Aalborg Universitet

AALBORG UNIVERSITY

On the trace problem for Lizorkin-Triebel spaces with mixed norms

Johnsen, Jon; Sickel, Winfried

[Link to publication from Aalborg University](#)

AALBORG UNIVERSITY

On the trace problem for
Lizorkin-Triebel spaces with mixed norms

by

Jon Johnsen and Winfried Sickel

R-2006-31 September 2006

ON THE TRACE PROBLEM FOR LIZORKIN–TRIEBEL SPACES WITH MIXED NORMS

JON JOHNSEN AND WINFRIED SICKEL

ABSTRACT. The subject is traces of Sobolev spaces with mixed Lebesgue norms on Euclidean space. Specifically, restrictions to the hyperplanes given by $x_1 = 0$ and $x_n = 0$ are applied to functions belonging to quasi-homogeneous, mixed norm Lizorkin–Triebel spaces $F_{\vec{p},q}^{s,\vec{a}}$; Sobolev spaces are obtained from these as special cases. Spaces admitting traces in the distribution sense are characterised up to the borderline cases; these are also covered in case $x_1 = 0$. For x_1 the trace spaces are proved to be mixed norm Lizorkin–Triebel spaces with a specific sum exponent; for x_n they are similarly defined Besov spaces. The treatment includes continuous right-inverses and higher order traces. The results rely on a sequence version of Nikol'skij's inequality, Marschall's inequality for pseudodifferential operators (and Fourier multiplier assertions), as well as dyadic ball criteria.

1. INTRODUCTION

The motivation for this paper comes from parabolic boundary problems. To settle ideas we consider a simple problem, say for a domain $\Omega \subset \mathbb{R}^n$ with C^∞ boundary $\Gamma := \partial\Omega$, and with $\Delta = \partial_1^2 + \cdots + \partial_n^2$ denoting the Laplacian,

$$\partial_t u - \Delta u = f \quad \text{in } \Omega \times {]}0, T[, \tag{1}$$

$$u_{|\Gamma} = \varphi \quad \text{on } \Gamma \times {]}0, T[, \tag{2}$$

$$u_{|t=0} = u_0 \quad \text{at } \Omega \times \{0\}. \tag{3}$$

Among the data, $f(x, t)$ may have different integrability properties with respect to the x and t-directions. E.g. there may be given $p_1 \neq p_2$ in $[1, \infty]$ such that

$$\left(\int_0^T \left(\int_\Omega |f(x,t)|^{p_1} \, dx \right)^{p_2/p_1} dt \right)^{1/p_2} < \infty. \tag{4}$$

(It is throughout understood that an L_∞-norm applies whenever $p_j = \infty$.)

Correspondingly, any solution $u(x, t)$ is expected to belong to this $L_{\vec{p}}$ space, $\vec{p} = (p_1, p_2)$, at least if $\varphi = 0$ and $u_0 = 0$. It is well known that this can have various interpretations such as a bounded kinetic energy of the associated physical system for $\vec{p} = (2, \infty)$. When $Q_T = \Omega \times {]}0; T[$, a more precise information on u will be that

$$u, \; \partial_t u, \; \partial_{x_1}^2 u, \ldots, \partial_{x_n}^2 u \in L_{\vec{p}}(Q_T). \tag{5}$$

1

The set of such u is denoted $W_{\vec{p}}^{2,1}(Q_T)$. That in this case $u \in W_{\vec{p}}^{2,1}(Q_T)$ is a result of the maximal regularity theory, that has been intensively studied since the 1980s; the reader may consult [Ama95, Ch. III,4.10] as a reference to this development.

In case $\varphi \neq 0$ and $u_0 \neq 0$, a natural question is of course in which spaces it is possible to prescribe φ and u_0, such that $u \in W_{\vec{p}}^{2,1}(Q_T)$ still holds. Even for the above problem, the answer is somewhat delicate for $p_1 \neq p_2$.

This investigation was seemingly begun by Weidemaier [Wei98, Wei02, Wei05], but other works have been devoted to this area, cf. the paper by Denk, Hieber and Prüss [DHP].

To give a brief account of what can be expected, let γ_0 denote the operator of restriction to the lateral surface, so that the boundary condition (2) may be written $\gamma_0 u = \varphi$, and let r_0 stand for the restriction to the initial surface at $t = 0$ (i.e. $r_0 u = u_0$).

However, we simplify by taking the flat case in which $\Omega = \mathbb{R}^n$ and $t \in \mathbb{R}$. The initial data u_0 should then be given in the Besov space $B_{p_1,p_2}^{2-2/p_2}(\mathbb{R}^n)$, as r_0 is a surjection

$$r_0 \colon W_{\vec{p}}^{2,1}(\mathbb{R}^n \times \mathbb{R}) \to B_{p_1,p_2}^{2-2/p_2}(\mathbb{R}^n). \tag{6}$$

For φ the situation is different, for if $\mathbb{R}_x^{n-1} \times \mathbb{R}_t$ is equipped with mixed-norm spaces $L_{p'}(\mathbb{R}_x^{n-1} \times \mathbb{R}_t)$ for $p' = (p_1, \ldots, p_1, p_2)$ ($n-1$ copies of p_1), γ_0 is a surjection

$$\gamma_0 \colon W_{\vec{p}}^{2,1}(\mathbb{R}^n \times \mathbb{R}) \to F_{p',p_1}^{2-1/p_1,a'}(\mathbb{R}_x^{n-1} \times \mathbb{R}_t). \tag{7}$$

Here the range space is a Lizorkin–Triebel space with mixed norms (due to p') and with its sum exponent equal to p_1 (so in general this is not a Besov space). In addition the space has an anisotropy related to the smoothness index s; this is handled via weights a_j assigned to each coordinate axis, so that $a' = (1, \ldots, 1, 2)$. The resulting quasi-homogeneity of the space is well known, so the exact definitions are given in Section 3 below.

Motivated by the above outline, we shall study the general trace problem for the quasi-homogeneous, mixed-norm Lizorkin–Triebel spaces $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$. This problem was first studied by Berkolaiko [Ber84, Ber85, Ber87b, Ber87a]. The fact that γ_0 has a Lizorkin–Triebel space as the range was discovered by him for spaces with $1 < p_k < \infty$ for all k, $1 < q < \infty$.

Like Berkolaiko, our point of departure is a Littlewood–Paley decomposition of the functions, $u = \sum u_j$, but this we combine with a rather straightforward L_∞–$L_{\vec{p}}$-estimate, using the maximal functions u_j^* of Fefferman–Stein type. More precisely, if $\vec{p} = (p_1, p'')$,

$$\sup_{z \in \mathbb{R}} \big\| \big(\sum_{j=0}^\infty 2^{j(s - \frac{a_1}{p_1})p_1} |u_j(z, \cdot)|^{p_1} \big)^{\frac{1}{p_1}} \big| L_{p''} \big\| \leq c \big\| \sup_{j=0,1,2,\ldots} 2^{sj} |u_j^*(\cdot)| \big| L_{\vec{p}}(\mathbb{R}^n) \big\|. \tag{8}$$

The expression to the right is estimated by $\|u\|$ in $F_{\vec{p},q}^{s,\vec{a}}$, so most of the conclusions can be drawn from this L_∞–L_p-estimate. With this method, there are extensions to arbitrary $p_k \in]0, \infty[$, for all k, $0 < q \leq \infty$. In particular we settle the cases when $p_k = 1$ for one or more $k = 1, \ldots, n$, which the previous works on the subject [Ber84, Ber85, Ber87b, Ber87a, DHP, Wei05] were unable to cover.

Moreover, the trace of $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ is treated for all s above a certain limit. The isotropic condition $s > \frac{1}{p}$ is for mixed norms replaced by $s > \frac{1}{p_k}$ for the trace at $x_k = 0$, when all

$p_j \in]1, \infty[$. As a minor novelty a shift of the borderline is necessary if $0 < p_j < 1$ holds for one the tangential variables x_j. This is evident from (i) in Theorem 1 and Figure 1 below.

The paper is organised as follows: In Section 2 our results on the trace problems are presented. The definition of $F_{\vec{p},q}^{s,\vec{a}}$ is recalled in Section 3, together with the properties needed for the spaces. In the definition we follow Triebel's book [Tri83], though the conventions for the quasi-homogeneity given by $\vec{a}$ are the same as in [Yam86b] (and as in our joint work with Farkas on the unmixed cases [FJS00]); mixed norms are treated as in works of Schmeisser, Schmeisser and Triebel [Sch84, ST87], but here we also draw on a joint work [JS] proving a crucial Nikol'skij inequality for vector-valued functions. In addition dyadic corona and ball criteria for the $F_{\vec{p},q}^{s,\vec{a}}$ are established in the applicable style known at least since [Yam86b]; a pointwise estimate of pseudo-differential operators is also shown, inspired by a work of Marschall [Mar96]. Section 4 then proceeds to give the proofs, using maximal functions (based on an estimate of Bagby [Bag75]); Section 5 contains a few final remarks.

2. Traces of quasi-homogeneous mixed-norm Lizorkin–Triebel spaces

2.1. **The main theorems.** In the following vectors $\vec{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ may be split in groups like $\vec{x} = (x', x_k, x'')$. E.g. when restriction to the hyperplane Γ_k given by $x_k = 0$ is considered, $x' = (x_1, \ldots, x_{k-1})$ and $x'' = (x_{k+1}, \ldots, x_n)$ will be convenient; because x' and x'' both indicate tuples, vector arrows are suppressed. These conventions are also used for $\vec{a}$ and $\vec{p}$.

In general one can define many standard traces, say for $f \in C^\infty(\mathbb{R}^n)$,

$$\gamma_{j,k} f(x', x'') = \frac{\partial^j f}{\partial x_k^j}(x', x_k, x'') \big|_{x_k = 0}. \tag{9}$$

Here we shall mainly treat $\gamma_{0,k}$ for $k = 0$ and $k = n$. However, for general f, the operator $\gamma_{0,k}$ should be understood as the distributional trace defined in the natural way as $\gamma_{0,k} f = f|_{x_k=0}$ when f in its dependence of x_k defines a continuous map from $\mathbb{R}$ to $\mathcal{D}'(\mathbb{R}^{n-1})$; that is, $\gamma_{0,k}$ is defined for f in the subspace

$$C(\mathbb{R}_{x_k}, \mathcal{D}'(\mathbb{R}^{n-1})) \subset \mathcal{D}'(\mathbb{R}^n). \tag{10}$$

Here we recall that any $g \in C(\mathbb{R}_{x_k}, \mathcal{D}'(\mathbb{R}^{n-1}))$ defines a distribution Λ_g in n variables, with its action on arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$ given by integration of the continuous function $x_k \mapsto \langle g(x_k), \varphi(\cdot, x_k, \cdot) \rangle$; more precisely, $\langle \Lambda_g, \varphi \rangle = \int_{\mathbb{R}} \langle g(x_k), \varphi(\cdot, x_k, \cdot) \rangle \, dx_k$. For topological vector spaces X, Y, the set of continuous bounded maps $f \colon X \to Y$ is denoted by $C_b(X, Y)$.

All mapping properties of $\gamma_{0,k}$ are meant as restrictions, for example $\gamma_{0,k} \colon X \to Y$ means that for the distributional trace, X is contained in the preimage $\gamma_{0,k}^{-1}(Y)$.

Similarly $\gamma_{j,k} f$ is defined for $f \in \mathcal{D}'(\mathbb{R}^n)$ when the distributional derivative $\partial_{x_k}^j f$ is in $C(\mathbb{R}_{x_k}, \mathcal{D}'(\mathbb{R}^{n-1}))$.

As our first main result, we determine the $F_{\vec{p},q}^{s,\vec{a}}$ that belong to the domain of the trace in the inner variable:

Theorem 1. *For the trace $\gamma_{0,1}$ on the hyperplane $\{x_1 = 0\}$, and for a given anisotropy $\vec{a} = (a_1, \ldots, a_n)$, the following properties of a triple $(s, \vec{p}, q)$ are equivalent:*

(i) *$(s, \vec{p}, q)$ satisfies the inequality*

$$s \geq \frac{a_1}{p_1} + \sum_{k>1} (\frac{a_k}{p_k} - a_k)_+, \tag{11}$$

and, in addition, $s = \frac{a_1}{p_1}$ only holds if also $p_1 \leq 1$;

(ii) *the operator $\gamma_{0,1}$ is continuous from $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^{n-1})$.*

In the affirmative case there is a continuous embedding $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}_{x_1}, L_{r''}(\mathbb{R}^{n-1}))$, with the integral exponents given by $r_k = \max(1, p_k)$ for $k = 2, \ldots, n$.

The co-domain $\mathcal{D}'$ above is of course not optimal. Indeed, it is a main point for $\gamma_{0,1}$ that the range space is a *Lizorkin–Triebel* space; cf. (7). This result is established here under the condition that

$$s > \frac{a_1}{p_1} + \sum_{k \geq 2} (\frac{a_k}{\min(1, p_2, \ldots, p_k, q)} - a_k). \tag{12}$$

This is stronger than the sharp inequality in (i), but e.g. when $q, p_k \geq 1$ for all $k > 1$ it gives the same borderline as (i); in general it does so if $q \geq p_1 \geq \cdots \geq p_n$.

Theorem 2. *When $(s, \vec{p}, q)$ fulfils (12), then $\gamma_{0,1}$ is a bounded* surjection $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to F_{p'',p_1}^{s-\frac{a_1}{p_1}, a''}(\mathbb{R}^{n-1})$.

The implication (ii) $\implies$ (i) in Theorem 1 is actually a consequence of the following result, that we obtain from specific counterexamples.

Lemma 1. *Let $m \in \{1, \ldots, n\}$. If $\gamma_{0,m}$ is continuous $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n-1})$, then it holds that $s \geq \frac{a_m}{p_m} + \sum_{k \neq m} (\frac{a_k}{p_k} - a_k)_+$. In the case $s_m = \frac{a_m}{p_m}$ (so that $p_k \geq 1$ for all $k \neq m$) continuity of $\gamma_{0,m}$ implies $p_m \leq 1$.*

In connection with restriction to the hyperplane given by $x_n = 0$, our result corresponding to Theorem 1 leaves a borderline case open in the quasi-Banach space case.

Theorem 3. *For the trace $\gamma_{0,n}$ on $\{x_n = 0\}$, and for a given anisotropy $\vec{a}$, it holds for the following properties of a triple $(s, \vec{p}, q)$ that (i) $\implies$ (ii):*

(i) *$(s, \vec{p}, q)$ satisfies*

$$s \geq \frac{a_n}{p_n} + \sum_{k<n} (\frac{a_k}{p_k} - a_k)_+ \tag{13}$$

and, in addition, equality only holds if $p_n \leq 1$;

(ii) *the operator $\gamma_{0,n}$ is continuous from $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^{n-1})$.*

Conversely (ii) $\implies$ (i) *in case $p_k \geq 1$ for all $k < n$; and if $0 < p_k < 1$ for some $k \in \{1, \ldots, n-1\}$, then* (ii) *implies the inequality* (13).

When (i) *holds, then* $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}, L_{r'}(\mathbb{R}^{n-1}))$ *for $r_k = \max(1, p_k)$, $k = 1, \ldots, n-1$.*

Here the implications of (ii) are obtained from Lemma 1 for $m = n$.

For the trace $\gamma_{0,n}$, that acts in the outer integration variable, the range is generically a *Besov* space:

Theorem 4. *When the triple $(s, \vec{p}, q)$ fulfils*

$$s > \frac{a_n}{p_n} + \sum_{k<n} \left(\frac{a_k}{\min(1, p_1, \ldots, p_k)} - a_k \right), \tag{14}$$

then $\gamma_{0,n}$ is a bounded surjection $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to B_{p',p_n}^{s-\frac{a_n}{p_n},a'}(\mathbb{R}^{n-1})$.

Since $F_{p,p}^s = B_{p,p}^s$ in the isotropic case, we get for $s > \frac{1}{p}$, $1 < p < \infty$ that

$$\gamma_{0,1}(F_{p,q}^s) = F_{p,p}^{s-1/p} = B_{p,p}^{s-1/p} = \gamma_{0,n}(F_{p,q}^s). \tag{15}$$

In this way the present results give back the isotropic trace theory, and they show how things split up qualitatively (with F- and B-spaces as ranges) and quantitatively (with p_1 and p_n as sum exponents) when mixed norms are introduced.

In Theorems 2 and 4 the surjectivity was just a convenient way to express the optimality of taking $F_{p'',p_1}^{s-\frac{a_1}{p_1},a''}$ and $B_{p',p_n}^{s-\frac{a_n}{p_n},a'}$, respectively, as co-domains. But not surprisingly the stronger fact that $\gamma_{0,1}$ and $\gamma_{0,n}$ have everywhere defined right-inverses also holds in the present context.

Theorem 5. *There exist continuous operators K_1, $K_n \colon \mathcal{S}'(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$, both with range in the space $C_{\mathrm{b}}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$, such that for every $v \in \mathcal{S}'(\mathbb{R}^{n-1})$,*

$$\gamma_{0,1}(K_1 v) = v, \qquad \gamma_{0,n}(K_n v) = v. \tag{16}$$

Moreover, for any $\vec{p} = (p_1, \ldots, p_n)$ in $]0, \infty[$ and any $\vec{a}$,

$$K_1 \colon F_{p'',p_1}^{s,a''}(\mathbb{R}^{n-1}) \to F_{\vec{p},q}^{s+\frac{a_1}{p_1},\vec{a}}(\mathbb{R}^n) \quad \text{for} \quad 0 < q \leq \infty, \tag{17}$$

$$K_n \colon B_{p',p_n}^{s,a'}(\mathbb{R}^{n-1}) \to F_{\vec{p},q}^{s+\frac{a_n}{p_n},\vec{a}}(\mathbb{R}^n) \quad \text{for} \quad 0 < q \leq \infty, \tag{18}$$

are bounded maps for arbitrary $s \in \mathbb{R}$.

Let us also briefly describe results for higher order traces $\gamma_{j,k}$. Because they are composites of the trace $\gamma_{0,k}$ and differentiation $\partial_{x_k}^j$, both in the sense of distributions, and since $\partial_{x_k}^j$ has order ja_k in the $F_{\vec{p},q}^{s,\vec{a}}$-scale, the continuity properties of $\gamma_{j,k}$ are straightforward consequences of the above theorems.

As usual, the surjectivity of $\gamma_{j,k}$ is implied by that of the matrix-formed operator $\rho_{m,k}$ used for posing Cauchy problems,

$$\rho_{m,k} = \begin{pmatrix} \gamma_{0,k} \\ \gamma_{1,k} \\ \vdots \\ \gamma_{m-1,k} \end{pmatrix}. \tag{19}$$

Corollary 1. *When* $s > (m-1)a_1 + \frac{a_1}{p_1} + \sum_{k>1}\left(\frac{a_k}{\min(1,p_2,\ldots,p_k,q)} - a_k\right)$ *then* $\rho_{m,1}$ *is a bounded surjection*

$$\rho_{m,1}\colon F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to \prod_{j=0}^{m-1} F^{s-ja_1-\frac{a_1}{p_1},a''}_{p'',p_1}(\mathbb{R}^{n-1}). \tag{20}$$

There is a continuous operator $K_1^{(m)}\colon \mathcal{S}'(\mathbb{R}^{n-1})^m \to \mathcal{S}'(\mathbb{R}^n)$, *which maps* $\mathcal{S}'(\mathbb{R}^{n-1})^m$ *into the domain of* $\rho_{m,1}$ *and is a right-inverse of* $\rho_{m,1}$; *and* $K_1^{(m)}$ *is furthermore continuous with respect to the spaces in* (20) *for the specified* s.

Corollary 2. *When* $s > (m-1)a_n + \frac{a_n}{p_n} + \sum_{k<n}\left(\frac{a_k}{\min(1,p_1,\ldots,p_k)} - a_k\right)$ *then* $\rho_{m,n}$ *is a bounded surjection*

$$\rho_{m,n}\colon F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to \prod_{j=0}^{m-1} B^{s-ja_n-\frac{a_n}{p_n},a'}_{p',p_n}(\mathbb{R}^{n-1}). \tag{21}$$

There is a continuous operator $K_n^{(m)}\colon \mathcal{S}'(\mathbb{R}^{n-1})^m \to \mathcal{S}'(\mathbb{R}^n)$, *which maps* $\mathcal{S}'(\mathbb{R}^{n-1})^m$ *into the domain of* $\rho_{m,n}$ *and is a right-inverse of* $\rho_{m,n}$; *and* $K_n^{(m)}$ *is furthermore continuous with respect to the spaces in* (21) *for the specified* s.

2.2. **Remarks on the borderlines.** As illustrated in Figure 1, the mixed-norm spaces $F^{s,\vec{a}}_{\vec{p},q}$ give borderline phenomena differing a good deal from the well-known isotropic, un-mixed L_p-theory (we take $\vec{a} = (1,\ldots,1)$ for simplicity): as a similarity q plays no role, so we take $q = 2$; then the spaces reduce to Sobolev spaces $H^s_{\vec{p}} = F^s_{\vec{p},2}$ when $1 < p_k < \infty$ for all k. Moreover, beginning with $\gamma_{0,1}$, it is by (i) of Theorem 1 necessary that $s \geq 1/p_1$, with $s = 1/p_1$ being possible only for $p_1 \leq 1$. This requires in addition that

$$\sum_{k>1}\left(\frac{1}{p_k} - 1\right)_+ = 0, \tag{22}$$

hence $p_k \geq 1$ for all $k \geq 2$. However, $p_1 \leq 1$ excludes the identification with a Sobolev space (but any u in $F^s_{\vec{p},2}(\mathbb{R}^n)$ is then at least a continuous function of x_1 valued in the Banach space $L_{p''}(\mathbb{R}^{n-1})$).

When $\sum_{k>1}(\frac{1}{p_k} - 1)_+ > 0$, i.e. at least one $p_k < 1$ there is a marked difference to the non-mixed case because the borderline is displaced upwards, cf. Figure 1. This is not unnatural, though, since there is a Sobolev embedding, with $r_k = \max(1,p_k)$ for $k > 1$,

$$F^s_{\vec{p},2}(\mathbb{R}^n) \hookrightarrow F^{\frac{1}{p_1}}_{(p_1,r''),2}(\mathbb{R}^n) \quad \text{for} \quad s = \frac{1}{p_1} + \sum_{k>1}\left(\frac{1}{p_k} - 1\right)_+, \tag{23}$$
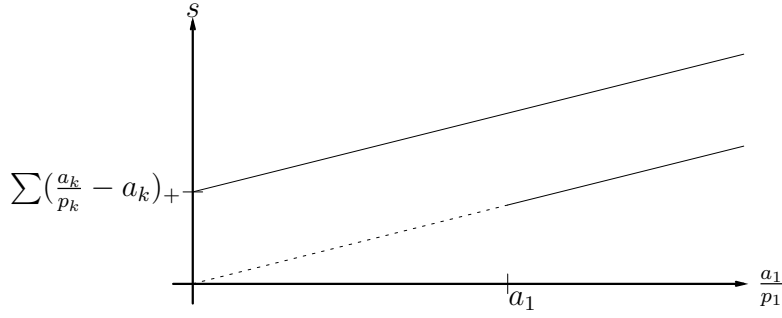
FIGURE 1. The $\gamma_{0,1}$-borderlines for s, for different values of p''; dashes indicate that s must be strictly larger than at the borderline

where the last space is located at the borderline for the Banach case. For $p_1 \leq 1$ it is therefore clear that $\gamma_{0,1}$ is defined on $F^s_{\vec{p},2}$, whereas for $p_1 > 1$ this might look contradictory. But the meaning of Theorem 1 is that the subspace to the left in (23) is *barely* small enough to be in the domain of $\gamma_{0,1}$, even for $p_1 > 1$ (cf. the proof, where (23) is sharpened by a precise application of the vector-valued Nikol′skij inequality, cf. (52) below, that allows a decisive shift to a sum exponent $q \leq 1$).

2.3. **The working definition of the trace.** For an overview of the methods, it is noted that we work with a quasi-homogeneous Littlewood–Paley decomposition $1 = \sum_{j=0}^{\infty} \Phi_j$ such that, for $j \geq 1$,
$$\xi \in \operatorname{supp}\Phi_j \implies 2^{j-1} \leq |\xi|_{\vec{a}} \leq 2^{j+1}. \tag{24}$$
Hereby $|\cdot|_{\vec{a}}$ stands for a quasi-homogeneous distance function, with level sets given by n-dimensional ellipsoids of varying eccentricity; cf. Section 3.1 for details.

Decomposing $u = \sum \Phi_j(D)u$ there is an obvious candidate for the trace, say $\gamma_{0,1}$, for since the $\Phi_j(D)u$ are C^∞-functions by the Paley–Wiener–Schwartz theorem, one can set
$$\tilde{\gamma}_{0,1}u = \sum_{j=0}^{\infty} \Phi_j(D)u \big|_{x_1=0}. \tag{25}$$

We adopt this as a working definition for $\gamma_{0,1}$. In fact, the proof of (i) $\implies$ (ii) in Theorem 1 shows that under the condition (i), the series in (25) converges in $L_{r''}$. But as the value $x_1 = 0$ does not play a special role, a further argument yields $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}, L_{r''}(\mathbb{R}^{n-1}))$. The argument also shows that $\tilde{\gamma}_{0,1}$ is a map $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n-1})$ that is a *restriction* of the distributional trace $\gamma_{0,1}$.

Similar remarks apply to the outer trace $\gamma_{0,n}$.

Remark 1. *Nikol′skij* [Nik75] *assigned a trace on e.g. $\{x_n = 0\}$ to any $f(x', x_n)$ behaving as an $L_{p'}$-function in x' and depending continuously (near $x_n = 0$) on the parameter x_n, i.e. to any f in $C(\mathbb{R}, L_{p'}(\mathbb{R}^{n-1}))$. The trace is of course defined on the larger space $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$, but by Theorems 1 and 3, the $F^{s,\vec{a}}_{\vec{p},q}$ that admit traces are regular enough to fulfil Nikol′skij's requirement, at least when the components of r' or r'' are equal.*

2.4. **Anisotropic Sobolev spaces.** For comparison's sake, we collect the relation to the anisotropic counterparts of the well-known Bessel potential and Sobolev spaces.

Proposition 1. *Let $1 < \vec{p} < \infty$ and $s \in \mathbb{R}$ be arbitrary.*

(i) *Then $F_{\vec{p},2}^{s,\vec{a}}(\mathbb{R}^n) = H_{\vec{p}}^{s,\vec{a}}(\mathbb{R}^n)$ where $H_{\vec{p}}^{s,\vec{a}}$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which*

$$\left\| \mathcal{F}^{-1}\left[(1 + |\xi|_{\vec{a}}^2)^{s/2}\, \mathcal{F}u\right](\,\cdot\,) \,\big|\, L_{\vec{p}}(\mathbb{R}^n) \right\| < \infty. \tag{26}$$

(ii) *When $m_k = \frac{s}{a_k} \in \mathbb{N}_0$ for each $k = 1, \ldots, n$, then $F_{\vec{p},2}^{s,\vec{a}}(\mathbb{R}^n) = W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n)$ for $\vec{m} = (m_1, \ldots, m_n)$, where $W_{\vec{p}}^{\vec{m}}$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| u \,\big|\, L_{\vec{p}}(\mathbb{R}^n) \right\| + \sum_{i=1}^{n} \left\| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \,\big|\, L_{\vec{p}}(\mathbb{R}^n) \right\| < \infty. \tag{27}$$

In both cases the norms are equivalent to that of $F_{\vec{p},q}^{s,\vec{a}}$.

The essential part of this result goes back to Lizorkin [Liz70], who introduced and discussed the above spaces.

Conversely to Proposition 1, one often needs to identify a given Sobolev space $W_{\vec{p}}^{\vec{m}}$ with a Lizorkin–Triebel space. While this can be done in many ways, we first recall the convention, preferred in the Russian school, e.g. [BIN79, Liz70], of taking the smoothness s as the harmonic mean of the given orders,

$$\frac{1}{s} = \frac{1}{n}\left(\frac{1}{m_1} + \cdots + \frac{1}{m_n}\right). \tag{28}$$

Then, by setting $a_k = s/m_k$ for $k = 1, \ldots, n$, Proposition 1 clearly gives

$$W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n) = F_{\vec{p},2}^{s,\vec{a}}(\mathbb{R}^n), \quad \text{and} \quad a_1 + \cdots + a_n = |\vec{a}| = n. \tag{29}$$

This yields the following trace results for Sobolev spaces.

Proposition 2. *Let $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ and $1 < p_k < \infty$ for $k = 1, \ldots, n$, and define s by (28) and $a_k = s/m_k$ for all k. Then there are bounded surjections*

$$\gamma_{0,1} \colon W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n) \to F_{p'',p_1}^{s-\frac{a_1}{p_1},a''}(\mathbb{R}^{n-1}) \quad \textit{for} \quad m_1 > \frac{1}{p_1}, \tag{30}$$

$$\gamma_{0,n} \colon W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n) \to B_{p',p_n}^{s-\frac{a_n}{p_n},a'}(\mathbb{R}^{n-1}) \quad \textit{for} \quad m_n > \frac{1}{p_n}. \tag{31}$$

Note that substitution of e.g. $a_1 = s/m_1$ entails $s - \frac{a_1}{p_1} = s(1 - \frac{1}{m_1 p_1})$, where the last expression is used by some authors.

However, as an alternative to (28)-(29), there is also an identification

$$W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n) = F_{\vec{p},2}^{s,\vec{a}}(\mathbb{R}^n) \quad \text{with} \quad s = \max(m_1, \ldots, m_n). \tag{32}$$

Indeed, it is verified in Lemma 10 below that $F_{\vec{p},q}^{s,\vec{a}} = F_{\vec{p},q}^{\lambda s,\lambda \vec{a}}$ with equivalent quasi-norms, for every $\lambda > 0$. So (32) follows from (29) for $\lambda = \frac{1}{n}(\frac{1}{m_1} + \cdots + \frac{1}{m_n}) \max(m_1, \ldots, m_n)$. Then the weigths in (32) fulfill

$$a_k = \frac{1}{m_k} \max(m_1, \ldots, m_n) \quad \text{for } k = 1, \ldots, n; \qquad \min(a_1, \ldots, a_n) = 1. \qquad (33)$$

In particular this gives the normalisation $\min(a_1, \ldots, a_n) = 1$, instead of $|\vec{a}| = n$.

Another virtue of (32)–(33) is that every $m_k \in [0, s]$. Moreover, in (7) the space $W_{\vec{p}}^{2,1}(\mathbb{R}^n \times \mathbb{R})$ stands for $W_{\vec{p}}^{2,\ldots,2,1}(\mathbb{R}^n \times \mathbb{R})$, so (33) clearly gives $\vec{a} = (1, \ldots, 1, 2)$; cf. (7).

We prefer to adopt the convention that $\min(a_1, \ldots, a_n) = 1$ throughout, since it makes some estimates simpler and gives direct reference to e.g. [Yam86b, Joh96, FJS00, JS].

Remark 2 (related work). *Traces of mixed norm Sobolev spaces $W_{\vec{p}}^{\vec{m}}$ were covered by Bugrov [Bug71]. In a series of papers [Ber84, Ber85, Ber87b, Ber87a] Berkolaiko proved Theorems 2–4 with all p_k and q in $]1, \infty[$. He also obtained the necessary condition $s > \frac{a_k}{p_k}$ for these cases (whereas corrections for $0 < p_k < 1$ can be found in the present paper).*

Moreover, Berkolaiko showed that for $k = 2, \ldots, n-1$ the ranges of $\gamma_{0,k}$ are given neither by Besov nor Lizorkin–Triebel spaces; instead the relevant norms will have the discrete ℓ_q-norm 'replacing' that of L_{p_k} (as is shown here for $k = 1$ and $k = n$). We have refrained from going into this, since $\gamma_{0,1}$ and $\gamma_{0,n}$ should suffice for most parabolic problems.

It was seemingly first realised by Weidemaier [Wei98] that it is relevant for the fine theory of parabolic problems to have Lizorkin–Triebel spaces as trace spaces. Among the other works on this application we can mention [DHP, Wei02, Wei05].

3. Lizorkin–Triebel spaces $F_{\vec{p},q}^{s,\vec{a}}$ based on mixed norms

3.1. **Notation and preliminaries.** For a given $\vec{p} = (p_1, \ldots, p_n)$ with $p_k \in]0, \infty]$, $k = 1, \ldots, n$, we denote by $L_{\vec{p}}(\mathbb{R}^n)$ the set of all equivalence classes of measurable functions $u : \mathbb{R}^n \to \mathbb{C}$ such that

$$\big\| u \,\big| L_{\vec{p}}(\mathbb{R}^n) \big\| := \left(\int_{\mathbb{R}} \ldots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x_1, \ldots, x_n)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \ldots dx_n \right)^{\frac{1}{p_n}} \qquad (34)$$

is finite (modification if some of the p_i are equal to ∞). With this quasi-norm $L_{\vec{p}}(\mathbb{R}^n)$ is complete, and a Banach space if $\min(p_1, \ldots, p_n) \geq 1$. Furthermore, for $0 < q \leq \infty$, we shall use the abbreviation $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ for the set of all sequences $\{u_k\}_{k=0}^{\infty}$ of measurable functions $u_k : \mathbb{R}^n \to \mathbb{C}$ such that (with $\sup_k$ for $q = \infty$)

$$\big\| \{u_k\}_{k=0}^{\infty} \,\big| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \big\| := \left\| \left(\sum_{k=0}^{\infty} |u_k|^q \right)^{1/q} \bigg| L_{\vec{p}}(\mathbb{R}^n) \right\| < \infty. \qquad (35)$$

For brevity $\big\| u_k \,\big| L_{\vec{p}}(\ell_q) \big\|$ may replace $\big\| \{u_k\}_{k=0}^{\infty} \,\big| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \big\|$. If $\max(p_1, \ldots, p_n, q) < \infty$, then C_0^{∞} is dense in $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$. $L_{\vec{p}}$ was studied by Benedek and Panzone [BP61].

In general we adopt standard notation from distribution theory. E.g. $\mathcal{D}'(\mathbb{R}^n)$ stands for the space of distributions on $\mathbb{R}^n$, while $\mathcal{S}'(\mathbb{R}^n)$ is the subspace of tempered distributions.

The Fourier transformation is denoted by $\mathcal{F}u = \hat{u}$, where $\mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\,x\cdot\xi} u(x)\,dx$ for $u \in \mathcal{S}(\mathbb{R}^n)$ with $\mathcal{S}(\mathbb{R}^n)$ being the Schwartz space of rapidly decreasing C^∞-functions on $\mathbb{R}^n$.

On $\mathbb{R}^n$ we use an anisotropic distance function $|\cdot|_{\vec{a}}$ of a quasi-homogeneous type, given by $\vec{a} = (a_1, \ldots, a_n)$. First $\vec{a}$ is used for the quasi-homogeneous dilation $t^{\vec{a}}x := (t^{a_1}x_1, \ldots, t^{a_n}x_n)$ for $t \geq 0$, and $t^{s\vec{a}}x := (t^s)^{\vec{a}}x$ for $s \in \mathbb{R}$, whence $t^{-\vec{a}}x = (t^{-1})^{\vec{a}}x$. Then $|x|_{\vec{a}}$ is the unique $t > 0$ such that $t^{-\vec{a}}x \in S^{n-1}$ ($|0|_{\vec{a}} = 0$), i.e.

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1. \tag{36}$$

It is seen directly that $|t^{\vec{a}}x|_{\vec{a}} = t|x|_{\vec{a}}$ ($|\cdot|_{\vec{a}}$ is not a norm), and one has

$$|x + y|_{\vec{a}} \leq |x|_{\vec{a}} + |y|_{\vec{a}}. \tag{37}$$

$$\max(|x_1|^{1/a_1}, \ldots, |x_n|^{1/a_n}) \leq |x|_{\vec{a}} \leq |x_1|^{1/a_1} + \cdots + |x_n|^{1/a_n}. \tag{38}$$

We set $B_{\vec{a}}(x, R) := \{\, y \mid |x - y|_{\vec{a}} \leq R \,\}$. A review of $|\cdot|_{\vec{a}}$ can be found in [JS, Yam86b].

Along with $|\cdot|_{\vec{a}}$, a quasi-homogeneous Littlewood–Paley decomposition $1 = \sum \Phi_j$ will be chosen as follows: based on some $\psi \in C^\infty(\mathbb{R})$ such that $0 \leq \psi(t) \leq 1$ for all t, $\psi(t) = 1$ if $\leq 11/10$, and $\psi(t) = 0$ if $t > 13/10$, we set $\Psi_j(\xi) := \psi(2^{-j}|\xi|_{\vec{a}})$ for $j \in \mathbb{N}_0$ ($\Psi_j \equiv 0$ for $j < 0$) so that $\Phi_j := \Psi_j - \Psi_{j-1}$ gives $1 = \sum_{j=0}^\infty \Phi_j(\xi)$ for all $\xi \in \mathbb{R}^n$. Clearly

$$\operatorname{supp}\Phi_j \subset \{\, \xi \mid \tfrac{11}{20}2^j \leq |\xi|_{\vec{a}} \leq \tfrac{13}{10}2^j \,\}. \tag{39}$$

This choice is indicated by the uppercase letters Ψ, Φ throughout. Whenever $1 < p_k < \infty$ for $k = 1, \ldots, n$, then a Littlewood–Paley inequality holds for all $u \in L_{\vec{p}}(\mathbb{R}^n)$:

$$c_1 \left\| u \,\middle|\, L_{\vec{p}} \right\| \leq \left\| \left(\sum_{j=0}^\infty |\mathcal{F}^{-1}[\Phi_j \mathcal{F}u]|^2\right)^{\frac{1}{2}} \middle|\, L_{\vec{p}} \right\| \leq c_2 \left\| u \,\middle|\, L_{\vec{p}} \right\|. \tag{40}$$

In fact the right-hand side inequality follows directly from a theorem of Krée [Kré67, Th. 4]; then the inequality to the left is obtained from the completeness of $L_{\vec{p}}$ and duality (cf. a similar proof in [Yam86a, Prop. 3.3]).

3.2. **Lizorkin–Triebel spaces with mixed norms.** Let Φ_j, $j \in \mathbb{N}_0$, be our anisotropic dyadic decomposition of unity.

Definition 1. *Let* $0 < p_1, \ldots, p_n < \infty$, $s \in \mathbb{R}$, *and* $0 < q \leq \infty$. *Then the* mixed norm *(quasi-homogeneous) Lizorkin–Triebel space* $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ *is the set of* $u \in \mathcal{S}'(\mathbb{R}^n)$ *such that*

$$\left\| u \,\middle|\, F_{\vec{p},q}^{s,\vec{a}} \right\| := \left\| \left(\sum_{j=0}^\infty 2^{jsq}|\mathcal{F}^{-1}[\Phi_j \mathcal{F}u](\cdot)|^q\right)^{\frac{1}{q}} \middle|\, L_{\vec{p}}(\mathbb{R}^n) \right\| < \infty. \tag{41}$$

The $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ are quasi-Banach spaces, and Banach spaces if $p_1, \ldots, p_n, q$ all belong to $[1, \infty]$. Instead of the quasi-triangle inequality, it is useful that for all $u, v \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ the number $\tau = \min(1, p_1, \ldots, p_n, q)$ gives rise to the estimate

$$\| u + v \,|\, F_{\vec{p},q}^{s,\vec{a}} \|^\tau \leq \| u \,|\, F_{\vec{p},q}^{s,\vec{a}} \|^\tau + \| v \,|\, F_{\vec{p},q}^{s,\vec{a}} \|^\tau. \tag{42}$$

Up to equivalent quasi-norms, the spaces $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ do not depend on the chosen anisotropic dyadic decomposition of unity

We shall also need the corresponding Besov spaces. They have properties like the above-mentioned for the $F_{\vec{p},q}^{s,\vec{a}}$, so we just give the definition.

Definition 2. *For $0 < p_1, \ldots, p_n, q \leq \infty$ and $s \in \mathbb{R}$ the quasi-homogeneous mixed norm Besov space $B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\left\| u \,\middle|\, B_{\vec{p},q}^{s,\vec{a}} \right\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \mathcal{F}^{-1}(\Phi_j \mathcal{F}u) \,\middle|\, L_{\vec{p}}(\mathbb{R}^n) \right\|^q \right)^{\frac{1}{q}} < \infty. \tag{43}$$

Proposition 3. *$F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ is translation invariant; and for $q < \infty$ and every $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$, the translations $\tau_h u := u(\cdot - h) \to u$ in $F_{\vec{p},q}^{s,\vec{a}}$ for $h \to 0$. Analogously $u \in B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ implies $\tau_h u \in B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$, with $\tau_h u \to u$ when q and all p_k are finite.*

Proof. Since $\Phi_j(D)\tau_h = \tau_h \Phi_j(D)$, the norm of $F_{\vec{p},q}^{s,\vec{a}}$ is translation invariant, as that of $L_{\vec{p}}(\mathbb{R}^n)$ is so. Hence both u, $\tau_h u$ may be approximated in $F_{\vec{p},q}^{s,\vec{a}}$ to within an ε, by choosing a suitable $\psi \in \mathcal{S}$, when $q < \infty$. And $\| \tau_h \psi - \psi \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \to 0$ for $h \to 0$, because $\tau_h \psi \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$ and the injection $\mathcal{S} \hookrightarrow F_{\vec{p},q}^{s,\vec{a}}$ is continuous. (Clearly $B_{\vec{p},q}^{s,\vec{a}}$ can replace $F_{\vec{p},q}^{s,\vec{a}}$ here.) $\square$

Remark 3. *For $a = (1, 1, \ldots, 1)$ these spaces fits into the general scheme developed by Hedberg and Netrusov, cf. [HN]. So in the isotropic situation we have a lot of properties at hand for these classes like characterization by atoms, characterization by oscillations (local approximation by polynomials) and characterization by differences. We envisage that most of the material presented there has a counterpart for the anisotropic spaces.*

3.3. Embedding results.

For a continuous linear injection of X into Y we throughout write $X \hookrightarrow Y$. A proof of the next result is given further below.

Lemma 2. *There are continuous embeddings*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{44}$$

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{45}$$

$\mathcal{S}(\mathbb{R}^n)$ is dense in $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for $q < \infty$, and dense in $B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for $q, p_1, \ldots, p_n < \infty$.

The definitions at once give part (i) of the next result; and (iii) follows from (ii), that holds by Minkowski's inequality.

Lemma 3. *When $p_k < \infty$ holds for all k in the F-spaces one has:*

(i) *For $s' < s$ and $q, q' \in]0, \infty]$,*

$$F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow F_{\vec{p},q'}^{s',\vec{a}}(\mathbb{R}^n); \qquad B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow B_{\vec{p},q'}^{s',\vec{a}}(\mathbb{R}^n). \tag{46}$$

(ii) *For $r_1 \leq \min(p_1, \ldots, p_n, q)$ and $\max(p_1, \ldots, p_n, q) \leq r_2$,*

$$\Big(\sum_{j=0}^{\infty} \| u_j \, | L_{\vec{p}}(\mathbb{R}^n) \|^{r_2} \Big)^{\frac{1}{r_2}} \leq \big\| u_j \, | L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \big\| \leq \Big(\sum_{j=0}^{\infty} \| u_j \, | L_{\vec{p}}(\mathbb{R}^n) \|^{r_1} \Big)^{\frac{1}{r_1}}, \qquad (47)$$

for an arbitrary sequence (u_j) of measurable functions.

(iii) *With r_1 and r_2 as in (ii),*

$$B_{\vec{p},r_1}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}_n) \hookrightarrow B_{\vec{p},r_2}^{s,\vec{a}}(\mathbb{R}^n). \qquad (48)$$

Let $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ such that $b_k > 0$, $k = 1, \ldots, n$. As a convenient notation we introduce the cube

$$Q_b := \Big\{ (x_1, \ldots x_n) \;\big|\; |x_k| \leq b_k, \; k = 1, \ldots, n \Big\} \qquad (49)$$

The symbol $x \cdot y$ refers to the scalar product of x, y in $\mathbb{R}^n$. For a vector $\vec{r}$ we shall as a convention set

$$\frac{1}{\vec{r}} = \Big(\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n} \Big). \qquad (50)$$

In our proofs the vector-valued Nikol'skij inequality will play a major role. This inequality concerns sequences (f_j) in $\mathcal{S}'(\mathbb{R}^n)$ that fulfill a *geometric rectangle condition*,

$$\operatorname{supp} \mathcal{F} f_j \subset [-AR_1^j, AR_1^j] \times \cdots \times [-AR_n^j, AR_n^j]. \qquad (51)$$

Here $A > 0$ is a constant, while the fixed numbers $R_1, \ldots, R_n > 1$ define the rectangles.

Theorem 6. *When $0 < p_k \leq r_k < \infty$ for $k = 1, \ldots, n$ and $\vec{r} \neq \vec{p}$, then there is for $0 < q \leq \infty$ a number $c > 0$ such that*

$$\Big\| \big(\sum_{j=0}^{\infty} |f_j(\cdot)|^q \big)^{\frac{1}{q}} \, | L_{\vec{r}} \Big\| \leq c \, \Big\| \sup_{j \in \mathbb{N}_0} \big(\prod_{k=1}^{n} R_k^{j(\frac{1}{p_k} - \frac{1}{r_k})} |f_j(\cdot)| \big) \, | L_{\vec{p}} \Big\| \qquad (52)$$

for all sequences (f_j) in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling (51).

For the proof the reader is referred to [JS, Thm. 5]. As noted there, this vector-valued Nikol'skij inequality at once gives Sobolev embeddings for the $F_{\vec{p},q}^{s,\vec{a}}$, where by virtue of (52) it suffices to increase only a single component p_k of $\vec{p}$:

Corollary 3. *When $0 < p_k \leq r_k < \infty$ for all k and $\vec{r} \neq \vec{p}$, then*

$$F_{\vec{p},q_1}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow F_{\vec{r},q_2}^{t,\vec{a}}(\mathbb{R}^n) \qquad (53)$$

holds for $t = s - \vec{a} \cdot \big(\frac{1}{\vec{p}} - \frac{1}{\vec{r}} \big)$.

The classical Nikol'skij inequality deals with a single function with compact spectrum. This results by applying (52) to a sequence with a single non-trivial element; then also $r_k = \infty$ is allowed (cf. [JS, Thm. 4]). This will, by the definition of $B_{\vec{p},q}^{s,\vec{a}}$, give

Corollary 4. *Suppose $0 < p_k \leq r_k \leq \infty$ for all k; $\vec{r} \neq \vec{p}$. Then*

$$B_{\vec{p},q_1}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow B_{\vec{r},q_2}^{t,\vec{a}}(\mathbb{R}^n) \qquad (54)$$

holds if $t - \vec{a} \cdot \frac{1}{\vec{r}} < s - \vec{a} \cdot \frac{1}{\vec{p}}$ or both $t - \vec{a} \cdot \frac{1}{\vec{r}} = s - \vec{a} \cdot \frac{1}{\vec{p}}$ and $q_1 \leq q_2$ do so.

By definition, every $u \in B_{\infty,1}^{0,\vec{a}}$, has finite norm series in L_∞, whence $B_{\infty,1}^{0,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^n)$. Therefore Lemma 3 and Corollary 4 give $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s-\vec{a}\cdot\frac{1}{\vec{p}},\vec{a}}(\mathbb{R}^n)$, so

$$F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^n) \quad \text{for} \quad s > \vec{a} \cdot \frac{1}{\vec{p}}. \tag{55}$$

Remark 4. *The embeddings and inequalities of this section have been extensively studied, in many versions, over several decades. It would be outside of our topic to recall this here,* [BIN96] *or* [ST87] *may be consulted as a general reference;* [JS] *has remarks on the development, as well as proofs pertaining to the anisotropic framework used here.*

3.4. **Maximal inequalities.** As usual we let Mf denote the Hardy–Littlewood maximal function, defined for a locally integrable function on $\mathbb{R}^n$ by

$$Mf(x) = \sup_{r>0} \frac{1}{\mathrm{meas}(B(0,r))} \int_{B(0,r)} |f(x+y)|, dy. \tag{56}$$

When the definition of M is applied only in the variable x_k, we shall via the splitting $x = (x', x_k, x'')$ use the abbreviation

$$M_k u(x_1, \ldots, x_n) := (Mu(x', \cdot, x''))(x_k) \tag{57}$$

Using this, we can formulate an important inequality due to Bagby [Bag75]. Let $1 < p_n < \infty$, and let $1 < q, p_k \le \infty$ for $k < n$. Then there exists a constant c such that every sequence in $L_{\vec{p}}(\ell_q)$ fulfils the inequality

$$\left\| M_n u_j \,\big|\, L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\| \le c \left\| u_j \,\big|\, L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\|. \tag{58}$$

It is well known that this allows the iterated maximal function $M_n(\ldots M_2(M_1 f)\ldots)(x)$ to be estimated in the mixed norm space $L_{\vec{p}}$.

However, we shall also use the maximal function of Fefferman–Stein type,

$$u^*(\vec{r}, b; x) = \sup_{z \in \mathbb{R}^n} \frac{|u(x-z)|}{(1+|b_1 z_1|^{1/r_1})\ldots(1+|b_n z_n|^{1/r_n})}. \tag{59}$$

In our cases the function u will have compact spectrum, and then u^* is majorised by the iterated Hardy–Littlewood maximal function. As a first step one has the next result.

Proposition 4. *Suppose* $0 < \vec{r} < \infty$. *Let* $b \in \mathbb{R}^n$ *such that* $b_i > 0$ *for all* i. *Then there exist a constant* $c > 0$ *such that*

$$\sup_{z \in \mathbb{R}^n} \frac{|u(x-z)|}{(1+|z_1|^{1/r_1})\ldots(1+|z_n|^{1/r_n})} \le c\big(M_n(\ldots M_2(M_1|u|^{r_1})^{r_2/r_1}\ldots)^{r_n/r_{n-1}}\big)^{1/r_n}(x) \tag{60}$$

holds whenever $\mathrm{supp}\,\mathcal{F}u \subset Q_b$ *and* $u \in L_{\vec{p}}(\mathbb{R}^n)$ *for* $0 < p_k < \infty$ *for all* k.

The proof given in [ST87, Thm. 1.6.4] for $n = 2$ is easily extended to arbitrary dimensions. Combined with a dilation, Proposition 4 gives, as in [ST87, 1.10.2], a vector-valued estimate for the Fefferman–Stein maximal function, which will be central to our trace estimates in Section 4:

Proposition 5. *Let $0 < \vec{p} < \infty$, $0 < q \leq \infty$, and suppose every component of $\vec{r}$ satisfies*

$$0 < r_k < \min(p_1, \ldots, p_k, q). \tag{61}$$

Then there exists a $c > 0$ such that, whenever (b^j) is a sequence in $]0, \infty[^n$,

$$\left\| u_j^*(\vec{r}, b^j, \cdot) \left| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\| \leq c \left\| u_j \left| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\| \right. \tag{62}$$

holds for all sequences (u_j) in $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ such that $\operatorname{supp} \mathcal{F} u_j \subset Q_{b^j}$ for all $j \in \mathbb{N}_0$.

Proof. We apply Proposition 4 to

$$g_j(x) = u_j(x_1/b_1^j, \ldots, x_n/b_n^j). \tag{63}$$

Obviously $\operatorname{supp} \mathcal{F} g_j \subset Q_{(1,\ldots,1)}$ for every j, and we have

$$g_j^*(x) \leq c_1 \big(M_n(\ldots M_2(M_1 |g_j|^{r_1})^{r_2/r_1} \ldots)^{r_n/r_{n-1}} \big)^{1/r_n}(x), \tag{64}$$

where c_1 is independent of j. Now (63) and $x = (b_1^j y_1, \ldots, b_n^j y_n)$ give

$$g_j^*(x) = \sup_{z \in \mathbb{R}^n} \frac{|g_j(b_1^j y_1 - b_1^j z_1, \ldots, b_n^j y_n - b_n^j z_n)|}{(1 + |b_1^j z_1|^{1/r_1}) \ldots (1 + |b_1^j z_n|^{1/r_n})} = u_j^*(\vec{r}, b; y). \tag{65}$$

Moreover, M commutes with dilation, i.e. $Mf(\delta x) = Mf(\delta \cdot)(x)$, so

$$\big(M_n(\ldots M_2(M_1 |g_j|^{r_1})^{\frac{r_2}{r_1}} \ldots)^{\frac{r_n}{r_{n-1}}} \big)^{\frac{1}{r_n}} (y)(b_1^j y_1, \ldots, b_n^j y_n)$$

$$= \big(M_n(\ldots M_2(M_1 |g_j(b_1^j \cdot, \ldots, b_n^j \cdot)|^{r_1})^{\frac{r_2}{r_1}} \ldots)^{\frac{r_n}{r_{n-1}}} \big)^{\frac{1}{r_n}} (y). \tag{66}$$

In view of (63) this means that

$$u_j^*(\vec{r}, b^j; y) \leq c_1 \big(M_n(\ldots M_2(M_1 |u_j|^{r_1})^{\frac{r_2}{r_1}} \ldots)^{\frac{r_n}{r_{n-1}}} \big)^{\frac{1}{r_n}} (y). \tag{67}$$

Applying Bagby's inequality (58) to $L_{(p_1/r_n, \ldots, p_n/r_n)}(\ell_{q/r_n})$ (using that all exponents belong to $]1, \infty[$, by the restriction on r_n), this gives

$$\left\| u_j^*(\vec{r}, b^j; \cdot) \left| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\| \leq c_2 \left\| \big(M_{n-1} \ldots M_2(M_1 |u_j|^{r_1})^{\frac{r_2}{r_1}} \ldots \big)^{\frac{1}{r_{n-1}}} \left| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \right\| \right. \tag{68}$$

By freezing x_n, Bagby's inequality (58) applies to $L_{(p_1/r_{n-1}, \ldots, p_{n-1}/r_{n-1})}(\ell_{q/r_{n-1}})(\mathbb{R}^{n-1})$. And by reiterating this, the statement follows. $\square$

3.5. **Marschall's inequality.** Inspired by Marschall's paper [Mar96], we shall give a version of his pointwise estimate of pseudo-differential operators $b(x, D)$, that is suitable for the mixed norm spaces.

In Marschall's inequality the symbol is estimated via the norm of a homogeneous Besov space $\dot{B}_{p,q}^{s,\vec{a}}(\mathbb{R}^n)$. To recall the definition of the norm, we need a dyadic partition of unity, $1 = \sum_{k=-\infty}^{\infty} \phi_k$ on $\mathbb{R}^n \setminus \{0\}$. This can be obtained from the previously introduced functions, by setting $\phi_j = \psi(2^{-j} | \cdot |_{\vec{a}}) - \psi(2^{1-j} | \cdot |_{\vec{a}})$ for all $j \in \mathbb{Z}$. With this,

$$\operatorname{supp} \phi_k \subset B_{\vec{a}}(0, 2^{k+1}) \setminus B_{\vec{a}}(0, 2^{k-1}) \subset Q_{2^{(k+1)\vec{a}}(1,\ldots,1)}. \tag{69}$$

Using $(\phi_j)_{j\in\mathbb{Z}}$, the norm $\|\cdot\,|\dot{B}_{p,q}^{s,\vec{a}}\|$ of $\dot{B}_{p,q}^{s,\vec{a}}(\mathbb{R}^n)$ is defined in analogy with that $B_{\vec{p},q}^{s,\vec{a}}$, simply by summing over $\mathbb{Z}$. It follows straightforwardly that

$$\| f(2^{k\vec{a}}\cdot)\,|\dot{B}_{\vec{p},q}^{s,\vec{a}}\| = 2^{k(s-\vec{a}\cdot\frac{1}{\vec{p}})}\,\| f\,|\dot{B}_{\vec{p},q}^{s,\vec{a}}\|, \qquad k\in\mathbb{Z}. \tag{70}$$

This scaling relation is the important property we need from this tool.

For the anisotropic weights, i.e. $\vec{a}$, the length is denoted by $|\vec{a}| = a_1 + \cdots + a_n$ for simplicity's sake.

Proposition 6. *Let a symbol $b \in C_0^\infty(\mathbb{R}^n)$ and a function $u \in C^\infty(\mathbb{R}^n)$ be given such that, for $A > 0$ and $R \geq 1$,*

$$\operatorname{supp}\mathcal{F}u \subset B_{\vec{a}}(0, AR) \quad and \quad \operatorname{supp} b \subset B_{\vec{a}}(0, A) \tag{71}$$

When $\vec{t} = (t_1, \ldots, t_n)$ satisfies $0 < t_k \leq 1$ for all k, then there exists $c > 0$ such that, for $d := \min(1, t_1, \ldots, t_n)$, the following inequality holds for all $x \in \mathbb{R}^n$,

$$|b(D)u(x)| \leq c(RA)^{\vec{a}\cdot\frac{1}{\vec{t}}-|\vec{a}|}\,\big\| b\,\big|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{\vec{t}},\vec{a}}\big\|\,\big(M_n(\ldots(M_1|u|^{t_1})^{t_2/t_1}\ldots)^{t_n}\big)^{1/t_n}(x). \tag{72}$$

Here c can be taken as a function of $\vec{a}$ and $\vec{t}$ only.

Proof. Since convolutions in $\mathcal{S}*\mathcal{S}'$ are mapped to products by the Fourier transformation,

$$b(D)u(x) = \mathcal{F}^{-1}(b\mathcal{F}u)(x) = \int \mathcal{F}^{-1}b(x-y)u(y)\,dy. \tag{73}$$

With x fixed, $y \mapsto \mathcal{F}^{-1}b(x-y)u(y)$ has, by the triangle inequality for $|\cdot|_{\vec{a}}$, its spectrum in

$$B_{\vec{a}}(0, A) + B_{\vec{a}}(0, RA) \subset B_{\vec{a}}(0, (R+1)A). \tag{74}$$

Therefore the Nikol'skij inequality (52) and an $L_{\vec{p}}$-version of (42) yields

$$\begin{aligned}
|b(D)u(x)| &\leq \int |\mathcal{F}^{-1}b(x-y)u(y)|\,dy \\
&\leq c_1(RA)^{\vec{a}\cdot\frac{1}{\vec{t}}-|\vec{a}|}\,\big\| \mathcal{F}^{-1}b(x-\cdot)u\,\big|L_{\vec{t}}\big\| \\
&\leq c_1(RA)^{(\vec{a}\cdot\frac{1}{\vec{t}}-|\vec{a}|)}\Big(\sum_{k\in\mathbb{Z}}\big\| \phi_k(x-\cdot)\mathcal{F}^{-1}b(x-\cdot)u\,\big|L_{\vec{t}}\big\|^d\Big)^{1/d}.
\end{aligned} \tag{75}$$

In this inequality it suffices for the $L_{\vec{t}}$-norm, by (69), to integrate over a cube on the right-hand side, and by the obvious estimate $\sup_y |\phi_k(y)\mathcal{F}^{-1}b(y)| \leq \int \big|\mathcal{F}_{y\to\eta}^{-1}(\phi_k\mathcal{F}^{-1}b)\big|\,d\eta =: b_k$, one finds

$$I_1 := \int_{B(x_1, 2^{(k+1)a_1})} |\phi_k(x-y)\mathcal{F}^{-1}b(x-y)u(y)|^{t_1}\,dy_1 \leq c_2 b_k^{t_1} 2^{ka_1} M_1|u|^{t_1}(x_1). \tag{76}$$

Proceeding iteratively by setting $I_j = \int_{-\infty}^\infty (I_{j-1})^{t_j/t_{j-1}}\,dy_j$, one finds analogously

$$\begin{aligned}
I_n &= \int_{B(x_n, 2^{(k+1)a_n})} (I_{n-1})^{t_n/t_{n-1}}\,dy_n \\
&\leq c_{n+1} b_k^{t_n} 2^{kt_n(\frac{a_1}{t_1}+\cdots+\frac{a_{n-1}}{t_{n-1}})} 2^{ka_n} M_n\big(\ldots(M_2(M_1|u|^{t_1})^{t_2/t_1})\ldots\big)^{t_n/t_{n-1}}(x_1, \ldots, x_n).
\end{aligned} \tag{77}$$

Raising to the power $1/t_n$ creates the factor $2^{k\vec{a}\cdot\frac{1}{t}}$, so the desired inequality follows from (75) by observing that $\sum_{k\in\mathbb{Z}} 2^{kd(\vec{a}\cdot\frac{1}{t})} \left\| \mathcal{F}^{-1}[\phi_k \mathcal{F}b] \,|L_1\right\|^d = \| b\,|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{t},\vec{a}}\|^d$. $\square$

Now we turn to a vector-valued version which will be of great service for us.

Proposition 7. *Suppose $0 < t_k < \min(1, p_1, \ldots, p_k, q)$ for $k = 1, \ldots, n$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset B_{\vec{a}}(0, 2)$, and set $\phi_j = \phi(2^{-ja}\cdot)$, $j \in \mathbb{N}$. Then there exists a constant c such that*

$$\left\| \mathcal{F}^{-1}[\phi_j \mathcal{F}u_j] \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\right\| \le cR^{\vec{a}\cdot\frac{1}{t}-|\vec{a}|} \, \| u_j \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\| \tag{78}$$

for all sequences $\{u_j\}_{j=1}^\infty$ in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling $\operatorname{supp}\mathcal{F}u_j \subset \{\xi \mid |\xi|_a \le R2^j\}$ for some $R \ge 1$.

Proof. Applying Proposition 6 with $A = 2^j$ to $\mathcal{F}^{-1}[\phi_j \mathcal{F}u_j]$, this is estimated by the iterated maximal function times $c(R2^j)^{\vec{a}\cdot\frac{1}{t}-|\vec{a}|} \| \phi(2^{-ja}\cdot)\,|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{t},\vec{a}}\|$. So by (70),

$$|\mathcal{F}^{-1}[\phi_j \mathcal{F}u_j](x)| \le cR^{\vec{a}\frac{1}{t}-|\vec{a}|} \| \phi\,|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{t},\vec{a}}\| \left(M_n(\ldots(M_1|u|^{t_1})^{t_2/t_1}\ldots)^{t_n}\right)^{1/t_n}(x). \tag{79}$$

The claim now follows by repeated use of (58), as in the proof of Proposition 5. $\square$

The above techniques also give a proof of the lift property for the $F_{\vec{p},q}^{s,\vec{a}}$ scale.

Proposition 8. *The map $\Lambda_r\colon \mathcal{S}' \to \mathcal{S}'$ given by $\Lambda_r u = \mathcal{F}^{-1}[(1+|\xi|_{\vec{a}}^2)^{r/2}\mathcal{F}u]$ is a linear homeomorphism $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to F_{\vec{p},q}^{s-r,\vec{a}}(\mathbb{R}^n)$ for every $r \in \mathbb{R}$.*

Proof. To show the boundedness of Λ_r, we let $1 = \sum \Phi_j$ denote the Littlewood–Paley decomposition; and take $\phi_j = \Phi_{j-1} + \Phi_j + \Phi_{j+1}$ such that $\phi_j\Phi_j = \Phi_j$ for all j. Moreover, $\phi_j = \phi(2^{-ja}\cdot)$ for $j \ge 1$ for a suitable ϕ. Then $\| \Lambda_r u \,|F_{\vec{p},q}^{s-r,\vec{a}}\|$ consists of terms like

$$2^{(s-r)j}\mathcal{F}^{-1}[\Phi_j(1+|\xi|_{\vec{a}}^2)^{r/2}\mathcal{F}u] = 2^{sj}\mathcal{F}^{-1}[g_j\Phi_j\mathcal{F}u], \tag{80}$$

with Fourier multipliers $g_j(\xi) := 2^{-rj}(1+|\xi|_{\vec{a}}^2)^{r/2}\phi_j(\xi)$. They fulfil $\operatorname{supp}g_j \subset \operatorname{supp}\phi_j \subset B_{\vec{a}}(0, R2^j)$ for a fixed $R \ge 1$. Hence Marschall's inequality gives a bound of $|2^{sj}g_j(D)u_j(x)|$ by the iterated maximal function on $2^{sj}\Phi_j(D)u$ times

$$\begin{aligned}
c2^{j(\vec{a}\cdot\frac{1}{t}-|\vec{a}|)} \| g_j\,|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{t},\vec{a}}\| &\le \left\| 2^{-rj}(1+|2^{ja}\xi|_{\vec{a}}^2)^{r/2}\phi(\xi)\,|\dot{B}_{1,d}^{\vec{a}\cdot\frac{1}{t},\vec{a}}\right\| \\
&\le c\left\| (2^{-2j}+|\xi|_{\vec{a}}^2)^{r/2}\phi(\xi)\,|W_1^m\right\| = C
\end{aligned} \tag{81}$$

Here we have used the scaling property, and taken some $m > \vec{a}\cdot\frac{1}{t}$ to get a uniform bound for all $j \ge 0$, which holds since $\phi = 0$ around the origin (the case $j = 0$ is obvious). Now boundedness of Λ_r follows from Bagby's inequality, similarly to the proof of Proposition 5. The estimates are valid for arbitrary $r \in \mathbb{R}$, so the boundedness of $\Lambda_r^{-1} = \Lambda_{-r}$ is also obtained. $\square$

Remark 5. *The lift property in Proposition 8 applies to the proof of Proposition 1. Indeed, for $H_{\vec{p}}^{s,\vec{a}}$ it will be enough to prove $H_{\vec{p}}^{0,\vec{a}}(\mathbb{R}^n) = F_{\vec{p},2}^{0,\vec{a}}(\mathbb{R}^n)$ with equivalent norms; but this holds by (40). (Krée's result [Kré67] was also used in [Liz70, Thm. 2] for the proof of a variant of (40) with a homogeneous, but non-smooth decomposition.) For $m_k = s/a_k$,*

$k = 1, \ldots, n$, *the identification* $W_{\vec{p}}^{\vec{m}}(\mathbb{R}^n) = H_{\vec{p}}^{s,\vec{a}}(\mathbb{R}^n)$, *with equivalent norms, has been proved by Lizorkin, cf. Theorem 3 and (20) ff. in* [Liz70].

3.6. **Convergence criteria.** It is a central theme to conclude the convergence in $\mathcal{S}'$ of a series $\sum_{j=0}^{\infty} u_j$, where $\operatorname{supp} \mathcal{F} u_j$ is compact for each j. More precisely the u_j are supposed to satisfy one of the following conditions, that can be imposed for each choice of $\vec{a}$:

(I) (The dyadic corona condition.) There exist an $A > 1$ such that for every $j \geq 1$,

$$\operatorname{supp} \hat{u}_j \subset \{\, \xi \mid \tfrac{2^j}{A} \leq |\xi|_{\vec{a}} \leq A 2^j \,\}, \tag{82}$$

whilst $\operatorname{supp} \hat{u}_0 \subset \{\, \xi \mid |\xi|_{\vec{a}} \leq A \,\}$.

(II) (The dyadic ball condition.) There exist an $A > 0$ such that for every $j \geq 0$,

$$\operatorname{supp} \hat{u}_j \subset \{\, \xi \mid |\xi|_{\vec{a}} \leq A 2^j \,\}. \tag{83}$$

The convergence of $\sum_{j=0}^{\infty} u_j$ will follow, if in addition to one of these conditions either some growth or integrability condition is fulfilled by the u_j in a uniform way. The resulting dyadic corona and dyadic ball *criteria* are summed up below.

To conclude the mere $\mathcal{S}'$-convergence, the following lemma was given by Coifman and Meyer albeit without arguments [MC91, Ch. 16]. We give a proof here, because some of the observations therein have additional consequences, that are useful for the present paper.

Lemma 4. $1°$ *Let* $(u_j)_{j \in \mathbb{N}_0}$ *be a sequence of* C^∞-*functions in* $\mathcal{S}'(\mathbb{R}^n)$ *that for suitable constants* $C \geq 0$, $m \geq 0$ *fulfils both* (I) *and*

$$|u_j(x)| \leq C 2^{jm} (1 + |x|)^m \text{ for all } j \geq 0. \tag{84}$$

Then $\sum_{j=0}^{\infty} u_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$ *to a distribution* u, *for which* $\hat{u}$ *is of order* m.

$2°$ *For every* $u \in \mathcal{S}'(\mathbb{R}^n)$ *the conditions* (I) *and* (84) *are fulfilled by the* u_j *defined from a Littlewood–Paley decomposition of* u.

Since any $u \in \mathcal{S}'$ is of finite order, the u_j in $2°$ have equal orders. Then there is some $m \geq 0$ such that $|u_j(x)| \leq c_j (1 + |x|)^m$, by the Paley–Wiener–Schwartz Theorem, which almost gives (84); but the j-dependence is by $2°$ not worse than $c_j = \mathcal{O}(2^{mj})$.

Proof. In $2°$ it is clear that $u_j(x) = c \langle \hat{u}, \Phi_j e^{\mathrm{i} x \cdot \xi} \rangle$ fulfils (I) and

$$|u_j(x)| \leq c \sup \{\, (1 + |\xi|)^m |D_\xi^\alpha (\Phi(2^{-j} \xi) e^{\mathrm{i} x \cdot \xi})| \mid \xi \in \mathbb{R}^n, \quad |\alpha| \leq m \,\}. \tag{85}$$

Invoking Leibniz' rule, the worst terms occurs when derivatives of order m fall on the exponential, and this is estimated by $C 2^{jm} (1 + |x|)^m$.

To prove $1°$, note that if $\psi \in C^\infty(\mathbb{R}^n)$ is supported for $\frac{1}{2A} \leq |\xi| \leq 2A$ and equalling 1 where $\frac{1}{A} \leq |\xi| \leq A$, any $\varphi \in \mathcal{S}$ fulfils

$$\langle u_j, \overline{\varphi} \rangle \leq \|(1 + |x|^2)^{-\frac{m+n}{2}} u_j\|_2 \|(1 + |x|^2)^{\frac{m+n}{2}} \mathcal{F}^{-1}(\psi(2^{-j} \cdot) \hat{\varphi})\|_2. \tag{86}$$

Here the first norm is $\mathcal{O}(2^{mj})$ by (84). For any $k > 0$ Parseval–Plancherel's identity gives

$$\|(1 + |x|^2)^{m+n}\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\hat{\varphi})\|_2$$

$$\leq \sum_{|\alpha+\beta| \leq 2m+2n} c_{\alpha,\beta} 2^{-j|\alpha|}\|D^\alpha\psi\|_\infty\|(1 + |\xi|)^{k+n}D^\beta\hat{\varphi}\|_\infty \int_{2^{j-1}/A}^\infty r^{-1-k}\,dr$$

$$= c(A, k, m, n, \varphi, \psi)2^{-jk}. \quad (87)$$

That is, $\langle u_j, \overline{\varphi} \rangle = \mathcal{O}(2^{(m-k)j})$ for $k > m$, so $\sum_{j=0}^\infty \langle u_j, \varphi \rangle$ converges, whence $\sum u_j$ does so in $\mathcal{S}'$. $\qquad\square$

Remark 6. *Littlewood–Paley decompositions $u = \sum_{j=0}^\infty u_j$ are rapidly convergent, in the following sense: if an arbitrary $u \in \mathcal{S}'$ is decomposed as in 2° above, the proof of 1° gives*

$$\langle u_j, \varphi \rangle = \mathcal{O}(2^{-Nj}) \quad \text{for every} \quad N > 0, \ \varphi \in \mathcal{S}(\mathbb{R}^n), \quad (88)$$

so $\langle u - \sum_{j<k} u_j, \varphi \rangle = \sum_{j \geq k}\langle u_j, \varphi \rangle = \mathcal{O}(2^{-Nk}) \to 0$, rapidly for $k \to \infty$.

For the $F_{\vec{p},q}^{s,\vec{a}}$ we have the following (quasi-homogeneous) dyadic ball criterion:

Lemma 5. *When $s > \sum_{k=1}^n \frac{a_k}{\min(1,p_1,\ldots,p_k,q)} - |\vec{a}|$ for $0 < \vec{p} < \vec{\infty}$ and $0 < q \leq \infty$, then there exists a $c > 0$ such that, for every sequence (u_j) in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling both the dyadic ball condition* (II) *and that*

$$F := \big\|\big(\sum_{j=0}^\infty 2^{sjq}|u_j|^q\big)^{\frac{1}{q}}\,\big|L_{\vec{p}}\big\| < \infty, \quad (89)$$

the series $\sum_{j=0}^\infty u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for which $\|u\,|F_{\vec{p},q}^{s,\vec{a}}\| \leq cF$.

Proof. By condition (II) there is a fixed $h \in \mathbb{N}$ such that $\Phi_j\mathcal{F}u_k = 0$ for $k < j - h$. So

$$\mathcal{F}^{-1}[\Phi_j \sum_{k=0}^M \mathcal{F}u_k] = \mathcal{F}^{-1}[\Phi_j \sum_{k=j-h}^M \mathcal{F}u_k] \quad \text{for} \quad M \geq j - h. \quad (90)$$

Setting $k = j + \ell$ and using that $\|\cdot|\ell_1\| \leq \|\cdot|\ell_\tau\|$ for $\tau = \min(1, p_1, \ldots, p_n, q)$, one obtains the first of the following inequalities, that also rely on Proposition 7 with $R = 2^{\ell_+}$,

$$\big\|\sum_{k \leq M} u_k \,\big|F_{\vec{p},q}^{s,\vec{a}}\big\|^\tau \leq \big\|\big(\sum_{j=0}^\infty (2^{sj\tau}\sum_{\ell=-h}^{M-j}|\mathcal{F}^{-1}[\Phi_j\mathcal{F}u_{j+\ell}]|^\tau)^{q/\tau}\big)^{\tau/q}\,\big|L_{\vec{p}/\tau}\big\|$$

$$\leq \sum_{\ell=-h}^M \big\|2^{js}\mathcal{F}^{-1}[\Phi_j\mathcal{F}u_{j+\ell}]\,\big|L_{\vec{p}}(\ell_q)\big\|^\tau \quad (91)$$

$$\leq c\sum_{\ell=-h}^\infty 2^{\ell_+\tau(\vec{a}\cdot\frac{1}{\vec{t}} - |\vec{a}|)}\big\|2^{js}u_{j+\ell}\,\big|L_{\vec{p}}(\ell_q)\big\|^\tau \leq c_1 F^\tau \sum_{\ell=-h}^\infty 2^{\ell_+\tau(-s+\vec{a}\cdot\frac{1}{\vec{t}} - |\vec{a}|)}.$$

Hereby $t_k < \min(1, p_1, \ldots, p_k, q)$ must be fulfilled. But the t_k can be taken with this property at the same time as $s > \vec{a}\cdot\frac{1}{\vec{t}} - |\vec{a}|$; cf. the conditions on s in the lemma.

With $\vec{t}$ as above, the sequence $(\sum_{k=0}^{M} u_k)_{M \in \mathbb{N}}$ is by (91) bounded in $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$. This yields at once that it is fundamental in $F_{\vec{p},1}^{s',\vec{a}}$ for $s' < s$, thence convergent to some u. By monotone convergence for $M \to \infty$ in (91), the estimate $\| u \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \leq cF$ is obtained. $\qquad \square$

In case $p_n \leq \ldots \leq p_2 \leq p_1$ the restriction for s reduces to $s > \sum_{k=1}^{n} a_k(\frac{1}{\min(1,p_k,q)} - 1)$. In case $p_1 = \ldots = p_n$ this gives back the unmixed version known since [Yam86b].

The above proof gives more, for if the series fulfils the stronger corona condition (I), then $\mathcal{F}^{-1}(\Phi_j \mathcal{F} u_k) = 0$ unless $j - h \leq k \leq j + h$. In this case the sums in (91) have $l \in \{-h, \ldots, h\}$, so the restriction on s is not needed. This proves

Lemma 6. *When $s \in \mathbb{R}$ and $0 < \vec{p} < \vec{\infty}$, $0 < q \leq \infty$, there exists $c > 0$ such that, for every sequence (u_j) in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling both the dyadic corona condition* (I) *and that*

$$F := \big\| \big(\sum_{j=0}^{\infty} 2^{sjq} |u_j|^q \big)^{\frac{1}{q}} \,\big|\, L_{\vec{p}} \big\| < \infty, \tag{92}$$

the series $\sum_{j=0}^{\infty} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for which $\| u \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \leq cF$.

For the Besov spaces, the dyadic ball and corona criteria follow by interchanging the order of the $L_{\vec{p}}$ and ℓ_q-norms in the proof Lemma 5, and by using Proposition 7 for sequences having only a single non-trivial term. Thus one has the next result.

Lemma 7. *When $s > \sum_{k=1}^{n} \frac{a_k}{\min(1,p_1,\ldots,p_k)} - |\vec{a}|$ for $0 < \vec{p} \leq \vec{\infty}$ and $0 < q \leq \infty$, there exists $c > 0$ such that, for every sequence (u_j) in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling both* (II) *and*

$$B := \big(\sum_{j=0}^{\infty} 2^{sjq} \| u_j \,|\, L_{\vec{p}} \|^q \big)^{\frac{1}{q}} < \infty, \tag{93}$$

the series $\sum_{j=0}^{\infty} u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a $u \in B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for which $\| u \,|\, B_{\vec{p},q}^{s,\vec{a}} \| \leq cB$.

If $B < \infty$ and (I) *hold, then the convergence and $\| u \,|\, B_{\vec{p},q}^{s,\vec{a}} \| \leq cB$ holds for all $s \in \mathbb{R}$.*

By Lemma 6 and 7, the choice of the Littlewood–Paley decomposition and the constants are without significance for the $F_{\vec{p},q}^{s,\vec{a}}$ and $B_{\vec{p},q}^{s,\vec{a}}$ spaces. For completeness the next result is given.

Lemma 8. *Every differential operator of the form $D^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$ gives continuous maps $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to F_{\vec{p},q}^{s-\alpha \cdot \vec{a}}(\mathbb{R}^n)$ and $B_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to B_{\vec{p},q}^{s-\alpha \cdot \vec{a}}(\mathbb{R}^n)$, for every $s \in \mathbb{R}$.*

Proof. For the scale $F_{\vec{p},q}^{s,\vec{a}}$, Lemma 6 and Proposition 7 applied to $D^\alpha u = \sum_{j=0}^{\infty} (D^\alpha \mathcal{F}^{-1} \Phi_j) * u$ give at once that D^α has order $\alpha \cdot \vec{a}$. The Besov case is similar. $\qquad \square$

As another consequence of the dyadic corona criterion, we sketch a

Proof of Lemma 2. The embeddings (44)–(45) were shown in [JS, Prop. 10]. The density of $\mathcal{S} \subset F_{\vec{p},q}^{s,\vec{a}}$ follows from Lemma 6: $u^N := \sum_{j=0}^{N} \Phi_j(D) u$ converges to u in $F_{\vec{p},q}^{s,\vec{a}}$, because for $u - u^N = \sum_{j>N} \Phi_j(D) u$ the number $F \to 0$ as $N \to \infty$ by dominated convergence

($q < \infty$). The set of $g \in L_{\vec{p}} \cap \mathcal{S}'$ with supp $\mathcal{F}g \subset B_{\vec{a}}(0, 2^{N+1})$ is embedded into $F_{\vec{p},q}^{s,\vec{a}}$, for $g = g + 0 + \dots$ fulfils (I) with $A = 2^{N+1}$. Therefore the convergence of $u^N \cdot c\mathcal{F}^{-1}\Psi_0(\varepsilon\cdot) \in \mathcal{S}$ to u^N in $L_{\vec{p}}$ for $\varepsilon \to 0$ implies $\| c\mathcal{F}^{-1}\Psi_0(\varepsilon\cdot)u^N - u^N \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \to 0$. A similar reasoning works for $B_{\vec{p},q}^{s,\vec{a}}$. $\qquad\square$

Occasionally it is useful to have a corona criterion based on powers of 2^λ for some $\lambda > 0$.

Lemma 9. *When $s \in \mathbb{R}$ and $0 < \vec{p} < \vec{\infty}$, $0 < q \le \infty$, there exists $c > 0$ such that, for every sequence (u_j) in $\mathcal{S}'(\mathbb{R}^n)$ fulfilling* supp $\mathcal{F}u_0 \subset B_{\vec{a}}(0, A)$ *and*

$$\text{supp}\,\mathcal{F}u_j \subset \{\,\xi \mid \tfrac{1}{A}2^{\lambda j} \le |\xi|_{\vec{a}} \le A2^{\lambda j}\,\} \quad \text{for} \quad j \ge 1, \tag{94}$$

$$F_\lambda := \| \, (\sum_{j=0}^\infty |2^{\lambda s j} u_j|^q)^{\frac{1}{q}} \,|\, L_{\vec{p}} \| < \infty, \tag{95}$$

the series $\sum_{j=0}^\infty u_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for which $\| u \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \le cF_\lambda$.

Proof. Note that (94) gives an $h \in \mathbb{N}$ such $\Phi_j \mathcal{F}u_k = 0$ unless $\frac{j}{\lambda} - h \le k \le \frac{j}{\lambda} + h$. With $k = [j/\lambda] + \nu$ ($[\cdot]$ is the integer part), a modification of (91) gives

$$\begin{aligned}
\Big\| \sum_{k \le M} u_k \,\big|\, F_{\vec{p},q}^{s,\vec{a}} \Big\|^\tau &\le \Big\| \, \big(\sum_{j=0}^\infty (2^{sj\tau} \sum_{\nu=-h}^{h+1} |\mathcal{F}^{-1}[\Phi_j \mathcal{F}u_{[j/\lambda]+\nu}]|^\tau)^{q/\tau} \big)^{\tau/q} \,\big|\, L_{\vec{p}/\tau} \Big\| \\
&\le c \sum_{|\nu| \le h+1} (A2^{|\nu|\lambda})^{\tau(\vec{a}\cdot\frac{1}{t} - |\vec{a}|)} \big\| \, 2^{js} u_{[j/\lambda]+\nu} \,\big|\, L_{\vec{p}}(\ell_q) \big\|^\tau.
\end{aligned} \tag{96}$$

Here the last inequality results from Proposition 7, for $\xi \in$ supp $u_{[j/\lambda]+\nu}$ entails $|\xi|_{\vec{a}} \le A2^{\lambda([j/\lambda]+\nu)} \le (A2^{\lambda|\nu|})2^j$. It is clear that $2^{sj} \le c2^{s\lambda[j/\lambda]}$. Therefore $m = [j/\lambda]$ gives $\| \, 2^{js} u_{[j/\lambda]+\nu} \,|\, \ell_q \| \le c_\lambda \| \, 2^{sm} u_m \,|\, \ell_q \|$, for the sequence $(2^{js} u_{[j/\lambda]+\nu})_{j \in \mathbb{N}_0}$ is either lacunary for $0 < \lambda < 1$ or, for $\lambda \ge 1$, it has every u_m repeated just $[\lambda]$ times. Consequently $\big\| \sum_{k \le M} u_k \,\big|\, F_{\vec{p},q}^{s,\vec{a}} \big\| \le cF_\lambda$ for all M, so that convergence and the estimate follow as in the proof of Lemma 6. $\qquad\square$

For example Lemma 9 gives invariance of $F_{\vec{p},q}^{s,\vec{a}}$ under the reparametrisation $(s, \vec{a}) \mapsto (\lambda s, \lambda \vec{a})$:

Lemma 10. $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) = F_{\vec{p},q}^{\lambda s, \lambda \vec{a}}(\mathbb{R}^n)$ *for every $\lambda > 0$, and the quasi-norms are equivalent.*

Proof. For $\vec{b} = \lambda\vec{a}$ the definition gives $|\xi|_{\vec{b}}^\lambda = |\xi|_{\vec{a}}$, so that the Littlewood–Paley decomposition $1 = \sum_{j=0}^\infty \Phi_j^{\vec{b}}$ associated with $\vec{b}$ yields functions that for $j \ge 1$ are equal to 1 in the set where $(\frac{13}{20})^\lambda 2^{\lambda j} \le |\xi|_{\vec{a}} \le (\frac{11}{10})^\lambda 2^{\lambda j}$. Hence Lemma 9 gives $\| u \,|\, F_{\vec{p},q}^{s,\vec{a}} \| \le c \| u \,|\, F_{\vec{p},q}^{\lambda s, \lambda \vec{a}} \|$. Since $\vec{a}$ and $\lambda > 0$ are arbitrary, the opposite inequality also holds. $\qquad\square$

In view of this, we may assume that all $a_k \ge 1$; this is convenient in Section 4 below.

Remark 7. *Since there are few general references to the mixed norm spaces $F_{\vec{p},q}^{s,\vec{a}}$, we note that the reader may find the necessary theory here and in [JS].*

4. Proofs

4.1. **The general necessary conditions.** We first give the proof of Lemma 1, since this just amounts to a calculation of some norms in $F_{\vec{p},q}^{s,\vec{a}}$ of suitably chosen functions. Recall our normalisation $\min(a_1,\ldots,a_n) = 1$.

4.1.1. *Examples.* To have a convenient set-up, we shall consider traces on the hyperplane $x_m = 0$ for arbitrary $m \in \{1,\ldots,n\}$. The remaining $n-1$ variables are split in two groups $x_\geq$ and $x_<$. The reason for this labelling will be clear later when a $\vec{p}$ is fixed: the components p_k with $k \neq m$ splits naturally into the groups $p_\geq$ and $p_<$ in which $p_k \geq 1$, respectively $p_k < 1$; accordingly $x_\geq$, $x_<$ are defined from the same indices.

Let f, $g \in \mathcal{S}(\mathbb{R})$ be fixed, as we may, such that $\int_{\mathbb{R}} f(t)\,dt = 1$, $g(0) = 1$ and, with $a_0 = \max(a_1,\ldots,a_n)$,

$$\operatorname{supp}\hat{f} \subset \{\,|\tau| < 1/(10n)^{a_0}\,\}, \qquad \operatorname{supp}\hat{g} \subset \{\,(\tfrac{8}{10})^{a_m} \leq |\tau| \leq 1\,\}. \tag{97}$$

Introducing the tensor product

$$w_l(x) = (\prod_{x_\geq} f(x_k)) \otimes g(2^{la_m}x_m) \otimes (\prod_{x_<} 2^{la_k}f(2^{la_k}x_k)) \tag{98}$$

we shall estimate the Schwartz function $v_j = \frac{1}{j}\sum_{l=j+1}^{2j} w_l$. Note first that for $\xi \in \operatorname{supp}\hat{w}_l$, one has for the vector $\eta = \xi - \xi_m e_m$ (formed by resetting the mth coordinate to 0) that, since $\frac{a_0}{a_k} \geq 1$ for all k,

$$|\eta|_{\vec{a}} \leq \sum_{k \neq m} |\xi_k|^{1/a_k} \leq \sum_{x_\geq} (10n)^{-\frac{a_0}{a_k}} + \sum_{x_<} 2^l(10n)^{-\frac{a_0}{a_k}} \leq \tfrac{n-1}{10n}\cdot 2^l. \tag{99}$$

Using the triangle inequality for $|\cdot|_{\vec{a}}$,

$$\tfrac{7}{10}2^l \leq |\xi_m|^{1/a_m} - |\eta|_{\vec{a}} \leq |\xi|_{\vec{a}} \leq |\xi_m|^{1/a_m} + |\eta|_{\vec{a}} < \tfrac{11}{10}2^l. \tag{100}$$

This means that every $\xi \in \operatorname{supp}\hat{w}_l$ satisfies $\Phi_l(\xi) = 1$, for this identity holds where $\frac{13}{20}2^l \leq |\xi|_{\vec{a}} \leq \frac{11}{10}2^l$. Consequently the Φ_l disappear from the norms of v_j, e.g.

$$\| v_j \,|F_{\vec{p},q}^{s,\vec{a}}\| = \tfrac{1}{j}\,\big\| \,(\sum_{l=j+1}^{2j} 2^{slq}|w_l(\cdot)|^q)^{1/q}\,\big|L_{\vec{p}}\big\|. \tag{101}$$

For certain triples $(s,\vec{p},q)$ this can be calculated precisely.

Lemma 11. *Let $\vec{p}$ be a vector from $]0,\infty]$, and let $p_\geq$ and $p_<$ be the above mentioned splitting corresponding to a fixed m.*

1° *For $s = \frac{a_m}{p_m} + \sum_{k \neq m}(\frac{a_k}{p_k} - a_k)_+$ it holds for every q that*

$$\| v_j \,|B_{\vec{p},q}^{s,\vec{a}}\| = c\cdot j^{\frac{1}{q}-1}. \tag{102}$$

2° *If $p_m > 1$ and $p_k \geq 1$ for $k \neq m$, then for $s = \frac{a_m}{p_m}$,*

$$\| v_j \,|F_{\vec{p},p_m}^{s,\vec{a}}\| = c\cdot j^{\frac{1}{p_m}-1}. \tag{103}$$

Proof. In analogy with (101) above, $\| v_j \, | B^{s,\vec{a}}_{\vec{p},q} \| = \frac{1}{j} (\sum_{l=j+1}^{2j} 2^{slq} \| w_l \, | L_{\vec{p}} \|^q)^{1/q}$. Since the $L_{\vec{r}}$-norm respects the tensor products entering w_l, and since $2^{l(\frac{a_m}{p_m} + \sum_{p_<}(\frac{a_k}{p_k} - a_k))}$ is absorbed by the dilations, $\| v_j \, | B^{s,\vec{a}}_{\vec{p},q} \| = \frac{1}{j} \prod_{k \neq m} \| f \|_{p_k} (\sum_{l=j+1}^{2j} \| g \|^q_{p_m})^{\frac{1}{q}} = cj^{\frac{1}{q} - 1}$.

In case $2°$, a similar procedure applies to (101); the group $x_<$ is empty by assumption, so

$$\| v_j \, | F^{s,\vec{a}}_{\vec{p},p_m} \| = \frac{1}{j} \prod_{k \neq m} \| f \|_{p_k} \Big(\sum_{l=j+1}^{2j} \int_{\mathbb{R}} 2^{la_m} |g(2^{la_m} x_m)|^{p_m} \, dx_m \Big)^{\frac{1}{p_m}} = c \cdot j^{\frac{1}{p_m} - 1} \tag{104}$$

since the factors involving f do not depend on the summation index. $\square$

The interest of Lemma 11 comes from the obvious fact that

$$\gamma_{0,m} v_j \to \delta_0(x_<) \otimes \prod_{x_\geq} f(x_k) \quad \text{for} \quad j \to \infty \tag{105}$$

(which means $f(x_1) \otimes \cdots \otimes f(x_n)$ if $x_<$ is empty). From this we get the

4.1.2. *Proof of Lemma 1.* Given that $\gamma_{0,m} \colon F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n-1})$ is continuous for some triple $(s, \vec{p}, q)$, we set $t = \frac{a_m}{p_m} + \sum_{p_k \neq m}(\frac{a_k}{p_k} - a_k)_+$.

Then $s < t$ cannot hold, for else $B^{t,\vec{a}}_{\vec{p},2} \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}$, and this embedding would be incompatible with the continuity of $\gamma_{0,m}$, since by Lemma 11 the v_j tend to 0 in $B^{t,\vec{a}}_{\vec{p},2}$ and a fortiori in $F^{s,\vec{a}}_{\vec{p},q}$ (whilst $\gamma_{0,m} v_j \not\to 0$, cf. (105)). Therefore the continuity implies $s \geq \frac{a_m}{p_m} + \sum_{p_k \neq m}(\frac{a_k}{p_k} - a_k)_+$.

Similarly $2°$ of Lemma 11 shows that in case $p_k \geq 1$ for $k \neq m$, the trace $\gamma_{0,m}$ is only continuous from $F^{s,\vec{a}}_{\vec{p},q}$ on the borderline (which is $s = a_m/p_m$ then) if $p_m \leq 1$.

4.2. **Proof of Theorem 5.** We shall proceed with Theorem 5, for later we draw on the properties of the extension operator, during the proof of the theorems on the trace.

The next well-known lemma plays a significant role in the proofs, e.g. because the property of K_1 and K_n that they map into $\bigcap_{0<q\leq\infty} F^{s,\vec{a}}_{\vec{p},q}$ is a consequence of the fact that both (106) and (107) hold for *any* ℓ_r-norm, $0 < r \leq \infty$.

Lemma 12. *If $(b_j)_{j\in N_0}$ is a sequence of complex numbers, $s > 0$ and q, $r \in \,]0,\infty]$, there is a constant $c = c(s,q,r)$ such that (with sup-norm over k for $r = \infty$)*

$$\big\| \{2^{sj}(\sum_{k=j}^{\infty} |b_k|^r)^{1/r}\}_{j=0}^{\infty} \, |\ell_q\| \leq c \, \big\| \{2^{sj}b_j\}_{j=0}^{\infty} \, |\ell_q\| \tag{106}$$

$$\big\| \{2^{-sj}(\sum_{k=0}^{j} |b_k|^r)^{1/r}\}_{j=0}^{\infty} \, |\ell_q\| \leq c \, \big\| \{2^{-sj}b_j\}_{j=0}^{\infty} \, |\ell_q\| . \tag{107}$$

For $r = 1$ this lemma is equivalent to [Yam86b, Lem. 3.8]; in general it may be proved in a similar fashion as noted in [Joh96, Lem. 2.5].

4.2.1. *The right-inverse K_1.* Note first that $\varphi_j(\xi'') := \Phi_j(0, \xi'')$ gives a Littlewood–Paley decomposition on $\mathbb{R}^{n-1}$, so any $v \in \mathcal{S}'(\mathbb{R}^{n-1})$ may be written $v = \sum v_j$ for $v_j = \varphi_j(D)v$.

To construct K_1 we introduce an auxiliary function $\psi \in C^\infty(\mathbb{R})$ such that $\psi(0) = 1$ and $\operatorname{supp} \mathcal{F}\psi \subset [1, 2]$. Then K_1 can be defined as

$$K_1 v(x) = \sum_{j=0}^\infty \psi(2^{ja_1}x_1)v_j(x''), \tag{108}$$

for the series converges in $\mathcal{S}'$ by Lemma 4. To verify this, note that $\mathcal{F}(\psi(2^{ja_1}\cdot)v_j)$ equals the product $2^{-ja_1}\hat{\psi}(2^{-ja_1}\xi_1)\varphi_j(\xi'')\hat{v}(\xi)$, where e.g. $1 \le |2^{-a_1 j}\xi_1| \le 2$ implies $2^{a_1 j} \le |\xi_1| \le 2^{a_1(j+1)}$ and

$$|\xi_1|^{1/a_1} \le |(\xi_1, \xi'')|_{\vec{a}} \le |(\xi_1, 0)|_{\vec{a}} + |(0, \xi'')|_{\vec{a}} \le |\xi_1|^{1/a_1} + |\xi''|_{a''}; \tag{109}$$

this immediately give the inclusions, valid for $j \ge 0$,

$$\operatorname{supp} \mathcal{F}(\psi(2^{ja_1}\cdot)v_j) \subset \{\, \xi \mid 2^j \le |\xi|_{\vec{a}} \le 4 \cdot 2^j \,\}. \tag{110}$$

Moreover, from $2°$ in Lemma 4 the growth condition (84) follows at once. Hence K_1 is a well defined linear map $\mathcal{S}'(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$.

Furthermore, $\Lambda \colon x_1 \to \sum_{j=0}^\infty \psi(2^{ja_1}x_1)v_j(x'')$ is in the set $C_{\mathrm{b}}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$ of continuous bounded maps $\mathbb{R} \to \mathcal{S}'(\mathbb{R}^{n-1})$. In fact, the functions $\psi(2^{ja_1}\cdot)$ are uniformly bounded, so that $g(x_1) = \sum \psi(2^{ja_1}x_1)\langle v_j, \varphi \rangle$ by (88) converges to a continuous and bounded function on $\mathbb{R}$. Hence $x_1 \mapsto \langle \Lambda(x_1), \varphi \rangle$ has these properties, so $\Lambda \in C_{\mathrm{b}}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$.

For every $\eta \in \mathcal{S}(\mathbb{R}^n)$ this implies the first identity in

$$\langle \Lambda, \eta \rangle = \int_{\mathbb{R}} \langle \Lambda(x_1), \eta(x_1, \cdot) \rangle_{\mathbb{R}^{n-1}} \, dx_1 = \int \sum_{j=0}^\infty \langle \psi(2^{ja_1}x_1)v_j, \eta(x_1, \cdot) \rangle dx_1$$
$$= \lim_{m\to\infty} \sum_{j=0}^m \langle \psi(2^{ja_1}\cdot)v_j, \eta \rangle_{\mathbb{R}^n} = \langle K_1 v, \eta \rangle. \tag{111}$$

Here passage to the last line is justified with the following majorisation,

$$\sup_{x_1} |\langle v_j, \eta(x_1, \cdot) \rangle| \le C_N 2^{-jN}(1 + x_1^2)^{-1}, \quad \text{for every } N > 0 \tag{112}$$

that follows analogously to (88), by taking for φ in the proof of (88) a function like $\varphi_t = (1 + t^2)\eta(t, x)$ depending on a parameter t.

By the above formula $K_1 v = \Lambda \in C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$, so since $\psi(0) = 1$,

$$\gamma_{0,1} K_1 v = \Lambda(0) = \sum_{j=0}^\infty \psi(0)v_j = v \quad \text{for every } v \in \mathcal{S}'(\mathbb{R}^{n-1}). \tag{113}$$

That is, K_1 maps all of $\mathcal{S}'(\mathbb{R}^{n-1})$ into the domain of $\gamma_{0,1}$, for which it acts as a right-inverse.

Continuity of $K_1\colon \mathcal{S}'(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$ results by proving that there exists an everywhere defined linear map $K_1^*\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$ given by

$$K_1^*\eta(x'') = \sum_{j=0}^{\infty} \int_{\mathbb{R}} \psi(2^{ja_1}y_1) \int_{\mathbb{R}^{n-1}} \mathcal{F}^{-1}\varphi_j(y'')\eta(y_1, x'' - y'') \, dy'' dy_1. \tag{114}$$

Indeed, using K_1^* one arrives at the following formula, where the right hand side depends continuously on $v \in \mathcal{S}'(\mathbb{R}^{n-1})$,

$$\langle\, K_1 v, \overline{\eta}\,\rangle = \langle\, v, \sum_{j=0}^{\infty} \big(\int_{\mathbb{R}} \psi(2^{ja_1}y_1)\mathcal{F}^{-1}_{\xi''\to x''}(\varphi_j\mathcal{F}_{x''\to\xi''}\eta) \, dy_1 \big)^- \rangle_{\mathbb{R}^{n-1}} = \langle\, v, \overline{K_1^*v}\,\rangle. \tag{115}$$

As for (114) it is noted that $\mathcal{S}(\mathbb{R}^n)$ contains

$$(\hat{\psi}(-\xi_1)\Phi_0(0, \xi'') + \sum_{j=1}^{\infty} 2^{-ja_1}\hat{\psi}(-2^{-ja_1}\xi_1)\Phi_1(0, 2^{-(j-1)a''}\xi''))\mathcal{F}\eta(\xi_1, \xi''), \tag{116}$$

since this is a product of $\mathcal{F}\eta \in \mathcal{S}$ and a C^∞-function with bounded derivatives. Applying $\mathcal{F}^{-1}$ and setting $x_1 = 0$, it results that the right-hand side of (114) is in $\mathcal{S}(\mathbb{R}^{n-1})$.

4.2.2. *Boundedness of K_1.* With $v \in F^{s - \frac{a_1}{p_1}, a''}_{p'', p_1}(\mathbb{R}^{n-1})$, for $s \in \mathbb{R}$, we obtain boundedness of K_1 by showing that the series defining $K_1 v$ converges in $F^{s, \vec{a}}_{\vec{p}, q}(\mathbb{R}^n)$. For this it suffices by Lemma 6 to show

$$\big\| \sum_{j=0}^{\infty} \psi(2^{ja_1}x_1)v_j(x'') \, \big| L_{\vec{p}}(\ell^s_q) \big\| \leq c \, \| \, v \, | F^{s - \frac{a_1}{p_1}, a''}_{p'', p_1} \|. \tag{117}$$

By embeddings this may be reduced to the case $q < p_1$. For the integral

$$I(x'') := \int_{\mathbb{R}} \big(\sum_{j=0}^{\infty} |2^{sj}\psi(2^{ja_1}x_1)v_j(x'')|^q \big)^{\frac{p_1}{q}} \, dx_1 \tag{118}$$

we take $N > \frac{1}{p_1}$ so that $|\psi(2^{ja_1}x_1)| \leq |2^{ja_1}x_1|^{-N} \sup_{\mathbb{R}} t^N|\psi(t)|$ for $x_1 \neq 0$. Then, if I_1 and I_0 denote the integrals over $|x_1| > 1$ and $|x_1| \leq 1$, respectively,

$$\begin{aligned} I_1 &\leq \int_{|x_1|>1} \big(\sum_{j=0}^{\infty} |2^{sj}v_j(x'')|^q 2^{-Na_1jq}c_\psi \big)^{\frac{p_1}{q}} x_1^{-Np_1} \, dx_1 \\ &\leq c_1(1 - 2^{(\frac{1}{p_1} - N)a_1q})^{-\frac{p_1}{q}} (\sup_j 2^{(s - \frac{a_1}{p_1})j}|v_j(x'')|)^{p_1}. \end{aligned} \tag{119}$$

By splitting the integration area for I_0 into intervals with $2^{-(k+1)a_1} \leq |x_1| \leq 2^{-ka_1}$, that are of length $(2 - 2^{1-a_1})2^{-ka_1}$, and by using the choice of N for $j > k$,

$$I_0 \leq \sum_{k=0}^{\infty} c2^{-ka_1}\big(\sum_{j=0}^{k} |2^{sj}v_j(x'')|^q\|\psi\|^q_\infty + \sum_{j=k+1}^{\infty} |v_j(x'')2^{(s-Na_1)j+N(k+1)a_1}c(\psi)|^q \big)^{\frac{p_1}{q}}. \tag{120}$$

At the cost of a factor of $2^{\frac{p_1}{q}}$ the two terms may be treated separately, so

$$I_0 \leq c_2 \sum_{k=0}^{\infty} 2^{-ka_1} \Big(\sum_{j=0}^{k} |2^{sj} v_j(x'')|^q\Big)^{\frac{p_1}{q}} + c_3 \sum_{k=0}^{\infty} 2^{k(Na_1 - \frac{a_1}{p_1})p_1} \Big(\sum_{j=k}^{\infty} |v_j(x'') 2^{(s-Na_1)j}|^q\Big)^{\frac{p_1}{q}}. \quad (121)$$

According to Lemma 12, the ℓ_q-norms over j may be "cancelled" since the weights have bases $2^{-a_1} < 1$ and $2^{(N-\frac{1}{p_1})a_1 p_1} > 1$, respectively, so

$$I_0 \leq (c_2 + c_3) \,\| \, 2^{(s-\frac{a_1}{p_1})j} v_j(x'') \, | \ell_{p_1} \|^{p_1}. \quad (122)$$

Altogether $I(x'') \leq c_4 \,\| \, 2^{(s-\frac{a_1}{p_1})j} v_j(x'') \, | \ell_{p_1} \|^{p_1}$, so by continued calculation of the $L_{\vec{p}}$-norm, (117) follows. Therefore K_1 is bounded $F_{p'',p_1}^{s-\frac{a_1}{p_1},a''}(\mathbb{R}^{n-1}) \to F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $q > 0$.

4.2.3. *The extension operator K_n.* This is in analogy with K_1 taken as

$$K_n v(x) = \sum_{j=0}^{\infty} \psi(2^{ja_n} x_n) v_j(x'). \quad (123)$$

By Lemma 4, this is also meaningful in $\mathcal{S}'$, and the above discussion, mutatis mutandis, gives that K_n is a right-inverse of $\gamma_{0,n}$.

To show that K_n is bounded from $B_{p',p_n}^{s-\frac{a_n}{p_n},a'}(\mathbb{R}^{n-1})$ to $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for all $q \in \,]0,\infty]$, we may assume that $q < \min(p_1, \ldots, p_n)$. For v belonging to the former space, we set

$$I := \int_{\mathbb{R}} \,\big\| \big(\sum_{j=0}^{\infty} |2^{sj} \psi(2^{ja_n} x_n) v_j(\cdot)|^q\big)^{\frac{1}{q}} \, | L_{p'} \big\|^{p_n} \, dx_n. \quad (124)$$

For the integral I_1 over $|x_n| \geq 1$, one use an $N > \frac{1}{p_n}$ (but otherwise as above) together with the triangle inequality for the mixed-norm with exponent $\frac{1}{q} p' = (\frac{p_1}{q}, \ldots, \frac{p_{n-1}}{q})$ to obtain that

$$I_1 \leq \int_{|x_n| \geq 1} \big(\sum_{j=0}^{\infty} \,\| \, 2^{sjq} |v_j|^q \, | L_{\frac{1}{q}p'} \| c_\psi 2^{-Na_n jq}\big)^{\frac{p_n}{q}} x_n^{-Np_n} \, dx_n. \quad (125)$$

Since $q < p_n$, Hölder's inequality gives $I_1 \leq c \,\| \, v \, | B_{p',p_n}^{s-\frac{a_n}{p_n},a'}(\mathbb{R}^{n-1}) \|^{p_n}$.

Correspondingly I_0 is split into regions with $2^{-(k+1)a_n} \leq |x_n| \leq 2^{-ka_n}$ and this yields, cf. the case for K_1 above,

$$I_0 \leq c_1 \sum_{k=0}^{\infty} 2^{-ka_n} \big(\sum_{j=0}^{k} \,\| \, 2^{sjq} |v_j|^q \, | L_{\frac{1}{q}p'} \|\big)^{\frac{p_n}{q}}$$

$$+ c_2 \sum_{k=0}^{\infty} 2^{k(Na_n - \frac{a_n}{p_n})p_n} \big(\sum_{j=k}^{\infty} \,\| \, |v_j|^q \, | L_{\frac{1}{q}p'} \| 2^{(s-Na_n)jq}\big)^{\frac{p_n}{q}}. \quad (126)$$

By passing to the $L_{p'}$-norms and applying Lemma 12, one can get rid of the sums over $j \lessgtr k$, hence $I \leq c \,\| \, v \, | B_{p',p_n}^{s-\frac{a_n}{p_n},a'} \|^{p_n}$. This shows that K_n is continuous $B_{p',p_n}^{s-\frac{a_n}{p_n},a'}(\mathbb{R}^{n-1}) \to F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ for $0 < q \leq \infty$, any $s \in \mathbb{R}$.

Remark 8. *Our treatment of K_1 and K_n was inspired by the isotropic estimates in* [Tri83, Thm. 2.7.2]. *We have preferred to use Lemma 12 and the dyadic corona criterion, that also give that the K_m map all of $\mathcal{S}'(\mathbb{R}^{n-1})$ into the domain of $\gamma_{0,m}$. The continuity $K_m\colon \mathcal{S}'(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$ followed from the existence of an adjoint $K_m^*\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$.*

4.3. **On Corollaries 1–2.** As noted prior to the corollaries, boundedness follows directly from the other results. But surjectivity of $\rho_{m,k}$ is conveniently established here, by means of some modifications of the right-inverses K_1, K_n. Details will be given for $k = 1$; to simplify notation, we treat $\rho_{m+1,1}$, so the trace of highest order is $\gamma_{m,1}$.

The auxiliary function $\psi \in \mathcal{F}^{-1}C_0^\infty(]1,2[)$ with $\psi(0) = 1$ can be taken such that also $\psi'(0) = \cdots = \psi^{(m)}(0) = 0$. Indeed, we may arrange that $\mathcal{F}\psi(\xi_1)$ is orthogonal in $L_2(]1,2[)$ to $W_m := \operatorname{span}(\xi_1,\ldots,\xi_1^m)$. (It is well known that if a Hilbert space H has a dense subspace U, it holds for every subspace W_m of dimension $m \in \mathbb{N}$ that $U \bigcap W_m^\perp$ is dense in the orthogonal complement $W_m^\perp$ (induction w.r.t. m). In our case $f(\xi_1) \equiv 1$ has projection $g \neq 0$ onto $W_m^\perp$, so the density implies the existence of $\phi \in C_0^\infty(]1,2[) \bigcap W_m^\perp$ such that $0 \neq \int_1^2 \phi\overline{g}\,d\xi_1 = \int_1^2 \phi\overline{f}\,d\xi_1 = \int_1^2 \phi\,d\xi_1 =:c$. Then we can take $\psi = \frac{2\pi}{c}\mathcal{F}^{-1}\phi$.)

Setting $\psi_k(x_1) = (k!)^{-1}x_1^k\psi(x_1)$ for $k \leq m$, we have $\gamma_{j,1}\psi_k = (\gamma_{j,1}x_1^k)\psi(0)/k! = \delta_{jk}$ (Kronecker delta). Using ψ_ν, we let

$$K_{\nu,1}v(x) = \sum_{j=0}^\infty 2^{-ja_1\nu}\psi_\nu(2^{ja_1}x_1)v_j(x'') \quad \text{for} \quad \nu = 0,1,\ldots,m. \tag{127}$$

It holds that $K_{\nu,1}v$ is in $C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$ and $K_{\nu,1}$ is continuous $\mathcal{S}'(\mathbb{R}^{n-1}) \to \mathcal{S}'(\mathbb{R}^n)$, for the arguments for K_1 apply verbatim, as ψ_ν amounts to a special choice of ψ. Moreover, since ∂_1^ν is $\mathcal{S}'$-continuous, it applies termwisely, which cancels the factor $2^{-ja_1\nu}$ and shows that $\partial_1^\nu K_{\nu,1}v$ is in $C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^{n-1}))$; i.e. $K_{\nu,1}$ maps into the domain of $\gamma_{\nu,1}$. Incorporation of the factor $2^{-ja_1\nu}$ into the K_1-estimates yield continuity of $K_{\nu,1}\colon F_{p'',p_1}^{s-\nu a_1-\frac{a_1}{p_1},a''} \to F_{\vec{p},q}^{s,\vec{a}}$ for all $s \in \mathbb{R}$, $0 < q \leq \infty$.

Finally $K_1^{(m+1)} = \begin{pmatrix} K_{0,1} & \cdots & K_{m,1} \end{pmatrix}$ maps $\mathcal{S}'(\mathbb{R}^{n-1})^{m+1}$ into the domain of $\rho_{m+1,1}$ and fulfils $\rho_{m+1,1} \circ K_1^{(m+1)} = I$, since $\gamma_{k,1}K_{\nu,1}v = \delta_{k\nu}v$; and $K_1^{(m+1)}$ is continuous with respect to the spaces in Corollary 1.

4.4. **Proof of Theorem 1.** Note first that (ii) $\implies$ (i) is the special case $m = 1$ of Lemma 1, proved above.

For brevity we use the following notation for maximal functions invoking the Littlewood–Paley decomposition,

$$u_j^*(\vec{t};x) = \sup_{y\in\mathbb{R}^n} |\Phi_j(D)u(x-y)| \prod_{k=1,\ldots,n} (1 + |2^{ja_k}y_k|^{\frac{1}{t_k}})^{-1}. \tag{128}$$

This applies via the estimate in Proposition 5, so it is once and for all assumed that $\vec{t}$ is chosen so that $t_j < \min(p_1,\ldots p_j,q)$ for all $j \geq 1$.

4.4.1. *The basic mixed-norm estimates.* To see that (i) $\implies$ (ii), let $u \in F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n)$ with $u = \sum_{j=0}^{\infty} u_j$ for $u_j = \Phi_j(D)u$, and let $\vec{t}$ be chosen as above. Then

$$|u_j(0, x'')| \leq c_1 \frac{u_j(x_1 - y_1, x'')}{1 + |2^{ja_1}y_1|^{\frac{1}{t_1}}}\Big|_{y_1 = x_1} \leq c_1 u_j^*(\vec{t}; (x_1, x'')), \tag{129}$$

since $1 + |2^{ja_1}x_1|^{\frac{1}{t_1}} \leq 1 + 2^{\frac{a_1}{t_1}} =: c_1$ for $x_1 \in [2^{-ja_1}, 2^{(1-j)a_1}]$. Next an integration yields

$$(2^{a_1} - 1)2^{-ja_1}|u_j(0, x'')|^{p_1} \leq c_1^{p_1} \int_{2^{-ja_1}}^{2^{(1-j)a_1}} |u_j^*(\vec{t}; x)|^{p_1}\, dx_1, \tag{130}$$

so after multiplication by 2^{sjp_1} and estimation by $\sup_k 2^{sk}|u_k^*(\vec{t}; x)|$ in the integral, a summation yields

$$\sum_{j=0}^{\infty} 2^{(s-\frac{a_1}{p_1})jp_1}|u_j(0, x'')|^{p_1} \leq c_1' \int_{\mathbb{R}} (\sup_k 2^{sk}|u_k^*(\vec{t}; x)|)^{p_1}\, dx_1. \tag{131}$$

Then Proposition 5 gives, since $F_{\vec{p},q}^{s,\vec{a}} \hookrightarrow F_{\vec{p},\infty}^{s,\vec{a}}$,

$$\Big\| \Big(\sum_{j=0}^{\infty} |2^{(s-\frac{a_1}{p_1})j}u_j(0, x'')|^{p_1}\Big)^{\frac{1}{p_1}} \,\big| L_{p''}\Big\| \leq c_1'' \,\| u \,| F_{\vec{p},q}^{s,\vec{a}}\|. \tag{132}$$

Moreover, by summing only over j between $N + 1$ and $N + m$ (and by applying the first part of (62) to a sequence of functions that vanish except for those j), one gets a sharper conclusion, with $v(x) := \sup_k 2^{sk}|u_k(x_1, x'')|$ and χ_N as the characteristic function of $]0, 2^{-Na_1}]$ for brevity,

$$\Big\| \Big(\sum_{j=N+1}^{N+m} |2^{(s-\frac{a_1}{p_1})j}u_j(0, \cdot)|^{p_1}\Big)^{\frac{1}{p_1}} \,\big| L_{p''}\Big\| \leq c_1'' \,\big\| \chi_N(x_1)v(x) \,\big| L_{\vec{p}}(\mathbb{R}^n)\big\| \searrow 0. \tag{133}$$

The behaviour for $N \to \infty$ follows by majorised convergence (with $v(\cdot, x'')$ as the first majorant), since c is independent of N.

For $s = \frac{a_1}{p_1} + \sum_{k>1}(\frac{a_k}{p_k} - a_k)_+$ we set $r_k = \max(1, p_k)$ so that

$$s - \frac{a_1}{p_1} = \sum_{k>1>p_k}\left(\frac{a_k}{p_k} - a_k\right) = \sum_{k>1}\left(\frac{a_k}{p_k} - \frac{a_k}{r_k}\right) =: \sigma. \tag{134}$$

We continue in the same way for $\sigma > 0$ and for $\sigma = 0$. The vector valued Nikol'skij inequality on $\mathbb{R}^{n-1}$, cf. Theorem 6, then implies

$$\Big\| \sum_{j=N+1}^{N+m} u_j(0, \cdot) \,\big| L_{r''}\Big\| \leq \Big\| \sum_{j=N+1}^{N+m} |u_j(0, \cdot)| \,\big| L_{r''}\Big\| \leq c_{r''} \Big\| \Big(\sum_{j=N+1}^{N+m} |2^{j\sigma}u_j(0, \cdot)|^{p_1}\Big)^{\frac{1}{p_1}} \,\big| L_{p''}\Big\| \tag{135}$$

$$\leq c_{r''}c_1'' \,\big\| \chi_N(x_1)v(x) \,\big| L_{\vec{p}}\big\|.$$

Consequently $\sum u_j(0, x'')$ converges in the Banach space $L_{r''}(\mathbb{R}^{n-1}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{n-1})$ in all the borderline cases. (For $p_1 \leq 1$ this can also be seen more directly, using that $\ell_{p_1} \hookrightarrow \ell_1$

instead of the Nikol′skij inequality.) By similar inequalities now with summation over $j \in \mathbb{N}_0$, it is in both cases seen from (132) that $\gamma_{0,1}$ is bounded $F_{\vec{p},q}^{s,\vec{a}} \to L_{r''}$.

The generic cases given by the sharp inequality $s > \frac{a_1}{p_1} + \sum_{k>1}(\frac{a_k}{p_k} - a_k)_+$ also give the desired $\mathcal{D}'$-continuity, as seen by restricting $\gamma_{0,1}$ to subspaces with higher values of s.

4.4.2. *Continuity in x_1.* To show that $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}, L_{r''}(\mathbb{R}^{n-1}))$ it is, by a simple embedding lowering s, enough to treat the case $s = \frac{a_1}{p_1} + \sigma$; cf (134). We may assume $q < \infty$, by passing to a larger space by means of a Sobolev embedding increasing a component of p''.

To evaluate at $x_1 = z$ for an arbitrary z one can extend the above estimates. Indeed, letting x_1 run in $[z + 2^{-ja_1}, z + 2^{(1-j)a_1}]$, and replacing y_1 by $y_1 - z$, one finds (130) with an integral over this interval (with the same constant).

This procedure gives the strengthened estimate

$$\sup_z \big\| \sum_{j=0}^{\infty} u_j(z,\cdot) \,|L_{r''}\big\| \le c_{r''} \sup_z \big\| \big(\sum_{j=0}^{\infty} |2^{j\sigma} u_j(z,\cdot)|^{p_1}\big)^{\frac{1}{p_1}} \,\big|L_{p''}\big\| \le c_{r''} c_1'' \, \| u \,|F_{\vec{p},q}^{s,\vec{a}}\|. \quad (136)$$

Redefining u_j to 0 for $j \notin [N+1, N+m]$, as before, this gives convergence of the series for every z, hence a function $z \mapsto f(z) = \sum u_j(z,\cdot)$, and (136) shows it is bounded $\mathbb{R} \to L_{r''}$.

The continuity of f follows because translations $\tau_h u \to u$ in $F_{\vec{p},q}^{s,\vec{a}}$ for $h \to 0$, since q is finite; cf. Proposition 3. Indeed, inserting $\tau_h u - u$ in (136),

$$\| f(z-h) - f(z) \,|L_{r''}\| \le c \, \| \tau_h u - u \,|F_{\vec{p},q}^{s,\vec{a}}\| \searrow 0. \quad (137)$$

To show that $\Lambda_f = u$, note first that by (136) there is an estimate uniformly over a compact interval containing every z appearing in $\operatorname{supp}\varphi$,

$$|\langle \sum_{j=0}^{N} u_j(z,\cdot), \, \varphi(z,\cdot) \rangle_{\mathbb{R}^{n-1}}| \le c \sup_z \| \varphi \,|(L_{r''})^*\| \, \| u \,|F_{\vec{p},q}^{s,\vec{a}}\|. \quad (138)$$

With this as a majorisation,

$$\langle \Lambda_f, \overline{\varphi} \rangle = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \langle u_j(z,\cdot), \overline{\varphi(z,\cdot)} \rangle_{\mathbb{R}^{n-1}} \, dz = \sum_{j=0}^{\infty} \iint u_j \overline{\varphi} \, dx'' dz = \sum_{j=0}^{\infty} \langle u, \overline{\varphi_j} \rangle = \langle u, \varphi \rangle. \tag{139}$$

Thence $u = \Lambda_f \in C_{\mathrm{b}}(\mathbb{R}, L_{r''}(\mathbb{R}^{n-1}))$ as desired.

4.5. **Boundedness in the F-scale (Theorem 2).** Departing from the proof of Theorem 1, note that in the subspaces where $s > \frac{a_1}{p_1} + \sum_{k>1}(\frac{a_k}{\min(1,p_2,\dots,p_k,q)} - a_k)$, the dyadic corona criterion applies, because $u_j(0,x'')$ by the Paley–Wiener–Schwartz Theorem has its spectrum where $|\xi''|_{a''} \le 2^{j+1}$; cf. [Joh00, Rem. 3.4]. Therefore (132) implies

$$\big\| \sum u_j(0,x'') \,\big|F_{p'',p_1}^{s-\frac{a_1}{p_1},a''}\big\| c \le \| u \,|F_{\vec{p},q}^{s,\vec{a}}\|. \quad (140)$$

The surjectivity follows from the already proved Theorem 5, in view of the formula $\gamma_{0,1} \circ K_1 v = v$, proved for all $v \in \mathcal{S}'(\mathbb{R}^{n-1})$, and the mapping properties of K_1.

4.6. **Proof of Theorems 3, 4.** The implications of (ii) were accounted for directly after the theorems by means of Lemma 1.

For the proof of (i) $\implies$ (ii) the argument from Theorem 1 applies, mutatis mutandis. Indeed, as in (129) one finds $|u_j(x', z)| \leq c'_1 u_j^*(\vec{t}; (x', x_n))$ for a constant c'_1 independent of z; then one can take the $L_{p'}(\mathbb{R}^{n-1})$-norm on both sides and proceed with the argument for (130)–(132). Setting $r_k = \max(1, p_k)$ for $k < n$ and $\sigma = \sum_{k<n}(\frac{a_k}{p_k} - \frac{a_k}{r_k})$, this gives for $s = \frac{a_n}{p_n} + \sigma$ and $p_n \leq 1$, when the Nikol'skij inequality is applied for each $j \geq 0$,

$$\sup_z (\sum \| u_j(\cdot, z) \, | L_{r'} \|^{p_n})^{\frac{1}{p_n}} \leq c' \sup_z (\sum 2^{j\sigma p_n} \| u_j(\cdot, z) \, | L_{p'} \|^{p_n})^{\frac{1}{p_n}}$$
$$\leq c'' \| u \, | F_{\vec{p},q}^{s,\vec{a}} \|. \tag{141}$$

Now $\| \cdot \, | \ell_1 \| \leq \| \cdot \, | \ell_{p_n} \|$ gives a finite norm series, hence convergence of $\sum_{j=0}^\infty u_j(\cdot, z)$ to some $f(z)$ in the Banach space $C_b(\mathbb{R}, L_{r'}(\mathbb{R}^{n-1}))$. Clearly $\sup_z \| f(z) \, | L_{r'} \| \leq c'' \| u \, | F_{\vec{p},q}^{s,\vec{a}} \|$.

As for $\gamma_{0,1}$ there is an identification $\Lambda_f = u$, which yields $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}, L_{r'}(\mathbb{R}^{n-1}))$. In particular the working definition of $\gamma_{0,n} u$ is defined by evaluation at $z = 0$.

In cases with $s = \varepsilon + \frac{a_n}{p_n} + \sigma$ for $\varepsilon > 0$, the inequality (141) is modified by having on its left-hand side a norm in $\ell_{p_n}^\varepsilon$. But since $\| \cdot \, | \ell_1 \| \leq \| \cdot \, | \ell_{p_n}^\varepsilon \|$ whenever $0 < p_n < \infty$, the inclusion into $C_b(\mathbb{R}, L_{r'})$ is seen in the same way. Altogether (i) $\implies$ (ii) holds in all cases.

When (14) holds, the dyadic ball criterion for Besov spaces, cf. Lemma 7, applies yielding continuity $F_{\vec{p},q}^{s,\vec{a}}(\mathbb{R}^n) \to B_{p',p_n}^{s-\frac{a_n}{p_n}, a'}(\mathbb{R}^{n-1})$; here the surjectivity is a consequence of the formula $\gamma_{0,n} \circ K_n = I$. This completes the proof of Theorem 3.

5. Final Remarks

To conclude, we note that also if we specialise to $\vec{a} = (1, \ldots, 1)$ and $\vec{p} = (p, \ldots, p)$, our results on the right-inverses K_j ($j = 1$ and $j = n$) supplement those previously available, say in [Tri83, 2.7.2], since the K_j are shown above to be well-defined continuous maps $S'(\mathbb{R}^{n-1}) \to S'(\mathbb{R}^n)$. Moreover, we show that all of $S'(\mathbb{R}^{n-1})$ is mapped into the domain of $\gamma_{0,j}$, i.e. into $C(\mathbb{R}, D'(\mathbb{R}^{n-1}))$. This makes sense because we consider the distributional trace.

We also estimate the norms in $C_b(\mathbb{R}, L_{r''})$ etc. directly in terms of the $F_{\vec{p},q}^{s,\vec{a}}$-norms.

Already Berkolaiko gave specific counterexamples for the trace problem of mixed-norm spaces with $1 < \vec{p} < \infty$. Our counterexamples show the necessity of raising the borderlines upwards when $0 < p_k < 1$ holds for one of the tangential variables x_k.

It should also be mentioned that we have a fairly complete theory, carrying over most of the well-known results for isotropic spaces to the quasi-homogeneous mixed-norm spaces $F_{\vec{p},q}^{s,\vec{a}}$. In particular, for fixed $\vec{p}$, we cover all s running in a maximal open half-line. (However, traces of $B_{\vec{p},q}^{s,\vec{a}}$ were not described, although we do not envisage any difficulties in doing so with the methods of the present paper.)

The works on parabolic problems with traces of mixed-norm spaces [DHP, Wei05] have for the lateral boundary data used spaces that are intersections of $F_{p,q}^{2-1/q}(\,]0, T[\,; L_q(\partial\Omega))$

and $L_p(\,]0, T[\,; W_q^2(\Omega))$; also vector-valued solutions have been treated. We have left both questions (identifications of $F_{\vec{p},q}^{s,\vec{a}}$ spaces with intersections and vector-valued versions) for the future.

References

[Ama95]   H. Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995, Abstract linear theory.

[Bag75]   R. J. Bagby, *An extended inequality for the maximal function*, Proc. Amer. Math. Soc. **48** (1975), 419–422.

[Ber84]   M. Z. Berkolaĭko, *Traces of generalized spaces of differentiable functions with mixed norm*, Dokl. Akad. Nauk SSSR **277** (1984), no. 2, 270–274, English translation: Soviet Math. Dokl. 30 (1984), no. 1, 60–64.

[Ber85]   _____, *Theorems on traces on coordinate subspaces for some spaces of differentiable functions with anisotropic mixed norm*, Dokl. Akad. Nauk SSSR **282** (1985), no. 5, 1042–1046, English translation: Soviet Math. Dokl. 31 (1985), no. 3, 540–544.

[Ber87a] _____, *Traces of functions in generalized Sobolev spaces with a mixed norm on an arbitrary coordinate subspace. I*, Trudy Inst. Mat. (Novosibirsk) **7** (1987), no. Issled. Geom. Mat. Anal., 30–44, 199.

[Ber87b] _____, *Traces of functions in generalized Sobolev spaces with a mixed norm on an arbitrary coordinate subspace. II*, Trudy Inst. Mat. (Novosibirsk) **9** (1987), no. Issled. Geom. "v tselom" i Mat. Anal., 34–41, 206.

[BIN79]   O. V. Besov, V. P. Il'in, and S. M. Nikol'skij, *Integral representations of functions and imbedding theorems. Vol. I+II*, V. H. Winston & Sons, Washington, D.C., 1978, 1979, Transalated from the Russian. Scripta Series in Mathematics, Edited by Mitchell H. Taibleson.

[BIN96]   _____, *Integralnye predstavleniya funktsii i teoremy vlozheniya*, second ed., Fizmatlit "Nauka", Moscow, 1996.

[BP61]   A. Benedek and R. Panzone, *The space L^p, with mixed norm*, Duke Math. J. **28** (1961), 301–324.

[Bug71]   Ya. S. Bugrov, *Function spaces with mixed norm*, Izv. Akad. Nauk. SSSR Ser. Mat. **35** (1971), 1137–1158, Engl. transl. Math USSR **5** (1971), 1145–1167.

[DHP]   R. Denk, M. Hieber, and J. Prüess, *Optimal L_p-L_q-regularity for parabolic problems with inhomogeneous boundary data*, Preprint, University of Konstanz, Germany,No. 205, 2005.

[FJS00]   W. Farkas, J. Johnsen, and W. Sickel, *Traces of anisotropic Besov-Lizorkin-Triebel spaces—a complete treatment of the borderline cases*, Math. Bohem. **125** (2000), 1–37.

[HN]   L. Hedberg and Y. Netrusov, *An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation*, To appear in Mem. Am. Math. Soc.

[Joh96]   J. Johnsen, *Elliptic boundary problems and the Boutet de Monvel calculus in Besov and Triebel-Lizorkin spaces*, Math. Scand. **79** (1996), no. 1, 25–85.

[Joh00]   _____, *Traces of Besov spaces revisited*, Z. Anal. Anwendungen **19** (2000), no. 3, 763–779.

[JS]   J. Johnsen and W. Sickel, *A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin–Triebel spaces with mixed norms*, Preprint R-2005-27, Aalborg University, 2005 (to appear in Journal of function spaces and their applications).

[Kré67]   P. Krée, *Propriétés de continuité dans L^p de certains noyaux*, Boll. Un. Mat. Ital. (3) **22** (1967), 330–344.

[Liz70]   P. I. Lizorkin, *Multipliers of fourier integrals and bounds of convolutions in spaces with mixed norms. applications*, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 225–255, Engl. transl. Math. USSR Izv. **4** (1970).

[Mar96]   J. Marschall, *Nonregular pseudo-differential operators*, Z. Anal. Anwendungen (1996), no. 1, 109–148.

[MC91]   Y. Meyer and R. R. Coifman, *Ondelettes et opérateurs. III*, Hermann, Paris, 1991, English version, Cambridge Univ. Press, 1997.

[Nik75]   S. M. Nikol′skij, *Approximation of functions of several variables and imbedding theorems*, Springer-Verlag, New York, 1975.

[Sch84]   H.-J. Schmeisser, *Maximal inequalities and Fourier multipliers for spaces with mixed quasi-norms. Applications*, Z. Anal. Anwendungen **3** (1984), 153–166.

[ST87]   H.-J. Schmeisser and H. Triebel, *Topics in Fourier analysis and function spaces*, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987, (Also John Wiley, 1987).

[Tri83]   H. Triebel, *Theory of function spaces*, vol. 78, Birkhäuser Verlag, Basel, 1983.

[Wei98]   P. Weidemaier, *Existence results in L_p-L_q spaces for second order parabolic equations with inhomogeneous Dirichlet boundary conditions*, Progress in partial differential equations, Vol. 2 (Pont-à-Mousson, 1997), Pitman Res. Notes Math. Ser., vol. 384, Longman, Harlow, 1998, pp. 189–200.

[Wei02]   _____, *Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm*, Electron. Res. Announc. Amer. Math. Soc. **8** (2002), 47–51 (electronic).

[Wei05]   _____, *Lizorkin-Triebel spaces of vector-valued functions and sharp trace theory for functions in Sobolev spaces with a mixed L_p-norm in parabolic problems*, Mat. Sb. **196** (2005), no. 6, 3–16.

[Yam86a] M. Yamazaki, *The L^p-boundedness of pseudodifferential operators with estimates of parabolic type and product type*, J. Math. Soc. Japan **38** (1986), 199–225.

[Yam86b] _____, *A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), no. 1, 131–174.

Department of Mathematics, Aalborg University, Fredrik Bajers Vej 7G, DK–9220 Aalborg East, Denmark
E-mail address: jjohnsen@math.aau.dk

Institute of Mathematics, Friedrich-Schiller-University Jena, Ernst-Abbe-Platz 1–2, D–07743 Jena, Germany
E-mail address: sickel@minet.uni-jena.de