Slowly-varying 2. order Wave Forces on Large Structures

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Slowly-varying 2. order Wave Forces on Large Structures

by

Michael Brorsen

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1 Drift forces

1.1 Introduction

The average over time of 1. order wave loads on a structure is always equal to zero. This is due to the fact that all load components are determined by 1. order wave theory and the integration of all wave loads stops at mean water level (MWL) i.e. the average over one wave period of the surface level. This corresponds to assuming a constant submerged area of the body and when the load averaged over time is zero at all points, the total load is of course also zero.

In reality, however, the wave loads are non-linear and the may be separated into the following 3 components:

1) forces at frequencies, f_{force} , equal to the wave frequencies $f_w \sim 1/T_p$, where T_p is the peak-period in the wave spectrum. The order of magnitude of these forces is $o(H/L)$, where H is the wave height and L is the wave length.

2) a constant drift force, defined as the average over a long time span of the wave forces. The order of magnitude is $o((H/L)^2)$.

3) slowly-varying drift forces, which may be determined from the total wave loads by filtering with a filter allowing only low frequent signals to pass. Typically the loads have frequencies in the range

$$\frac{1}{10} f_w < f_{force} < \frac{1}{5} f_w \tag{1}$$

The order of magnitude the slowly-varying drift forces is $o((H/L)^2)$.

In the following the non-linear drift forces are considered. Basically one would assume that the drift forces are of minor importance because they are small compared to the forces at wave frequencies. In practice, however, drift forces are of major importance for the motions of a floating body.

The combination of large masses and relative small spring forces (for example in slack moorings) results in natural (resonance) periods with an order of magnitude of one minute, and such periods are often coinciding with the periods of drift forces as indicated above.

This may result in resonance and because the damping is quite limited for ship motions at these frequencies, it may results in large motions with corresponding large mooring forces.

1.2 Physical reasons for drift forces

When a body with relatively large dimensions compared to the wave length is considered, it may be shown that the viscous forces are of marginal importance. The vorticity created in the boundary layer has in this case only marginal effect on the flow field outside the thin layer. With good approximation the flow field can be considered free of vorticity and consequently potential theory can be applied, where the relation between particle velocity, $\vec{v}$, and the velocity potential, φ, is:

$$\vec{v} = grad\,\varphi = \nabla\varphi \tag{2}$$

Furthermore, it is a very good approximation to consider the forces from the water as pressure forces, i.e. locally acting in a direction perpendicular to the surface of the body.

The total non-stationary force on the body can therefore be determined from:

$$\vec{F}(t) = \int_A p\,\vec{n}\,dA \tag{3}$$

where

$A = A(t)$ is the submerged area of the body

$\vec{n} = \vec{n}(t)$ is the normal to the body surface, positive from water towards the body. Varies in time, if the body moves.

The pressure, p, is calculated by Bernoulli's generalised equation :

$$p = -\rho\,g\,z - \rho\frac{\partial\varphi}{\partial t} - \frac{1}{2}\rho\,v^2 \tag{4}$$

The horizontal components of the drift force, which usually are the most interesting, may be determined by projecting the drift force on the horizontal x-axis:

$$F_x(t) = \int_A p\,n_x\,dA \tag{5}$$

The average of $F_x(t)$ over one wave period T is:

$$\overline{F_x} = \frac{1}{T}\int_0^T F_x(t)dt \neq 0 \tag{6}$$

where the symbol $\overline{}$ above a variable means average over time.

Several reasons leads to $\overline{F_x} \neq 0$, but the two major reasons are:

1) the pressure forces act up to the instantaneous position of free surface and not to MWL

2) the non-linear term $-\frac{1}{2}\rho\, v^2$ in Bernoullis equation, due to $\overline{v^2} > 0$.

As analytical solutions do not exist to non-linear problems, one must apply numerical methods to calculate drift forces.

If it is possible to calculate the non-linear flow close to the structure equation (3) may be used to calculate the drift force. The result is called a *near-field* solution.

Formerly it was impossible to obtain near-field solutions in the time domain due to lack of computer power. Instead solutions based on the momentum equation were developed. These solutions only require information on the flow field far from the structure. Accordingly they are called *far-field* solutions.

To day a near-field solution can be obtained, but it requires much larger computer power than a far-field solution. The pay off from this extra effort is time series of both local and total forces in contrast to the time average of the total force, which is the only result produced by a far-field solution.

2 Drift forces from regular waves

Under special conditions drift forces may be calculated approximately by means of 1. order theory, only by including the 2. order components, i.e. components with the order of magnitude $o((H/L)^2$. By the way, in many textbooks this method is applied to derive the expression for the energy transport velocity.

2.1 Time average of the momentum equation

Below is shown that the the average over one wave period of the momentum equation may be considered as a force equilibrium equation.

A fluid body between two fixed vertical cross sections, denoted A and B, is considered at time $t = 0$, see Figure 1. To simplify, we consider a wave train propagating on constant wave depth h.

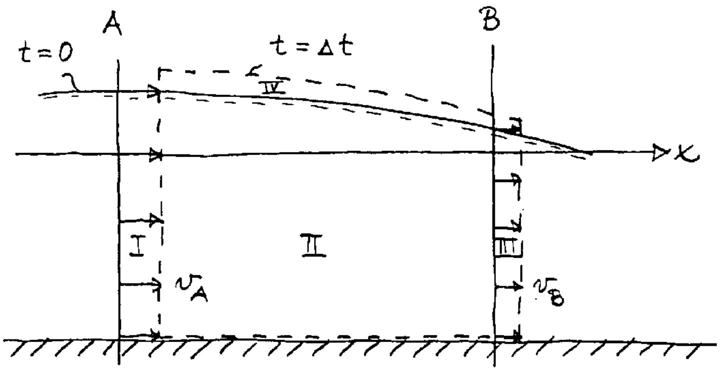

Figure 1: Momentum equation for a non-stationary flow

During the time span Δt the fluid body moves to the position shown with the broken line.

The horizontal component of the momentum equation reads:

$$\sum_i (F_x)_i \, \Delta t = \Delta B \tag{7}$$

where

$(F_x)_i$ is x-component (horizontal) of i'th external force on the body

$B = \int_X u \, \rho \, dX$ is the x-component of the momentum of the body

X is the volume of the body

The horizontal external forces acting on the body are solely pressure forces. However, other types of forces may be included. For example, the gravitational forces should be included when non-horizontal components of the momentum equation are considered.

The change in momentum may be determined in this way:

$$\begin{aligned}\Delta B &= B(\Delta t) - B(0) \\ &= B(\Delta t)_{II} + B(\Delta t)_{III} + B(\Delta t)_{IV} - B(0)_I - B(0)_{II}\end{aligned} \tag{8}$$

where the indices I, II, III og IV refer to the volumes indicated on Figure 1 1. As $B(0)_{IV} = 0$, we may reformulate equation (8) to:

$$\begin{aligned}\Delta B &= B(\Delta t)_{II} - B(0)_{II} + B(\Delta t)_{IV} - B(0)_{IV} + B(\Delta t)_{III} - B(0)_I \\ &= \frac{\partial(B_{II} + B_{IV})}{\partial t}\,\Delta t + \sum_B (\rho\, v_B\, \Delta t\, \Delta z)\, v_B - \sum_A (\rho\, v_A\, \Delta t\, \Delta z)\, v_A \\ &= \frac{\partial(B_{II} + B_{IV})}{\partial t}\,\Delta t + \int_B \rho\, v_B\, \Delta t\, v_B\, dz - \int_A \rho\, v_A\, \Delta t\, v_A\, dz\end{aligned} \tag{9}$$

Note that the size of volume II is finite, and the volume of I is going towards zero for $\Delta t \to 0$. We therefore only make a small error (which can be made arbitrarily small) by using the approximation

$$\frac{\partial(B_{II} + B_{IV})}{\partial t} = \frac{\partial B_{AB}}{\partial t}$$

Consequently the momentum equation, equation (7), reads:

$$\sum_i (F_x)_i\, \Delta t = \frac{\partial B_{AB}}{\partial t}\,\Delta t + \Delta t \left(\int_B \rho\, v_B^2\, dz - \int_A \rho\, v_A^2\, dz \right) \tag{10}$$

After division by Δt we obtain

$$\sum_i (F_x)_i = \frac{\partial B_{AB}}{\partial t} + \int_B \rho\, v_B^2\, dz - \int_A \rho\, v_A^2\, dz \tag{11}$$

Because the integrals in equation (11) have the dimension force (even though they represent changes of momentum) they are called *inertia forces*.

In the following these forces are denoted I in accordance with the general practice within hydraulics. If the expressions

$$I_A = \int_A \rho v_A^2 dz \tag{12}$$

$$I_B = \int_B \rho v_B^2 dz \tag{13}$$

are used for substitution into equation (11) and the equation subsequently are averaged over one wave period, the average of the momentum equation reads

$$\sum_i (\overline{F_x})_i = \frac{1}{T}\int_0^T \frac{\partial B_{AB}}{\partial t} dt + \overline{I_B} - \overline{I_A} \tag{14}$$

The integral in equation (14) may be reformulated to

$$\int_0^T \frac{\partial B_{AB}}{\partial t} dt = \int_0^T d\left(B_{AB}\right) = \left|B_{AB}\right|_0^T = B_{AB}(T) - B_{AB}(0) = 0 \tag{15}$$

due to the fact that a body between two fixed cross section has the same momentum at time $t = 0$ and time $t = T$.

Consequently the time average of the momentum equation reads

$$\boxed{\sum_i (\overline{F_x})_i + \overline{I_A} - \overline{I_B} = 0} \tag{16}$$

Equation (16) shows that formally there exists equilibrium between all horizontal forces averaged over one wave period. The momentum equation for a body in stationary flow can be interpreted the same way, but time averaging is of course not necessary.

Therefore a vertical cross section is exposed to the sum of the time averaged pressure force an the time averaged inertia force, when the time averaged momentum equation is applied as a force equilibrium equation.

2.2 Wave thrust

In a wave motion a vertical cross section is exposed by the following time averaged force:

$$\overline{F} = \overline{P} + \overline{I} \tag{17}$$

where

$$\begin{aligned}
\overline{P} &= \frac{1}{T}\int_0^T P\, dt \\
&= \frac{1}{T}\int_0^T \left(\int_{-h}^{\eta} p\, dz\right) dt \\
&= \frac{1}{T}\int_0^T \left(\int_{-h}^{\eta} (p_{hyd} + p^+) dz\right) dt \\
&= \frac{1}{T}\int_0^T \left(\int_{-h}^{\eta} (\rho\, g\, (-z) + p^+) dz\right) dt
\end{aligned} \tag{18}$$

og

$$\begin{aligned}\overline{I} &= \frac{1}{T}\int_0^T I\,dt \\ &= \frac{1}{T}\int_0^T \left(\int_{-h}^{\eta} \rho\, v^2\, dz\right) dt \qquad (19)\end{aligned}$$

where z is the *level*, i.e. the vertical distance above a datum. The datum may be chosen arbitrarily.

If only progressive waves are considered, the integrals in the equations (18) and equation (19) may be solved analytically, see for example Jonsson & Svendsen (1980) and Fredsøe (1990).

The expression for the force, P_0, corresponding to hydrostatic pressure, p_{hyd}, reads:

$$P_0 = \int_{-h}^{0} p_{hyd}\, dz = \int_{-h}^{0} \rho\, g\,(-z) dz = \frac{1}{2}\rho\, g\, h^2 \qquad (20)$$

and the time averaged forces may be expressed as shown in Table 1.

	shallow water	general	deep water
$\overline{P}$:	$P_0 + \frac{1}{16}\rho\, g\, H^2$	$P_0 + \frac{1}{16}\rho\, g\, H^2\, \frac{2\,k\,h}{\sinh 2\,k\,h}$	P_0
$\overline{I}$:	$\frac{1}{8}\rho\, g\, H^2$	$\frac{1}{16}\rho\, g\, H^2 \left(1 + \frac{2\,k\,h}{\sinh 2\,k\,h}\right)$	$\frac{1}{16}\rho\, g\, H^2$

Table 1: *Time averaged pressure forces and inertia forces*

The component of the force caused by the wave motion, i.e. the deviation from the hydrostatic pressure force, is denoted F_r and named *wave thrust* or *radiation stress* (in danish: bølgens reaktionskraft).

$$F_r = \overline{P} + \overline{I} - P_0 \qquad (21)$$

Insertion of the expressions from Tabel 1 gives:

$$\boxed{F_r = \frac{1}{16}\rho\, g\, H^2 \left(1 + 2\,\frac{2\,k\,h}{\sinh 2\,k\,h}\right)} \qquad (22)$$

By realizing the existence of wave thrust, it is possible to explain phenomena as wave induced current, changes in mean water level, bounded (long periodic) waves in irregular waves and drift forces on bodies.

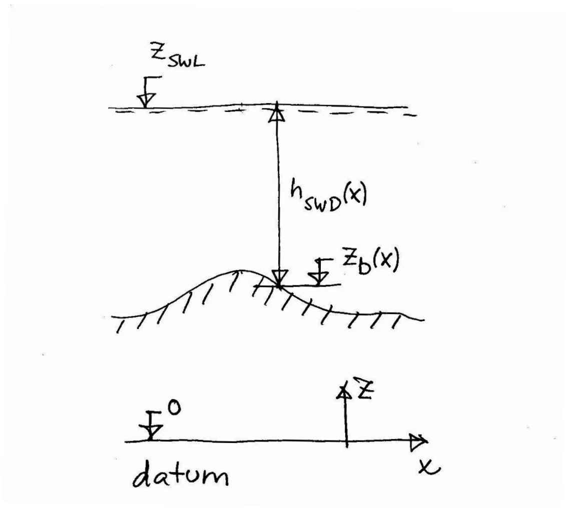

Figure 2: *Still Water. Definition sketch.*

In this context one should note some important definitions concerning water levels and depths.

In case of still water the level of the free surface is named *still water level* and denoted z_{SWL}. The corresponding water depth is named *still water depth*, and is defined as $h_{SWD}(x) = z_{SWL} - z_b(x)$. See Figure 2.

In case of wave motion the *local mean water level* is defined as

$$z_{MWL}(x) = \bar{z}_{surface}(x) = \frac{1}{T}\int_0^T z_{surface}(x,t)\,dt \tag{23}$$

The local wave surface elevation is defined as $\eta(x,t) = z_{surface}(x,t) - z_{MWL}(x)$ giving $\bar{\eta}(x) = 0$. See Figure 3.

The *local mean water depth* is defined as

$$h(x) = z_{MWL}(x) - z_b(x) \tag{24}$$

Note that $h(x)$ is the water depth that a wave will "feel" at position x, and therefore this water depth must be used in the previous expressions for wave thrust.

In general local mean water level and still water level are not equal, i.e. $z_{MWL} \neq z_{SWL}$! This is due to phenomena like shoaling and bounded long waves.

However, on *deep water* one finds that local mean water level and still water level are coinciding, i.e. $z_{MWL} = z_{SWL}$. This is shown indirectly by assuming that spatial changes in local mean water level are present. This assumption leads to a contradiction and must therefore be rejected.

A gradient of the mean water level $\partial z_{MWL}(x)/\partial x$ gives a horizontal gradient of the hydrostatic pressure of $\partial p_{hyd}/\partial x = \rho\, g \partial z_{MWL}(x)/\partial x$.

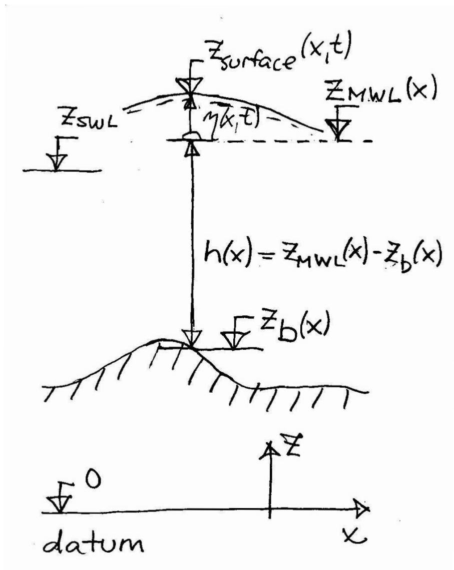

Figure 3: *Wave motion. Definition sketch.*

This pressure gradient gives a non-zero resulting hydrostatic pressure force on a body. The size of this pressure force reads:

$$
\begin{aligned}
P_{hyd} &= -\int_X \operatorname{grad} p_{hyd}\, dX \\
&= -\int_X \rho\, g \frac{\partial z_{MWL}}{\partial x} dX \\
&= -\rho\, g \frac{\partial z_{MWL}}{\partial x} \int_X dX \quad \text{, as the gradient is constant along a vertical} \\
&= -\rho\, g \frac{\partial (z_{MWL})}{\partial x} h\, A \quad \text{, as the volume reads } X = h\, A
\end{aligned}
$$

As $P_{hyd} \to \infty$ for $h \to \infty$, it is obvious that changes in the local mean water level can *not* be present at deep water, because there exist no other wave induced forces which could out balance P_{hyd} in the momentum equation.

2.3 Momentum in regular waves on deep water

Regular waves have a non-zero mean value of the momentum in the direction of propagation, though this momentum is of 2. order, i.e. the order of magnitude is $o((H/L)^2)$.

This mean momentum is due to a greater volume (with positive velocities) under the wave crest than the volume (with negative velocities) under the wave trough.

In practice the *mean momentum* reads

$$< B >= \frac{1}{L}\int_0^L B(x)\,dx$$

or

$$< B >= \frac{1}{L}\int_0^L \left(\int_{-h}^{\eta} \rho\, u\, dz\right)\, dx \tag{25}$$

It is by no means trivial to determine $< B >$ from this integral for finite water depths.

On *deep water* it is simpler as shown below. We consider 1. order waves with the surface elevation

$$\eta = \frac{H}{2}\cos(\omega t - kx) \tag{26}$$

and horizontal particle velocities:

$$u = \frac{\pi\, H}{T}\frac{\cosh k(z+h)}{\sinh kh}\cos(\omega t - kx) \tag{27}$$

Without loss of generality the momentum $B(x)$ is calculated at the time $t = 0$. The horizontal momentum pr. unit length (along the wave front) is:

$$\begin{aligned}
B(x) &= \int_{-h}^{\eta} \rho\, u(z)\, dz \\
&= \int_{-h}^{\eta} \rho\, \frac{\pi\, H}{T}\frac{\cosh k(z+h)}{\sinh kh}\cos kx\, dz \\
&= \frac{\rho\,\pi\, H}{T}\cdot\frac{\cos kx}{\sinh kh}\int_{-h}^{\eta}\cosh k(z+h)\, dz \\
&= \frac{\rho\,\pi\, H}{T}\cdot\frac{\cos kx}{\sinh kh}\cdot\frac{1}{k}\int_{-h}^{\eta}\cosh k(z+h)\, d(k(z+h)) \\
&= \frac{\rho\,\pi\, H}{T}\cdot\frac{\cos kx}{\sinh kh}\cdot\frac{1}{k}\left|\sinh k(z+h)\right|_{-h}^{\eta}
\end{aligned}$$

$$\begin{aligned} B(x) &= \frac{\rho\,\pi\,H}{T} \cdot \frac{\cos kx}{\sinh kh} \cdot \frac{1}{k} \sinh k(\eta + h) \\ &= \frac{\rho\,L\,H}{2\,T} \cdot \frac{\sinh k(\eta + h)}{\sinh kh} \cos kx \end{aligned}$$

On deep water the following is valid:

$$\frac{\sinh k(\eta + h)}{\sinh kh} \approx \frac{e^{k(\eta+h)}}{e^{kh}} = e^{k\eta} \approx 1 + k\,\eta$$

After insertion of this expression and the expression for η, the equation for $B(x)$ reads

$$B(x) = \frac{\rho\,L\,H}{2\,T} \left(1 + k\,\frac{H}{2} \cos kx\right) \cos kx$$

This expression is substituted into equation (25) giving:

$$\begin{aligned} < B > &= \frac{\rho\,H}{2\,T} \int_0^L \left(\cos kx + k\,\frac{H}{2} \cos^2 kx\right)\,dx \\ &= \frac{\rho\,H}{2\,T} \cdot k\,\frac{H}{2} \cdot \frac{L}{2} \\ &= \frac{1}{2} \rho\,a^2\,k\,c \end{aligned}$$

where $a = H/2$ and $c = L/T$.

On deep water one has

$$c^2 = \frac{g}{k} \quad \text{and} \quad c_g = c$$

where c_g is the energy transport velocty. The equation for mean momentum on *deep water* therefore reads:

$$\boxed{< B >= \frac{1}{2} \rho\,g\,a^2 \frac{1}{c} = \frac{1}{4} \rho\,g\,a^2 \frac{1}{c_g}} \tag{28}$$

2.4 Deep water. Constant drift force on a body totally *absorbing* the incident wave energy

In this section we assume that it is possible to construct a floating body, which can absorb all incident wave energy. In the following such a body is called an *absorber*.

The effect of the absorber may be compared with the removal of all incident momentum in the wave motion. Consequently the absorber has to act on the water with a force directed against the direction of wave propagation. According to Newton's 3. law the water acts on the absorber with a force of same size, but in the direction of propagation.

Because no energy is being reflected, the wave field in front of the absorber is equal the wave field in undisturbed progressive waves.

If the absorber is located on deep water we may calculate the the drift force $\overline{F_d}$ corresponding to incident regular waves with the amplitude $a_i = H_i/2$. See Figure 4.

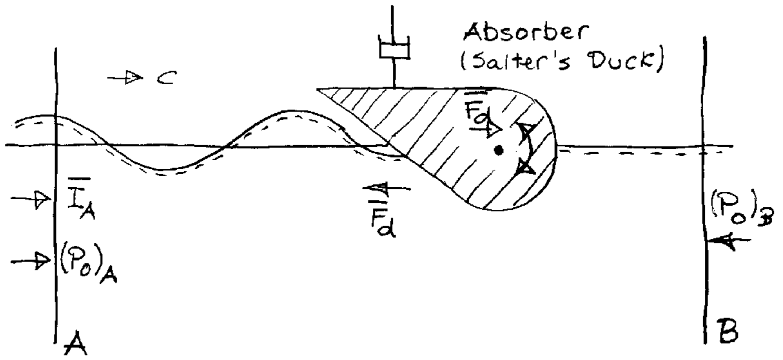

Figure 4: Ideal wave absorber on deep water.

The time averaged momentum equation is applied between section A and B. Because there is no increase in the momentum between A and B at time t and time $t + T$ one finds:

$$(P_0)_A + I_A - \overline{F_d} - (P_0)_B = 0 \tag{29}$$

Insertion of the expression for the inertia force on deep water and $(P_0)_A = (P_0)_B = P_0$ gives:

$$\overline{F_d} = \frac{1}{16}\rho\, g\, H_i^2 = \frac{1}{4}\rho\, g\, a_i^2 \tag{30}$$

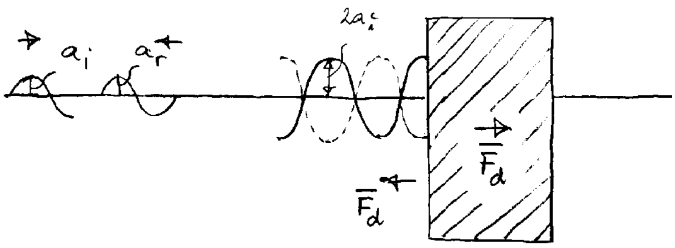

Figure 5: Deep water. Body reflects incident waves totally.

2.5 Deep water. Constant drift force on a body totally *reflecting* the incident wave energy

In this section we assume that the size of the floating body is so large that all incident wave energy is being reflected. Therefore standing waves are created in front of the absorber. See Figure 5.

If the body is located on deep water the drift force $\overline{F_d}$ may be calculated by arguments from analogy. The amplitude of the regular incident waves is again $a_i = H_i/2$.

In case of full reflection of the waves the body has to:

1) absorb the incident waves

2) generate similar waves moving in the opposite direction

The effect of 1) has been calculated at section 2.4. The reflected waves contains the same size of momentum but in *opposite* direction. The same mean force must therefore act on the water no matter if waves are absorbed or generated. Accordingly the necessary mean force acting on the water from the body is twice as large as the force necessary to absorb waves totally. The drift force on the body at complete reflection is therefore

$$\overline{F_d} = 2 \cdot \frac{1}{16} \rho \, g \, H_i^2 = \frac{1}{2} \rho \, g \, a_i^2 \tag{31}$$

In this case the amplitude of the reflected waves is $a_r = a_i$, and we may reformulate equation (31) to

$$\boxed{\overline{F_d} = \frac{1}{2} \rho \, g \, a_r^2} \tag{32}$$

In the literature this equation is often called *Maruo's* equation.

2.6 Finite water depth. Generalized expression of the constant drift force on a floating body

In general the size of the body is not large enough to cause complete reflection of the waves. Part of the incident energy normally passes under the body and this energy is called *transmitted* wave energy. The amplitude of the transmitted waves is denoted a_t.

The amount of transmitted energy increases if the floating body is allowed to move. The body is so to speak more transparent to the waves when it can move freely.

For linear 2-D waves on *finite* water depth, Longuet-Higgins (1977) has shown that on a long, floating body the drift force pr. m may by determined from:

$$\boxed{\overline{F_d} = \frac{1}{4}\,\rho\, g\,(a_i^2 + a_r^2 - a_t^2)\left(1 + \frac{2\,k\,h}{\sinh 2\,k\,h}\right)} \tag{33}$$

where a_t is the amplitude of the transmitted waves.

Equation (33) is a generalized version of Maruo's equation, but the derivation is by no means trivial. This is due to gradients of the local mean water levels caused by changes in local wave height.

Note that an increase of the wave height causes an increase of the wave thrust. Because the time averaged momentum equation corresponds to a force equilibrium, there has to occur a reduction in the hydrostatic pressure force when the wave height increases. This is only possible by a change in the local mean water level.

2.7 Constant drift force on a 3-D body in regular waves.

So far only drift forces on infinite long bodies has been treated. In these cases wave energy may only be transmitted *under* the body. When the length of the body is finite, wave energy may also be transmitted *around* the ends of the body, and the body is called a 3-D body.

Accordingly, diffraction are of importance, and even far-field solutions require that the wave field around the body at least is calculated by numerical methods based on 1. order wave theory.

It is out of the scope of this note to describe such calculations in details, and in the following we will simply assume that we are capable of calculating the constant drift force on a 3-D body with arbitrary shape.

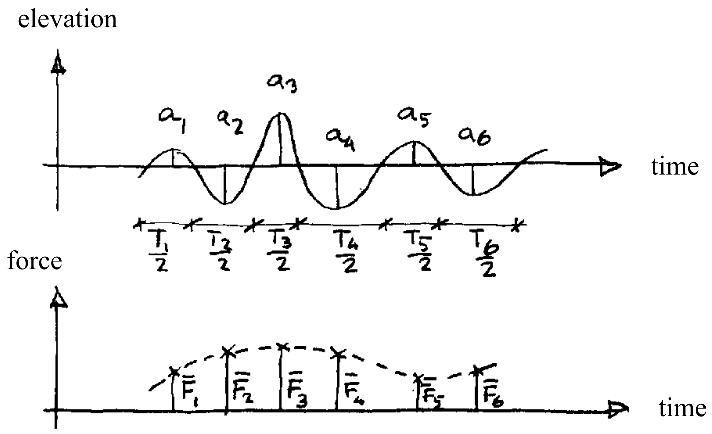

Figure 6: Approximate calculation of drift force from irregular waves

3 Drift forces from irregular waves

3.1 Approximate calculation of a slowly varying drift force

A direct approximate calculation of a drift force from irregular waves is based on the assumption that each crest and trough corresponds to one half of a regular wave.

This assumption is a complete analog to the determination of the maximum wave force on a body in a design storm. In the irregular waves one seek out the maximum wave height and corresponding period. Hereafter the forces are calculated under the assumption that the body is exposed to regular waves with this height and period.

As seen from Figure 6 it is possible from the elevation time series to find estimates of the mean drift force from each "half wave" as described in the chapters above. These force estimates varies in time and the corresponding periods shows up to be equal with the periods of the wave groups.

Unfortunately this method cannot take into account the effect from the so-called bounded waves, which in many cases are of major importance.

3.2 Bounded long waves in irregular waves

Bounded long waves (sometimes named set-down waves) are long periodic waves propagating with the so-called group velocity, i.e. they follow the wave groups and we say that they are bound to the wave groups.

Bounded long periodic waves are present as soon as a variation in wave thrust is present.

Wave thrust is proportional to the square of the wave height in regular waves. A gradient in wave height therefore gives a gradient in the wave thrust and according to the time averaged momentum equation this thrust gradient has to be balanced by a gradient in the hydrostatic pressure force. This is only possible if a gradient of the local mean level exists, i.e. the mean water level is sloping.

If the wave height in a wave group varies slowly in space, the waves may locally be considered as regular waves, and the previous expressions for wave thrust in regular waves may be applied.

For wave groups we should therefore expect the local mean water level to be highest under small waves (group nodes) and lowest under the highest waves (between the nodes). Consequently a long wave is created having a length equal to the length of the wave groups. It is obvious that this long wave must propagate with the same velocity as the wave groups do. This leads to the name *bounded* waves.

The most ordinary wave groups may be formed by super position of two regular waves having periods fairly close to each other. This wave condition is named *regular wave groups* or *pair of beating waves.*

For simplicity we consider first wave groups formed by super position of two regular waves having the same amplitude. The expressions for these regular waves η_n and η_m reads:

$$\eta_n = a\,\cos(\omega_n\,t - k_n\,x) \qquad \text{og} \qquad \eta_m = a\,\cos(\omega_m\,t - k_m\,x) \tag{34}$$

It is assumed that the periods are close to each other with

$$T_n < T_m \qquad \Leftrightarrow \qquad \omega_n > \omega_m$$

giving

$$L_n < L_m \qquad \Leftrightarrow \qquad k_n > k_m$$

as L increases monotonic with T.

By means of trigonometric relations the resulting surface elevation η_s reads:

$$\eta_s = 2a\,\cos\left(\frac{\omega_n+\omega_m}{2}\,t - \frac{k_n+k_m}{2}\,x\right)\cdot\cos\left(\frac{\omega_n-\omega_m}{2}\,t - \frac{k_n-k_m}{2}\,x\right) \tag{35}$$

The resulting surface elevation is therefore calculated as the product of two factors oscillating with very different frequencies. Relatively the frequency of the last factor is small compared to the frequency of the first factor.

Equation (35) may be reformulated to

$$\eta_s = a_s(x,t)\cdot\cos\left(\frac{\omega_n+\omega_m}{2}\,t - \frac{k_n+k_m}{2}\,x\right) \tag{36}$$

where

$$a_s(x,t) = 2a\,\cos\left(\frac{\omega_n - \omega_m}{2}\,t - \frac{k_n - k_m}{2}\,x\right) \tag{37}$$

Formally η_s may be considered as a *nearly* regular wave. The frequency of this regular wave is the mean value of the frequencies of the two superposed waves, and the nearly regular wave has an amplitude $a_s(x,t)$ varying very slowly in time and space.

If $T_n \to T_m$, the period of $a_s(x,t)$ increases infinitely, and $a_s(x,t)$ becomes a constant, i.e. η_s is a regular wave.

The distance between two nodes in the wave groups is equal to the distance between the points (at a given time), where $a_s(x,t) = 0$.

This distance L_b may be found from

$$\frac{k_n - k_m}{2} \cdot L_b = \pi$$

or

$$\frac{2\,\pi}{L_n} - \frac{2\,\pi}{L_m} = \frac{2\,\pi}{L_b}$$

giving

$$L_b = \frac{L_n\,L_m}{L_m - L_n} \tag{38}$$

If one stay at a fixed point, the nodes of the wave groups will arrive with a time lag of

$$T_b = \frac{T_n\,T_m}{T_m - T_n} \tag{39}$$

T_b is thereby the period of the bounded long waves.

I general the amplitudes of the superposed waves are not equal. By a non-trivial generalization of the time averaged momentum equation for regular waves Ottesen-Hansen (1978) determined a 2. order transfer function between two regular waves and the associated bounded long wave may be found, see also Sand (1982). It was *not* assumed that the amplitudes of the superposed waves were equal, and the two amplitudes are denoted a_n og a_m in the following.

The differences in wave number and cyclic frequency are denoted $\Delta\,k_{nm} = k_n - k_m$ and $\Delta\,\omega_{nm} = \omega_n - \omega_m$, respectively. The 2. order transfer function is denoted G_{nm}, and the elevation of the bounded long wave, denoted η_{nm}, is determined by:

$$\eta_{nm}(t) = G_{nm}\,a_n\,a_m\,\cos(\Delta\,\omega_{nm}\,t - \Delta\,k_{nm}\,x) \tag{40}$$

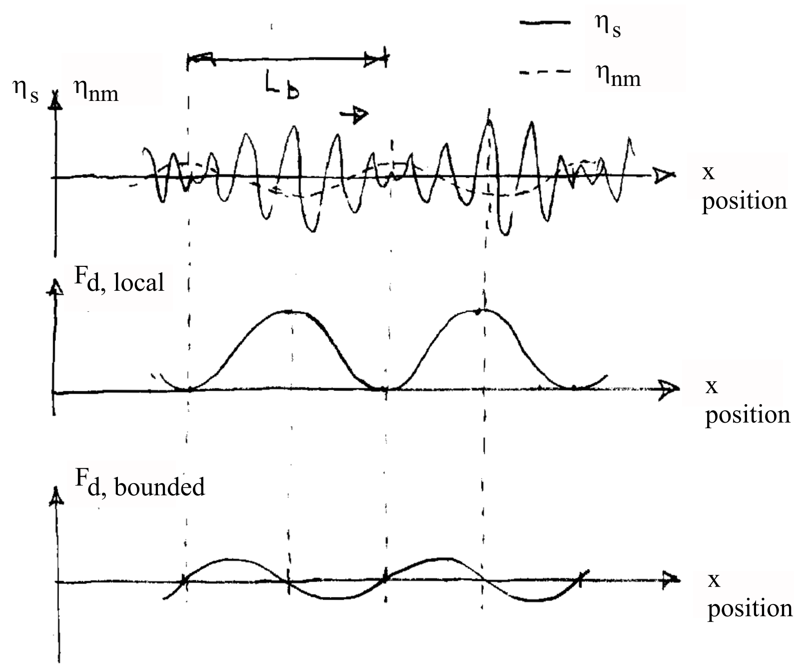

Figure 7: Long section of wave groups and associated drift forces on a body

The general, somewhat complicated, expression for G_{nm} is described most clearly by Sand (1982).

Here we just mention that on *shallow water* the transferfunction reads:

$$G_{nm} = \frac{-3g}{8\,(\,h\,\pi\,f_n)^2} \cdot \frac{1}{1 + \frac{f_n - f_m}{f_n}} \tag{41}$$

From equation (41) it is seen that the lower the water depth h and the frequency f_n are, the larger becomes the transfer function G_{nm} for a fixed value of the parameter $f_n/(f_n - f_m)$. This parameter is by the way approximately equal to the number of of waves per wave group.

The effect of the bounded long waves on a floating body may be estimated by considering the slope of the bounded waves.

Figure 7 shows a long section of the wave groups at a given time. As a floating body always will slide downward on a sloping surface, the forces from the bounded long wave will vary in space as depicted on Figure 7. This force is out of phase with the locally "constant" drift force (calculated e.g. as the "half wave" drift force described in the previous section) also shown on Figure 7.

The sum of the two forces is a slowly varying drift force having an amplitude and phase depending of the relative size of the two components. Usually the maximum drift force will occur shortly *after* the largest waves in a wave group reaches the body.

If the length of the bounded long wave is vary large compared to the body dimensions one might approximate the drift force form the bounded wave by integration of the pressure gradient over the submerged volume of the body (according to the ”minor gradient theorem”).

Notice that in laboratory experiments it is very difficult to avoid reflection of the bounded long waves. Due to the long wave length any physical possible spending beach will act nearly as a vertical wall corresponding to nearly 100% reflection.

The reflected waves are not bounded to any wave groups, because the individual waves in the groups were absorbed quite well. The reflected long wave will propagate as a free long wave, i.e. the propagation velocity fulfills the ordinary dispersion equation. In the lab basin we therefore have a mixture of bounded and free long waves propagating with different velocities, and in general the phase shift between wave groups and the resulting long wave will vary from place to place.

3.3 Principal structure of the expression for drift forces in irregular waves

We consider a fixed body exposed to irregular waves composed of N superposed 1. order waves. At a given place the expression for the surface elevation reads:

$$\eta(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t + \delta_i) \tag{42}$$

We want to determine the 2. order expression for the drift forces and include only components having the order of magnitude $o((H/L)^2)$.

Because all pair of regular waves form a wave group, the total wave field also includes all long periodic bounded waves corresponding to these pairs of regular waves.

The causes to drift forces reads:

1) pressure forces integrated up to the actual surface elevation (*not* to $z = 0$)
2) the term $-\frac{1}{2}\rho v^2$ is included in Bernoullis equation, when the pressure p is calculated
3) pressure forces from pressure gradients caused by bounded long waves. This term is sometimes named the 2. order term in p^+, i.e. the term $-\rho\, \partial\varphi^{(2)}/\partial t$.

Ad 1) This 2. order term may be calculated on the basis of 1. order expressions of surface elevation and velocity. From 1. order theory is known that the pressure deviation from hydrostatic pressure is

$$p^+ = \rho\, g\, \eta \qquad \text{for} \quad z = 0 \tag{43}$$

As p^+ varies slowly with z, we may apply the following approximation for the pressure:

$$p = p^+ - \rho\, g\, z \approx \rho\, g\, \eta - \rho\, g\, z = \rho\, g(\eta - z) \qquad \text{for} \quad \eta_{min} < z < \eta_{max} \tag{44}$$

This approximation corresponds to a hydrostatic pressure distribution with $p = 0$ for $z = \eta$.

The force pr. m from this pressure distribution, denoted P_η, reads:

$$P_\eta = \int_0^\eta p\, dz \approx \int_0^\eta \rho\, g(\eta - z)\, dz = \rho\, g \left| \eta\, z - \frac{z^2}{2} \right|_0^\eta = \frac{1}{2}\rho\, g\, \eta^2 \tag{45}$$

Insertion of the expression for η, equation (42), gives :

$$\begin{aligned}\eta^2 &= \left(\sum_{n=1}^{N} a_n \cos(\omega_n t + \delta_n)\right)\left(\sum_{m=1}^{N} a_m \cos(\omega_m t + \delta_m)\right) \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n\, a_m \cos(\omega_n t + \delta_n)\cdot\cos(\omega_m t + \delta_m)\end{aligned}$$

From the trigonometric expression

$$\cos\alpha\cdot\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta)) \tag{46}$$

is seen that η^2 has components with the frequencies $\omega_n + \omega_m$ and $\omega_n - \omega_m$. The importance of the high frequency terms is normally little for large bodies as they only result in small forces at frequencies far away from the natural frequencies of e.g. a moored ship. In the following only low frequency components are taken into account and in this case the expression for η^2 reads:

$$\eta^2 = \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{1}{2} a_n\, a_m \cos[(\omega_n - \omega_m)\, t + \delta_n - \delta_m] \tag{47}$$

The resultant of pressure forces in the waterline is found by integration of the components along the waterline of the body. This gives:

$$\begin{aligned}\vec{F}_\eta(t) &= \int_0^l \frac{1}{2}\rho\, g\, \eta^2(l,t)\, \vec{n}(l)\, dl \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n\, a_m\, (\vec{F}_{nm})_\eta \cos[(\omega_n - \omega_m)\, t + (\delta_{nm})_\eta]\end{aligned} \tag{48}$$

where

$\vec{n}(l)$ is the normal to body in the waterline at station l
$(\vec{F}_{nm})_\eta$ is the amplitude (= a 2. order transfer function) of the resultant of pressure forces in the waterline, caused by cosine-waves with unit amplitudes at the frequencies ω_n and ω_m
$(\delta_{nm})_\eta$ is phase shift.

Ad 2)

In a similar way it may be shown that the term $-\frac{1}{2}\rho\, v^2$ in the expression for pressure gives the resultant force:

$$\begin{aligned}\vec{F}_v(t) &= \int_A -\frac{1}{2}\rho\, v^2(t)\, \vec{n}\, dA \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n\, a_m\, (\vec{F}_{nm})_v \cos[(\omega_n - \omega_m)\, t + (\delta_{nm})_v]\end{aligned} \tag{49}$$

where

$(\vec{F}_{nm})_v$ is the amplitude (= 2. order transfer function) of the resultant of the nonlinear pressure forces caused by by cosine-waves with unit amplitudes at the frequencies ω_n and ω_m

$(\delta_{nm})_v$ is phase shift.

Ad 3)

The force from the bounded long waves, denoted $\vec{F}_b(t)$, reads:

$$\vec{F}_b(t) = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \, a_m \, (\vec{F}_{nm})_b \, \cos((\omega_n \, - \, \omega_m) \, t + (\delta_{nm})_b) \tag{50}$$

where

$(\vec{F}_{nm})_b$ is the amplitude (=2. order transfer function) of the resultant of the forces from the bounded long wave created by the cosine-waves with unit amplitudes at the frequencies ω_n and ω_m

$(\delta_{nm})_b$ is phase shift.

Summation of the three types of drift force components gives the *total* drift force reads:

$$\begin{aligned} \vec{F}_d(t) &= \\ &= \vec{F}_\eta(t) + \vec{F}_v(t) + \vec{F}_b(t) \\ &= \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \, a_m \, (\vec{F}_{nm})_d \cos[(\omega_n \, - \omega_m) \, t + (\delta_{nm})_d] \end{aligned} \tag{51}$$

hvor

$(\vec{F}_{nm})_d$ is the total 2. order transfer function between waves and drift force

$(\delta_{nm})_d$ is phase shift.

All in all it is seen that the total drift force in irregular waves may be interpreted as the sum of the drift forces from all the wave groups which may be formed by pairing of the N regular waves, which form the irregular sea state.

However, one thing is to realize the principal structure of the expression for the the drift force. Another thing is to calculate $(\vec{F}_{nm})_d$ and $(\delta_{nm})_d$ in practice. This turns out to be very difficult, and especially the component from the bounded long wave causes severe computational problems.

Often the expression for the component of the total drift force in a given direction is reformulated to

$$F_d(t) = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \, a_m \, P_{nm} \cos(\omega_n \, - \omega_m) \, t + a_n \, a_m \, Q_{nm} \sin(\omega_n \, - \omega_m) \, t \tag{52}$$

Here the terms with P_{nm} represents forces *in* phase with the wave groups and the terms with Q_{nm} represents forces *out of* phase with the wave groups.

The time average of $F_d(t)$ reads:

$$\overline{F_d} = \left(\frac{1}{T} \int_0^T F_d(t)\, dt \right)_{T \to \infty} = \sum_{i=1}^{N} a_i^2 \, P_{ii} \tag{53}$$

because all other terms are zero.

In the special case $\eta = a_1 \cos \omega_1 t$, equation (53) reads: $\overline{F_d} = a_1^2 \, P_{11}$. Previously this force was named the constant drift force in regular waves with frequency ω_i.

From equation (53) is therefore seen that the time average of the slowly-varying drift force in irregular waves is the sum of constant drift forces from the N regular waves, which form the sea state.

Example on 2. order transfer function.

On deep water we consider a very long body exposed to regular waves with amplitude a_1. The body completely reflects all incident wave energy. As shown in section 2.5 the constant drift force reads:

$$\overline{F_d} = \frac{1}{2} \rho \, g \, a_1^2 \tag{54}$$

With only one wave component ($N = 1$), equation (53) again reads

$$\overline{F_d} = a_1^2 \, P_{11} \tag{55}$$

If the equations (54) and (55) are compared, we find that

$$P_{11} = \frac{1}{2} \rho \, g \tag{56}$$

In this example the transfer function is independent of the frequency. This is only correct in case of complete reflection.

3.4 Newman's approximation

It is very difficult to calculate P_{nm} og Q_{nm} in equation (52), and in practice approximations are necessary.

In this context one should note that the importance of bounded long waves is low on deep water, and that their effect on shallow water mainly is a phase shift of the total drift force, see for example Figure 7.

Consequently Newman (1974) assumed the size of the forces from the bounded long waves to be negligible. This corresponds to assume the drift forces to be in phase with the wave groups, or to set $Q_{nm} = 0$ in the expression for $F_d(t)$.

Furthermore Newman assumed that $P_{nm} = P_{nn}$, which is a sound assumption if $\omega_n \approx \omega_m$, where the waves in the corresponding wave group are nearly regular waves with the frequency $\omega = (\omega_n + \omega_m)/2 \approx \omega_n$.

Through this it is possible to calculate an estimate of the slowly-varying drift force $F_d(t)$ based alone on the constant drift forces from the N regular waves forming the actual sea state.

If we furthermore assume $a_n = a_m$, the maximum value of the slowly-varying drift force may be approximated to the drift force in regular waves with amplitude $a_n = a_m$, giving

$$(F_d(t))_{max} = (2\,a_n)^2 P_{nn} \tag{57}$$

As the slowly-varying drift force in the wave group varies between this value and zero, the amplitude is equal to

$$a_d = \frac{1}{2}\,(2\,a_n)^2 P_{nn} = 2a_n^2 P_{nn} \tag{58}$$

As $\cos(\omega_n - \omega_m)\,t = \cos(\omega_m - \omega_n)\,t$, the correct expression for the drift force at the frequency $\omega_n - \omega_m$ reads:

$$\begin{aligned} F_d(t) &= (a_n\,a_m\,P_{nm} + a_m\,a_n\,P_{mn})\cos(\omega_n - \omega_m)\,t \\ &= a_n\,a_m(P_{nm} + P_{mn})\cos(\omega_n - \omega_m)\,t \end{aligned}$$

In practice it is only possible to determine the amplitude of $F_d(t)$ and thereby only $P_{nm} + P_{mn}$. We may therefore choose the relative size arbitrarily. It is decided to choose

$$P_{nm} = P_{mn} \tag{59}$$

A comparison between the estimated and the correct force amplitude gives in the case $a_n = a_m$:

$$a_d = a_n\,a_n(P_{nm} + P_{mn}) \approx 2\,a_n^2 P_{nn} \tag{60}$$

or

$$P_{nm} \approx P_{nn} \tag{61}$$

In case $a_n \neq a_m$ Newman's assumptions are not correct. If ω_n og ω_m deviate considerably it has to be expected that Newman's force estimate is rather uncertain.

The importance of this uncertainty is probably low in practice, because the frequency of the drift force is rather high and the corresponding dynamic amplification rather low. Only drift forces with frequencies close to the natural frequencies of the mooring system are important due to the large dynamic amplification.

Newman's assumptions therefore gives this expression for the slowly-varying drift force:

$$\boxed{F_d(t) = \sum_{n=1}^{N} \sum_{m=1}^{N} a_n \, a_m \, P_{nn} \cos(\omega_n - \omega_m) \, t} \tag{62}$$

Based on the physics mentioned above it has to be expected that equation (62) will give rather accurate force estimates on deep water and less accurate force estimates on shallow water.

This has been confirmed in the few cases, where it was possible to calculate the slowly-varying drift force correctly, see e.g. Faltinsen (1978).

Standing et. al. (1981) gave a slightly modified version of Newman's assumptions. As the modified version is physically reasonable and the results are in better agreement with the results in e.g. Faltinsen (1978), the modified version is described below.

In equation (62) it is surprising that P_{nm} is unaffected by the frequency ω_m.

This is achieved, however, if we instead apply this estimate:

$$\boxed{P_{nm} \approx P_{\frac{n+m}{2} \frac{n+m}{2}}} \tag{63}$$

where the terms the outside the diagonal in P_{nm}-matricen are estimated by the diagonal terms corresponding to the average frquency.

3.5 Calculation of the slowly-varying drift force by use of the Boundary Element Method

In general it is very difficult to calculate the slowly-varying drift forces correctly. By use of the Boundary Element Method (BEM) it is quite easy to fulfill the nonlinear boundary conditions at the free surface and at the surface of the body. It is also quite easy to integrate pressure forces up the the actual position of the free surface and to include the nonlinear velocity term in the calculation of pressure.

BEM may therefore include all nonlinearities that can be described by potential theory.

Unfortunately BEM is very demanding with respect to computer resources (time as well as memory), and on deep water there are unsolved problems with unwanted reflection of bounded long waves at the open boundaries of the computational area. However in a numerical model this reflection is of less importance than in a corresponding physical model.

Therefore it is still expected that BEM may be used to achieve accurate examples of drift forces. These examples may then be used to asses the quality of other approximate, less demanding methods. In Brorsen & Larsen (1987) and Brorsen & Bundgaard (1990) nonlinear BEM in 2D models have been used to calculate nonlinear waves and drift forces on floating bodies.

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