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by

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ON STABILITY OF SCHAUDER BASES OF INTEGER TRANSLATES

MORTEN NIELSEN[†] AND HRVOJE ŠIKIĆ[∗]

ABSTRACT. We apply the Garnett-Jones distance to the analysis of Schauder bases of translates. A special role is played by periodization functions p_ψ with $\ln p_\psi$ in the closure of L^∞ in $BMO(\mathbb{T})$. In particular, for Schauder bases with such periodization functions we study the corresponding coefficient space. We also use the Garnett-Jones distance approach to show the stability of bases of translates with respect to convolution powers. The case of democratic conditional Schauder bases of translates is emphasized, as well.

1. INTRODUCTION

For any $\psi \in L^2(\mathbb{R})$ one can generate an associated shift invariant space

$$\langle\psi\rangle := \overline{\text{span}}\{\psi(\cdot - k) : k \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}).$$

A very natural question is then to consider the stability of the family of integer translates

(1.1) $$\mathcal{E} := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$$

in $\langle\psi\rangle$. The answer will of course depend on the properties of ψ. The Fourier transform provides a very convenient tool to facilitate such an analysis, and it turns out that the so-called periodization function given by

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2, \qquad \xi \in \mathbb{R}.$$

plays a central role. As it was observed already in [15], the family of translates (properly ordered) forms a Schauder basis for $\langle\psi\rangle$ if and only if p_ψ is an A_2 weight in the sense of Muckenhoupt. We aim to study various properties of such bases in connection with the A_p weight properties of p_ψ. As it is well-known, A_p weights are closely connected to the corresponding BMO space; if w is an A_p weight, then $\ln w$ is in BMO. Obviously, the same approach can be applied to periodizations; considered as weights. In particular, observe (see [15]) that $\ln p_\psi$ is in L^∞ if and only if the corresponding $\mathcal{E}$ is a Riesz (i.e., Schauder unconditional) basis for $\langle\psi\rangle$. Therefore, the A_2 weights p_ψ, such that $\ln p_\psi$ is outside L^∞, provide us with examples of conditional Schauder bases, which in general are often difficult to construct. Since in BMO spaces

we can measure the distance from L^∞ (see [4]), we can employ this distance as a way of measuring "how far" is a particular Schauder basis from the family of Riesz bases. Indeed, as we argue in this article, such an approach enables us to learn more about coefficient spaces of such Schauder bases, about Schauder basis constants, and provides tools to show stability properties of convolution roots and powers of weights p_ψ. We find particularly intriguing the family of conditional Schauder bases of translates such that $\ln p_\psi$ is in the closure of L^∞ in $BMO(\mathbb{T})$ (i.e., the Garnett-Jones distance for such weights is zero). As we show, they have a very small subset of coefficient sequences for which the convergence is conditional.

Problems on translates of functions have a long history, let us mention Kolmogoroff [11] and Helson [5]. Stability of integer translates has been considered in many articles, see e.g. [1,8]. Schauder bases of integer translates and the connection to Muckenhoupt weights are considered in [13–15].

The paper is organized so that after this introduction we develop necessary notation and basic results in Section 2. Main results, about stability, about coefficient spaces and Schauder basis constants, are presented in Section 3. In Section 4 we consider democratic Schauder bases. As is well-known, both the greedy property and the democratic property are important in the study of bases. However, as we have shown in [16] for bases of translates the greedy property leads us into Riesz bases, while the democratic property may exist among conditional bases of translates, as well.

2. NOTATION AND RESULTS

Let us begin by introducing some notation and recalling some necessary results. We let the Fourier transform of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be normalized such that $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}dx$. We denote by $L^p(\mathbb{T})$, $1 \le p \le \infty$, the L^p space on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with respect to the Lebesgue measure. Functions on $\mathbb{T}$ are considered as 1-periodic functions on $\mathbb{R}$. When a 1-periodic measurable density (or weight) w is taken into account, we denote by $L^2(\mathbb{T}; w)$ the L^2 space on $\mathbb{T}$ with respect to the measure $w(\xi)\,d\xi$.

One observes that $\langle\psi\rangle$ is the smallest shift invariant space in $L^2(\mathbb{R})$ generated by ψ. Hence, it is known that

$$(2.1) \qquad\qquad t \mapsto (t \cdot \hat{\psi})^\vee$$

is an isometry between the weighted L^2-space $L^2(\mathbb{T}; p_\psi)$ and $\langle\psi\rangle$, see [8], where p_ψ is the periodization of $|\hat{\psi}|^2$, given by

$$(2.2) \qquad\qquad p_\psi(\xi) := \sum_{k\in\mathbb{Z}} |\hat{\psi}(\xi+k)|^2, \qquad \xi \in \mathbb{R}.$$

Since $\psi \in L^2(\mathbb{R})$, p_ψ is a 1-periodic function in $L^1(\mathbb{T})$ and

$$(2.3) \qquad\qquad \int_{\mathbb{T}} p_\psi(\xi)\,d\xi = \|\hat{\psi}\|_2^2 = \|\psi\|_2^2.$$

The isometry (2.1) maps the exponential $e_k(\xi) := e^{-2\pi i k\xi}$, $k \in \mathbb{Z}$, into $\psi(\cdot - k)$.

A family $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ of vectors in a Hilbert space $\mathbb{H}$ is a *Schauder basis* for $\mathbb{H}$ if for every $x \in \mathbb{H}$ there exists a unique sequence $\{\alpha_n := \alpha_n(x) : n \in \mathbb{N}\}$ of scalars

such that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n x_n = x$$

in the norm topology of $\mathbb{H}$. For a Schauder basis $\mathcal{B}$, and $n \in \mathbb{N}$, there exists a unique vector y_n such that $\alpha_n(x) = \langle x, y_n \rangle$. It follows that

$$(2.4) \qquad \langle x_m, y_n \rangle = \delta_{m,n}, \quad m, n \in \mathbb{N}$$

and that there exists a smallest constant $C = C(\mathcal{B}) \geq 1$ such that, for every $n \in \mathbb{N}$,

$$(2.5) \qquad 1 \leq \|x_n\| \cdot \|y_n\| \leq C.$$

A pair of sequences $(\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}})$ in $\mathbb{H}$ is a *bi-orthogonal system* if $\langle u_m, v_n \rangle = \delta_{m,n}$, $m, n \in \mathbb{N}$. We say that $\{v_n\}_{n\in\mathbb{N}}$ is a *dual sequence* to $\{u_n\}_{n\in\mathbb{N}}$, and vice versa. A dual sequence is not necessarily uniquely defined. In fact, it is unique if and only if the original sequence is complete in $\mathbb{H}$ (i.e., if the span of the original sequence is dense in $\mathbb{H}$). A complete sequence $\{x_n : n \in \mathbb{N}\}$ with dual sequence $\{y_n\}$ is a Schauder basis for $\mathbb{H}$ if and only if the partial sum operators $S_N(x) = \sum_{n=1}^{N} \langle x, y_n \rangle x_n$ are uniformly bounded on $\mathbb{H}$. Obviously, (2.4) shows that every (Schauder) basis $\{x_n : n \in \mathbb{N}\}$ for $\mathbb{H}$ has an associated bi-orthogonal system $(\{x_n\}, \{y_n\})$ with a uniquely determined dual sequence. Furthermore, the dual sequence $\{y_n\}$ is also a Schauder basis for $\mathbb{H}$.

For a Schauder basis $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ with dual sequence $\{y_n\}$, the number

$$\kappa(\mathcal{B}) := \sup_{N\in\mathbb{N}} \|S_N\|$$

is called the *basis constant* for $\mathcal{B}$. The coefficient space associated with $\mathcal{B}$ is the sequence space given by

$$\mathcal{C}(\mathcal{B}) := \big\{ \{y_n(x)\}_{n\in\mathbb{N}} : x \in \mathbb{H} \big\}.$$

The coefficient space $\mathcal{C}(\mathcal{B})$ inherits a norm from $\mathbb{H}$ in a natural way. Whenever $\mathcal{B}$ is normalized, it follows from (2.5) that we have the continuous embedding $\mathcal{C}(\mathcal{B}) \hookrightarrow \ell^\infty(\mathbb{N})$. However, for Schauder bases of integer translates, we will give an improved estimate in Theorem 3.4 that depends on certain finer properties of p_ψ.

It was demonstrated in [15] that $\mathcal{T} := \{e_k : k \in \mathbb{Z}\}$, with $\mathbb{Z}$ ordered the natural way as $0, 1, -1, 2, -2, \ldots$, forms a Schauder basis for $L^2(\mathbb{T}; p_\psi)$ if and only if the periodization function p_ψ satisfies the so-called $A_2(\mathbb{T})$ condition.

Definition 2.1. Let $\mathcal{I}$ be the collection of finite intervals of $\mathbb{R}$, and let $1 < p < \infty$. A measurable, 1-periodic function $w : \mathbb{R} \to (0, \infty)$ is an $A_p(\mathbb{T})$-weight provided that

$$[w]_{A_p} := \sup_{I\in\mathcal{I}} \left(\frac{1}{|I|} \int_I w(\xi)\, d\xi \right) \left(\frac{1}{|I|} \int_I w(\xi)^{-\frac{1}{p-1}}\, d\xi \right)^{p-1} < \infty.$$

The $A_2(\mathbb{T})$-condition will be of special importance to us, so we remark that the $A_2(\mathbb{T})$-weight condition simplifies to

$$[w]_{A_2} := \sup_{I\in\mathcal{I}} \left(\frac{1}{|I|} \int_I w(\xi)\, d\xi \right) \left(\frac{1}{|I|} \int_I w(\xi)^{-1}\, d\xi \right) < \infty.$$

It is known that the $A_2(\mathbb{T})$-condition is equivalent to the so-called Helson-Szegö condition, see [6,9].

We say that $w : \mathbb{R} \to (0, \infty)$ is an A_1-weight if there exists $C < \infty$ such that for any $x \in \mathbb{T}$ and $I \in \mathcal{I}$ with $x \in I$,

$$\frac{1}{|I|} \int_I w(t)\, dt \le Cw(x).$$

Let us state some well-known properties of A_p-weights that will be used throughout the paper. For proofs of the properties stated in Lemma 2.2 see e.g. [3].

Lemma 2.2. *The following holds true:*

a. *For any $A_2(\mathbb{T})$-weight w, we have $w, 1/w \in L^1(\mathbb{T})$.*
b. *For $w \in A_1(\mathbb{T})$ there exists $c := c(w) > 0$ such that $c \le w(x)$ for a.e. $x \in \mathbb{T}$.*
c. *For $1 \le p \le q < \infty$, $A_p(\mathbb{T}) \subseteq A_q(\mathbb{T})$ with $[w]_{A_q(\mathbb{T})} \le [w]_{A_p(\mathbb{T})}$ for any $w \in A_p(\mathbb{T})$.*
d. *Suppose $w \in A_p(\mathbb{T})$, $1 \le p < \infty$. Then there exists $\delta := \delta(w) > 0$ such that $w^{1+\eta} \in A_p(\mathbb{T})$.*
e. *For $w \in A_p(\mathbb{T})$, $1 \le p < \infty$ and $0 < \theta \le 1$, $w^\theta \in A_p(\mathbb{T})$ with $[w^\theta]_{A_p(\mathbb{T})} \le [w]_{A_p(\mathbb{T})}^\theta$.*

3. The distance to L^∞ in BMO and the connection to Schauder bases

The A_p classes are closely related to the functions of bounded mean oscillation. The space of such functions will play a central role in our analysis below.

Definition 3.1. Let $f \in L^1_{\text{loc}}(\mathbb{R})$ be 1-periodic, and let $\mathcal{I}$ be the collection of finite intervals of $\mathbb{R}$. We say that $f \in BMO(\mathbb{T})$ provided that

$$(3.1) \qquad \|f\|_{BMO(\mathbb{T})} := \sup_{I \in \mathcal{I}} \frac{1}{|I|} \int_I |f(x) - f_I|\, dx < \infty,$$

where $f_I := \frac{1}{|I|} \int_I f(x)\, dx$.

It is easy to check that $L^\infty(\mathbb{T}) \hookrightarrow BMO(\mathbb{T})$. For $f \in BMO(\mathbb{T})$ we can therefore consider the distance to L^∞ given by

$$(3.2) \qquad \text{dist}(f, L^\infty(\mathbb{T})) := \inf_{g \in L^\infty(\mathbb{T})} \|f - g\|_{BMO(\mathbb{T})}.$$

The distance to L^∞ in BMO is of interest to our study of Schauder bases since L^∞ is exactly the class of BMO-functions coming from logarithms of periodization functions associated with Riesz bases. Hence one may speculate that the distance to L^∞ could, with a suitable intrepretation, measure "closeness" to a Riesz basis. Theorems 3.4 and 3.8 below give two such intrepretations.

It is known that $L^\infty(\mathbb{T})$ is not a closed subset of $BMO(\mathbb{T})$. In fact, one has (see [3, p. 474])

$$\{f \in BMO(\mathbb{T}) : \mathrm{dist}(f, L^\infty) = 0\} = \left\{f \in BMO(\mathbb{T}) : e^f, e^{-f} \in \bigcap_{p>1} A_p(\mathbb{T})\right\}$$

$$= \left\{f \in BMO(\mathbb{T}) : e^{mf} \in A_2(\mathbb{T}), \quad \forall m \in \mathbb{Z}\right\}.$$

Example 3.2. An example of an unbounded BMO function in $\{\mathrm{dist}(f, L^\infty) = 0\}$ is given by

$$f(x) = \ln\left(\ln(2 + |x|^{-1})\right), \qquad x \in \mathbb{T}.$$

This is a consequence of the fact that $\ln^N(2 + |x|^{-1}) \in A_2(\mathbb{T})$ for any $N \in \mathbb{N}$, which follows by a direct calculation.

For $f \in BMO(\mathbb{T})$ we can also introduce the following quantity

(3.3) $$\varepsilon(f) = \inf\{\lambda > 0 : [e^{f/\lambda}]_{A_2(\mathbb{T})} < \infty\}.$$

The John-Nirenberg inequality ([10]) implies that for $f \in BMO(\mathbb{T})$ there is some $\alpha > 0$ such that $e^{\alpha f} \in A_2(\mathbb{T})$, so we always have $0 \le \lambda(f) < \infty$.

The celebrated result by Garnett and Jones ([4]) asserts that $\mathrm{dist}(f, L^\infty)$ and $\varepsilon(f)$ are in fact equivalent independent of $f \in BMO(\mathbb{T})$.

Theorem 3.3 ([4]). *There exist positive constants C_1 and C_2 such that for $f \in BMO(\mathbb{T})$,*

(3.4) $$C_1 \varepsilon(f) \le dist(f, L^\infty(\mathbb{T})) \le C_2 \varepsilon(f).$$

The Reverse Hölder inequality shows that if $w \in A_2(\mathbb{T})$ then always $\varepsilon(\ln w) < 1$ since there exists $\delta > 0$ such that $w^{1+\delta} \in A_2(\mathbb{T})$.

We now prove that for a Schauder basis of integer translates, a small distance to L^∞ of $\ln p_\psi$ gives added control of the coefficient space for the Schauder basis. Indeed, if the distance is zero then the coefficient spaces is very close to being contained in ℓ^2.

Theorem 3.4. *Let $\psi \in L^2(\mathbb{R})$ and suppose that the periodization function $p_\psi \in A_2(\mathbb{T})$. We let $\mathcal{C}(\mathcal{E})$ denote the coefficient space associated with the Schauder basis $\mathcal{E} = \{\psi(\cdot - k)\}_k$ for $\langle \psi \rangle$. Define $\varepsilon = \varepsilon(\ln p_\psi) := \inf\{\lambda > 0 : [p_\psi^{1/\lambda}]_{A_2} < \infty\}$. Then the following inclusion holds*

$$\mathcal{C}(\mathcal{E}) \subset \bigcap_{p_0 < p < \infty} \ell^p(\mathbb{Z}),$$

where $p_0 := \frac{2}{1-\varepsilon}$.

Proof. Let $p_\psi \in A_2(\mathbb{T})$ so in particular $p_\psi, 1/p_\psi \in L^1(\mathbb{T})$. We have the isomorphic isometry between $J_\psi : L^2(\mathbb{T}, p_\psi) \to \langle \psi \rangle$ given by

$$J_\psi m := (m\hat{\psi})^\vee$$

We notice that we have the continuous embedding

(3.5) $$L^2(\mathbb{T}, p_\psi) \hookrightarrow L^1(\mathbb{T}),$$

which follows from the Cauchy-Schwarz inequality using that

$$\|h\|_{L^1(\mathbb{T})} = \|h \cdot \sqrt{p_\psi} \cdot p_\psi^{-1/2}\|_{L^1(\mathbb{T})} \leq \|h\|_{L^2(\mathbb{T}, p_\psi)} \|p_\psi^{-1}\|_{L^1(\mathbb{T})}^{1/2}, \qquad h \in L^2(\mathbb{T}, p_\psi).$$

Now we take any

$$f = \lim_{N \to \infty} \sum_{|k| \leq N} \langle f, T_k \tilde\psi \rangle T_k \psi \in \langle \psi \rangle,$$

and let $m_f = J_\psi^{-1}(f) \in L^2(\mathbb{T}, p_\psi)$. The trigonometric system $\mathcal{T} := \{e_k\}_{k \in \mathbb{Z}}$ forms a Schauder basis for $L^2(\mathbb{T}, p_\psi)$ since $p_\psi \in A_2(\mathbb{T})$. In particular, the expansion of m_f relative to $\mathcal{T}$ is norm convergent in $L^2(\mathbb{T}, p_\psi)$ and thus in $L^1(\mathbb{T})$ by the embedding (3.5). An easy calculation shows that

$$m_f = \lim_{N \to \infty} \sum_{|k| \leq N} \langle m_f, \tilde{e}_k \rangle_{L^2(\mathbb{T}, p_\psi)} e^{2\pi i k x}$$

$$= \lim_{N \to \infty} \sum_{|k| \leq N} \int_{\mathbb{T}} m_f(\xi) \frac{e^{-2\pi i k \xi}}{p_\psi(\xi)} p_\psi(\xi) d\xi \cdot e^{2\pi i k x}$$

$$= \lim_{N \to \infty} \sum_{|k| \leq N} \langle m_f, e_k \rangle_{L^2(\mathbb{T})} e^{2\pi i k x},$$

with convergence in $L^2(\mathbb{T}, p_\psi)$ and in $L^1(\mathbb{T})$. Hence, using the isometry J_ψ,

$$\langle f, T_k \tilde\psi \rangle = \langle m_f, e_k \rangle_{L^2(\mathbb{T})},$$

with $m_f \in L^1(\mathbb{T})$. Hence, $\mathcal{C}(\mathcal{E})$ is exactly the family of Fourier coefficient sequences of the periodic functions m_f, $f \in \langle \psi \rangle$.

Next we use the $A_2(\mathbb{T})$ properties of p_ψ to obtain additional information about m_f. It holds true that $\varepsilon < 1$ since $p_\psi^{1+\eta} \in A_2(\mathbb{T})$ for some $\eta > 0$ by the Reverse Hölder Inequality. We notice that for any $1 < \eta < 1/\varepsilon$, $p_\psi^\eta \in A_2(\mathbb{T})$. In particular, $p_\psi^{-\eta} \in L^1(\mathbb{T})$ so $p_\psi^{-1/2} \in L^{q_0}(\mathbb{T})$ for $q_0 := 2\eta > 2$. By the generalized Hölder inequality, for $1/r = 1/2 + 1/q_0 \Rightarrow r = 2\eta/(1+\eta)$,

$$\|m_f\|_{L^r(\mathbb{T})} = \|(m_f \sqrt{p_\psi}) \cdot p_\psi^{-1/2}\|_{L^r(\mathbb{T})} \leq \|m_f\|_{L^2(\mathbb{T}, p_\psi)} \cdot \|p_\psi^{-1/2}\|_{L^{q_0}(\mathbb{T})} < +\infty.$$

By the Hausdorff-Young inequality, the Fourier coefficients of m_f are contained in $\ell^{r'}(\mathbb{Z})$ where $r' = r/(r-1) = 2\eta/(\eta - 1)$. Finally we let $\eta \to 1/\varepsilon$ to conclude. $\qquad \square$

Remark 3.5. It is perhaps not surprising that the most restrictive inclusion in Theorem 3.4 happens when $\operatorname{dist}(\ln p_\psi, L^\infty) = 0$, which happens if and only if $\varepsilon(\ln p_\psi) = 0$. In this case, we have

$$\mathcal{C}(\mathcal{E}) \subset \bigcap_{2 < p < \infty} \ell^p(\mathbb{Z}).$$

Remark 3.6. It is known that the Fourier transform $\mathcal{F} : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$, $1 \leq p \leq 2$, fails to be onto $\ell^{p'}(\mathbb{Z})$ unless $p = 2$, so it is in fact not possible that the stronger conclusion $\mathcal{C}(\mathcal{E}) = \cap_{p_0 < p < \infty} \ell^p(\mathbb{Z})$ can hold in Theorem 3.4.

Example 3.7. Define $\psi \in L^2(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \sqrt{\ln\left(\ln(2 + |\xi|^{-1})\right)} \cdot \chi_{[0,1)}(\xi).$$

It follows that $p_\psi(\xi) = \ln\left(\ln(2 + |\xi|^{-1})\right), \xi \in [-1/2, 1/2]$. A direct calculation shows that $p_\psi \in A_2(\mathbb{T})$, so $\mathcal{E} := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$ forms a Schauder basis for $\langle \psi \rangle$. However, p_ψ is not bounded and consequently $\mathcal{E}$ fails to be an unconditional Riesz basis for $\langle \psi \rangle$. However, according to Example 3.2, $\text{dist}(\ln p_\psi, L^\infty) = 0$, so the coefficient space for $\mathcal{E}$ is controlled by $\mathcal{C}(\mathcal{E}) \subset \bigcap_{2 < p < \infty} \ell^p(\mathbb{Z})$.

3.1. Improved conditioning of Schauder bases.

A well-known conditioning step in the construction of multiresolution analysis based orthonormal wavelets is to transform a Riesz basis of the form

$$\mathcal{S} := \{g(\cdot - k)\}_{k \in \mathbb{Z}}$$

to an orthonormal basis for the same space by switching to a new improved generator given by $\hat{\varphi} = \hat{g}/\sqrt{p_g}$, with $p_g = \sum_{k \in \mathbb{Z}} |\hat{g}(\cdot - k)|^2$. One can show that $c_1 \leq p_g(\xi) \leq c_2$ for two positive constants c_1 and c_2, so the tranformation is actually carried out by the bounded and bijective multiplier operator on $L^2(\mathbb{R})$ defined by $f \to (\frac{\hat{f}}{\sqrt{p_g}})^\vee$. In particular, the old Riesz basis $\mathcal{S}$ and the new orthonormal basis generated by φ are equivalent bases.

Next we study what happens if the starting point $\mathcal{S}$ is "only" a conditional Schauder basis and we are restricted to transforming the system by a bounded and invertible multiplier operator on $L^2(\mathbb{R})$. It is clearly not possible to transform such a basis to an orthonormal system by a bounded and bijective map since a conditional basis can never be equivalent to an unconditional one, but we will demonstrate that improvement on the original basis can still be obtained. We have the following result.

Theorem 3.8. *Let $\psi \in L^2(\mathbb{R})$ with periodization function $p_\psi \in A_2(\mathbb{T})$. Suppose p_ψ satisfies $\text{dist}(\ln p_\psi, L^\infty) = 0$. We let $\mathcal{E}$ be given by (1.1). Then we have*

   i. *If $\ln p_\psi \in L^\infty(\mathbb{T})$ then $\mathcal{E}$ forms a Riesz basis for $\langle \psi \rangle$.*
   ii. *If $\ln p_\psi \notin L^\infty(\mathbb{T})$ then for every $\eta > 0$ there exists $b \in L^\infty(\mathbb{T})$ such that $\tilde{\mathcal{E}} = \{\varphi(\cdot - k)\}_k$, with $\hat{\varphi} := \frac{\hat{\psi}}{e^b}$, forms a Schauder basis for $\langle \psi \rangle$ with Schauder basis constant at most $3 + \mathcal{O}(\eta)$. The Schauder bases $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent.*

Proof. For (i) we immediately obtain positive constants c_1, c_2 such that $c_1 \leq p_\psi(\xi) \leq c_2$ for $\xi \in \mathbb{T}$. Then the Schauder basis property then follows from a standard result, see e.g. [17].

For (ii), we let $\eta > 0$ be given, where we may assume $\eta \ll 1$. We pick $b \in L^\infty$ such that $\|\ln p_\psi - b\|_{BMO(\mathbb{T})} < \eta$. It follows from the John-Nirenberg inequality, by exponentiating, that $[p_\psi e^{-b}]_{A_2(\mathbb{T})} = 1 + \mathcal{O}(\eta)$. The best possible constant for this estimate is studied in [12]. It now follows from [18] that the Riesz projection P_+ onto H^2 for $f \in L^2(\mathbb{T}; p_\psi e^{-b})$ has norm $1 + \mathcal{O}(\eta)$. As is well-known, we can write for any

$f \in L^2(\mathbb{T}; p_\psi e^{-b})$,

$$S_{2N+1}(f) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ik\cdot} = e^{-2\pi iN\cdot}P_+(e^{2\pi iN\cdot}f) - e^{2\pi i(N+1)\cdot}P_+(e^{-2\pi i(N+1)\cdot}f),$$

so $\|S_{2N+1}\| \leq 2 + \mathcal{O}(\eta)$, $N \in \mathbb{N}$. This takes care of partial sums with an odd number of terms.

Finally we notice that $S_{2N+2} = S_{2N+1} + \langle f, \tilde{e}_{-N-1}\rangle e_{-N-1}$, with $\tilde{e}_k = \frac{e_k}{p_\psi e^b}$ and $\langle \cdot, \cdot \rangle$ the standard inner product on $L^2(\mathbb{T})$. The linear operators $f \mapsto \langle f, \tilde{e}_k\rangle e_k$ are uniformly bounded on $L^2(\mathbb{T}; p_\Psi)$ and their operator norms are given by

$$\|e_k\|_{L^2(\mathbb{T};p_\psi e^{-b})} \cdot \|\tilde{e}_k\|_{L^2(\mathbb{T};p_\psi e^{-b})} = \sqrt{\int_{\mathbb{T}} p_\psi(\xi)e^{-b(\xi)}\,d\xi} \cdot \sqrt{\int_{\mathbb{T}} \frac{1}{p_\psi(\xi)e^{-b(\xi)}}\,d\xi}$$
$$\leq [p_\psi e^{-b}]_{A_2}^{1/2}$$
$$= 1 + \mathcal{O}(\eta),$$

for every $k \in \mathbb{Z}$. Hence, we have the bound $\|S_N\| \leq 3 + \mathcal{O}(\eta)$ for any $N \in \mathbb{N} \cup \{0\}$.

The equivalence of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ follows immediately from the fact that the multiplier operator $f \to (e^{-b}\hat{f})^\vee$ on $L^2(\mathbb{R})$ is bounded and bijective since $e^{-b}, e^b \in L^\infty(\mathbb{T})$. $\square$

3.2. Convolution roots and powers.
The Garnett-Jones formula (see Theorem 3.3) shows that an $A_2(\mathbb{T})$-weight w remains in $A_2(\mathbb{T})$ for any power less than $1/\varepsilon(\ln w) \asymp 1/\text{dist}(\ln w, L^\infty)$. In this section we propose one interpretation of this fact related to stability of Schauder bases of integer translates under certain "convolution powers" of the Schauder basis generator ψ.

For $\psi \in L^2(\mathbb{R})$ we formally define the convolution powers of ψ by

$$\psi^{(1)} := \psi, \quad \psi^{(2)} := \psi * \psi^{(1)}, \dots, \psi^{(k+1)} := \psi * \psi^{(k)}, \dots.$$

The convolution Nth root of ψ denoted by $\psi^{(1/N)}$ is defined by $\hat{\psi}^{(1/N)}(\xi) := |\hat{\psi}|^{1/N}(\xi)$, $\xi \in \mathbb{R}$. We notice that the "power and root" operations are not, in general, bounded on $L^2(\mathbb{R})$, so we can only perform these mappings on "nice" ψ.

We now focus on bandlimited $\psi \in L^2(\mathbb{R})$. Suppose there exists $K \in \mathbb{N}$ such that $\text{Supp}(\hat{\psi}) \subseteq [-K, K]$. Notice in particular that

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi - k)|^2 = \sum_{k \in F_\xi} |\hat{\psi}(\xi - k)|^2,$$

with $F_\xi = \{k \in \mathbb{Z} : \hat{\psi}(\xi - k) \neq 0\}$. By the assumption on the support of $\hat{\psi}$ we deduce that $\#F_\xi \leq 2K + 1$ for a.e. $\xi \in \mathbb{T}$. For $0 < \beta < \infty$ we use the equivalence of any two norms on $\mathbb{R}^{2K+1}$ to obtain constants c_1^β, c_2^β such that

$$(3.6) \qquad c_1^\beta p_\psi^\beta(\xi) = c_1\Big(\sum_{k \in F_\xi} |\hat{\psi}(\xi - k)|^2\Big)^\beta \leq \sum_{k \in F_\xi} |\hat{\psi}(\xi - k)|^{2\beta} \leq c_2^\beta p_\psi^\beta(\xi), \quad a.e.$$

The estimate (3.6) can be used to deduce the following result.

Proposition 3.9. *Let $\psi \in L^2(\mathbb{R})$ be bandlimited and suppose that the periodization function $p_\psi \in A_2(\mathbb{T})$. Define $\varepsilon = \varepsilon(\ln p_\psi) := \inf\{\lambda > 0 : [p_\psi^{1/\lambda}]_{A_2} < \infty\}$. Then*

- i. *For any $N \in \mathbb{N}$, $\psi^{(1/N)}$ generates a Schauder basis for $\langle \psi^{(1/N)} \rangle$.*
- ii. *Suppose $\varepsilon > 0$. Then for $N \in \mathbb{N}$ such that $N < \varepsilon^{-1}$, the convolution power $\psi^{(N)}$ generates a Schauder basis for $\langle \psi^{(N)} \rangle$.*
- iii. *If $\varepsilon = 0$ then the convolution power $\psi^{(N)}$ generates a Schauder basis for $\langle \psi^{(N)} \rangle$ for any $N \in \mathbb{N}$.*

Proof. For (i), we notice that the periodization function for $\psi^{(1/N)}$ is given by

$$p_{\psi^{(1/N)}}(\xi) = \sum_{k \in \mathbb{Z}} |\hat{\psi}^{(1/N)}(\xi - k)|^2 = \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi - k)|^{2/N}.$$

Using (3.6) we immediately obtain that uniformly for a.e. $\xi \in \mathbb{T}$,

$$p_{\psi^{(1/N)}}(\xi) \asymp p_\psi^{1/N}(\xi).$$

However, since $p_\psi \in A_2(\mathbb{T})$ we also have $p_\psi^{1/N} \in A_2(\mathbb{T})$ according to Lemma 2.2 and the result follows.

For (ii) and (iii) the proofs are very similar. We use fact that $\hat{\psi}^{(N)}(\cdot) = \hat{\psi}^N(\cdot)$ to obtain

$$p_{\psi^{(N)}}(\xi) \asymp p_\psi^N(\xi).$$

Then we notice that $p_\psi^N \in A_2(\mathbb{T})$ for $N < \varepsilon^{-1}$ in case $\varepsilon > 0$ [or that $p_\psi^N \in A_2(\mathbb{T})$ for any $N \in \mathbb{N}$ when $\varepsilon = 0$]. $\qquad\square$

4. DEMOCRATIC SCHAUDER BASES OF INTEGER TRANSLATES

A Schauder basis $\mathcal{B} = \{e_n\}_{n \in \mathbb{N}}$ for a Hilbert space $\mathbb{H}$ is said to be democratic if there exists $C > 0$ such that

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{H}} \leq C \left\| \sum_{k \in \Gamma'} e_k \right\|_{\mathbb{H}}, \tag{4.1}$$

for all finite sets $\Gamma, \Gamma' \subset \mathbb{N}$ with the same cardinality. It is known that for democratic Schauder bases in a Hilbert space, we necessarily have the uniform estimate (see [7, 20])

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{H}} \asymp \sqrt{|\Gamma|}, \qquad \Gamma \subset \mathbb{N}. \tag{4.2}$$

Democratic systems of integer translates $\mathcal{E}$ as defined by (1.1) were studied in [7]. It was shown in [7, Theorem 4.7] that for a democratic system $\mathcal{E}$ there is a finite constant B such that $p_\psi(\xi) \leq B$ a.e. In particular, if $\mathcal{E}$ is a democratic system then the dual system $\tilde{\mathcal{E}}$ is a Besselian system. Moreover, for a democratic system $\mathcal{E}$, it follows from Lemma 2.2.(b) that $p_\psi \in A_1(\mathbb{T})$ implies $p_\psi, 1/p_\psi \in L^\infty(\mathbb{T})$, which is equivalent to $\mathcal{E}$ being a Riesz basis.

We now improve the result given by Theorem 3.4 in the case we have a democratic system. The Lorentz sequence space $\ell^{2,1}(\mathbb{Z})$ is defined by

$$(4.3) \qquad \|\{c_k\}_k\|_{\ell^{2,1}(\mathbb{Z})} := \sum_{m=1}^{\infty} \frac{|c_m^\star|}{m^{1/2}} < \infty.$$

where $\{|c_k^\star|\}_k$ denotes a decreasing rearrangement of $\{c_k\}_k$. If we let $\Lambda_j = \{k : |c_k^\star| \geq 2^{-j}\}$ then it can be shown that

$$\|\{c_k\}_k\|_{\ell^{2,1}(\mathbb{Z})} \asymp \sum_{j\mathbb{Z}} 2^{-j} |\Lambda_j|^{1/2},$$

see [2].

Proposition 4.1. *Suppose $\mathcal{E}$ defined by (1.1) is a democratic Schauder basis of integer translates. Then we have the continuous embedding $\ell^{2,1}(\mathbb{Z}) \hookrightarrow \mathcal{C}(\mathcal{B})$, i.e., there exists a constant C such that*

$$\Big\| \sum_{k\in\mathbb{Z}} c_k T_k \psi \Big\|_{L^2(\mathbb{R})} \leq \|\{c_k\}_k\|_{\ell^{2,1}}.$$

Proof. An extremal point argument (see [7, Lemma 4.6]) shows that for any finite set $\Lambda \subset \mathbb{Z}$, and any set of coefficients $\{c_k\}_{k\in\mathbb{Z}}$,

$$\Big\| \sum_{k\in\Lambda} c_k T_k \psi \Big\|_{L^2(\mathbb{R})} \leq \max_{k\in\Lambda} |c_k| \max_{\varepsilon_k\in\{-1,1\}} \Big\| \sum_{k\in\Lambda} \varepsilon_k T_k \psi \Big\|_{L^2(\mathbb{R})} \leq C \max_{k\in\Lambda} |c_k| |\Lambda|^{1/2}.$$

Let $f = \sum_{k\in\mathbb{Z}} c_k T_k \psi$ be any finite expansion and denote by $\Lambda_j = \{k : |c_k^\star| \geq 2^{-j}\}$, where $\{c_k^*\}_k$ is a decreasing rearrangement of $\{c_k\}_k$ performed by the index-permutation $\pi : \mathbb{Z} \to \mathbb{Z}$. We have

$$\|f\|_{L^2(\mathbb{R})} = \Big\| \sum_{j=-\infty}^{\infty} \sum_{k\in\Lambda_j \setminus \Lambda_{j-1}} c_k^\star T_{\pi(k)} \psi \Big\|_{L^2(\mathbb{R})}$$

$$\leq \sum_{j=-\infty}^{\infty} \Big\| \sum_{k\in\Lambda_j \setminus \Lambda_{j-1}} c_k^\star T_{\pi(k)} \psi \Big\|_{L^2(\mathbb{R})}$$

$$\leq \sum_{j\in\mathbb{Z}} C 2^{-(j-1)} |\Lambda_j \setminus \Lambda_{j-1}|^{1/2}$$

$$\leq \tilde{C} \sum_{j\in\mathbb{Z}} 2^{-j} |\Lambda_j|^{1/2}$$

$$\leq C' \|\{c_k(f)\}\|_{\ell^{2,1}}.$$

The claim now follows. $\qquad\square$

We can now combine Thorem 3.4 and Proposition 4.1.

Corollary 4.2. *Let $\psi \in L^2(\mathbb{R})$ and suppose that the periodization function $p_\psi \in A_2(\mathbb{T})$. We let $\mathcal{C}(\mathcal{E})$ denote the coefficient space associated with the Schauder basis $\mathcal{E} = \{\psi(\cdot - k)\}_k$*

for $\langle\psi\rangle$. *Define* $\varepsilon = \varepsilon(\ln p_\psi) := \inf\{\lambda > 0 : [p_\psi^{1/\lambda}]_{A_2} < \infty\}$. *If* $\mathcal{E}$ *is democratic, then the following inclusions hold*

$$\ell^{2,1}(\mathbb{Z}) \subset \mathcal{C}(\mathcal{E}) \subset \bigcap_{p_0 < p < \infty} \ell^p(\mathbb{Z}),$$

where $p_0 := \frac{2}{1-\varepsilon}$. *In particular, when* $\varepsilon = 0$,

$$\ell^{2,1}(\mathbb{Z}) \subset \mathcal{C}(\mathcal{E}) \subset \bigcap_{2 < p < \infty} \ell^p(\mathbb{Z}).$$

Let us complete this article by the following observation. Since in the democratic case the periodization function p_ψ is bounded above, it is not difficult to prove, using the Orlicz theorem on unconditional convergence (see [19]), that infinite linear combinations of vectors from our basis will converge unconditionally if and only if the coefficients belong to $\ell^2(\mathbb{Z})$. Hence, if we have a democratic conditional Schauder basis of translates, with $\ln p_\psi$ in the closure of L^∞ in $BMO(\mathbb{T})$ (see Example 4.3 below), then the only conditional sums that such bases will produce will consist of sums with coefficients which are in all $\ell^p(\mathbb{Z})$ spaces, for $p > 2$, but are not in $\ell^2(\mathbb{Z})$.

Example 4.3. For $0 < a < 1/2$, we define $\psi_a \in L^2(\mathbb{R})$ by

$$\hat{\psi}_a(\xi) = \frac{\chi_{[0,1)}(\xi)}{\sqrt{\ln\left(\ln(2 + |\xi - a|^{-1})\right)}}.$$

It follows that $p_{\psi_a}(\xi) = \ln\left(\ln(2 + |\xi - a|^{-1})\right)^{-1}$, $\xi \in [-1/2, 1/2)$. A direct calculation shows that $p_{\psi_a} \in A_2(\mathbb{T})$, so $\mathcal{E}_a := \{\psi_a(\cdot - k) : k \in \mathbb{Z}\}$ forms a Schauder basis for $\langle\psi_a\rangle$, and by [7, Corollary 4.20] $\mathcal{E}_a$ is democratic. We notice that p_{ψ_a} is not bounded from below so $\mathcal{E}_a$ is a conditional Schauder basis for $\langle\psi_a\rangle$. However, by a direct calculation (see Example 3.2), dist$(\ln p_{\psi_a}, L^\infty) = 0$ so the coefficient space for $\mathcal{E}_a$ is controlled by $\ell^{2,1}(\mathbb{Z}) \subset \mathcal{C}(\mathcal{E}_a) \subset \bigcap_{2 < p < \infty} \ell^p(\mathbb{Z})$.

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