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Publication date:
1990

Document Version
Publisher's PDF, also known as Version of record

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITETSCENTER • AUC • AALBORG • DANMARK

STRUCTURAL RELIABILITY THEORY

PAPER NO. 82

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OCTOBER 1990

ISSN 0902-7513 R9039

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Abstract

Reliability-based design of structural systems is considered. Especially systems where the reliability model is a series system of parallel systems are analysed. A sensitivity analysis for this class of problems is presented. Direct and sequential optimization procedures to solve the optimization problems are described. Numerical tests indicate that a sequential technique called the bounds iteration method (BIM) is particularly fast and stable.

1. Introduction

A realistic reliability modelling of complex structural systems, such as offshore platforms and bridges, often requires that a series system of parallel systems model is used. It is therefore very interesting to analyse whether already developed techniques for optimization/design of systems where failure is related to single element failure can be extended to more complex systems. Element reliability based design is considered in Murotsu, Kishi, Okada, Yonezawa & Taguchi [12], Frangopol [4] and Sørensen [17].

The calculation of the reliability of failure elements (modelling potential failure modes) by first order reliability methods (FORM) is well established in practical reliability-based analysis and design, see e.g. Madsen, Krenk & Lind [11]. An approximate evaluation of the reliability of series systems of parallel systems, where each parallel system is approximated by an equivalent component, is considered in e.g. Gollwitzer & Rackwitz [6] and in Sørensen & Enevoldsen [15].

In reliability-based systems design and optimization and other types of design it is important to have the possibility to estimating sensitivities with respect to the design parameters either for a purely numerical use or for design evaluation.

In reliability-based optimization based on the use of non-linear optimization algorithms the requirement for a low calculation time and high precision of the sensitivities can be determining for whether it is practically possible to perform the optimization process or not.

In this paper reliability-based optimization of a series system of parallel systems is considered. In section 2 the optimization problem is defined and the system reliability

methods are outlined. Semi-analytical sensitivity analyses are presented in section 3. In section 4 some approximations in the sensitivity calculations are discussed both for reliability-based optimization and less restrictive design evaluation cases. In section 5 two sequential optimization techniques are presented and finally the paper is exemplified with computional aspects in sections 6 and 7.

2. Formulation of the Optimization Problems

The optimization problem for optimization of series systems of parallel systems can be formulated as

$$\min \quad W(\overline{z}) \tag{1}$$

$$s.t. \quad \beta^S(\overline{z}) \geq \beta^S_{min} \tag{2}$$

$$z_i^l \leq z_i \leq z_i^u \quad , \quad i = 1, 2, \cdots, N \tag{3}$$

where W is the objective function, $\overline{z}$ the optimization/design variables with $\overline{z}^l$ as lower and $\overline{z}^u$ as upper bounds. β^S is the generalized systems reliability index with the minimum requirement β^S_{min}. In the following it is described how β^S can be estimated for a series system of parallel systems. It is assumed that the elements in the parallel systems are identified by a suitable technique.

2.1 Reliability of a Series System of Parallel Systems

Consider a series system consisting of M_P parallel systems. Parallel system no. i has M_i elements. β^S can then be estimated by

$$\beta^S = -\Phi_1^{-1}(1 - \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)) \tag{4}$$

where Φ_M is the M-dimensional normal distribution function. $\overline{\beta}^P$ is an M_P-vector of generalized reliability indices for the individual parallel systems and $\overline{\overline{\rho}}^P$ is a matrix of approximate correlation coefficients between the parallel systems.

The elements in $\overline{\beta}^P$ are calculated by

$$\beta_i^P = -\Phi_1^{-1}(\Phi_{M_{A_i}}(-\overline{\beta}_i^J, \overline{\overline{\rho}}_i)) \tag{5}$$

where $\overline{\beta}^J$ is an M_A-vector of element indices calculated in the joint design point $\overline{u}^*$ (and thus not the usual FORM element reliability index β^e) where M_{A_i} is the number of active elements in the ith parallel system. $\overline{\overline{\rho}}$ is the corresponding matrix of correlation coefficients, see next section.

2.2 Calculation of $\overline{\beta}^J$ and $\overline{\overline{\rho}}$

Let the uncertain quantities be modelled by n stochastic variables $\overline{X} = (X_1, X_2, \cdots, X_n)^T$ m failure elements are assigned to the system each modelling a specific potential failure mode at a specific location in the structure and described by failure functions $g_i(\overline{x}) = 0, i = 1, 2, \cdots, m$. For a realization $\overline{x}$ of $\overline{X}$ the failure function divides the space into failure states ($g_i(\overline{x}) \leq 0$) and safe states ($g_i(\overline{x}) > 0$). Further, a $\overline{U}$-space of standardized and normally distributed variables is introduced by the transformation $\overline{X} = \overline{T}(\overline{U})$.

It is assumed that at least one of the $i = 1, 2, \cdots, M$ failure functions in the parallel systems is greater than 0 in the origo of the $\overline{U}$-space, for further explanation see Hohenbichler, Gollwitzer, Kruse & Rackwitz [8].

The so-called joint design point is then defined and determined as the solution of the following optimization problem

$$\begin{aligned} \min_{\overline{u}} \quad & \gamma = \frac{1}{2}\overline{u}^T\overline{u} \\ s.t. \quad & g_1(\overline{u}) \leq 0 \\ & g_2(\overline{u}) \leq 0 \\ & \vdots \\ & g_M(\overline{u}) \leq 0 \end{aligned} \tag{6}$$

For further explanation and more detailed calculation of the joint design point, see Enevoldsen & Sørensen [3].

When the joint design point $\overline{u}^*$ is determined the safety margins $M_i = g_i(\overline{T}(\overline{U}))$ of the M_A active constraints/failure elements (i.e. $g_i(\overline{u}) = 0$, $i = 1, 2, \cdots, M_A$) are linearized at the joint design point

$$M_i \simeq -\overline{\alpha}_i^T\overline{U} + \beta_i^J \quad , \quad i = 1, 2, \cdots, M_A \tag{7}$$

where $\overline{\alpha}_i$ and β_i^J are determined as

$$\overline{\alpha}_i = \frac{-\overline{\nabla} g_i(\overline{u}^*)}{|\overline{\nabla} g_i(\overline{u}^*)|} \qquad \beta_i^J = \overline{\alpha}_i^T\overline{u}^* \tag{8}$$

The elements in the correlation coefficient matrix $\overline{\overline{\rho}}$ are determined by

$$\rho_{ij} = \overline{\alpha}_i^T\overline{\alpha}_j \tag{9}$$

From the formulas in this section it is now possible to calculate β_i^P for all $i = 1, 2, \cdots, M$ parallel systems from (5).

2.3 Calculation of the Correlations Between the Parallel Systems $\overline{\overline{\rho}}^P$

For calculation of the elements in the correlation matrix $\overline{\overline{\rho}}^P$ each of the parallel systems are equivalated by a linear failure element with a linear failure margin, see Gollwitzer & Rackwitz [6] or Sørensen & Enevoldsen [15]

$$M_{P_i} = -\overline{\alpha}_i^{P^T}\overline{U} + \beta_i^P \quad , \quad i = 1, 2, \cdots, M \tag{10}$$

where the vectors $\overline{\alpha}_i^P$ are determined such that $\overline{\nabla}_{u^*}\beta^P$ from (5) and (10) are equivalent and normalized for calculation of the correlations

$$\overline{\alpha}_i^P = \frac{\overline{a}_i^P}{|\overline{a}_i^P|}, \quad i = 1, 2, \cdots, M \tag{11}$$

where the elements of $\overline{\alpha}_i^P$ are obtained from

$$\begin{aligned} a_{ij}^P = \frac{1}{\varphi(-\beta_i^P)} \sum_{k=1}^{M_{A_i}} \{ & (\alpha_{kj}^i + \frac{d\overline{\alpha}_k^{i^T}}{du_j^*}\overline{u}^{i^*})\varphi(\beta_k^{J^i})\Phi_{M_{A_i}-1}(-\overline{\beta}^{J_a^i}, \overline{\overline{\rho}}^{a^i}) \\ & + 2\sum_{l=1}^{k-1}\varphi(-\beta_k^{J^i}, -\beta_l^{J^i}, \rho_{kl}^i)\Phi_{M_{A_i}-2}(-\overline{\beta}^{J_b^i}, \overline{\overline{\rho}}^{b^i})\frac{d\rho_{kl}}{du_j^*}\} \end{aligned} \tag{12}$$

Here, M_{A_i} is the number of active constraints in the i-th parallel system, $\overline{\beta}^{J_a^i}$, $\overline{\beta}^{J_b^i}$ and $\overline{\overline{\rho}}^{a^i}$, $\overline{\overline{\rho}}^{b^i}$ are conditional indices and correlation coefficient matrices obtained from $\overline{\beta}^{J^i}$ and $\overline{\overline{\rho}}^i$ in (8) and (9), see Sørensen [17] and Johnson & Kotz [9].

$d\rho_{kl}/du_j^*$ is calculated as

$$\frac{d\rho_{kl}}{du_j^*} = \frac{d\overline{\alpha}_k^T}{du_j^*}\overline{\alpha}_l + \overline{\alpha}_k^T\frac{d\overline{\alpha}_l}{du_j^*} \tag{13}$$

where

$$\frac{d\overline{\alpha}_k}{du_j^*} = \left(\frac{-\overline{\overline{I}}}{|\overline{\nabla} g_k|} + \frac{\overline{\nabla} g_k \overline{\nabla} g_k^T}{|\overline{\nabla} g_k|^3}\right) \frac{\partial \overline{\nabla} g_k}{\partial u_j^*} \tag{14}$$

$\overline{\overline{I}}$ is the identity matrix. The elements in the matrix of correlations between the parallel systems are then calculated from

$$\rho_{mn}^P = \overline{\alpha}_m^{P^T} \overline{\alpha}_n^P \tag{15}$$

Now β^S can be estimated from (4). In a crude FORM analysis of the parallel system the joint point is not determined. Instead, the usual element reliability indices and $\overline{\alpha}$-vectors obtained from the individual elements in the parallel system are used to calculate the generalised reliability index for the parallel system by (5). In this case a_{ij}^P becomes much easier to calculate because $d\rho_{kl}/du_j^* = 0$ and $d\overline{\alpha}_k^i/du_j^* = 0$. Further, it is not necessary to calculate $d\overline{\alpha}_k/d\overline{u}^*$ which implies estimation of the Hessian matrix $\partial \overline{\nabla} g_k/\partial u_j^*$.

However, in reliability-based optimization the Hessian matrix is needed in the sensitivity analyses (see next section), so estimation of $d\overline{\alpha}_k^i/du_j^*$ by (14) does not give additional computational costs.

3. Sensitivity Analyses

In most structural reliability analyses response calculations are rather expensive so the number of failure function evaluations must be as low as possible, i.e. an efficient optimization algorithm must be used. Numerical tests indicates that the lowest amount of response calculations is obtained by using 1. order optimization methods which require sensitivities of the objective function and the constraints with respect to the optimization variables.

Usually it is easy to calculate $\overline{\nabla}_z W$ by finite differences, whereas $\overline{\nabla}_z \beta^S$ will not only be very expensive but also in many cases inaccurate to calculate by finite differences because it will be necessary to calculate differences between results from iterative solutions of (6) and the linearizations in (10).

Also sensitivities obtained from analytical solutions can give problems mainly because in general, Φ_M for $M > 2$ can only be calculated approximately. Another reason is that the formulas become very extensive so it can be numerically more attractive to use various levels of quasi-analytical methods as pointed out in the following.

The analytical derivatives obtained from the formulas in section (2) can be written as follows

From (4)

$$\frac{d\beta^S}{dz_k} = \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{M_P} \{ \frac{\partial \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)}{\partial \beta_i^P} \frac{d\beta_i^P}{dz_k} + 2 \sum_{j=1}^{i-1} \frac{\partial \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)}{\partial \rho_{ij}^P} \frac{d\rho_{ij}^P}{dz_k} \} \tag{16}$$

where

$$\frac{\partial \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)}{\partial \beta_i^P} = \varphi(\beta_i^P) \Phi_{M_P-1}(\overline{\beta}^{P_a}, \overline{\overline{\rho}}^{P_a}) \tag{17}$$

and

$$\frac{\partial \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)}{\partial \rho_{ij}^P} = \varphi_2(\beta_i^P, \beta_j^P, \rho_{ij}^P) \Phi_{M_P-2}(\overline{\beta}^{P_b}, \overline{\overline{\rho}}^{P_b}) \tag{18}$$

where $\overline{\beta}^{P_a}$, $\overline{\beta}^{P_b}$ and $\overline{\overline{\rho}}^{P_a}$, $\overline{\overline{\rho}}^{P_b}$ are again conditional indices and correlation coefficient matrices obtained from $\overline{\beta}^P$ and $\overline{\overline{\rho}}^P$, see Sørensen [17] and Johnson & Kotz [9].

Sensitivities For Parallel Systems

$d\beta_i^P/dz_k$ in (16) is obtained on the basis of (5). The following formulas are for one parallel system, i.e. the index i is suppressed, see also Madsen [10].

$$\frac{d\beta^P}{dz_k} = \frac{1}{\varphi(\beta^P)} \sum_{j=1}^{M_A} \{ \frac{\partial \Phi_{M_A}(-\overline{\beta}^J, \overline{\overline{\rho}})}{\partial \beta_j^J} \frac{d\beta_j^J}{dz_k} + 2 \sum_{p=1}^{j-1} \frac{\partial \Phi_{M_A}(-\overline{\beta}^J, \overline{\overline{\rho}})}{\partial \rho_{jp}} \frac{d\rho_{jp}}{dz_k} \} \tag{19}$$

$\partial \Phi_{M_A}(-\overline{\beta}^J, \overline{\overline{\rho}})/\partial \beta_j^J$ and $\partial \Phi_{M_A}(-\overline{\beta}^J, \overline{\overline{\rho}})/\partial \rho_{jp}$ are calculated as in (17) and (18)

$d\beta_j^J/dz_k$ is calculated from (8)

$$\frac{d\beta_j^J}{dz_k} = \frac{d\overline{\alpha}_j^T}{dz_k} \overline{u}^* + \overline{\alpha}_j^T \frac{d\overline{u}^*}{dz_k} \tag{20}$$

If the auxiliary quantity

$$d\overline{\nabla} g_j = \frac{d\overline{\nabla} g_j}{dz_k} = \frac{\partial \overline{\nabla} g_j}{\partial z_k} + \overline{\overline{\Delta}} g_j \frac{d\overline{u}^*}{dz_k} \tag{21}$$

where $\{\overline{\overline{\Delta}} g_j\}_{lm} = \partial^2 g_j / \partial u_l \partial u_m$ is introduced, then $d\overline{\alpha}_j/dz_k$ is calculated from (8)

$$\frac{d\overline{\alpha}_j}{dz_k} = \frac{-d\overline{\nabla} g_j}{|\overline{\nabla} g_j|} + \frac{\overline{\nabla} g_j d\overline{\nabla} g_j^T \overline{\nabla} g_j}{|\overline{\nabla} g_j|^3} \tag{22}$$

$d\overline{u}^*/dz_k$ is calculated using of the Kuhn-Tucker conditions of the joint design point optimization problem in (6)

$$u_q^* + \sum_{j=1}^{M} \lambda_j^* \frac{dg_j}{du_q^*} = 0 \ , \quad q = 1, 2, \cdots, n \tag{23}$$

$$g_j(\overline{u}^*) \leq 0 \ , \quad j = 1, 2, \cdots, M \tag{24}$$

$$\lambda_j^* g_j(\overline{u}^*) = 0 \ , \quad \lambda_j^* \geq 0 \ , \quad j = 1, 2, \cdots, M \tag{25}$$

where $\overline{\lambda}^*$ is the vector of Lagrange multipliers. It is then required that the Kuhn-Tucker conditions must remain valid when a small change in z_k is introduced

$$\frac{du_q^*}{dz_k} + \sum_{j=1}^{M} \left(\frac{d\lambda_j^*}{dz_k} \frac{dg_j}{du_q^*} + \lambda_j^* \left(\frac{\partial^2 g_j}{\partial u_q^* \partial z_k} + \sum_{l=1}^{n} \frac{\partial^2 g_j}{\partial u_q^* \partial u_l^*} \frac{du_l^*}{dz_k} \right) \right) = 0 \ , \quad q = 1, 2, \cdots, n \tag{26}$$

$$\frac{\partial g_j}{\partial z_k} + \sum_{l=1}^{n} \frac{\partial g_j}{\partial u_l^*} \frac{du_l^*}{dz_k} \leq 0 \ , \quad j = 1, 2, \cdots, M \tag{27}$$

$$\frac{d\lambda_j^*}{dz_k} g_j(\overline{u}^*) + \lambda_j^* \left(\frac{\partial g_j}{\partial z_k} + \sum_{l=1}^{n} \frac{\partial g_j}{\partial u_l^*} \frac{du_l^*}{dz_k} \right) = 0 \ , \quad \lambda_j^* \geq 0 \ , \quad j = 1, 2, \cdots, M \tag{28}$$

If it is assumed that the active set of constraints remain active after the perturbation the following system of equations is obtained from (26) to (28)

$$\begin{bmatrix} \overline{\overline{I}} + \sum_{j=1}^{M_A} \lambda_j^* \overline{\overline{\Delta}} g_j & \overline{\overline{D}} \\ \overline{\overline{D}}^T & \overline{\overline{0}} \end{bmatrix} \begin{bmatrix} d\overline{u}^*/dz_k \\ d\overline{\lambda}^*/dz_k \end{bmatrix} = \begin{bmatrix} -\overline{\overline{E}}\,\overline{\lambda}^* \\ -\partial \overline{g}/\partial z_k \end{bmatrix} \tag{29}$$

where $\overline{\overline{D}} = \{\overline{\nabla}_u g_1, \overline{\nabla}_u g_2, \cdots, \overline{\nabla}_u g_{M_A}\}$, $\overline{\overline{E}} = \{\partial \overline{\nabla}_u g_1/\partial z_k, \partial \overline{\nabla}_u g_2/\partial z_k, \cdots, \partial \overline{\nabla}_u g_{M_A}/\partial z_k\}$ and $\overline{\lambda}^*$ is the vector of Lagrange multipliers of the active constraints.

The last term in (19) is calculated from (9)

$$\frac{d\rho_{jp}}{dz_k} = \frac{d\overline{\alpha}_j^T}{dz_k}\overline{\alpha}_p + \overline{\alpha}_j^T\frac{d\overline{\alpha}_p}{dz_k} \tag{30}$$

where the derivatives of the α-vectors are calculated from (22).

From the equations in this section it is now possible to obtain the sensitivities of the parallel systems.

Now return to (16) where the explanation of $d\rho_{ij}^P/dz_k$ remains. From (11) to (15) it is seen that to carry on with analytical sensitivities will cause a very large swelling of the formulas, where especially the derivatives of the conditional indices and correlation coefficients make problems. Instead, a semi-numerical solution with parts calculated by numerical differentiation is suggested. If $\overline{\overline{\rho}}^P$ is written

$$\overline{\overline{\rho}}^P(\overline{z}) = \overline{\overline{\rho}}^P(\overline{\beta}_p^J(\overline{z}), \overline{\overline{\rho}}_p(\overline{z})) \quad , \quad p = 1, 2, \cdots, M \tag{31}$$

then

$$\frac{d\rho_{ij}^P}{dz_k} = \sum_{p=1}^{M}\sum_{q=1}^{M_{A_P}}\{\frac{d\rho_{ij}^P}{d\beta_{pq}^J}\frac{d\beta_{pq}^J}{dz_k} + 2\sum_{r=1}^{q-1}\frac{d\rho_{ij}^P}{d\rho_{pqr}}\frac{d\rho_{pqr}}{dz_k}\} \tag{32}$$

the parts $d\rho_{ij}^P/d\beta_{pq}^J$ and $d\rho_{ij}^P/d\rho_{pqr}$ are calculated by numerical differences, whereas the two other parts in (32) are already known from the sensitivity analysis of the parallel systems. It is seen that this formula requires no additional response calculations and structural analyses.

As mentioned in section 2 the second order derivatives of g with respect to $\overline{u}$, $\overline{\overline{\nabla}}g$ is evaluated in order to determine $d\beta^P/d\overline{z}$, namely in (29). The same second order derivatives can thus be used to determine the equivalent linear element in (12) without any additional computional costs.

4. Approximations

For small and moderate correlation coefficients the last part in both (16) and (19) containing the derivatives of the correlations usually, only contribute insignificantly to $d\beta^S/d\overline{z}$ and $d\overline{\beta}^P/d\overline{z}$, respectively. Furthermore, they are very computer time consuming to estimate(especially the numerical calculation of $d\overline{\overline{\rho}}^P/d\overline{z}$ in (32)).

A natural approximation is then to neglect the derivatives of the correlations and to use the two following formulas in the optimization instead

$$\frac{d\beta^S}{dz_k} = \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{M_P} \frac{\partial \Phi_{M_P}(\overline{\beta}^P, \overline{\overline{\rho}}^P)}{\partial \beta_i^P} \frac{d\beta_i^P}{dz_k} \tag{33}$$

$$\frac{d\beta^P}{dz_k} = \frac{1}{\varphi(\beta^P)} \sum_{j=1}^{M_A} \frac{\partial \Phi_{M_A}(-\overline{\beta}^J, \overline{\overline{\rho}})}{\partial \beta_j^J} \frac{d\beta_j^J}{dz_k} \tag{34}$$

From (34) it is seen that calculation of $d\overline{u}^*/d\overline{z}$ from (29) is still necessary.

5. Sequential Optimization Methods

Reliability-based optimization of series systems of parallel systems can be performed directly by using the sensitivities of the systems reliability index β^S. A directly formulated optimization problem can, however, cause problems and give instabilities, see section 7.

In the following, two optimization methods using another so-called sequential strategy are presented. Instead of the direct formulation in (1) - (3) with a requirement for β^S, a problem with requirements for each of the M_P parallel systems indices is introduced. These requirements are then adjusted in a sequence until the underlying requirement for the β^S is finally fulfilled.

For both methods the reliability-based optimization problem in (1) - (3) is solved by a sequence of parallel system reliability based problems denoted $k = 1, 2, \cdots$

$$\begin{aligned} \min \quad & W(\overline{z}) && (35)\\ s.t. \quad & \beta_i^P(\overline{z}) \geq \beta_i^{P^k}, i = 1, 2, \cdots, M_P && (36)\\ & z_i^l \leq z_i \leq z_i^u \quad , \quad i = 1, 2, \cdots, N && (37) \end{aligned}$$

where $\beta_i^{P^k}, i = 1, 2, \cdots, M_P$ is a sequence of lower bounds determined as

$$\overline{\beta}^{P^{k+1}} = \overline{\beta}^{P^k} + \Delta\overline{\beta}^{P^k} \tag{38}$$

Initially, for $k = 1$, $\beta_i^{P^0} = \beta_{min}^S$, $i = 1, 2, \cdots, M_P$ and all the elements in $\Delta\overline{\beta}^{P^0}$ are selected as a good guess of the difference between β_{min}^S and the lowest final $\beta_i^P, i = 1, 2, \cdots, M_P$, for example $\Delta\beta_i^{P^0} = 5$ % of β_{min}^S.

The difference between the two sequential techniques lies in the requirements for and the calculation of $\Delta\overline{\beta}^{P^k}$.

5.1 Sequential Method I: Constant Objective Function Method (COFM)

In this method (see also Sørensen [18]) $\Delta\overline{\beta}^{P^k}$ is calculated as

$$\Delta\overline{\beta}^{P^k} = c\,\overline{d} \tag{39}$$

where c and the unit vector $\overline{d}$ are determined on the basis of a linearization of β^S

$$\beta^S \approx \beta^{S^k} + \frac{d\beta^{S^k}{}^T}{d\overline{\beta}^{P^k}}\Delta\overline{\beta}^{P^k} = \beta^{S^k} + \overline{\alpha}_\beta^{k\,T}\Delta\overline{\beta}^{P^k} = \beta^S_{min} \tag{40}$$

where the auxiliary quantity $\overline{\alpha}_\beta^k$ is introduced. c is calculated by

$$c = (\beta^S_{min} - \beta^{S^k})/\overline{\alpha}_\beta^{k\,T}\overline{d} \tag{41}$$

c is the step length in the $\overline{d}$ direction which is obtained by maximizing a linear approximation of the systems reliability index β^S subject to the requirement that $W(\overline{z})$ is kept approximately constant

$$\min \quad -\beta^{S^k} - \overline{\alpha}_\beta^{k\,T}\overline{d} \tag{42}$$

$$s.t. \quad \overline{\alpha}_W^{k^T}\overline{d} = 0 \tag{43}$$

$$\overline{d}^T\overline{d} = 1 \tag{44}$$

where

$$\alpha^k_{W_i} = \sum_{j=1}^{N}\frac{dW(\overline{z}^k)}{dz_j}\frac{dz_j}{d\beta_i^P} \tag{45}$$

The first constraint (43) signifies that the objective function is held approximately constant while the linear approximation of the system reliability index is maximized. The second constraint (44) is a normalization of the search direction.

From the Kuhn-Tucker conditions of the optimization problem (42) - (44) the following solution can be obtained

$$\overline{d} = \frac{\overline{\gamma}}{\sqrt{\overline{\gamma}^T\overline{\gamma}}}, \qquad \overline{\gamma} = \overline{\alpha}_\beta^k - \frac{\overline{\alpha}_\beta^{k\,T}\overline{\alpha}_W^k}{\overline{\alpha}_W^{k^T}\overline{\alpha}_W^k}\overline{\alpha}_W^k \tag{46}$$

Hereby c can be calculated from (41) the new bounds from (39) and the problem in (35) - (37) is again solved for $k = k + 1$.

The iteration is stopped when

$$|\beta^{S^k}(\overline{z}^k) - \beta^S_{min}| \leq \epsilon_1 \tag{47}$$

where ϵ_1 is the desired requirement for the accuracy.

5.2 Sequential Method II: Bounds Iteration Method (BIM)

In this method it is not required that $W(\overline{z})$ is kept approximately constant but instead that $\beta^S(\overline{\beta}^{P^k}, \overline{\overline{\rho}}^P) = \beta^S_{min}$ in the calculation of $\Delta\overline{\beta}^{P^k}$. Further, only the bounds on the active constraints from the solution of (35) - (37) are changed. The elements in $\Delta\overline{\beta}^{P^k}$ are calculated as in (39) but $\overline{d} = \{1, 1, \cdots, 1, 0, 0, \cdots, 0\}$, and c is an unknown coefficient. The constraints in (35) - (37) are rearranged so the N_A active constraints are the first constraints, i.e. 1 in $\overline{d}$ corresponds to an active constraint, whereas 0 corresponds to an inactive constraint.

The following problem is then solved for with respect to the variable c for each of the k sequences for fixed $\overline{\overline{\rho}}^P$

$$\beta^{S^l} = -\Phi_1^{-1}(1 - \Phi_{M_P}(\overline{\beta}^{P^{k_l}} + \Delta\overline{\beta}^{P^{k_l}}, \overline{\overline{\rho}}^P)) = \beta^S_{min} \tag{48}$$

c can e.g. be determined by using Newton's method for a given sequence number k as iterations $l = 1, 2, \cdots$ with

$$c^l = \frac{\beta^S_{min} - \beta^{S^l}}{\sum_{i=1}^{N_A} d\beta^{S^l}/d\beta_i^{S^l}} \tag{49}$$

This Newton sub-iteration is stopped when

$$|\beta^S_{min} - \beta^{S^l}| \leq \epsilon_2 \tag{50}$$

where ϵ_2 is the desired accuracy of the bounds problem. when $\Delta\overline{\beta}^{P^k}$ is determined (35) - (37) can be solved with new bounds for $k = k + 1$. The optimization is again stopped when

$$|\beta^{S^k}(\overline{z}^k) - \beta^S_{min}| \leq \epsilon_1 \tag{51}$$

where again ϵ_1 is the desired requirement for the accuracy.

The Newton sub-problem is very easy and fast to solve because only work on the active constraints is performed. From a stability convergence point of view it is required that $\epsilon_2 < \epsilon_1$.

6. Computional Aspects

6.1 Optimization procedures

The optimization problems in (1) - (3) and (35) - (37) can in principle be solved using any general non-linear optimization algorithm. In this paper the VMCWD algorithm is used, see Powell [13].

The VMCWD algorithm is based on the sequential quadratic optimization method by Han, Powell and Wilson, see Gill, Murray & Wright [5]. Generally it is a very effective method where each iteration consists of two steps. The first step is determination of the search direction by solving a quadratic optimization problem formed by a quadratic approximation of the Lagrange function of the non-linear optimization problem and a linearization of the constraints at the current design point. The second step is a line search with an augmented Lagrangian merit function using the so-called Watchdog technique by Chamberlain, Lemarechal, Petersen & Powell [2].

6.2 Calculation of the Joint Design Point $\overline{u}^*$

The joint design point can be calculated by any general non-linear optimization algorithm applied to the optimization problem in (6). However, the special structure of this optimization problem implies that optimality criteria based algorithms can be used. In Enevoldsen and Sørensen [3] such a method with a strong active set technique is described. The algorithm is in general more stable and fast than e.g. VMCWD.

6.3 Calculation of Φ_M

Except for special cases the multi-dimensional normal distribution function Φ_M must be calculated by an approximation, e.g. Hohenbichler's approximation, see Hohenbichler [7]. This is a source of errors in the reliability-based optimization. From a numerical point of view the most stable optimization is obtained by using the derivatives belonging to the approximation of Φ_M and not to the actual problem. By a careful implementation of Hohenbichler's approximation the errors can be significantly reduced.

Another possibility is to use the average correlation coefficient approximation method, see Sørensen [17] and Johnson & Kotz [9], which, even though it is a coarser approximation of the system probability of failure, gives more consistent sensitivity coefficients.

By this method Φ_M is calculated as

$$\Phi_M(\overline{\beta}, \rho_{av}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^{M} \left(\frac{\beta_i - \sqrt{\rho_{av}}\, t}{\sqrt{1-\rho_{av}}}\right) dt \tag{52}$$

where ρ_{av} is the average correlation coefficient of the correlation coefficients in the correlation matrix $\overline{\overline{\rho}}$ corresponding to $\overline{\beta}$.

6.4 Solution of the Optimization Problem in (1) - (3)

In this section two algorithms are presented to solve the general optimization problem (1) - (3). The first algorithm solves the optimization problem directly and the second algorithm uses a sequential technique.

Model Algorithm 1: Direct Solution of Optimization Problem

In the following a model algorithm is presented which solves the optimization problem in (1) - (3) directly

Step 0 (Initialization)
Set the overall iteration number $j = 1$ and the parallel system number $i = 1$
Input $\overline{z}^0$

Step 1 (Calculation of β_i^P)
Calculate $\overline{u}_i^*$ (6), $\overline{\alpha}_i$ (8), $\overline{\overline{\rho}}_i$ (9) and $\overline{\beta}_i^J$ (8)
Calculate β_i^P (5)

Step 2 (Calculation of $d\beta_i^P/d\overline{z}$)
Calculate $d\overline{u}_i^*/d\overline{z}$ by assembling and solution of (29)
Calculate $d\overline{\overline{\alpha}}_i/d\overline{z}$ (22), $d\overline{\beta}_i^J/d\overline{z}$ (20) and $d\overline{\overline{\rho}}_i/d\overline{z}$ (30)
Calculate $d\beta_i^P/d\overline{z}$ (19)

Step 3 (Calculation of equivalent linear failure element)
Calculate $d\overline{\overline{\alpha}}/d\overline{u}_i^*$ (14) and $d\overline{\overline{\rho}}/d\overline{u}_i^*$ (13)
Calculate $\overline{\alpha}_i^P$ (11)
IF $i < M_p$ set $i = i + 1$ and **GOTO Step 1**

Step 4 (Calculation of β^S)
Calculate $\overline{\overline{\rho}}^P$ (15)
Calculate β^S (4)

Step 5 (Calculation of $d\beta^S/d\overline{z}$)
Calculate $d\overline{\overline{\rho}}^P/d\overline{z}$ (32) and $d\beta^S/d\overline{z}$ (16)

Step 6 (New point and Convergence test)
Calculate new point $\overline{z}^{j+1}$ with VMCWD
IF $\overline{z}^{j+1}$ Optimal **EXIT**
ELSE $j = j + 1$,$i = 1$ and **GOTO Step 1**

Model Algorithm 2: Sequential Solution of Optimization Problem

In the following a model algorithm is presented which solves the optimization problem in (1) - (3) sequentially by the formulation in (35) - (37) and the bounds iteration technique (BIM).

Step 0 (Initialization)
Set the iteration number $j = 1$, the sequence number $k = 1$, the parallel system number $i = 1$ and $\beta_i^{P^1} = 1.05 \cdot \beta_{min}^S, i = 1, \cdots M_P$. Input $\overline{z}^0$

Step 1 (Calculation of β_i^P)
Calculate $\overline{u}_i^*$ (6), $\overline{\alpha}_i$ (8), $\overline{\overline{\rho}}_i$ (9) and $\overline{\beta}_i^J$ (8)
Calculate β_i^P (5)

Step 2 (Calculation of $d\beta_i^P/d\overline{z}$)
Calculate $d\overline{u}_i^*/d\overline{z}$ by assembling and solution of (29)
Calculate $d\overline{\overline{\alpha}}_i/d\overline{z}$ (22), $d\overline{\beta}_i^J/d\overline{z}$ (20) and $d\overline{\overline{\rho}}_i/d\overline{z}$ (30)
Calculate $d\beta_i^P/d\overline{z}$ (19) **IF** $i < M_p$ set $i = i + 1$ and **GOTO Step 1**

Step 3 (New point and convergence test of sub-problem)
Calculate new point $\overline{z}^{j+1}$ with VMCWD
IF $\overline{z}^{j+1}$ Optimal for sub-problem **GOTO Step 4**
ELSE $j = j + 1$,$i = 1$ and **GOTO Step 1**

Step 4 (Calculation of equivalent linear failure element)
for $j = 1, \cdots, M_P$
Calculate $d\overline{\overline{\alpha}}/d\overline{u}_j^*$ (14), $d\overline{\overline{\rho}}/d\overline{u}_j^*$ (13) and $\overline{\alpha}_i^P$ (11)

Step 5 (Calculation of β^{S^k})
Calculate $\overline{\overline{\rho}}^P$ (15)
Calculate β^{S^k} (4)

Step 6 (Convergence test of total problem)
IF $\overline{z}^{j+1}$ satisfies the total convergence criteria (51) **EXIT**

Step 7 (Adjust bounds by BIM)
Calculate c (49) - (50) and $\overline{\beta}^{P^{k+1}}$ (38)
Set $j = j + 1$, $k = k + 1$, $i = 1$ and **GOTO Step 1**

7. Example

Optimization of a Series System of 6 Parallel Systems

Consider a system of 3 elements where failure is defined as failure of all 3 elements. Each of the possible failure paths is then modelled in a parallel system which again is placed in a series system modelling the total system. With 3 elements 6 possible failure paths can be found. The model is then a series system consisting of 6 parallel systems each with 3 failure elements described by a failure function

$$g_i = S_i^a z_i - \eta_i P^b, \quad , i = 1,2,3 \tag{53}$$

where S_i^a and P^b can be considered as the strength of element i and the load on the parallel system, respectively. z_i is an optimization variable which can be considered as the area of element i. η_i is a load division parameter for a brittle system for a given failure path

$$\eta_i = \frac{z_i}{\sum_{j=i}^{3} z_j} \tag{54}$$

The strengths of the 3 elements are modelled as 3 independent log-normally distributed stochastic variables S_i, $i = 1,2,3$ raised to the power of $a = 2.5$. The load on the system is modelled as a normally distributed stochastic variable raised to the power of $b = 1.2$. The expected values of the strengths are $\overline{\mu}_s = \{25, 27, 30\}$ and of the load μ_P=2700, all with 0.1 as the coefficient of variation.

The problem is to optimize the series system of parallel systems described above. The optimization problem is formulated as, see (1) to (3)

$$\min \quad W(\overline{z}) = z_1^2 + 1.2z_2^2 + 1.3z_3^2 \tag{55}$$

$$s.t. \quad \beta^S \geq 3.5 \tag{56}$$

$$z_i^l \leq z_i \leq z_i^u \tag{57}$$

The optimization problem is solved by 4 different methods:

I Directly by using the sensitivity techniques described in section 3.

II Directly by using the approximate sensitivity techniques described in section 4.

III Sequentially by using COFM described in section 5.

IV Sequentially by using BIM described in section 5.

The starting point is selected as $\overline{z}^0 = \{2.0, 2.5, 3.5\}$ with ± 0.5 as simple bounds. The elements in the bounds on $\overline{\beta}^P$ in $\overline{\beta}^{P^1}$ are selected as $\beta^S_{min} + 5\ \% = 3.675$ and the overall

convergence parameter is set as $\epsilon_1 = 0.0005$. The sub-iteration convergence parameter in BIM is chosen as $\epsilon_2 = 0.1\epsilon_1$. The optimization problems are solved by VMCWD [13] and the joint design point problems are solved by the algorithm described in Enevoldsen & Sørensen [3].

Solution Characteristics

It was possible to solve the optimization problem by solution methods I, III and IV, whereas method II broke down in the line searches which is probably due to that the sensitivities are not precise enough. The solution is found as $\overline{z}^* = \{1.74, 2.62, 3.73\}$ with $W(\overline{z}^*) = 29.3$. Further, the following table 1 of solution characteristics can be outlined.

Method	N_g	T_∇	I_t	F_c	C_{pu}
I	24080	2107	13	14	36.6
III	37440	16	9	21	2.1
IV	15986	7	3	9	0.8

Table 1: Solution Characteristics.

N_g is the number of failure function calculations (all derivatives of failure functions are calculated by numerical differences), T_∇ the CPU-time in seconds on a VAX8700 used for calculation of $d\beta^S/d\overline{z}$ in (16) for method I and $d\overline{\beta}^P/d\overline{z}$ in (19) for methods I, III and IV, respectively. I_t is the number of iterations which for method III and IV is the number of sequential solutions of (35) - (37). F_c is the number of functional calls which implies a calculation of the objective function, the constraints and their derivatives. C_{pu} is the total CPU calculation time in minutes on the VAX8700.

The two most important figures are N_g and T_∇. In this example it is seen from T_∇ that the sensitivity calculation in method I is very time consuming compared to the two other methods which is due to the numerical calculation of $d\overline{\rho}^P/d\overline{z}$ in (32). It is also seen that nearly all C_{pu}-time in method I is used for sensitivity calculations.

In this example the calculation of the failure function in (53) is relatively cheap. In a more realistic model, e.g. using finite element methods, the time for calculating the failure functions will dominate the calculation time totally. Therefore, the selection of method must be made from two considerations, 1) the number of failure function evaluations and 2) stability of the methods. N_g is clearly lowest for method IV with a very fast convergence. The stability of methods III and IV is nearly the same but method III is more slowly converging and will be dependent on the initial guess of $\Delta\overline{\beta}^{P^0}$ in (38), whereas method IV adjusts the bounds in the first Newton iteration in (49). The stability of the methods III and IV is much better than for method I where the

starting point and the simple bounds on $\overline{z}$ are very important because large zig-zagging in the optimization variables gives stability problems.

Furthermore, it must be expected that $d\overline{\beta}^P/d\overline{z}$ is more precise than $d\beta^S/d\overline{z}$. Hohenbichler's approximation of Φ_M is used at two levels when estimating β^S and equivalent linear failure elements are introduced in the calculation of β^S.

Finally, it must be mentioned that there is no guarantee that methods I, III and IV always find the same solution of the problem but all methods will find allowable solutions.

Sensitivity Analyses

In optimization the requirements for the precision of the gradients are very high. When the topic is to use sensitivities for design evaluation the order and mutual size of the sensitivities are often of more interest. In Table 2 the differences between the sensitivities calculated by methods I and II are shown (calculated for the sensitivities at the starting point $\overline{z}^0$ and the optimum point $\overline{z}^*$)

Method	$\Delta_{sensi}(\%)$	C_{pu} (sec.)
II	5 - 12	5
$d\overline{\rho}^P/d\overline{z}$ in (32) neglected	3 - 5	8

Table 2: Sensitivities compared to sensitivities calculated from (16) and calculation times.

In method II all derivatives of correlations are neglected. In the second row $d\beta^S/d\overline{z}$ are calculated by neglecting $d\overline{\rho}^P/d\overline{z}$ in (32), whereas the sensitivities of the correlations in (12) and (19) are still taken into account. C_{pu} is CPU-seconds on the VAX8700 for one calculation of $d\beta^S/d\overline{z}$. The times must be compared with 148 CPU- seconds in method I. Δ_{sensi} is the range of differences in % between the sensitivities calculated by method I and the two other methods. It is seen that neglecting the derivatives of the correlations is satisfactory for most practical design evaluations.

8. Conclusions

Reliability-based optimization problems of series systems of parallel systems is formulated and sensitivity analyses are presented. It is discussed whether it is possible to neglect derivatives in sensitivity analysis and it is concluded that this is possible when the sensitivities are used for design evaluation.However, this is not possible when the sensitivities are used in optimization.

Among the suggested direct optimization method and two sequential methods, especially the sequential method called the bounds iteration method (BIM), which adjusts the bounds in each sub-optimization problem, seems fast and stable.

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