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An upper bound on the domination number of a graph with minimum degree two

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Abstract

A set S of vertices in a graph G is a dominating set of G if every vertex of $V(G) \setminus S$ is adjacent to some vertex in S. The minimum cardinality of a dominating set of G is the domination number of G, denoted $\gamma(G)$. Let P_n and C_n denote a path and a cycle, respectively, on n vertices. Let $k_1(F)$ and $k_2(F)$ denote the number of components of a graph F that are isomorphic to a graph in the family $\{P_3, P_4, P_5, C_5\}$ and $\{P_1, P_2\}$, respectively. Let $\mathcal{L}$ be the set of vertices of G of degree more than 2, and let $G - \mathcal{L}$ be the graph obtained from G by deleting the vertices in $\mathcal{L}$ and all edges incident with $\mathcal{L}$. McCuaig and Shepherd [5] showed that if G is a connected graph of order $n \ge 8$ with $\delta(G) \ge 2$, then $\gamma(G) \le 2n/5$, while Reed [7] showed that if G is a graph of order n with $\delta(G) \ge 3$, then $\gamma(G) \le 3n/8$. As an application of Reed's result, we show that if G is a graph of order $n \ge 14$ with $\delta(G) \ge 2$, then $\gamma(G) \le \frac{3}{8}n + \frac{1}{8}k_1(G - \mathcal{L}) + \frac{1}{4}k_2(G - \mathcal{L})$.

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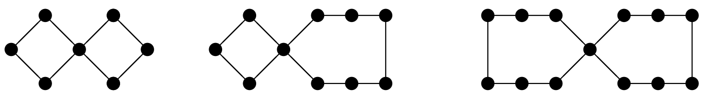
1 Introduction

In this paper, we continue the study of domination in graphs. Domination in graphs is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [3, 4].

For notation and graph theory terminology we in general follow [3]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and let v be a vertex in V. The *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighbourhood* of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, the open neighbourhood of S is defined by $N(S) = \cup_{v \in S} N(v)$, and the closed neighbourhood of S by $N[S] = N(S) \cup S$. If $X, Y \subseteq V$, then the set X is said to *dominate* the set Y if $Y \subseteq N[X]$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$ while the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all edges incident with S. We denote the degree of v in G by $d_G(v)$, or simply by $d(v)$ if the graph G is clear from context. The minimum degree among the vertices of G is denoted by $\delta(G)$.

We denote a path on n vertices by P_n and a cycle on n vertices by C_n. We call a component of a graph a path-component if it is isomorphic to a path and a cycle-component if it is isomorphic to a cycle. A path-component isomorphic to a path P_i we call a P_i-component, and a cycle-component isomorphic to a cycle C_i we call a C_i-component.

We define a *daisy* to be a connected graph that can be constructed from two disjoint cycles by identifying a set of two vertices, one from each cycle, into one vertex. In particular, if the two cycles have lengths n_1 and n_2, we denote the daisy by $D(n_1, n_2)$. The daisies $D(4, 4)$, $D(4, 7)$ and $D(7, 7)$ are shown in Figure 1.



$D(4,4)$ $D(4,7)$ $D(7,7)$

Figure 1: The daisies $D(4, 4)$, $D(4, 7)$ and $D(7, 7)$.

A *dominating set* of a graph $G = (V, E)$ is a set S of vertices of G such that every vertex $v \in V$ is either in S or adjacent to a vertex of S. (That is, $N[S] = V$.) The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. The domination number of a cycle or a path is easy to compute.

Theorem 1 *For* $n \geq 3$, $\gamma(P_n) = \gamma(C_n) = \lceil n/3 \rceil$.

Let G be a graph with $\delta(G) \geq 2$. We define a vertex as *small* if it has degree 2 and *large* if it has degree more than 2. Let $\mathcal{S}$ be the set of all small vertices of G and $\mathcal{L}$ the set of all large vertices of G. Let C be any component of $G - \mathcal{L}$. If C is a component of G, then C is a cycle; otherwise, if C is not a component of G, then it is a path.

For $i \in \{0, 1, 2, 3\}$, we denote the number of components of $G - \mathcal{L}$ of order congruent to i modulo 4 by $p_i(G)$, or simply by p_i if the graph G is clear from context. If G' is a graph, then for $i \in \{0, 1, 2, 3\}$ we denote $p_i(G')$ simply by p'_i, and we denote the order and size of G' by n' and m', respectively. Further, we denote the set of large vertices in G' by $\mathcal{L}'$.

Let $\mathcal{B}_1 = \{C_4, C_7, D(4,4)\}$ and $\mathcal{B}_2 = \mathcal{B}_1 \cup \{C_{10}, C_{13}, D(4,7), D(7,7)\}$ be two families consisting of cycles and daisies. For $i = 1, 2$, we say that a component is a $\mathcal{B}_i$-component if it is isomorphic to a graph in the family $\mathcal{B}_i$.

We call a component a *type-1 component* if it is a P_i-component for some $i \in \{3, 4, 5\}$ or a C_5-component, and we call a component a *type-2 component* if it is a P_1-component or a P_2-component. For $i = 1, 2$, we denote the number of type-i components in a graph G by $k_i(G)$.

2 Known Results

The decision problem to determine the domination number of a graph is known to be NP-complete. Hence it is of interest to determine upper bounds on the domination number of a graph. Upper bounds have been established in [1, 2, 5, 6, 7, 8, 9] and elsewhere.

McCuaig and Shepherd [5] showed that the domination number of a connected graph with minimum degree at least 2 is at most two-fifths its order except for seven exceptional graphs (one of order four and six of order seven). More precisely, they defined a collection $\mathcal{B}$ of "bad" graphs shown in Figure 2, and proved the following result.

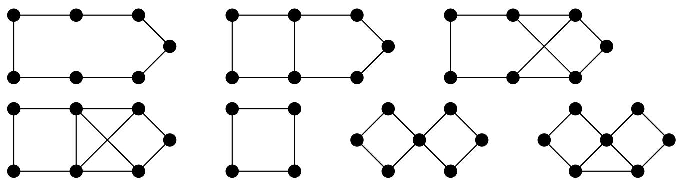


Figure 2: The family $\mathcal{B}$ of "bad" graphs.

Theorem 2 (McCuaig and Shepherd [5]) *If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \mathcal{B}$, then $\gamma(G) \leq 2n/5$.*

In 1996, Reed [7] presented the important and useful result that if we restrict the minimum degree to be at least three, then the upper bound in Theorem 2 can be improved from two-fifths its order to three-eights its order.

Theorem 3 (Reed [7]) *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.*

3 Main Result

Our aim in the paper is to generalize Theorem 3 by relaxing the degree condition to minimum degree at least two. For notational convenience, for a graph G of order n and a graph G' of order n' we let

$$\begin{aligned}\psi(G) &= \frac{3}{8}n + \frac{1}{8}(p_0 + p_3) + \frac{1}{4}(p_1 + p_2),\\ \psi(G') &= \frac{3}{8}n' + \frac{1}{8}(p'_0 + p'_3) + \frac{1}{4}(p'_1 + p'_2),\\ \varphi(G) &= \frac{3}{8}n + \frac{1}{8}k_1(G - \mathcal{L}) + \frac{1}{4}k_2(G - \mathcal{L}), \text{ and}\\ \varphi(G') &= \frac{3}{8}n' + \frac{1}{8}k_1(G' - \mathcal{L}') + \frac{1}{4}k_2(G' - \mathcal{L}').\end{aligned}$$

We shall prove:

Theorem 4 *If G is a graph of order n with $\delta(G) \geq 2$ that has no $\mathcal{B}_1$-component, then $\gamma(G) \leq \psi(G)$.*

Theorem 5 *If G is a graph of order n with $\delta(G) \geq 2$ that has no $\mathcal{B}_2$-component, then $\gamma(G) \leq \varphi(G)$.*

3.1 Preliminary Observations

Let G be an arbitrary graph. By *attaching a G_8-unit* to a specified vertex v of G, we mean adding a (disjoint) copy of the graph G_8 of Figure 3 and identifying any one of its vertices that is in a triangle with v.

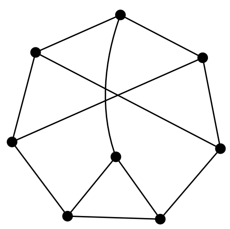


Figure 3: A cubic graph G_8 with domination number 3

We will frequently use the following observation in the proof of Theorem 4.

Observation 1 *If G' is obtained from a graph G by attaching a G_8-unit to a vertex v, then there exists a $\gamma(G')$-set that contains v and two other vertices in the G_8-unit.*

We define an *elementary* 3*-subdivision* of a nonempty graph G as a graph obtained from G by subdividing some edge three times. The following observation will prove to be useful.

Observation 2 *If G is obtained from a nontrivial graph G' by an elementary 3-subdivision, then $\gamma(G) = \gamma(G') + 1$.*

We will refer to a graph G as a *reduced graph* if G has no induced path on five vertices, the internal vertices of which have degree 2 in G. Hence if u, v_1, v_2, v_3, v is a path in a reduced graph G, then $d_G(v_i) \geq 3$ for at least one i, $1 \leq i \leq 3$, or $uv \in E(G)$.

3.2 Proof of Theorem 4

It suffices to prove that if G is a *connected* graph of order n with $\delta(G) \geq 2$ and $G \notin \mathcal{B}_1$, then $\gamma(G) \leq \psi(G)$. We proceed by induction on the order of the lexicographic sequence $(p_0+p_1+p_2+p_3, n, m)$, where $p_0+p_1+p_2+p_3 \geq 0$, $n \geq 3$ and $m \geq 3$. We remark that the order of the considered graphs does not always have to drop when applying an inductive argument. For notational convenience, for a graph G of order n and size m and a graph G' of order n' and size m', we denote the sequence $(p_0 + p_1 + p_2 + p_3, n, m)$ by $s(G)$ and the sequence $(p'_0 + p'_1 + p'_2 + p'_3, n', m')$ by $s(G')$. Further, we denote the set of small vertices of G and G' by $\mathcal{S}$ and $\mathcal{S}'$, respectively, and the set of large vertices of G and G' by $\mathcal{L}$ and $\mathcal{L}'$, respectively.

When $p_0+p_1+p_2+p_3 = 0$, the graph G has only large vertices. Thus, $\delta(G) \geq 3$ and the desired result follows from Theorem 3. This establishes the base case. Let $p_0+p_1+p_2+p_3 \geq 1$, $n \geq 3$ and $m \geq 3$. Assume that for all connected graphs $G' \notin \mathcal{B}_1$ of order n' with $\delta(G') \geq 2$ that have lexicographic sequence $s(G')$ smaller than s, $\gamma(G') \leq \psi(G')$. Let $G \notin \mathcal{B}_1$ be a connected graph of order n, size m with $\delta(G) \geq 2$ and with lexicographic sequence $s(G) = s$. Let $G = (V, E)$. We proceed further with a series of claims that we may assume the graph G satisfies.

Claim A *G is a reduced graph.*

Proof. Assume that G is not a reduced graph. Then, G contains an induced path u, v_1, v_2, v_3, v on five vertices, the internal vertices of which have degree 2 in G and $uv \notin E$ (possibly, u or v or both u and v are large vertices in G). Let $G' = (G - \{v_1, v_2, v_3\}) \cup \{uv\}$. Then, $\delta(G') \geq 2$ and G is obtained from G' by an elementary 3-subdivision. By Observation 2, $\gamma(G) = \gamma(G') + 1$. If $G' = C_4$, then $G = C_7$ and $G \in \mathcal{B}_1$, a contradiction. If $G' = C_7$, then $G = C_{10}$, while if $G' = D(4, 4)$, then $G = D(4, 7)$. In both cases, $\gamma(G) = 4 = \psi(G)$, and the desired bound holds. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$ and $n' = n - 3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 = \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. Hence we may assume that G is a reduced graph. □

Claim B *G is not a cycle.*

Proof. Assume that G is a cycle. By Claim A, either $G = C_3$ or $G = C_5$. On the one hand, if $G = C_3$, then $\gamma(G) = 1 < (3/8) * 3 + 1/8 = \psi(G)$. On the other hand, if $G = C_5$, then $\gamma(G) = 2 < (3/8) * 5 + 1/4 = \psi(G)$. In both cases, $\gamma(G) < \psi(G)$. $\square$

Note that if $G - \mathcal{L}$ has a cycle-component C, then C is also a cycle-component of G, implying that $G = C$ since G is connected. Hence by Claim B, every component of $G - \mathcal{L}$ is a path-component. By Claim A, every path-component has order 1, 2, 3 or 4.

Claim C $p_0 = 0$.

Proof. Suppose that $p_0 \geq 1$. Let $P\colon v_1, v_2, v_3, v_4$ be a P_4-component of $G[\mathcal{S}]$. Since G is a reduced graph, the two ends of P are adjacent in G to the same large vertex. Let v be the common large neighbor of v_1 and v_4. Then, v, v_1, v_2, v_3, v_4, v is a cycle in G. Let G' be the graph obtained from $G - V(P)$ by attaching a G_8-unit to the vertex v. Then, G' is a graph of order $n' = n + 3$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further $p'_0+p'_1+p'_2+p'_3 = p_0+p_1+p_2+p_3-1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set D' that contains v and a set D_v of two other vertices in the attached G_8-unit. Hence, $D = (D' \setminus D_v) \cup \{v_2\}$ is a dominating set in G. Thus, $\gamma(G) \leq |D| = |D'| - 1 = \gamma(G') - 1$. Therefore, $\gamma(G) + 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + (3/8) * 3 - 1/8 = \psi(G) + 1$. Consequently, $\gamma(G) \leq \psi(G)$. $\square$

Claim D $p_3 = 0$.

Proof. Suppose that $p_3 \geq 1$. Let $P\colon v_1, v_2, v_3$ be a P_3-component of $G[\mathcal{S}]$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_3 not on P. We consider two possibilities.

Case 1. $u = v$. Then, v, v_1, v_2, v_3, v is a cycle in G. Suppose $d_G(v) \geq 4$. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$. If $G' = C_4$, then $G = D(4,4)$ and $G \in \mathcal{B}_1$, a contradiction. If $G' = C_7$ or if $G' = D(4,4)$, then $\gamma(G) = 4 = (3/8) * 7 + 3/8 \leq \psi(G)$, and the desired bound holds. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$ and $n' = n - 3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of G by adding to it the vertex v_2, and so $\gamma(G) \leq \gamma(G') + 1$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. Hence we may assume that $d_G(v) = 3$.

Let w be the neighbor of v not on P. If $d_G(w) = 2$, let x be the neighbor of w different from v. If $d_G(x) = 2$, let y be the neighbor of x different from w. Let G' be the graph obtained from $G - \{v, v_1, v_2, v_3\}$ by attaching a G_8-unit to the vertex w. Then, G' is a graph of order $n' = n + 3$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$.

If $d_G(w) \geq 3$, then $p'_3 = p_3 - 1$ and $p'_i = p_i$ for $i \in \{0, 1, 2\}$. If $d_G(w) = 2$ and $d_G(x) \geq 3$, then, $p'_1 = p_1 - 1$, $p'_3 = p_3 - 1$ and $p'_i = p_i$ for $i \in \{0, 2\}$. If $d_G(w) = d_G(x) = 2$, then since G is a reduced graph, we have that $d_G(y) \geq 3$, and so $p'_0 = p_0$, $p'_1 = p_1 + 1$, $p'_2 = p_2 - 1$, and $p'_3 = p_3 - 1$. Therefore in all three cases, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Further, $\psi(G') \leq \psi(G) + (3/8) * 3 - 1/8 = \psi(G) + 1$.

Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set D' that contains w and a set D_w of two other vertices in the attached G_8-unit. Hence, $D = (D' \setminus D_w) \cup \{v_2\}$ is a dominating set in G. Thus, $\gamma(G) \leq |D| = |D'| - 1 = \gamma(G') - 1$. Consequently, $\gamma(G) + 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + 1$, whence $\gamma(G) \leq \psi(G)$.

Case 2. $u \neq v$. Since G is a reduced graph, we must have $uv \in E$. Let $G' = G - V(P)$. Then, $\delta(G') \geq 2$. If $G' = C_4$, then $\gamma(G) = 3$. Further $n = 7$, and $p_2 = p_3 = 1$ and $p_0 = p_1 = 0$, and so $\psi(G) = (3/8) * 7 + 1/8 + 1/4 = 3$. Thus if $G' = C_4$, then $\gamma(G) = \psi(G)$. If $G' = C_7$, then G would not be a reduced graph, contrary to assumption. If $G' = D(4, 4)$, then $\gamma(G) = 3$. Further $n = 10$, and $p_3 = 2$ and $p_1 + p_2 \geq 1$, and so $\psi(G) = (3/8) * 10 + 2/8 + 1/4 > 3$. Thus, if $G' = D(4, 4)$, then $\gamma(G) < \psi(G)$. Hence we may assume that $G' \notin \mathcal{B}_1$. Since $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$ and $n' = n - 3$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of G by adding to it the vertex v_2, and so $\gamma(G) \leq \gamma(G') + 1$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8)*3 - 1/8 + 1/4 = \psi(G) - 1$, and so $\gamma(G) \leq \psi(G)$. □

Claim E $p_2 = 0$.

Proof. Suppose that $p_2 \geq 1$. Let $P\colon v_1, v_2$ be a P_2-component of $G[\mathcal{S}]$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_2 not on P.

If the one hand, suppose that $u = v$. Let G' be the graph obtained from $G - V(P)$ by attaching a G_8-unit to the vertex v. Then, G' is a graph of order $n' = n + 5$ with $\delta(G') \geq 2$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_2 = p_2 - 1$ and $p'_i = p_i$ for $i \in \{0, 1, 3\}$. Hence, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set D' that contains v and a set D_v of two other vertices in the attached G_8-unit. Hence, $D = D' \setminus D_v$ is a dominating set in G. Thus, $\gamma(G) \leq |D| = |D'| - 2 = \gamma(G') - 2$. Therefore, $\gamma(G) + 2 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + (3/8) * 5 - 1/4 < \psi(G) + 2$. Consequently, $\gamma(G) \leq \psi(G)$.

If the other hand, suppose that $u \neq v$. If $uv \in E$, then let $G' = G - uv$. Then, $\delta(G') \geq 2$. By our structure of G, $G' \notin \{C_4, D(4, 4)\}$. If $G' = C_7$, then $p_3 = 1$, contrary to our assumption in Claim D. Hence, $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3$. Thus since G' has order $n' = n$ and size $m' = m - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G') \leq \psi(G)$. Since the domination number of a graph cannot decrease if edges are removed, $\gamma(G) \leq \gamma(G')$, implying that $\gamma(G) \leq \psi(G)$. Hence we may assume that $uv \notin E$.

Let G' be obtained from $G - V(P)$ by adding the edge uv. Then, $\delta(G') \geq 2$ and both u and v are large vertices in G'. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further $p'_2 = p_2 - 1$ while $p'_i = p_i$ for $i \in \{0, 1, 3\}$. Thus since $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Every $\gamma(G')$-set can be extended to a dominating set of G by adding to it v_1 or v_2, and so $\gamma(G) \leq \gamma(G') + 1$. Therefore, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \leq \psi(G)$. □

By Claims C, D and E, we have $p_0 = p_2 = p_3 = 0$ and $p_1 \geq 1$. Thus, by our earlier assumptions, every component of $G[\mathcal{S}] = G - \mathcal{L}$ is a P_1-component. Let P be a P_1-component of $G[\mathcal{S}]$ with $V(P) = \{v_1\}$. Let u and v be the two neighbors of v_1. Then, $\{u, v\} \subseteq \mathcal{L}$.

Claim F *$uv \notin E$.*

Proof. Suppose that $uv \in E$. Let $G' = G - uv$. Then, $\delta(G') \geq 2$ and $\gamma(G) \leq \gamma(G')$. If $G' = C_4$, then $\gamma(G) = 1 < \psi(G)$. Since G is a reduced graph, $G' \neq C_7$. If $G' = D(4, 4)$, then $n = 7$ and $\gamma(G) = 2 < \psi(G)$. Hence, we may assume that $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$. Thus since G' has order $n' = n$ and size $m' = m - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Since $\gamma(G) \leq \gamma(G')$ and $\psi(G') \leq \psi(G)$, we have that $\gamma(G) \leq \psi(G)$. □

Claim G *The vertices u and v have only one common degree-2 neighbor.*

Proof. Suppose that u and v have a common degree-2 neighbor v_2 that is different from v_1. Let G' be obtained from $G - \{v_1, v_2\}$ by adding the edge uv. Then, $\delta(G') \geq 2$ and $\gamma(G) \leq \gamma(G') + 1$. If $G' = C_4$, then $n = 6$ and $\gamma(G) = 2 < \psi(G)$. Since G is a reduced graph, $G' \neq C_7$. If $G' = D(4, 4)$, then $n = 9$ and $\gamma(G) = 3 < \psi(G)$. Hence, we may assume that $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 2 - (1/4) * 2 + 1/4 = \psi(G) - 1$. Thus, $\gamma(G) \leq \psi(G)$. □

Claim H *The vertices u and v have at least one common neighbor different from v_1, and each such common neighbor is a degree-3 vertex in G.*

Proof. Suppose that v_1 is the only common neighbor of u and v. Let G' be obtained from $G - \{u, v, v_1\}$ by adding a new vertex w and joining it to all vertices in $(N(u) \cup N(v)) \setminus \{v_1\}$. Then, $d_{G'}(w) \geq 4$, $\delta(G') \geq 2$ and $\gamma(G) \leq \gamma(G') + 1$. If $G' \in \mathcal{B}_1$ then $G' = D(4, 4)$, $n = 9$, and $\gamma(G) = 3 < \psi(G)$. Hence $G' \notin \mathcal{B}_1$. Further, $p'_1 = p_1 - 1$ and $p'_i = p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \leq \psi(G)$.

Now suppose that w is a common neighbor of u and v different from v_1. Suppose that $d_G(w) \geq 4$. Let $G' = G - vw$. Then, $\delta(G') \geq 2$. Further, $G' \notin \mathcal{B}_1$ and $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$. Thus since G' has order $n' = n$ and size $m' = m - 1$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Since $\gamma(G) \leq \gamma(G')$ and $\psi(G') \leq \psi(G)$, we have that $\gamma(G) \leq \psi(G)$. Hence we may assume that $d_G(w) \leq 3$. By Claim G, $d_G(w) \geq 3$. Consequently, $d_G(w) = 3$. □

Claim I *Both u and v are degree-3 vertices in G.*

Proof. Suppose that u or v has degree greater than 3. Without loss of generality, we may assume that $d_G(u) \geq 4$. Let G' be the graph obtained from $G - v_1$ by attaching a G_8-unit to the vertex v. Then, G' is a graph of order $n' = n + 6$ with $\delta(G') \geq 2$. Note that both u and v are large vertices in G'. Since $n' > 7$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_1 = p_1 - 1$ and $p'_i = p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 = p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. By Observation 1, there exists a $\gamma(G')$-set D' that contains v and a set D_v of two other vertices in the attached G_8-unit. Hence, $D = D' \setminus D_v$ is a dominating set in G. Thus, $\gamma(G) \leq |D| = |D'| - 2 = \gamma(G') - 2$. Therefore, $\gamma(G) + 2 \leq \gamma(G') \leq \psi(G') \leq \psi(G) + (3/8) * 6 - 1/4 = \psi(G) + 2$. Consequently, $\gamma(G) \leq \psi(G)$. □

By Claim H, we may assume that there is a degree-3 vertex y that is adjacent to both u and v. By Claim I, we may assume that both u and v are degree-3 vertices in G. Let $N(u) = \{v_1, y, w\}$ and let $N(v) = \{v_1, y, z\}$.

Claim J *$w = z$.*

Proof. Suppose that $w \neq z$.

Since $d_G(y) = 3$ and $\{u, v\} \subset N(y)$, the vertex y is adjacent to at most one of w and z. Without loss of generality, we may assume that $yz \notin E$. Let G' be obtained from $G - \{v, v_1\}$ by adding the two edges uz and yz. Then, $\delta(G') \geq 2$ and each of u, y and z is a large vertex in G'. Let D' be a $\gamma(G')$-set. If $z \in D'$, then $D' \cup \{u\}$ is a dominating set of G, while if $z \notin D'$, then $D' \cup \{v\}$ is a dominating set of G. Hence every $\gamma(G')$-set can be extended to a dominating set of G by adding to it either u or v. Thus, $\gamma(G) \leq \gamma(G') + 1$. Since $G \notin \mathcal{B}_1$, we have that $G' \notin \mathcal{B}_1$. Further, $p'_1 \leq p_1 - 1$ and $p'_i \leq p_i$ for $i \in \{0, 2, 3\}$. Thus, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3 - 1$, and so the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \leq \psi(G')$. Hence, $\gamma(G) - 1 \leq \gamma(G') \leq \psi(G') \leq \psi(G) - (3/8) * 2 - 1/4 = \psi(G) - 1$. Consequently, $\gamma(G) \leq \psi(G)$. □

By Claim J, we may assume that $w = z$, and so w is a common neighbor of u and v different from v_1. By Claim H, $d_G(w) = 3$. Let $G' = G - uy - vw$. Then, $\delta(G') \geq 2$. If $G' \in \mathcal{B}_1$, then $G' = C_7$. But then $G - \mathcal{L}$ would contain a P_2-component, contrary to our earlier assumption. Hence, $G' \notin \mathcal{B}_1$. Further, $p'_0 + p'_1 + p'_2 + p'_3 \leq p_0 + p_1 + p_2 + p_3$. Thus since G' has order $n' = n$ and size $m' = m - 2$, the lexicographic sequence $s(G')$ is smaller than $s(G)$. Applying the inductive hypothesis to G', $\gamma(G') \le \psi(G')$. Since $\gamma(G) \le \gamma(G')$ and $\psi(G') \le \psi(G)$, we have that $\gamma(G) \le \psi(G)$. This completes the proof of Theorem 4. □

3.3 Proof of Theorem 5

Assume the theorem is false. Among all counterexamples, let G be one of minimum order n. Then, G is a connected graph with $\delta(G) \ge 2$, $G \notin \mathcal{B}_2$, and $\gamma(G) > \varphi(G)$. We proceed further with three claims.

Claim K *The graph $G - \mathcal{L}$ has no cycle-component.*

Proof. Assume, to the contrary, that $G - \mathcal{L}$ has a cycle-component C. Then, C is also a cycle-component of G, implying that $G = C = C_n$ and $\gamma(G) = \lceil n/3 \rceil$. Since $G \notin \mathcal{B}_2$, $n \notin \{4, 7, 10, 13\}$. If $n \equiv 0 \,(\mathrm{mod}\, 3)$, then $\gamma(G) = n/3 < 3n/8 \le \varphi(G)$. If $n \equiv 1 \,(\mathrm{mod}\, 3)$, then $n \ge 16$ and $\gamma(G) = (n+2)/3 \le 3n/8 = \varphi(G)$. If $n \equiv 2 \,(\mathrm{mod}\, 3)$, then either $n = 5$ and $\gamma(G) = 2 = 3n/8 + 1/8 = \varphi(G)$ or $n \ge 8$ and $\gamma(G) = (n+1)/3 \le 3n/8 = \varphi(G)$. In all three cases, $\gamma(G) \le \varphi(G)$, contradicting our assumption that G is a counterexample to Theorem 5. □

By Claim K, the graph $G - \mathcal{L}$ has no cycle-component. Thus, $|\mathcal{L}| \ge 1$ and every component of $G - \mathcal{L}$ is a path-component.

Claim L *The graph $G - \mathcal{L}$ has no path-component of order $k \ge 8$.*

Proof. Assume, to the contrary, that $P \colon v_1, v_2, \dots, v_k$ is a P_k-component of $G - \mathcal{L}$ where $k \ge 8$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_k not on P. (Possibly, $u = v$.) Let $G' = (G - \{v_1, v_2, \dots, v_6\}) \cup \{uv_7\}$ and let $P' = P - \{v_1, v_2, \dots, v_6\}$. Then, G' is a connected graph of order $n' = n - 6$ with $\delta(G') \ge 2$. It follows from Observation 2 that $\gamma(G) = \gamma(G') + 2$. Note that the set of large vertices of G' is the set $\mathcal{L}$.

If $G' \in \mathcal{B}_2$, then $G' \in \{D(4,4), D(4,7), D(7,7)\}$. Since $G \notin \mathcal{B}_2$, this implies that $G \in \{D(4,10), D(4,13), D(7,10), D(7,13)\}$. In all cases, $\gamma(G) \le \varphi(G)$, a contradiction. Hence, $G' \notin \mathcal{B}_2$. Since G' is not a counterexample to our theorem, $\gamma(G') \le \varphi(G')$.

Note that the type-1 or type-2 components of $G' - \mathcal{L}$ and $G - \mathcal{L}$ are the same, except that $G' - \mathcal{L}$ may contain one additional type-1 or type-2 component, namely the component P'. Hence, $\varphi(G') \le \varphi(G) - (3/8) * 6 + 1/4 = \varphi(G) - 2$. Thus, $\gamma(G) = \gamma(G') + 2 \le \varphi(G') + 2 \le \varphi(G)$, a contradiction. □

Claim M *The graph $G - \mathcal{L}$ has no path-component of order* 5, 6 *or* 7.

Proof. Assume, to the contrary, that $P \colon v_1, v_2, \dots, v_k$ is a P_k-component of $G - \mathcal{L}$, where $k \in \{5, 6, 7\}$. Let u be the neighbor of v_1 not on P and let v be the neighbor of v_k not on

P. (Possibly, $u = v$.) Let $G' = (G - \{v_1, v_2, v_3\}) \cup \{uv_4\}$ and let $P' = P - \{v_1, v_2, v_3\}$. Then, G' is a connected graph of order $n' = n - 3$ with $\delta(G') \geq 2$. By Observation 2, $\gamma(G) = \gamma(G') + 1$. Note that the set of large vertices of G' is the set $\mathcal{L}$.

Suppose $k = 5$. Since G has no $\mathcal{B}_2$-component, neither does G'. Note that the type-1 or type-2 components of $G' - \mathcal{L}$ and $G - \mathcal{L}$ are the same, except for the type-1 component P of $G - \mathcal{L}$ which becomes the type-2 component P' of $G' - \mathcal{L}$. Hence, $k_1(G' - \mathcal{L}) = k_1(G - \mathcal{L}) - 1$ and $k_2(G' - \mathcal{L}) = k_2(G - \mathcal{L}) + 1$, and so $\varphi(G') = \varphi(G) - (3/8) * 3 - 1/8 + 1/4 = \varphi(G) - 1$. Since G' is not a counterexample to our theorem, $\gamma(G') \leq \varphi(G')$. Hence, $\gamma(G) = \gamma(G') + 1 \leq \varphi(G') + 1 = \varphi(G)$, a contradiction.

Suppose $k \in \{6, 7\}$. If $G' \in \mathcal{B}_2$, then $k = 6$ and $G \in \{D(4, 10), D(7, 10)\}$ and $\gamma(G) \leq \varphi(G)$, a contradiction. Hence, $G' \notin \mathcal{B}_2$. Note that the type-1 or type-2 components of $G' - \mathcal{L}$ and $G - \mathcal{L}$ are the same, except that $G' - \mathcal{L}$ contains one additional type-1 component, namely the component P'. Hence, $k_1(G' - \mathcal{L}) = k_1(G - \mathcal{L}) + 1$ and $k_2(G' - \mathcal{L}) = k_2(G - \mathcal{L})$, and so $\varphi(G') = \varphi(G) - (3/8) * 3 + 1/8 = \varphi(G) - 1$. Since G' is not a counterexample to our theorem, $\gamma(G') \leq \varphi(G')$. Hence, $\gamma(G) = \gamma(G') + 1 \leq \varphi(G') + 1 = \varphi(G)$, a contradiction. $\Box$

By Claims L and M, every path-component of $G - \mathcal{L}$ has order at most 4. Hence, $k_1(G - \mathcal{L}) = p_0 + p_3$ and $k_2(G - \mathcal{L}) = p_1 + p_2$, and so $\psi(G) = \varphi(G)$. Thus, by Theorem 4, $\gamma(G) \leq \varphi(G)$, a contradiction. This completes the proof of Theorem 5. $\Box$

That the bound of Theorem 5 is in a sense best possible, may be seen as follows. Let v be a specified vertex of some graph. By *attaching a C_n-unit* to v, we mean adding a (disjoint) copy of an n-cycle and identifying any one of its vertices with v. By *attaching a key-unit* to v, we mean adding a (disjoint) copy of a 4-cycle and joining with an edge one of its vertices to v. Let $\mathcal{G}$ denote the family of all graphs that can be obtained from a connected graph F by attaching to each vertex v of F a G_8-unit, a C_5-unit, a C_8-unit, or if $d_F(v) \geq 2$, a key-unit. A graph in the family $\mathcal{G}$ with one key-unit, one C_5-unit and one G_8-unit that is obtained from a complete graph $F = K_3$ on three vertices is illustrated in Figure 4.

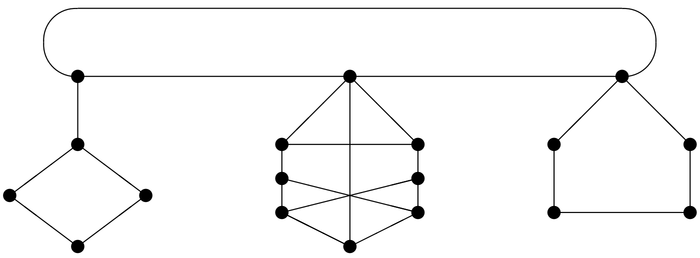


Figure 4: A graph in the family $\mathcal{G}$.

If $G \in \mathcal{G}$, then each key-unit and each C_5-unit of G contributes two to $\gamma(G)$, five to $|V(G)|$, one to $k_1(G - \mathcal{L})$, and zero to $k_2(G - \mathcal{L})$, while each C_8-unit and each G_8-unit contributes three to $\gamma(G)$, eight to $|V(G)|$ and zero to $k_1(G - \mathcal{L}) + k_2(G - \mathcal{L})$. Thus, if $G \in \mathcal{G}$ has order n with a key-unit, b C_5-units, c C_8-units, and d G_8-units, then $n = 5(a+b) + 8(c+d)$, $k_1(G - \mathcal{L}) = a + b$ and $\gamma(G) = 2(a + b) + 3(c + d) = \psi(G)$.

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