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Abstract: The notions of controllability and observability play an important role in different problems within feedback control analysis and design. To verify the controllability and observability of a system, several techniques have been introduced. However, often it is not only important to verify if the system is controllable or observable, but also it is required to know the degree of controllability or observability of the system. Gramian matrices were introduced to address this issue by providing a quantitative measure for controllability and observability. In many applications, the information on the controllability and observability properties of a system is needed within a specific frequency interval rather than the whole frequency-domain. The frequency interval gramians provide such information. While this concept were originally introduced for fixed known systems, it needs to be investigated for the case of uncertain systems. In this paper, we derive upper and lower bounds of frequency interval gramians under perturbations of an *A*-matrix in the state-space form. These bounds are obtained by solving algebraic Riccati equations. The results are further used to obtain upper and lower bounds of the frequency interval Hankel singular values for perturbed systems.

1. INTRODUCTION

The controllability and observability are two major and fundamental notions in modern control theory. These concepts play a key role in different problems such as model reduction, optimal control, state estimation. See Dullerud and Paganini [2005]. Standard methods exist in the literature to verify the controllability and/or observability of a system. However, often the quantitative information on the level of controllability and observability are required. In other words, it is important to know how much controllable or observable a system is. Gramian matrices were introduced to address this issue by providing a quantitative measure for controllability and observability. See Antoulas [2005], Gugercin and Antoulas [2004]. These matrices are very popular in model reduction methods such as well-known balanced truncation. The reason is that to apply balanced reduction, first the system needs to be represented in a basis where the states which are difficult to control are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians. See Gugercin and Antoulas [2004] for more details. The gramians are related to the Hankel operator and the Hankel singular values of a dynamical system. The Hankel singular values of dynamical systems play key roles in many fields. One of the practically important relevant fields of application for Hankel singular value is signal processing , in which the Hankel singular values are called the second-order modes. The Hankel singular values provide the optimal dynamic range of analog filters. See e. g. Groenewold [1991]. The optimal dynamic range is the highest ratio of the maximal and minimal signal levels that can be processed in the filters. Furthermore, in digital signal processing that

the Hankel singular values shows the minimum attainable value of roundoff noise and statistical coefficient sensitivity of digital filters. See Mullis and Roberts [1976] and Iwatsuki et al. [1990]. Other interesting application of gramians and the Hankel singular values are in control configuration selection as it is described in Khaki-Sedigh and Moaveni [2009], Salgado and Conley [2004]. The ordinary gramians and Hankel singular values have been extensively studied for both fixed known systems and uncertain systems. See Auba and Funahashi [1992], Xu et al. [1990] , Sojoudi et al. [2009] . However,in many applications, the information on the controllability and observability properties of a system is needed within a specific frequency interval rather than the whole frequency range. The frequency interval gramians provide such information. See Gawronski and Juang [1990], Antoulas [2005] , Gugercin and Antoulas [2004]. The frequency interval gramians and the frequency interval Hankel singular values were introduced and studied for fixed known systems in Gawronski and Juang [1990], Ghafoor and Sreeram [2008], Shaker [2008]. These notions are investigated for the case of uncertain systems in this paper. We derive a bound of frequency interval gramians under perturbations of an *A*-matrix in the state-space representation. For a class of perturbations, both upper and lower bounds are obtained by solving algebraic Riccati equations.The results are further used to obtain upper and lower bounds of the frequency interval Hankel singular values for perturbed systems.

The notation used in this paper is as follows: M^* denotes transpose of matrix if $M \in R^{n \times m}$ and complex conjugate transpose if $M \in C^{n \times m}$. The $\lambda_i(M)$ denotes the i'th eigen value of M. The standard notation $> , \geq (< , \leq)$ is used

to denote the positive (negative) definite and semidefinite ordering of matrices.

2. FREQUENCY INTERVAL CONTROLLABILITY AND OBSERVABILITY GRAMIANS

Consider a dynamic system with minimal realization:

$$G(s) := (A, B, C, D) \tag{1}$$

where $G(s)$ is the transfer matrix with the state-space representation:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t) \in R^n$$
$$y(t) = Cx(t) + Du(t). \tag{2}$$

The ordinary gramians are given by the solutions of the Lyapunov equations,(Antoulas [2005] , Gugercin and Antoulas [2004]):

$$AP + PA^* + BB^* = 0$$
$$A^*Q + QA + C^*C = 0 \tag{3}$$

where P is the ordinary controllability gramian and Q is the ordinary observability gramian. These gramians are related to the so-called Hankel singular values (Antoulas [2005] , Gugercin and Antoulas [2004]):

$$\sigma_i = \sqrt{\lambda_i(PQ)} \tag{4}$$

The controllability gramian shows how much controllable a system is. This is based on the fact that the input energy required for controlling the system is, roughly speaking, proportional to the inverse of the matrix P. The dual energy interpretion holds for the observability gramian Q. The ordinary gramians are quantitative measures for the controllability and observability over the whole frequency domain. The frequency interval gramians focus on a desired frequency interval and in such a way encode more information on the controllability and the observability of the system within the desired frequency interval.

For dynamical system (1) the controllability gramian $P(\omega_1, \omega_2)$ and observability gramians $Q(\omega_1, \omega_2)$ within frequency range $[\omega_1, \omega_2]$ are defined as (Gawronski and Juang [1990] , Gugercin and Antoulas [2004]):

$$P(\omega_1, \omega_2) = P(\omega_1) - P(\omega_2)$$
$$Q(\omega_1, \omega_2) = Q(\omega_1) - Q(\omega_2) \tag{5}$$

where:

$$P(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{+\omega} (Ij\omega - A)^{-1} BB^* (-Ij\omega - A^*)^{-1} d\omega$$

$$\tag{6}$$

$$Q(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{+\omega} (-Ij\omega - A^*)^{-1} C^*C (Ij\omega - A)^{-1} d\omega$$

The frequency interval gramians are the solutions to particular Lyapunov equations. In order to show these Lyapunov equations, more notations are introduced from Gawronski and Juang [1990] , Gugercin and Antoulas [2004]:

$$S(\omega) =: \frac{1}{2\pi} \int_{-\omega}^{+\omega} (Ij\omega - A)^{-1} d\omega \tag{7}$$

$$W_c(\omega) = S(\omega)BB^* + BB^*S^*(-\omega)$$
$$W_o(\omega) = C^*CS(\omega) + S^*(-\omega)C^*C \tag{8}$$

$$W_c(\omega_1, \omega_2) = W_c(\omega_2) - W_c(\omega_1)$$
$$W_o(\omega_1, \omega_2) = W_o(\omega_2) - W_o(\omega_1) \tag{9}$$

The frequency-interval gramians satisfy the following Lyapunov equations (Gawronski and Juang [1990] , Gugercin and Antoulas [2004]):

$$AP(\omega_1, \omega_2) + P(\omega_1, \omega_2)A^* + W_c(\omega_1, \omega_2) = 0$$
$$A^*Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)A + W_o(\omega_1, \omega_2) = 0 \tag{10}$$

The frequency interval Hankel singular values are related to the frequency-interval gramians by:

$$\sigma_i(\omega_1, \omega_2) = \sqrt{\lambda_i(P(\omega_1, \omega_2)Q(\omega_1, \omega_2))} \tag{11}$$

It should be noted that when $(\omega_1, \omega_2) = (-\infty, +\infty)$, we have:

$$P(\omega_1, \omega_2) = P$$
$$Q(\omega_1, \omega_2) = Q$$
$$\sigma_i(\omega_1, \omega_2) = \sigma_i \tag{12}$$

This can be shown using the definitions of the frequency interval gramians and the frequency interval Hankel singular values.

3. BOUNDS OF FREQUENCY INTERVAL GRAMIANS FOR A CLASS OF PERTURBED LINEAR SYSTEMS

Let a class of perturbed linear system be described by:

$$\dot{x}(t) = (A + \Delta A)x(t) + Bu(t), \quad x(t) \in R^n$$
$$y(t) = Cx(t) + Du(t). \tag{13}$$

where (A, B) is a stabilizable pair and ΔA is the uncertainty which belongs to :

$$\Omega := \{\Delta A : \quad \Delta A = G\Sigma H , \bar{\sigma}(\Sigma) < 1\} \tag{14}$$

where G and H are the given matrices and $\bar{\sigma}(\Sigma)$ denotes the largest singular value of Σ .

The frequency interval gramians for the perturbed system are solutions to the following Lyapunov equations:

$$(A + \Delta A)P(\omega_1, \omega_2) + P(\omega_1, \omega_2)(A + \Delta A)^*$$
$$+W_c(\omega_1, \omega_2) = 0$$

$$\tag{15}$$

$$(A + \Delta A)^*Q(\omega_1, \omega_2) + Q(\omega_1, \omega_2)(A + \Delta A)$$
$$+W_o(\omega_1, \omega_2) = 0$$

In the sequel, Theorem 1 introduces an upper bound for the frequency interval controllability gramian of system(13).

Theorem 1: Assume that the algebraic Riccati equation:

$$A\bar{P} + \bar{P}A^* + W_c(\omega_1, \omega_2) + \beta GG^* + \frac{\bar{P}H^*H\bar{P}}{\beta} = 0 \tag{16}$$

has nonnegative stabilizing solution $\bar{P}$ for some $\beta > 0$, then:

$$P(\omega_1, \omega_2) \leq \bar{P} \tag{17}$$

Proof: From (15) and (16), we have the following Lyapunov equation:

$$(A+\Delta A)(\bar{P}-P(\omega_1,\omega_2))+(\bar{P}-P(\omega_1,\omega_2))(A+\Delta A)^*$$
$$+\left(\frac{\bar{P}H^*}{\sqrt{\beta}}-\sqrt{\beta}G\Sigma\right)\left(\frac{\bar{P}H^*}{\sqrt{\beta}}-\sqrt{\beta}G\Sigma\right)^*$$
$$+\beta G(I-\Sigma\Sigma^*)G^*=0$$

It can be easily shown that $\lambda_i(A+\Delta A)<0$. Hence:
$$\bar{P}-P(\omega_1,\omega_2)>0 \qquad (18)$$
□

The equation (16) has a nonnegative definite solution iff the following algebraic Riccati equation:
$$A\hat{P}+\hat{P}A^*+\frac{W_c(\omega_1,\omega_2)}{\beta}+GG^*+\hat{P}H^*H\hat{P}=0 \quad (19)$$

has a nonnegative definite solution. In this case: $\quad\bar{P}=\beta\hat{P}$.

In Theorem 2, a lower bound of the frequency-interval controllability gramian for the perturbed system is obtained.

Theorem 2: Assume that the algebraic Riccati equation:
$$A\underline{P}+\underline{P}A^*+W_c(\omega_1,\omega_2)-\gamma GG^*-\frac{\underline{P}H^*H\underline{P}}{\gamma}=0 \quad (20)$$

has nonnegative stabilizing solution $\underline{P}$ for some $\gamma>0$, then:
$$\underline{P}\leq P(\omega_1,\omega_2) \qquad (21)$$

Proof: From (15) and (20), we have the following Lyapunov equation:
$$(A+\Delta A)(P(\omega_1,\omega_2)-\underline{P})+(P(\omega_1,\omega_2)-\underline{P})(A+\Delta A)^*$$
$$+\left(\frac{\underline{P}H^*}{\sqrt{\gamma}}+\sqrt{\gamma}G\Sigma\right)\left(\frac{\underline{P}H^*}{\sqrt{\gamma}}+\sqrt{\gamma}G\Sigma\right)^*=0$$

Since $\lambda_i(A+\Delta A)<0$. We have:
$$P(\omega_1,\omega_2)-\underline{P}>0 \qquad (22)$$
□

Dually the lower and upper bound on the frequency interval observability gramian are found through Theorem 3 and Theorem 4.

Theorem 3: Let $\bar{Q}$ for some $\tilde{\beta}>0$ be the nonnegative stabilizing solution to the algebraic Riccati equation:
$$A^*\bar{Q}+\bar{Q}A+W_o(\omega_1,\omega_2)+\tilde{\beta}H^*H+\frac{\bar{Q}GG^*\bar{Q}}{\tilde{\beta}}=0 \quad (23)$$

, then:
$$Q(\omega_1,\omega_2)\leq\bar{Q}. \qquad (24)$$

Theorem 4: Assume that the algebraic Riccati equation:
$$A^*\underline{Q}+\underline{Q}A+W_o(\omega_1,\omega_2)-\tilde{\gamma}H^*H-\frac{\underline{Q}GG^*\underline{Q}}{\tilde{\gamma}}=0 \quad (25)$$

has nonnegative stabilizing solution $\underline{Q}$ for some $\tilde{\gamma}>0$, then:
$$\underline{Q}\leq Q(\omega_1,\omega_2). \qquad (26)$$

In the sequel using the results which have been presented earlier in this section, a lower and upper bound on the frequency interval Hankel singular values are derived:

Theorem 5: Assume that (16),(20),(23) and (25) have nonnegative definite solutions for some $\beta>0,\gamma>0,\tilde{\beta}>0$, $\tilde{\gamma}>0$ respectively. Then:
$$\sqrt{\lambda_i(\underline{P}\underline{Q})}\leq\sigma_i(\omega_1,\omega_2)\leq\sqrt{\lambda_i(\bar{P}\bar{Q})} \qquad (27)$$

Proof:
we have:
$$\sigma_i{}^2(\omega_1,\omega_2)=\lambda_i(P(\omega_1,\omega_2)Q(\omega_1,\omega_2))= \\ \lambda_i(Q^{1/2}(\omega_1,\omega_2)P(\omega_1,\omega_2)Q^{1/2}(\omega_1,\omega_2)) \qquad (28)$$

On the other hand, from Theorem (1) and Theorem (3):
$$P(\omega_1,\omega_2)\leq\bar{P} \qquad (29)$$
$$Q(\omega_1,\omega_2)\leq\bar{Q}. \qquad (30)$$

Hence:
$$\sigma_i{}^2(\omega_1,\omega_2)=\lambda_i(P(\omega_1,\omega_2)Q(\omega_1,\omega_2))= \\ \lambda_i(Q^{1/2}(\omega_1,\omega_2)P(\omega_1,\omega_2)Q^{1/2}(\omega_1,\omega_2)) \\ \leq\lambda_i(Q^{1/2}(\omega_1,\omega_2)\bar{P}Q^{1/2}(\omega_1,\omega_2)) \\ =\lambda_i(\bar{P}^{1/2}Q(\omega_1,\omega_2)\bar{P}^{1/2})\leq\lambda_i(\bar{P}^{1/2}\bar{Q}\bar{P}^{1/2}) \\ \leq\lambda_i(\bar{P}\bar{Q}) \qquad (31)$$

The proof for the lower bound is similar.

□

4. ILLUSTRATIVE EXAMPLE

In this section the lower and upper bound for the frequency interval gramian and the frequency interval Hankel singular values are computed for a numerical example. The numerical example is the same as the one in Auba and Funahashi [1992] and Xu et al. [1990].
Let a linear system be described by:
$$A=\begin{bmatrix}-6.5000 & 1 & 0 \\ 0 & -4.8333 & 0.6667 \\ 0 & 0.3333 & -5.1667\end{bmatrix}, C=I,$$
$$B=\begin{bmatrix}0.0148 & 0.2892 & 0.7524 \\ -0.2132 & -0.9320 & 0.2294 \\ 1.4303 & -0.1419 & 0.0264\end{bmatrix}$$

, and the perturbation is given by: $G=0.1A, H=I$. For $\beta=0.26, \quad\gamma=0.22, \quad\tilde{\beta}=0.06, \quad\tilde{\gamma}=0.05$ and $[\omega_1,\omega_2]=[10,100]$, the bounds are:

$$\bar{P}=\begin{bmatrix}3.3741 & -0.2158 & 0.0099 \\ -0.2158 & 2.4707 & -0.2577 \\ 0.0099 & -0.2577 & 2.6156\end{bmatrix},$$
$$\underline{P}=\begin{bmatrix}0.0090 & -0.0016 & -0.0001 \\ -0.0016 & 0.0200 & -0.0031 \\ -0.0001 & -0.0031 & 0.0474\end{bmatrix},$$
$$\bar{Q}=\begin{bmatrix}1.8171 & 0.1611 & 0.0153 \\ 0.1611 & 2.4935 & 0.2452 \\ 0.0153 & 0.2452 & 2.3347\end{bmatrix},$$
$$\underline{Q}=\begin{bmatrix}0.0061 & 0 & 0 \\ 0 & 0.0059 & -0.0002 \\ 0 & -0.0002 & 0.006\end{bmatrix}$$

The bounds on the frequency interval Hankel singular values are:

$$0.0073\leq\sigma_1(\omega_1,\omega_2)\leq 2.4704$$
$$0.0108\leq\sigma_2(\omega_1,\omega_2)\leq 2.4572$$
$$0.0169\leq\sigma_3(\omega_1,\omega_2)\leq 2.4622$$

5. CONCLUSION

The frequency interval gramians were introduced in the literature inspired by many applications in which the information on the controllability and observability properties of a system is needed within a specific frequency interval rather than the whole frequency domain. This notion is investigated for the case of uncertain systems in this paper. We derive a bound of frequency interval gramians under perturbations of an A-matrix in the state-space representation. For a class of perturbations, both upper and lower bounds are obtained by solving algebraic Riccati equations.The results are further used to obtain upper and lower bounds of the frequency interval Hankel singular values for perturbed systems.

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