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by

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All P_3-equipackable graphs

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Abstract

A graph G is P_3-equipackable if any sequence of successive removals of edge-disjoint copies of P_3 from G always terminates with a graph having at most one edge. All P_3-equipackable graphs are characterised. They belong to a small number of families listed here.

Keywords: Packing, equipackable, randomly packable, covering, factor, decomposition, equiremovable

2000 Mathematics Subject Classification: 05C70, 05C35

1 Introduction

Let H be a subgraph of a graph G. An *H-packing* in G is a partition of the edges of G into disjoint sets, each of which is the edge set of a subgraph of G isomorphic to H, and possibly a remainder set. For short, $E(G)$ is partitioned into copies of H and maybe a remainder set. We list some references to an extensive literature at the back. A graph is called *H-packable* if G is the union of edge disjoint copies of H. An H-packing is *maximal* if the remainder set of edges is empty or contains no copy of H. An H-packing is *maximum* if $E(G)$ has been partitioned into a maximum number of sets isomorphic to H and a possible remainder set. A graph is called *H-equipackable* if any maximal H-packing is also a maximum H-packing, i.e., a graph G is H-equipackable if successive removals of copies of H from G can be done the same number of times regardless of the particular choices of edge sets for H in each step. If every maximal H-packing of a graph G uses all edges of G, then G is called *randomly H-packable*. Equivalently, G is randomly H-packable if each H-packing can be extended to an H-packing containing all edges of G, see e.g. [1, 2, 5, 6].
Zhang and Fan [9] have studied H-equipackable graphs for the case $H = 2K_2$. We shall consider path packing and in the following H will always be assumed to be the graph P_3, the path of length two, and we may omit explicit reference to it. A graph G is then (P_3-) equipackable if successive removals of two adjacent edges from G can be done the same number of times

regardless of the particular choices of edge pairs in each step. A component consisting of one vertex is called *trivial*, a *non-trivial* component contains an edge. A graph has *order* $|V(G)|$ and *size* $|E(G)|$. A graph of odd (even) size is called *odd* (even). A vertex of valency one is called a *leaf*. A star is called even if its size is even, and by $K_{1,2k}$ we denote the even star with $2k$ leaves.

Observation 1 *A graph is randomly H-packable if and only if it is H-packable and H-equipackable.*

S. Ruiz [7] characterised randomly P_3-packable graphs.

Theorem 2 *A connected graph G is randomly packable if and only if $G \cong C_4$ or $G \cong K_{1,2k}$, $k \geq 1$.*

Y. Caro, J. Schönheim [3] and S. Ruiz [7] stated the following result.

Lemma 3 *A connected graph is packable if and only if it has even size.*

This immediately implies Corollary 4 below.

Corollary 4 *If a connected graph is equipackable, a maximal packing either contains all edges or all but one edge of the graph.*

From B.L. Hartnell, P.D. Vestergaard [4] and P.D. Vestergaard [8] we have the following observation.

Observation 5 *Let G be an equipackable graph. Then any sequence of P_3-removals from G will produce an equipackable graph.*

From Corollary 4 and Observation 5 we obtain

Corollary 6 *Let G be a connected graph. If there is a sequence of P_3-removals from G that creates more than one component of odd size, then G is not equipackable.*

We now state our main result, a characterisation of all equipackable graphs with at most one non-trivial component:

Theorem 7 *Let $G = (V, E)$ be a graph with at most one non-trivial component. Then G is equipackable if and only if its non-trivial component belongs to one of the thirteen families listed in Figure 1 or can be obtained by a sequence of P_3-removals from such a graph.*

Clearly, we wish those thirteen families listed to be *maximal w.r.t. P_3-removals*, i.e., no graph from one of the families can be obtained as a subgraph of a larger equipackable graph by removing a P_3 from it.
In the figures below we indicate by an arrow from which family of graphs we may obtain the given graph by a sequence of P_3-deletions. The shaded vertex sets may vary in cardinality.

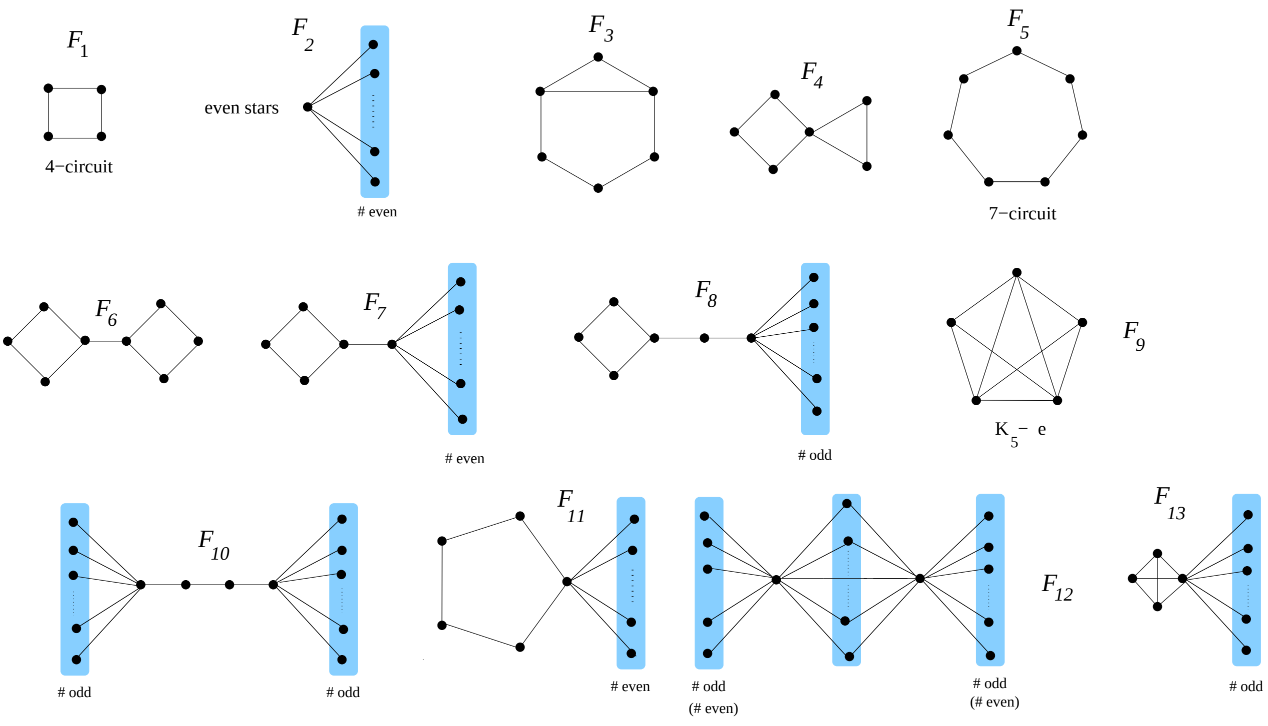

Figure 1: All connected, maximal with respect to P_3–removal, P_3–equipackable graphs

We will prove this characterisation in the following section.

2 Proof of Theorem 7

By Lemma 3 and Theorem 2 a graph with at most one non-trivial component, which has even size, is equipackable if and only if its non-trivial component is a 4-circuit or an even star (Figure 2). Thus it only remains to characterise equipackable graphs with at most one non-trivial component of odd size.

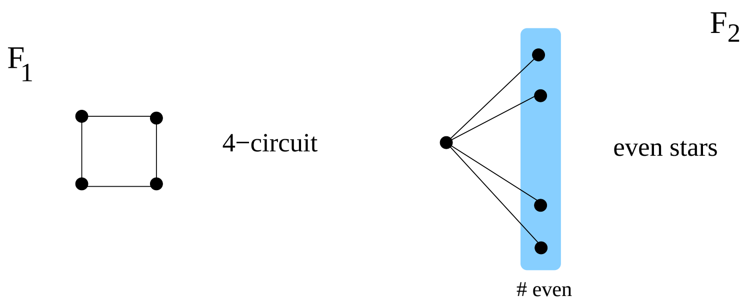

Figure 2: Connected P_3–equipackable graphs of even size (Ruiz graphs)

In [8] P.D. Vestergaard examined equipackable graph with all degrees ≥ 2 and stated the following result.

Theorem 8 *A connected graph G with all degrees ≥ 2 is equipackable if and only if G is one of the graphs listed in Figure 3.*

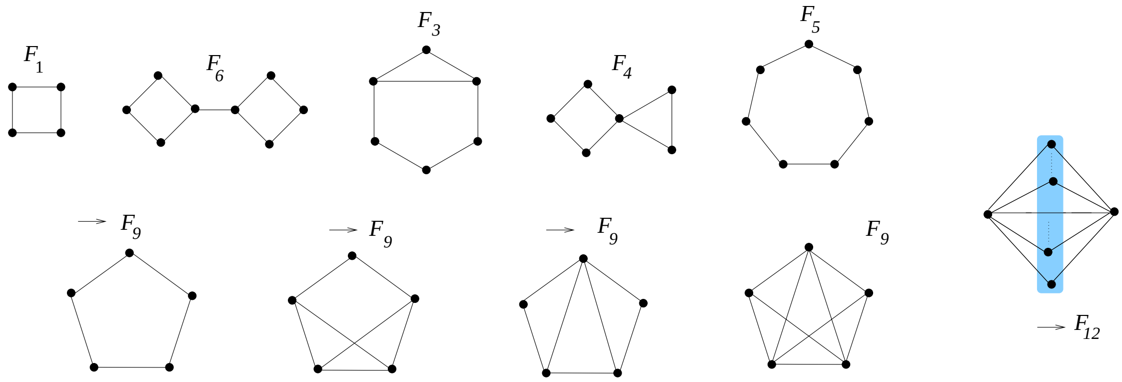

Figure 3: All connected P_3–equipackable graphs G without leaves

Observe that this solution contributes to our characterisation five graphs $(F_6, F_3, F_4, F_5, F_9)$ maximal with respect to P_3-removals. All other graphs of this solution are obtained by a sequence of P_3-removals from graphs of the thirteen graph families of our characterisation. Thus it now remains to characterise equipackable graphs G which have only one non-trivial component, say H, where H has odd size and contains a leaf.

Since H has a leaf, it also has a bridge. Let $b = xy$ be a bridge of H. Throughout we shall denote the two components of $H - xy$ by H_1 and H_2 with $x \in V(H_1), y \in V(H_2)$. We shall first treat the case that G has a non-leaf bridge, then the case that all bridges are leaf bridges.

Case 1: *Assume $b = xy$ is a non-leaf bridge of G, i.e., $deg(x) \geq 2$, $deg(y) \geq 2$.*
Subcase 1.1: *Assume further that H has a maximum P_3-packing $\mathcal{P}$ which does not contain b.*
Since $\mathcal{P}$ by Corollary 4 contains all but one edge of G and $b \notin \mathcal{P}$, we have for $i = 1, 2$ that $\mathcal{P} \cap H_i$ is a P_3-packing of H_i and therefore H_i has even size ≥ 2.
Let $z \in N(x) \setminus \{y\}$. By P_3-removal of zxy we obtain an equipackable graph which has an odd component contained in $H_1 - xz$, and $H - \{zx, xy\}$ also has the even component H_2 which is connected, randomly packable and hence, by Observation 1, is either a 4-circuit or an even star. By symmetry also H_1 is a 4-circuit or an even star. Therefore H belongs to one of the families of graphs depicted in Figure 4.

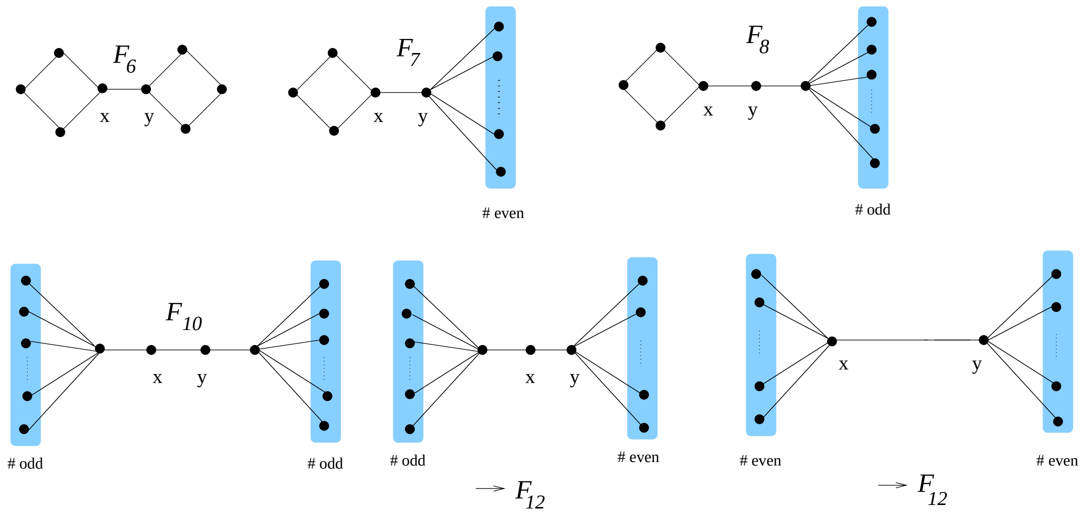

Figure 4: Connected, P_3–equipackable graphs in Case 1.1

Note that only three new graph families (F_7, F_8, F_{10}) maximal with respect to P_3-removals contribute in this case to our characterisation. All other graph families of this subcase are obtained by a sequence of P_3-removals from graphs of the thirteen graph families of our characterisation.

Subcase 1.2: *Assume now that each non-leaf bridge of H is contained in every maximum P_3-packing.*
With notation as above let $b = xy$ be a non-leaf bridge of H, the components of $H - xy$ are H_1, H_2. Their sizes have the same parity since H has odd size. If H_1, H_2 both had even size they would be P_3-packable and H would have a maximum P_3-packing not containing b in contradiction to assumption. Therefore H_1, H_2 both have odd size.

Claim: At least one of H_1, H_2 is an odd star.
Proof. P_3-removal from H of zxy, $z \in N(x) \setminus \{y\}$, creates an odd size component, namely H_2. If H_2 is an odd star we are finished. Otherwise, we can isolate an odd component inside H_2: If $\deg_{H_2}(y)$ is even we P_3-remove all edges incident to y in pairs and if $\deg_{H_2}$ is odd we P_3-remove all but one edge incident to y in pairs and that remaining edge $yw, w \in N(y)$, together with $wr, r \in N(w) \setminus \{y\}$ (Since H_2 is not an odd star there has to exist at least one such vertex w). Then $H_1 \cup \{xy\}$ is even, connected, randomly packable and hence is either a 4-circuit or an even star. Since $H_1 \cup \{xy\}$ contains a leaf, it is an even star and hence H_1 is an odd star. That proves the claim.□

Suppose H_1 and H_2 are both odd stars. Now assume that, say x, is not the center of H_1 and let v be the center of H_1. Since vx is a non-leaf bridge and there obviously exists a maximum P_3-packing $\mathcal{P}$ which does not contain vx, we obtain a contradiction to the assumption of Subcase 1.2. Hence we find that H is obtained by adding an edge between the centers of H_1 and H_2 (see Figure 5). Consequently H can be obtained from one of the graphs of the family F_{12} in our characterisation by P_3-deletions.

If, say, H_2 is an odd star and H_1 is not, then P_3-removal of zxy from H, $z \in N(x) \setminus \{y\}$, gives that $H_1 - xz$ has even size.
Now assume that zx is a leaf bridge of H (and likewise of H_1), i.e., $\deg_H(z) = 1$.
Then P_3-removal of zxy leaves the odd component H_2 and $H_1 - xz$ with one non-trivial even component. Thus the non-trivial even component of $H_1 - xz$ is either a 4-circuit or an even star. The former yields easily a non-equipackable graph, the latter gives that H_1 is an odd star, a contradiction to assumption on H_1.
Suppose now that zx is a non-leaf bridge of H (and likewise for H_1).
The two components of $H_1 - xz$ have sizes of same parity. That cannot be odd since $G - zxy$ would then have three odd components in contradiction to Corollary 6. It cannot be even either because then we could easily construct a maximum P_3-packing $\mathcal{P}$ which does not contain the non-leaf bridge xz, a contradiction to the basic assumption of this subcase.
So we may for all $z \in N(x) \setminus \{y\}$ assume that xz is not a bridge of H (and H_1).
P_3-removal of zxy for $z \in N(x) \setminus \{y\}$ produces the connected, even component $H_1 - xz$ which is then randomly P_3-packable and hence is either an even star or a 4-circuit. If $H_1 - xz$ is a 4-circuit we are immediately led to H not being equipackable because, if a, b, c, d are the edges of this 4-circuit (in cyclic order) then the packing $\{xy, a\}, \{xz, c\}$ cannot be extended to a maximum packing of H. Observe that we have $N(x) \setminus \{y\} = \{z_1, z_2, ..., z_p\}$ with $z_1 = z$ and $p \geq 2$. Thus for all $z_i \in N(x) \setminus \{y\}$ the connected graph $H_1 - xz_i$ is an even star.It follows that $p = 2$ and $H_1 - xz_i$ must always be isomorphic to a $P_3 = K_{1,2}$ with a center vertex z_{3-i} having neighbours x and z_i for $i = 1, 2$. Thus H_1 is a 3-circuit with vertices x, z_1, z_2 with x joined to y, and y has an odd number of leaves attached (see Figure 5).

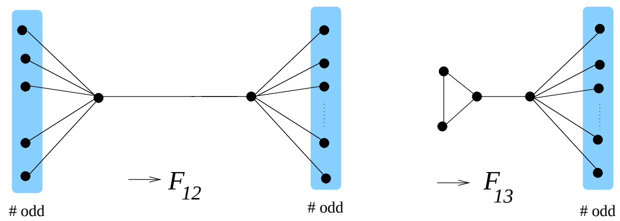

Figure 5: Connected P_3–equipackable graphs in Case 1.2

Observe that none of these equipackable graph families are new families maximal with respect to P_3-removals for our characterisation. Both graph families of this subcase are obtained by a sequence of P_3-removals from graphs of the graph families (F_{12}, F_{13}) of our characterisation. We may now assume that there exist no non-leaf bridge of H.

Case 2: *All bridges of H are leaf bridges and there exists at least one bridge $b = xy$ of H, i.e. $H_2 = \{y\}$.*
If all xz, $z \in N(x) \setminus \{y\}$, are bridges of H, then they are leaf bridges and H is an odd star, derivable from a member of our characterisation by a sequence of P_3-removals. Thus we may assume that x is contained in at least one cycle of H_1 and there exist at least two edges incident to x, which are not bridges.
If x has an even number of neighbours in H_1 we can isolate xy by pairing up and P_3-removing all $xz, z \in N(x) \setminus \{y\}$. If x has an odd number of neighbours in H_1 we isolate xy by P_3-removing

all xz, $z \in N(x) \setminus \{y\}$, and one further edge $zw, w \in N(z) \setminus \{x\}$ (observe that such an edge has to exist). For simplicity, let E' be the set of edges of all P_3's necessary to remove in order to isolate the bridge xy and $H' = H - E'$. Since xy is isolated in H' and H' is equipackable, we obtain by Lemmas 3, Observation 5 and Corollaries 4, 6 that all non-trivial components D not containing x and y are randomly packable and therefore of even size ≥ 2. Thus every such non-trivial component D is either a 4-circuit or an even star.
Assume that one of these components is a 4-circuit C with vertices $\{c_0, c_1, c_2, c_3\}$ and edges $\{c_i c_{(i+1) mod 4} | 0 \leq i \leq 3\}$.
As all bridges of H are leaf bridges, with $E_C = \{xc | c \in N_{H_1}(x) \cap V(C)\}$ we have $|E_C| \geq 2$. It is easy to see that we can remove two (if $|E(C)| = 2$) or three $P_3's$ from the subgraph of H induced by $\{x\} \cup V(C)$ to produce two (if $|E(C)| = 3$) or three isolated edges (including xy) in contradiction with Corollary 6.
If $|E_C| = 2$ there exist $i, j, k, \ell = \{0, 1, 2, 3\}$ such that $xc_i, xc_j \in E_C$ and $xc_i c_k, xc_j c_\ell$ are P_3's of H that isolate the two independent edges $c_i c_k, c_j c_\ell$ remaining in C. By Corollary 6 then H is not equipackable, a contradiction. If $|E_C| = 3$, without loss of generality we may assume that $E_C = \{xc_0, xc_1, xc_2\}$ and in that case $E' \cup \{c_0 c_3, c_1 c_2\}$ is an edge set of even size, which can paired up in P_3's whose removal isolate two edges $c_0 c_1$ and $c_2 c_3$ on C, by Corollary 6 that contradicts H being equipackable. If $|E_C| = 4$, again $E' \cup \{c_0 c_3, c_1 c_2\}$ has even size and can be paired up and P_3-removed to leave two independent edges $c_0 c_1$ and $c_2 c_3$ on C, giving a contradiction to H being equipackable.
Hence every such non-trivial component D not containing x and y is an even star.
Now suppose there exist two different components R_1 and R_2 of this kind. Analogously to the previous argumentation let E_{R_i} be the subset of E' of edges incident to the vertices of R_i for $i = 1, 2$. Since H is connected, and all bridges of H are leaf bridges there has to exist for each $i = 1, 2$ at least two edges f_i', f_i'' of E_{R_i} adjacent to an edge of R_i. Pairing up f_i' with one edge of $E(R_i)$, say f_i, $i = 1, 2$, and P_3-removing all remaining edges of E' (their number is even, recall that $f_i \notin E'$) will isolate two odd stars $E_{R_1} - f_1$ and $E_{R_2} - f_2$, a contradiction to Corollary 6. Thus there exists only one non-trivial component R of H' not containing x and y, and that is an even star.
We now distinguish between two cases depending on the parity of $\deg_{H_1}(x)$. Assume $\deg_{H_1}(x)$ is even. Then obviously H is, regardless of whether the centre r of R is adjacent to x or not, a member of the graph family F_{12} or can be obtained by a sequence of P_3-removals from a member of F_{12}.
Now it remains to consider that $\deg_{H_1}(x)$ is odd, i. e. $d_H(x)$ is even. As already noted at the beginning of Case 2 the vertex x must be contained in at least one cycle of H_1 and there exist at least two edges incident to x, which are not bridges. Since R is an even star $K_{1,2l}$ with $l \geq 1$ it is not difficult to deduce that the cycle has length ≤ 5. First let R be a star with at least four branches. Recall that E' is the set of edges of all P_3's necessary to remove in order to isolate the bridge xy and let $H' = H - E'$. Moreover, since x has an odd number of neighbours in H_1 we isolate xy by P_3-removing all xz, $z \in N(x) \setminus \{y\}$, and one further edge $zw, w \in N(z) \setminus \{x\}$. Regardless of the choice of this additional edge zw the remainder will be an even star with at least four edges. Concatenation of all ingredients builds up a member of F_{12} or a graph that can be obtained by a sequence of P_3-removals from a member of F_{12}. Therefore we conclude that R is always an even star with two branches regardless of the choice of the additional edge zw. By

inspection we obtain that H is either the graph F_{11} or F_{13} depicted in Figure 6.

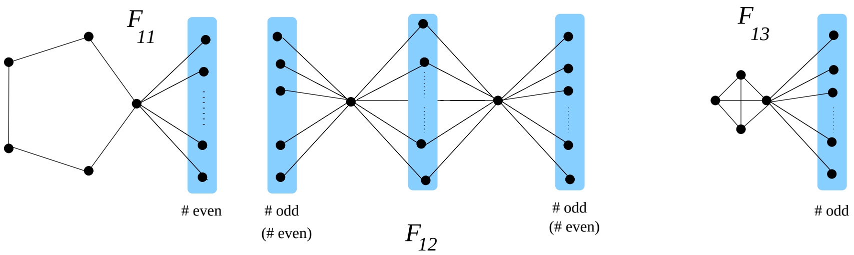

Figure 6: Connected P_3–equipackable graphs in Case 2

This completes the proof of our main result. □

The proof can also be done by induction on $|E(G)|$, but the arguments are not shorter.

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