A_j = \left(\frac{a}{b}\right)^2 \cdot \frac{1}{(n-1) \cdot \sinh r_j a} \cdot \cos(n-2)\kappa_j \cdot j \quad j = 1,2, \ldots (n-1) \tag{7.68}$$

In this way, for an arbitrary load $\phi_n = f_s$ the solution for the velocity potential within the layer assumes the form:

$$(\varphi) = (\frac{a}{b})^2 \cdot \frac{1}{n-1} \cdot f_s \cdot \sum_{j=1}^{n-1} \frac{\cos(n-2)\kappa_j}{\sinh r_j a} \cdot e^{-r_j |x_r - x_s|} \cdot (M_j) \qquad (7.69)$$

The last expression describes the vector of the potential at x_r = const. Having the solution it is easy to construct a solution for the case of prescribed values of the velocity potential at points z = h. Following the principle of superposition we have:

$$(\phi)_{(x)} = f_0 \cdot \sum_{j=1}^{n-1} A_j \cdot e^{-r_j x} \cdot (M_j) + \sum_{s=1} f_s \cdot \sum_{j=1}^{n-1} A_j \cdot [e^{-r_j |x-x_s|} + e^{-r_j (x+x_s)}] \cdot (M_j) \, , \qquad (7.70)$$

where x and x_s assume the discrete set of values: o, a, 2a, ... The last term in the square brackets on the right hand side of (7.70) results from symmetrical extension of the solution on the region $x < 0$. A remark is needed. In practical calculations the summation over s should be finite. This depends on the dimension L of the region under consideration (see Fig. 7). At the beginning of the motion it is plausible to assume $\phi/x = L$ and $\phi_{,x}/x = L$ are both equal to zero. With passing time, however, such assumption cannot be held and it is necessary to employ transmitting boundary conditions on the boundary x = L

8. TRANSMITTING CONDITIONS FOR A LAYER OF FLUID

In the previous section the method of separation of variables was applied to the Laplace equation in discrete formulation. The result of such a procedure was the set of equations corresponding to nodal points belonging to one vertical line x_j = const. (Eqs. 7.37). In this case the equations were written for points z = 0, b, . . ., h − b. The same procedure may be applied directly to the equations (7.14) and (7.18) which include points on the boundary z = h. Following the procedure, the set of equations (7.14) may be written in the form :

$$[H] \cdot (\phi) = (RS) \, , \qquad (8.1)$$

where (ϕ) corresponds to the fixed time $t_m = m \cdot \Delta t$, and:

$$(RS)^T = (0, 0, \ldots, 0, -\beta \cdot \phi_n^{m-1} - \beta \cdot \phi_n^{m+1} - 2 \cdot b \cdot \frac{\partial \Psi^m}{\partial z} /_{z=h}) \qquad (8.2)$$

The number of equations (8.1) exceeds the number of equations (7.39) by one. The square n x n matrix [H] is indicated in table 8.

Table 8.

$$[H] = \begin{bmatrix} K-\lambda^{\star} & -2 & & & & \\ -1 & K-\lambda^{\star} & -1 & & & \\ & & \ddots & & & \\ & & & -1 & K-\lambda^{\star} & -1 \\ & & & & -2 & R-\lambda^{\star} \end{bmatrix}$$

The terms that appear in the table are defined by (7.12), (7.13), and (7.38). The matrix does not depend on the discrete time sequence. Therefore, we will seek for a solution of (ϕ) in the form of linear combination of eigenvectors of the matrix. Thus, consider the homogeneous set of equations corresponding to (8.1):

$$[P] \cdot (\phi) = 0 \, , \tag{8.3}$$

where the matrix [P] is obtained by dividing the first and last rows of the matrix [H] by two. Substitution of

$$(\phi) = [D] \cdot (\varphi) \tag{8.4}$$

into (8.3) and pre-multiplication of the result by the diagonal matrix [D] give:

$$[D] \cdot [P] \cdot [D] \cdot (\varphi) = [Q] \cdot (\varphi) = 0 \tag{8.5}$$

The matrices [D] and [Q] are shown in the subsequent tables 9 and 10, respectively. The matrix [Q] is symmetric and it has the standard form for the eigenvalue problem. To find the eigenvalues and eigenvectors of the matrix substitutions (7.43) are used.

Table 9.

$$[D] = \begin{bmatrix} \sqrt{2} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \sqrt{2} \end{bmatrix}$$

Table 10.

$$[Q] = \begin{bmatrix} K-\lambda^{\star} & -\sqrt{2} & & & & \\ -\sqrt{2} & K-\lambda^{\star} & -1 & & & \\ & & \ddots & & & \\ & & & -1 & K-\lambda^{\star} & -\sqrt{2} \\ & & & & -\sqrt{2} & R-\lambda^{\star} \end{bmatrix}$$

The first substitution (7.43) leads to the solution:

$$\varphi_r^{\star} = \sqrt{2} \cdot \varphi_1 \cdot \cosh(r-1)\kappa \ , \quad r = 2,3,\ldots,n-1 \qquad (8.6)$$

$$\varphi_n = \varphi_1 \cdot \cosh(n-1)\kappa$$

From substitution of (8.6) into equation (8.5) it follows that

$$\beta = \sinh\kappa \cdot \tanh(n-1)\kappa \qquad (8.7)$$

Equation (8.7) has only one root κ_o, which leads to the eigenvalue

$$\lambda_o = K - 2 \cdot \cosh\kappa_o = 2 \cdot [1 + (\frac{b}{a})^2 - \cosh\kappa_o] \qquad (8.8)$$

of the matrix. The eigenvalue is followed by the relevant eigenvector:

$$(N_o)^T = (\frac{1}{\sqrt{2}}, \cosh\kappa_o, \cosh 2\kappa_o, \ldots, \cosh(n-2)\kappa_o, \frac{1}{\sqrt{2}} \cdot \cosh(n-1)\kappa_o) \qquad (8.9)$$

The square of the vector magnitude is described by the formula:

$$\|N_o\|^2 = \frac{1}{2} + \cosh^2\kappa_o + \ldots + \cosh^2(n-2)\kappa_o + \frac{1}{2} \cdot \cosh^2(n-1)\kappa_o =$$

$$= \frac{n-1}{2} + \frac{1}{2\beta} \cdot \cosh\kappa_o \cdot \sinh^2(n-1)\kappa_o \qquad (8.10)$$

The second substitution (7.43) gives the solutions:

$$\varphi_r = \sqrt{2} \cdot \varphi_1 \cdot \cos(r-1)\kappa \ , \quad r = 2,3,\ldots,n-1$$

$$\varphi_n = \varphi_1 \cdot \cos(n-1)\kappa \qquad (8.11)$$

Upon inserting the equations (8.11) into equation (8.5) we get:

$$\beta = -\sin\kappa_j \cdot \tan(n-1)\kappa_j\,, \quad j = 1,2,\ldots \tag{8.12}$$

The equation obtained has an infinite number of roots, but the relevant eigenvalues of the matrix [Q] form the finite set:

$$\lambda_j^{\star} = K - 2\cdot\cos\kappa_j = 2\cdot[1 + (\frac{b}{a})^2 - \cos\kappa_j]\,, \quad j = 1,2\ldots \tag{8.13}$$

It is seen that it is sufficient to confine our attention to the interval $0 < \kappa_j < \pi$ where there are (n – 1) roots of (8.12). The eigenvectors corresponding to the eigenvalues (8.13) are:

$$(N_j)^T = (\frac{1}{\sqrt{2}}, \cos\kappa_j, \cos 2\kappa_j, \ldots, \cos(n-2)\kappa_j, \frac{1}{\sqrt{2}}\cos(n-1)\kappa_j) \tag{8.14}$$

The square of magnitudes of the eigenvectors is described by the formula:

$$\|N_j\|^2 = \frac{1}{2} + \cos^2\kappa_j + \cdots + \cos^2(n-2)\kappa_j + \frac{1}{2}\cdot\cos^2(n-1)\kappa_j =$$

$$= \frac{n-1}{2} - \frac{1}{2\beta}\cdot\cos\kappa_j\cdot\sin^2(n-1)\kappa_j\,, \quad j = 1,2,\ldots,n-1 \tag{8.15}$$

The third substitution (7.43) provides the solutions:

$$\begin{aligned} \varphi_r &= (-1)^{r-1}\cdot\sqrt{2}\cdot\varphi_1\cdot\cosh(r-1)\kappa\,, \quad r = 2,3,\ldots,n-1 \\ \varphi_n &= (-1)^n\cdot\varphi_1\cdot\cosh(n-1)\kappa\,, \end{aligned} \tag{8.16}$$

which upon insertion into equation (8.5) give:

$$\beta = -\sinh\kappa\cdot\tanh(n-1)\kappa \tag{8.17}$$

It is seen that (8.17) has no solution. Thus, the number of eigenvalues and eigenvectors of the matrix [Q] is, as it should be, equal to n. The eigenvalues are distinct. Thus, the eigenvectors (8.9) and (8.14) are mutually orthogonal. Using the eigenvectors it is easy to calculate the eigenvectors of the matrix [H] (the eigenvalues of the matrices [Q] and [H] are the same). Simple manipulations lead to the result:

$$\begin{aligned} (M_0)^T &= (1, \cosh\kappa_o, \ldots, \cosh(n-2)\kappa_o, \cosh(n-1)\kappa_o) \\ (M_j) &= (1, \cos\kappa_j\,, \ldots, \cos(n-2)\kappa_j, \cos(n-1)\kappa_j), \quad j = 1,2,\ldots, n-1 \end{aligned} \tag{8.18}$$

It should be stressed, however, that the latter eigenvectors are not mutually orthogonal, – they are linearly independent only.

From substitution of (8.8) and (8.13) into (7.38) it follows that:

$$\lambda_o = \cosh r_o a = 1 + (\frac{a}{b})^2 \cdot (1 - \cosh\kappa_o)$$
$$\lambda_j = \cosh r_j a = 1 + (\frac{a}{b})^2 \cdot (1 - \cos\kappa_j), \quad j = 1,2, \ldots, n-1, \tag{8.19}$$

where κ_o and κ_j are roots of the dispersion relations (8.7) and (8.12), respectively. To describe the distribution of the solutions along the x-axis it is necessary to calculate $r_o a$ and $r_j a$ ($j = 1,2, \ldots, n-1$). It is seen from the latter equations that $r_o a$ is a complex number while $r_j a$ ($j = 1,2, \ldots, n-1$) are all real numbers. Moreover, if

$$\cosh\kappa_o < 1 + 2(\frac{b}{a})^2 \tag{8.20}$$

then:

$$-1 < \lambda_o < 1, \tag{8.21}$$

and $r_o a$ is a pure imaginary number. The number is given by the formula:

$$r_o a \overset{i}{=} \cos^{-1}[1 + (\frac{a}{b})^2 \cdot (1 - \cosh\kappa_o)] \tag{8.22}$$

The second equation (8.19) leads to the result:

$$r_j a = \cosh^{-1}[1 + (\frac{a}{b})^2 \cdot (1 - \cos\kappa_j)], \quad j = 1,2, \ldots, n-1 \tag{8.23}$$

The case (8.20) corresponds to solutions in continuum for surface propagating waves. The condition (8.20) and the solutions (8.18), (8.22) and (8.23) lead to the expression for the velocity potential:

$$(\Phi) = (A_o \cdot \cos r_o x + B_o \cdot \sin r_o x) \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j x} \cdot (M_j), \tag{8.24}$$

where A_o, B_o and A_j ($j = 1,2, \ldots, n-1$) are constants of the solution and $x = m \cdot a$ ($m = 0,1,2 \ldots$) is the discrete set of values of x. For the case $\lambda_o < -1$, the first equation (8.19) gives:

$$r_o a = \cosh^{-1}[(\frac{a}{b})^2 \cdot (\cosh\kappa_o - 1) - 1] + i \cdot (2 \cdot l - 1) \cdot \pi, \tag{8.25}$$

where $i = \sqrt{-1}$ and l is an arbitrary natural number. The relevant potential for the case is given by the formula:

$$(\Phi) = (-1)^m \cdot A_o \cdot e^{-r_o x} \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j x} \cdot (M_j), \tag{8.26}$$

where $x = m \cdot a$ ($m = 0,1,2, \ldots$) and $r_o a$ is the real part of the complex number (8.25). The latter solution does not correspond to that for continuum. It is seen that the first term in (8.26) changes sign from one nodal point to the next in the x- direction.

Having the solutions (8.24) and (8.26), the values of the velocity potential at points $x > L$ (Fig. 7) may be expressed in terms of the values of the potential at points $x = L$ and $x = L - a$. For the case $\lambda_o > -1$, the appropriate potentials are:

$$\Phi(x = L) = (\Phi_L) = (A_o \cdot \cos r_o L + B_o \cdot \sin r_o L) \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j L} \cdot (M_j)$$

$$(\Phi_{L-a}) = [A_o \cdot \cos r_o (L-a) + B_o \cdot \sin r_o (L-a)] \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j (L-a)} \cdot (M_j) \tag{8.27}$$

$$(\Phi_{L+a}) = [A_o \cdot \cos r_o (L+a) + B_o \cdot \sin r_o (L+a)] \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j (L+a)} \cdot (M_j)$$

The case $\lambda_o < -1$ provides the solutions:

$$(\Phi_L) = A_o \cdot e^{r_o L} \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j L} \cdot (M_j)$$

$$(\Phi_{L+a}) = -A_o \cdot e^{-r_o (L+a)} \cdot (M_o) + \sum_{j=1}^{n-1} A_j \cdot e^{-r_j (L+a)} \cdot (M_j) \tag{8.28}$$

Now, consider again the case $\lambda_o > -1$. Substitution of (8.4) into the first and second equations (8.27) and pre-multiplication of the result subsequently by $[D]^{-1}$ and $(N_o)^T$ give:

$$A_o \cdot \cos r_o L + B_o \cdot \sin r_o L = \frac{(N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot (\Phi_L)$$

$$A_o \cdot \cos r_o (L-a) + B_o \cdot \sin r_o (L-a) = \frac{(N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot (\Phi_{L-a}) \tag{8.29}$$

Inserting the latter results into the third equation (8.27) the following expression is obtained:

$$A_o \cdot \cos r_o (L+a) + B_o \cdot \sin r_o (L+a) = \frac{(N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot [2 \cdot \cos r_o a \cdot (\Phi_L) - (\Phi_{L-a})] \tag{8.30}$$

The similar operations with the series terms in (8.27) give:

$$A_j \cdot e^{-r_j (L+a)} = e^{-r_j a} \cdot \frac{(N_j)^T \cdot [D]^{-1}}{\|N_j\|^2} \cdot (\Phi_L) \tag{8.31}$$

Substitution of the equations (8.30) and (8.31) into the third of (8.27) leads to the solution:

$$(\Phi_{L+a}) = \frac{(N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot [2 \cdot \cos r_o a \cdot (\Phi_L) - (\Phi_{L-a})] \cdot (M_o) + \sum_{j=1}^{n-1} e^{-r_j a} \cdot \frac{(N_j)^T \cdot [D]^{-1}}{\|N_j\|^2} \cdot (\Phi_L) \cdot (M_j) \tag{8.32}$$

The same procedure is applied to the case $\lambda_o < -1$. The final result of the procedure for the case is:

$$(\Phi_{L+a}) = -e^{-r_o a} \cdot \frac{(N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot (\Phi_L) \cdot (M_o) + \sum_{j=1}^{n-1} e^{-r_j a} \cdot \frac{(N_j)^T \cdot [D]^{-1}}{\|N_j\|^2} \cdot (\Phi_L) \cdot (M_j) \quad (8.33)$$

The solutions (8.32) and (8.33) may be transformed into the following:

$\lambda_o > -1.$

$$(\Phi_{L+a}) = \frac{(M_o) \cdot (N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot [2 \cdot \cos r_o a \cdot (\Phi_L) - (\Phi_{L-a})] + \sum_{j=1}^{n-1} e^{-r_j a} \cdot \frac{(M_j) \cdot (N_j)^T \cdot [D]^{-1}}{\|N_j\|^2} \cdot (\Phi_L) \quad (8.34)$$

$\lambda_o < -1$

$$(\Phi_{L+a}) = -e^{-r_o a} \cdot \frac{(M_o) \cdot (N_o)^T \cdot [D]^{-1}}{\|N_o\|^2} \cdot (\Phi_L) + \sum_{j=1}^{n-1} e^{-r_j a} \cdot \frac{(M_j) \cdot (N_j)^T \cdot [D]^{-1}}{\|N_j\|^2} \cdot (\Phi_L)$$

The latter formulae describe the non-local transmitting boundary conditions which should be superimposed on the boundary $x = L$.

9. NUMERICAL COMPUTATIONS

Following the considerations and procedures developed in the previous sections the numerical calculations have been made. The main aim of the calculations was to find the free surface elevation of the water layer at subsequent time steps $t_m = m \cdot \Delta t$ $(m = 1,2, \ldots)$. The computations were performed for the excitement (1.1) and the following data input:

$h = 40$ cm, $\quad d = 1$ cm, $\quad \lambda = h, \quad L = 240 \text{ cm} = 120 \cdot a$

$a = b = 2$ cm, $\quad k_o = \frac{2\pi}{\lambda} = 0.157080 \text{ cm}^{-1}, \quad \omega = \sqrt{g \cdot k_o \cdot \tanh k_o h} = 12.413 \text{ sec}^{-1},$

$c_f = \frac{\omega}{k_o} = 79.0266 \text{ cm} \cdot \text{sec}^{-1}, \quad \Delta t = \frac{a}{c_f} = 0.025308 \text{ sec},$

$T = \frac{2\pi}{\omega} = 20 \cdot \Delta t = 0.506159 \text{ sec}, \quad \nu = 0.0131 \text{ cm}^2 \cdot \text{sec}^{-1}.$

Some of the results obtained are indicated in the Fig. 11, where the plots represent the free surface elevation at instants $t_m = m \cdot \Delta t$. The ordinates of the plots give the free surface elevation which is measured in cm from the level of undisturbed fluid. The abscissas of the plots express the distance from the rigid wall OA (Fig. 1) and are indicated in a-units ($a = 2$ cm).

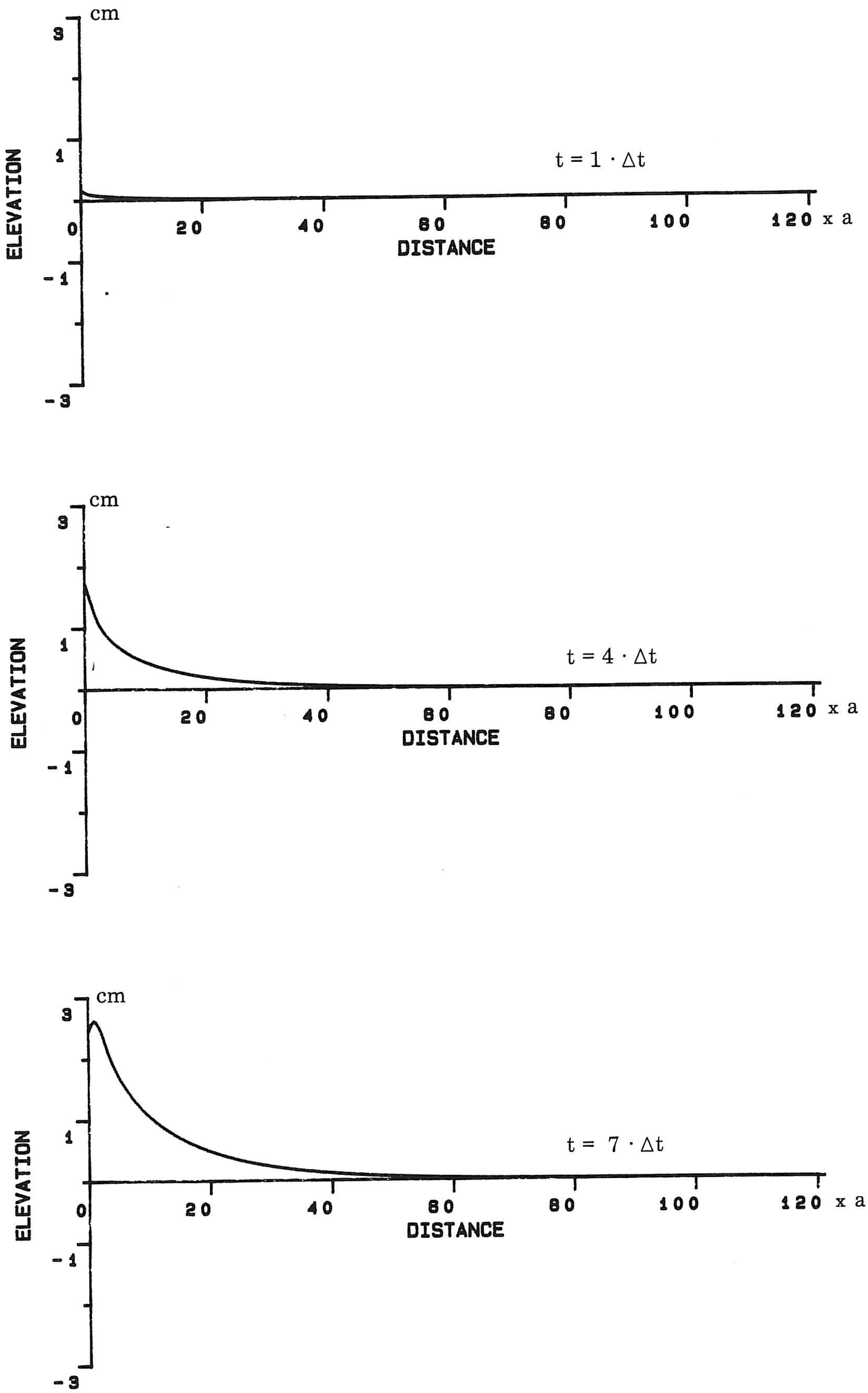


Fig. 11. The free-surface elevations for a sequence of time steps.

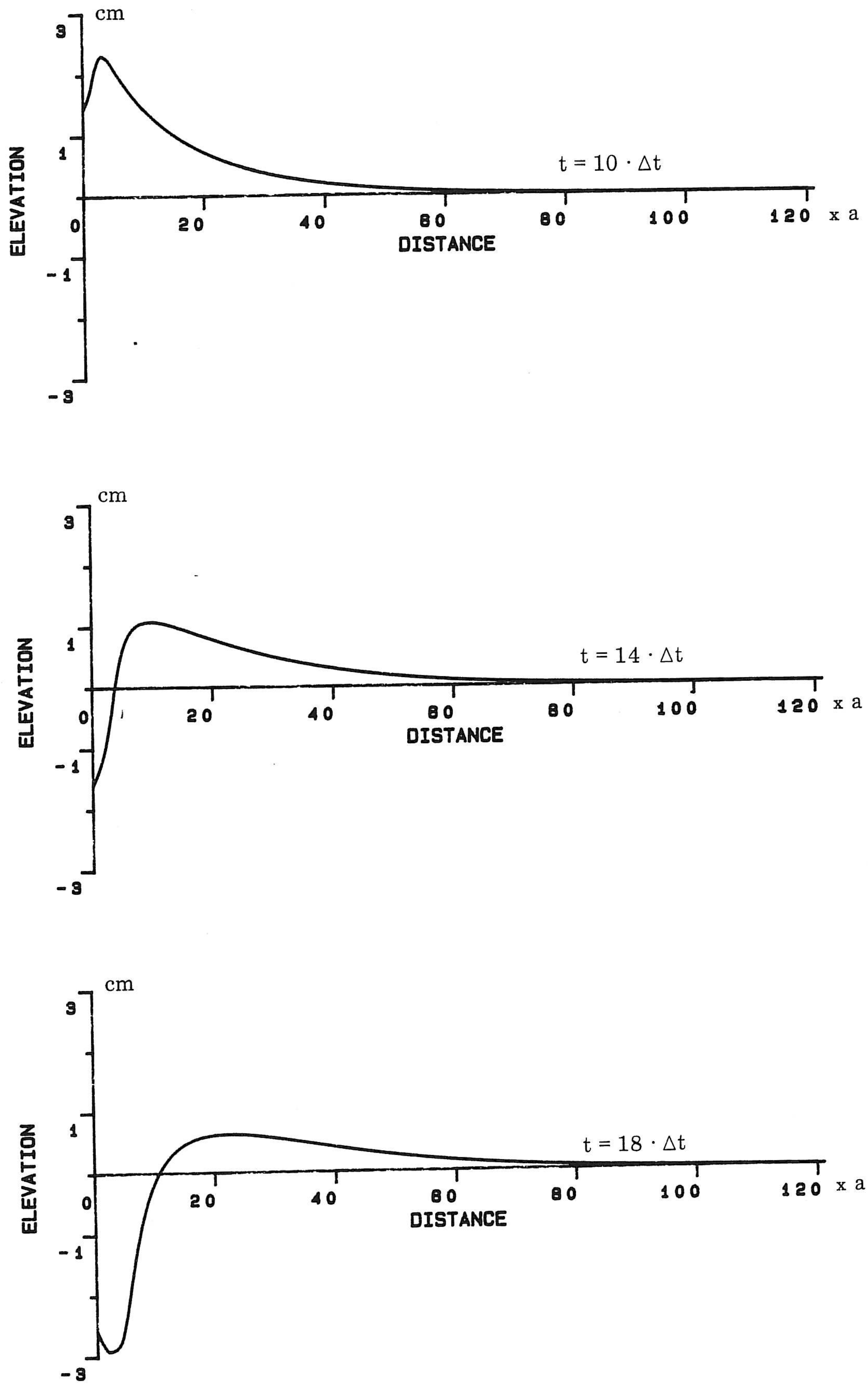


Fig. 11. (Continued)

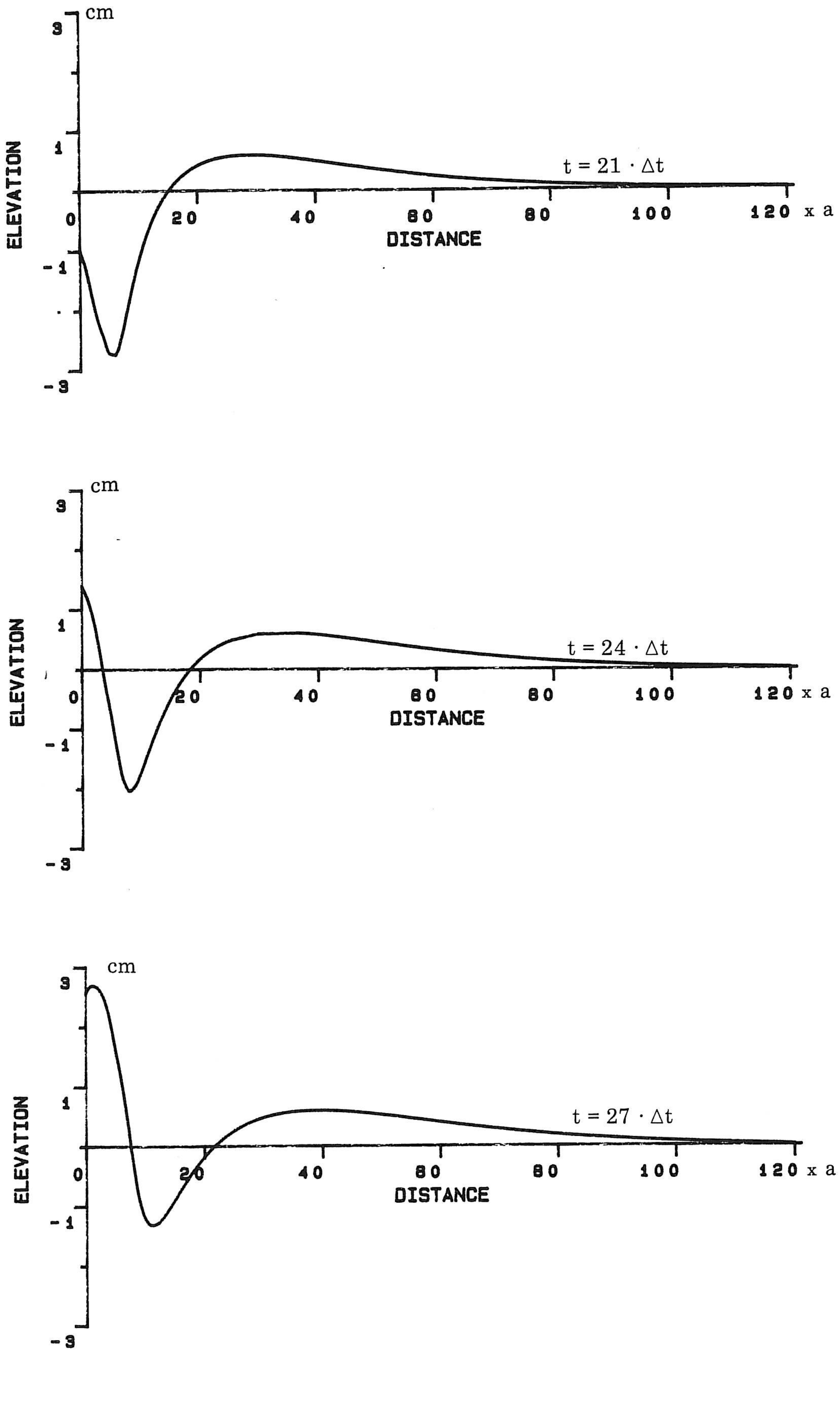
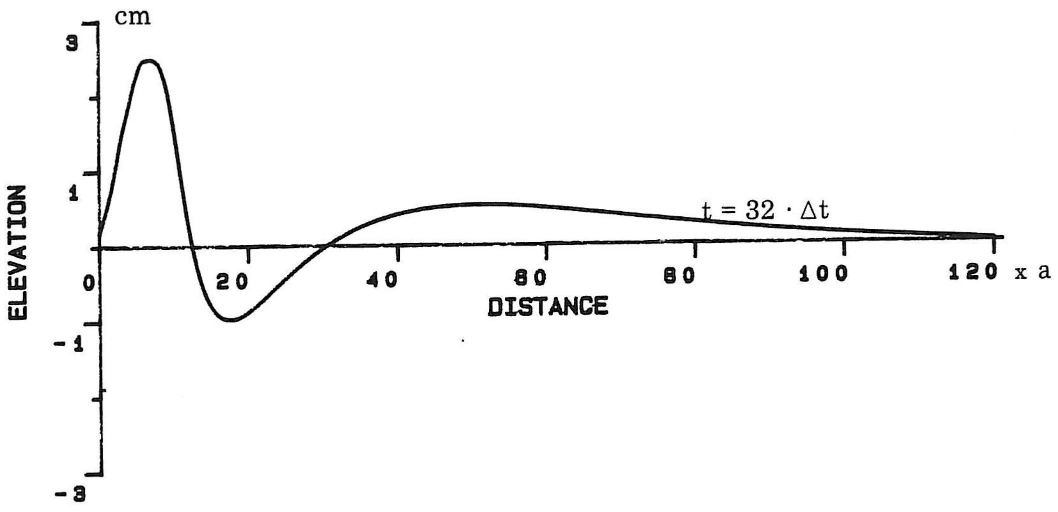
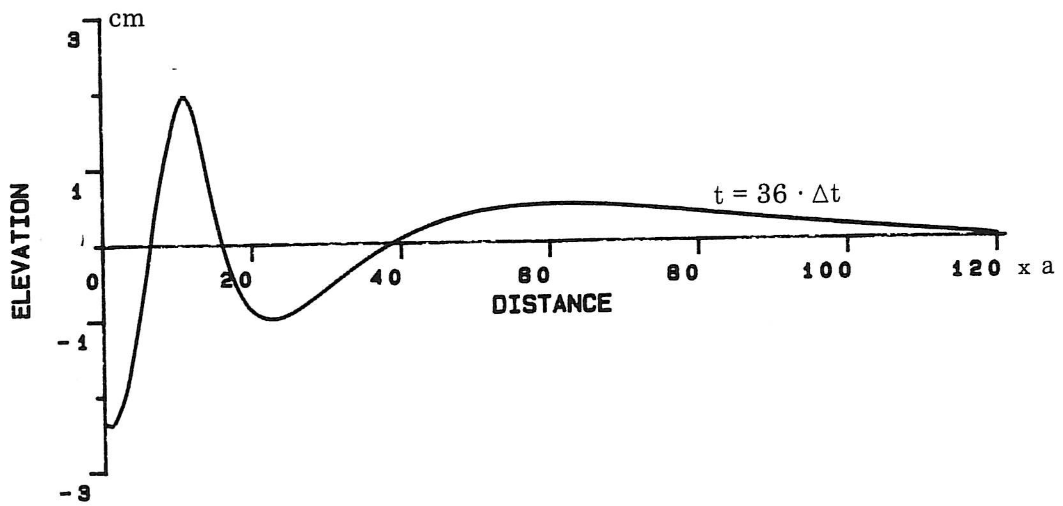


Fig. 11. (Continued)





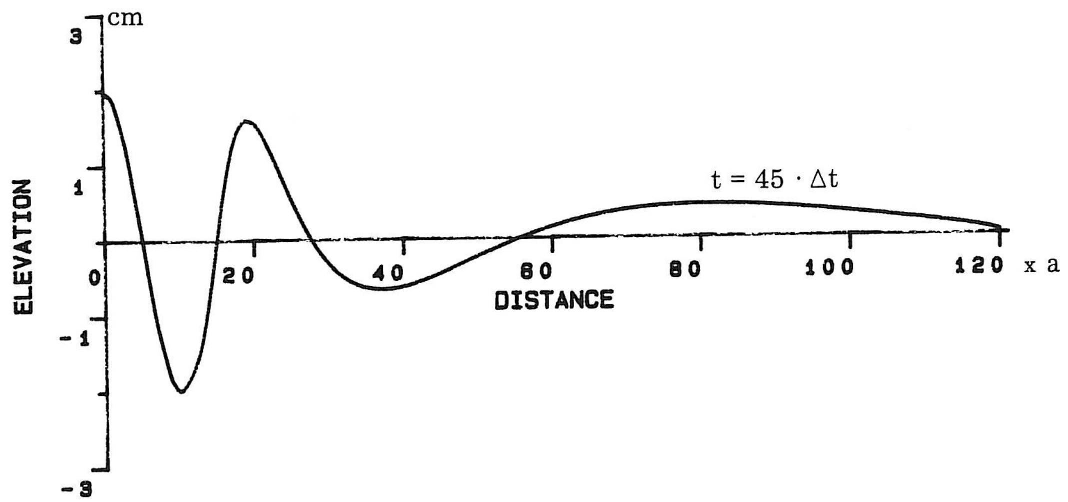


Fig. 11. (Continued)

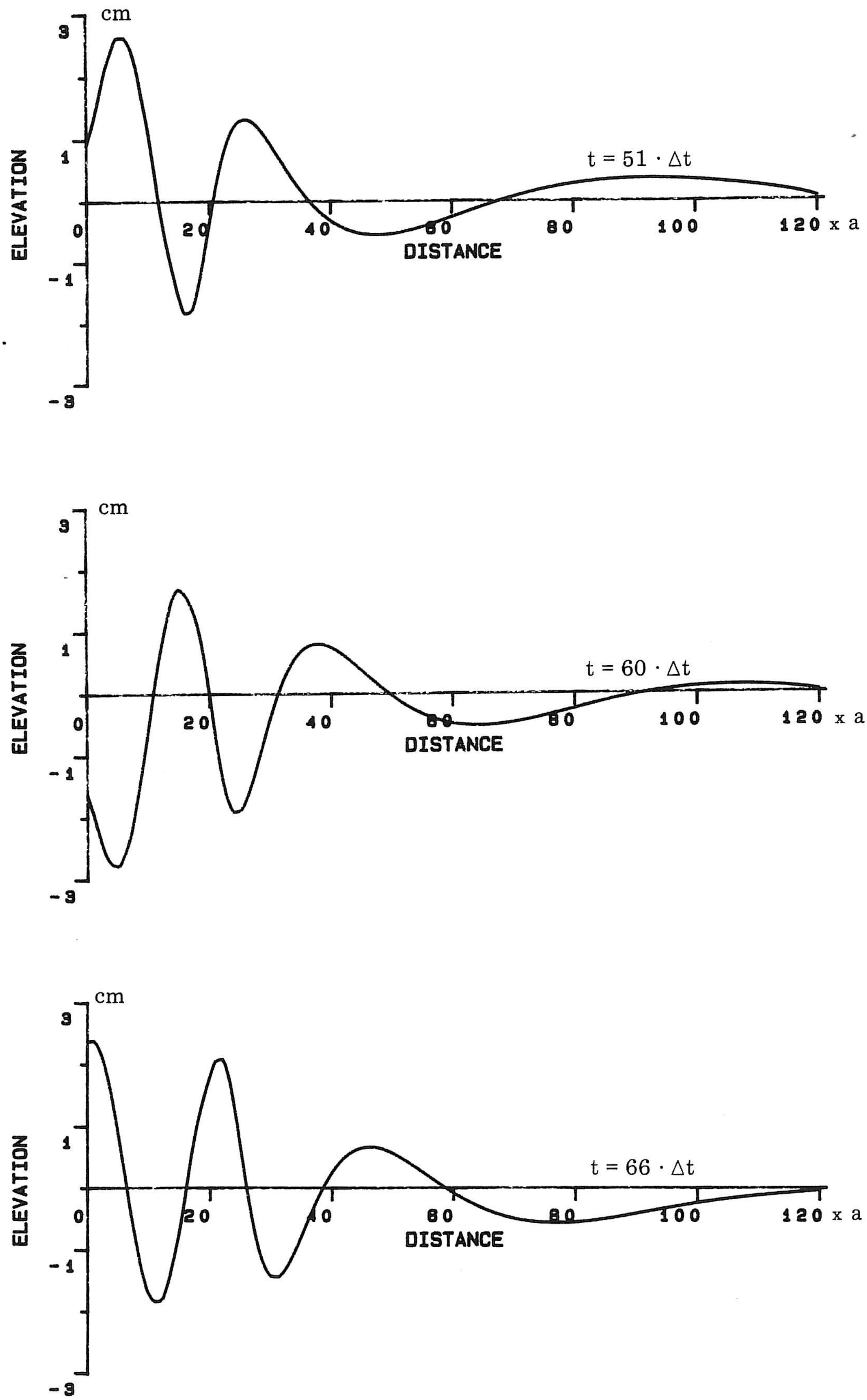


Fig. 11. (Continued)

10. CONCLUSIONS

The discrete formulation of the initial problem of generation of surface waves starting from rest may be useful in practical applications. Although the considerations presented are confined to the plane linear problem and the finite difference method, they provide some information and conclusions that may be important in discrete analysis of the problems mentioned. One of the most significant difficulties arising in a discrete description of the wave propagation phenomenon is the fact that the discrete system is the dispersive one. This is particularly important in analysing the surface waves which are, as we know, the dispersive waves. Therefore, it may be difficult to distinguish the two kinds of dispersion and to estimate the accuracy of the discrete solution. Hence, care should be taken in the estimation of the discrete results. Special attention should be paid to the proper spacing of points in both time and space. In some cases it is possible to choose the mesh-size in such a way that the result is the elimination of the dispersion associated with the net assumed. As far as the linear problem is concerned it is possible to apply the principle of superposition and to reduce the problem to the eigenvalues and eigenvectors problem for a matrix corresponding to the Laplace equation written for nodal points on the layer depth. Such a procedure of solving the problem is developed in the paper. It enables us to reduce the number of resulting algebraic equations to a relative minimum. The method cannot be used directly for the case of non-linear boundary conditions. In the latter case the more promising method seems to be the finite element method, especially in variational formulation. The non-linearity when taken into account will result in a large system of non-linear algebraic equations which should be solved in each time step.

The results obtained in computations and indicated in Fig. 11 are plausible, but to estimate the accuracy of the method proposed it would be desirable to confirm them with experimental results. Such confirmation is important, because the wave obtained in computations is not a symmetric wave. The lack of symmetry of the wave is suspected to be caused by improper solution of the boundary condition at $x = 0$, which does not correspond to the current position of the moving wall OA (Fig. 1).

The expansion of the velocity potential into Taylor series with respect to time increment has shown that the first-order approximation in the description of the potential leads to the second-order approximation in the description of the free surface elevation. This may explain the well-known phenomenon that the linearized theory of water waves gives surprisingly good results compared to results of experiments in hydraulic channels where the free surface elevation is obviously finite. The method of analysis developed here seems to be useful in cases of given forms of excitation of the fluid flow by wave-makers in the channels. The discrete method needs further investigations, especially in applications to the problems with non-linear boundary conditions.

REFERENCES

[1] COLE G.H.A., Fluid Dynamics, PWN, Warszawa 1964 (in Polish).

[2] KINSMAN B., Wind Waves, Prentice-Hall, New York, 1965.

[3] LAMB H., Hydrodynamics, Cambridge University Press, 1975.

[4] MARCZUK G.I., Analiza numeryczna zagadnień fizyki matematycznej, PWN, Warszawa, 1983.

[5] PHILLIPS P.M., - The Dynamics of the Upper Ocean, Cambridge University Press, 1969.

[6] STOKER I.I., Water Waves, Interscience Publishers, New York, 1957.

[7] SZMIDT J.K., Finite difference method and a spurious reflection of waves in a layer of fluid, Comp. Meth. in Appl. Mech. and Engrg., 60 (1987), 31–44.

[8] SZMIDT J.K., Problemy dyskretyzacji plaskich zagadnień hydrosprezystosci, prace IBW–PAN, Nr. 12, Gdańsk, 1984.

[9] TIMOSHENKO S.P., GOODIER J.N., Theory of elasticity, McGraw-Hill, 1982.

[10] TÖLKE F., Praktische Funktionenlehre, Springer, 1950.

[11] WILDE P., Metody numeryczne w problemach geotechnicznych, Mech. i Komputer, T.4, 103-120, Warszawa, 1981.

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THE UNIVERSITY OF AALBORG
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TELEPHONE: Int. + 45 - 8 - 14 23 33

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SOHNGAARDSHOLMSVEJ 57, DK-9000 AALBORG
TELEPHONE: Int. + 45 - 8 - 14 23 33