New mixed Moore graphs and directed strongly regular graphs

Jørgensen, Leif Kjær

Publication date:
2013

Document Version
Early version, also known as pre-print

AALBORG UNIVERSITY

New mixed Moore graphs and directed strongly regular graphs

by

Leif Kjær Jørgensen

R-2013-13 December 2013

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 99 40 ▪ Telefax: +45 99 40 35 48
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

New Mixed Moore graphs and Directed Strongly Regular Graphs

L. K. Jørgensen
Dept. of Mathematical Sciences
Aalborg University
Fr. Bajers Vej 7, 9220 Aalborg, Denmark.
leif@math.aau.dk

December 11, 2013

Abstract

A directed strongly regular graph with parameters (n, k, t, λ, μ) is a k-regular directed graph with n vertices satisfying that the number of walks of length 2 from a vertex x to a vertex y is t if $x = y$, λ if there is an edge directed from x to y and μ otherwise. If $\lambda = 0$ and $\mu = 1$ then we say that it is a mixed Moore graph. It is known that there are unique mixed Moore graphs with parameters $(k^2 + k, k, 1, 0, 1)$, $k \geq 2$, and $(18, 4, 3, 0, 1)$. We construct a new mixed Moore graph with parameters $(108, 10, 3, 0, 1)$ and also new directed strongly regular graphs with parameters $(36, 10, 5, 2, 3)$ and $(96, 13, 5, 0, 2)$. This new graph on 108 vertices can also be seen as an example of a so called multipartite Moore digraph. Finally we consider the possibility that mixed Moore graphs with other parameters could exist, in particular the first open case which is $(40, 6, 3, 0, 1)$.

1 Introduction

Directed strongly regular graphs were introduced by Duval [5] in 1988. A directed strongly regular graph with parameters (n, k, t, λ, μ) is a directed graph with n vertices and with adjacency matrix A satisfying

$$A^2 = tI + \lambda A + \mu(J - I - A), \quad AJ = JA = kJ,$$

where I is the identity matrix and J is the all 1 matrix.

The special case with $\lambda = 0$ and $\mu = 1$ was considered earlier by Bosák [1] in 1979. Bosák defined a T-graph as a mixed graph for which there exists a number d (the diameter) so that for every ordered pair (x, y) of vertices there is a unique path from x to y of length at most d. Bosák showed that for a T-graph which is not an undirected tree, there exist numbers t and z so that every vertex is incident to t undirected edges and has z edges directed into it and z edges directed out from it. Such a graph is called a mixed Moore graph. Nguyen, Miller and Gimbert [14] showed that every mixed Moore graph has diameter 2, and thus it is a directed strongly regular graph with $\lambda = 0$, $\mu = 1$, $k = t + z$ and $n = k^2 + k - t + 1$.

Using eigenvalues of the adjacency matrix Bosák proved that

Proposition 1 *If a mixed Moore graph other than a directed triangle or an undirected 5-cycle exists then $t = \frac{c^2+3}{4}$ for some odd positive integer c where c divides $(4z - 3)(4z + 5)$.*

For $t = 1$ the so-called Kautz digraph (of diameter 2) on $(z + 2)(z + 1)$ vertices, i.e, the line-digraph $L(K_{z+2})$ of the complete directed graph $z + 2$ vertices, is a mixed Moore graph for every $z \geq 1$. Gimbert [8] proved that the Kautz digraphs are the unique mixed Moore graphs with $t = 1$.

For a group X and connection set $S \subset X$ we let $\mathrm{Cay}(X, S)$ denote the (directed) Cayley graph with vertex set X and edges (x, y) when $yx^{-1} \in S$.

If $q = z + 2$ is a prime power then the Kautz digraph $L(K_q)$ is a Cayley graph of the affine group $AGL_1(q)$ that consists of matrices $\begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$ over the field of q elements where $x \neq 0$. The connection set consists the matrices with $y = 1$.

In general $L(K_q)$ is a Cayley graph of a group X if and only if X has a sharply 2-transitive action on the vertices of K_q. It is known that such a group exists exactly when q is a prime power, see Dixon and Mortimer [4]. However, the group may belong to a larger class than those described above. Thus

Proposition 2 *The Kautz digraph $L(K_q)$ is a Cayley graph if and only if q is a prime power.*

For $t > 1$ the only known mixed Moore graph is the Bosák graph [1] with 18 vertices, $t = 3$ and $z = 1$, i.e., it is a directed strongly regular graph with

parameters $(18, 4, 3, 0, 1)$. Nguyen, Miller and Gimbert [14] proved that it is unique with these parameters.

In section 2 we describe a new mixed Moore graph denoted by G_{108} with 108 vertices and with $t = 3$, $z = 7$. In fact there are (at least) two such graphs: a second graph can be obtained by reversing the direction of all edges of G_{108}. We also indicate how G_{108} is related to the Bosák graph and a new directed strong regular graph with parameters $(36, 10, 5, 2, 3)$.

In section 3 we show that G_{108} is 6-partite and can be seen as an example of a multipartite Moore digraph, a concept introduced by Fiol, Gimbert and Miller [6].

In section 4 we describe a new directed strongly regular graph G_{96} with parameters $(96, 13, 5, 0, 2)$. G_{96} and G_{108} were both discovered in a computer search for directed strongly regular graph with $\lambda = 0$ appearing as Cayley graphs. The groups are non-abelian. In fact, in [9] we observed that it follows from a theorem by Klin, Munemasa, Muzychuk and Zieschang [11] that a directed strongly regular graph with $0 < t < k$ can not be a Cayley graph of an abelian group. A nice proof of this result was found by Lyubshin and Savchenko [12].

In section 4 we consider other feasible parameter sets for mixed Moore graphs. In particular, the first case which has $n = 40$ and $t = z = 3$. We consider the possibility that it contains a smaller mixed Moore graph as a subgraph and we consider some restrictions on the automorphism group of a mixed Moore graph with 40 vertices.

2 A mixed Moore graph with 108 vertices

We first consider the Bosák graph as it is related to the graph on 108 vertices.

In general we have the following.

Lemma 3 *The directed edges of a mixed Moore graph are partitioned in directed triangles.*

Proof If there is an edge directed from x to y then there must be a path of length 2 from y to x. This path has only directed edges. □

For mixed Moore graphs with directed degree $z = 1$ we have more information.

Lemma 4 *A mixed Moore graph G with $z = 1$ satisfies:*

- *The vertex set is partitioned in directed triangles, say with vertex sets* $V_1, \ldots, V_m$, $m = \frac{n}{3}$.
- *Either there are no edges between* V_i *and* V_j *or there is a matching between the sets.*
- *Let* C *be the undirected graph with vertices* $V_1, \ldots, V_m$ *and edges* V_iV_j *if there is a matching in* G *between* V_i *and* V_j*. Then* C *is a* t*-regular triangle-free graph in which any two non-adjacent vertices have exactly three common neighbours, i.e.,* C *is strongly regular with* $\lambda = 0$ *and* $\mu = 3$.
- *If* $t > 1$ *then the subgraph of* G *with only the undirected edges is an antipodal distance regular graph with diameter 4.*

Proof Suppose that $u_1 \to u_2 \to u_3 \to u_1$ and $w_1 \to w_2 \to w_3 \to w_1$ are directed triangles. If they are joined by an edge, say u_1w_1, then there is a path u_1, w_1, w_2 using a directed edge. Thus the path of length at most 2 from w_2 to u_1 also uses a directed edge. Since u_1 can have only one neighbour in $\{w_1, w_2, w_3\}$, the path from w_2 to u_1 uses an edge w_2u_3. Similarly w_3u_2 is present.

If V_i and V_j are non-adjacent then in order to have three paths of length 2 from a given vertex in V_i to the vertices of V_j, V_i and V_j have exactly three common neighbours. □

From Proposition 1 it follows that if $z = 1$ then t is either 1, 3, or 21. For $t = 1$ we have the Kautz digraph on six vertices and the graph C in the Lemma is K_2. If there exists a mixed Moore graph with $t = 21$ then C is an unknown strongly regular graph with parameters $(162, 21, 0, 3)$, se Brouwer [2]. For $t = 3$, C is $K_{3,3}$, G is the Bosák graph and the graph obtained by deleting the directed edges is an antipodal (bipartite) distance regular graph known as the Pappus graph, see [3].

The Pappus graph is distance transitive with automorphism group of order 216. The automorphism group of the Bosák graph is a subgroup of order 108, with 4 conjugacy classes of transitive subgroups of order 18. I.e., the Bosák graph is a Cayley in four different ways. We want to describe it as a Cayley graph of the group $(S_3 \times S_3) \cap A_6$. In $S_3 \times S_3$, the first S_3 permutes $\{1, 2, 3\}$ and the second S_3 permutes $\{4, 5, 6\}$. The even permutations form a group of order 18.

The Bosák graph can be described as follows.

Proposition 5 *The Cayley graph*

$$Cay((S_3 \times S_3) \cap A_6, \{(1,2,3)(4,5,6), (1,2)(4,5), (1,2)(4,6), (1,3)(4,5)\})$$

is a mixed Moore graph with $t = 3$ *and* $z = 1$.

This proved by showing that every non-identity group element is either in the connection set S or it is in a unique way a product of two elements in S

We will introduce a directed strongly regular graph containing the Bosák graph.

Proposition 6 *The Cayley graph*

$$\begin{aligned} G_{36} = Cay(S_3 \times S_3, \{&(1,2,3)(4,5,6), (1,2)(4,5), (1,2)(4,6), (1,3)(4,5), \\ &(1,3), (1,2)(4,5,6), (1,3)(4,5,6), (4,6), (1,2,3)(4,5), (1,2,3)(4,6)\}) \end{aligned}$$

is a directed strongly regular graph with parameters $(36, 10, 5, 2, 3)$.

Clearly, the subgraph of G_{36} spanned by the even permutations and also subgraph spanned by the odd permutation are isomorphic to the Bosák graph. These are in fact the only two such subgraphs.

If we let $\alpha = (1,2), \beta = (4,5), \sigma = (4,5,6), \tau = (1,2,3)$, we see that $S_3 \times S_3$ is isomorphic to

$$\begin{aligned} \langle \alpha, \beta, \sigma, \tau \mid \alpha^2 = \beta^2 = \sigma^3 = \tau^3 = 1, \alpha\beta = \beta\alpha, \sigma\tau = \tau\sigma, \\ \alpha\sigma = \sigma\alpha, \beta\tau = \tau\beta, \tau\alpha = \alpha\tau^2, \sigma\beta = \beta\sigma^2 \rangle. \end{aligned}$$

In this notation the connection set in Proposition 6 is

$$\{\sigma\tau, \alpha\beta, \alpha\beta\sigma, \alpha\beta\tau, \alpha\tau, \alpha\sigma, \alpha\sigma\tau, \beta\sigma, \beta\tau, \beta\sigma\tau\}.$$

Group number 17 in the GAP [7] catalogue of groups of order 108 can be realized as the full automorphism group of the Bosák graph, but we use the following describtion:

$$\begin{aligned} \Gamma := \langle a, b, s, t, x \mid a^2 = b^2 = s^3 = t^3 = x^3 = 1, ab = ba, as = sa, bt = tb, \\ sb = bs^2, ta = at^2, ts = stx, axa^{-1} = bxb^{-1} = x^2, sxs^{-1} = txt^{-1} = x \rangle. \end{aligned}$$

We see that $\langle x \rangle$ is a normal subgroup and that the factorgroup is isomorphic to $S_3 \times S_3$.

Theorem 7 *The Cayley graph*

$$G_{108} = Cay(\Gamma, \{stx^2, ab, absx, abtx^2, at, asx^2, astx, bs, btx, bst\})$$

is a mixed Moore graph with $n = 108, t = 3, z = 7$.

Let G_{108}^T be the graph obtain from G_{108} by reversing the direction of all edges, i.e. with the transposed adjacency matrix.

Corollary 8 G_{108}^T *is a mixed Moore graph with* $n = 108, t = 3, z = 7$. G_{108} *and* G_{108}^T *are not isomorphic.*

This was proved on a computer using GAP [7] including GRAPE [16] and Nauty [13].

G_{108} and G_{108}^T also appear as Cayley graphs of group number 15 in the GAP catalogue of groups of order 108. The full automorphism group has order 216.

The adjacency matrix of G_{108} and two other directed strongly regular graphs in the paper can be obtained from the author's web page [10].

3 G_{108} and G_{108}^T are multipartite Moore digraphs

Fiol, Gimbert and Miller [6] considered a Moore-like bound for multipartite directed graphs. An r-partite digraph is said to be δ-equioutregular if every vertex has exactly δ outneighbours in each of the other $r-1$ partite classes. The following is a special case of a bound for digraphs with arbitrary diameter.

Proposition 9 (Fiol, Gimbert and Miller [6]) *A* δ*-equioutregular* r*-partite digraph with outdegree* $d = \delta(r-1)$ *and diameter 2 has at most* $d^2 + d - \delta^2 + \delta$ *vertices.*

A digraph attaining this bound is said to be a multipartite Moore digraph. Fiol et al. considered multipartite Moore digraphs that are also weakly distance regular digraphs. A weakly distance regular digraph of diameter 2 is exactly a directed strongly regular graph. And if a multipartite Moore digraph is a directed strongly regular graph then it is necessarily a mixed

Moore graph. One of the feasible cases is that a mixed Moore digraph of order 108 is a 2-equioutregular 6-partite Moore graph. We therefore consider the digraphs G_{108} and G_{108}^T.

We first consider G_{36} using GAP [7] with GRAPE [16]. The largest independent sets of vertices in this graph has size 6. It has 18 independent sets I of 6 vertices with the property that every vertex not in I has exactly two (out-) neighbours in I. These sets are

$$\{(), (4,5,6), (2,3)(5,6), (2,3)(4,6), (1,2,3), (1,3)(5,6)\}$$

and its images under the action of the automorphism group.

The graph with these 18 sets as vertices and with an edge between two disjoint sets is the graph $\overline{K_{3,3,3} \cup K_{3,3,3}}$, which has 9 copies of K_6. So there are 9 partitions of the vertex set into such sets of 6 vertices. But these are all in the same orbit under the automorphism group of G_{36}.

Considering the group-homomorphism from Γ to $S_3 \times S_3$ described in Section 2 and the corresponding graph-homomorpism from G_{108} to G_{36} we find that the preimage of the abovemention partitions of G_{36} are 9 equioutregular partitions, all in the same orbit under the automorpism group.

G_{108} has two additional orbits of such partitions.

G_{36} does not have any independent set I of 6 vertices so that every vertex not in I has exactly two in-neighbours in I. However G_{108}^T has one orbit of equioutregular partitions.

Thus we have:

Theorem 10 *G_{108} is a 6-partite Moore digraph in three ways and G_{108}^T is a 6-partite Moore graph in one way.*

We note that the largest independent sets in G_{108} has size 30. There are 18 independent set of size 30. The intersection of two of these sets has size 0, 6 or 12. The graph with these 18 sets as vertices and with an edge between disjoint sets is isomorphic to the Pappus graph.

4 A directed strongly regular graph with parameters $(96, 13, 5, 0, 2)$

Group number 64 in the GAP [7] catalogue of groups of order 96 can be described as follows:

$$\Delta := \langle a, t, x, y \mid a^2 = t^3 = x^4 = y^4 = 1, ta = at^2, yx = xy, \\ axa = y, aya = x, txt^2 = x^3y^3, tyt^2 = x\rangle.$$

We see that $\langle x, y\rangle$ is a normal subgroup of Δ of order 16 and that the factorgroup is $S_3 = \langle \alpha, \tau \mid \alpha^2 = \tau^3 = 1, \tau\alpha = \alpha\tau^2\rangle$.

The Kautz digraph on six vertices is a Cayley graph $\mathrm{Cay}(S_3, \{\tau^2, \alpha\tau^2\})$. The complement $G_6 = \mathrm{Cay}(S_3, \{\tau, \alpha, \alpha\tau\})$ is a directed strongly regular graph with parameters $(6, 3, 2, 1, 2)$. As $t = \mu$, Duval [5] proved that a directed strongly regular graph G_{24} with parameters $(24, 12, 8, 4, 8)$ can be obtained by from G_6 by a 4-coclique extension.

Theorem 11 *The Cayley graph*

$$G_{96} = Cay(\Delta, \{x^2, ty^2, txy^2, tx^2y, tx^3y, a, ax, ax^2y, ax^3y, aty, atx, atxy, atx^2\})$$

is a directed strongly regular graph with parameters $(96, 13, 5, 0, 2)$.

Furthermore G_{96} and G_{96}^T are isomorphic and have full automorphism group of order 192.

Let $\phi : \Delta \mapsto \Delta/\langle x, y\rangle = S_3$ be the natural homomorphism. Considered as a map from the graph G_{96} to G_6, ϕ covers each edge of G_6 exactly four times. Note however that $\langle x, y\rangle$ is not an independent set, but a matching. Similarly, the homomorphism $\Delta \mapsto \Delta/\langle x^2, y^2\rangle$ maps G_{96} to G_{24}.

5 Open cases for mixed Moore graphs

As mentioned above the mixed Moore graphs with $t = 1$ are completely characterized. The following table is a list of all feasible parameter sets for mixed Moore graphs with $t > 1$ and at most 200 vertices. Only in two cases a graph is known. In the remaining 7 cases existence is still open.

n	t	z	Graph
18	3	1	Bosák
40	3	3	
54	3	4	
84	7	2	
88	3	6	
108	3	7	G_{108} and G_{108}^T
150	7	5	
154	3	9	
180	3	10	

Proposition 12 *Let G be a mixed Moore graph with $n = 40$ and $t = z = 3$. Then G does not contain an undirected 5-cycle as a subgraph.*

Proof Suppose that $x_0, x_1, x_2, x_3, x_4, x_0$ is an undirected 5-cycle in G. Let y_i be the unique undirected neighbour of x_i not on this cycle, and let S_i be the set consisting of y_i and the out-neighbours and in-neighbours of x_i. Then S_i and S_j are disjoint for $i \neq j$, as otherwise there are two vertices joined by two paths of length at most 2. Thus every vertex of G is either on the 5-cycle or in one of the sets S_i.

There are no (directed or undirected) edges between y_i and vertices in the sets S_{i-1}, S_i or S_{i+1}, as otherwise there are to many paths of length at most 2. This means that a path from y_1 to y_3 or from y_3 to y_1 can not have length 2. Thus y_1y_3 is an undirected edge, and similarly y_1y_4 is an undirected edge. But then there can not be any directed edge between y_1 and $S_3 \cup S_4$, a contradiction. □

For a mixed Moore graph with $n = 84$ and $t = 7$ a result like Proposition 12 does not hold. In fact every 7-regular undirected graph with girth 6 has at least 90 vertices, see [15].

Proposition 13 *Let G be a mixed Moore graph with $n = 40$ and $t = z = 3$. Then G does not contain the Kautz digraph on 12 vertices as a subgraph.*

Proof Suppose that H is a subgraph of G isomorphic to the Kautz graph on 12 vertices. Then there are 24 undirected edges between H and $G - H$. These edges have distinct endvertices in $G - H$, as H has diameter 2. The remaining four vertices have in-degree 3. The in-neighbours are exactly the vertices of H. But now there can not be any edge directed from $G - H$ to H, a contradiction. □

Proposition 14 *Let G be a mixed Moore graph with $t = 3$ and $z > 1$. Suppose that H is a subgraph of G isomorphic to the Bosák graph. Then $z \geq 10$.*

Proof From the introduction we know that $n = k^2+k-t+1 = z^2+7z+10$. Let A be one of the bipartition classes of the undirected subgraph of H. Then all $9 \cdot 2(z-1)$ in-neighbours and out-neighbours of A in $G - H$ are distinct. We find that $n \geq 18z$ and therefore $z \geq 10$. □

Proposition 15 *Suppose that G is a mixed Moore graph $n = 40$ and $t = z = 3$. Then G does not have an automorphism of order 5.*
In particular G is not vertex-transitive.

Proof Suppose that ϕ is an automorphism of G of order 5. If some vertex is fixed by ϕ then, as $t = z = 3$ all its neighbours are also fixed. Thus ϕ can not fix any vertices and so ϕ have 8 orbits $V_1, \ldots, V_8$ of length 5. There exist numbers c_{ij} so that every vertex in V_i has exactly c_{ij} (out-) neighbours in V_j. Let $C = [c_{ij}]$ be an 8×8 matrix. The number of paths of length 2 from a vertex in V_i to the vertices in V_j is $\sum_\ell c_{i\ell}c_{\ell j}$. Thus

$$C^2 + C = 5J + 2I. \tag{1}$$

Furthermore every entry in C is an integer from 0 to 5, and in fact by Propsition 12 the diagonal entries are at most 1. A computer search shows that no such matrix C exists. □

References

[1] J. Bosák, Partially directed Moore graphs, *Math. Slovaca* **29** (1979) 181–196.

[2] A. E. Brouwer, Strongly regular graphs, in: Handbook of Combinatorial Designs, eds: C. J. Colbourn and J. H. Dinitz, Chapman & Hall, 852–867, (2006).

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-regular graphs, *Springer-Verlag, Berlin* (1989).

[4] J. D. Dixon and B. Mortimer, Permutation Groups, Springer, (1996).

[5] A. M. Duval, A Directed Version of Strongly Regular Graphs, *Journal of Combininatorial Theory (A)* **47** (1988) 71–100.

[6] M. A. Fiol, J. Gimbert and M. Miller, Multipartite Moore digraphs, *Linear Algebra Appl.* **419** (2006) 234–250.

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.6.4, (2013), (`http://www.gap-system.org`)

[8] J. Gimbert, Enumeration of almost Moore digraphs of diameter two, *Discrete Math.* **231** (2001) 177–190.

[9] L. K. Jørgensen, Directed strongly regular graphs with $\mu = \lambda$, *Discrete Math.* **231** (2001) 289–293.

[10] L. K. Jørgensen, Adjacency matrices of some directed strongly regular graphs, `http://people.math.aau.dk/~leif/research/dsrg-matrices/`

[11] M. Klin, A. Munemasa, M. Muzychuk, P.-H. Zieschang, Directed strongly regular graphs obtained from coherent algebras. Linear Algebra Appl. 377 (2004), 83–109.

[12] D. S. Lyubshin, S. V. Savchenko, Cayley digraphs with normal adjacency matrices. Discrete Math. 309 (2009), 4343–4348.

[13] B.D. McKay, nauty user's guide (version 1.5), Technical report TR-CS-90-02, Australian National University, Computer Science Department, ANU, 1990.

[14] M. H. Nguyen, M. Miller and J. Gimbert, On mixed Moore graphs, *Discrete Math.* **307** 964–970.

[15] M. O'Keefe and P. K. Wong, The smallest graph of girth 6 and valency 7, *J. Graph Theory* **5** (1981) 79–85.

[16] L. H. Soicher. GRAPE: a system for computing with graphs and groups. In: *Groups and Computation* (eds.: Finkelstein and Kantor), volume 11 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, pages 287–291. American Mathematical Society, 1993.