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Aalborg University
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ABSTRACT

The main objectives of this paper are to study procedures to decompose the aggregate technological progress down to the industry level, explicitly taking into account both the direct effects from within the industry as well as the indirect effects from the supporting industries, and to apply these procedures to Danish industry-level data. Among the results are evidence of a strong convergence in the industry-level productivity and the identification of, for economic policy making, important structural characteristics of the Danish economy. This includes a subsystem based CO_2 accounting that identifies the origin of the specific demand that eventually led to the production that caused the emission, e.g., it is shown that the public sector is responsible for four times the CO_2 emission reported in the official (direct) statistics.

Keywords: Structural change, Input–output analysis, Subsystems, Production prices, Productivity accounting

JEL classifications: C67, O30, O47, L16

*Department of Economics, Politics and Public Administration, University of Aalborg; Fibigerstræde 3, 9220 Aalborg Ø, Denmark; E-mail: thomasfredholm@gmail.com

†Interdepartmental Centre for Research Training in Economics and Management CIFREM), University of Trento; via Rosmini 70, 38100 Trento, Italy; E-mail: stefanozamb@gmail.com

1 Introduction

The purpose of this paper is to study structural and technological changes in and among Danish industries, and its effects on the economy as a whole. Furthermore, we show how the approach can be utilised to evaluate economic policy. This is accomplished by applying the subsystem approach introduced by Sraffa (1960) to the unique Danish input–output tables 1966–2005. The subsystem approach allows us to decompose the effects from industry-level technological change to the overall industry interdependences.

Structural change and the link between aggregate and industry-level productivity can be measured in different ways, but one method has since its introduction by Domar (1961) and generalization by Hulten (1978) dominated the productivity accounting carried out by major statistical bureaus. This method uses *Domar weights* to—supposedly—capture the combined effect of productivity growth within the individual industries and indirect effects through the supporting industries.[1]

From a fundamentally different methodological point of view we have the subsystem approach introduced by Sraffa (1960) and further developed by Gossling (1972), Pasinetti (1973), and others. The idea behind the notion of subsystems is to construct technically autarkic subsystems that as a final (net) output produce only one industry's output. This enables us to compute all the intermediate goods and labour directly and indirectly needed to produce this single commodity. A subsystem can be thought of as a isolated complex supply chain, including all commodities, producing only one final product.

One advantage of this approach is that changes in methods of production, interdependences, and structural change can be detected by the study of subsystems alone. Hence relative changes in the importance of the different industries can be detected. Another important property of the individual subsystems is that they are additive, i.e., the individual subsystems can be combined into meso level groups of autarkic subsystems producing as a final output any subset of the basket of final products from the entire system.

The notion of subsystems was introduced by Sraffa (1960, p. 89) in a typically concise three-quarter page appendix that is worth quoting in full:[2]

> Consider a system of industries (each producing a different commodity) which is in a self-replacing state. The commodities forming the gross product can be unambiguously distinguished as those which

[1] Domar weights are computed as the ratio of industry gross output to total deliveries to final demand. For further information see OECD (2001, 2008) and Hulten et. al. (2001).

[2] See Velupillai (2008) for a discussion on the intrinsic algorithmic content of Sraffa's arguments, including Sraffa's description of subsystems. This fundamentally algorithmic way of posing the problem, and the procedures by which they can be solved, is instrumental in our quest to utilise this powerful tool in an unambiguous fashion.

> go to replace the means of production and those which together form the net product of the system.
>
> Such a system can be subdivided into as many parts as there are commodities in its net product, in such a way that each part form a smaller self-replacing system the net product of which consists of only one kind of commodity. These parts we shall call 'subsystems'.
>
> This involves subdividing each of the industries of the original system (namely, the means of production, the labour and the product of each) into parts of such size as will ensure self-replacements for each subsystem.
>
> Although only a fraction of the labour of a subsystem is employed in the industry which directly produces the commodity forming the net product, yet, since all other industries merely provide replacement of the means of production used up, the whole of the labour employed can be regarded as directly or indirectly going to produce that commodity.
>
> Thus in the subsystem we see at a glance, as an aggregate, the same quantity of labour that we obtain as the sum of the series of terms when we trace back the successive stages of the production of the commodity.

Sraffa is here pointing to the possibility of using subsystems as units of measurement in a way which is both theoretically relevant and useful for empirical analysis.

Empirical applications of the subsystem approach were originally developed by Gossling (1972) to study the American agricultural industry. Other empirical applications of the subsystem approach, on the measurement of productivity and on the relation among market prices, production prices, and labour values, include: Juan and Febrero (2000), Dietzenbacher et. al. (2000), Miller and Gowdy (1998), Tsoulfidis and Mariolis (2007), Tsoulfidis (2008), and Alcántara and Padilla (2009). This paper is a contribution to the literature on empirical subsystem analysis, that both empirically and computationally will go deeper into the practical applications of this powerful tool.

A major advantage, of all the indices that will be presented, is that they do not change as a consequence of changes in the scale of production alone, even if it is asymmetric across industries. Hence, the indices will only change, when real technological innovations take place. It is exactly the consequences of such changes we want to capture in the indices, since they can influence both the structural relationship among industries and the productivity level in the single industries. It cannot be stressed enough, that this is not based on an assumption of *constant returns to scale*. If a changes occur in the scale of production in one or more industries, without changing the proportional use of the means of production (including labour), the indices remain unaffected. If on the other hand the relative proportions of the means of production are affected, then it will influence both the decomposition into subsystems and the vector of production prices.

A general property of this approach is that it circumvents many of the theoretical problems, innate to neoclassical studies of structural change and technological progress. In particular, thus related to the use of aggregate production function, see Pasinetti (2000); Cohen and Harcourt (2003); and Felipe and Fisher (2003).

Consequently, this approach has huge potentials, not only, as an analytical and descriptive tool, but also to provide procedures to evaluate economic policy. To accomplice this, we combine algorithmic reasoning with a naturalistic approach to the theory of production. We consider only practically observable phenomena, and from this work our way through the problems applying only mathematical statements, for which we can actually provide procedures to compute.

This paper is structured as follows: Section 2 presents the theoretical framework, Section 3 the data, Section 4 the results associated with structural and technological change, and Section 5 the applications to economic policy. Section 6 concludes the paper. Appendix A and B contain a both technical and practical introduction to the construction of subsystems and Appendix C and D respectively a list of symbols and details on the data used. Appendix E in the statistical companion contains a comprehensive collection of the empirical results obtained.

2 On Subsystems

Let $[\boldsymbol{A}_t,\boldsymbol{l}_t,\boldsymbol{B}_t]$ be a set of data variables measured in physical quantities. The entries are respectively, the non-singular indecomposable semi-positive $n \times n$ *input-matrix*, the $n \times 1$ column *vector of labour inputs*, and the $n \times n$ semi-positive diagonal *gross output-matrix*. $\boldsymbol{A}_t$ is composed of row vectors of intraindustry inputs and column vectors of interindustry flows. Furthermore, let $\boldsymbol{e}$ be a $n \times 1$ unit vector. It is necessary to introduce a rather cumbersome mathematical notation, *viz.*[3]

$a_{(i,j,t)}$	the ijth entry of $\boldsymbol{A}$ at time t
$\boldsymbol{a}_{(i,:,t)}$	the ith row of $\boldsymbol{A}$ at time t
$\boldsymbol{A}_{(\neg i,t)}$	$\boldsymbol{A}$ at time t, but without its ith row and column
$\boldsymbol{a}_{(i,\neg j,t)}$	the ith row of $\boldsymbol{A}$ at time t, but without its jth entry

Only single production systems will be considered, but most—if not all—results are valid in the more general case of *joint production*. Furthermore, only circulating and not fixed capital will be considered. At this point it is not only a matter of convenience, but also one of deep theoretical and empirical considerations. Both the standard way of approximated fixed capital

[3]Matrices, vectors, and scalars are respectively represented by bold capital letters, bold non-capital letters, and non-bold non-capital letters. Furthermore, single entries and vectors from a given matrix are represented by the corresponding non-capital letter.

in value terms by invoking the *ad hoc* and partly stochastic *perpetual inventory method*, and the theoretical sound, but empirical intractable, method of treating fixed capital in the framework of joint production, are so problematic that we *pro tempore* have choosing to abstract from fixed capital.[4]

2.1 The subsystem multipliers

Two additional assumptions are necessary and sufficient for the following results to hold; i) fixed production techniques, over the accounting period and ii) viable economic systems in a self-replacing state, i.e., systems capable of and actually producing at least the commodities required to replace the circulating capital goods.

To construct the technically autarkic *gross output subsystem* associated with the ith industry at time t from the parent system $[\boldsymbol{A}_t, \boldsymbol{l}_t, \boldsymbol{B}_t]$, the system must be rescaled such that the entire subsystem as a gross output produce the gross output of the ith industry in the original system, while the final output in all supporting industries are zero. This can be done, applying the following intuitive and computational direct procedure, that as an auxiliary tool use multipliers to decompose the parent system into subsystems.[5]

To obtain the ith *gross output subsystem multiplier* at time t, first compute the non-trivial strictly positive unique solution, $\overline{\boldsymbol{q}}_t^i$, of the following system of equations:[6]

$$\left[\boldsymbol{B}_{(\neg i,t)} - \boldsymbol{A}'_{(\neg i,t)}\right]\overline{\boldsymbol{q}}_t^i = \boldsymbol{a}'_{(i,\neg i,t)} \qquad i = 1,2,...,n \quad t = 1,2,...,T \qquad (2.1)$$

Second, on the ith entry of the vector $\overline{\boldsymbol{q}}_t^i$ *squeeze in* a single "1". The intuition behind this procedure is that the final output from the supporting industries (the LHS of 2.1) must be equal to the interindustry flow into the industry associated with the subsystem (the RHS of 2.1).

From this it is straightforward to compute the *final output subsystem multipliers*, henceforth called the *subsystem multipliers*, $\tilde{\boldsymbol{q}}_t^i$. Rescale the gross subsystem multipliers, such that the net products of the individual subsystems are equal to the corresponding sectoral net products in the par-

[4]For a discussion on the consequence of not including fixed capital, see Han and Schefold (2006, p. 752).

[5]Appendix A contains an introduction to numerical equivalent, but conceptually different, procedures to construct subsystems.

[6]A unique non-trivial solution requires that $\left[\boldsymbol{B}_{(\neg i,t)} - \boldsymbol{A}'_{(\neg i,t)}\right]$ is non-singular for all $t = 1,2,....,T$ and $i = 1,2,...,n$. The strictly positive solution with $0 < \overline{\boldsymbol{q}}_t^i \leq 1$ is guaranteed if the original system is in a self-replacing state, since the supporting industries, in the ith gross output subsystem, as a final output only produce what is needed to produce the gross output in the ith industry, which is necessarily less than the same plus a 'non-negative' surplus.

ent system, *viz.*

$$\tilde{\boldsymbol{q}}_t^i = \overline{\boldsymbol{q}}_t^i \frac{b_{(i,i,t)} - \boldsymbol{e}'\boldsymbol{a}_{(:,i,t)}}{b_{(i,i,t)} - \overline{\boldsymbol{q}}_t^{i\prime}\boldsymbol{a}_{(:,i,t)}} \tag{2.2}$$

Using the subsystem multipliers obtained above, the matrices forming the ith final output subsystem at time t measured in physical quantities are given by $[\tilde{\boldsymbol{A}}_t^i, \tilde{\boldsymbol{l}}_t^i, \tilde{\boldsymbol{B}}_t^i] = [\boldsymbol{A}_t \otimes \tilde{\boldsymbol{q}}_t^i \boldsymbol{e}', \boldsymbol{l}_t \otimes \tilde{\boldsymbol{q}}_t^i, \boldsymbol{B}_t \otimes \tilde{\boldsymbol{q}}_t^i \boldsymbol{e}']$.

The following two subsections are respectively devoted to indices based on physical quantities and prices of production. Moreover, a distinction is made between *productivity indices*, i.e., a measure of output divided by a measure of inputs, and *structural change indices* that try to capture changing interdependence among industries.

Appendix B contains detailed numerical examples on how to compute the indices presented in the following.

2.2 Indices based on physical quantities

Two simple and intuitive measures of productivity, derived from the final output subsystems, are the following σ- and ξ-indices:

$$\sigma_t^i = \frac{\tilde{b}_{(i,i,t)}^i - \tilde{a}_{(i,i,t)}^i}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i} = \frac{\text{external output}}{\text{direct + indirect labour}} \tag{2.3}$$

$$\xi_t^i = \frac{\tilde{b}_{(i,i,t)}^i - \boldsymbol{e}'\tilde{\boldsymbol{a}}_{(:,i,t)}^i}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i} = \frac{\text{final output}}{\text{direct + indirect labour}} \tag{2.4}$$

The ξ-index is also known as the *Gossling I* index, see Gossling (1972, p. 45). It is numerical equal to reciprocal of Pasinetti's vertically integrated labour coefficients. Itself a measure of labour productivity, *viz.*

$$\boldsymbol{v}_t = (\boldsymbol{B}_t - \boldsymbol{A}_t)^{-1} \boldsymbol{l}_t \tag{2.5}$$

The vertically integrated labour coefficients provide the units of direct and indirect labour needed to produce one unit of the ith industry's final output. See also Appendix A.1 and B.2.

In Equation 2.3 and 2.4 the external and final output in the numerators refer to the external and final output in the single subsystems. The external output is by definition the gross output minus the industry's sales of its own output to itself. The denominator consists of the total labour employed in the ith subsystem, i.e., the direct labour employed in the ith industry and the indirect employed in the supporting industries.

To be precise, a proportion of the labour employed in the ith industry of the ith subsystem, should be accounted as indirect, since a proportion of the industry's output in the subsystem is sold as means of production and hence enters indirect into the industry's own production. In most cases this complication is not important, since we are mainly interested in the total of the direct and indirect labour, but in one of the following indices the distinction is explicitly taken into account.

The α-, β-, and ρ-indices below are measures of structural change, the last two of which are derived from the subsystems. They all provide measures of the integration of the single industries with the system as a whole. The three indices are bounded within the unit interval and the closer the indices are to unity the more isolated is the industry.

$$\alpha_t^i = \frac{b_{(i,i,t)} - \boldsymbol{e}'\boldsymbol{a}_{(:,i,t)}}{b_{(i,i,t)}} = \frac{\text{final output*}}{\text{gross output*}} \tag{2.6}$$

$$\beta_t^i = \frac{\tilde{b}^i_{(i,i,t)} - \boldsymbol{e}'\tilde{\boldsymbol{a}}^i_{(:,i,t)}}{\tilde{b}^i_{(i,i,t)}} = \frac{\text{final output}}{\text{gross output}} \tag{2.7}$$

$$\rho_t^i = \frac{\beta_t^i \tilde{l}^i_{(i,t)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i} = \frac{\text{direct labour}}{\text{direct + indirect labour}} \tag{2.8}$$

The asterisk '*' here denotes 'for the system as a whole'. The β-index is computed as the ratio of final to gross output for the ith industry in the ith subsystem. The analogue α-index is computed from the system as a whole and is a common measure for the integration of the industry with the system as a whole. If there is a large difference between an industry's final and gross output (for the whole system) it implies that a large amount of the industry's output is sold as means of production, and *vice versa*.

The interpretation of the β-index is different. It provides a measure of the importance of the ith commodity within the ith subsystems, i.e., the importance of the single commodities within their own supply chain. This should be interesting for detailed inter- and intraindustry studies.

The β-index can be used to compute the intraindustry direct and indirect labour discussed above. This is done in the numerator of the ρ-index which is computed as the ratio of direct to direct and indirect labour. Think of $(1 - \beta_t^i)\tilde{l}^i_{(i,t)}$ as the amount of labour employed in the ith industry in the ith subsystem, that is producing commodities that are eventually used as means of production within the subsystem.

The ρ-index is therefore a properly generated measure of the ratio between direct labour and the total amount of labour employed throughout the supply chain.

2.3 Indices based on production prices

A physical production system $[\boldsymbol{A}_t, \boldsymbol{l}_t, \boldsymbol{B}_t]$ has, for a given distribution of the Net National Product (NNP) between wages and profits, a unique vector of Sraffian *production prices*, $\boldsymbol{p}_t(r)$, measured in terms of a given *numéraire.*

Following the *Non-substitution theorem* these production prices are unaffected by any rescaling of the system.[7] Consequently, also of the transformation into subsystems. As usual production prices and the associated *wage-profit frontier* are for $t = 1, 2, ..., T$ given by:

$$\boldsymbol{p}_t(r) = \frac{\big(\boldsymbol{B}_t - \boldsymbol{A}_t(1+r)\big)^{-1}\boldsymbol{l}_t}{\boldsymbol{\eta}'\big(\boldsymbol{B}_t - \boldsymbol{A}_t(1+r)\big)^{-1}\boldsymbol{l}_t} \qquad r = \{r \in \mathbb{Q} : 0 \leq r \leq R_t\} \tag{2.9}$$

$$w_t(r) = \Big[\boldsymbol{\eta}'\big(\boldsymbol{B}_t - \boldsymbol{A}_t(1+r)\big)^{-1}\boldsymbol{l}_t\Big]^{-1} \tag{2.10}$$

Where $\boldsymbol{\eta}$ is a *pro tempore* unspecified *numéraire* and R_t the maximum rate of profit.[8]

Consequently, the value in terms of production prices of the net products for the system as a whole and the final output subsystems, are given by the following accounting identities:

$$\boldsymbol{\zeta}_t(r) = \big(\boldsymbol{B}_t - \boldsymbol{A}_t\big)\boldsymbol{p}_t(r) \tag{2.11}$$

$$\tilde{\zeta}_t^i(r) = \boldsymbol{e}'\big(\tilde{\boldsymbol{B}}_t^i - \tilde{\boldsymbol{A}}_t^i\big)\boldsymbol{p}_t(r) \tag{2.12}$$

An obvious property, following the additivity of the final output subsystems, is that $\mathrm{NNP}_t(r) = \boldsymbol{e}'\boldsymbol{\zeta}_t(r) = \sum_{i=1}^n \tilde{\zeta}_t^i(r)$.

Following Degasperi and Fredholm (2010) a procedure to construct a distribution free measure of labour productivity from the above net products is to compute the following definite integrals (by means of computational methods):

$$\mu_t^i = \frac{1}{l_{(i,t)}R_t}\int_0^{R_t} \zeta_{(i,t)}(r)\,dr \tag{2.13}$$

$$\psi_t^i = \frac{1}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i R_t}\int_0^{R_t} \tilde{\zeta}_t^i(r)\,dr \tag{2.14}$$

The maximum rate of profit, R_t, associated with the system, which like the production prices is unaffected by the rescaling into subsystems, is used to normalise the indices, such that systems with different maximum profit rates better can be compared. The ψ-index takes into account the effect from the supporting industries, while the μ-index does not.

[7] See Kurz and Salvadori (1995, p. 26–28) for discussion of the origin and implications of this peculiar result.

[8] The maximum rate of profit can be computed as $R_t = \lambda_t^{-1} - 1$, where λ_t is the maximum eigenvalue of the matrix of interindustry coefficients, $\boldsymbol{B}_t^{-1}\boldsymbol{A}_t$, at time t, see Pasinetti (1977, p. 76).

Two measures which can be used to study structural change are:

$$\gamma_t^i = \frac{1}{R_t}\int_0^{R_t} \frac{\big(\tilde{b}^i_{(i,i,t)} - \tilde{a}^i_{(i,i,t)}\big)p_{(i,t)}(r)}{\boldsymbol{e}'\tilde{\boldsymbol{A}}^i_t\boldsymbol{p}_t(r) + \boldsymbol{e}'\tilde{\boldsymbol{l}}^i_t w_t(r)}\,dr \tag{2.15}$$

$$\delta_t^i = \frac{1}{R_t}\int_0^{R_t} \frac{\big(\tilde{b}^i_{(i,i,t)} - \tilde{a}^i_{(i,i,t)}\big)p_{(i,t)}(r)}{\tilde{\boldsymbol{a}}^i_{(i,:,t)}\boldsymbol{p}_t(r) + \tilde{l}^i_{(i,t)} w_t(r)}\,dr \tag{2.16}$$

The γ-index is based on the ratio of the 'value of external output' over the 'social costs', where the social costs are the total costs for the subsystem as a whole in terms of capital goods and wages. The δ-index is based on the ratio of the 'value of external output' over the 'local costs', where the local costs are the total costs, in terms of capital goods and wages, for the ith industry in the ith final output subsystem. The γ-index takes into account the effect from the supporting industries, while the δ-index does not.

A major advantage of all the indices presented, is that they do not change alone as a consequence of changes in the scale of production in the original system, even if it is asymmetric across the single industries. Hence, the indices will only change when real technological innovations take place. It is exactly the consequence of such changes we want to capture in the indices, since they can influence both the structural relationship among industries and the productivity level in the single industries. This very convenient property follows from the Non-substitution theorem (the invariance of the production prices) and the fact that the reproportioning into subsystems, likewise independently of the vector of final consumption, determines the relative proportions of inputs to outputs.

2.4 Policy implications

The subsystem approach has several useful features for both for ex ante and ex post evaluations of economic policy. Ex post, the indices presented here can be use to better separate structural, technological, and scale effects from a given economic policy. Of course with the usual reservations about the *ceteris paribus* assumption in such analysis.

Ex ante, it is possible to provide a first approximation of the total (direct plus indirect) effect on, e.g., labour demand, emission of greenhouse gasses, or the balance of trade following a change in the scale of production in a single or group of industries. Not only is it possible to provide an estimate on the aggregate effect, but also how these effects are distributed across industries. Remember that the final output subsystems are additive such that it is possible to move freely between local and social effects.

Here the subsystem multipliers, obtained in Section 2.1, emerge as a very convenient auxiliary tool. Why this is so, is shown together with a few example in Section 5.

3 Data and the Choice of *Numéraire*

The Danish input–output tables cover the entire period 1966–2005 for 130 industries following international standards of national accounting and include the flow among Danish industries as well as industry/commodity specific imports. The data are available in both current and fixed prices with base-period 2000.[9]

The fixed base-period denominated tables are used as a proxy for the physical inter-industrial flow of commodities. It must be stressed that by so doing we only see the "shadow" of the physical flow, but if empirical studies on this are to be carried out, this is as good as it gets.[10] In effect, what we are using is data measured in *Leontief Units*, i.e., a volume of physical goods worth one Danish krone, but the data is treated as if it were heterogeneous physical quantities, e.g., we do not sum distinct commodities.

Furthermore, detailed employment data are used on the total hours worked in each industry in each accounting period. Note, that labour is treated as a homogeneous input, both over time and across industries. This is likewise a very strong assumption, but again necessary given the data availability.

The 130 industries must be aggregated down to 123, to ensure non-singular matrices for all periods. For convenience in presenting the results—and only for that reason—the tables are aggregated into 52 industries. The full list of industries and details on the aggregation are found in Appendix D.

As we will see in the results, there are cases where it is seems more plausible to be residuals from monetary shocks left in the data, rather than real technological phenomena, that are causing the dynamics observed in the computed indices. Examples of this are the oil price shocks in 1970s, the financial turmoil in 1987, and a breakdown of an international monetary system.

A choice has been made to exclude the 1970 and 1971 tables from the dataset. The 1970 and 1971 tables can be seen as outliers, especially the 1971 table is extreme, in the sense that the system has a maximum interest rate very close to zero.[11] Hence, the economic system is close to *non-viable*, see Section 3.1. Excluding tables from the dataset does not in any way influence the other results, since the production prices, wage-profit frontiers, and the decomposition into subsystems are fully determined within each period. This is another practical feature of this approach compared with

[9] Statistics Denmark, www.dst.dk/inputoutput

[10] For a discussion on monetary vs. physical denominated input–output data, see Han and Schefold (2006, p. 750).

[11] An possible explanation of this phenomenon, is the economic turmoil around the collapse of the Bretton Woods system of monetary management, i.e., monetary and not technological phenomena that are not properly deflated from the data.

standard econometric exercises.

As a *numéraire*, for the computation of production prices, we choose the vector of domestic net products from the base year 2000 normalised with the total hours worked. We use the domestic net product, because if imported means of production were subtracted we would not necessarily obtain a vector of non-negative entries. This naturally leads to the next section.

3.1 Viability, self-replacing, and imports

All the systems 1966–2005 are viable, i.e., there exist the possibility (by rescaling) for the system to reproduce itself, but not all systems are in a self-replacing state.[12] Not being in a self-replacing state implies that the system in its current state does not produce a strictly positive vector of final (net) products.

In the Danish data, this is mainly a consequence of the inclusion of imported capital goods. The input-matrix, $\boldsymbol{A}_t$, is the sum of the matrix of domestic interindustry flow and the matrix of industry specific imports. In hindsight, this is not surprising for a small open economy, as the Danish, where exports account for roughly half of the NNP.

This constitutes a problem, since the subsystems per construction are set to produce and only produce the final output of the given industry. Nevertheless, the practical implications of this problem do not seem critical. All the $52 \cdot 40 = 2080$ (52 industries in 40 time periods) gross output subsystem multipliers (Equation 2.1) are strictly positive. The final output subsystem multipliers (Equation 2.2) on the other hand are strictly negative for the subsystems associated with commodities for which the system as a whole produces a negative final output.

This is still conceptually a problem, but since all the subsystem based indices are scale-independent, also negative numerators and denominators will cancel out. Hence, the interpretation of the subsystem based indices are not affected by non-self-replacement. The full extend of the non-self-replacement can be seen in the α-index (Figure E.34–E.41 in the statistical companion), i.e., the ratio of final to gross output for the system as a whole.

4 On Structural and Technological Change

Computing all the presented indices produces a huge number of time-series (52 series for each index). Therefore, only a small subset of the results is presented and analysed. The full set of results based on the Danish input–output tables is collected in the statistical companion.

[12] See Chiodi (1998) for a theoretical discussion on the notion of viability and non-self-replacing states.

Since the final output subsystems are additive, the 52 industries can be grouped in seven meso-sectors, as summarised in Table 4.1.

Meso-sector	industry classification	industry #
1	Agriculture, fishing, and quarrying	1–7
2	Manufacturing	8–20
3	Electricity, gas, and water supply; and Construction	21–22
4	Wholesale-, retail trade, hotels, restaurants	23–30
5	Transport, storage, and communication	31–35
6	Financial intermediation, business activities	36–43
7	Public and personal services	44–52

Table 4.1: *Grouping the 52 industries into seven meso-sectors*

4.1 Technological progress

Grouping the industries allows us to construct figures such as Figure 4.1 for the σ-index, which can be used to illustrate the decomposition of the aggregate technological change. The σ-index shows a clear positive but cyclical trend in the (aggregate) technological progress in Denmark since the 1960s. Figure 4.1 also shows that the primary forces behind this technological progress is to be found within the meso-sectors of 'Agriculture, fishing, and quarrying' and 'Manufacturing'.

Figure E.1–E.18 in the statistical companion show the industry-level technological progress within the single meso groups measured by the σ- and ξ-index. From this it is clear that the main sources of the progress within 'Agriculture, fishing, and quarrying' and 'Manufacturing' are respectively 'Extr. of crude petroleum, natural gas etc.' and 'Mfr. of refined petroleum products etc.'. These two industries are so important that, if their subsystems were removed from the productivity accounting the level of technological progress for 2005 would be similar to that of the mid 1990s, see Figure E.10 in Appendix E.1 in the statistical companion.

The same pattern is found using the μ- and ψ-index, see Appendix E.7 and E.8. As noted in Section 3, it is possible that some of these effects are caused by monetary phenomena (change in the price of oil etc.) which are not sufficiently deflated from the interindustrial flow. See also Figure E.41 for the α-index which show for the aggregate economy a drastic development in the ratio of final to gross output in 'Extr. of crude petroleum, natural gas etc.'.

Figure E.4 in the statistical companion, the σ-index for 'Wholesale-, retail trade, hotels, restaurants', shows that the σ-index for 'Retail trade of food etc.' increased steadily from the 1960s until 1994/95 where after it decreased for almost a decade returning to the level of 1985. From around 2003 it again increased. Unfortunately, since the food crisis of 2007-2008 would be interesting to follow, the series ends in 2005. Nevertheless, these

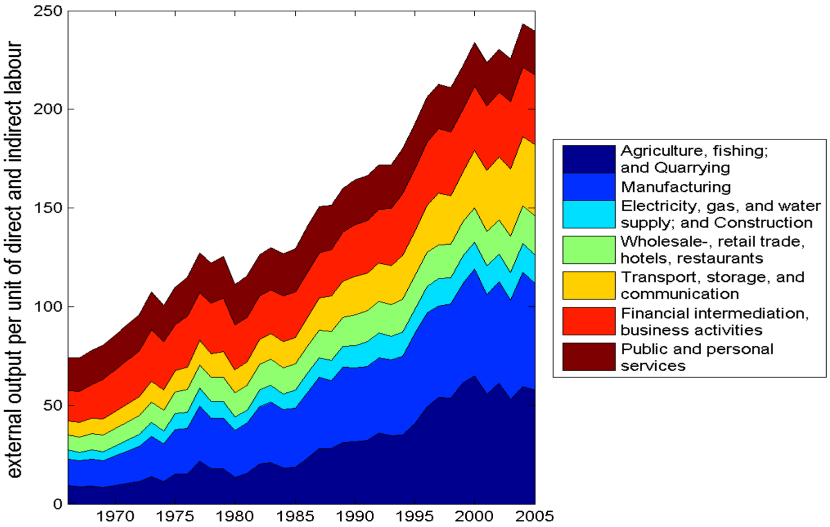

Fig. 4.1: *The σ-index for the seven meso-sectors*

results could, supported from the other indices and the evolution in similar industries, such as 'Agriculture' and 'Mfr. of food, beverages and tobacco', be interesting in the quest of understanding the local effects from apparent global phenomena. Moreover, the study could be extended to a industry-level comparative study among countries. Future studies should also try to go deeper into the identification of possible 'monetary residue' in the fixed base-period denominated tables, because it seems unlikely that such monetary phenomenon, as a global food crises, could be fully deflated in fixed base-period inter-industrial data.

Figure E.6 in the statistical companion shows that the σ-index for 'Financial intermediation' increased slowly until around 1982 where after it accelerated to a higher rate of growth which was maintained until around 2000, where the rate of growth further increased. The series ends in 2005. Without going into detail the *structural break* around 1982 corresponds with a strong deregulations of the Danish financial sector, a trend which continued until the outbreak of the present financial crisis. The increased growth rate from around 2000 coincides with initiatives for further European integration of financial markets, e.g., the 'Financial Services Action Plan' of 1999, see Kurek (2004). Whether or not there is an actual causal relationship between the institutional changes and the observed development, calls for further research, again together with the search for monetary residue in the data.

4.2 Direct and indirect labour

The ρ-index (direct over direct plus indirect labour), Figure E.27–E.33, for the primary and secondary industries: 'Agriculture, fishing, and quarrying' and 'Manufacturing' in general fall between two-third and one, while the tertiary industries (the industries in Meso-sector 3–7) between 0.95 and 1. This implies that the tertiary industries are the more isolated, in the sense that they use relatively less amounts of indirect labour compared with the primary and secondary industries.

The results also show the relative use of direct and indirect labour is fairly stable over time.

4.3 The Great Convergence

The evolution of the vertically integrated labour coefficients are collected in Figure 4.2 for the industries collected in the meso groups of 'Agriculture, fishing, and quarrying'; 'Manufacturing'; and 'Wholesale-, retail trade, hotels, restaurants'. From this a peculiar result emerges; the vertically integrated labour coefficients tend to converge to a common factor around one-half.

This might be a consequence of investment allocation towards low productive industries/subsystems with the expectations of higher returns, i.e., the usual explanation. Independently of the causes behind this convergence,

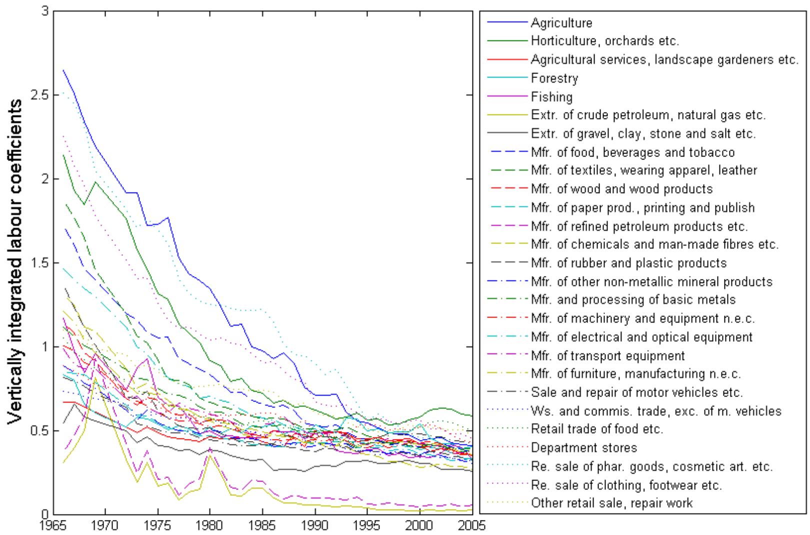

Fig. 4.2: *The v-index, Agriculture, fishing, and quarrying; Manufacturing; and Wholesale-, retail trade, hotels, restaurants*

it is a very clear and strong empirical observation that calls for further research to establish the causes and consequences of this economic phenomenon.

The only two industries in Figure 4.2 that do not converge to a value close to one-half are *the usual suspects* 'Extr. of crude petroleum, natural gas etc.' and 'Mfr. of refined petroleum products etc.' that steadily approach zero. For the industries in the other meso-sectors this convergence is less clear, but still a tendency is observed for most industries, see Figure E.20–E.26.

4.4 Comparing the different indices

Two different measures are used to access the quantitative differences among the indices. The first is the correlation coefficients for simple linear regression, where to obtain one measure for each par of indices, a simple average is computed across the 52 time series.

The second measure is a modified Mean Absolute Percentage Error (MAPE), computed as follows: First, normalise all indices, such that the index for the base-period 2000 is equal to unity. This is done to better abstract from differences in levels. Second, compute the following mean of the modified MAPE, *viz.*

$$m_{(k,\kappa)} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\left|\frac{x_t^{i,k} - x_t^{i,\kappa}}{\max\left\{x_t^{i,k}, x_t^{i,\kappa}\right\}}\right|\right) \quad \forall\ k < \kappa \tag{4.1}$$

Where $x_t^{i,k}$ is the kth index for the ith subsystem at time t. As with the correlation coefficients a simple average is computed across the 52 time series (the first summation). The denominator is chosen as the maximum of the two entries in the numerator, because when more than two time series are compared it can make a huge difference which is chosen as the base. A value of $m_{(k,\kappa)} = 0.10$ should be read as an average absolute deviation of 10 percent.

These measures are collected in Table 4.2 where the upper triangles show the means of the correlation coefficients and the lower triangles the means of the modified MAPEs. The productivity indices are collected in the left

Indices of productivity						Indices of structural change					
	σ	ξ	v	μ	ψ		α	β	ρ	γ	δ
σ		0.99	0.68	0.32	0.30	α		0.50	0.50	0.52	0.45
ξ	0.01		0.69	0.32	0.30	β	1.40		0.61	0.63	0.48
v	0.48	0.48		0.49	0.58	ρ	0.91	0.15		0.88	0.73
μ	0.27	0.27	0.45		0.93	γ	1.30	0.06	0.11		0.80
ψ	0.26	0.26	0.46	0.07		δ	1.50	0.05	0.18	0.10	

Table 4.2: *Comparing the different indices – correlation coefficients and mean modified MAPEs*

hand side matrix of Table 4.2 and the indices of structural change in the right hand side matrix.

The σ-index (external output over direct and indirect labour) and the ξ-index (final output over direct and indirect labour) are almost identical. The difference between the two measures, is the amount of the ith commodities in the ith subsystem used in the supporting industries, which consequently must be relatively small. Hence, choosing either two does not seem to make much of an difference.

The two measures of productivity based on production prices are also closely correlated, i.e., the μ-index (not subsystem based) and the ψ-index (subsystem based).

The measures based on production prices and those based on physical quantities are also correlated, but not to the same extent as indices based on production prices and physical quantities, respectively.

What cannot be seen in the table is however that the v-index (the vertically integrated labour coefficients) and the ξ-index are perfectly correlated, but not linear so. The ξ-index is equal to the reciprocal of the v-index, see Appendix A and B.

For the indices of structural change, in the right hand side matrix of Table 4.2, the correlation coefficients and the modified MAPEs report evidence of some co-evolution among the indices. The α-index, which is not based on the subsystems, is compared with the other indices clearly the most distinct.

This exercise should be taken into consideration, when one index is chosen over another.

5 On Economic Policy

5.1 A subsystem based CO_2 accounting

Table 5.1 shows a subsystem based CO_2 accounting for Denmark 2005. For practical reasons the results are presented only for the seven meso-sectors, but could easily be constructed at any level of aggregating. Table 5.1 is constructed by multiplying element-by-element the vector of industry specific emission of CO_2 found in The Danish Air Emissions Accounts[13] with the subsystem multipliers for 2005. The outcome is the matrix below which decompose the total CO_2 emission for 2005, 48,731 units, into not only where the pollution occurred, but also to the industries in which the demand that let to this pollution was created. In a sense this is a distinction between cause and effect, i.e., what drives the production (cause) and where the production/pollution take place (effect). The columns represent cause and the rows effect, e.g., the sum of the second row, 7,833 units of CO_2, is the total

[13]www.dst.dk/inputoutput, which includes data on the eight groups of greenhouse gasses; Carbon dioxide (CO_2), Sulphur dioxide (SO_2), Nitrogen oxides (NO_x), Carbon oxide (CO), Laughing gas (N_2O), Ammonia (NH_3), Methane (CH_4), and Non methane volatile organic compounds (NMVOC) all covering 130 industries 1990–2006.

emission occurring in Meso-sector 2 'Manufacturing' and the single entries in the second row show where the demand which let to this pollution was created, e.g., the 736 units in last entry of the second row is the demand created in Meso-sector 7 'Public and personal services'. Consequently, the

demand from/emission in	1	2	3	4	5	6	7	totals
1. Agriculture, fishing, and quarrying	1444	2166	286	497	360	95	382	5222
2. Manufacturing	76	2877	74	3144	406	532	736	7833
3. Electricity, gas, and water supply; and Construction	243	3522	13699	3666	972	799	3100	25996
4. Wholesale-, retail trade, hotels, restaurants	9	113	17	2055	42	183	135	2555
5. Transport, storage, and communication	32	455	19	1188	2811	167	659	5322
6. Financial intermediation, business activities	1	20	2	44	10	152	48	277
7. Public and personal services	4	69	6	153	24	59	1222	1533
totals	1800	9211	14102	10732	4622	1999	6277	48731

Table 5.1: *The decomposition of the total Danish CO_2 emission for 2005, all measured in units of 1000 tonnes of CO_2*

entries on the main diagonal show the emission caused by and occurred in the single industries.

An interesting observation is the differences between the totals found in last row and column, e.g., the emission directly created in the public sector is 1,533 units, but the indirect from the supporting industries, necessary to maintain the activity level in the public sector, is 6,277 units, i.e., a difference of a factor four.

This observation is in line with a similar empirical study of the Spanish economy by Alcántara and Padilla (2009). Based on the evidence from the Danish economy, we support both the general conclusion and policy recommendation stated by Alcántara and Padilla (2009, p. 913):

> The results of our work refute the idea that a services economy is necessarily a less polluting economy. Although industrial productive processes are more directly linked to energy consumption, the final responsibility of their emissions rests on the industries that demand their production. [...]
>
> A policy designed to control and mitigate emissions should consider the importance of the consumption of energy, and the emissions needed to facilitate these industries' production.

5.2 Direct and indirect employment effects

Imaging that the government was to implement a policy that would increase construction activities by 5 percent. What is the effect on aggregate employment and how is the increased employment distributed across industries? Assuming constant returns to scale, we can use the subsystem approach to provide an answer which takes into account the fact that employment must increase in the supporting industries as well as the supporting industries of the supporting industries and so on and so forth.

This way of reasoning is closely related to the seminal work by Kahn (1931) on 'The relation of home investment to unemployment', upon which Keynes based the theory of his famous multiplier (Keynes, 1936 ch. 10). We compute what Kahn called the *primary employment*, i.e., the sum of direct and indirect employment.[14]

Table 5.2 shows the decomposition of the aggregate Danish employment for 2005, 430 million hours, into where they are employed (the rows) and where the demand which let to this employment originated (the columns). Consequently, to see the direct and indirect effect of an increased activity

demand from/employment in	1	2	3	4	5	6	7	totals
1. Agriculture, fishing, and quarrying	307	851	17	86	100	19	109	1490
2. Manufacturing	55	2963	48	1652	290	370	765	6144
3. Electricity, gas, and water supply; and Construction	2	24	122	27	7	6	25	213
4. Wholesale-, retail trade, hotels, restaurants	31	416	46	7884	163	480	467	9487
5. Transport, storage, and communication	20	298	12	779	2025	116	420	3670
6. Financial intermediation, business activities	21	290	29	549	137	2058	618	3701
7. Public and personal services	37	661	50	1500	219	566	15313	18347
totals	474	5504	324	12477	2941	3615	17717	43052

Table 5.2: *The decomposition of the total Danish employment 2005, all measured in 10000 hours*

in 'Manufacturing', simply increase all entries in the second column by the desired proportion and compute the new sums to see, in the rightmost column, how much the employment *ceteris paribus* will increase in the different industries. Hence, we obtain not only the total effect on employment, but also how the employment is distributed across industries. This is convenient, if for example the economy is close to full employment for groups primarily employment in specific industries.

To make the computation for the construction sector all you need to do is to go to the disaggregated table (available upon request) and follow the procedure presented above.

6 Concluding Remarks

One of the main advantages of the subsystem approach over conventional methods is according to Gossling (1972, pp. 40–41) that

[14]The *secondary employment*, *viz.* the convergent infinite series of effects caused by the initial increase in wages and profits, lies outside what can be explained in this theoretical framework.

> partial productivity measures [...] are always bedevilled by the reservation that the interdependence of the industry with the economy is changing from year to year. It occurs to us that productivity indexes for the sub-system are always free from this reservation because its interdependence with the economy is permanently nought.

It is for that reason, we in this paper have tried to collect, organize, define, compute, and assess indices based on the subsystem approach. In our opinion this approach has huge potentials not only as a descriptive tool, but also since it provides procedures to evaluate economic policy. In particular, considering that the subsystem multiplier approach is easy to implement from both a computational and intuitive point of view. By following this approach we have been able to study the evolution in, and integration among, Danish industries.

First, we observe very strong evidence of a convergence in the levels of productivity as measured by Pasinetti's vertically integrated labour coefficients, i.e., a convergence in the amount of direct and indirect labour needed to produce one unit of the final output.

Second, by inspecting the industry-specific contributions to the aggregate technological progress, we observe that two industries: 'Extr. of crude petroleum, natural gas etc.' and 'Mfr. of refined petroleum products etc.' have a surprisingly high impact on the overall development. Without these two industries in the productivity accounting the level of technological progress for 2005 would be similar to that of the mid 1990s.

Third, the subsystem approach has been shown to provide important insight for policy making. A fairly simple method have been presented, to measure direct and indirect emission of CO_2 which enables us to identify the origin of the specific demand that causes the emission. Such a method could prove instrumental, in the current discussion among world leaders on how to reduce the emission of greenhouse gasses. If focus is kept on only the direct (observable) emission of greenhouse gasses, the treatment of the problem can never be on more than on the symptoms.

Likewise, the direct and indirect employment effects from industry specific expansions should prove useful when policy makers are faced with different initiatives to increase (industry-specific) employment.

To sum up, the subsystem approach can both be used as a powerful descriptive tool and as a complexity-reducing tool for policy making. We can only endorse the following statement by Gossling (1972, p. 28):

> It is now to be hoped that whenever the reader sees a tableau of interdependent single-product activities, whether for a firm or an economy, he may also visualize the corresponding sets of independent isolated sectors (or sub-systems), and thereby may *abstract* from interdependence.

Acknowledgements

We would like to thank Charlotte Bruun, Carsten Heyn-Johnsen, Matteo Degasperi, and K. Vela Velupillai for their much appreciated help and support. Also the participants at the Second Schumpeter Summer School in Graz 2009 deserve our gratitude for suggestions, well placed critique, and words of encouragement.

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A On the Construction of Subsystems

There are at least four conceptually different procedures to construct subsystems, Gossling's (1972) iterative method, Pasinetti's (1973) notion of vertically integrated sectors, Sraffa's reduction to dated quantities of labour, and the direct multiplier method presented in this paper.[15] They all yield quantitatively identical results, but some applications follow more direct from one procedure than another.[16]

Let $[\boldsymbol{A}_t, \boldsymbol{B}_t, \boldsymbol{l}_t, \boldsymbol{O}_t, \boldsymbol{z}_t, \boldsymbol{U}_t, \boldsymbol{\tau}_t]$ be a set of data variables measured in physical quantities. The entries are respectively, the non-singular indecomposable semi-positive $n \times n$ *input-matrix*, the diagonal *gross output-matrix*, the column *vector of labour inputs*, the diagonal *final output-matrix*, the row *vector of the industries' total sales to means of production*, the *matrix of market share coefficients to other activities*, and the *vector of market share coefficients to final buyer.* $\boldsymbol{A}_t$ is composed of row vectors of intraindustry inputs and column vectors of interindustry flow. Furthermore, let $[\check{\boldsymbol{A}}_t, \check{\boldsymbol{l}}_t, \boldsymbol{I}]$ be the associated matrix of interindustry coefficients, the vector of direct labour coefficients, and the identity matrix, respectively.

Connected with this we have the following accounting identities, where $\boldsymbol{e}$ is a $n \times 1$ unit vector and $\widehat{\boldsymbol{b}}$ the diagonal from the corresponding $\boldsymbol{B}$ matrix, now a $n \times 1$ vector.

$$\boldsymbol{z}_t = \boldsymbol{e}'\boldsymbol{A}_t \tag{A.1}$$
$$\widehat{\boldsymbol{o}}_t = \boldsymbol{B}_t\boldsymbol{e} - \boldsymbol{z}_t' = \big(\boldsymbol{B}_t - \boldsymbol{A}_t'\big)\boldsymbol{e} \tag{A.2}$$
$$\boldsymbol{U}_t = \boldsymbol{A}_t\boldsymbol{B}_t^{-1} \tag{A.3}$$
$$\boldsymbol{\tau}_t = \widehat{\boldsymbol{o}}_t \big/ \widehat{\boldsymbol{b}}_t \tag{A.4}$$
$$\boldsymbol{e} = (\boldsymbol{e}'\boldsymbol{U}_t)' + \boldsymbol{\tau}_t \tag{A.5}$$
$$\check{\boldsymbol{A}}_t = \boldsymbol{B}_t^{-1}\boldsymbol{A}_t \tag{A.6}$$
$$\check{\boldsymbol{l}}_t = \boldsymbol{l}_t \big/ \widehat{\boldsymbol{b}}_t \tag{A.7}$$

In a paper by Gossling and Dovring (1966) and a book by Gossling (1972), both based on Gossling's unpublished PhD thesis from 1964, several years before Pasinetti's (1973) paper, it is shown how to construct *gross* and *final output subsystems* from input–output data measured in physical quantities as well as market prices. As will be seen later in this appendix Gossling's

[15]The fifth guise of the subsystems is found in Goodwin's Normalized General Coordinates approach, see e.g. Goodwin (1976). Conceptually, this approach seems very different and it should be checked whether or not there is a numerical difference.

[16]A note on terminology; the terms *incorporated* labour will be used when referring to the reduction to dated quantities of labour (equivalent to *embodied* labour), i.e., the classical notion used by Ricardo and Marx. The terms *direct* and *indirect* (e.g. labour) refer to the contemporary use of that input from respectively the industry producing the output and the supporting industries.

final output subsystems based on physical input–output data and Pasinetti's vertically integrated sectors yield quantitatively identical results.[17]

The following three sections provide an introduction to Pasinetti's vertically integrated sectors, Gossling's gross and final output subsystems, and on the reduction to dated quantities of labour.

A.1 Pasinetti's vertically integrated sectors

In the case of no joint production and excluding fixed capital. The *vector of vertically integrated labour coefficients* is given by:

$$\boldsymbol{v}_t = \left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1} \breve{\boldsymbol{l}}_t \tag{A.8}$$

Where $\left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1}$ is the well known Leontief inverse. The ith entry of $\boldsymbol{v}_t$ constitutes the direct and indirect labour needed to produce one unit of the ith final output. Consequently, the aggregated labour directly and indirectly required to produce the final output of the n commodities is given by:

$$\boldsymbol{\iota}_t = \boldsymbol{O}_t \left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1} \breve{\boldsymbol{l}}_t = \boldsymbol{O}_t \boldsymbol{v}_t \tag{A.9}$$

Furthermore, the total quantities of the n commodities as respectively gross output and total outlays in each vertically integrated sector are given by:

$$\boldsymbol{\Pi}_t = \boldsymbol{O}_t \left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1} \tag{A.10}$$

$$\boldsymbol{\Upsilon}_t = \boldsymbol{O}_t \breve{\boldsymbol{A}}_t \left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1} = \boldsymbol{O}_t \boldsymbol{H}_t \tag{A.11}$$

Where $\boldsymbol{H}_t = \breve{\boldsymbol{A}}_t \left(\boldsymbol{I} - \breve{\boldsymbol{A}}_t\right)^{-1}$ is the so-called *vertically integrated technical coefficient matrix*. For further details on vertically integrated sectors see Pasinetti (1973).

[17] The more historical oriented reader might add that the basic concepts behind the subsystems can be traced back to Petty, Smith, Ricardo, and more recently Hicks. See Pasinetti (1973), Scazzieri (1990), and Kurz and Salvadori (1995, pp. 175–80) on the origin of subsystems. However, the point we want to stress is that the procedure by which the subsystems can be constructed, first appears in Gossling's writings, but is always credited to Pasinetti. If this is because Gossling's work is unknown to most economists or that it is hitherto unknown that the two procedures yield identical results, is not clear to us.

This is not meant to discredit Pasinetti's work on vertically integrated sectors. The procedure presented in Pasinetti (1973) is much easier to apply than Gossling's iterative method, and the subsequent work on *dynamic subsystem* based on Pasinetti (1988) has developed much further the theoretical models.

A.2 Gossling's gross and final output subsystems

Following Gossling (1972) there are two distinct, but closely related subsystems, *viz.* the gross and the final output subsystem. The ith *gross output subsystem* is the system that produces the gross output associated with the ith industry from the original system and no final output in all other industries than the ith. The ith *final output subsystem* is the system that produces (and only produces) the final output associated with the ith industry from the original system. Only the final output subsystems are additive.

The matrix used by Gossling to transform the original system into gross output subsystems is the $[\boldsymbol{I} + \boldsymbol{P}_t]$ matrix, where $\boldsymbol{P}_t$ is the following matrix sum of an infinite series of restricted matrix products, where $\boldsymbol{U}_t$ (Equation A.3) is the matrix of market share coefficients to other activities, common for both physical and value denominated systems, *viz.*

$$\boldsymbol{P}_t = \boldsymbol{U}'_{t\boldsymbol{B}}\boldsymbol{I} + \boldsymbol{U}'_{t\boldsymbol{\Gamma}}\boldsymbol{U}'_t + \boldsymbol{U}'_{t\boldsymbol{\Gamma}}[\boldsymbol{U}'_{t\boldsymbol{\Gamma}}\boldsymbol{U}'_t] + \boldsymbol{U}'_{t\boldsymbol{\Gamma}}\big[\boldsymbol{U}'_{t\boldsymbol{\Gamma}}[\boldsymbol{U}'_{t\boldsymbol{\Gamma}}\boldsymbol{U}'_t]\big] + \cdots \tag{A.12}$$

The B-operation is the usual matrix multiplication for square matrices, but with the main diagonal entries replaced with zeros. The Γ-operation is like the B-operation except that, in forming the scalar products associated with the ijth entry, the jth element from the jth column vector is replaced with a zero.[18]

The jth column in the $[\boldsymbol{I}+\boldsymbol{P}_t]$ is equal to the ith gross output subsystem multiplier $\overline{\boldsymbol{q}}_t^i$ presented in Section 2.1.

This procedure to construct subsystems, i.e., also the subsystem multiplier approach, is independent of the units of which the input–output data is denominated (physical quantities, current, or fixed market prices) as long as each industry's output has its own unique price.[19] The reason for this peculiar property is that Gossling's iterative procedure is based on the price independent market shares to other activities, and not the matrix of interindustry coefficients as in the Pasinetti's representation. Consequently, the procedure above is not only applicable on data measured in physical quantities as well as (constant) market prices, but yields identical multipliers.

[18] This iterative procedure is similar to Sraffa's construction of the *standard commodity*. See Velupillai (2008) for a discussion about the *constructive* mathematical logic behind this intrinsically algorithmic approach to the *mathematics of economics*. More general this is an excellent example of how mathematics more often should be applied in economics. As Gossling (1972, p. 29) writes "These matrices have been produced—some as 'by-products' in the quest for sub-systems' definition—using a direct method that bends the mathematics to the economic logic rather than *vice versa*." Too much economics is bending to fit into conventional mathematical logic, in particular *classical real analysis*.

[19] In real input–output tables each industry's output is itself a composite commodity. Consequently, each entry possesses innate index number problems.

A.3 Reduction to dated quantities of labour

Closely related to the concept of subsystems we have the reduction to dated quantities of labour (Sraffa 1960, ch. 6).[20] The quantities of labour needed to produce one unit of final output of the individual commodities can be traced back in time as:

$$\check{\boldsymbol{l}} + \check{\boldsymbol{A}}\check{\boldsymbol{l}} + \check{\boldsymbol{A}}^2\check{\boldsymbol{l}} + \check{\boldsymbol{A}}^3\check{\boldsymbol{l}} + \cdots = \left(\boldsymbol{I} - \check{\boldsymbol{A}}\right)^{-1}\check{\boldsymbol{l}} \tag{A.13}$$

This is exactly Pasinetti's vector of vertically integrated labour coefficients.

Consequently, the total labour incorporated in each final and gross output is respectively given by:

$$\boldsymbol{\Pi} = \boldsymbol{O}\left(\boldsymbol{I} - \check{\boldsymbol{A}}\right)^{-1}\check{\boldsymbol{l}} \tag{A.14}$$

$$\boldsymbol{\Theta} = \boldsymbol{B}\left(\boldsymbol{I} - \check{\boldsymbol{A}}\right)^{-1}\check{\boldsymbol{l}} \tag{A.15}$$

Hence, the total non-discounted labour incorporated is equal to the total of the contemporary direct and indirect labour computed using the subsystem approach. It is a strange property that two conceptually so different procedures yield identical results. This Sraffa (1960, p. 89) could "see at a glance"!

This is however only the pure flow of physical quantities of labour. The discounted value of the dated labour is derived from the following identity.

$$\boldsymbol{p}(r) = w\check{\boldsymbol{l}} + (1+r)\check{\boldsymbol{A}}\boldsymbol{p}(r) \tag{A.16}$$

By recursively substituting the right-hand side $\boldsymbol{p}(r)$ with the right-hand side of the equation. The value in labour terms as a function of the distribution of the net nation product is given by:

$$\begin{aligned}
\boldsymbol{p}(r) &= w\check{\boldsymbol{l}} + (1+r)\check{\boldsymbol{A}}\big(w\check{\boldsymbol{l}} + (1+r)\check{\boldsymbol{A}}\boldsymbol{p}(r)\big) = w\check{\boldsymbol{l}} + w(1+r)\check{\boldsymbol{A}}\check{\boldsymbol{l}} + w(1+r)^2\check{\boldsymbol{A}}^2\check{\boldsymbol{l}}\boldsymbol{p}(r)\\
&= w\big[\check{\boldsymbol{l}} + (1+r)\check{\boldsymbol{A}}\check{\boldsymbol{l}} + w(1+r)^2\check{\boldsymbol{A}}^2\check{\boldsymbol{l}} + \cdots + w(1+r)^t\check{\boldsymbol{A}}^t\check{\boldsymbol{l}} + \cdots\big]\\
&= w\big[\boldsymbol{I} + (1+r)\check{\boldsymbol{A}} + w(1+r)^2\check{\boldsymbol{A}}^2 + \cdots + w(1+r)^t\check{\boldsymbol{A}}^t + \cdots\big]\check{\boldsymbol{l}}\\
&= w\big[\boldsymbol{I} - (1+r)\check{\boldsymbol{A}}\big]^{-1}\check{\boldsymbol{l}} \qquad \forall \;\; 0 \le r < R
\end{aligned} \tag{A.17}$$

Note that in the special case where $r = 0$ and the wage rate is chosen as the *numéraire*, the two measures (A.13) and (A.17) coincide.[21]

[20]See also Pasinetti (1977, ch. 4) and Kurz and Salvadori (1995, ch. 6).

[21]Two indices based on the reduction to dated quantities of labour seems natural to consider.

$$\boldsymbol{\varrho}_t = \check{\boldsymbol{l}}_t + \check{\boldsymbol{A}}_t\check{\boldsymbol{l}}_t + \check{\boldsymbol{A}}^2_{t-1}\check{\boldsymbol{l}}_{t-1} + \check{\boldsymbol{A}}^3_{t-2}\check{\boldsymbol{l}}_{t-2} + \cdots + \check{\boldsymbol{A}}^k_{t-k-1}\check{\boldsymbol{l}}_{t-k-1} \tag{A.18}$$

$$\boldsymbol{\varphi}_t = \frac{1}{R_t}\int_0^{R_t} \boldsymbol{p}(r)\,dr \tag{A.19}$$

The ϱ-index is an approximation of A.13, but instead of using the same techniques of production when the inputs are traced back in time, the actual techniques used in each of the $k-1$ preceding periods are used. The φ-index is an attempt extract on scalar for each industry the time t.

B Subsystems Explained Using Examples

If otherwise not explicitly stated, the entries in the following examples are considered as physical quantities and all non-natural numbers in the following are rounded, i.e., minor discrepancies must be expected and "=" should be read as "≈".

	input-matrix				labour	gross	final
Industry 1	120	80	240	45	160	520	90
Industry 2	40	210	330	100	250	670	105
Industry 3	160	225	75	125	80	900	80
Industry 4	110	50	175	25	350	510	215
total use	430	565	820	295	840		

Table B.1: *Original system*

	input-matrix				labour	gross	final
Industry 1	120	80	240	45	160	520	255
Industry 2	19	101	159	48	120	323	0
Industry 3	89	125	42	69	44	499	0
Industry 4	37	17	59	8	117	171	0
total use	265	323	499	171	441		

Table B.2: *Gross output subsystem for Industry 1*

	input-matrix				labour	gross	final
Industry 1	42	28	85	16	56	183	90
Industry 2	7	36	56	17	42	114	0
Industry 3	31	44	15	24	16	176	0
Industry 4	13	6	21	3	41	60	0
total use	93	114	176	60	155		

Table B.3: *Final output subsystem for Industry 1*

	input-matrix				labour	gross	final
Industry 1	55	37	111	21	74	240	0
Industry 2	40	210	330	100	250	670	263
Industry 3	99	140	47	78	50	559	0
Industry 4	45	20	72	10	143	209	0
total use	240	407	559	209	517		

Table B.4: *Gross output subsystem for Industry 2*

	input-matrix				labour	gross	final
Industry 1	22	15	44	8	29	96	0
Industry 2	16	84	132	40	100	267	105
Industry 3	40	56	19	31	20	223	0
Industry 4	18	8	29	4	57	83	0
total use	96	162	223	83	206		

Table B.5: *Final output subsystem for Industry 2*

	input-matrix				labour	gross	final
Industry 1	70	47	141	26	94	305	0
Industry 2	26	134	211	64	160	428	0
Industry 3	160	225	75	125	80	900	396
Industry 4	49	22	78	11	155	226	0
total use	305	428	504	226	489		

Table B.6: *Gross output subsystem for Industry 3*

	input-matrix				labour	gross	final
Industry 1	14	9	28	5	19	62	0
Industry 2	5	27	43	13	32	87	0
Industry 3	32	45	15	25	16	182	80
Industry 4	10	4	16	2	31	46	0
total use	62	87	102	46	98		

Table B.7: *Final output subsystem for Industry 3*

	input-matrix				labour	gross	final
Industry 1	66	44	132	25	88	285	0
Industry 2	19	101	158	48	120	321	0
Industry 3	90	127	42	70	45	507	0
Industry 4	110	50	175	25	350	510	342
total use	285	321	507	168	603		

Table B.8: *Gross output subsystem for Industry 4*

	input-matrix				labour	gross	final
Industry 1	41	28	83	16	55	179	0
Industry 2	12	63	100	30	75	202	0
Industry 3	57	80	27	44	28	319	0
Industry 4	69	31	110	16	220	321	215
total use	179	202	319	106	378		

Table B.9: *Final output subsystem for Industry 4*

B.1 The subsystem multiplier method

Using equation 2.1 the gross output multiplier for the first gross output subsystem can be computed as the non-trivial solution of the following system.

$$\left(\begin{bmatrix} 670 & 0 & 0 \\ 0 & 900 & 0 \\ 0 & 0 & 510 \end{bmatrix} - \begin{bmatrix} 210 & 225 & 50 \\ 330 & 75 & 175 \\ 100 & 125 & 25 \end{bmatrix}\right) \bar{\boldsymbol{q}}^1 = \begin{bmatrix} 80 \\ 240 \\ 45 \end{bmatrix}$$

The solution of which is $[\ 0.482 \quad 0.555 \quad 0.335\ ]'$. Next *squeeze in* a single "1" on the ith entry. Following this procedure the four gross output multipliers can be computed as.

$$\bar{\boldsymbol{q}}^1 = \begin{bmatrix} 1.000 \\ 0.482 \\ 0.555 \\ 0.335 \end{bmatrix}, \bar{\boldsymbol{q}}^2 = \begin{bmatrix} 0.461 \\ 1.000 \\ 0.621 \\ 0.409 \end{bmatrix}, \bar{\boldsymbol{q}}^3 = \begin{bmatrix} 0.586 \\ 0.639 \\ 1.000 \\ 0.444 \end{bmatrix}, \bar{\boldsymbol{q}}^4 = \begin{bmatrix} 0.548 \\ 0.480 \\ 0.564 \\ 1.000 \end{bmatrix}$$

Using equation 2.2 the final output multiplier for the first subsystem can be computed as.

$$\tilde{\boldsymbol{q}}^1 = \begin{bmatrix} 1.000 \\ 0.482 \\ 0.555 \\ 0.335 \end{bmatrix} \frac{520 - (120 + 40 + 160 + 110)}{520 - (120 + 0.482 \cdot 40 + 0.555 \cdot 160 + 0.335 \cdot 110)} = \begin{bmatrix} 0.353 \\ 0.170 \\ 0.200 \\ 0.118 \end{bmatrix}$$

The full set of final output multipliers is given by.

$$\tilde{\boldsymbol{q}}^1 = \begin{bmatrix} 0.353 \\ 0.170 \\ 0.120 \\ 0.118 \end{bmatrix}, \tilde{\boldsymbol{q}}^2 = \begin{bmatrix} 0.184 \\ 0.399 \\ 0.248 \\ 0.163 \end{bmatrix}, \tilde{\boldsymbol{q}}^3 = \begin{bmatrix} 0.119 \\ 0.130 \\ 0.202 \\ 0.090 \end{bmatrix}, \tilde{\boldsymbol{q}}^4 = \begin{bmatrix} 0.345 \\ 0.302 \\ 0.354 \\ 0.629 \end{bmatrix}$$

Now it is straightforward to compute the σ-, ξ-, α-, β-, and ρ-indices for the final output subsystems. Here computed for the first industry, Table B.3:

$$\begin{aligned}
\sigma^1 &= \frac{\tilde{b}_{(1,1)} - \tilde{a}_{(1,1)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}^1} = \frac{\text{external output}}{\text{direct + indirect labour}} = \frac{183 - 42}{155} = 0.91 \\
\xi^1 &= \frac{\tilde{b}_{(1,1)} - \boldsymbol{e}'\tilde{\boldsymbol{a}}^1_{(:,1)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}^1} = \frac{\text{final output}}{\text{direct + indirect labour}} = \frac{90}{155} = 0.58 \\
\alpha^1 &= \frac{b_{(1,1)} - \boldsymbol{e}'\boldsymbol{a}_{(:,1)}}{b_{(1,1)}} = \frac{\text{final output*}}{\text{gross output*}} = \frac{90}{520} = 0.17 \\
\beta^1 &= \frac{\tilde{b}_{(1,1)} - \boldsymbol{e}'\tilde{\boldsymbol{a}}^1_{(:,1)}}{\tilde{b}_{(1,1)}} = \frac{\text{final output}}{\text{gross output}} = \frac{90}{183} = 0.49 \\
\rho^1 &= \frac{\beta^1\tilde{l}_1}{\boldsymbol{e}'\tilde{\boldsymbol{l}}^1} = \frac{\text{direct labour}}{\text{direct + indirect labour}} = \frac{0.49 \cdot 56}{155} = 0.18
\end{aligned}$$

The asterisk '*' here denotes for the system as a whole, i.e., Table B.1.

	σ^i	ξ^i	α^i	β^i	ρ^i
subsystem 1	0.91	0.58	0.17	0.18	0.49
subsystem 2	0.89	0.51	0.16	0.19	0.39
subsystem 3	1.69	0.81	0.09	0.07	0.44
subsystem 4	0.80	0.57	0.42	0.39	0.67

Table B.10: *Collection of indices based on physical quantities*

The indices based on physical quantities and production prices are collected in Table B.10 and B.11, respectively. The indices based production prices are all computed with the first commodity as a *numéraire*.

	μ^i	ψ^i	γ^i	δ^i
subsystem 1	0.58	0.59	0.32	0.85
subsystem 2	0.57	0.56	0.36	0.76
subsystem 3	0.60	1.09	0.41	1.02
subsystem 4	0.50	0.38	0.32	1.04

Table B.11: *Collection of indices based on production prices*

B.2 Pasinetti's vertically integrated sectors

The Leontief inverse associated with the system is given by:

$$\left(\boldsymbol{I}-\check{\boldsymbol{A}}\right)^{-1}=\left(\boldsymbol{I}-\begin{bmatrix}0.23 & 0.15 & 0.46 & 0.09\\ 0.06 & 0.31 & 0.49 & 0.15\\ 0.18 & 0.25 & 0.08 & 0.14\\ 0.22 & 0.10 & 0.34 & 0.05\end{bmatrix}\right)^{-1}=\begin{bmatrix}2.04 & 1.26 & 1.96 & 0.67\\ 0.91 & 2.55 & 2.12 & 0.79\\ 0.77 & 1.08 & 2.27 & 0.57\\ 0.93 & 0.94 & 1.48 & 1.49\end{bmatrix}$$

The vector of vertically integrated labour coefficients is consequently given by:

$$\boldsymbol{v}=\left(\boldsymbol{I}-\check{\boldsymbol{A}}\right)^{-1}\check{\boldsymbol{l}}=\begin{bmatrix}2.04 & 1.26 & 1.96 & 0.67\\ 0.91 & 2.55 & 2.12 & 0.79\\ 0.77 & 1.08 & 2.27 & 0.57\\ 0.93 & 0.94 & 1.48 & 1.49\end{bmatrix}\begin{bmatrix}0.31\\ 0.37\\ 0.09\\ 0.69\end{bmatrix}=\begin{bmatrix}1.73\\ 1.96\\ 1.24\\ 1.76\end{bmatrix}$$

Furthermore, the labour directly and indirectly required to produce the final output of the ith commodity is given by:

$$\boldsymbol{\iota}=\boldsymbol{O}\left(\boldsymbol{I}-\check{\boldsymbol{A}}\right)^{-1}\check{\boldsymbol{l}}=\boldsymbol{O}\boldsymbol{v} \tag{B.1}$$

Where $\boldsymbol{O}$ is a diagonal matrix of final outputs.

$$\boldsymbol{\iota}=\boldsymbol{O}\boldsymbol{v}=\begin{bmatrix}90 & 0 & 0 & 0\\ 0 & 105 & 0 & 0\\ 0 & 0 & 80 & 0\\ 0 & 0 & 0 & 215\end{bmatrix}\begin{bmatrix}1.73\\ 1.96\\ 1.24\\ 1.76\end{bmatrix}=\begin{bmatrix}155\\ 206\\ 98\\ 378\end{bmatrix}$$

Note, that this is exactly the total use of labour in the final output subsystems found in Table A.3, A.5, A.7, and A.9. Furthermore, the total quantities of the n commodities as respectively gross output and total outlays in each vertically integrated sector are given by:

$$\boldsymbol{\Pi} = \boldsymbol{O}\left(\boldsymbol{I} - \breve{\boldsymbol{A}}\right)^{-1} = \begin{bmatrix} 90 & 0 & 0 & 0 \\ 0 & 105 & 0 & 0 \\ 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 215 \end{bmatrix} \begin{bmatrix} 2.04 & 1.26 & 1.96 & 0.67 \\ 0.91 & 2.55 & 2.12 & 0.79 \\ 0.77 & 1.08 & 2.27 & 0.57 \\ 0.93 & 0.94 & 1.48 & 1.49 \end{bmatrix}$$

$$= \begin{bmatrix} 183 & 114 & 176 & 60 \\ 96 & 267 & 223 & 83 \\ 62 & 87 & 182 & 46 \\ 179 & 202 & 319 & 321 \end{bmatrix}$$

$$\boldsymbol{\Upsilon} = \boldsymbol{O}\boldsymbol{H} = \begin{bmatrix} 90 & 0 & 0 & 0 \\ 0 & 105 & 0 & 0 \\ 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 215 \end{bmatrix} \begin{bmatrix} 1.04 & 1.26 & 1.96 & 0.67 \\ 0.91 & 1.55 & 2.12 & 0.79 \\ 0.77 & 1.08 & 1.27 & 0.57 \\ 0.93 & 0.94 & 1.48 & 0.49 \end{bmatrix}$$

$$= \begin{bmatrix} 93 & 114 & 176 & 60 \\ 96 & 162 & 223 & 83 \\ 62 & 87 & 102 & 46 \\ 179 & 202 & 319 & 106 \end{bmatrix}$$

Compare with the gross output listed in Table A.3, A.5, A.7, and A.9 to see that these are the same.

B.3 Gossling's iterative method

The matrix of market share coefficients to other activities associated with the system presented in Table B.1 is given by:

$$\boldsymbol{U} = \boldsymbol{A}\boldsymbol{B}^{-1} = \begin{bmatrix} 0.231 & 0.119 & 0.267 & 0.088 \\ 0.077 & 0.313 & 0.367 & 0.196 \\ 0.308 & 0.336 & 0.083 & 0.245 \\ 0.212 & 0.0746 & 0.194 & 0.049 \end{bmatrix}$$

From this the $\boldsymbol{P}$ and $[\boldsymbol{I}+\boldsymbol{P}]$ matrices are computed as:

$$\begin{aligned}
\boldsymbol{P} &= \boldsymbol{U}'_{\boldsymbol{B}}\boldsymbol{I} + \boldsymbol{U}'_{\Gamma}\boldsymbol{U}' + \boldsymbol{U}'_{\Gamma}[\boldsymbol{U}'_{\Gamma}\boldsymbol{U}'] + \boldsymbol{U}'_{\Gamma}[\boldsymbol{U}'_{\Gamma}[\boldsymbol{U}'_{\Gamma}\boldsymbol{U}']] + \cdots \\
&= \begin{bmatrix} 0 & 0.077 & 0.308 & 0.212 \\ 0.119 & 0 & 0.336 & 0.075 \\ 0.267 & 0.367 & 0 & 0.194 \\ 0.088 & 0.196 & 0.245 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0.172 & 0.149 & 0.114 \\ 0.134 & 0 & 0.160 & 0.114 \\ 0.083 & 0.089 & 0 & 0.100 \\ 0.093 & 0.106 & 0.105 & 0 \end{bmatrix} + \\
&\quad \begin{bmatrix} 0 & 0.090 & 0.069 & 0.066 \\ 0.077 & 0 & 0.076 & 0.083 \\ 0.074 & 0.074 & 0 & 0.081 \\ 0.051 & 0.042 & 0.050 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0.052 & 0.032 & 0.046 \\ 0.053 & 0 & 0.036 & 0.061 \\ 0.044 & 0.038 & 0 & 0.055 \\ 0.036 & 0.028 & 0.023 & 0 \end{bmatrix} + \\
&\quad \begin{bmatrix} 0 & 0.030 & 0.015 & 0.032 \\ 0.034 & 0 & 0.017 & 0.043 \\ 0.030 & 0.023 & 0 & 0.039 \\ 0.023 & 0.015 & 0.011 & 0 \end{bmatrix} + ... = \begin{bmatrix} 0 & 0.461 & 0.586 & 0.548 \\ 0.482 & 0 & 0.639 & 0.480 \\ 0.555 & 0.621 & 0 & 0.564 \\ 0.335 & 0.409 & 0.444 & 0 \end{bmatrix}
\end{aligned}$$

$$[\boldsymbol{I}+\boldsymbol{P}] = \begin{bmatrix} 1 & 0.461 & 0.586 & 0.548 \\ 0.482 & 1 & 0.639 & 0.480 \\ 0.555 & 0.621 & 1 & 0.564 \\ 0.335 & 0.409 & 0.444 & 1 \end{bmatrix}$$

Compare with the gross subsystem multipliers $\overline{\boldsymbol{q}}^1$, ..., $\overline{\boldsymbol{q}}^4$ to see that these are identical to the ith columns in the $[\boldsymbol{I}+\boldsymbol{P}]$ matrix. The Γ-operation is here presented as a directly applicable M-code.

```
function [Z]=gossling_gamma(X,Y);
n=size(X,1); e=ones(n,1);
for i=1:1:n;
    for j=1:1:n;
        a=e; a(j)=0;
        Z(i,j)=X(i,:)*(Y(:,j).*a);
        if i==j;
            Z(i,j)=0;
        end;
    end;
end;
```

For a full description including proofs see Gossling (1972).

C Core Variables

$\boldsymbol{A}$	input-matrix	$n \times n$	physical quantities
$\boldsymbol{\breve{A}}$	matrix of interindustry coefficients	$n \times n$	
$\boldsymbol{B}$	output-matrix	$n \times n$	physical quantities
$\boldsymbol{O}$	final output	$n \times n$	physical quantities
$\boldsymbol{H}$	vertically integrated technical coefficient matrix	$n \times n$	
$\boldsymbol{I}$	identity matrix	$n \times n$	
$\boldsymbol{P}$	Gossling's P	$n \times n$	
$\boldsymbol{U}$	market shares coefficients to other activities	$n \times n$	
$\boldsymbol{e}$	unit vector	$n \times 1$	
$\boldsymbol{l}$	direct labour inputs	$n \times 1$	physical quantities
$\boldsymbol{\breve{l}}$	direct labour input coefficients	$n \times 1$	
$\boldsymbol{v}$	vertically integrated labour coefficients	$n \times 1$	
$\boldsymbol{p}$	production prices	$n \times 1$	
$\boldsymbol{q}$	subsystem multiplier	$n \times 1$	
$\boldsymbol{\tau}$	market share coefficients to final buyer	$n \times 1$	
$\boldsymbol{z}$	industries total sales to means of production	$1 \times n$	physical quantities
w	uniform wage-rate	1×1	
R	maximum rate of profit	1×1	

D Data

The industry classification used in this study is as follows, where the numbers in the brackets refer to their original classification used by Statistics Denmark, www.dst.dk/inputoutput.

(1) Agriculture {1}
(2) Horticulture, orchards etc. {2}
(3) Agricultural services, landscape gardeners etc. {3}
(4) Forestry {4}
(5) Fishing {5}
(6) Extr. of crude petroleum, natural gas etc. {6}
(7) Extr. of gravel, clay, stone and salt etc. {7}
(8) Mfr. of food, beverages and tobacco {8–18}
(9) Mfr. of textiles, wearing apparel, leather {19–21}
(10) Mfr. of wood and wood products {22}
(11) Mfr. of paper prod., printing and publish {23–26}
(12) Mfr. of refined petroleum products etc. {27}
(13) Mfr. of chemicals and man-made fibres etc. {28–35}
(14) Mfr. of rubber and plastic products {36–38}
(15) Mfr. of other non-metallic mineral products {39–41}
(16) Mfr. and processing of basic metals {42–47}
(17) Mfr. of machinery and equipment n.e.c. {48–52}
(18) Mfr. of electrical and optical equipment {53–56}
(19) Mfr. of transport equipment {57–59}
(20) Mfr. of furniture, manufacturing n.e.c. {60–62}
(21) Electricity supply {63}
(22) Gas and water supply {64–66}
(23) Construction {67–70}
(24) Sale and repair of motor vehicles etc. {71–73}
(25) Ws. and commis. trade, exc. of m. vehicles {74}
(26) Retail trade of food etc. {75}
(27) Department stores {76}
(28) Re. sale of phar. goods, cosmetic art. etc. {77}
(29) Re. sale of clothing, footwear etc. {78}
(30) Other retail sale, repair work {79}
(31) Hotels and restaurants {80–81}
(32) Land transport, transport via pipelines {82–85}
(33) Water transport {86}
(34) Air transport {87}
(35) Support. trans. activities, travel agencies {88–89}
(36) Post and telecommunications {90}
(37) Financial intermediation {91–92}
(38) Insurance and pension funding {93–94}
(39) Activities auxiliary to finan. intermediat. {95}
(40) Real estate activities {96–98}
(41) Renting of machinery and equipment etc. {99}
(42) Computer and related activities {100–101}
(43) Research and development {102–103}
(44) Consultancy etc. and cleaning activities {104–109}
(45) Public administration etc. {110–113}
(46) Education {114–118}
(47) Health care activities {119–120}
(48) Social institutions etc. {121–122}
(49) Sewage and refuse disp. and similar act. {123–125}
(50) Activities of membership organiza. n.e.c. {126}
(51) Recreational, cultural, sporting activities {127–128}
(52) Other service activities {129–130}

E Additional Results

For some of the indices, extreme cases have been collected in separate figures to better be able to study the evolution in the single industries.

E.1 The σ-index (Productivity Index, Physical Quantities)

$$\sigma_t^i = \frac{\tilde{b}^i_{(i,i,t)} - \tilde{a}^i_{(i,i,t)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}^i_t} = \frac{\text{external output}}{\text{direct + indirect labour}}$$

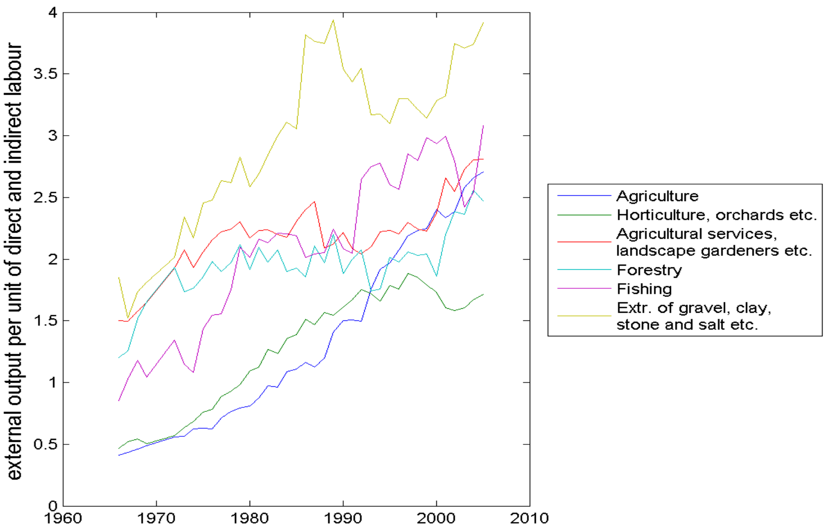

Fig. E.1: *The σ-index, Agriculture, fishing, and quarrying*

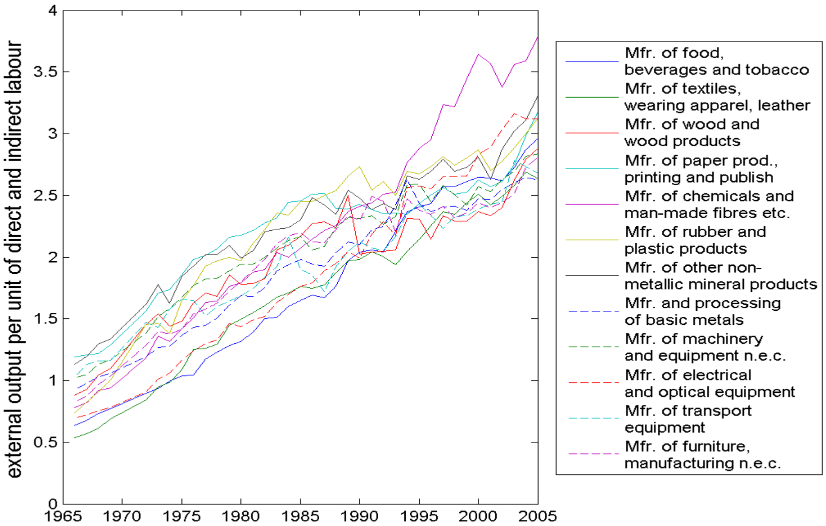

Fig. E.2: *The σ-index, Manufacturing*

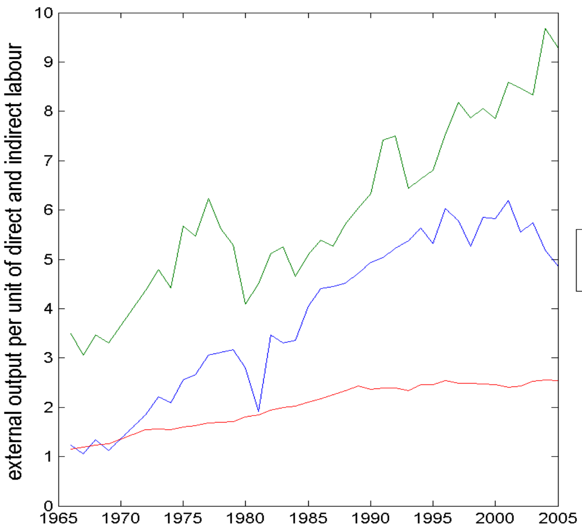

Fig. E.3: *The σ-index, Electricity, gas, and water supply; and Construction*

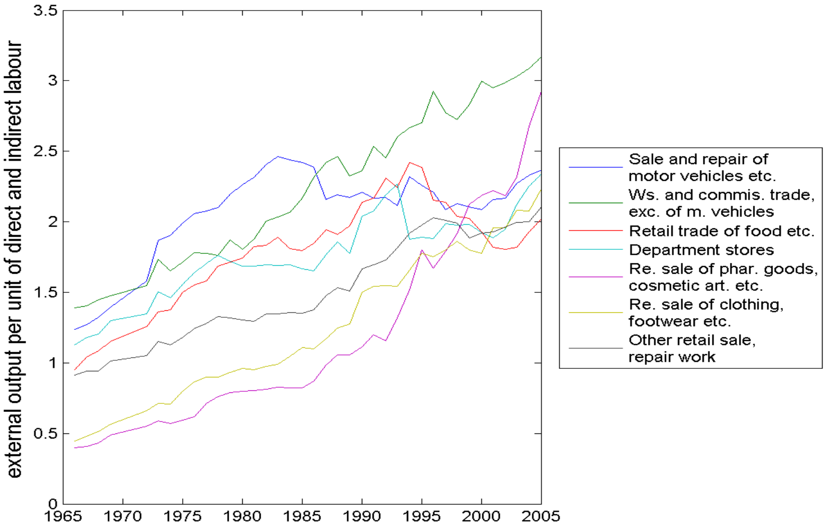

Fig. E.4: *The σ-index, Wholesale-, retail trade, hotels, restaurants*

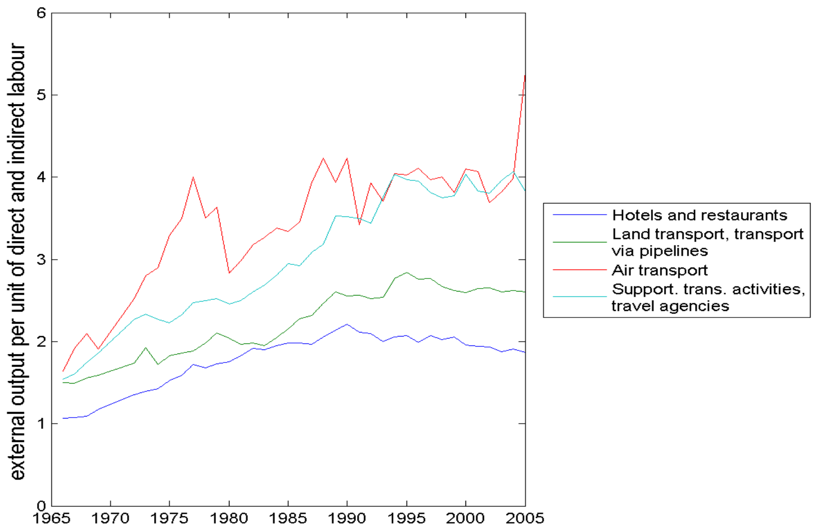

Fig. E.5: *The σ-index, Transport, storage, and communication*

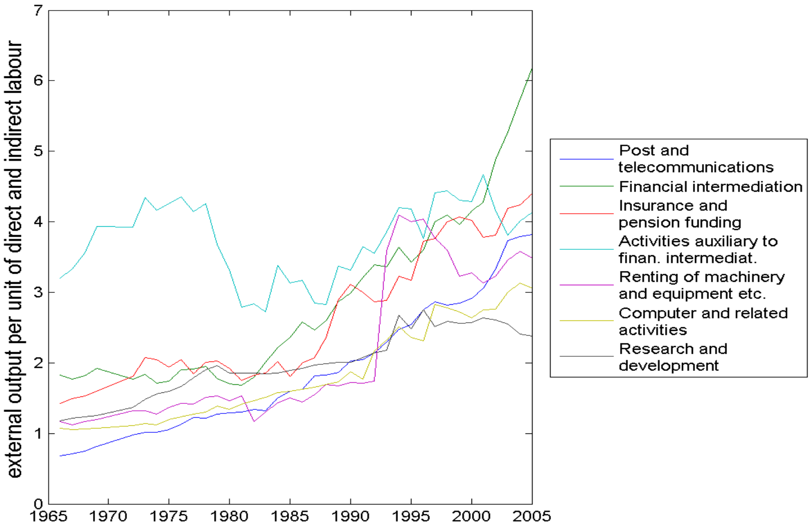

Fig. E.6: *The σ-index, Financial intermediation, business activities*

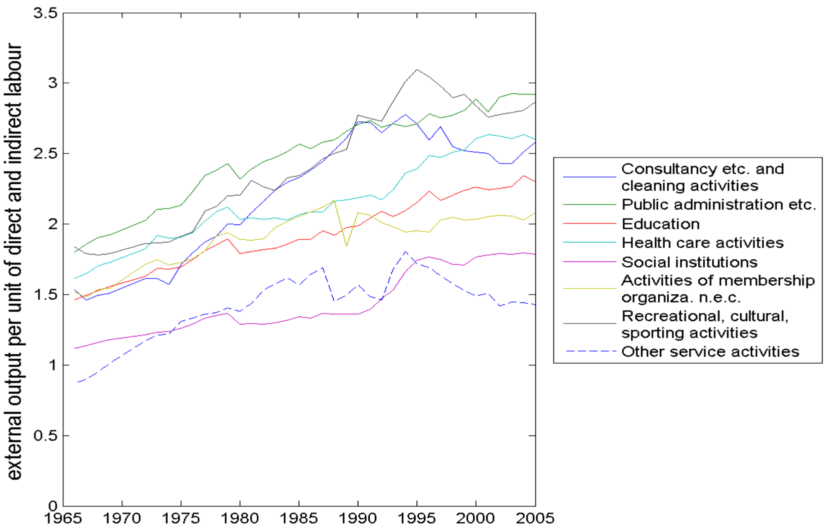

Fig. E.7: *The σ-index, Public and personal services*

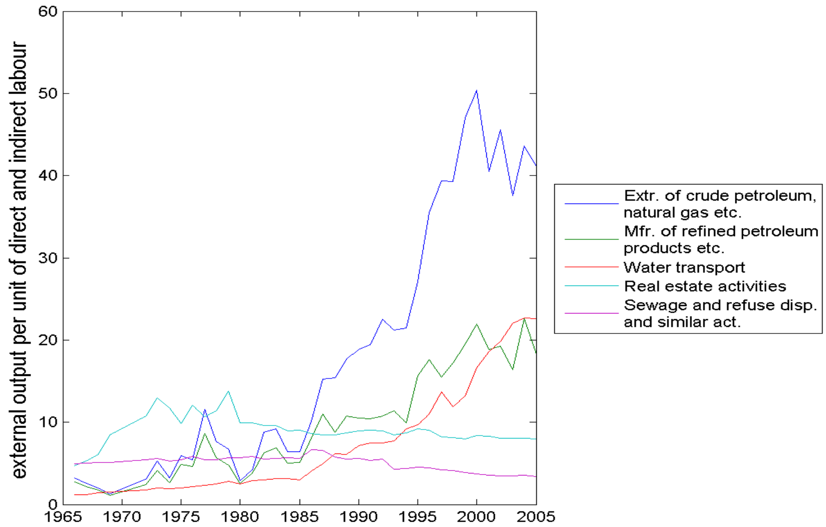

Fig. E.8: *The σ-index, extreme cases*

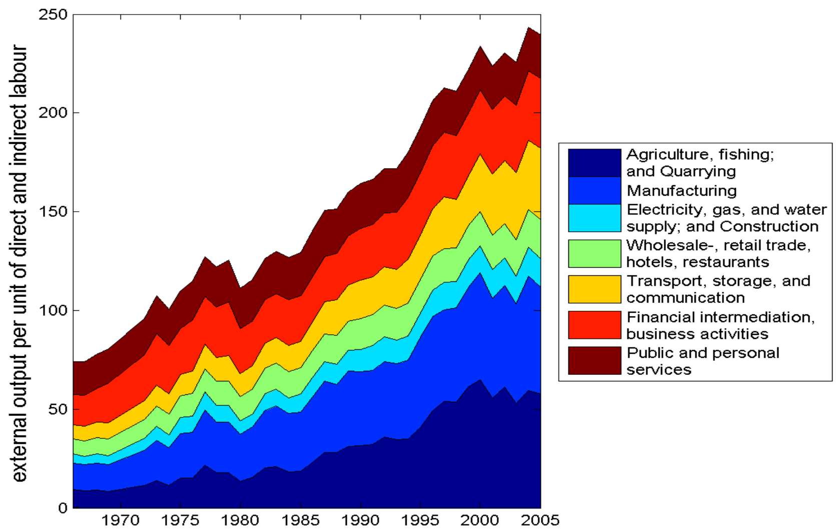

Fig. E.9: *The σ-index for the meso-sectors*

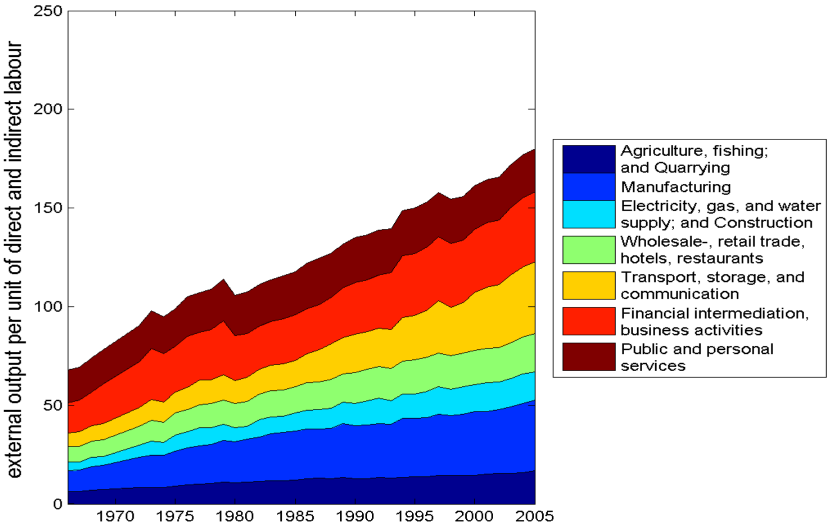

Fig. E.10: *The σ-index for the meso-sectors, excluding 'Extr. of crude petroleum, natural gas etc.' and 'Mfr. of refined petroleum products etc.'*

E.2 The ξ-index (Productivity Index, Physical Quantities)

$$\xi_t^i = \frac{\tilde{b}^i_{(i,i,t)} - \boldsymbol{e}'\tilde{\boldsymbol{a}}^i_{(:,i,t)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}^i_t} = \frac{\text{final output}}{\text{direct + indirect labour}}$$

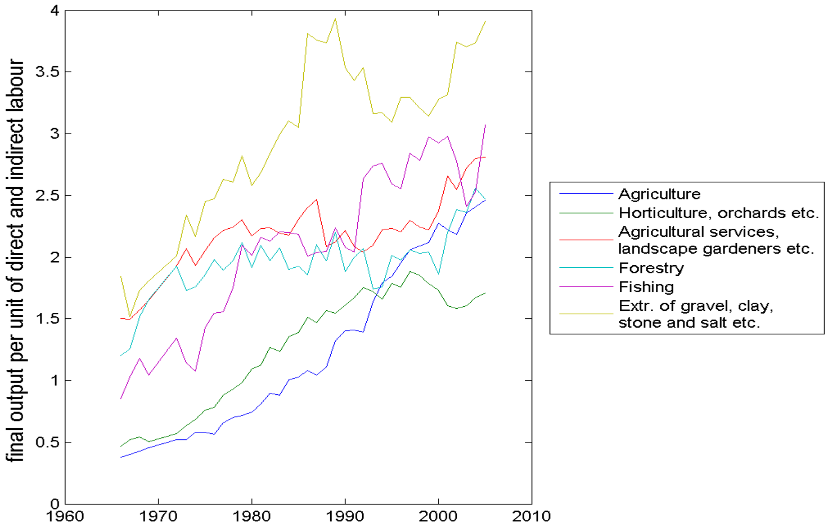

Fig. E.11: *The ξ-index, Agriculture, fishing, and quarrying*

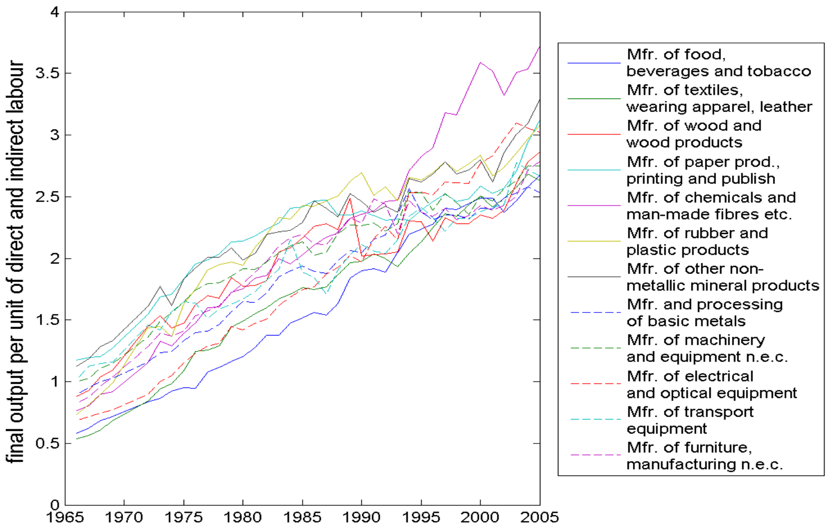

Fig. E.12: *The ξ-index, Manufacturing*

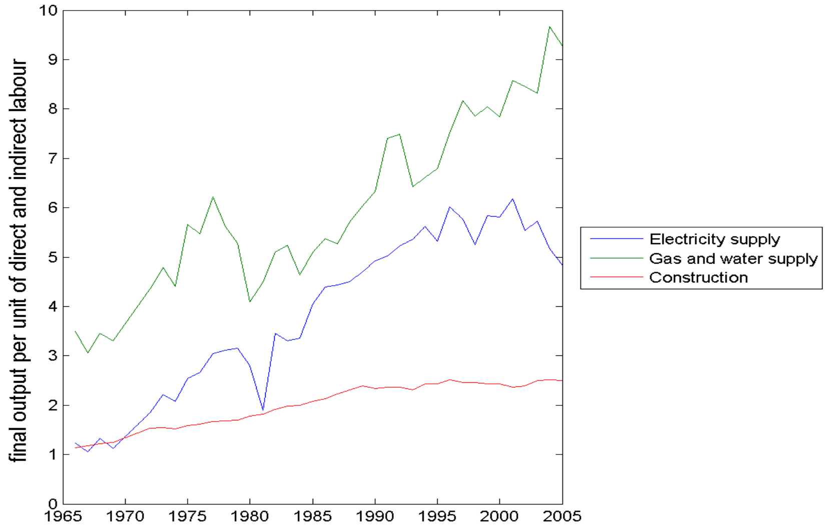

Fig. E.13: *The ξ-index, Electricity, gas, and water supply; and Construction*

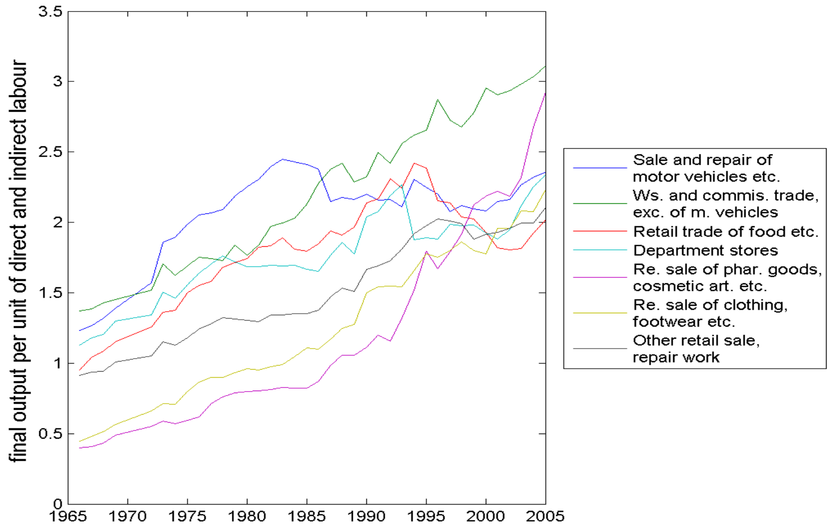

Fig. E.14: *The ξ-index, Wholesale-, retail trade, hotels, restaurants*

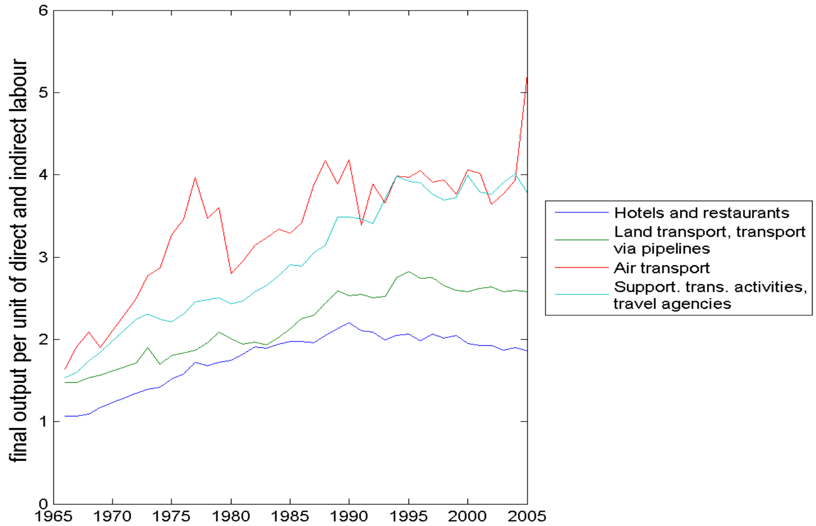

Fig. E.15: *The ξ-index, Transport, storage, and communication*

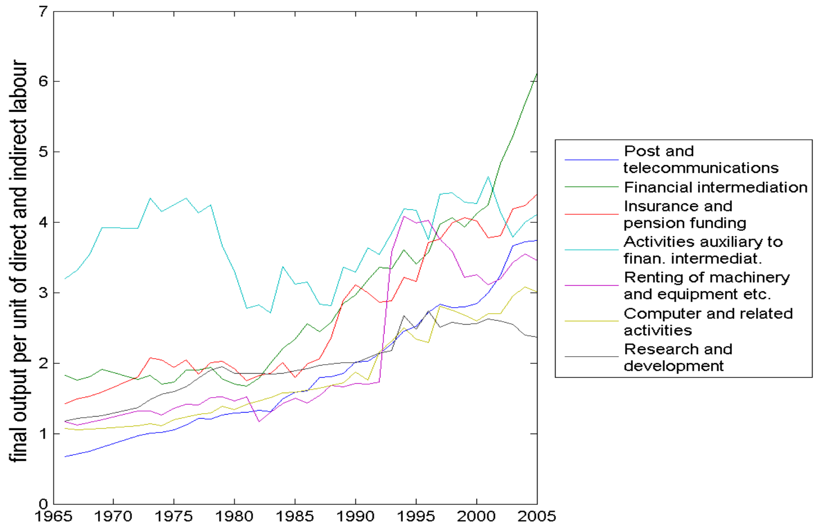

Fig. E.16: *The ξ-index, Financial intermediation, business activities*

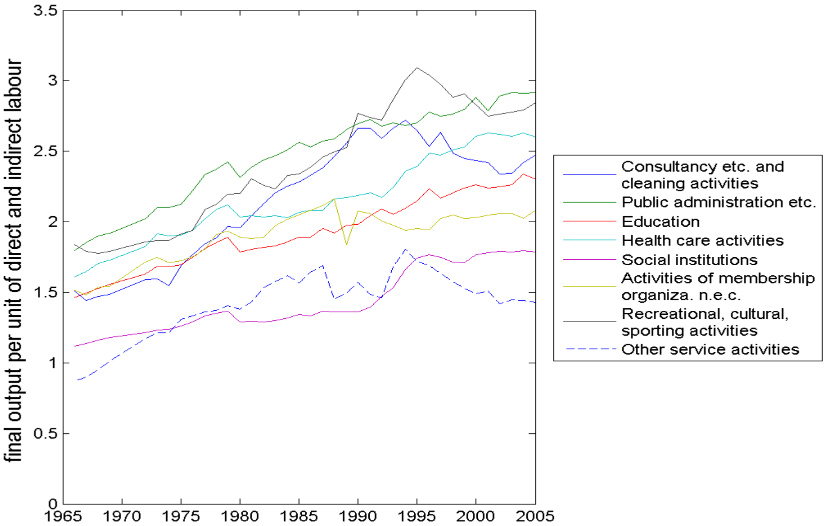

Fig. E.17: *The ξ-index, Public and personal services*

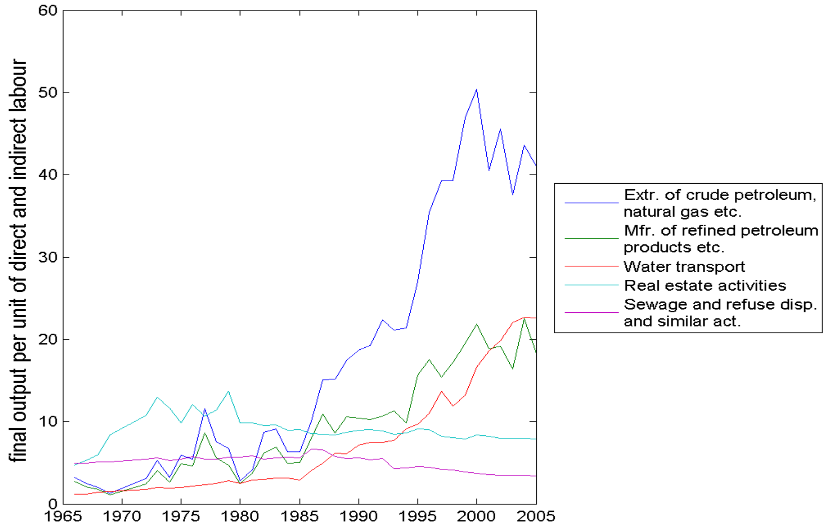

Fig. E.18: *The ξ-index, extreme cases*

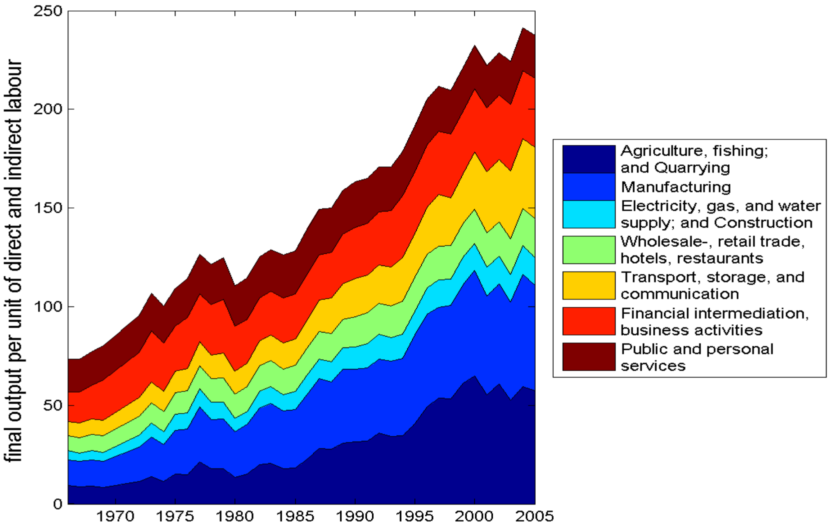

Fig. E.19: *The ξ-index for the meso-sectors*

E.3 The v-index (Productivity Index, Physical Quantities)

$$\boldsymbol{v}_t = (\boldsymbol{B}_t - \boldsymbol{A}_t)^{-1}\, \boldsymbol{l}_t$$

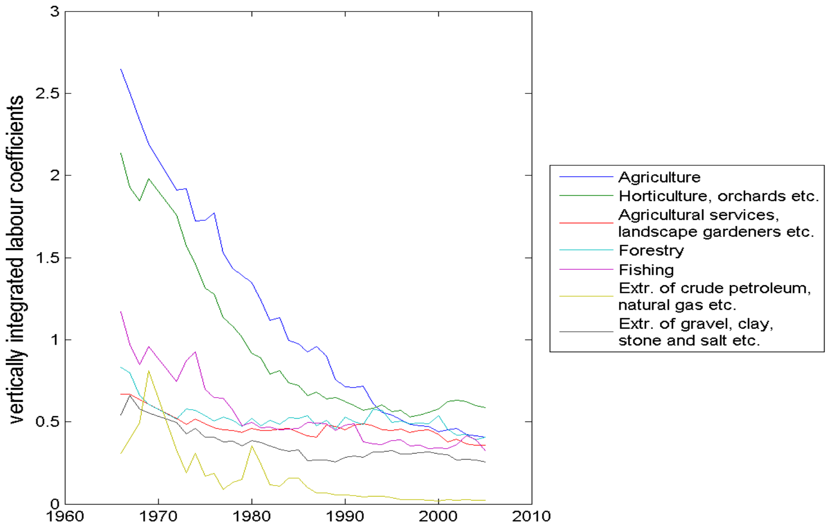

Fig. E.20: *The v-index, Agriculture, fishing, and quarrying*

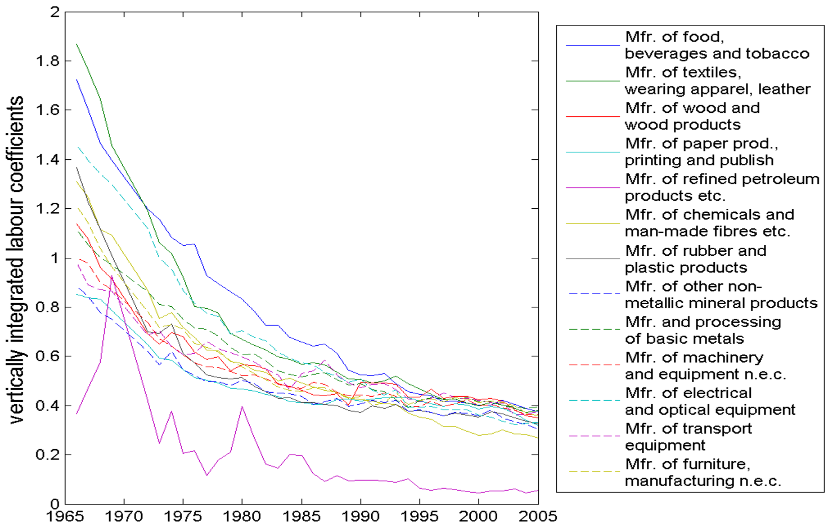

Fig. E.21: *The v-index, Manufacturing*

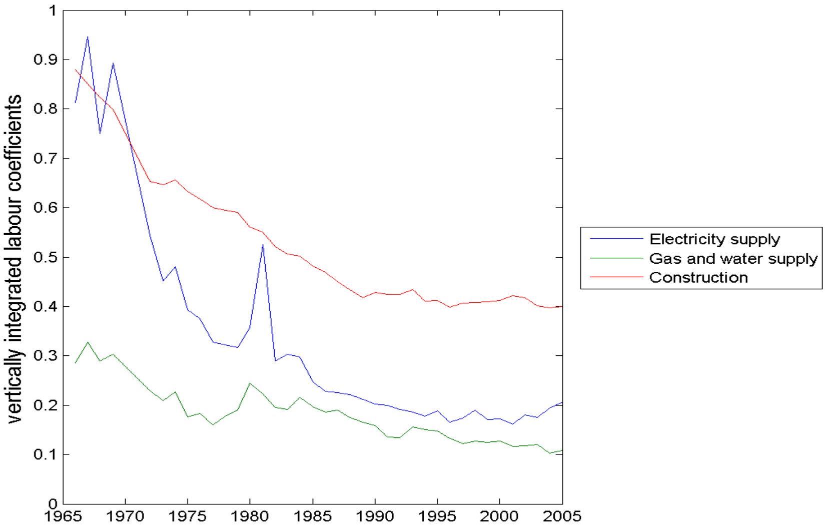

Fig. E.22: *The v-index, Electricity, gas, and water supply; and Construction*

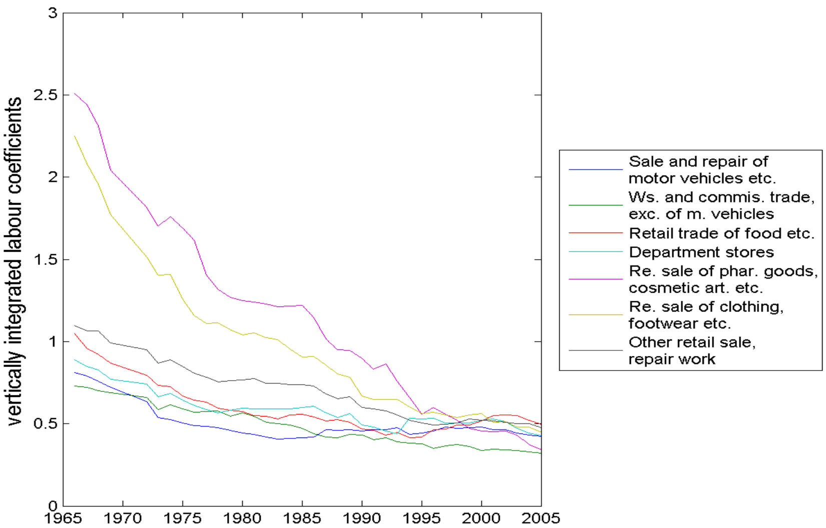

Fig. E.23: *The v-index, Wholesale-, retail trade, hotels, restaurants*

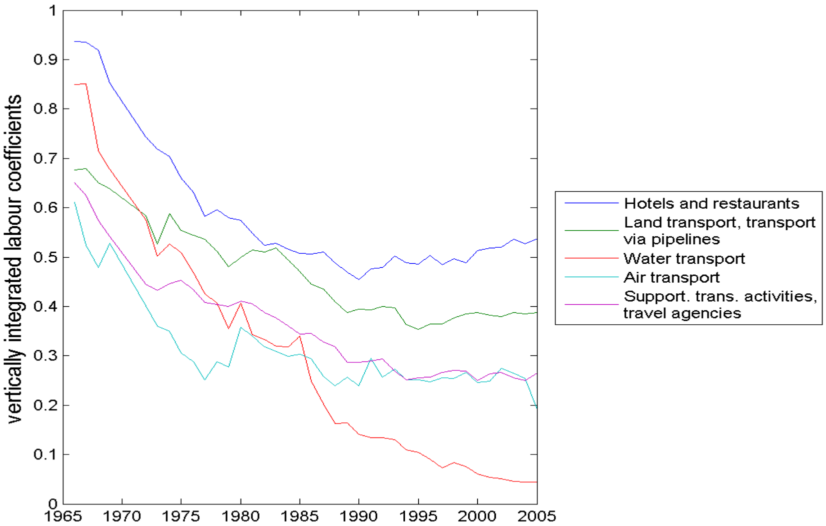

Fig. E.24: *The v-index, Transport, storage, and communication*

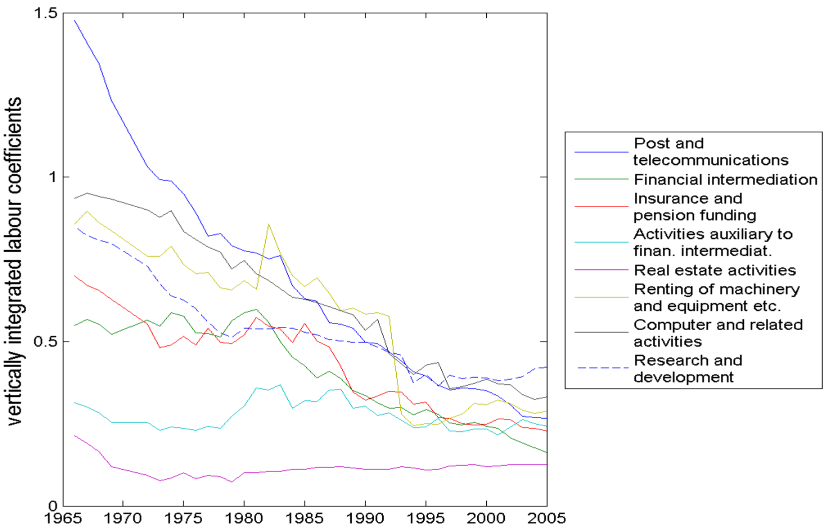

Fig. E.25: *The v-index, Financial intermediation, business activities*

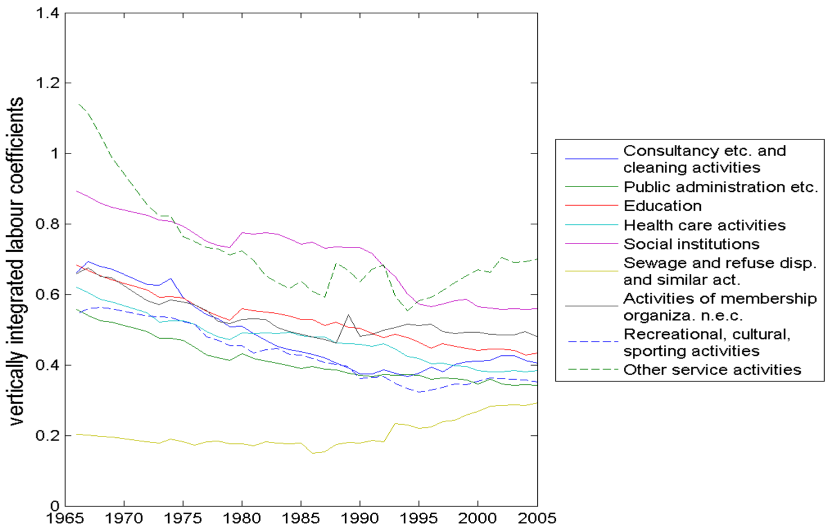

Fig. E.26: *The v-index, Public and personal services*

E.4 The ρ-index (Index of Structural Change, Physical Quantities)

$$\rho_t^i = \frac{\beta_t^i \tilde{l}^i_{(i,t)}}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i} = \frac{\text{direct labour}}{\text{direct + indirect labour}}$$

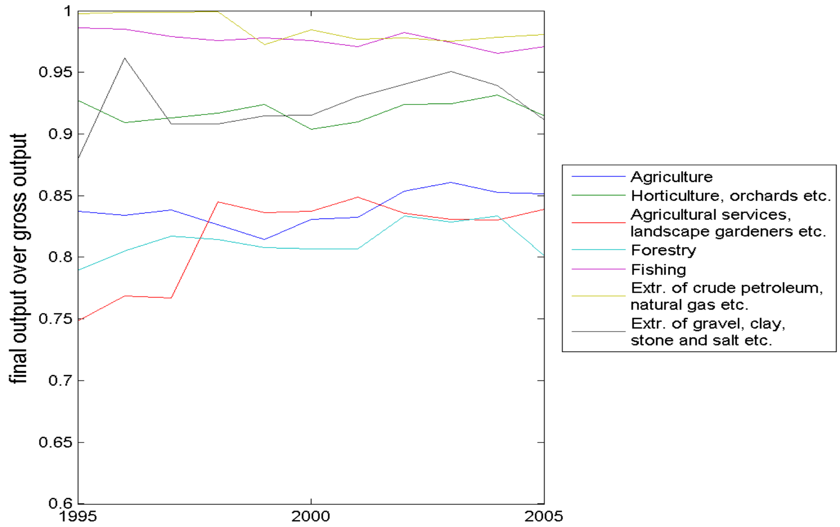

Fig. E.27: *The ρ-index, Agriculture, fishing, and quarrying*

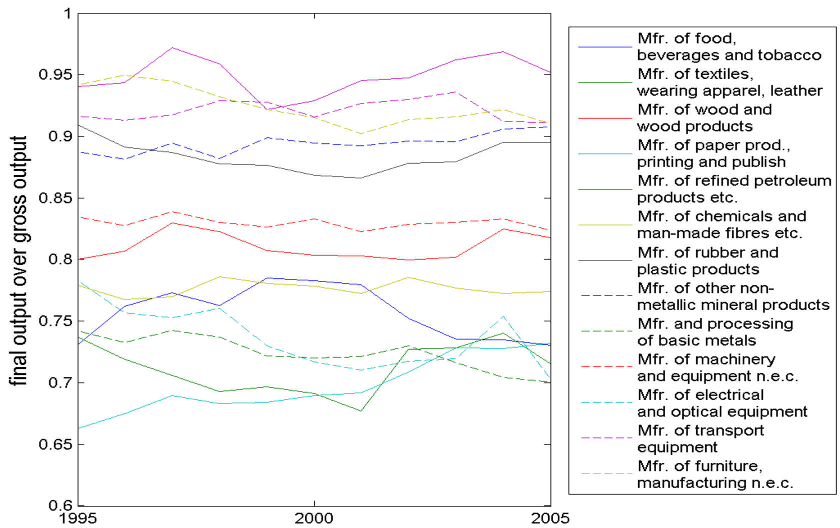

Fig. E.28: *The ρ-index, Manufacturing*

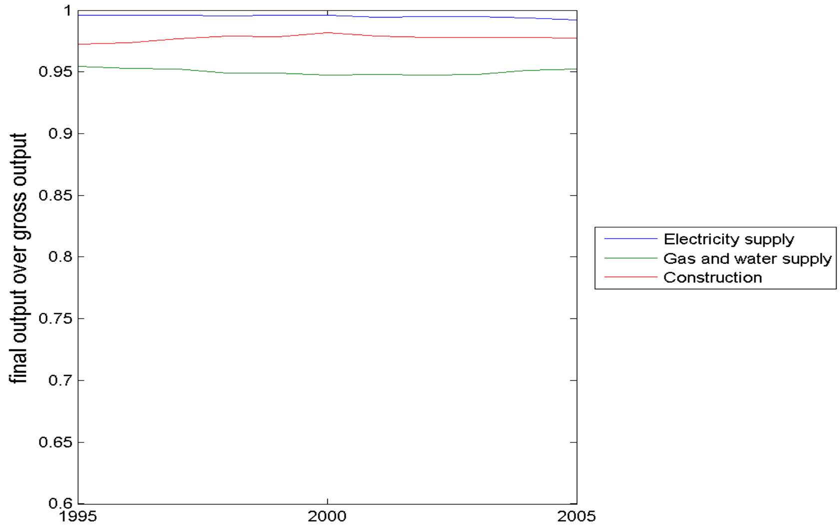

Fig. E.29: *The ρ-index, Electricity, gas, and water supply; and Construction*

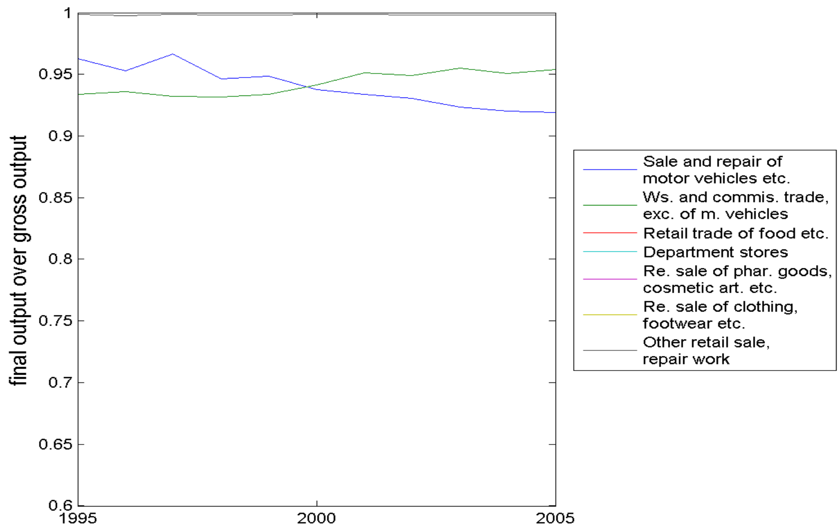

Fig. E.30: *The ρ-index, Wholesale-, retail trade, hotels, restaurants*

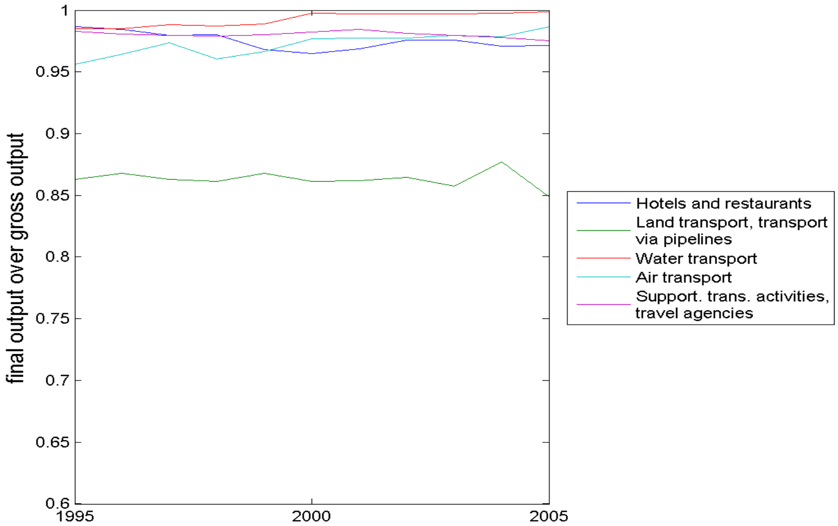

Fig. E.31: *The ρ-index, Transport, storage, and communication*

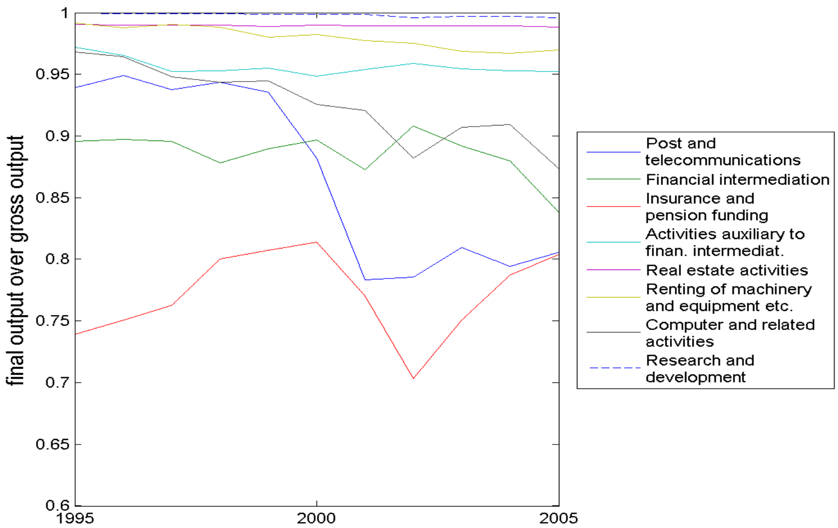

Fig. E.32: *The ρ-index, Financial intermediation, business activities*

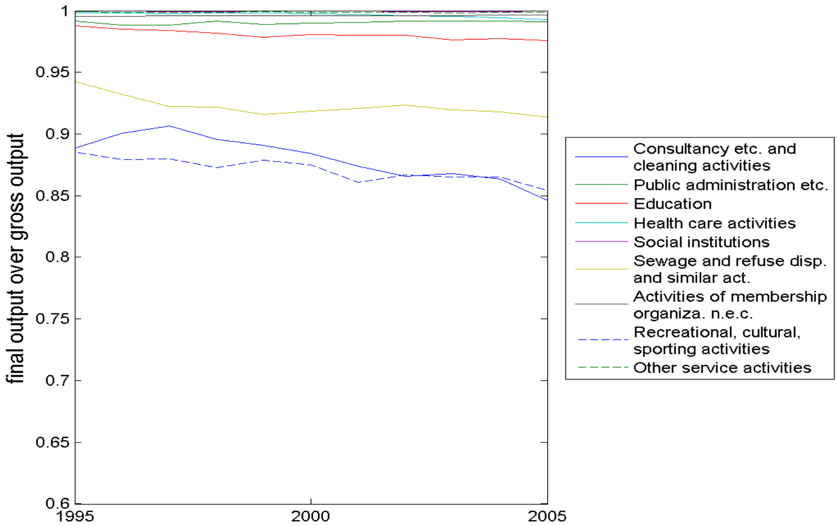

Fig. E.33: *The ρ-index, Public and personal services*

E.5 The α-index (Index of Structural Change, Physical Quantities)

$$\alpha_t^i = \frac{b_{(i,i,t)} - \boldsymbol{e}'\boldsymbol{a}_{(:,i,t)}}{b_{(i,i,t)}} = \frac{\text{final output*}}{\text{gross output*}}$$

Fig. E.34: *The α-index, Agriculture, fishing, and quarrying*

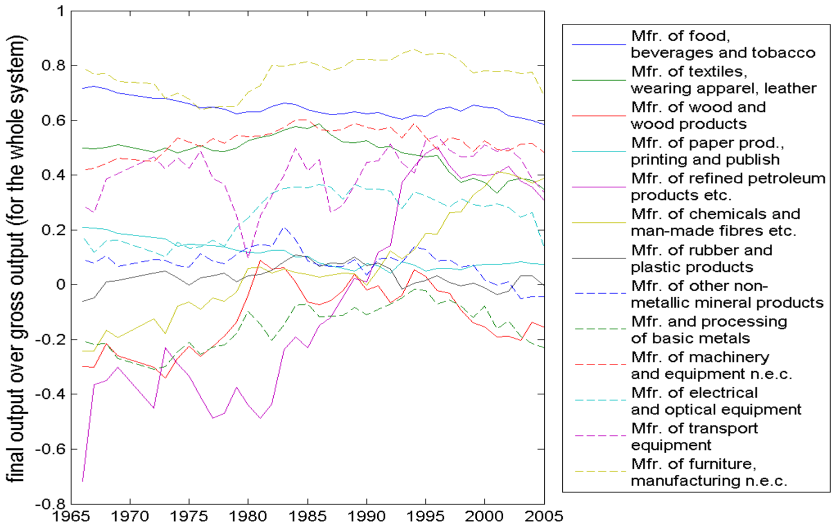

Fig. E.35: *The α-index, Manufacturing*

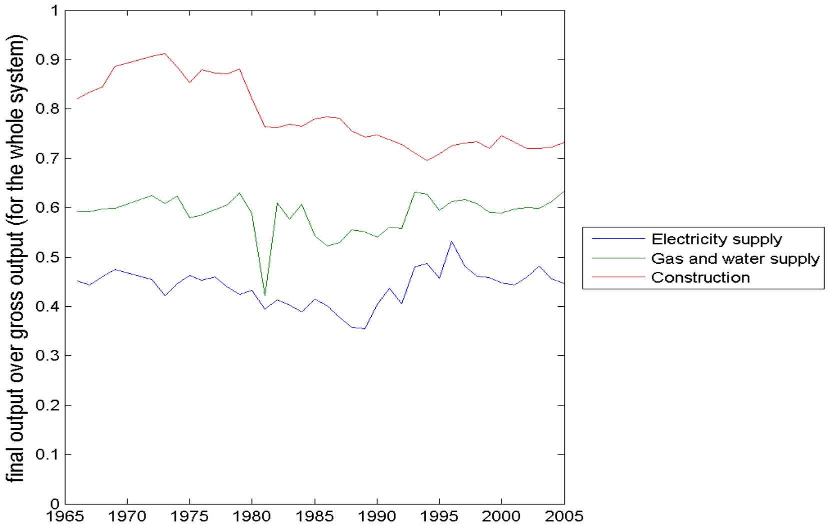

Fig. E.36: *The α-index, Electricity, gas, and water supply; and Construction*

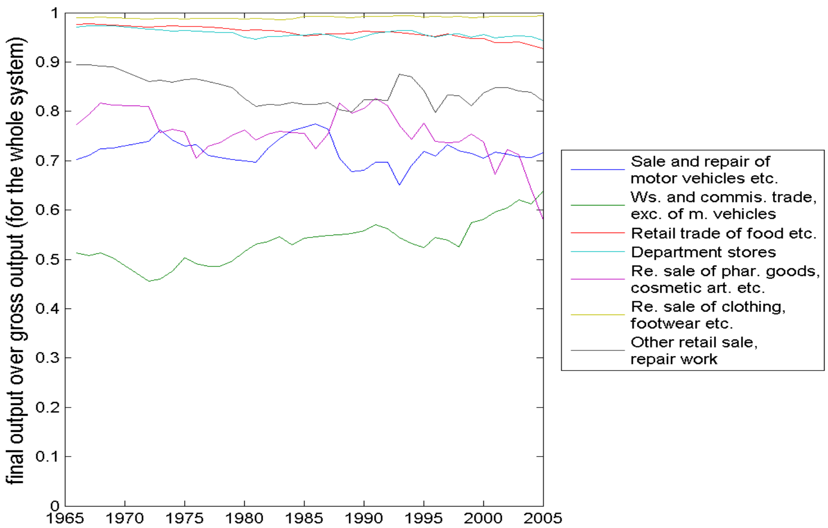

Fig. E.37: *The α-index, Wholesale-, retail trade, hotels, restaurants*

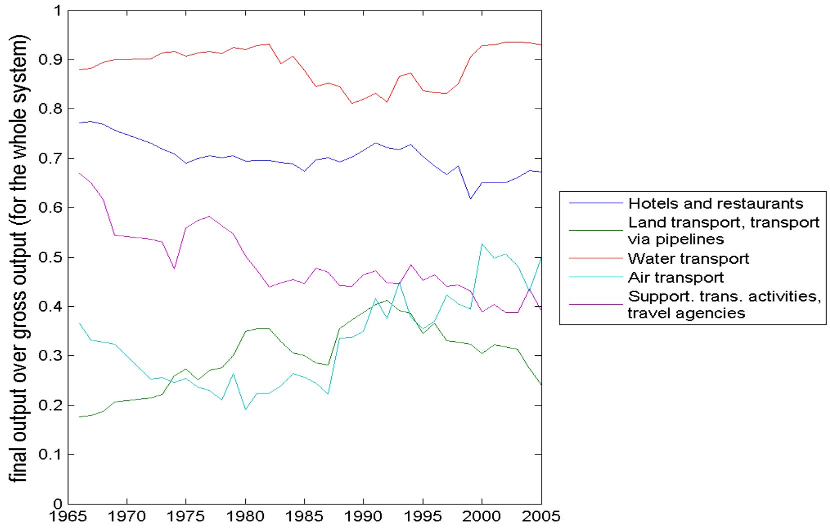

Fig. E.38: *The α-index, Transport, storage, and communication*

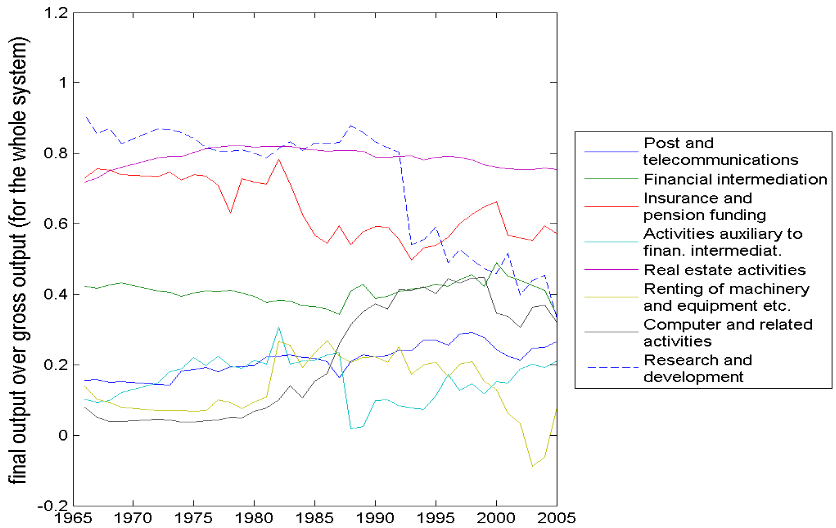

Fig. E.39: *The α-index, Financial intermediation, business activities*

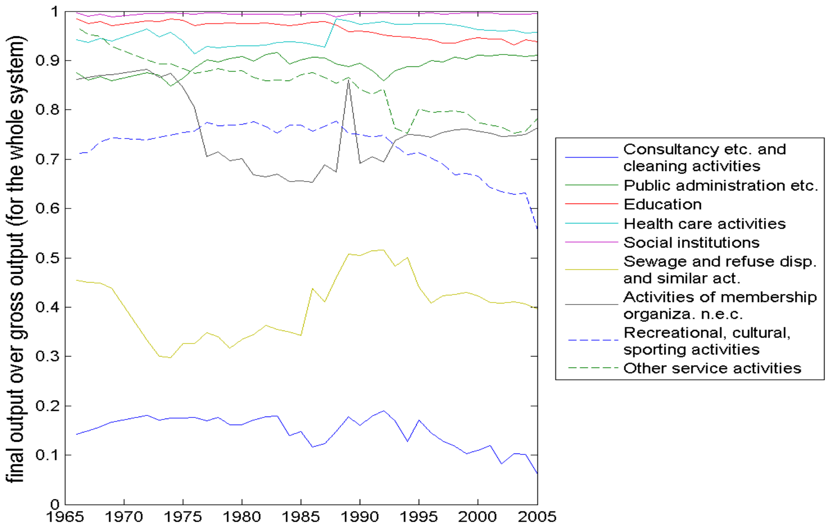

Fig. E.40: *The α-index, Public and personal services*

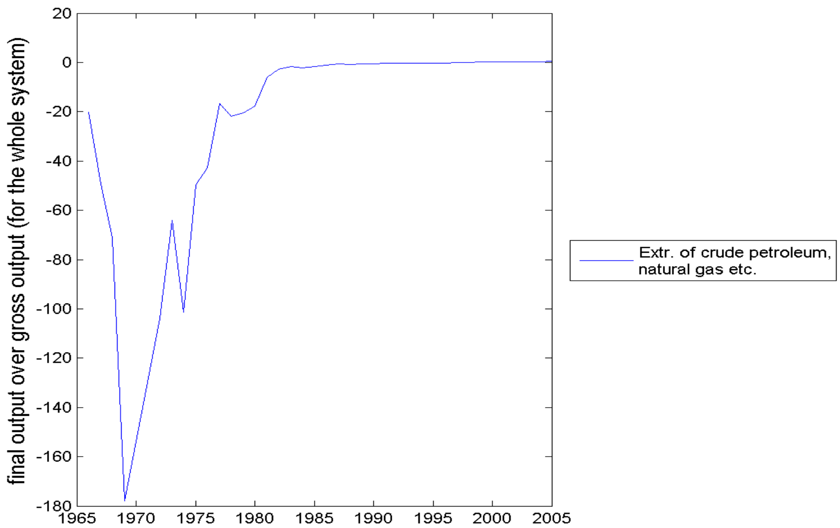

Fig. E.41: *The α-index, extreme cases*

E.6 The β-index (Index of Structural Change, Physical Quantities)

$$\beta_t^i = \frac{\tilde{b}^i_{(i,i,t)} - \boldsymbol{e}'\tilde{\boldsymbol{a}}^i_{(:,i,t)}}{\tilde{b}^i_{(i,i,t)}} = \frac{\text{final output}}{\text{gross output}}$$

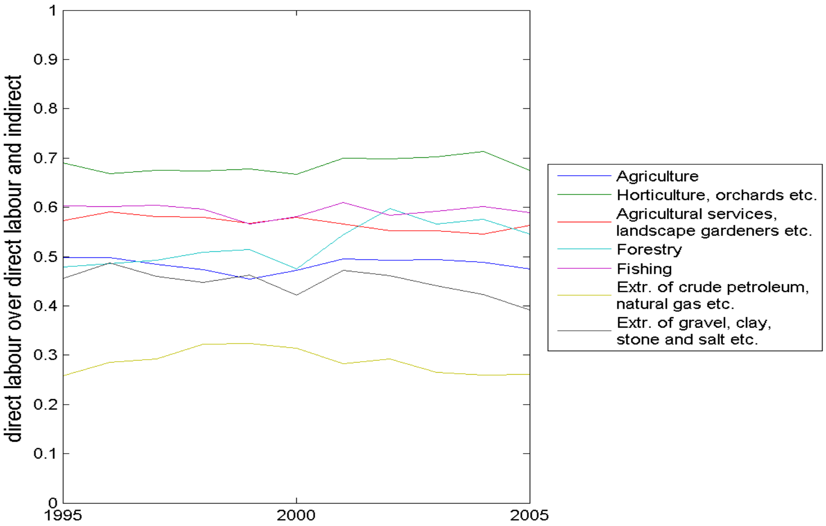

Fig. E.42: *The β-index, Agriculture, fishing, and quarrying*

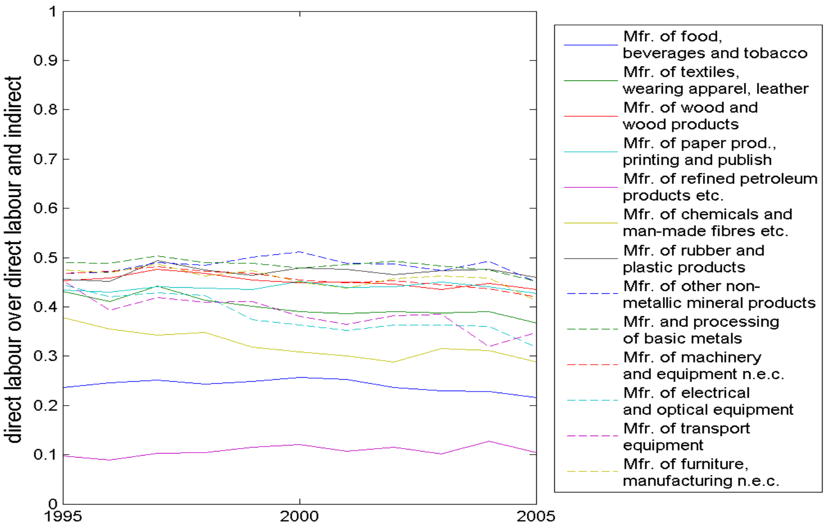

Fig. E.43: *The β-index, Manufacturing*

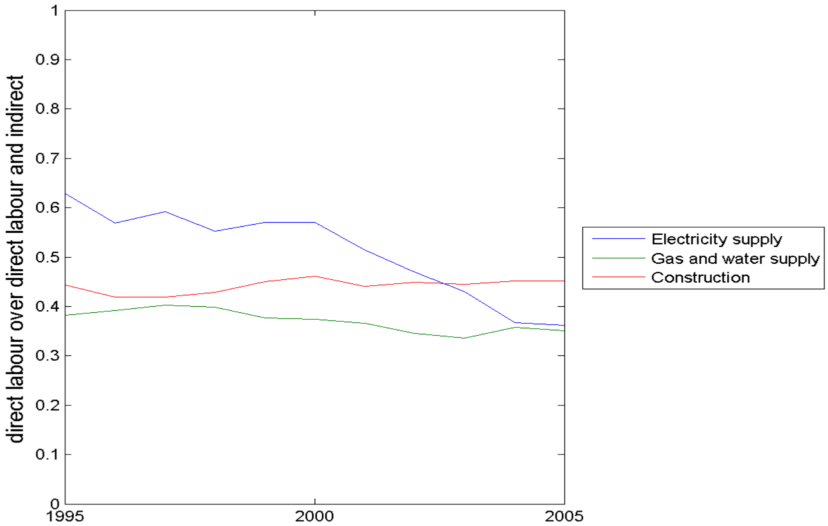

Fig. E.44: *The β-index, Electricity, gas, and water supply; and Construction*

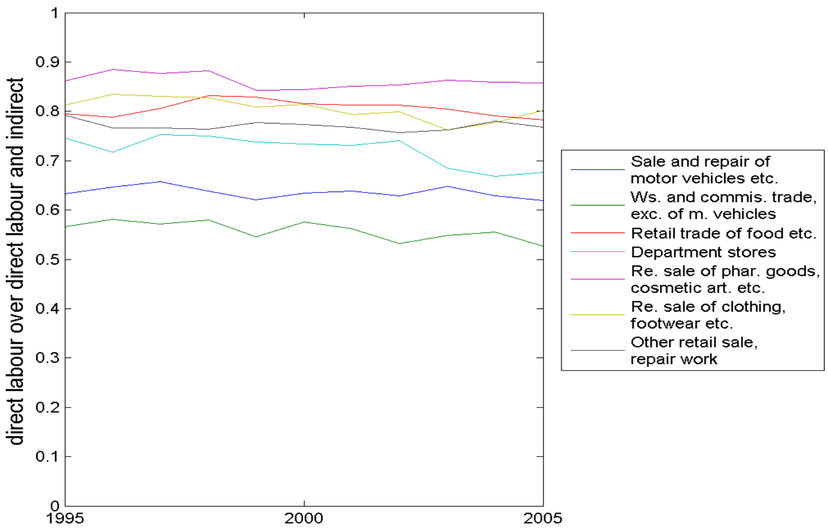

Fig. E.45: *The β-index, Wholesale-, retail trade, hotels, restaurants*

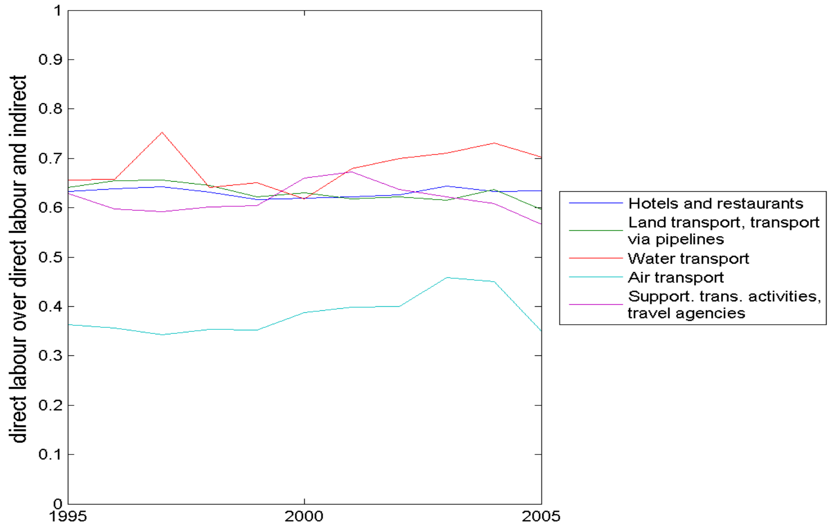

Fig. E.46: *The β-index, Transport, storage, and communication*

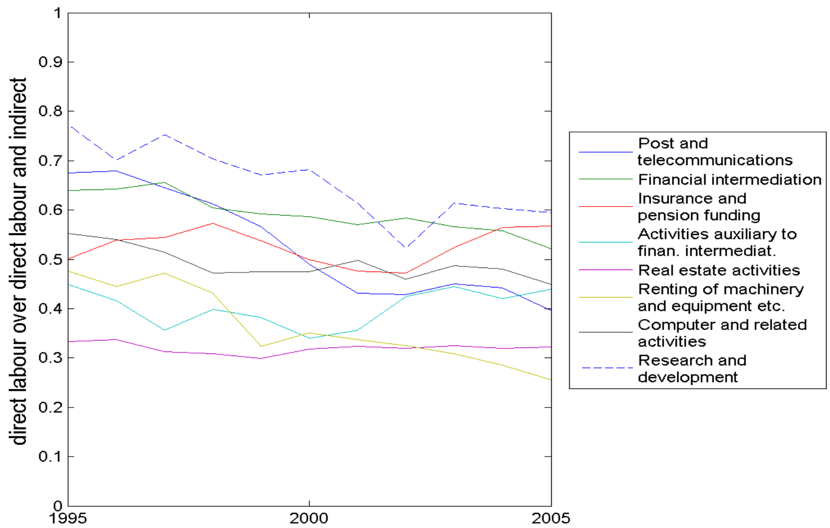

Fig. E.47: *The β-index, Financial intermediation, business activities*

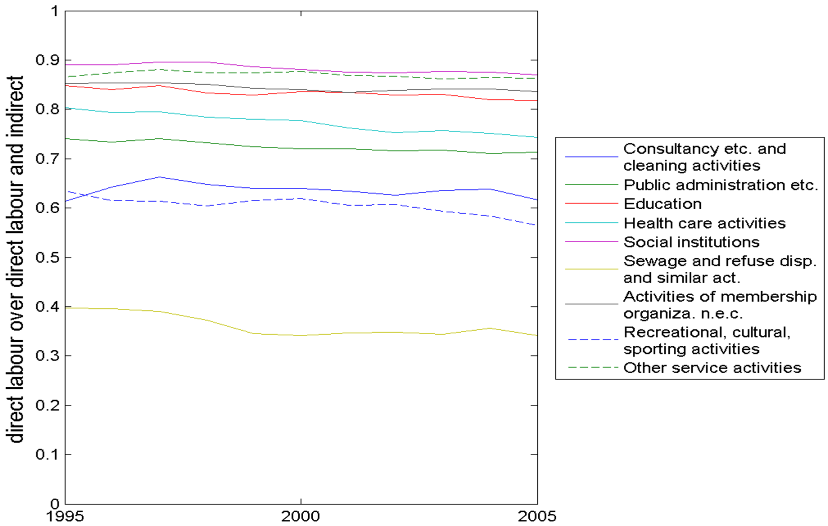

Fig. E.48: *The β-index, Public and personal services*

E.7 The μ-index (Productivity Index, Production Prices)

$$\mu_t^i = \frac{1}{l_{(i,t)} R_t} \int_0^{R_t} \zeta_{(i,t)}(r)\, dr$$

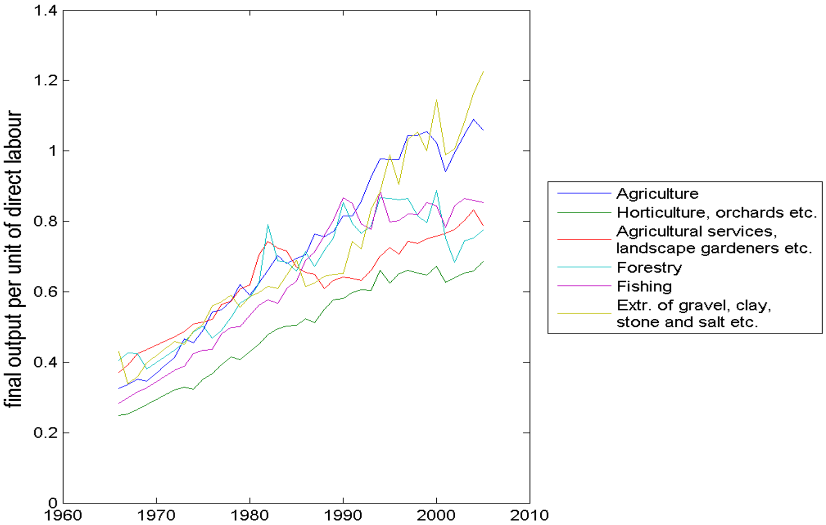

Fig. E.49: *The μ-index, Agriculture, fishing, and quarrying*

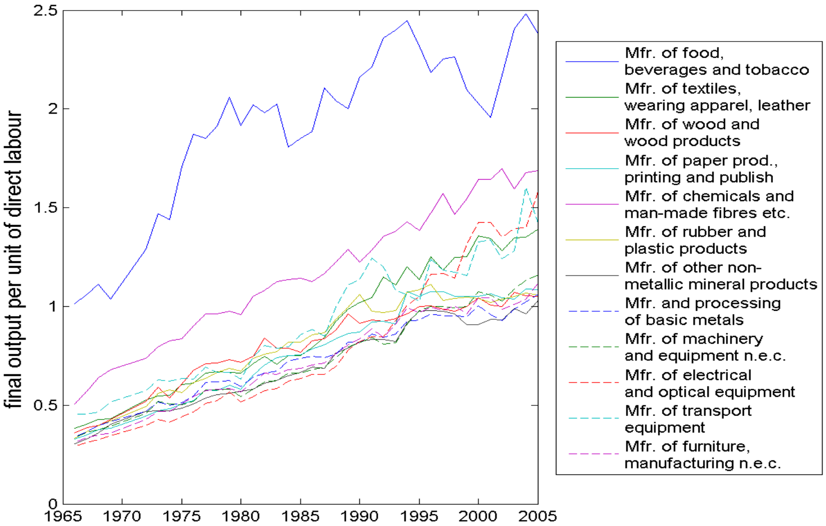

Fig. E.50: *The μ-index, Manufacturing*

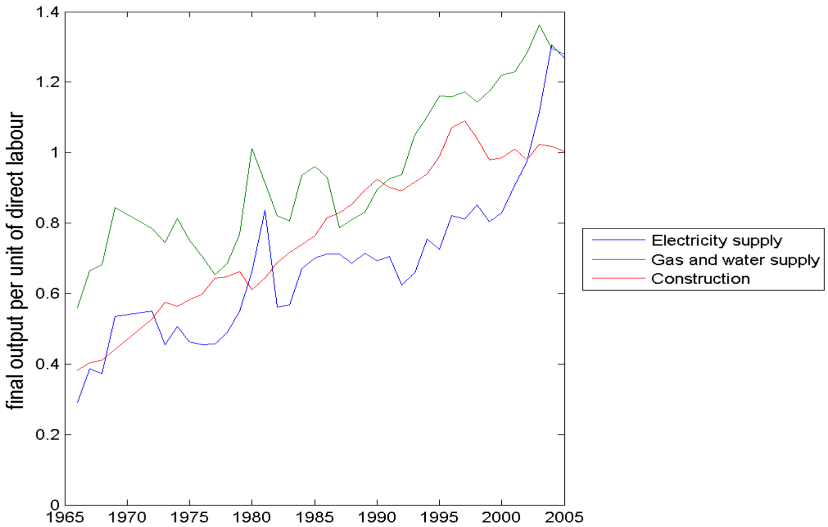

Fig. E.51: *The μ-index, Electricity, gas, and water supply; and Construction*

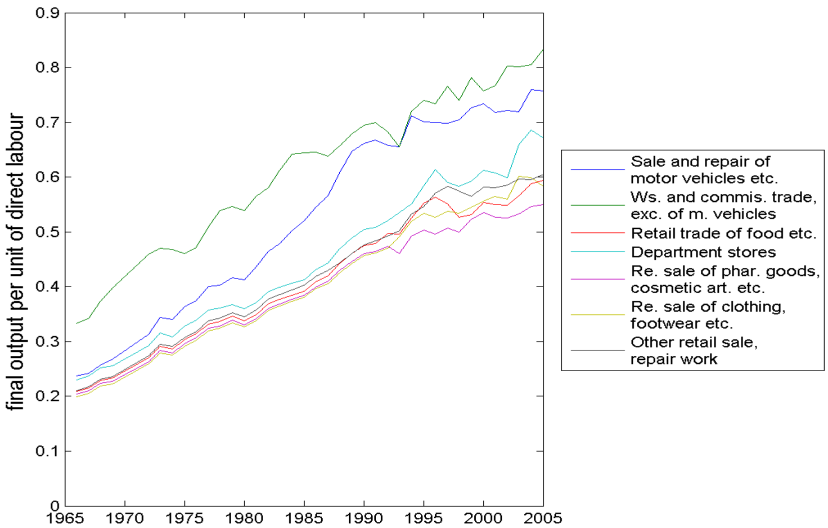

Fig. E.52: *The μ-index, Wholesale-, retail trade, hotels, restaurants*

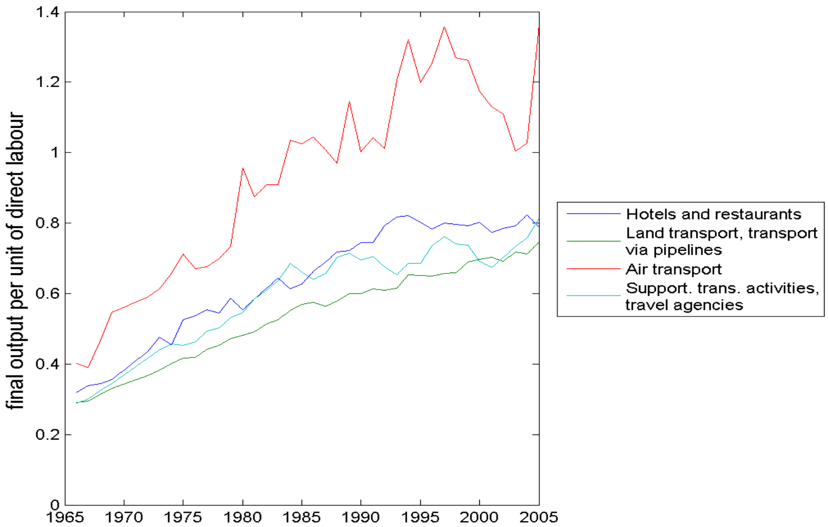

Fig. E.53: *The μ-index, Transport, storage, and communication*

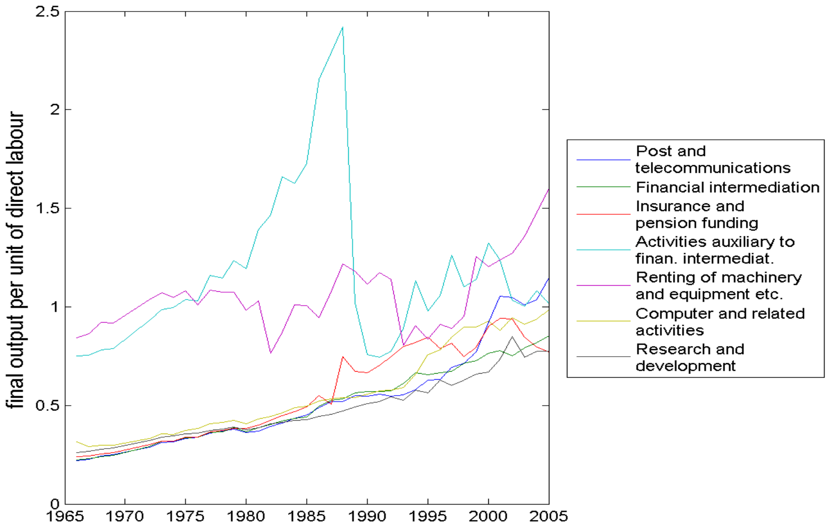

Fig. E.54: *The μ-index, Financial intermediation, business activities*

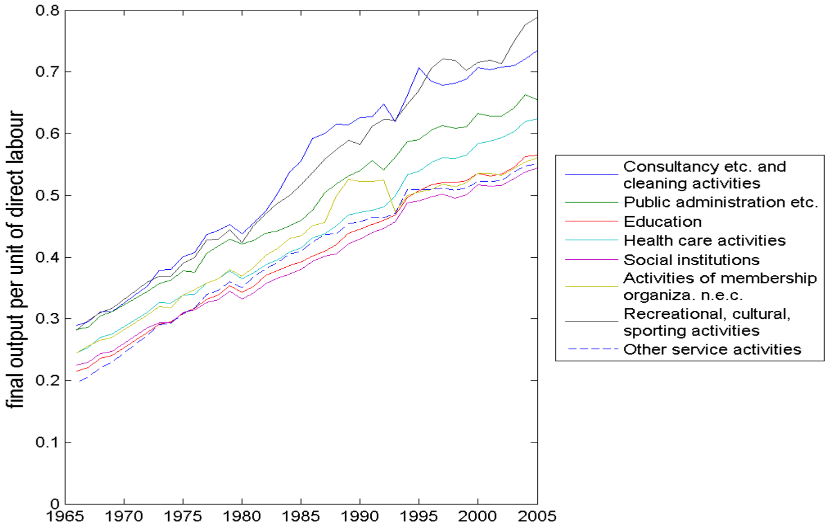

Fig. E.55: *The μ-index, Public and personal services*

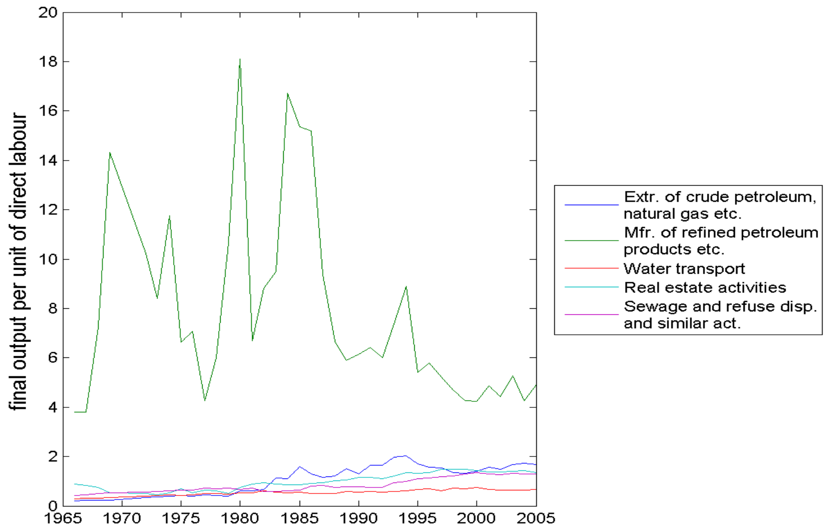

Fig. E.56: *The μ-index, Extreme cases*

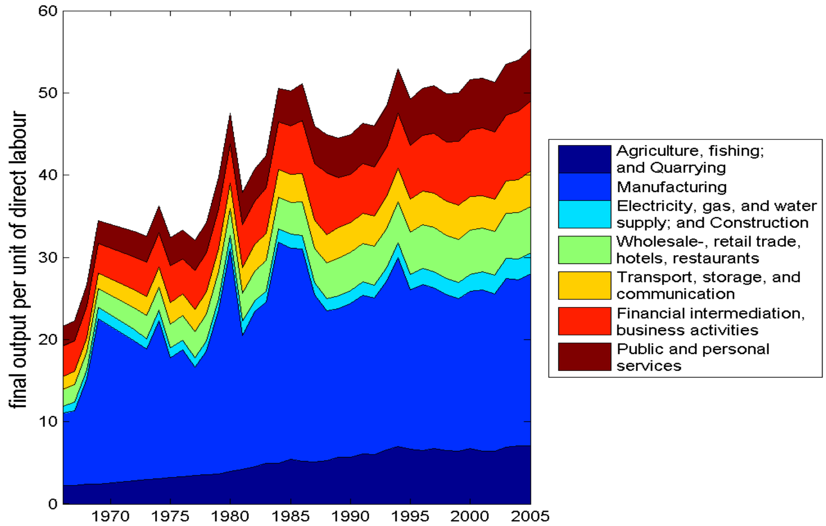

Fig. E.57: *The μ-index for the meso-sectors*

E.8 The ψ-index (Productivity Index, Production Prices)

$$\psi_t^i = \frac{1}{\boldsymbol{e}'\tilde{\boldsymbol{l}}_t^i R_t} \int_0^{R_t} \tilde{\zeta}_t^i(r)\, dr$$

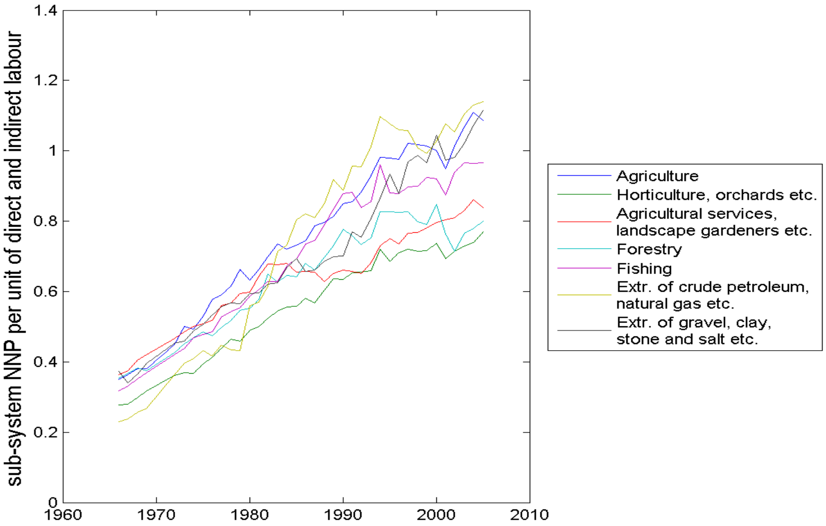

Fig. E.58: *The ψ-index, Agriculture, fishing, and quarrying*

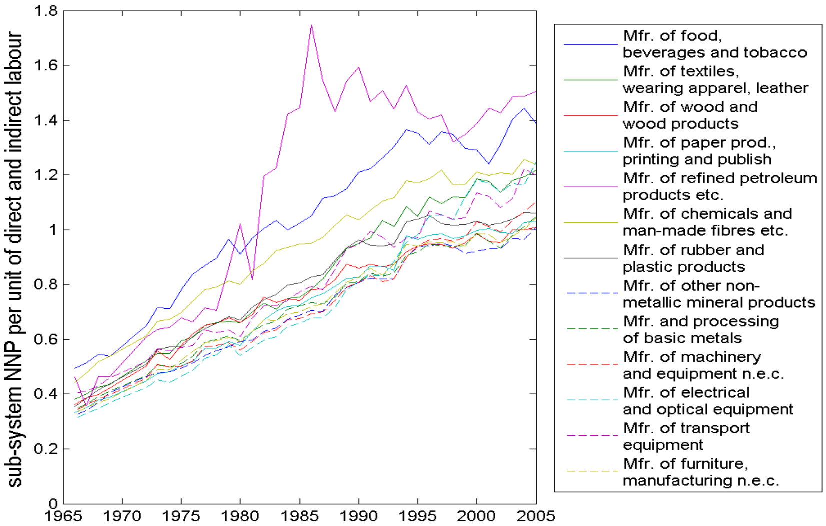

Fig. E.59: *The ψ-index, Manufacturing*

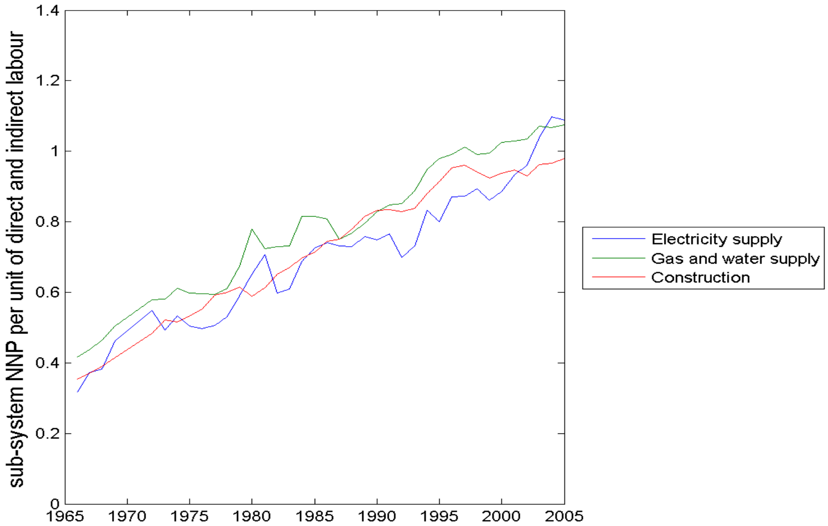

Fig. E.60: *The ψ-index, Electricity, gas, and water supply; and Construction*

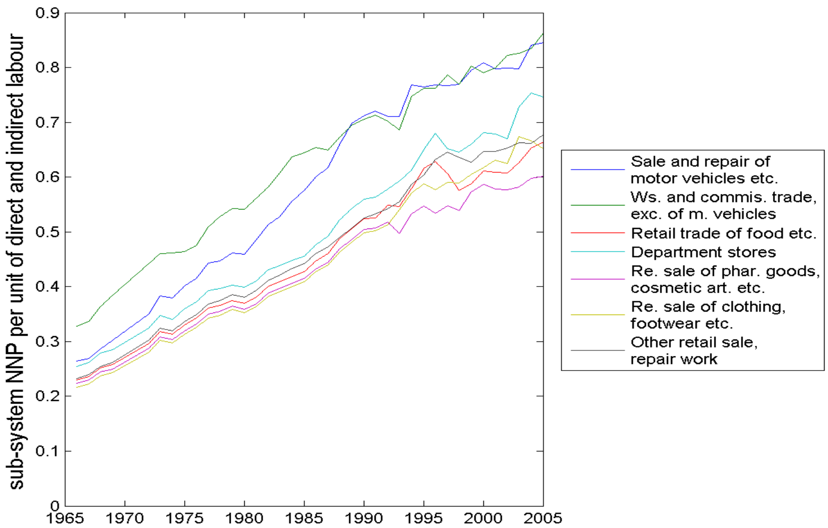

Fig. E.61: *The ψ-index, Wholesale-, retail trade, hotels, restaurants*

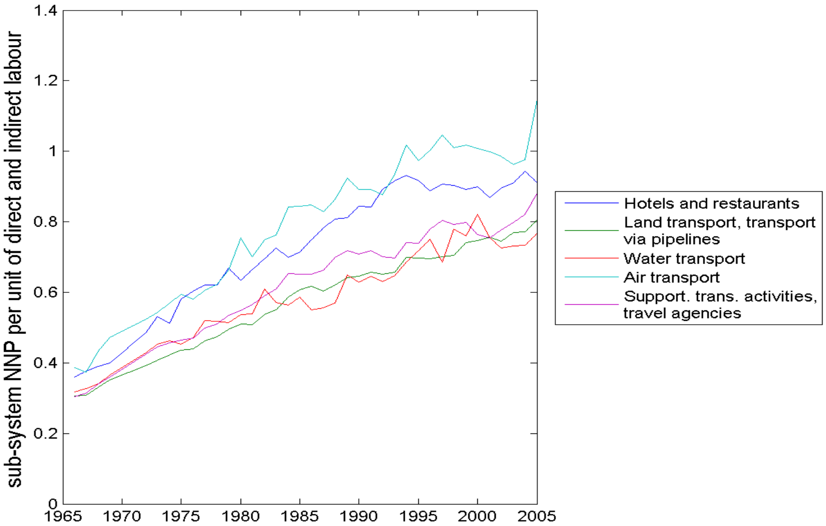

Fig. E.62: *The ψ-index, Transport, storage, and communication*

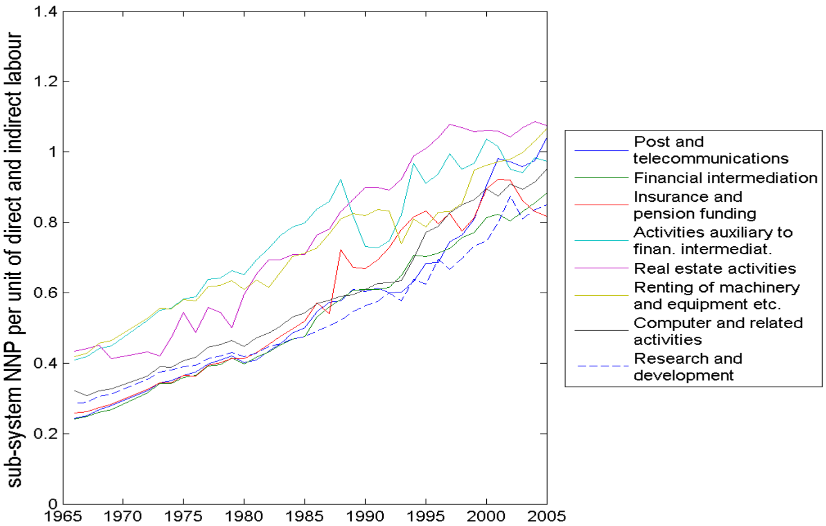

Fig. E.63: *The ψ-index, Financial intermediation, business activities*

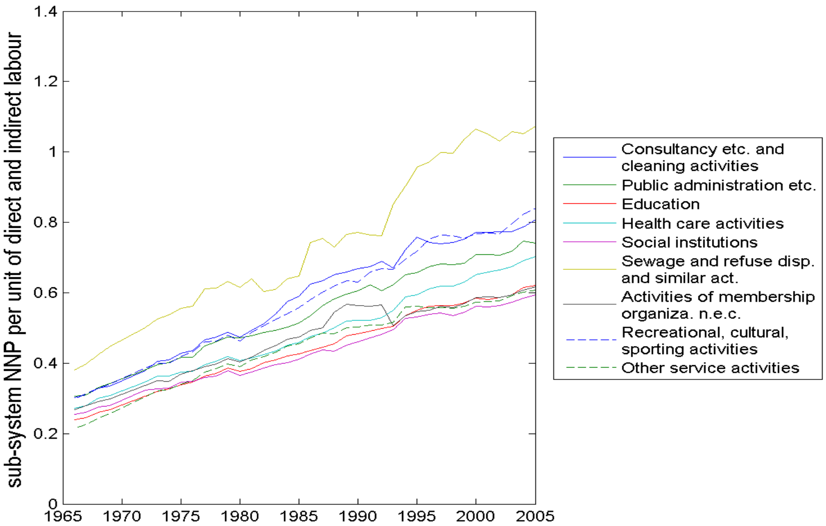

Fig. E.64: *The ψ-index, Public and personal services*

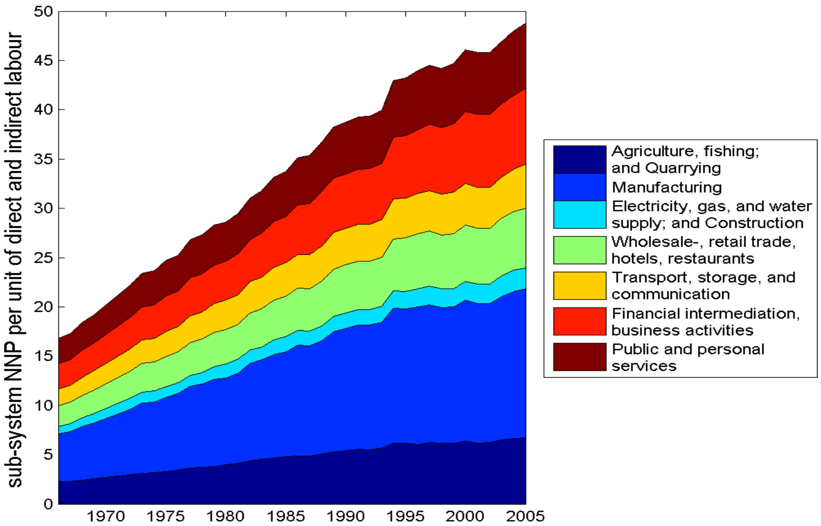

Fig. E.65: *The ψ-index for the meso-sectors*

E.9 The γ-index (Index of Structural Change, Production Prices)

$$\gamma_t^i = \frac{1}{R_t}\int_0^{R_t} \frac{(\tilde{b}^i_{(i,i,t)} - \tilde{a}^i_{(i,i,t)})p_{(i,t)}(r)}{\boldsymbol{e}'\tilde{\boldsymbol{A}}^i_t \boldsymbol{p}_t(r) + \boldsymbol{e}'\tilde{\boldsymbol{l}}^i_t w_t(r)}\, dr = \frac{\text{value of external output}}{\text{social costs}}$$

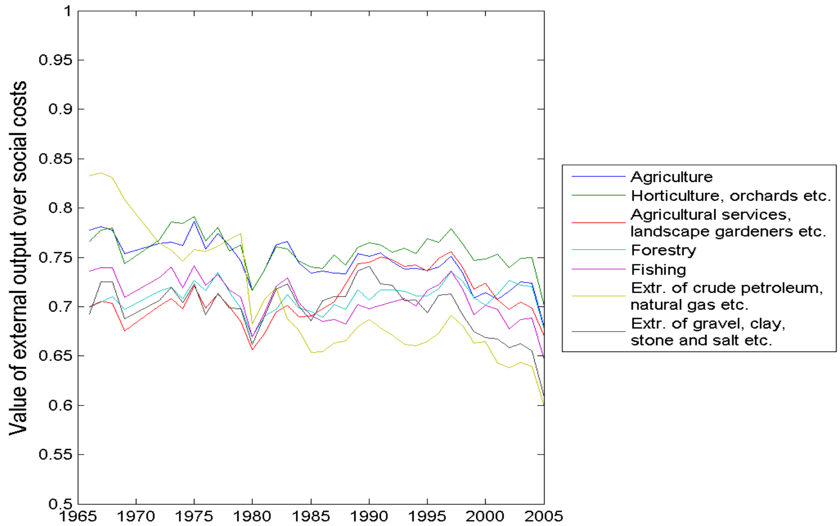

Fig. E.66: *The γ-index, Agriculture, fishing, and quarrying*

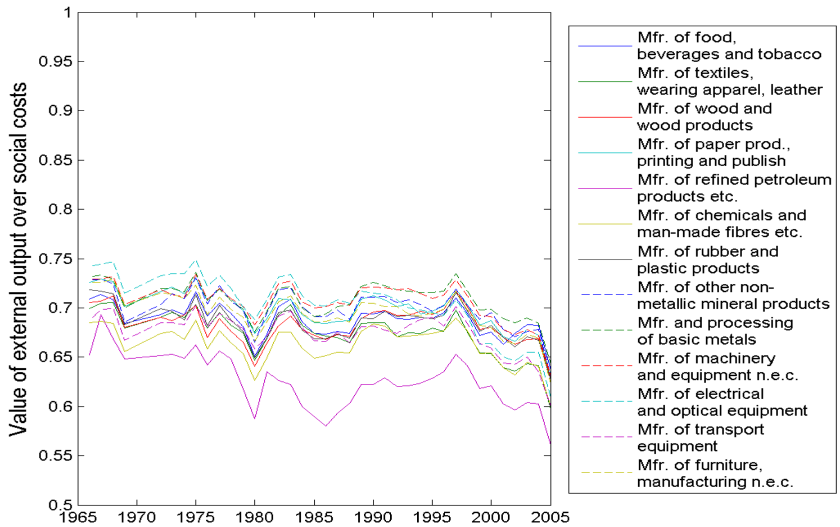

Fig. E.67: *The γ-index, Manufacturing*

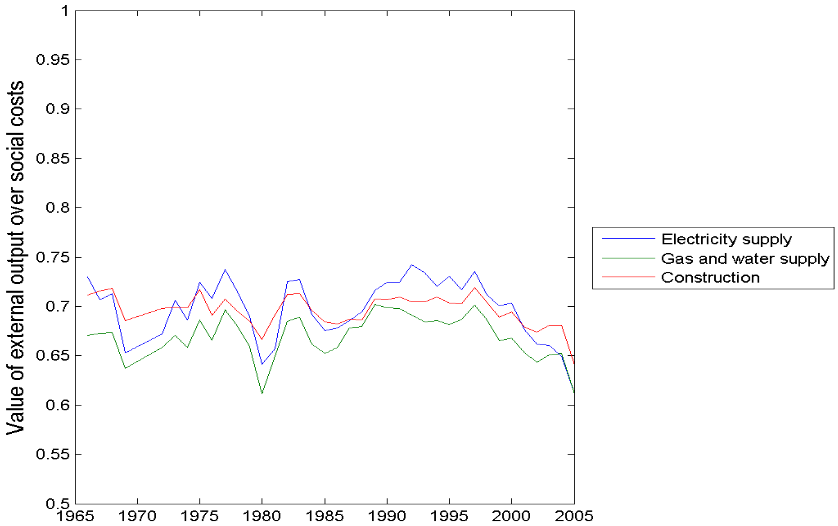

Fig. E.68: *The γ-index, Electricity, gas, and water supply; and Construction*

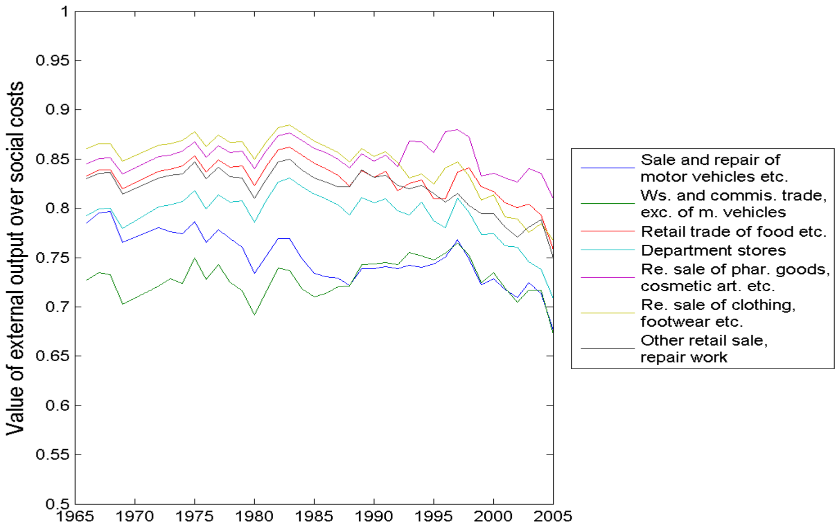

Fig. E.69: *The γ-index, Wholesale-, retail trade, hotels, restaurants*

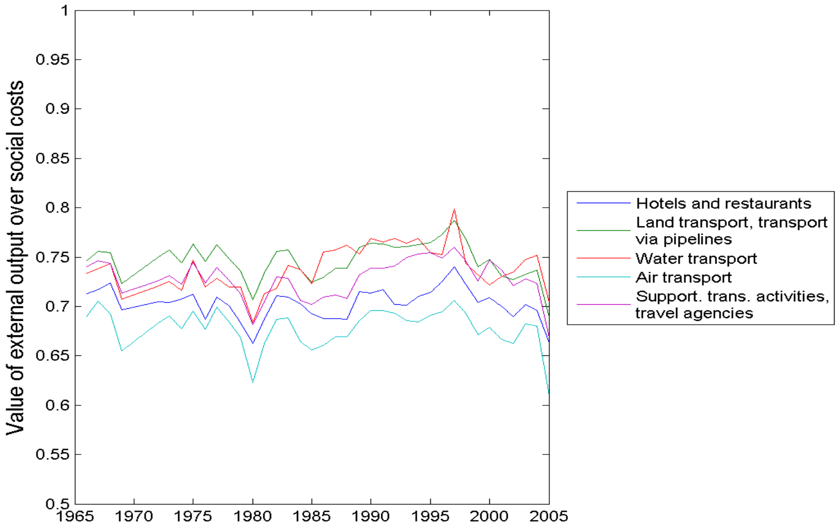

Fig. E.70: *The γ-index, Transport, storage, and communication*

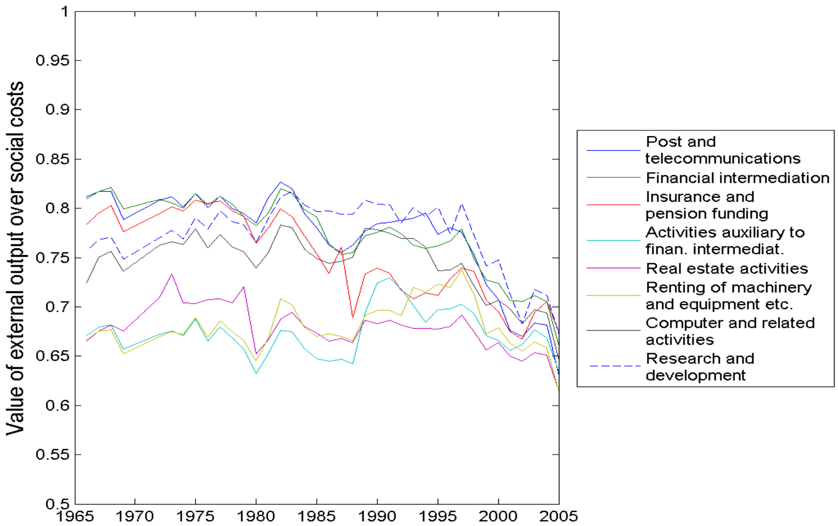

Fig. E.71: *The γ-index, Financial intermediation, business activities*

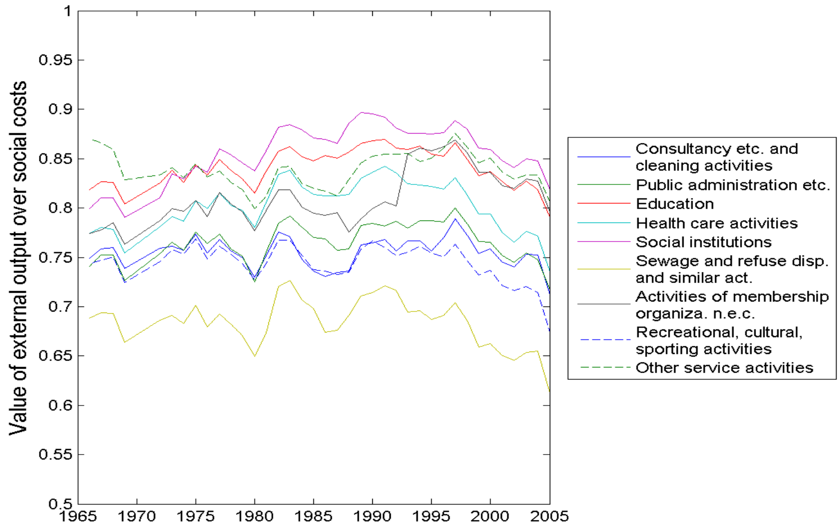

Fig. E.72: *The γ-index, Public and personal services*

E.10 The δ-index (Index of Structural Change, Production Prices)

$$\delta_t^i = \frac{1}{R_t}\int_0^{R_t} \frac{(\tilde{b}^i_{(i,i,t)} - \tilde{a}^i_{(i,i,t)})p_{(i,t)}(r)}{\tilde{\boldsymbol{a}}^i_{(i,:,t)}\boldsymbol{p}_t(r) + \tilde{l}^i_{(i,t)}w_t(r)}\,dr = \frac{\text{value of external output}}{\text{local costs}}$$

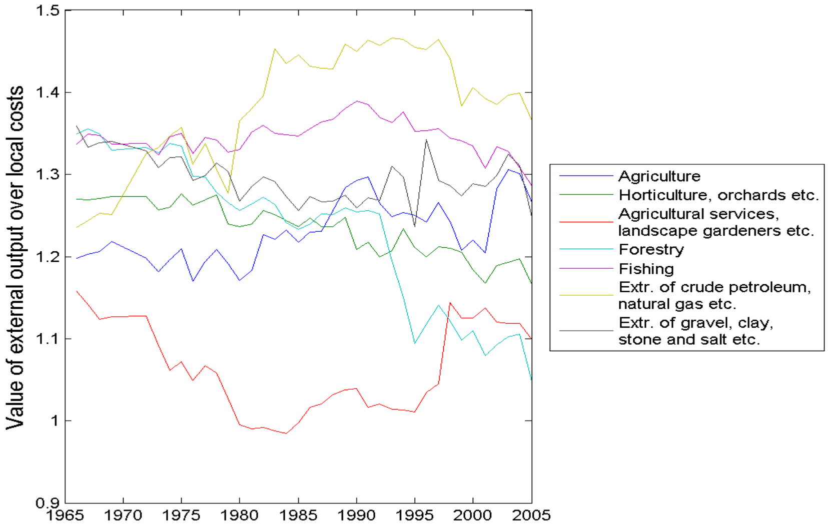

Fig. E.73: *The δ-index, Agriculture, fishing, and quarrying*

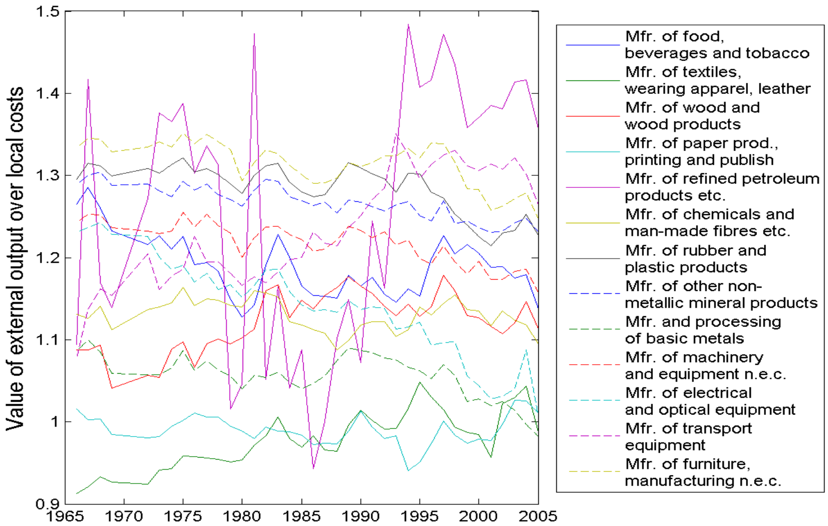

Fig. E.74: *The δ-index, Manufacturing*

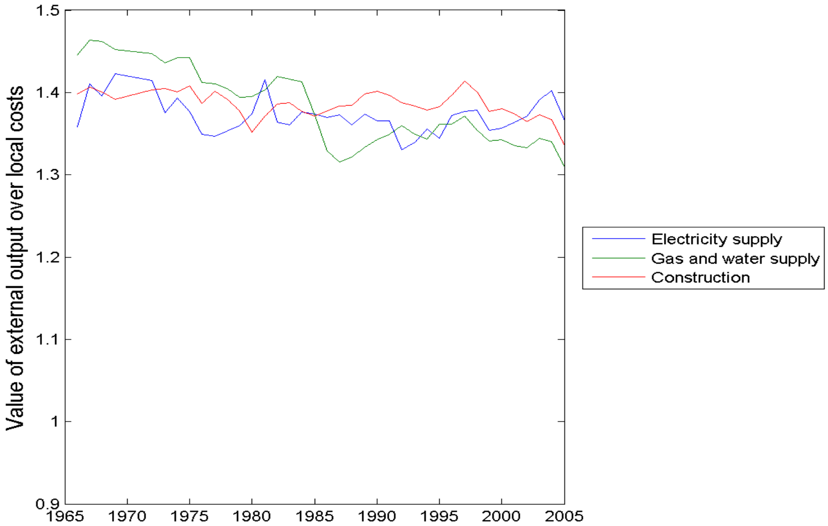

Fig. E.75: *The δ-index, Electricity, gas, and water supply; and Construction*

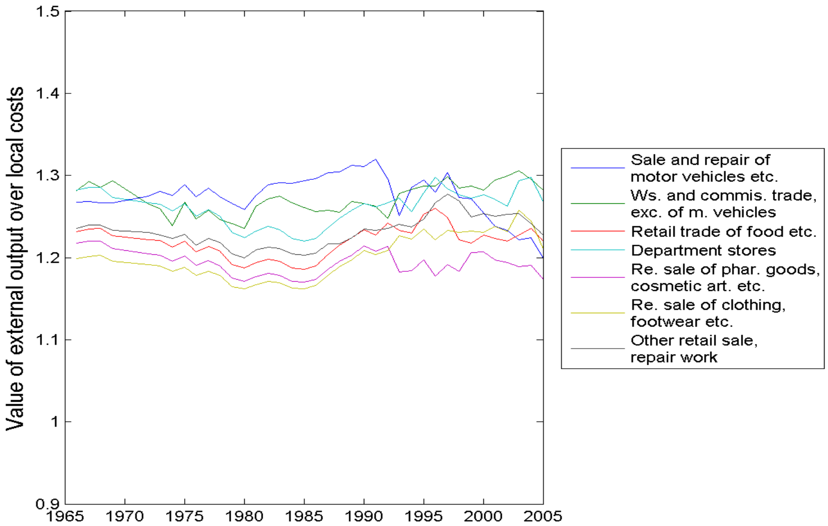

Fig. E.76: *The δ-index, Wholesale-, retail trade, hotels, restaurants*

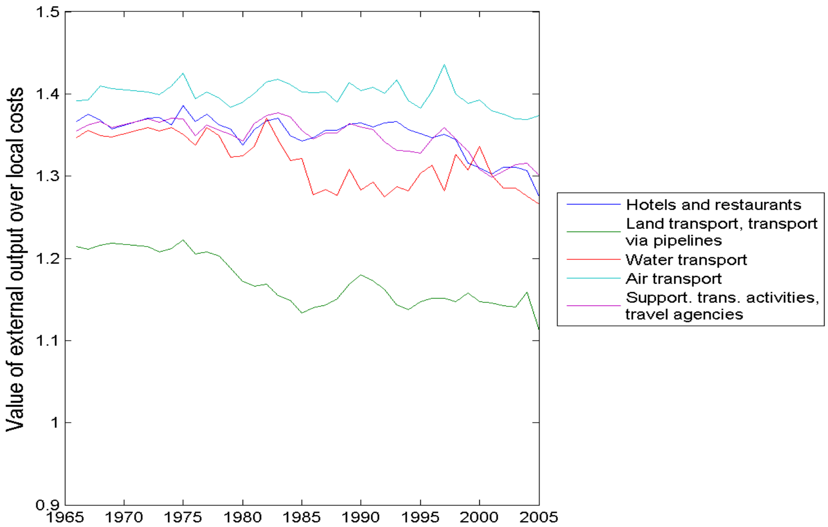

Fig. E.77: *The δ-index, Transport, storage, and communication*

Fig. E.78: *The δ-index, Financial intermediation, business activities*

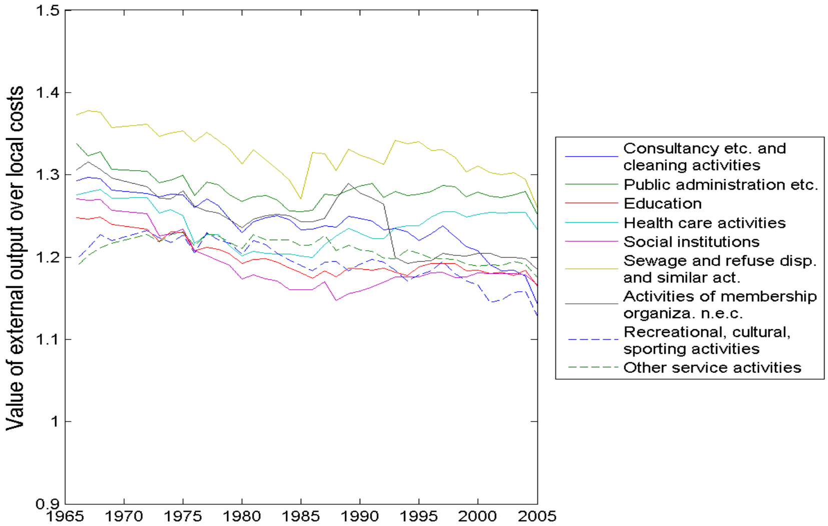

Fig. E.79: *The δ-index, Public and personal services*