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Robust Structured Control Design
via LMI Optimization *

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Abstract: This paper presents a new procedure for discrete-time robust structured control design. Parameter-dependent nonconvex conditions for stabilizable and induced $\mathcal{L}_2$-norm performance controllers are solved by an iterative linear matrix inequalities (LMI) optimization. A wide class of controller structures including decentralized of any order, fixed-order dynamic output feedback, static output feedback can be designed robust to polytopic uncertainties. Stability is proven by a parameter-dependent Lyapunov function. Numerical examples on robust stability margins shows that the proposed procedure can obtain less conservative results than traditional stability criteria.

Keywords: Robust control, discrete-time systems, structure systems, iterative methods.

1. INTRODUCTION

The most important topics on the field of robust and structured control theories are related to bilinear matrix inequalities (BMI). Some BMI problems can be readily recast as linear matrix inequalities (LMI) by the use of standand techniques (e.g. congruence transformation, linearizing change of variables, projections) and thus are convex problems. For many others, reformulations are unknown and maybe will never be addressable by the LMI framework [Mesbahi et al. (2000) and references therein].

The numerical exact solution of a BMI for arbitrary problem size is still beyond the state of the art in nonconvex programming algorithms. Global optimization algorithms such as branch and bound have been proposed [Goh et al. (1995), VanAntwerp and Braatz (2000)] but they do not have polynomial time worst case performance bounds and are effective only for small problems. On the other hand, local algorithms consist of iteratively execute efficient polynomial-time algorithms and thus can provide solutions for medium to large size problems in a reasonable computational time, with the drawback that its solution is suboptimal. They have been applied to structured control and robust control synthesis problems [Ghaoui et al. (1997), Grigoriadis and Beran (1999), Ghaoui and Balakrishnan (1994), Dussy (2000)]. Most of the local algorithms try to compute a stabilizing controller or a controller that satisfies a number of specifications, rather than finding one with optimal performance. Their convergence properties is also of concern. Some algorithms can even fail to converge, others converge when the starting point is in a neighborhood of a feasible solution, while some guarantees at least a stationary point.

Local algorithms initially relied on a single Lyapunov variable for proving stability and thus could lead to con-

servative results, especially for robust and multiobjective control problems. Initiatives followed to reduce conservativeness by extending them to synthesis conditions based on parameter-dependent Lyapunov functions [Feron et al. (1996), Coutinho et al. (2002)].

Another local algorithm for structured linear control design approximates the nonconvex problem by a convex one with the addition of a convexifying function [de Oliveira et al. (2000)]. It is suitable to optimize control performance and has interesting properties like convergence to a local minima, easy implementation, and encompass a large class of structured control problems. Application of this algorithm to robust control would be of interest, if it does not relied on a single Lyapunov variable.

This paper presents a new local algorithm for robust control design. Parameter-dependent nonconvex conditions for synthesis of stabilizable and induced $\mathcal{L}_2$-norm performance controllers are solved by an iterative LMI optimization. A wide class of controller structures including decentralized of any order, fixed-order dynamic output feedback, static output feedback can be robustly designed to polytopic uncertainties. Moreover, a parameter-dependent Lyapunov function proves system stability. The proposed algorithm can be seen as an extension to uncertain parameter-dependent systems of de Oliveira et al. (2000) and other two subsequent works [Han and Skelton (2003a), Han and Skelton (2003b)]. A numerical example on robust stability margins shows that the algorithm can obtain less conservative results than traditional stability criteria. Another example deals with the compromise between robust margins and performance for a decentralized feedback.

The paper is organized as follows. Section 2 and 3 presents the nonconvex inequality conditions for analysis and synthesis, respectively. The iterative LMI algorithm to solve the nonconvex problem is presented on Section 4. The numerical examples are shown on Section 5.

2. ANALYSIS

The class of linear discrete-time systems in state-space form given by

$$x(k+1) = \mathcal{A}x(k) \tag{1}$$

is considered for analysis purposes, where the state vector $x \in \mathbb{R}^n$. For an uncertain system, $\mathcal{A} = A(\alpha)$ belongs to the following bounded convex set

$$\left\{ A(\alpha) : A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\}. \tag{2}$$

The Lyapunov stability theory states that, for a LTI system, a quadratic and constant Lyapunov function is a necessary and sufficient condition and does not introduce conservativeness. In the case of uncertain systems this choice usually leads to very consevative results. A parameter-dependant Lyapunov variable $\mathcal{P} = P(\alpha)$ contained in a convex set

$$\left\{ P(\alpha) : P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\} \tag{3}$$

reduces conservativeness by allowing the variation of $\mathcal{P}$ with respect to α.

The next lemma relates Lyapunov stability theory with uncertain systems.

Lemma 1. (Robust Stability). System (1) is robustly stable in the uncertainty domain (2) if there exists $P(\alpha) = P(\alpha)' > 0$, such that,

i.
$$\begin{bmatrix} P(\alpha) & A(\alpha)P(\alpha) \\ \star & P(\alpha) \end{bmatrix} > 0. \tag{4}$$

ii.
$$\begin{bmatrix} P(\alpha) & A(\alpha) \\ \star & P(\alpha)^{-1} \end{bmatrix} > 0. \tag{5}$$

Proof. Condition (4) directly results from the application of Lyapunov theory to (1)-(3). Multiplying (4) to the right by $T := \operatorname{diag}(I, P(\alpha)^{-1})$ and to the left by $T' = T$ results in (5). $\square$

The main advantage of formulation (5) is the decoupling of the state matrix $A(\alpha)$ from any matrix variable. The major drawback is the nonconvex entry $P(\alpha)^{-1}$. The focus is to pursue an alternative parametrization of this nonconvex variable and derive an alternative stability condition. The relation between the inverse of a matrix and a polynomial matrix function, given in the next lemma, will be useful for this purpose.

Lemma 2. (Convexifying Inequality). For $P(\alpha) = P(\alpha)' > 0$ and $G(\alpha)$ the matrix inequality

$$P(\alpha)^{-1} \geq -G(\alpha)'P(\alpha)G(\alpha) + G(\alpha)' + G(\alpha) \tag{6}$$

always hold.

With the previous lemma at hand, the theorem in the sequel states an equivalence between the Lyapunov stability condition and a polynomial matrix inequality.

Theorem 3. The following conditions are equivalent:

i. There exists $P_i = P_i' > 0$ such that (5) is satisfied.
ii. There exist $P_i = P_i' > 0$ and $G(\alpha)$ such that

$$\begin{bmatrix} P(\alpha) & A(\alpha) \\ \star & -G(\alpha)'P(\alpha)G(\alpha) + G(\alpha)' + G(\alpha) \end{bmatrix} > 0. \tag{7}$$

Proof. A direct substitution of the $P(\alpha)^{-1}$ entry located at the (2,2) position of (5) by the right hand side of (6) results in (7). The equivalence occurs when $G(\alpha) = P(\alpha)^{-1}$. $\square$

In this paper, $G(\alpha)$ is assumed to belong to a bounded convex set,

$$\left\{ G(\alpha) : G(\alpha) = \sum_{i=1}^{N} \alpha_i G_i, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\}. \tag{8}$$

All before mentioned matrix functions are infinite dimensional in α. Conditions in a finite dimensional polytopic convex set, taken in the set of vertices of the parameter space, are desired. Notice that all entries of (7), except the convexifying inequality, are affine in α and thus could be checked at the vertices of the parameter space.

A multi-convexity property for matrix functions polynomially dependent on the parameters [Apkarian and Tuan (2000)] come as a support for attaining a finite-dimensional condition.

Lemma 4. (Multi-convexity). [Apkarian and Tuan (2000)] Consider a polynomially α-dependent LMI of the form,

$$\mathcal{F}(\alpha, z) := \sum_{v \in J} \alpha^{[v]} M_v(z) > 0,$$

where M_v denote symmetric matrix-valued linear functions of the decision variable z. The notation $[v]$ is the vector of partial degrees $[v] = [v_1, \ldots, v_N]$ associated with the lexicographically ordered term,

$$\alpha^{[v]} = \alpha_1^{v_1} \alpha_2^{v_2} \ldots \alpha_N^{[v_N]},$$

with the convention $\alpha^{[0]} = 1$. J is a set of N-tuples of partial degrees describing the polynomial expansion. The symbols d_k and d designate the partial and total degrees in the matrix polynomial expansion. Then the LMI condition over the hyperectangle H,

$$\mathcal{F}(\alpha, z) > 0, \quad H := \underline{\alpha_i} \leq \alpha_i \leq \overline{\alpha_i}, \quad i = 1, 2, \ldots, N,$$

hold for some z, whenever the finite set of LMI,

$$\mathcal{F}(\alpha, z) > 0, \quad \alpha \in \operatorname{Vert} H, \tag{9a}$$

$$(-1)^m \frac{\partial^{2m}}{\partial \alpha_{l_1}^2 \ldots \partial \alpha_{l_m}^2} \mathcal{F}(\alpha, z) \geq 0, \quad \forall \alpha \in \operatorname{Vert} H, \tag{9b}$$

where $1 \leq l_1 \leq l_2 \leq \ldots \leq l_m \leq N$, $1 \leq m \leq d/2$, $2\sharp\{l_j = k : j \in \{1, \ldots, m\}\} \leq d_k$, $k = 1, 2, \ldots, N$.

Theorem 5. (Finite-Dimensional Analysis). System (1) is robustly stable in the uncertain domain (2) if there exists $P_i = P_i' > 0$ and G_i such that,

$$\begin{bmatrix} P_i & A_i \\ \star & -G_i'P_iG_i + G_i' + G_{i,} \end{bmatrix} > 0, \quad i = 1, \ldots, N, \tag{10a}$$

$$3G_i'P_iG_i + G_i'P_iG_j + G_i'P_jG_i + G_j'P_iG_i \geq 0, \\ i \neq j = 1, 2, \ldots, N. \tag{10b}$$

Proof. Inequality (10a) is a direct result of (9a) applied to (7), (8). It is of interest to note that the right hand side of the convexifying inequality,

$$-\left(\sum_{i,j,k=1}^{N} \alpha_i\, \alpha_j\, \alpha_k\, G_i'P_jG_k\right) + \sum_{i=1}^{N}\alpha_iG_i' + \sum_{i=1}^{N}\alpha_iG_i,$$

is cubically α-dependent ($d = 3$). The multi-convexity condition (9b) results in,

$$\Phi_{ii} = 6\,G_i'P_iG_i,$$

$$\Phi_{ij} = 6\,G_i'P_iG_i + 2\left(G_i'P_iG_j + G_i'P_jG_i + G_j'P_iG_i\right),$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \Phi_{ii} \end{bmatrix} \geq 0, \quad \begin{bmatrix} 0 & 0 \\ 0 & \Phi_{ij} \end{bmatrix} \geq 0, \quad i \neq j = 1,2,\ldots,N,$$

and is fully respected by (10b). $\square$

3. SYNTHESIS

An open-loop, discrete-time uncertain system with state-space realization of the form

$$\begin{aligned} x(k+1) &= A(\alpha)x(k) + B_w(\alpha)w(k) + B_u(\alpha)u(k) \\ z(k) &= C_z(\alpha)x(k) + D_{zw}(\alpha)w(k) + D_{zu}(\alpha)u(k) \\ y(k) &= C_y(\alpha)x(k) + D_{yw}(\alpha)w(k) \end{aligned} \quad (11)$$

is considered for synthesis purposes, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{n_w}$ is the vector of exogenous perturbation, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $z(k) \in \mathbb{R}^{n_z}$ is the controlled output, and $y(k) \in \mathbb{R}^{n_y}$ is the measured output. The system matrices belong to a polytopic convex set,

$$\left\{ \begin{bmatrix} A(\alpha) & B_w(\alpha) & B_u(\alpha) \\ C_z(\alpha) & D_{zw}(\alpha) & D_{zu}(\alpha) \\ C_y(\alpha) & D_{yw}(\alpha) & 0 \end{bmatrix} : \right.$$

$$\begin{bmatrix} A(\alpha) & B_w(\alpha) & B_u(\alpha) \\ C_z(\alpha) & D_{zw}(\alpha) & D_{zu}(\alpha) \\ C_y(\alpha) & D_{yw}(\alpha) & 0 \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_{w,i} & B_{u,i} \\ C_{z,i} & D_{zw,i} & D_{zu,i} \\ C_{y,i} & D_{yw,i} & 0 \end{bmatrix},$$

$$\left. \sum_{i=1}^{N}\alpha_i = 1,\ \alpha_i \geq 0 \right\}$$

$$(12)$$

Also consider a controller of the form

$$\begin{aligned} x_c(k+1) &= A_c\, x_c(k) + B_c\, y(k) \\ u(k) &= C_c\, x_c(k) + D_c\, y(k) \end{aligned} \quad (13)$$

where $x_c(k) \in \mathbb{R}^{n_c}$. Representing the controller matrices in a compact way,

$$K \overset{\text{def}}{=} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}, \quad (14)$$

the closed-loop system interconnection of system (11)-(12) and controller (13) leads to the following closed-loop system,

$$\begin{aligned} x(k+1) &= \mathcal{A}(\alpha,K)x_{cl}(k) + \mathcal{B}(\alpha,K)w(k) \\ z(k) &= \mathcal{C}(\alpha,K)x_{cl}(k) + \mathcal{D}(\alpha,K)w(k) \end{aligned} \quad (15)$$

where the closed-loop system matrices are [Skelton et al. (1999)],

$$\begin{aligned} \mathcal{A}(\alpha,K) &= \mathbf{A}(\alpha) + \mathbf{B}(\alpha)\,K\,\mathbf{M}(\alpha) \\ \mathcal{B}(\alpha,K) &= \mathbf{D}(\alpha) + \mathbf{B}(\alpha)\,K\,\mathbf{E}(\alpha) \\ \mathcal{C}(\alpha,K) &= \mathbf{C}(\alpha) + \mathbf{H}(\alpha)\,K\,\mathbf{M}(\alpha) \\ \mathcal{D}(\alpha,K) &= \mathbf{F}(\alpha) + \mathbf{H}(\alpha)\,K\,\mathbf{E}(\alpha) \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\alpha) &= \begin{bmatrix} A(\alpha) & 0 \\ 0 & 0 \end{bmatrix} & \mathbf{B}(\alpha) &= \begin{bmatrix} B_u(\alpha) & 0 \\ 0 & I \end{bmatrix} \\ \mathbf{M}(\alpha) &= \begin{bmatrix} C_y(\alpha) & 0 \\ 0 & I \end{bmatrix} & \mathbf{E}(\alpha) &= \begin{bmatrix} D_{yw}(\alpha) \\ 0 \end{bmatrix} \\ \mathbf{H}(\alpha) &= [D_{zu}(\alpha)\ 0] & \mathbf{D}(\alpha) &= \begin{bmatrix} B_w(\alpha) \\ 0 \end{bmatrix} \\ \mathbf{C}(\alpha) &= [C_z(\alpha)\ 0] & \mathbf{F}(\alpha) &= D_{zw}(\alpha) \end{aligned} \quad (16)$$

The problem becomes a static output feedback (SOF) if $n_c = 0$. The static state feedback (SSF) is a particular case of SOF in which $C_y(\alpha) = I$. The full-order dynamic output feedback arises when $n = n_c$. When $n_c < n$, the resulting structure is the fixed-order dynamic output feedback. Decentralized controllers of arbitrary order occurs when K has a block diagonal structure $A_c = \text{diag}(A_{ci})$, ..., $D_c = \text{diag}(D_{ci})$.

For ease of exposure, a quadratic α-dependence of the closed-loop system matrices is avoided by assuming B_u and D_{zu}, or C_y and D_{yw}, parameter-independent. If they are parameter-dependent, inequality (10b) will differ to satisfy multi-convexity arguments of Lemma 4.

3.1 Stabilizing Controllers

In the design of stabilizing controllers, the exogenous perturbations $w(k)$ and the controlled output $z(k)$ are not considered, thus $B_w, D_{zw}, D_{yw}, C_z, D_{zu} = 0$. The condition for finding a stabilizing robust controller,

$$\begin{bmatrix} P(\alpha) & \mathcal{A}(\alpha,K) \\ \mathcal{A}(\alpha,K)' & -G(\alpha)'P(\alpha)G(\alpha) + G(\alpha)' + G(\alpha) \end{bmatrix} > 0, \quad (17)$$

is obtained by similar arguments of the analysis section. Notice that the closed-loop state matrix $\mathcal{A}(\alpha,K)$ appears only in the off-diagonal terms of the inequality (17). Also recall that it is assumed affine in α and K. Similarly to the analysis case, the multi-convexity property is the required tool to reduce the infinite dimensional inequality (17) to a finite-dimensional one.

Theorem 6. (Finite-Dimensional Stabilization). System 15 is robustly stabilizable in the uncertain domain (12) if there exists P_i, K and G_i such that,

$$\begin{bmatrix} P_i & \mathcal{A}_i(K) \\ \mathcal{A}_i(K)' & -G_i'P_iG_i + G_i' + G_i \end{bmatrix} > 0, \quad i = 1, \ldots, N, \quad (18a)$$

$$3G_i'P_iG_i + G_i'P_iG_j + G_i'P_jG_i + G_j'P_iG_i \geq 0, \quad i \neq j = 1,2,\ldots,N. \quad (18b)$$

Proof. The proof follows the same lines of Theorem 5 and will be ommited for brevity. $\square$

3.2 Induced $\mathcal{L}_2$-norm performance controllers

Controller synthesis may also consider performance level specifications. Defining $T_{zw}(\alpha,K)$ as the input-output operator that provides the forced response of system (15) to an input signal $w(k) \in \mathcal{L}_2$ for zero initial conditions, the next lemma states a condition for guaranteed upper-bound on the induced $\mathcal{L}_2$-norm of $T_{zw}(\alpha,K)$.

Lemma 7. ($\mathcal{L}_2$-norm performance). $\|T_{zw}(\alpha, K)\|_{i,2}^2 < \gamma$ holds in the uncertain domain (12) if the following equivalent conditions are satisfied.

i. There exists $P(\alpha) = P(\alpha)' > 0$ and K such that,

$$\begin{bmatrix} P(\alpha) & \mathcal{A}(\alpha, K)P(\alpha) & \mathcal{B}(\alpha, K) & 0 \\ \star & P(\alpha) & 0 & P(\alpha)\mathcal{C}(\alpha, K)' \\ \star & \star & I & \mathcal{D}(\alpha, K)' \\ \star & \star & \star & \gamma I \end{bmatrix} > 0$$

(19)

ii. There exists $P_i = P_i' > 0$, $G(\alpha)$ and K such that,

$$\begin{bmatrix} P(\alpha) & \mathcal{A}(\alpha, K) & \mathcal{B}(\alpha, K) & 0 \\ \star & \begin{matrix} -G'(\alpha)P(\alpha)G(\alpha) \\ +G(\alpha)' + G(\alpha) \end{matrix} & 0 & \mathcal{C}(\alpha, K)' \\ \star & \star & I & \mathcal{D}(\alpha, K)' \\ \star & \star & \star & \gamma I \end{bmatrix} > 0$$

(20)

Proof. The proof follows similar arguments of the analysis section and will be ommited for brevity. □

A finite dimensional condition can also be derived for (20).

Theorem 8. (Finite-Dimensional Synthesis). System (15) has performance level γ in the uncertain domain (12) if there exists $P_i = P_i' > 0$, G_i and K such that,

$$\begin{bmatrix} P_i & \mathcal{A}_i(\alpha, K) & \mathcal{B}_i(\alpha, K) & 0 \\ \star & -G_i' P_i G_i + G' + G & 0 & \mathcal{C}_i(\alpha, K)' \\ \star & \star & I & \mathcal{D}_i(\alpha, K)' \\ \star & \star & \star & \gamma I \end{bmatrix} > 0,$$
$$i = 1, \ldots, N,$$

(21a)

$$3G_i' P_i G_i + G_i' P_i G_j + G_i' P_j G_i + G_j' P_i G_i \geq 0,$$
$$i \neq j = 1, \ldots, N.$$

(21b)

Proof. The proof follows the same arguments of Theorem 5 and is omitted for brevity.

4. ITERATIVE ALGORITHM

If P_i and G_i are considered variables, the constraints (10), (18), (21) are non-convex functions, Therefore, optimization problems wih such constraints cannot be solved directly by semidefinite programming. An LMI iterative algorithm is proposed to overcome such fact. At each iteraton, G_i is kept in a constant value. The solution of the optimization problem in the current iteration is utilized to update the value of G_i for the next iteration.

The right hand side of (6) has a particular interpretation if we restrict the set of admissible values of $G(\alpha)$. It can be seen as a parametrization of multivariate Taylor expansions of the function $f(P(\alpha)) = P(\alpha)^{-1}$. A multivariate linearization of $f(P(\alpha))$ at a particular point $P^*(\alpha)$ can be obtained if $G(\alpha) = P^*(\alpha)^{-1}$. Notice that (6) turns into an equality when $G(\alpha) = P^*(\alpha)^{-1}$ and $P(\alpha) = P^*(\alpha)$. This is taken into account to decide for an update rule,

$$G_i^{\{l+1\}} = \left(P_i^{\{l\}}\right)^{-1}$$

(22)

where $^{\{\cdot\}}$ is the iteration index and l is the current iteration number. The iterative algorithm is conceptually described by Algorithm 1.

Algorithm 1. (Conceptual). Set initials $G_i^{\{0\}}$, $l = 0$ and start to iterate:

(1) Find $P_i^{\{l\}} > 0$ and $K^{\{l\}}$ that solve the LMI problem (10), or (18), or (21) with constant $G_i^{\{l\}}$.
(2) If a stopping criterion is satisfied, exit. Else, compute $G_i^{\{l+1\}} = \left(P_i^{\{l\}}\right)^{-1}$. Update possible terms of interest. Increment $l = l + 1$ and go to step 1.

The general description of Algorithm 1 can be particularized to deal with different analysis and synthesis problems. Algorithm 2 compute the stability bounds with respect to an uncertain parameter α_1. It resembles a bissection method combined with LMI feasibility problems.

Algorithm 2. (Stability Bounds). Choose a tolerance ϵ, initials $G_i^{\{0\}}$, $\alpha_1^{\{0\}} = 0$, set $l = 0$ and start to iterate:

(1) If $l \neq 0$, update $\alpha_1^{\{l\}} = \alpha_1^{\{l-1\}} + \delta\alpha_1^{\{l\}}$.
(2) Try to find $P_i^{\{l\}}$ (and K if synthesis) for the LMI problem (10) ((18) if synthesis) with constant $G_i^{\{l\}}$.
(3) If unfeasible, set $\delta\alpha_i^{\{l\}} = 1/2\,\delta\alpha_1^{\{l\}}$ and go to step 1. Else compute $G_i^{\{l+1\}} = \left(P_i^{\{l\}}\right)^{-1}$ and $\delta\alpha_i^{\{l+1\}} = \delta\alpha_1^{\{l\}}$.
(4) If $|\alpha^{\{l\}} - \alpha^{\{l-1\}}| < \epsilon$, stop. Else, set $l = l + 1$ and go to step 1.

Algorithm 3 synthesizes a robust controller with minimum performance level γ.

Algorithm 3. (Performance level γ). Choose a tolerance ϵ, initials $G_i^{\{0\}}$, set $l = 0$ and start to iterate:

(1) Find $P_i^{\{l\}}$ and K that solve the optimization problem:
    Minimize γ subject to (21) with constant $G_i^{\{l\}}$.
(2) Compute $G_i^{\{l+1\}} = \left(P_i^{\{l\}}\right)^{-1}$.
(3) If $|\gamma^{\{l\}} - \gamma^{\{l-1\}}| < \epsilon$, stop. Else, set $l = l + 1$ and go to step 1.

Algorithms 1 to 3 assume the LMI problem at the first iteration feasible for the chosen values of $G_i^{\{0\}}$. The solution of standard LMI formulations can give appropriate values of the Lyapunov variable to compute $G_i^{\{0\}}$ using (22). The same initial value can also be attributed to all $G_i^{\{0\}}$.

5. NUMERICAL EXAMPLES

5.1 Stabilizing Static Output Feedback

The results of the following problem borrowed from [de Oliveira et al. (1999)] illustrate the effectiveness of Algorithm 2. The problem is to find the largest scalar α^* such that $A(\alpha)$ is robustly stable for all $|\alpha| < \alpha^*$. An initial $G^{\{0\}} = I$ was not able to make the first iteration feasible, therefore a resulting P computed with (4) for the open-loop nominal plant $A(\alpha = 0)$ was utilized as $G^{\{0\}} = P^{-1}$.

$$A(\alpha) = \begin{bmatrix} 0.8 & -0.25 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.8 & -0.5 & 0 & 1 \end{bmatrix}$$

Convergence tolerance is $\epsilon = 1^{-4}$ and the initial $\delta\alpha^{\{0\}} = 0.1$. The algorithm converged after 23 iterations. Figures (1a)(1b) illustrate the convergence of α and $\delta\alpha$ for the analysis problem. The final $\alpha^{(23)} = 0.4619$ is identical to the exact maximum value presented at [de Oliveira et al. (1999)]. The stability bounds for a full state feedback with control input uncertain matrix,

$$B_u(\beta) = \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (1 - \beta) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad 0 \le \beta \le 1$$

is also determined. Figures (1c)(1d) ilustrates the convergence towards the bound $\alpha^{(41)} = 0.9833$. The stability bounds of a static output feedback is computed considering a measured output matrix,

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

as well as the previous uncertain $B_u(\beta)$. Algorithm 2 computes the bound $\alpha^{(32)} = 0.7665$. Figures (1e)(1f) illustrate the evolution of $\alpha^{\{l\}}$ and $\delta\alpha^{\{l\}}$. Table 1 brings a summary of the bounds for the different situations as well as results for usual stability conditions.

Table 1. Summary of stability bounds.

Method	Analysis	FS	SO (C)
Exact Value	0.4619	-	-
de Oliveira, 1999	0.4619	0.8892	-
Quadr. Stability	0.4279	-	-
$\mathcal{RH}_\infty$	0.2956	-	-
Algorithm 2	0.4619	0.9833	0.7665

5.2 Decentralized Feedback with Performance Level

In this example a system is robustly controlled using two decentralized and strictly proper dynamic output feedbacks. The discrete-time system matrices are,

$A(\alpha) =$
$$\begin{bmatrix} 0.8189 & 0.0863 & 0.0900 & 0.0813 \\ 0.2524 & 1.0033 & 0.0313 & 0.2004 \\ -0.0545 & 0.0102 & 0.7901 & -0.2580 \\ -0.1918 & -0.1034 & 0.1602 & 0.8604 \end{bmatrix} + \alpha \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B_w = \begin{bmatrix} 0.0953 & 0 & 0 \\ 0.0145 & 0 & 0 \\ 0.0862 & 0 & 0 \\ -0.0011 & 0 & 0 \end{bmatrix}, B_u = \begin{bmatrix} 0.0045 & 0.0044 \\ 0.1001 & 0.0100 \\ 0.0003 & -0.0136 \\ -0.0051 & 0.0936 \end{bmatrix},$$

$$C_y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}', C_w = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, D_{zu} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D_{yw} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D_{zw} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

The control objective is to minimize the upper bound γ on the induced $\mathcal{L}_2$-norm from $w(k)$ to $z(k)$. One controller

(a) $\alpha^{(k)}$ (Analysis).

(b) $\delta\alpha^{(k)}$ (Analysis).

(c) $\alpha^{(k)}$ (Full-State).

(d) $\delta\alpha^{(k)}$ (Full-State).

(e) $\alpha^{(k)}$ (Static-Output).
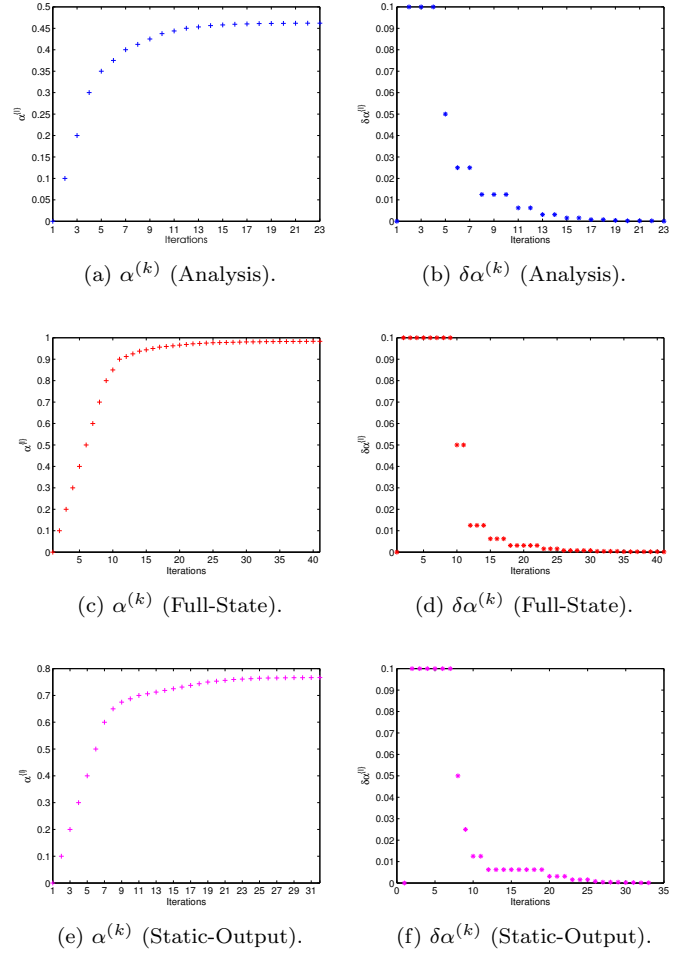

(f) $\delta\alpha^{(k)}$ (Static-Output).

Fig. 1. Convergence of stability bounds.

measures the first output and manipulates the first input, and another controller measures the second output and manipulates the second input. Initials $G_i^{\{0\}}$ have to be chosen in order to initialize the algorithm. The nominal system ($\alpha = 0$) in closed-loop with the following decentralized controller [de Oliveira et al. (2000)],

$$K = \begin{bmatrix} D_{c1} & 0 & C_{c1} & 0 \\ 0 & D_{c2} & 0 & C_{c2} \\ B_{c1} & 0 & A_{c1} & 0 \\ 0 & B_{c2} & 0 & A_{c2} \end{bmatrix}, A_{c1} = \begin{bmatrix} 1.2700 & -0.6660 \\ 0.5000 & 0 \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} 0.6300 & 0.3956 \\ 0.5000 & 0 \end{bmatrix}, B_{c1} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}, B_{c2} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix},$$

$$C_{c1} = \begin{bmatrix} -0.2720 & 0.2013 \end{bmatrix}, C_{c2} = \begin{bmatrix} -0.1760 & -0.0810 \end{bmatrix},$$

is used in the LTI version of (19) to compute a single Lyapunov matrix P. The same initial $G_i^{\{0\}} = P^{-1}$ is applied to $i = 1, \dots N$. The aim is to evaluate how γ varies with respect to the uncertainty range of α. It is expected that larger uncertainties will result in worse performance levels. The iterative algorithm in this example is a mixture of the stability bounds and performance level algorithms. During the course of the optimization, $\alpha^{\{l\}}$ is gradually increased according to Algorithm 2 with $\delta\alpha^{\{0\}} = 0.001$ until a target value α_{tg} is achieved, remaining fixed throughout the subsequent iterations. The induced $\mathcal{L}_2$-norm LMI of Algorithm 3 is in place of the feasibility LMI

of Algorithm 2. The stopping criteria is convergence of γ to a tolerance 10^{-3}. γ was computed for uncertainty target values of $\alpha_{tg} = \{0.01, 0.02, \ldots, 0.06\} \cup \{0.001\}$ (Fig. 2a). The number of iterations to reach convergence varied from 76 to 92.

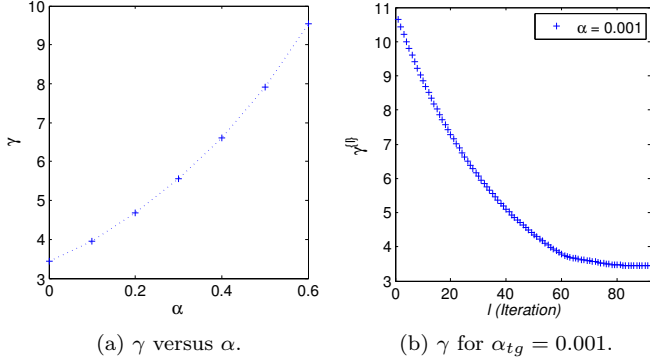(a) γ versus α. (b) γ for $\alpha_{tg} = 0.001$.

Fig. 2. Performance versus uncertainty.

The evolution of $\gamma^{\{l\}}$ per iteration for $\alpha_{tg} = 0.001$ is depicted on Fig. 2b. In this case the uncertainty is very small, $\alpha^{\{l\}}$ is kept constant throughout the whole optimization, and thus the obtained controller is nearly the same as the optimal controller for the nominal plant. Notice that $\gamma^{\{l\}}$ is monotonically decreasing.

The same behaviour does not occur when $\alpha^{\{l\}}$ is changing through the optimization process. The evolution of $\alpha^{\{l\}}$ and $\gamma^{\{l\}}$ per iteration, for $\alpha_{tg} = 0.04$, is shown on Fig.(3a) and Fig.(3b), respectively. Figure (3c) only includes values of $\gamma^{\{l\}} < 15$, in order to better visualize the convergence.
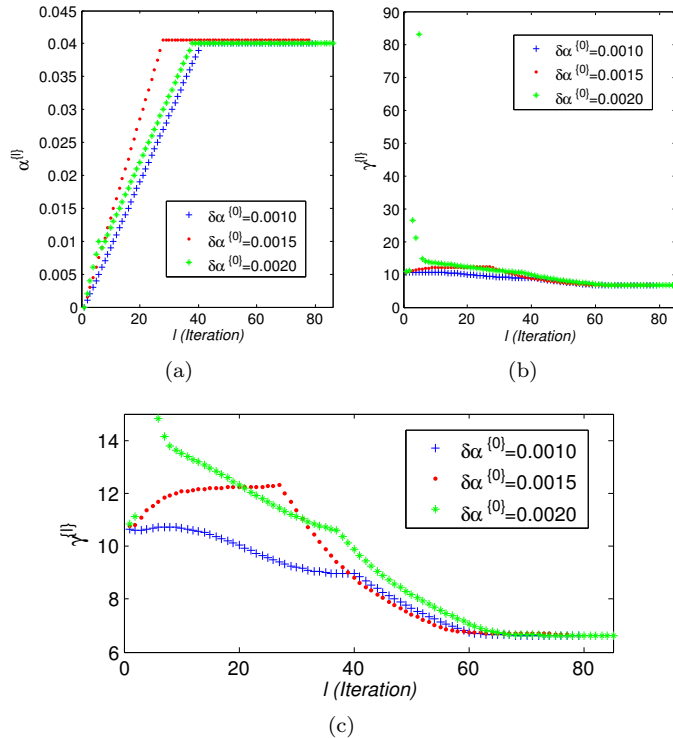

(a) (b)

(c)

Fig. 3. Evolution of $\delta\alpha^{\{l\}}$ and $\gamma^{\{l\}}$ for $\alpha_{tg} = 0.4$.

These plots depict results for three different values of initial $\delta\alpha^{\{0\}}$. Firstly, notice that $\gamma^{\{l\}}$ is no longer monotonically decreasing at every iteration, although this behaviour shows up after the target value α_{tg} is attained.

Large values of $\gamma^{\{l\}}$ during the first iterations for $\delta\alpha^{\{0\}} = 0.002$ can be noticed on Fig. (3b). The value of $\delta\alpha^{\{l\}}$ was reduced from 0.002 to 0.001 at $l = 7$ (Fig. (3a)), due to infeasibility of the induced $\mathcal{L}_2$-norm LMI problem for such increment on the uncertainty. Despite the different solution paths for each value of initial $\delta\alpha^{\{0\}}$, Fig. (3c), the convergent value of γ differ less than 1%.

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