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IDENTIFICATION OF DYNAMICAL PROPERTIES FROM CORRELATION FUNCTION ESTIMATES

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1. INTRODUCTION

Identification of physical properties from the dynamic response of structural systems - often called experimental modal analysis or system identification - is an area where a huge amount of research has been carried out, and where the interest for research results and practical applications is still increasing.

The growing interest for these techniques can be explained in different ways. One explanation is that computational possibilities in structural dynamics are getting better and new structural designs are introduced calling for a better and more detailed knowledge about the physical

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properties of the structures and how these properties are affected by damage and changes in load conditions. Another explanation is that by introduction of the computer in the measurement system, the possibility of handling large amounts of data became available, and the potential of the techniques were revealed.

The many possibilities of practical applications can be illustrated by studying one of the latest conference proceedings about experimental modal analysis, for instance one of the latest IMAC proceedings, see [20]. Only a few examples of applications will mentioned here.

One of the first applications of structural dynamic measurement was in the 1940's where the problem of describing the loads on aircraft wings was studied and where especially the problems of flutter gave rise to experimental studies of the dynamical properties of aircraft structures. Masts, chimneys and wind turbines are examples of structures where experimental studies of flutter and dynamic wind load might be wanted. Measurements on offshore structures loaded by sea waves have been performed in many locations for determination of sea loads and structural response, see e.g. Jensen [10].

One of the interesting applications is damage detection. When a specimen or even a large complex structure is damaged, the damage will cause a change of the dynamic properties. For instance if a structural member is cracked, the crack will decrease the stiffness and thereby decrease the eigenfrequencies of the structure and it may increase the damping due to local plasticity and thereby change the energy flow and the overall damping in the structure.

When dynamical properties of a structure are to be identified from an experiment, one would expect that the best way to perform the measurements would be to excite the structure by artificial loading and make the measurements when the natural loads are small. In this case the loading and the response is approximately deterministic, and the

dynamical parameters can be obtained by simple methods. However, there is several reasons for not using this approach. Using artificial loading is expensive and it is difficult to get a reasonable signal to noise ratio without introducing non-linearities that are non-relevant to the conditions of operation. I practice therefore, one will often have to deal with the more complex situation of accepting the natural loading as the input. In this case the parameters has to estimated from stochastic processes.

The modern techniques for identification of system parameters from the stochastic system responses are quite powerful. Very small system changes can be detected relatively fast. For instance using the two-stage techniques proposed in this paper on a specimen with an eigenfrequency of 1 kHz excited by white noise, the average Youngs modulus can be obtained from the eigenfrequency estimate in about one second with no systematic errors and with a coefficient of variation of less than 10^{-3}. This includes the time for data aquisition, correlation function estimation and system parameter identification.

This example introduces the two most important concepts in parameter identification: systematic and random errors - in estimation theory denoted bias and variance. Let the physical properties of the system be described by a limited number of scalars θ_i forming the vector $\underline{\theta} = \{\theta_i\}$ and let the values $\underline{\theta}^*$, $\hat{\underline{\theta}}$ be the true value and the value estimated by parameter identification respectively. Since we are always dealing with limited information, the estimated parameters $\hat{\underline{\theta}}$ will not be identical to the true properties $\underline{\theta}^*$. However we want the deviation to be as small as possible. Therefore the method of system identification should be chosen in such a way that the systematic errors defined by the expected value $E[\hat{\underline{\theta}} - \underline{\theta}^*]$ - also denoted bias - and the elements of the covariance matrix $Cov[\hat{\underline{\theta}}\hat{\underline{\theta}}^T]$ are minimum. If the bias is zero and the covariences are minimum then we are dealing with an unbiased and effective estimator, Papoulis [4].

When parameter identification techniques are used in structural applications, and the parameters are to be identified from the time series obtained by measuring the loads and the structural responses, different strategies can be followed. There are two main strategies: one-stage techniques and two-stage techniques.

In one-stage techniques the parameter identification is performed by fitting a model directly to the time series obtained from the experiment. In this case "blackbox" models in discrete time like Auto Regressive Moving Average (ARMA) models or oversized Auto Regressive (AR) models (also denoted method of maximum entropy) are frequently used, Ljung [1], Söderström and Stoica [2], Pandit and Wu [3]. These techniques have been developed mainly for applications in electrical engineering, but they are considered to be very accurate - in practice the closest one can get to unbiased effective estimators. For applications in structural engineering se e.g. Jensen [10]. In these techniques the parameter identification is based on nonlinear optimization and therefore the techniques require a relatively large computation power. However if the computation time and the time for transferring and storage of the large amounts of data can be accepted, these techniques will be an obvious choice.

If it is essential to keep the estimation time down to a minimum, other and less accurate techniques like two-stage techniques can be used. In two-stage techniques the basic idea is to transfer the most important porperties of the time series by small characteristic sequences like spectral density functions or correlation functions. The characteristic sequences acts as interface functions. They carry the most important information hidden in the time series in compressed form and thereby provide a basis for simple parameter identification by extracting this information from the interface function estimates. The advantage is that significantly smaller amounts of data has to be tranferred and stored, and that the system identification proces becomes much faster because models are fitted to a significantly smaller number of data points.

However there is a price to pay for this data compression: information is lost, and bias might be introduced.

Information will be lost in a data compression process, because it is not possible to contain all the detailed information hidden in the time series in the estimates of correlation functions or spectral density functions. Therefore, system parameters estimated from interface functions, will show larger variance than parameters estimated by effective fitting of models directly to the time series as mentioned above. In practice, however, the choice is governed by a trade-off between accuracy and speed, and sometimes it is beneficial to accept a small incease in variance for a large decrease in the time used in the estimation process.

It is well known that spectral density functions estimated by Fourier transform techniques will be biased, and that bias in the frequency domain can be reduced, but not prevented or removed. The bias problems are much easier to handle in the time domain, in fact it is possible to prevent bias in the time domain, e.g. by estimation of correlation functions. This is the major advantage of representing the properties of the time series by correlation functions instead of spectral density functions.

In this paper a simple and self-contained presentation of three known techniques for non-parametric correlation function estimation is given: the direct method where the correlation function is estimated by direct calculation of the correlation integral, the FFT technique, where the correlation function is estimated by the Fast Fourier Transform, and finally the Random Decrement technique, where the correlation function is estimated by simple averaging. The basic ideas of the three techniques are reviewed, sources of bias are pointed out, methods to prevent the bias problems are presented, and the methods are evaluated by comparing the speed and the accuracy estimating the auto correlation function for a single-degree-of-freedom (SDOF) system loaded by white noise.

The main aim of this paper is to investigate and illustrate the possibilities of using a two-stage parameter identification technique where correlation functions are used as interface functions. Since the Random Decrement technique is the fastest technique for correlation function estimation, this technique is used in the two-stage identification method. The results from the two-stage techniques are compared to the results of fitting an Auto Regressive Moving Average (ARMA) model directly to the original time series. In the investigation auto correlation functions are estimated on the response from a SDOF system, and the system parameters are then identified from the auto correlation function estimates. Three identification techniques are used: a simple non-parametric technique, calibration of Auto Regressive (AR) models by solving the overdetermined set of Yule-Walker equations by linear regression and finally least square fitting of the theoretical correlation function.

The simple SDOF problem treated in this paper does not reflect the complex situations of nonlinearities, overlapping resonance peaks and broad-banded but non-white excitation often present in practical applications. However, if a single and well separated resonance peak is considered, and if the load spectrum is approximately constant in the neighbourhood of the considered resonance peak, the signal can be band-pass filtered and the presented methods can be used as an approximation.

In this paper the response from the SDOF system loaded by white noise is simulated using an ARMA (2,1) model.

2. ESTIMATION OF CORRELATION FUNCTIONS

Estimating correlation functions for use in two-stage parameter identification technique one would like not to assume very much about system properties. This means that only non-parametric estimates are relevant for a two-stage technique.

As mentioned in the introduction, the accuracy of the non-parametric correlation function estimation is essential. Therefore the variance and the bias errors should be as small as possible. Only a few techniqes for non-parametric correlation function estimation are known: the direct technique, the Fast Fourier Transform (FFT) technique and finally the Random Decrement (RDD) technique. In this section these three techniques are reviewed with special emphasis on bias problems.

2.1 The Direct Method

The direct method was applied for estimation of correlation functions up to the mid sixties where the modern methods of Fast Fourier Transfomation were introduced. The algorithm is simple and easy to use and programme, but slow in most cases.

For the stationary stochastic processes $X(t)$, $Y(t)$ the correlation function $R_{XY}(\tau)$ is defined by, Bendat and Piersol [6],

$$R_{XY}(\tau) = E\,[X\,(t+\tau)Y(t)] \tag{1}$$

In practice, however, it is not possible to perform ensemble averaging. Instead the processes are assumed to be ergodic, and the ensemble averaging is replaced by time averaging on the realizations $x(t)$, $y(t)$.

$$R_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t+\tau)y(t)dt \tag{2}$$

Real time series are always limited, they are known over some finite time domain, say $[0;T]$. To keep the arguments inside the definition set, for positive time lags the upper integration limit cannot exceed $T-\tau$, and the natural modification of eq. (2) is then

$$\hat{R}_{XY}(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t+\tau)y(t)dt, \quad \tau \in [0;T] \tag{3}$$

forming a correlation function estimate. The corresponding result for negative time lag is found by modifying the lower integration limit, or using that $R_{XY}(-\tau) = R_{YX}(\tau)$. The estimate given by eq. (3) is unbiased since the expectation of the estimator is equal to the definition given by eq. (1),

$$E[\hat{R}_{XY}(\tau)] = \frac{1}{T-\tau}\int_0^{T-\tau} E[X(t+\tau)Y(t)]dt = R_{XY}(\tau) \tag{4}$$

The corresponding biased estimate is defined by

$$\hat{R}^w_{XY}(\tau) = \frac{1}{T}\int_0^{T-\tau} x(t+\tau)y(t)dt, \tau \in [0;T] \tag{5}$$

It is seen that the estimates are related by $\hat{R}^w_{XY}(\tau) = w(\tau)\hat{R}_{XY}(\tau)$, where $w(\tau)$ is a triangular window, the so-called "basic lag window", given by $w(\tau) = (T-\tau)/T$, Newland [9].

2.2 The Fast Fourier Transforms Method

The Fast Fourier Transform is an effective algorithm for calculation of Fourier coefficients, Brigham [5]. The basic ideas were discovered in the forties by Danielson and Lanczos, [11], but the technique became known by the work of Cole and Tukey [12] and was implemented in larger scale from the the mid-sixties.

Application of the Fast Fourier Transform (FFT) technique is based on the periodic estimate

$$\hat{R}^p_{XY}(\tau) = \frac{1}{T}\int_0^{T} x(t+\tau)y(t)dt \tag{6}$$

where the data segment pair $x(t), y(t)$ originally defined on $[0;T]$ is made periodic by setting $x(t+T) = x(t), y(t+T) = y(t)$. Using the results from appendix A, it is easy to see that the periodic estimate can be written as a convolution

$$\hat{R}^p_{XY}(\tau) = x(\tau) * y(-\tau) \tag{7}$$

and that $\hat{R}^p_{XY}(\tau)$ is periodic with period T. Therefore the length of the periodic estimate is only T and not $2T$ as for the direct estimates introduced in the preceeding section. Let $y(t)$ and Y_n be a Fourier Transform pair, $y(t) \leftrightarrow Y_n$, then $y(-t) \leftrightarrow \overline{Y}_n$ where the overbar denotes complex conjugation, se (A.3) in appendix A, and the Fourier Transform S_n of $\hat{R}^p_{XY}(\tau)$ is therefore given by

$$\hat{R}^p_{XY}(\tau) \leftrightarrow X_n\overline{Y}_n = S_n \tag{8}$$

where $x(t) \leftrightarrow X_n$. This equation, together with the FFT algorithm, form the basis of correlation function estimation by the Fast Fourier Transform.

The idea is to take the time series, divide them into smaller data segment pairs of the length T, estimate the spectral values S_n for all the data segment pairs, average over the pairs to reduce random uncertainty and finally transform back to the time domain to obtain the correlation function estimate by inverse FFT.

The advantage is, that this technique - because of the efficiency of the FFT algorithm - is faster than the direct method. The problem is that the estimate is biased.

The bias is introduced in the calculation of the correlation integral (6) by the assumption of periodicity of the data segments. If the argument $t+\tau$ is larger than T, then the factor $x(t+\tau)$ in the correlation integral is replaced by $x(t+\tau-T)$, and the ends of the periodic estimate of the correlation function is polluted by "wrap around bias".

The bias might be large or small depending on how well the assumption of periodicity is satisfied, but once the bias is there, it cannot be removed. The bias might be reduced in the frequency domain by introduction of suitable windows, Brigham [5].

In the time domain, however, the bias can be prevented by padding zeroes, Bendat and Piersol [6]. From the original data segment pair $x(t), y(t)$ defined on $[0;T]$ a new pair is defined

$$[x_0(t), y_0(t)] = \begin{cases} [x(t), y(t)] & t \in [0;T] \\ 0 & t \in [T;2T] \end{cases} \qquad (9)$$

This completely removes the wrap around bias on the periodic estimate. The reason is that the added zeroes make sure that the integrand in the convolution integral vanishes outside the original definition set $[0;T]$. Thus the zero padding doubles the length of the correlation function estimate, and makes it correspond exactly to the biased estimate $\hat{R}^{w}_{XY}(\tau)$.

The effect of padding zeroes is therefore that the more or less arbitrary wrap around bias is replaced by the well defined window bias. As it appears from the previous section, the window bias is simply removed by division by the basic lag window $w(\tau) = (T-\tau)/T$.

This technique of unbiased correlation function estimation by FFT is illustrated in figure 1. Since the variance increases at the ends of the estimate due to decreasing information, it might be considered to discard

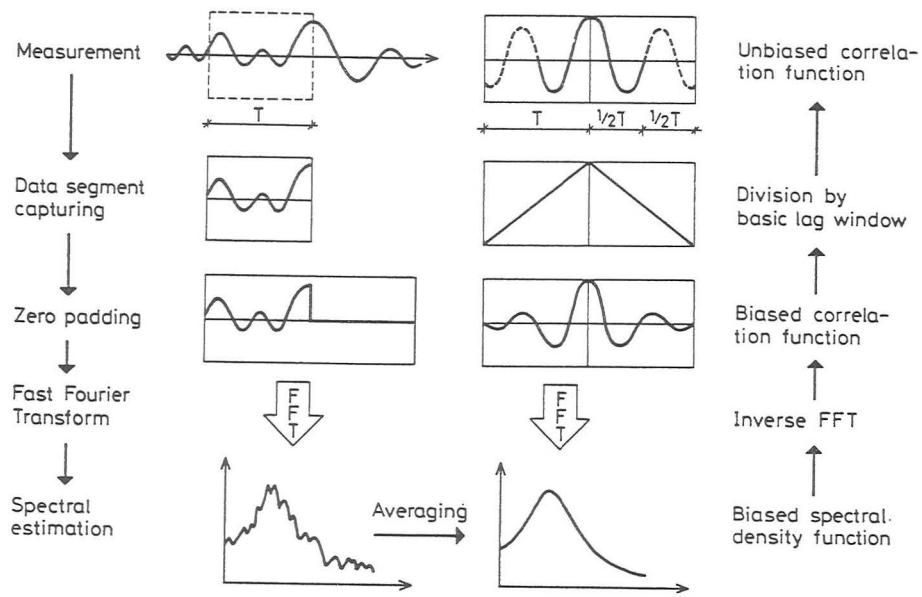

Figure 1: Unbiased correlation function estimation by FFT.

the outermost parts of the correlation function estimate. In figure 1 the possibility of discarding the last half part of the estimate is indicated.

2.3 The Random Decrement Method

The Random Decrement (RDD) technique is a fast technique for estimation of correlation functions for Gaussian processes by simple averaging.

The RDD technique was developed at NASA in the late sixties and early seventies by Henry Cole and co-workers [13-16], just a little later than the development of the FFT technique.

The basic idea of the technique is to estimate a co-called RDD signature. If the time series $x(t)$, $y(t)$ are given, then the RDD signature estimate $\hat{D}_{XY}(\tau)$ is formed by averaging N segments of the time series $x(t)$

$$\hat{D}_{XY}(\tau) = \frac{1}{N}\sum_{i=1}^{N} x(\tau + t_i) \mid C_{y(t_i)} \tag{10}$$

where the time series $y(t)$ at the times t_i satisfies the trig condition $C_y(t_i)$, and N is the number of trig points. The trig condition might be for instance that $y(t_i) = a$ (the level crossing condition) or some similar condition. The algorithm is illustrated in figure 2. In eq. (10) a cross signature is estimated since the accumulated average calculation and the trig condition are applied to two different time series. If instead the trig condition is applied to the same time series as the data segments are taken from, an auto signature is estimated.

One of the problems of the technique is that the theoretical basis has been unclear and is still being discussed. Most of old references on the RDD Technique, including the original papers by Cole and co-workers argue on a more or less heuristic basis that the RDD signature formed by averaging time series segments from the output of a stochasic loaded system should describe system properties only.

This interpretation was modified by Vandiver et al, [17], who defined the RDD auto signature as the conditionale expectation $D_{XX}(\tau) = E[X(\tau) \mid X(o) = a]$, and proved that in the case of the level crossing trig condition applied to a Gaussian process, the RDD signature is simply proportional to the auto correlation function

$$D_{XX}(\tau) = E[X(\tau) \mid X(0) = a] = \frac{R_{XX}(\tau)}{\sigma_X^2} a \tag{11}$$

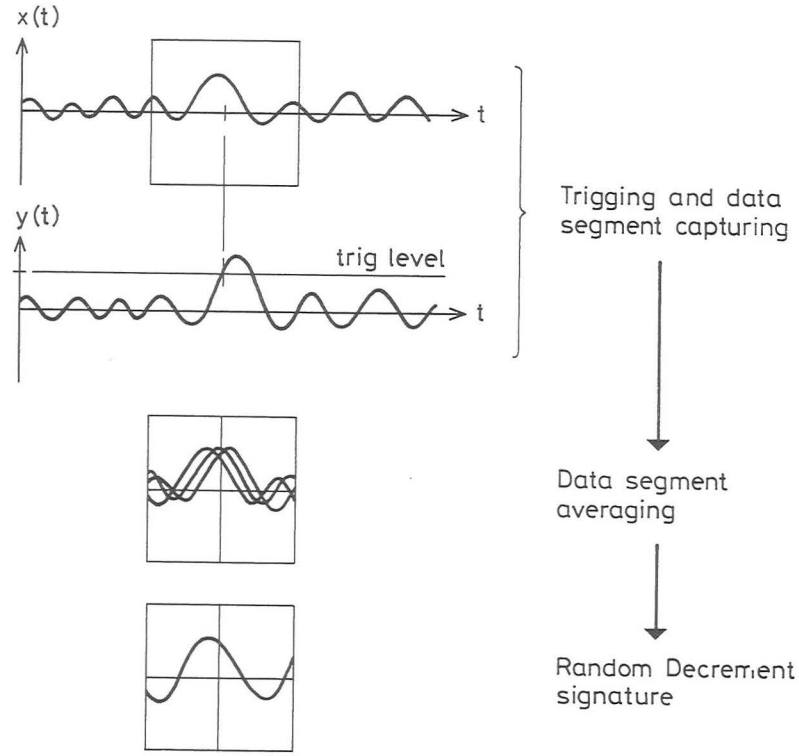

Figure 2. Determination of the Random Decrement signature.

where a is the trig level and σ_X^2 is the variance of the proces. The original interpretation of the RDD signature is therefore correct only under the asumption of white noise loading where the free decay is proportional to the correlation function.

It is not difficult to generalize Vandiver's result to the cross signature case. Let $X(t)$ and $Y(t)$ be stationary, zero mean, Gausian processes. The dependency between the processes is then completely described by the covariance matrix

$$Cov\left[\begin{bmatrix} X(\tau) \\ Y(0) \end{bmatrix} [X(\tau)\ Y(0)]\right] = \begin{bmatrix} \sigma_X^2 & R_{XY}(\tau) \\ R_{XY}(\tau) & \sigma_Y^2 \end{bmatrix} \tag{12}$$

Now, using the definition of the RDD signature as a conditional expectation and using (B.4) in appendix B, taking the Gaussian vectors $\underline{X}$ and $\underline{Y}$ as scalars equal to $X(\tau)$ and $Y(0)$ respectively, the generalization of eq. (11) is easily obtained

$$D_{XY}(\tau) = E\,[X(t) \mid Y(0) = a] = \frac{R_{XY}(\tau)}{\sigma_Y^2}\, a \tag{13}$$

One of the qualities of the RDD technique is that the RDD estimate is "born unbiased". This is easily verified

$$E\,[\hat{D}_{XY}(\tau)] = \frac{1}{N} \sum_{i=1}^{N} E\,[X(\tau + t) \mid Y(t) = a] = D_{XY}(\tau) \tag{14}$$

The result given by eq. (13) is limited to the case of the level trig condition. It is possible however, to derive a kind of fundamental solution, forming a basis for application of different trig conditions. Conditioning on both Y and $\dot{Y}$ defining the Gaussian vectors $\underline{X} = X(\tau)$ and $\underline{Y}^T = [Y(0)\ \dot{Y}(0)] = [a\ v]$ and using eq. (B.4) in appendix B we obtain, Krenk and Brincker [18]

$$D_{XY}(\tau) = E\,[X(\tau) \mid Y(0) = a, \dot{Y}(0) = v] = \frac{R_{XY}(\tau)}{\sigma_Y^2}\, a - \frac{R'_{XY}(\tau)}{\sigma_{\dot{Y}}^2}\, v \tag{15}$$

where $R'_{XY}(\tau)$ is the derivative of the correlation function and where $\sigma_{\dot{Y}}^2$ is the variance of the derivative proces $\dot{Y}(t)$, for a Gaussian proces given by $\sigma_{\dot{Y}}^2 = -R''_{YY}(0)$. From this fundamental solution it is possible to explain the meaning of the RDD signature for several trig condition of practical interest

$$\begin{array}{lll} A: & \dot{Y}(0) = a & \Rightarrow D_{XY}(\tau) \propto R_{XY}(\tau) \\ B: & \dot{Y}(0) = v & \Rightarrow D_{XY}(\tau) \propto R'_{XY}(\tau) \\ C: & \dot{Y}(0) = 0,\ Y(0) > a & \Rightarrow D_{XY}(\tau) \propto R_{XY}(\tau) \\ D: & Y(0) = 0,\ \dot{Y}(0) > v & \Rightarrow D_{XY}(\tau) \propto R'_{XY}(\tau) \end{array} \tag{16}$$

The result for trig condition A is found using that the distribution of $\dot{Y}(t)$ is symmetrical and independent of $Y(t)$. The last term in eq. (15) will therefore vanish, and the result becomes proportional to the correlation function in agreement with eq. (13). The result for condition B is obtained by a similar argument, and the results for the conditions C and D follows directly from eq. (15).

An estimate of the variance on the RDD signature in closed form might be found by obtianing the conditional variance using eq. (B.5) and assuming the averaged data segments to be independent, Krenk and Brincker [18].

Even though the RDD estimates are "born" unbiased as shown above, the implemetation might introduce bias. At least two sources of bias are known at present, but if these bias errors are prevented, the remaining bias errors will be small at least for ligthly damped systems. The two bias sources are "trig point sorting" bias and "trig window" bias, both illustrated and discussed in Brincker et al. [19].

"Trig point sorting" bias means bias introduced by non-representative data segments, typically obtained by sorting the trig points. The bias is prevented by using time series containing a large number of trig points, and by using all the trig points in the time series.

The "trig window" bias is introduced by improper implementation of the trig condition. A sampled time series is only known at certain times t_i. This means that it is not possible for instance to obtain the exact time t the realization $y(t_i)$ crosses through the level a, and therefore the theoretical level crossing condition cannot be realized on sampled data. The condition must be modified by introduction of a finite size trig

window. Vertical, horizontal and general slanted trig windows might be used, and it is possible to show that the bias as a first approximation will give rise only to a time shift of the RDD estimate, Krenk and Brincker [18]. The bias can be removed by shifting the signature back to the right origin, or prevented by using self-compensating trig windows.

Some of the important elements of the theory of the RDD technique, the fundamental solution, closed form solutions for the variance on the estimate and bias introduced by different kind of windows are developed in Krenk and Brincker [18], and applications are illustrated in Brincker et al [19], [20].

3. EVALUATION OF CORRELATION FUNCTION ESTIMATION TECHNIQUES

In this section the three techniques are evaluated comparing the speed and accuracy for estimation of auto correlation functions. The autocorrelation functions are estimated on the output from a single degree-of-freedom (SDOF) system loaded by white noise.

The output x_n formed by sampling the continuous output signal from a SDOF system at the fixed sampling rate $1/\Delta T$ is simulated by an ARMA (2,1) model as explained in appendix C. The autocorrelation function of the simulated proces is therefore known to be $R_{XX}(m\Delta t)$ where $R_{XX}(.)$ is the auto correlation function for the system output in continuous time given by eq. (C.2). Therefore, if an estimate of the correlation function $\hat{R}_{XX}(t_m)$ is obtained from the simulated time series, the estimation error ε can be calculated directly

$$\varepsilon^2 = \frac{1}{M\sigma_X^2} \sum_{m=0}^{M} (R_{XX}(m\Delta T) - \hat{R}_{XX}(m\Delta T))^2 \qquad (17)$$

where M is the number of points in the one-sided auto correlation estimate.

All the simulations were performed modelling a system with a period of 0.5 seconds corresponding to $\omega_0 = 12.57$ rad/s, the time spacing ΔT between sample points was 0.051 seconds and the length of all the time series was 4000 points. The simulations were performed using the PC version of the MATLAB software package, [21], except the algorithm for estimation of the RDD signature which was programmed in the C programming language and linked to the MATLAB software by the MATLAB user function interface. All calculations were made on a 33 MHz 386-based PC with a 387 mathematical co-processor.

Only unbiased versions of the techniques were used as explained in the preceeding section, and the RDD technique was used with the trig condition C in eq. (16) and $a = \sigma_X$.

Figure 3 illustrates the accuracy of the three techniques. M denotes the number of points in the one-sided correlation function. This corresponds to an estimate length of $T = (M-1)\Delta T$. For the FFT estimates, M denotes the number of points in the one-sided correlation function estimate after the last half part of the estimate has been discarded. The reduced estimate is a little more accurate and a little faster to calculate. The difference in accuracy between the reduced FFT estimates and the estimates obtained by the direct technique is small, and the FFT results can be considered representative for the direct unbiased technique also. Each point in the figure is the average of ten values, each estimated from time series of 4000 points.

In figures 3a and 3b the estimation error is shown as a function of M for three different damping ratios, heavy damping $\zeta = 0.1$, moderate damping $\zeta = 0.01$, and light damping $\zeta = 0.001$. It appears from the results that for heavy damping and short estimates the FFT technique is more accurate

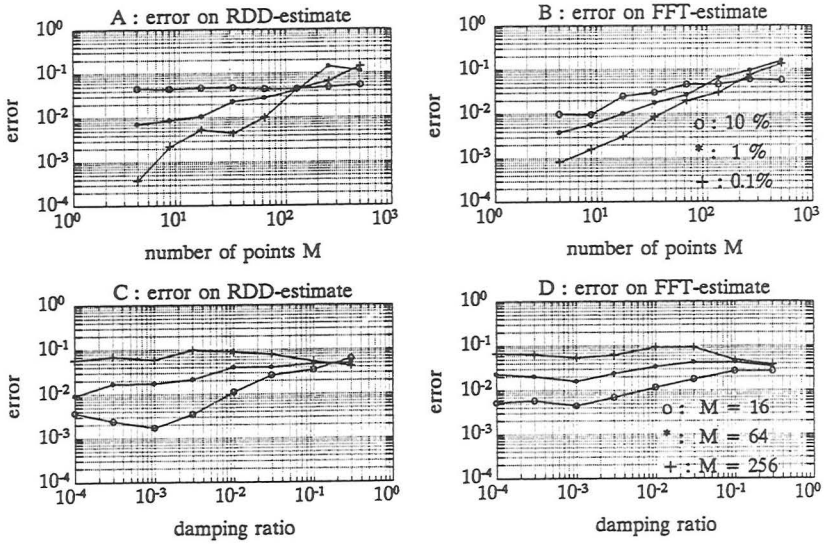

Figure 3. Accuracy of autocorrelation function estimation by the Random Decrement technique (RDD), fig 3a and 3c and by the Fast Fourier Transform (FFT) technique, 3b and 3d.

than the RDD technique. For moderate and light damping there does not seem to be any significant difference.

In figure 3.c and 3.d the estimation error is shown as a function of the damping ratio for three different lengths, short estimates $M = 16$, medium length estimates $M = 64$ and long estimates $M = 256$. The estimation errors for the long and the medium lenghts estimates are about the same for the two techniques, but for the short estimates, the estimation error for the RDD estimates is smaller than the estimation error for the FFT estimates in the low damping region $\zeta \approx 0.001$.

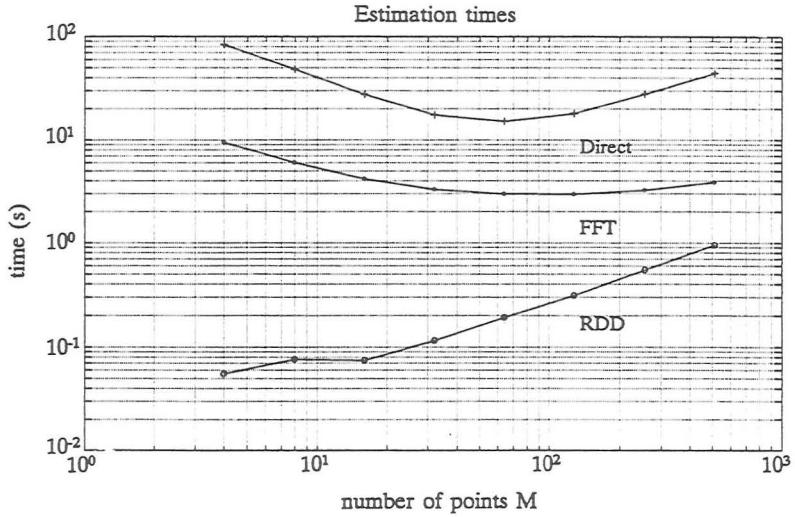

Figure 4. Times for calculation of autocorrelation function estimates by the Random Decrement technique (RDD), the Fast Fourier Transform (FFT) technique and the direct technique for a time series of 4000 data points using a 33 MHz 386 PC with 387 co-processor.

Figure 4 shows the observed estimation times for the three techniques as a function of the number of points M in the one-sided correlation function estimates. As it appears from the results, the direct technique is always the slowest and the RDD technique is always the fastest of the three. This migth change for extremely long estimates, $M > 512$, but for all the investiagted values of M the RDD technique is significantly faster than the two others. For the short estimates the RDD technique is about a factor of 100 faster than the FFT technique.

However, the speed results reflects not only the efficiency of the techniques, but also the way the computer treats numbers. The results reflects the ratio between computation time for addition (RDD estimation) and multiplication (FFT estimation and direct estimation). This ratio is different from computer to computer, and therefore the results given in figure 4 are not universal. The results are a strong indication that in the most cases the RDD technique will be the most efficient for estimation of correlation functions of moderate length.

4. IDENTIFICATION OF SYSTEM PARAMETERS

From the results in the previous sections, the RDD technique seems to be the fastest and in some cases the most accurate technique for estimation of correlation functions. The RDD technique is therefore used in the further investigations.

After the response simulation and auto correlation estimation by the RDD technique, the parameter estimate $\underline{\hat{\theta}} = [\hat{T}\ \hat{\zeta}]$ is obtained from the auto correlation estimates. A typical RDD auto correlation estimate is shown in figure 5a.

In this section three methods of identifying system properties from correlation function estimates are described: a simple non-parametric technique, estimation of AR models by linear regression and fitting of the theoretical auto correlation function by non-linear optimization. Finally it is described how system parameters can be estimated directly from the original time series by calibration of an ARMA model.

4.1 A Simple Non-Parametric method

In the non-parametric method denoted RDD-NP the damping and the eigenperiod are found simply by estimating crossing times and the logarithmic decrement.

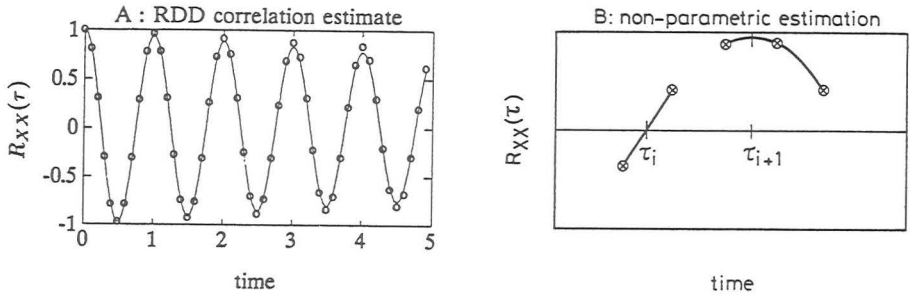

Figure 5. 5a: a typical auto correlation function estimated by the Random Decrement technique for M = 50, $\Delta t = 0.1T$ and $\zeta = 0.01$. o: RDD estimate, solid line: exact solution. 5b: estimation of the extreme values (peaks and valleys) and crossing times on the auto correlation function and the crossing times of the derivative.

First all the extremes r_i - both peaks and valleys - on the correlation function are found. The logarithmic decrement δ can then be expressed by the initial value r_0 of the correlation function and the i'th extreme

$$\delta = \frac{2}{i} ln \left(\frac{r_0}{|r_i|} \right) \tag{18}$$

The logarithmic decrement and initial value of the correlation function can then be found by linear regression on $i\delta$ and $2ln(|r_i|)$, and the damping ratio then is, Thomson [8],

$$\zeta = \delta / \sqrt{\delta^2 + 4\pi^2} \tag{19}$$

A similar procedure was adopted for determination of the eigenperiod. The estimated times corresponding to the estremes and the zero crossing times

of the correlation function forms an number of time points τ_i equally spaced by a quarter of the damped eigenperiod. The damped eigenperiod T_d and the time shift of the correlation function can then be found by linear regression on the crossing times τ_i, and the eigenperiod is then given by

$$T = T_d \sqrt{1-\zeta^2} \tag{20}$$

The extreme values and the corresponding times were found by quadratic interpolation, fig 5b. The crossing times of the correlation function itself were found by linear interpolation, figure 5b.

The extremes are a part of the envelope of the auto correlation function. The envelope only depends on the damping, and therefore even though the technique might seem simple, it provides a direct way of separating the problems of estimation of the damping and the eigenperiod. This observation is still true when nonlinear damping is present, and therefore the simple non-parametric technique might be especially usefull in situations where the other techniques break down.

The simple non-parametric algorithm for estimation of the damping ratio and the eigenperiod of the auto correlation function was programmed in the C programming language and linked to the MATLAB environment by the MATLAB user function interface, [21].

4.2 Auto Regressive (AR) Model Estimation

For a SDOF system loaded by white noise the stationary output in discrete time space is given by (C.3, appendix C). Using the white noise assumption $E[a_i a_j] = \sigma_a^2 \delta_{ij}$ where δ_{ij} is the Kronecker delta and the definition of the auto correlation function $R_{XX}(\tau) = E[X(t+\tau)X(t)]$, a difference equation for the auto correlation function can be obtained by

multiplying both sides of (C.3) by x_{m-k} and then taking the expectation. If the estimated auto correlation function is substituted for the theoretical auto correlation function, the result is

$$\hat{R}_{XX}(m\Delta t) = \Phi_1 \hat{R}_{XX}((m-1)\Delta) + \Phi_2 \hat{R}_{XX}((m-2)\Delta t)\ ;\ 1 < m < M \tag{21}$$

where M is the number of points in the estimated one-sided auto correlation function. This is a system of linear equations in the Auto Regressive (AR) Parameters Φ_1, Φ_2 often referred to as the Yule-Walker equations. When the number of equations are larger than the number of parameters to be estimated, the system becomes overdetermined, and the equations may not have a solution. For such situations, however, standard methods exist for determination of approximate solutions. One possibility is to solve the system of equations by least square linear regression. In that case the estimate $\hat{\Phi}_1, \hat{\Phi}_2$ is called an overdetermined Yule-Walker estimate, Söderström and Stoica [2].

From the estimates $\hat{\Phi}_1, \hat{\Phi}_2$ found by linear regression calibration of the AR model, the eigenperiod and damping ratio are found using the inverse closed form solutions given by eq. (C.4) and (C.5). The method of estimating the system parameters by AR calibration is denoted RRD-AR.

All the estimations were performed using the standard estimation function for the AR model in the MATLAB system identification toolbox, [21].

4.3 Fitting the Theoretical Correlation Function

Since we know that we are dealing with a SDOF system, the form of the auto correlation function is known. An obvious way of estimating the system parameters $\underline{\theta}$ is therefore to minimize the difference between the theoretical auto correlation function $R_{XX}(\underline{\theta}, m\Delta t)$ given by eq. (C.2) and the estimated auto correlation function $\hat{R}_{XX}(m\Delta t)$. Using a least square

approach, the method is denoted RDD-FIT and the parameter estimate $\hat{\underline{\theta}} = [\hat{T}\ \hat{\zeta}]$ is found as the solution to the following set of nonlinear equations

$$\frac{\partial}{\partial \theta_i} \sum_{m=1}^{M} (R_{XX}(\underline{\theta}, m\Delta t) - \hat{R}_{XX}(m\Delta t))^2 = 0 \qquad (22)$$

Typically the error function has many local minima, and therefore to prevent false solutions (not corresponding to the global minimum) a good initial estimate is essential. The nonlinear set of equations were solved using a quasi-Newton algorithm with the parameter estimate from the non-parametric method as initial values.

All the estimations were performed using the standard optimization function FMINU in the MATLAB optimization toolbox, [21].

4.4 Identification on the Original Time Series

From the simulation it is known that the true model is an ARMA (2,1). The ARMA models were calibrated using a predictor error method, Ljung, [1], and all estimations were performed using the standard calibration function in the MATLAB system identification toolbox.

5. EVALUATION OF IDENTIFICATION METHODS

The accuracy of the techniques were investigated by identifying the system parameters for different sampling times ΔT, different length's M of the auto correlation function estimate and for different damping ratios ζ.

For each combination 20 time series of 4000 points each were simulated, the parameters were estimated as explained in the preceeding sections, and the coeffient of variation $\overline{std}[\hat{\theta}_i] / \theta_i^*$ and the empirical bias $\overline{E}[\hat{\theta}_i - \theta_i^*] / \theta_i^*$ were calculated for the eigenperiod and the damping ratio,

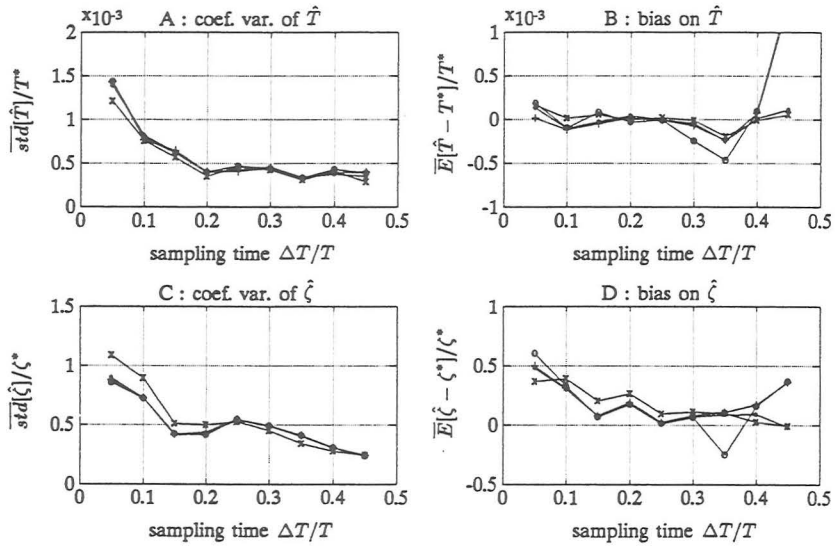

*Figure 6. Coefficient of variation and bias on the estimated eigenperiod $\hat{T}$ and the estimated damping ratio $\hat{\zeta}$ as a function of the sampling time ΔT. For all curves $M = 50$, $\zeta^* = 0.001$ and the symbols mean: o: RDD-NP, *: RDD-AR, +: RDD-FIT, x: ARMA.*

where *std* [ ] is the empirical standard deviation and $\overline{E}$ [ ] is the empirical expectation. The results are shown in figure 6, 7 and 8.

Figure 6 shows the identification results as a function of the sampling time Δt for $\zeta = 0.001$ and $M = 50$. As it appears from the results the optimal samling time is relatively close to the Nyquist frequency $\Delta T = T/2$. It is not surprising that all four techniques manage to give reasonable estimates for the two well defined system parameters. The differences in accuracy between the four techniques are small. If the

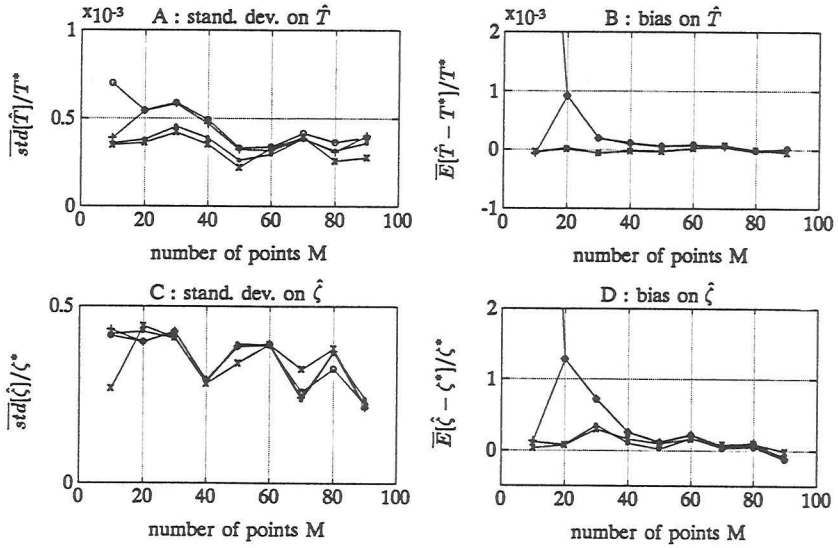

Figure 7. *Coefficient of variation and bias on the estimated eigenperiod $\hat{T}$ and the estimated damping ratio $\hat{\zeta}$ as a function of the number of points M. For all curves $\Delta T = 0.4T^*$, $\zeta^* = 0.001$ and the symbols mean: o: RDD-NP, *: RDD-AR, +: RDD-FIT, x: ARMA.*

sampling time becomes too long then the RDD-NP technique becomes more unreliable than the other techniques. This was to be expected because then the simple algorithm cannot identify the extremes on the correlation function estimates. There is a tendency for all the methods to over estimate the damping, propably this is due to insufficient information in the relatively short time series.

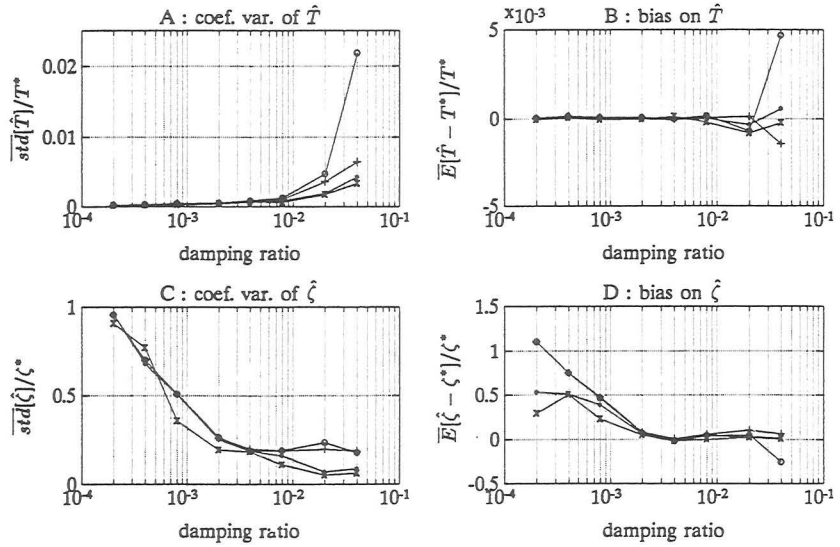

Figure 8. *Coefficient of variation and bias on the estimated eigenperiod $\hat{T}$ and the estimated damping ratio $\hat{\zeta}$ as a function of the damping ratio ζ. For all curves $M = 50$, $\Delta t = 0.4T$ and the symbols mean: o: RDD-NP, *: RDD-AR, +: RDD-FIT, x: ARMA.*

Figure 7 shows the identification results as a function of the number of points M in the one-sided auto correlation function estimate for $\Delta t = 0.4T$ and $\zeta = 0.001$. Again there is only a small difference in accuracy, allthough there is a tendency for the RDD-AR and ARMA estimates to be more accurate for short correlation function estimates. The RDD-AR estimates are very close to the ARMA estimates, figure 7a, 7b and 7d. For short auto correlation function estimates the NP estimates seem to be biased, and because of problems with formulating a flexible termination criterion for the nonlinear optimization this affects the RDD-FIT estimates through

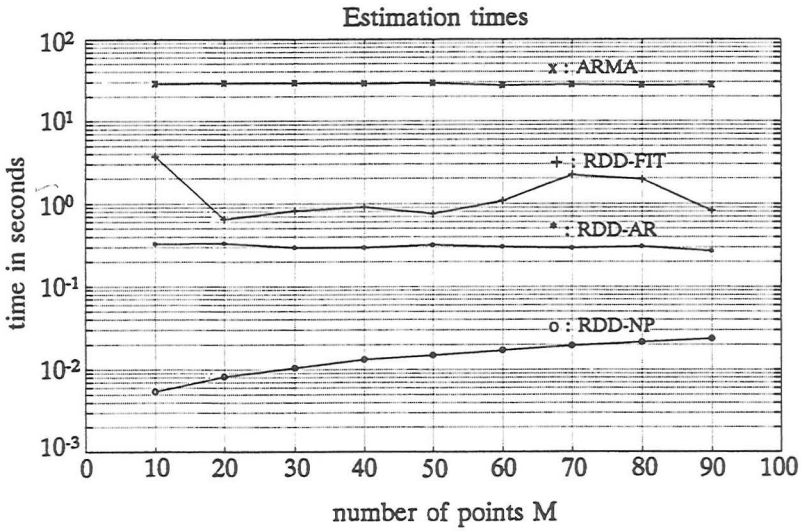

Figure 9. Estimation times for the different techniques as a function of the length M of the one-sided auto correlation function estimate using a 33 MHz 386-based PC with 387 co-processor.

the initial values. Therefore nothing seems to be gained by trying to fit the correlation function after the non-parametric estimates has been obtained.

Figure 8 shows the results as a function of the damping ratio ζ. As expected small damping ratios give a large coefficient of variation on the damping estimates, and a small coeficient of variation on the eigenperiod estimates. For small damping ratios all damping estimates seem to be biased. This is propably due to limited information about the damping in the time series - propably the time series becomes too short measured in correlation times. Again it appears from the results that there is only a marginal difference between the accuracy of the four investigated

techniques. For large damping ratios, the ARMA and the RDD-AR estimates are systematically better than the RDD-NP and RDD-FIT estimates, but for damping ratios in the structural range $\zeta \in [0.01; 0.001]$ there seems, to be no big difference.

Figure 9 shows the estimation times as a function of the length M of the correlation functions estimates. The depency of M is small compared to mutual differences and the estimation times for the ARMA calibration was about 30 s, for the RDD-FIT technique about 1 - 2 s (10 - 20 iterations), for the AR estimation about 0.3 s and for the simple non-parametric technique about 10 - 20 ms. The time for RDD estimation of the auto correlation function was in section 3 found to about 0.1 s. This means that the total estimation times for the two-stage techniques are about 1-2 s for the RDD-FIT technique, about 0.4 s for RDD-AR estimation and about 0.1 s for the RDD-NP technique.

6. CONCLUSIONS

Three techniques are known for non-parametric estimation of correlation functions: the direct technique, the Fourier Transform technique and the Random Decrement technique.

All three techniques can be implemented to yield unbiased correlation function estimates, but there are differences in the efficiency and accuracy.

The results in this investigation show that if correlation functions are estimated on output from systems with heavy damping then the FFT technique seem to be the most accurate. If correlation function estimates are short and the correlation functions are estimated on output from systems with light damping the accuracy of the Random Decrement technique seem to be better than for the two others.

The results in this investigation show - in agreement with common knowledge - that the direct technique is the slowest of the three, and the results indicate that for calcution of short estimates the Random Decrement technique will be significantly faster than the Fast Fourier Transform.

The use of different two-stage system parameter identification methods has been illustrated on a simple SDOF system loaded by white noise.

In this case the use of interface functions only results in a relativily small loss of information. In this investigation about 50 points in the auto correlation function estimates were enough to ensure about the same estimation uncertainties as an effective one-stage technique (ARMA calibration).

If an unbiased correlation function estimator like the Random Decrement technique is used for interface function estimation, then the observed empirical bias on the parameter estimates is small compared to random errors in the most cases. When bias is present, it seems to be introduced by limited information in the relatively short time series used in the investigation.

Three techniques were used for identification of system properties from the auto correlation function estimates: a simple non-parametric technique, calibration of Auto Regressive models and finally least square fitting of the theoretical auto correlation function. The three techniques did not differ very much in accuracy, allthough a detailed examination shows that AR estimation is the most accurate and the most reliable of the three techniques.

However the three techniques showed a significant difference in speed, the fit technique being slowest with about 1-2 s per estimation and the non-parametric being the fastest with about 10-20 ms per estimation.

The simple non-parametric technique can be recommended. It is extremely fast and pretty reliable. Furthermore if nonlinear damping is present it might be useful in situations where other methods might break down.

Calibration of an AR model by solving the overdetermined set of Yule-Walker equations by least square linear regression seems to be a fast, accurate and reliable technique. It is nearly as accurate as calibration of an ARMA model directly on the original time series. Furthermore, in situations where the system is only aproximately an SDOF system, for instance while an output from a multi-degree-of-freedom system is band-pass filtered, an oversized AR model can be applied in a similar way.

APPENDIX A. Fourier transforms of finite length data

The Fourier series of a function $f(t)$ defined on $[0;T]$ is defined by

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{in\omega t}, \ \omega = \frac{2\pi}{T} \tag{A.1}$$

Using the ortogonality relation

$$\int_0^T e^{i(n-m)\omega t} dt = T\delta_{nm} \tag{A.2}$$

where δ_{nm} is the Kronecker delta, the Fourier coefficients or the Fourier transform of $f(t)$ is obtained

$$F_n = \frac{1}{T} \int_0^T f(t) e^{-in\omega t} dt \tag{A.3}$$

The functions $f(t)$ and F_n is said to form a Fourier transform pair

$$f(t) \leftrightarrow F_n \tag{A.4}$$

Since the base functions $e^{in\omega t}$ are all defined on $]-\infty;\infty[$ it is natural to use eq. (A.1) to extend the definition set for $f(t)$ to $]-\infty;\infty[$. In that case $f(t)$ becomes periodic with period T. Now let $g(t)$ and $h(t)$ be periodic functions with period T. If the convolution is defined as

$$g(t) * h(t) = \frac{1}{T} \int_0^T g(t-\tau)\, h(\tau)\, d\tau \tag{A.5}$$

then by using eq. (A.2) it is easy to show that

$$g(t) * h(t) \leftrightarrow G_n H_n \tag{A.6}$$

The corresponding theorem for convolution in the frequency domain

$$G_n * H_n = \sum_{k=-\infty}^{\infty} G_{n-k}\, H_k \tag{A.7}$$

is found to

$$g(t)\, h(t) \leftrightarrow G_n * H_n \tag{A.8}$$

APPENDIX B. Condition on Gaussian variables via regression

Let $\underline{X}$ and $\underline{Y}$ be Gaussian vectors, i.e. the elements $X_1, X_2, \ldots X_n$; $Y_1, Y_2, \ldots Y_m$ are jointly normal distributed and are therefore completely described by the expectation and the covariance

APPENDIX C. System Response Simulation by ARMA (2.1) Model

The response $X(t)$ from a single degree-of-freedoms-system loaded by stationary Gaussian white noise is the solution to the second order differential equation

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2 = Q(t) \qquad \text{(C.1)}$$

where ω_0 is the undamped natural angular frequency, ζ is the damping ratio and $Q(t)$ stationary zero mean Gaussian white noise. For this case the normalized (corresponding to variance one) autocorrelation function is given by, Crandall and Mark [7]

$$R_{XX}(\tau) = \exp(-\zeta\omega_0\tau)\left(\cos(\omega_d\tau) + \frac{\zeta\omega_0}{\omega_d}\sin(\omega_d\tau)\right); \quad \tau \geq 0 \qquad \text{(C.2)}$$

where ω_d is the damped natural frequency $\omega_d = \omega_0\sqrt{1-\zeta^2}$. The most accurate way to perform simulations of a system formulated in continuous time, is to transform the system model to the discrete time space. This can be done by using an ARMA model. It can be shown, Pandit and Wu [3], that a second order system formulated in continuous time may be represented in the discrete time space by a (2,1) ARMA model given by

$$x_m = \Phi_1 x_{m-1} + \Phi_2 x_{m-2} + a_m - \Theta a_{m-1} \qquad \text{C.3)}$$

where m is the discrete time ($t_m = m\Delta t$), Φ_1, Φ_2 are the Auto Regressive (AR) parameters, Θ is the Moving Average (MA) parameter and a_m is a time series of independent Gaussian distributed numbers with zero mean and variance σ_a^2. The model is denoted (2, 1) since it has 2 AR parameters and 1 MA parameter. If the ARMA parameters are chosen as

$$\Phi_1 = 2\exp(-\zeta\omega_0\Delta t)\cos(\omega_d\Delta t) \qquad \text{(C.4)}$$

$$\Phi_2 = -\exp(-2\zeta\omega_0\Delta t) \qquad \text{(C.5)}$$

$$\Theta = -P \pm \sqrt{P_2 - 1}\,; \quad |\Theta| < 1 \qquad \text{(C.6)}$$

where

$$P = \frac{\omega_d\sinh(2\zeta\omega_0\Delta t) - \zeta\omega_0\sin(2\omega_d\Delta t)}{2\zeta\omega_0\sin(\omega_d\Delta t)\cosh(\zeta\omega_0\Delta t) - 2\omega_d\sinh(\zeta\omega_0\Delta t)\cos(\omega_d\Delta t)} \qquad \text{(C.7)}$$

then the ARMA model given by eq. (C.3) is the representation of the continuous system given in eq. (C.1) in the discrete time space. It can be shown, Pandit and Wu [3], that the discrete auto correlation function of the time series x_m is equal to the sampled auto correlation function of the continuous process $X(t)$.

ACKNOWLEDGEMENTS

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RESUME

Identifikation af bærende konstruktioner ved anvendelse af korrelationsfunktioner betragtes. En to-trins identifikationsmetode er foreslået. Første trin er bestemmelse af korrelationsfunktioner medens andet trin består i bestemmelse af systemparametrene udfra korrelationsfunktionsestimaterne. Tre metoder til bestemmelse af korrelationsfunktioner præsenteres: den direkte metode, FFT efterfulgt af invers FFT samt Random Decrement teknikken. For alle metoderne er der givet en gennemgang af mulige årsager til samt metoder til forebyggelse af fejl på korrelationsfunktionsestimaterne. Herefter præsenteres metoder til bestemmelse af systemparametre fra korrelationsfunktionsestimaterne bestemt ved Random Decrement teknikken: en simpel ikke-parametrisk metode, estimering ved en Auto Regressive (AR) model og ved at fitte et analytisk udtryk for korrelationsfunktionen. Ved en simuleringsundersøgelse sammenlignes resultaterne opnået ved disse to-trins metoder med resultater opnået ved at fitte en Auto Regressive Moving Average (ARMA) model direkte til accelerationerne fra et system med en frihedsgrad belastet med hvid støj.

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