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Abstractions for Mechanical Systems [*]

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Abstract: This paper proposes a method for discretizing the state space of mechanical systems. This is a first attempt in using reduction techniques for mechanical systems in the partitioning of the state space. The method relies on a combination of transversal and tangential manifolds for the conservative mechanical system. The tangential manifolds are generated using constants of motion, which can be derived from Noether's theorem. The transversal manifolds are subsequently generated on a reduced space given by the Routhian, via action-angle coordinates. The method fully applies for integrable systems.

We focus on a particular aspect of abstraction - partitioning the state space, as existing methods can be applied on the discretized state space to obtain an automata-based model. The contribution of the paper is to show that well-known reduction methods can be used to generate abstract models, which can be used for formal verification.

Keywords: Mechanical Systems, Formal Verification, Reduction, Reachability, Timed Automata.

1. INTRODUCTION

In the design of a safety-critical system, it is vital to formally verify the system before its deployment. Formal verification can be used to prove the safety of a system, i.e., that no admissible solution trajectories reach a forbidden subset of the state space, see Guéguen et al. (2009) for a survey.

There exist lots of methods for verifying different properties of systems, and the choice of a method should be done based on the dynamics of the considered system and the properties that should be verified. A method for verifying timed and temporal specifications of timed automata is presented in Alur and Dill (1994). In addition, a framework for verifying the safety of more general stochastic hybrid systems, by the use of Lyapunov-like functions called barrier certificates, is presented in Prajna et al. (2007).

To verify the system, we generate an abstract model of a mechanical system based on a partition of the state space. We follow the ideas of Broucke (1998) and generate the partition using invariant sets. However, in contrast to Broucke (1998), we provide a method for generating the tangential and transversal manifolds used in the partitioning for mechanical systems. To allow the verification of timed and temporal specifications, we abstract the system by a timed automaton instead of a directed graph, which is most commonly used.

The contribution of this paper is to show that symmetry reduction techniques from mechanics can be utilized for realizing the partition used for abstracting dynamical systems presented in Section 2. We apply Lagrange-

D'Alembert's principle to model the mechanical system. At first, we remove all dissipation and discretize the resulting conservative system. Using the Lagrangian, we identify cyclic coordinates, and generate tangential manifolds given by constants of motion of the system found via Noether's theorem. Subsequently, we generate transversal manifolds on a reduced system, given by the Routhian, via the use of action-angle coordinates. Afterwards, we add the dissipation and obtain a so-called transversal partition. This gives a finite discretization of the state space for *integrable mechanical systems.* For more general mechanical systems, the effectiveness of the method depends on the symmetries of the system. Note that our approach is elementary as it is accomplished in coordinates, in contrast to abstract coordinate free formalism for reductions in mechanical systems Marsden and Ratiu (1999). The generated abstract model is a timed automaton that can be checked in existing tools; hence, allowing the verification of timed and temporal properties of the mechanical system. To delimit the content of the paper, we only present the method for generating the partition, as the abstract model subsequently can be generated using Sloth and Wisniewski (2011).

This paper is organized as follows. Section 2 contains preliminary definitions, Section 3 explains how to make abstractions for a mechanical system, and Section 4 applies the proposed partition on a model of the inverted pendulum on a cart. Finally, Section 5 comprises conclusions.

2. PRELIMINARIES

The purpose of this section is to provide definitions related to dynamical systems and partitions of state spaces.

2.1 Dynamical System

An autonomous dynamical system $\Gamma = (X, f)$, with state space $X \subseteq \mathbb{R}^n$ and continuous map $f : X \to \mathbb{R}^n$, has dynamics described by ordinary differential equations

$$\dot{x} = f(x). \tag{1}$$

Let $\phi_\Gamma : [0, \epsilon] \times X_0 \to X$, $\epsilon > 0$ be the flow map satisfying

$$\frac{d\phi_\Gamma(t, x_0)}{dt} = f(\phi_\Gamma(t, x_0)) \tag{2}$$

for all $t \in [0, \epsilon]$ and $x_0 = \phi_\Gamma(0, x_0)$.

For a map $f : A \to B$, and a subset $C \subseteq A$, $f(C) \equiv \{f(x) | x \in C\}$. Thus, the reachable set of a system Γ from a set of initial states $X_0 \subseteq X$ on the time interval $[t_1, t_2]$ is defined as

$$\phi_\Gamma([t_1, t_2], X_0). \tag{3}$$

2.2 Partitioning

A definition of a cell, generated by a collection of functions is given below, where $\boldsymbol{k}$ denotes the set $\{1, \ldots, k\}$.

Definition 1. (Cell). Let $\Phi = \{\varphi_i : \mathbb{R}^n \to \mathbb{R} | i \in \boldsymbol{k}\}$ be a collection of functions, let $\mathcal{A} = \{A_i | i \in \boldsymbol{k}\}$ be a collection of sets of regular values, where $A_i = \{a_i^j \in \mathbb{R} | j \in I_i \subseteq \mathbb{N}\}$ is a set of regular values of φ_i and $a_i^j < a_i^{j'}$ if $j < j'$. Assume that the level sets $\varphi_i^{-1}(a_i^j)$ and $\varphi_{i'}^{-1}(a_{i'}^{j'})$ intersect transversally for all $i \neq i'$, $j \in I_i$, and $j' \in I_{i'}$. Then a connected component of

$$\bigcap_{i=1}^{k} \varphi_i^{-1}([a_i^j, a_i^{j'}]) \tag{4}$$

with $a_i^j, a_i^{j'} \in A_i$ and $j < j'$ is called a cell.

In the definition, I_i indexes regular values of φ_i. A finite partition $E(\Phi, \mathcal{A})$ is defined to be the collection of all cells generated by Φ and $\mathcal{A}$ according to Definition 1.

Definition 2. (Transversal Partition). Let X be an open connected subset of $\mathbb{R}^n$. Suppose $f : X \to \mathbb{R}^n$ is continuous and let $\mathrm{Cr}(f)$ be the set of critical points of f (equilibria). Let $\Phi = \{\varphi_i : X \to \mathbb{R} | i \in \boldsymbol{k}\}$ be a set of real differentiable functions, where

$$L_f \varphi(x) \equiv \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(x) f^j(x) \tag{5a}$$

and let $\mathcal{A} = \{A_i | i \in \boldsymbol{k}\}$ be a collection of sets of regular values. Then the finite partition $E(\Phi, \mathcal{A})$ is said to be transversal (we call it a transversal partition) if for each cell $e \in E(\Phi, \mathcal{A})$ there is a partitioning function $\varphi_i \in \Phi$ such that

$$L_f \varphi_i(x) \neq 0 \ \forall x \in e \backslash \mathrm{Cr}(f) \tag{5b}$$

and for all $i \in \boldsymbol{k}$

$$L_f \varphi_i(x) = 0 \ \forall x \in \mathrm{Cr}(f). \tag{5c}$$

It is seen from (5b) that at least one partitioning function has to have nonzero gradient in each cell (5b); hence, the vector field should be transversal to the level sets of at least one partitioning function. This is important in the generation of time information for the abstraction.
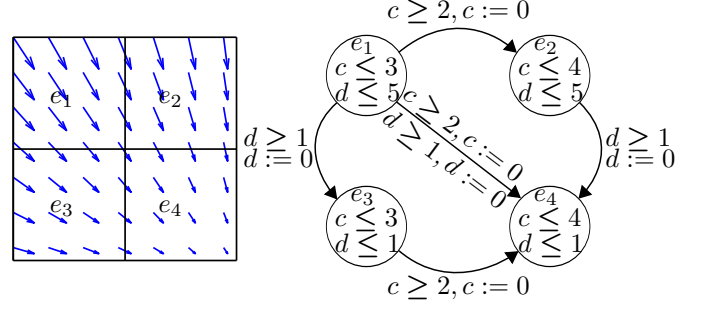

Fig. 1. A phase plot of a system and a partition of its state space. The behavior of the system is abstracted by a timed automaton.

2.3 Abstraction

To motivate the proposed abstraction procedure, it is briefly explained in the following. For details see Sloth and Wisniewski (2011).

Consider a state space partitioned into a number of cells, as shown in Fig. 1. The abstraction procedure consists of first generating discrete locations E, representing the cells. Second, an edge is added between two locations if there exists a trajectory initialized in one cell that eventually reaches the adjacent cell. Finally, time information is added as guards and invariants to quantify "eventually reached". A timed automaton is illustrated in Figure 1. The locations are denoted by $e_1, \ldots, e_4$; there are two clocks denoted by c and d. The transition between location e_1 and e_2 may happen whenever the clock $c \geq 2$ and must take place before the clock c exceeds 3. Once this transition occurs, the clock c resets to 0. The timed automaton $\mathcal{T}$ models the system by bounding the time that a trajectory can stay within a cell. Note that the clock valuations of c and d should be initialized appropriately to represent the initial configuration. We denote the reachable locations from initial locations E_0 on the time interval $[t_1, t_2]$ by

$$\phi_\mathcal{T}([t_1, t_2], E_0). \tag{6}$$

To the partitioning E, we associate an abstraction function, which to each point in the state space associates the cells that this point belongs to.

Definition 3. (Abstraction Function). Let $\Lambda \subset \mathbb{N}$ be a finite index set, and $E \equiv \{e_\lambda | \lambda \in \Lambda\}$ be a finite partition of the state space X. An abstraction function for E is the multivalued function defined by

$$\alpha_E : X \to 2^E, \ \alpha_E(x) = \{e \in E | x \in e\}.$$

To be able to draw conclusions about the original system Γ based on the abstraction $\mathcal{T}$, it is essential to determine how the two models are related. This relation is given in terms of reachable sets in the following.

Definition 4. Let $\Gamma = (X, f)$ be a dynamical system, and suppose its state space X is partitioned by $E = \{e_1, \ldots, e_k\}$. Let the initial states $X_0 = \bigcup_{i \in \mathcal{I}} e_i$, with $\mathcal{I} \subseteq \boldsymbol{k}$. Then an abstraction $\mathcal{T}$ with locations E and initial locations $E_0 = \{e_i | i \in \mathcal{I}\}$ is said to be

(1) **sound** on an interval $[t_1, t_2]$ if
$$\alpha_E \circ \Phi_\Gamma(t, X_0) \subseteq \Phi_\mathcal{T}(t, \alpha_E(X_0)), \text{ for all } t \in [t_1, t_2]$$
(2) **complete** on an interval $[t_1, t_2]$ if
$$\alpha_E \circ \Phi_\Gamma(t, X_0) = \Phi_\mathcal{T}(t, \alpha_E(X_0)) \text{ for all } t \in [t_1, t_2].$$

A sound abstraction $\mathcal{T}$ of Γ reaches all locations reached by $\Gamma = (X, f)$, and a complete abstraction $\mathcal{T}$ reaches exactly the same locations as Γ.

3. METHOD

In this section, the method for discretizing mechanical systems is presented. The discretization is accomplished in the following steps

(A) Discard all dissipation of the system and partition the state space of the conservative system using tangential and transversal manifolds.
(B) Add dissipation and select level sets to obtain a transversal partition, see Definition 2.
(C) Generate a timed automaton abstracting the system, according to Wisniewski and Sloth (2011).

We show in Proposition 2 that a transversal partition generated by n partitioning functions can be realized for integrable systems using the presented procedure.

First, we consider the mechanical system without dissipation, as this enables the identification of cyclic coordinates and first integrals or constants of motion. The constants of motion are functions with level sets being tangential manifolds; hence, they are used as partitioning functions. Then the model is reduced using Routh reduction Goldstein (1960). This reduced space is partitioned using transversal manifolds, which are generated via action-angle coordinates. Finally, we add dissipation to the system. This implies that the system trajectories no longer are confined to a certain constant of motion. Instead, the system trajectories traverse the manifolds according to dynamics described by the dissipation. This partition is shown to be transversal.

3.1 Discretizing Conservative Mechanical System

The aim of this subsection is to provide guidance for finding $2n$ mutually transversal partitioning functions for a conservative mechanical system with n degrees of freedom. It is required to find $2n$ mutually transversal partitioning functions, i.e., functions whose gradients are linearly independent at each point (except of critical points), to obtain arbitrary accuracy of the abstraction. The method consists of the following steps

(1) Identify cyclic coordinates from the Lagrangian.
(2) Find tangential manifolds via Noether's theorem.
(3) Reduce the system using Routh reduction.
(4) Find transversal manifolds for the reduced system using action-angle coordinates.

We assume that the mechanical system with n degrees of freedom is described by n Euler-Lagrange equations of motion in generalized coordinates.

Identification of Cyclic Coordinates   Recall that a coordinate q^i is said to be cyclic if the Lagrangian of a system does not depend on it.

From the Lagrangian of a system, it is seen that $\partial L/\partial q^i = 0$ if q^i is cyclic; hence, the generalized momentum $\partial L/\partial \dot{q}^i$ is constant. This means that cyclic coordinates identify symmetries of the system, where a symmetry is a transformation that generates a displacement under which the

system is invariant, e.g., a translation along a cyclic coordinate. Therefore, each cyclic coordinate should be partitioned independently of the other coordinates, i.e., if q^i is a cyclic coordinate then

$$\varphi : (q, \dot{q}) \mapsto q^i \qquad (7)$$

should be used as partitioning function. The cyclic coordinate should be discarded in the remainder of the partitioning procedure.

Identification of Tangential Manifolds   Broucke (1998) proposes a partition based on tangential and transversal manifolds, generated by foliations. This method is based on the local existence of $m - 1$ linear independent tangential manifolds on $\mathbb{R}^m$.

Definition 5. Suppose $N_1, \ldots, N_k$ are co-dimension 1 submanifolds of $\mathbb{R}^m$, and let $\nu(N_i, \mathbb{R}^m)$ be normal bundles of N_i (in $\mathbb{R}^m$) (Lee, 2000, p. 253). Then $N_1, \ldots, N_k$ are said to be linear independent manifolds, if for any $x \in \bigcap_{i \in \boldsymbol{k}} N_i$ there exist $(x, v_i) \in \nu(N_i, \mathbb{R}^m)$, $i \in \boldsymbol{k}$ with $v_i \neq 0$, v_i are linearly independent.

It is seen that the normal to the linear independent manifolds are linearly independent at each point of their intersection. From Flow Box Theorem, see Junior and de Melo (1980), it is seen that locally there exist $m - 1$ tangential manifolds and one transversal manifold in the neighborhood of a regular point.

We are interested in constructing the tangential manifolds without the use of solutions of the differential equations. This motivates the identification of tangential manifolds, via the Euler-Lagrange equations. In contrary to the local analysis, the presented method may not identify $2n - 1$ constants of motion; however, the tangential manifolds are identified globally. The number of constants of motion that one can find for a given system is not a priori known.

Following (Arnold, 1989, p. 207), the function H is a first integral of the Hamiltonian phase flow with Hamiltonian function H. This implies that we can always find one constant of motion: the Hamiltonian. The Hamiltonian function should be used as a tangential partitioning function

$$\varphi(q, \dot{q}) = H(q, \dot{q}). \qquad (8)$$

An integrable system has, per definition, n linear independent tangential manifolds. These are also called functionally independent constants of motion. The Poisson bracket is used in the definition of an integrable system. Recall that given two smooth real-valued functions A and B defined on the phase space of a Hamiltonian system, the canonical Poisson bracket of A and B is defined by

$$\{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q^i} \frac{\partial A}{\partial p_i} \right), \qquad (9)$$

where (q^i, p_i) are conjugate pairs of canonical coordinates Marsden (1992).

Definition 6. (Integrable System). A Hamiltonian systems in a $2n$-dimensional symplectic manifold is said to be integrable (in the Arnold-Liouville sense) if there exist n functionally independent constants of motion that are in involution, meaning that they pairwise satisfy

$$\{F_i, F_j\} = 0 \quad \forall i, j. \qquad (10)$$

Constants of motion can be found, by using the symmetries of the system, given by the cyclic coordinates, according to the following theorem (Arnold, 1989, p. 88).

Theorem 1. (Noether's Theorem). Let M be a smooth manifold, $L : TM \to \mathbb{R}$ a smooth function on its tangent bundle TM. If the system (M, L) admits the one-parameter group of diffeomorphisms $h_s : M \to M$, $s \in \mathbb{R}$, then the lagrangian system of equations corresponding to L has a first integral $I : TM \to \mathbb{R}$. In local coordinates q on M the integral I is written in the form

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{dh_s(q)}{ds}\bigg|_{s=0}. \tag{11}$$

From the theorem, it is seen that we can find one constant of motion per cyclic coordinate, as the generalized momentum $\partial L / \partial \dot{q}^i$ is constant if q^i is cyclic; hence, the generalized momentum should be used as a tangential partitioning function

$$\varphi(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}. \tag{12}$$

In relation to Theorem 1, let $M = \mathbb{R}^n$ and let the first coordinate be a cyclic coordinate, then $h_s : (q^1, \ldots, q^n) \mapsto (q^1 + s, \ldots, q^n)$ is a one-parameter group. Note that symmetry under translation corresponds to momentum conservation, symmetry under rotation to angular momentum conservation, symmetry in time to energy conservation Butterfield (2005).

Reduction of the System   The remaining partition should be conducted on a reduced state space, given by the following theorem, which can be used to restrict the dynamics of a system to a lower dimensional surface using constants of motion, Langerock et al. (2010).

Theorem 2. (Routh Reduction).
Let $L : \mathbb{R}^{2n} \to \mathbb{R}$ be the Lagrangian for a system with n degrees of freedom. Assume that q^1 is a cyclic coordinate and that locally $\partial^2 L / \partial \dot{q}^1 \partial \dot{q}^1 \neq 0$ so that $\dot{q}^1$ can be expressed as $\dot{q}^1 = \varrho(q^2, \ldots, q^n, \dot{q}^2, \ldots, \dot{q}^n)$. Consider the Routhian $R^\mu : \mathbb{R}^{2(n-1)} \to \mathbb{R}$ defined as the function $R^\mu = L - \dot{q}^1 \mu$, where all instances of $\dot{q}^1$ are substituted by the function ϱ and momentum $p_1 = \mu$. The Routhian is now interpreted as the Lagrangian for a system with $(n-1)$ degrees of freedom $(q^2, \ldots, q^n)$.

Any solution $(q^1(t), \ldots, q^n(t))$ of the Euler-Lagrange equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, i = 1, \ldots, n \tag{13}$$

with momentum $p_1 = \mu$, projects onto a solution $(q^2(t), \ldots, q^n(t))$ of the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial R^\mu}{\partial \dot{q}^k} - \frac{\partial R^\mu}{\partial q^k} = 0, k = 2, \ldots, n. \tag{14}$$

Conversely, any solution of the Euler-Lagrange equations for R^μ can be lifted to a solution of the Euler-Lagrange equations for L with momentum $p_1 = \mu$.

Using Theorem 2, we can obtain Euler-Lagrange equations of reduced dimension, which should be used in the generation of the transversal manifolds.

The idea of Routh reduction is to use the constants of motion as coordinates in the system description. This enables the system to be analyzed using fewer coordinates,

as the system has no dynamics in the coordinates given by the constants of motion. The concept is shown in Fig. 2.
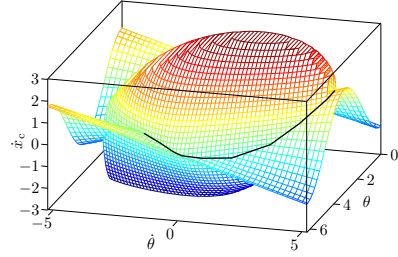

Fig. 2. The two surfaces are level sets of the constants of motion. The black line is the simulated trajectory.

The figure illustrates two constants of motion and a solution trajectory (black line) that is located at their intersection; hence, the solution can be described using only one coordinate (apart from the constants of motion).

Identification of Transversal Manifolds   We have not found general method for finding transversal manifolds; however, for integrable systems, we can find transversal manifolds via the use of action-angle coordinates.

Theorem 3. (Jose and Saletan (1998)). Consider a completely integrable Hamiltonian system with constants of motion $C_1(q, p) = H(q, p)$, $C_1(q, p), \ldots, C_n(q, p)$ which are in involution. The hypersurfaces given by sets of constants $c = \{c_i | i \in \boldsymbol{n}\}$

$$S(c) = \{(q, p) \in T^* \mathbb{Q} | C_i(q, p) = c_i, i = 1, \ldots, n\} \tag{15}$$

are invariant under the flow of the Hamiltonian system. If $S(c)$ is compact and connected, then $S(c)$ can be mapped in a diffeomorphic way on a n-torus $T^n = S^1 \times \cdots \times S^1$. Each circle can be described by an angle coordinate $\theta_i(t)$ with dynamics

$$\frac{d\theta_i}{dt} = \Omega_i(c), i = 1, \ldots, n. \tag{16}$$

From the theorem we see that for integrable systems one can find a coordinate system, where n coordinates are given by constants of motion and n coordinates which are independent of each other and are given by trivial dynamics. For each action-angle θ_i, a transversal partition function

$$\varphi_i(q, p) = \theta_i \tag{17}$$

should be used in the partitioning of the state space. For details in the synthesis of the coordinate transformation, see Jose and Saletan (1998).

Note that the proposed method does not provide $2n$ linear independent partitioning functions for all systems; however, for integrable systems they can be found via Theorem 3. Therefore, the proposed partition can be applied to partly partition a state space, and then the remaining part of the state space can be partitioned using, e.g., hypercubes as used in most other abstraction procedures.

3.2 Obtaining Transversal Partition

The final step of the partitioning procedure is to check if the partition is transversal. We show that a transversal partition can always be found for integrable systems.

Proposition 1. Let the system (M, L) be defined as shown in Theorem 1 with first integral in local coordinates (U, ς)

$$I(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{dh_s(q)}{ds}. \tag{18}$$

Then by adding external forces Q to the system, the time derivative of I becomes

$$\frac{d}{dt} I(q, \dot{q}) = Q \frac{dh_s(q)}{ds}. \tag{19}$$

Proof 1. Let $\phi : \mathbb{R} \to U$, $q = \phi(t)$ be a local solution to the Lagrange equation. Since h_s preserves L, the translation of a solution, $h_s \circ \phi : \mathbb{R} \to U$ also satisfies the Lagrange equations for any s.

Let the mapping $\Upsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$, given by $q = \Upsilon(s, t) = h_s(\phi(t))$. We will denote the derivatives with respect to t by dots and with respect to s by primes. By hypothesis

$$\frac{\partial L(\Upsilon, \dot{\Upsilon})}{\partial s} = \frac{\partial L}{\partial q} \Upsilon' + \frac{\partial L}{\partial \dot{q}} \dot{\Upsilon}' = 0 \tag{20}$$

where the partial derivatives of L are taken at the point $q = \Upsilon(s, t)$, $\dot{q} = \dot{\Upsilon}(s, t)$.

For any fixed s, $\Upsilon : \mathbb{R} \to \mathbb{R}^n$ satisfies

$$\frac{\partial L}{\partial q}(\Upsilon(s,t), \dot{\Upsilon}(s,t)) = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}(\Upsilon(s,t), \dot{\Upsilon}(s,t))\right) \\ - Q(\Upsilon(s,t), \dot{\Upsilon}(s,t)) \tag{21}$$

By inserting (21) into (20) we get

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\Upsilon' - Q\Upsilon' + \frac{\partial L}{\partial \dot{q}}\dot{\Upsilon}' = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}q'\right) - Q\Upsilon' = 0 \tag{22a}$$

$$\frac{dI}{dt} = Q\Upsilon'. \tag{22b}$$

Proposition 2. Let (M, L) be an integrable system, and let $k = 2n$; thus, $\boldsymbol{k} = \{1, \ldots, 2n\}$. Then there exists a collection of nonempty sets of regular values $\mathcal{A} = \{A_i | i \in \boldsymbol{k}\}$ for the partitioning functions $\Phi = \{\varphi_i(q, \dot{q}) | i \in \boldsymbol{k}\}$ (see Theorem 3), such that the generated partition $K = (\Phi, \mathcal{A})$ is transversal.

Proof 2. Consider cells containing no critical points. Only one of the $2n$ partitioning functions should have a nonzero gradient in each cell. We look at points, where $dI/dt = 0$, i.e., $Q\Upsilon' = 0$. Let $\mathcal{G} = \{(q, \dot{q}) | Q\Upsilon' = 0\}$ and let $\mathrm{Cr}(f)$ be the set of critical points. By the definition of transversality, the transversal partitioning functions $\varphi_i(q, \dot{q})$ are transversal to the vector field at each point in $\mathcal{G} \backslash \mathrm{Cr}(f)$, i.e., for each transversal partitioning function given by (17)

$$L_f \varphi_i(q, \dot{q}) \neq 0 \ \forall (q, \dot{q}) \in \mathcal{G} \backslash \mathrm{Cr}(f). \tag{23}$$

This implies that there exists $\mathcal{A}$ such that the partition is transversal.

4. EXAMPLE

The proposed abstraction is applied to an inverted pendulum. The units are omitted to clarify the presentation.

The cart has a point mass m_c, a position x_c, velocity $\dot{x}_c$, and acceleration $\ddot{x}_c$. The pendulum is modeled as a point mass m_p extended from the cart in a massless rod of length l and has inertia I_p with respect to the point, where the pendulum is attached to the cart. The angle of

the pendulum with respect to the vertical axis is θ. Finally, the cart is affected by a frictional force $F_f = -k\dot{x}_c$. A state space model of the system is shown in (Khalil, 2002, p. 28), and we use the following parameter values: $g = 9.82$, $l = 1$, $m_p = 1$, $m_c = 2$, $I_p = 1$, $k = 1$. The Lagrangian is

$$L(x) = \frac{1}{2}(m_c + m_p)\dot{x}_c^2 + \frac{1}{2}(I_p + m_p l^2)\dot{\theta}^2 + m_p \dot{x}_c \dot{\theta} l \cos\theta \\ - m_p g l \cos\theta. \tag{24}$$

4.1 Discretizing Conservative Mechanical System

Identification of Cyclic Coordinates   The cart position x_c is a cyclic coordinate, as $\partial L / \partial x_c = 0$. The partitioning function for the cyclic coordinate is given by

$$\psi_1(x) = x_c. \tag{25}$$

Identification of Tangential Manifolds   The Hamiltonian is a constant of motion. Therefore, we get the following partitioning function

$$\psi_2(x) = \frac{1}{2}(m_c + m_p)\dot{x}_c^2 + \frac{1}{2}(I_p + m_p l^2)\dot{\theta}^2 + m_p \dot{x}_c \dot{\theta} l \cos\theta \\ + m_p g l \cos\theta. \tag{26}$$

Second, Theorem 1 is used to identify $\partial L / \partial \dot{x}_c$ as a constant of motion, since this is the conjugate momentum corresponding to x_c

$$\psi_3(x) = (m_c + m_p)\dot{x}_c + m_p \dot{\theta} l \cos\theta. \tag{27}$$

The dimension of both $\psi_2^{-1}(a_2)$ and $\psi_3^{-1}(a_3)$ is two, and their intersection is one dimensional (Bredon, 1993, p. 114); hence, there exist no more linearly independent tangential manifolds. Level sets of $\psi_2(x)$ and $\psi_3(x)$ are shown in Fig. 2.

Reduction of the System   The reduced system only depends on the variables θ and $\dot{\theta}$, and has Routhian

$$R^\mu(\theta, \dot{\theta}) = -\frac{1}{2}\frac{\left(\mu - m_p \dot{\theta} l \cos\theta\right)^2}{m_c + m_p} + \frac{1}{2}(I_p + m_p l^2)\dot{\theta}^2 \\ - m_p g l \cos\theta \tag{28}$$

where $\psi_3(x) = \mu$.

Identification of Transversal Manifolds   From $R^\mu(\theta, \dot{\theta})$, the momentum $p_\theta \equiv \partial R^\mu(\theta, \dot{\theta})/\partial \dot{\theta}$ can be expressed in terms of the value of the hamiltonian function $H^\mu(\theta, p_\theta) = E^\mu$ corresponding to $R^\mu(\theta, \dot{\theta})$ and θ

$$p_\theta(\theta, E^\mu) = \pm\sqrt{2b\left(E^\mu + \frac{1}{2}\frac{\mu^2}{m_c + m_p} - m_p g l \cos\theta\right) + a^2} \tag{29}$$

where

$$a = \frac{\mu m_p l \cos\theta}{m_c + m_p}, b = (I_p + m_p l^2) - \frac{(m_p l \cos\theta)^2}{m_c + m_p}.$$

The action variable is defined as

$$J = \oint p \, dq, \tag{30}$$

where the integration is over an entire period. In particular, if the pendulum takes a full rotation, then the integration is from 0 to 2π. Otherwise the integration is from the minimum angle to the maximum angle and

back again. This angle can be found from (29), as the momentum p_θ is zero, when the pendulum reaches its minimum and maximum angle. This expression is solved in Maple, and is a third degree polynomial in $\cos(\theta)$. The expression is not explicitly shown, as it is very long, but the graph of the hamiltonian function H^μ as a function of θ and μ for $p_\theta = 0$ is shown in Fig. 3.
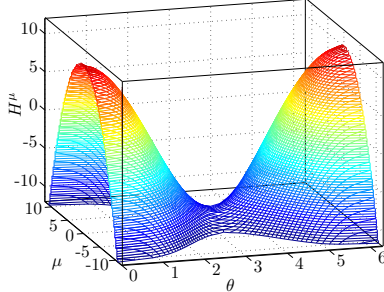

Fig. 3. Graph of H^μ as a function of θ and μ for $p_\theta = 0$.

It is seen that $\theta = \pi$ (hanging downwards), when $\mu = 0$ and $H^\mu = -m_\mathrm{p}gl$ and there are no solutions with lower H^μ when $\mu = 0$. Furthermore, the pendulum swings between θ and $-\theta + 2\pi$ until the value of the hamiltonian function gets greater that $m_\mathrm{p}gl$, where the pendulum starts to do full rotations; hence, $p_\theta \neq 0$ everywhere.

The action variable J only depends on E^μ, not θ. Therefore, $J = J(H^\mu, \mu) = J(E^\mu, \mu)$ and

$$\dot{w} = \frac{\partial H^\mu(J)}{\partial J} = v(J). \tag{31}$$

The value of v as a function of E^μ and μ can be calculated in Maple, and relates to frequency at which the pendulum oscillates. Finally, we get

$$w = v(J)t + \beta. \tag{32}$$

The action variable is not used as a partitioning function, as it is a constant, but $\psi_4(x) = w$ is a transversal partitioning function.

4.2 Generation of Abstraction

We do not show the abstraction of for the inverted pendulum, but a detailed description of the abstraction procedure, including a method for calculating the invariants and guards is shown in Wisniewski and Sloth (2011). We use $L_f\psi_i(x)$

$$L_f\psi_1(x) = \dot{x}_\mathrm{c}, \quad L_f\psi_2(x) = -k\dot{x}_\mathrm{c}, \quad L_f\psi_3(x) = -k\dot{x}_\mathrm{c}^2. \tag{33}$$

The value of $L_f\psi_4(x)$ is again calculated in Maple. Now regular values can be chosen to generate E in accordance with Proposition 2.

The generated timed automaton is a sound abstraction and can be automatically verified by a tool. Therefore, we can verify timed temporal properties of the mechanical system via the verification of the generated timed automaton in tools such as Uppaal or Kronos.

5. CONCLUSION

In this paper, we have provided a constructive method for partitioning the state space of integrable mechanical systems. This partitioning can be used in the abstraction of the mechanical system by a combinatorial model such as a timed automaton. The partition is generated by intersecting tangential and transversal manifolds. The generation of the manifolds is based on reduction techniques for mechanical systems, via the Euler-Lagrange equations and Noether's theorem. The method is applied to the inverted pendulum on a cart, showing its applicability for a nonlinear system. Furthermore, it is shown that a transversal partition can always be obtained for integrable systems using the proposed partition.

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