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Publication date:
1998

Document Version
Early version, also known as pre-print

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITET • AAU • AALBORG • DANMARK

FRACTURE & DYNAMICS
PAPER NO. 108

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MARCH 1998 ISSN 1395-7953 R9816

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Printed at Aalborg University

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ESTIMATION OF MODAL PARAMETERS AND THEIR UNCERTAINTIES

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ABSTRACT: In this paper it is shown how to estimate the modal parameters as well as their uncertainties using the prediction error method of a dynamic system on the basis of output measurements only. The estimation scheme is assessed by means of a simulation study. As a part of the introduction, an example is given showing how the uncertainty estimates can be used in applications such as damage detection.

1 INTRODUCTION

The estimation of the dynamic properties of linear and time-invariant systems has been applied for various reasons in a variety of engineering fields. This extraction of dynamic information is known as modal analysis since the dynamics is usually represented by the so-called modal parameters. However, modal analysis is usually confined to an investigation of the expected values of the modal parameters. In such an analysis they have often been treated as deterministic parameters instead of a realization of some stochastic variables.

In this paper, it is the intention to show how to obtain information about the uncertainties of estimated modal parameters. This can be accomplished by assuming the estimated modal parameters to be realizations of Gaussian stochastic variables. The modal parameter estimates are then assumed to correspond to the mean values of these stochastic variables. It will be shown how to estimate these mean values by calibration of a linear and time-invariant discrete-time parametric model to time series data. The associated covariance matrix can then be estimated afterwards. Two methods for estimation of the covariance matrix will be presented. Finally, the performance of these methods will be tested by a simulation study.

1.1 *An Example - Damage Detection*

In applications where the change of the modal parameters is of interest, the extra information about the quality of the estimates is essential. Such an application could e.g. be damage detection, where damage is believed to be detected if one or more of the estimated modal parameters change significantly with time, and if this change cannot be explained by e.g. a change of extra mass on a structure or changes in the ambient environment.

However, the problem is how to define a significant change of a modal parameter. This problem can be solved in a statistical sense if the estimates are treated as realizations of stochastic variables, see Andersen et al. [1], Kirkegaard et al. [2] and Doebling et al. [3].

Different statistical tests exist for testing whether two realizations with a certain probability are outcomes of the same stochastic variables. If, with a significant confidence, it can be rejected that two realizations are outcomes of the same stochastic variable then it can be concluded that a significant change has occurred.

1.2 *The ARMAV Model*

The estimation of modal parameters is based on the assumption that the dynamic system shows linear and time-invariant behaviour. The dynamic behaviour of an ambient excited system is usually modelled by a second-order differential equation system, see e.g. Andersen [4]

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t)+\boldsymbol{C}\dot{\boldsymbol{y}}(t)+\boldsymbol{K}\boldsymbol{y}(t) = \boldsymbol{w}(t)\ , \quad \boldsymbol{w}(t)\in NID(\boldsymbol{0},\boldsymbol{W}) \tag{1}$$

$\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the mass, viscous damping and stiffness matrices. $\boldsymbol{y}(t)$ is the displacement vector. $\boldsymbol{w}(t)$ is continuous-time Gaussian white noise with zero mean and an intensity described by the matrix $\boldsymbol{W}$.

In Andersen [4] and Andersen et al. [5], it is shown how to represent such a system by a discrete-time Auto-Regressive Moving Average Vector (ARMAV) model. Assume that the continuous-time system is observed at discrete time instances k using a sampling interval T. If the continuous-time system consists of $np/2$ degrees of freedom (DOF) and if p displacements are observed and external measurement noise is present, a model having covariance equivalent system response at all discrete time steps $t_k=kT$ is of the form

$$\begin{aligned}&\boldsymbol{y}(t_k)+\boldsymbol{A}_1\boldsymbol{y}(t_{k-1})+\ldots+\boldsymbol{A}_n\boldsymbol{y}(t_{k-n}) = \\ &\boldsymbol{e}(t_k)+\boldsymbol{C}_1\boldsymbol{e}(t_{k-1})+\ldots+\boldsymbol{C}_n\boldsymbol{e}(t_{k-n})\ , \quad \boldsymbol{e}(t_k)\in NID(\boldsymbol{0},\Lambda)\end{aligned} \tag{2}$$

This model consists of an nth order auto-regressive matrix polynomial, and a moving average matrix polynomial of similar order, where $n=np/p$. All coefficient matrices of the polynomials are of the dimension $p \times p$. The discrete-time displacement vector $\boldsymbol{y}(t_k)$ is of dimension $p \times 1$. The discrete-time Gaussian white noise $\boldsymbol{e}(t_k)$ has the same dimension. Further, $\boldsymbol{e}(t_k)$ has zero mean and a second-order moment described be the covariance matrix Λ.

This particular model is referred to as an ARMAV(n,n) model. It can be represented equivalently by a stochastic state space system of the form

$$\begin{aligned}\boldsymbol{x}(t_{k+1}) &= \boldsymbol{A}\boldsymbol{x}(t_k)+\boldsymbol{B}\boldsymbol{e}(t_k)\ , \quad \boldsymbol{e}(t_k)\in NID(\boldsymbol{0},\Lambda) \\ \boldsymbol{y}(t_k) &= \boldsymbol{C}\boldsymbol{x}(t_k)+\boldsymbol{e}(t_k)\end{aligned} \tag{3}$$

where $\boldsymbol{x}(t_k)$ isan $np \times 1$ dimensional state vector. The state matrix $\boldsymbol{A}$, the stochastic input matrix $\boldsymbol{B}$ and the observation matrix $\boldsymbol{C}$ are defined as, see Andersen [4]

$$A = \begin{bmatrix} 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \\ -A_n & -A_{n-1} & -A_{n-2} & . & -A_1 \end{bmatrix}$$

$$B = \begin{bmatrix} I & 0 & 0 & . & 0 \\ A_1 & I & 0 & . & 0 \\ . & . & . & . & . \\ A_{n-2} & A_{n-3} & A_{n-4} & . & 0 \\ A_{n-1} & A_{n-2} & A_{n-3} & . & I \end{bmatrix}^{-1} \begin{bmatrix} C_1 - A_1 \\ C_2 - A_2 \\ C_3 - A_3 \\ . \\ C_n - A_n \end{bmatrix} \quad (4)$$

$$C = [I \ 0 \ 0 \ . \ 0]$$

The relation between the auto-regressive system matrices and the state space matrices A and C are given by

$$\begin{aligned} &[A_n \ A_{n-1} \ . \ . \ A_1] = -CA^n O^{-1}(n) \\ &O(n) = [C^T \ (CA)^T \ . \ (CA^{n-2})^T \ (CA^{n-1})^T]^T \end{aligned} \quad (5)$$

where $O(n)$ is the reduced observability matrix, see Andersen [1].

1.3 *Modal Parameter Estimation*

The modal parameters can be extracted from the modal decomposed state matrix A.

$$A = \Psi \mu \Psi^{-1}, \quad \mu = diag\{\mu_i\} \quad (6)$$

The modal decomposition is described by the np eigenvectors, which are the columns of the matrix Ψ, and by the np eigenvalues μ_i located in the diagonal of the matrix μ. The eigenvectors Ψ_j are constructed from the mode shapes Φ_j and the eigenvalues μ_j as

$$\Psi = \begin{bmatrix} \Phi_1 & . & . & \Phi_{np} \\ \mu_1 \Phi_1 & . & . & \mu_{np} \Phi_{np} \\ . & . & . & . \\ \mu_1^{n-1} \Phi_1 & . & . & \mu_{np}^{n-1} \Phi_{np} \end{bmatrix} \quad (7)$$

The mode shapes, natural eigenfrequencies and damping ratios of the continuous-time system can therefore be extracted as

$$\Phi_j = C\Psi_j, \quad \{\mu_j, \mu_j^*\} = e^{\left(-2\pi f_j \zeta_j \pm i 2\pi f_j \sqrt{1-\zeta_j^2}\right)T} \quad (8)$$

where $j=1, \ldots, s$. Since all modes are assumed underdamped, $s = np/2$.

1.4 *The Prediction Error Method*

The parameter estimates, based on N samples, and returned in $\hat{\theta}_N$ can be obtained as the global minimum point of the criterion function

$$V_N(\theta) = det\left(\frac{1}{N} \sum_{k=1}^{N} \varepsilon(t_k, \theta) \varepsilon^T(t_k, \theta) \right) \quad (9)$$

The model parameter vector θ is determined so that the prediction error, defined as

$$\varepsilon(t_k, \theta) = y(t_k) - \hat{y}(t_k | t_{k-1}; \theta) \quad (10)$$

is as small as possible. $\hat{y}(t_k | t_{k-1}; \theta)$ is the one-step ahead predicted system response. The parameter vector θ can be separated into the two $m \times 1$ parameter vectors θ^A and θ^C which are organised in the following way

$$\begin{aligned} &\theta^A = col([A_1 \ . \ . \ A_n]), \quad \theta^C = col([C_1 \ . \ . \ C_n]) \\ &\theta = col([\theta^A \ \theta^C]) \end{aligned} \quad (11)$$

where *col* means stacking of all columns of the argument matrix. The total number of adjustable parameters in θ is as such $2m = 2np^2$.

The predictor of the ARMAV(n,n) model is defined as

$$\begin{aligned} \hat{y}(t_k | t_{k-1}; \theta) = &-A_1(\theta) y(t_{k-1}) - \ldots - A_n(\theta) y(t_{k-n}) + \\ &C_1(\theta) \varepsilon(t_{k-1}, \theta) + \ldots + C_n(\theta) \varepsilon(t_{k-nc}, \theta) \end{aligned} \quad (12)$$

This relation reveals that the predictor of the ARMAV model is non-linear, since the prediction errors themselves depend on the parameter vector θ. This implies that an iterative minimization procedure such as the following Gauss-Newton search scheme has to be applied.

$$\begin{aligned} &\hat{\theta}_N^{i+1} = \hat{\theta}_N^i + \mu_i R^{-1}(\hat{\theta}_N^i) F(\hat{\theta}_N^i) \\ &R(\theta) = \sum_{k=1}^{N} \psi(t_k, \theta) Q_N^{-1}(\theta) \psi^T(t_k, \theta) \\ &F(\theta) = \sum_{k=1}^{N} \psi(t_k, \theta) Q_N^{-1}(\theta) \varepsilon(t_k, \theta) \\ &Q_N(\theta) = \frac{1}{N} \sum_{k=1}^{N} \varepsilon(t_k, \theta) \varepsilon^T(t_k, \theta) \\ &\psi(t_k, \theta) = \frac{\partial \hat{y}^T(t_k | t_{k-1}; \theta)}{\partial \theta} \end{aligned} \quad (13)$$

The dimensions of $R(\theta)$ and $F(\theta)$ are $2m \times 2m$ and $2m \times 1$, respectively. μ_i is a bisection constant that adjusts the step size. $\psi(t_k, \theta)$ is the gradient of the predictor (12), i.e. the derivative of (12) with respect to each of the adjustable parameters of the ARMAV model. At each time step this gradient forms an $2m \times p$ dimensional matrix.

The estimate of the parameters of the ARMAV model can as such be calculated by supplying an initial parameter estimate. On the basis of this the prediction errors can be calculated, the matrix $R(\theta)$ and the vector $F(\theta)$ can be calculated. An updated estimate can then be calculated

using (12). This method is called the prediction error method (PEM) since it is the prediction errors that are minimized, see Ljung [6].

For Gaussian distributed prediction errors this method is asymptotically efficient. A standard for the estimation errors of such an estimator is provided by the Cramer-Rao lower bound of variance.

This standard is utilized by the model parameter covariance matrix of the difference between the true parameters θ_0 and estimated parameters $\hat{\theta}_N$ as N tends to infinity, i.e. $\boldsymbol{P}_\theta(\hat{\theta}_N) = E[(\theta_0 - \hat{\theta}_N)(\theta_0 - \hat{\theta}_N)^T]$. An estimate of $\boldsymbol{P}_\theta(\hat{\theta}_N)$ is provided by the Hessian matrix as, Andersen [4] and Ljung [6]

$$\hat{\boldsymbol{P}}_\theta(\theta) = \boldsymbol{R}^{-1}(\theta) \tag{14}$$

The covariance of the auto-regressive parameters depends on the estimation uncertainties of the auto-regressive parameters as well as the moving average parameters. This is easy to realise from the following block matrix formulation of (14)

$$\begin{aligned} \hat{\boldsymbol{P}}(\theta) &= \begin{bmatrix} \hat{\boldsymbol{P}}_{AA}(\theta) & \hat{\boldsymbol{P}}_{AC}(\theta) \\ \hat{\boldsymbol{P}}_{CA}(\theta) & \hat{\boldsymbol{P}}_{CC}(\theta) \end{bmatrix} \\ \boldsymbol{R}(\theta) &= \begin{bmatrix} \boldsymbol{R}_{AA}(\theta) & \boldsymbol{R}_{AC}(\theta) \\ \boldsymbol{R}_{CA}(\theta) & \boldsymbol{R}_{CC}(\theta) \end{bmatrix} \\ \hat{\boldsymbol{P}}_{AA}(\theta) &= \left(\boldsymbol{R}_{AA}(\theta) - \boldsymbol{R}_{AC}(\theta)\boldsymbol{R}_{CC}^{-1}(\theta)\boldsymbol{R}_{CA}(\theta)\right)^{-1} \end{aligned} \tag{15}$$

It is therefore important to estimate the moving average correctly.

2 ESTIMATION OF MODAL UNCERTAINTIES

In general, the change of parameterization from a set of auto-regressive parameters, given in an $m \times 1$ dimensional vector θ^A, to another set of physical parameters, given in an $r \times 1$ dimensional vector κ, can be performed by a known r-dimensional functional relation

$$\kappa = f(\theta^A) \tag{16}$$

The functional relationship between the auto-regressive parameters and the modal parameters is given by the eigenvalue problem followed by the calculation of the modal parameters. This means that the resulting functional relation between θ^A and κ is highly non-linear.

2.1 *Approximation using a 1st. Order Taylor Expansion*

To obtain a practically applicable approach, (16) is usually linearized using a first-order generalized Taylor expansion at the operating point $(\hat{\kappa}_N, \hat{\theta}_N^A)$, Andersen [5].

This linearization can either be performed as

$$\begin{aligned} \kappa &= \hat{\kappa}_N + \left(\frac{\partial f(\theta^A)}{\partial \theta^A} \right)_{|\theta^A = \hat{\theta}_N^A} \left(\theta^A - \hat{\theta}_N^A\right) \\ &= \hat{\kappa}_N + \boldsymbol{J}(\hat{\theta}_N^A)\left(\theta^A - \hat{\theta}_N^A\right) \end{aligned} \tag{17}$$

or as

$$\begin{aligned} \theta^A &= \hat{\theta}_N^A + \left(\frac{\partial f^{-1}(\kappa)}{\partial \kappa} \right)_{|\kappa = \hat{\kappa}_N} \left(\kappa - \hat{\kappa}_N\right) \\ &= \hat{\theta}_N^A + \boldsymbol{G}(\hat{\kappa}_N)\left(\kappa - \hat{\kappa}_N\right) \end{aligned} \tag{18}$$

For simplicity, it is assumed that all modes s are underdamped and that mode shapes are normalized with respect to their pth element. There will therefore only be p-1 real and p-1 imaginary mode shape elements. The elements of $\hat{\kappa}_N$ can therefore be defined as

$$\hat{\kappa}_N = \begin{bmatrix} \hat{\kappa}_N^1 \\ \hat{\kappa}_N^2 \\ . \\ . \\ \hat{\kappa}_N^s \end{bmatrix}, \quad \hat{\kappa}_N^j = \begin{bmatrix} f_j \\ \zeta_j \\ re(\Phi_{j,1}) \\ . \\ re(\Phi_{j,p-1}) \\ im(\Phi_{j,1}) \\ . \\ im(\Phi_{j,p-1}) \end{bmatrix} \tag{19}$$

$\boldsymbol{J}(\hat{\theta}_N^A)$ and $\boldsymbol{G}(\hat{\kappa}_N)$ are Jacobian matrices of partial derivatives

$$\begin{aligned} \boldsymbol{J}(\theta^A) &= \begin{bmatrix} \frac{\partial f_1(\theta^A)}{\partial \theta_1^A} & . . & \frac{\partial f_1(\theta^A)}{\partial \theta_m^A} \\ . & . . & . \\ \frac{\partial f_r(\theta^A)}{\partial \theta_1^A} & . . & \frac{\partial f_r(\theta^A)}{\partial \theta_m^A} \end{bmatrix} \\ \boldsymbol{G}(\kappa) &= \begin{bmatrix} \frac{\partial f_1^{-1}(\kappa)}{\partial \kappa_1} & . . & \frac{\partial f_1^{-1}(\kappa)}{\partial \kappa_r} \\ . & . . & . \\ \frac{\partial f_m^{-1}(\kappa)}{\partial \kappa_1} & . . & \frac{\partial f_m^{-1}(\kappa)}{\partial \kappa_r} \end{bmatrix} \end{aligned} \tag{20}$$

which should both be evaluated at the operating point $(\hat{\kappa}_N, \hat{\theta}_N^A)$.

2.2 *A Simple but Slow Approach*

The covariance matrix $\boldsymbol{P}_{\kappa}(\hat{\kappa}_N)$ of the deviation of $\hat{\kappa}_N$ from the true modal parameters can be estimated by

$$\begin{aligned}\hat{\boldsymbol{P}}_{\kappa}(\hat{\kappa}_N) &= E\left[\left(\kappa_0 - \hat{\kappa}_N\right)\left(\kappa_0 - \hat{\kappa}_N\right)^T\right] \\ &= \boldsymbol{J}(\hat{\theta}_N^A)\boldsymbol{P}_{AA}(\hat{\theta}_N)\boldsymbol{J}^T(\hat{\theta}_N^A)\end{aligned} \tag{21}$$

The estimated covariance matrix $\hat{\boldsymbol{P}}_{AA}(\hat{\theta}_N)$ obtained from (14) can then be inserted instead of $\boldsymbol{P}_{AA}(\hat{\theta}_N)$. What remains is to calculate the Jacobian matrix $\boldsymbol{J}(\hat{\theta}_N^A)$. The first step is therefore to define the function $\kappa = f(\theta^A)$.

Definition 1

□ The function $\kappa = f(\theta^A)$ can be divided into 5 steps as

Step 1. $\left[\boldsymbol{A}_1 \;\; \boldsymbol{A}_2 \;\; . \;\; . \;\; \boldsymbol{A}_{n-1} \;\; \boldsymbol{A}_n\right] = assem1(\theta^A)$

Step 2. $$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} & . & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} & . & \boldsymbol{0} \\ . & . & . & . & . \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & . & \boldsymbol{I} \\ -\boldsymbol{A}_n & -\boldsymbol{A}_{n-1} & -\boldsymbol{A}_{n-2} & . & -\boldsymbol{A}_1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & . & \boldsymbol{0} \end{bmatrix} \tag{22}$$

Step 3. $\boldsymbol{\Psi}\boldsymbol{\mu}\boldsymbol{\Psi}^{-1} = \boldsymbol{A}$

Step 4. $\boldsymbol{\Phi}_j = \boldsymbol{C}\boldsymbol{\Psi}_j$, $\{\mu_j, \mu_j^*\} = e^{\left(-2\pi f_j\zeta_j \pm i2\pi f_j\sqrt{1-\zeta_j^2}\right)T}$

Step 5. $\kappa = assem2(\left[f_1 \;\; \zeta_1 \;\; \boldsymbol{\Phi}_1 \;\; . \;\; . \;\; f_s \;\; \zeta_s \;\; \boldsymbol{\Phi}_s\right])$

where *assem1* is a virtual function that should return the auto-regressive parameters when given the vector θ^A as input. *assem2* is another virtual function that should return the vector κ when given the modal parameters as input. □

Analytical calculation of the partial derivatives of this function is in general impossible even for small model structures, since the calculations include solution of a series of eigenvalue problems. However, a simple way to calculate the Jacobian is by numerical differentiation using the central difference theorem. The ith column of $\boldsymbol{J}(\hat{\theta}_N^A)$ can then be calculated by

$$\boldsymbol{J}_i(\hat{\theta}_N^A) = \frac{f(\hat{\theta}_N^A + \boldsymbol{P}) - f(\hat{\theta}_N^A - \boldsymbol{P})}{2\boldsymbol{P}_i} \tag{23}$$

where $\boldsymbol{P}$ is an $m \times 1$ vector whose elements all are zero except for the ith element $\boldsymbol{P}_i$ that contains a small number. This number results in a small perturbation of the ith element of $\hat{\theta}_N^A$. The modal decomposition and the calculation of the modal parameters must therefore be repeated $2m$ times which makes this a slow approach. A more comprehensive description of this approach is given in Andersen [4].

2.3 *An Advanced and Fast Approach*

The estimated covariance matrix $\hat{\boldsymbol{P}}_{\kappa}(\hat{\kappa}_N)$ can also be obtained in another way which does not rely on numerical differentiation. The idea is to construct $\boldsymbol{G}(\hat{\kappa}_N)$ analytically instead of constructing $\boldsymbol{J}(\hat{\theta}_N^A)$ numerically.

Definition 2

□ The function $\theta^A = f^{-1}(\kappa)$ can be divided into 7 steps as

Step 1. $\left[f_1 \;\; \zeta_1 \;\; \boldsymbol{\Phi}_1 \;\; . \;\; . \;\; f_s \;\; \zeta_s \;\; \boldsymbol{\Phi}_s\right] = assem(\kappa)$

Step 2. $\{\mu_j, \mu_j^*\} = e^{\left(-2\pi f_j\zeta_j \pm i2\pi f_j\sqrt{1-\zeta_j^2}\right)T}$

Step 3. $$\boldsymbol{\Psi} = \begin{bmatrix} \boldsymbol{\Phi}_1 & . & . & \boldsymbol{\Phi}_{np} \\ \mu_1\boldsymbol{\Phi}_1 & . & . & \mu_{np}\boldsymbol{\Phi}_{np} \\ . & . & . & . \\ \mu_1^{n-1}\boldsymbol{\Phi}_1 & . & . & \mu_{np}^{n-1}\boldsymbol{\Phi}_{np} \end{bmatrix}, \quad \boldsymbol{\mu} = diag(\mu_j)$$

Step 4. $\boldsymbol{A} = \boldsymbol{\Psi}\boldsymbol{\mu}\boldsymbol{\Psi}^{-1}$, $\boldsymbol{C} = \left[\boldsymbol{I} \;\; \boldsymbol{0} \;\; . \;\; . \;\; \boldsymbol{0}\right]$

Step 5. $\boldsymbol{O}(n) = \left[\boldsymbol{C}^T \;\; (\boldsymbol{CA})^T \;\; . \;\; (\boldsymbol{CA}^{n-2})^T \;\; (\boldsymbol{CA}^{n-1})^T\right]^T$

Step 6. $\left[\boldsymbol{A}_n \;\; \boldsymbol{A}_{n-1} \;\; . \;\; . \;\; \boldsymbol{A}_1\right] = -\boldsymbol{CA}^n\boldsymbol{O}^{-1}(n)$,

Step 7. $\theta^A = col(\left[\boldsymbol{A}_1 \;\; . \;\; . \;\; \boldsymbol{A}_n\right])$ (24)

where *assem* is a virtual function that should return the modal parameters when given the vector κ as input. □

From (15) and by using the chain rule the Hessian matrix of the modal parameter vector κ and the moving average parameters is given by

$$\begin{aligned}\boldsymbol{R}(\kappa, \theta^C) &= \begin{bmatrix} \boldsymbol{G}(\kappa)\boldsymbol{R}_{AA}(\theta)\boldsymbol{G}^T(\kappa) & \boldsymbol{G}(\kappa)\boldsymbol{R}_{AC}(\theta) \\ \boldsymbol{R}_{CA}(\theta)\boldsymbol{G}^T(\kappa) & \boldsymbol{R}_{CC}(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{R}_{\kappa\kappa}(\theta) & \boldsymbol{R}_{\kappa C}(\theta) \\ \boldsymbol{R}_{C\kappa}(\theta) & \boldsymbol{R}_{CC}(\theta) \end{bmatrix}\end{aligned} \tag{25}$$

The covariance of the modal parameter vector is therefore obtained from (14) as

$$\hat{\boldsymbol{P}}_{\kappa}(\kappa, \theta) = \left(\boldsymbol{R}_{\kappa\kappa}(\theta) - \boldsymbol{R}_{\kappa C}(\theta)\boldsymbol{R}_{CC}^{-1}(\theta)\boldsymbol{R}_{C\kappa}(\theta)\right)^{-1} \tag{26}$$

Essentially, $\boldsymbol{G}(\hat{\kappa}_N)$ is constructed by partial differentiation of steps 2 to 6 with respect to the parameters κ, which for the jth eigenvalue are f_j, ζ_j, $\boldsymbol{\Phi}_{jl,R}$ and $\boldsymbol{\Phi}_{jl,I}$. The mode shape subscript of the jth mode shape signifies the real or imaginary parts of the lth row coordinate.

Differentiation of Step 2

The derivatives of step 2 with respect to f_j, ζ_j, $\boldsymbol{\Phi}_{jl,R}$ and $\boldsymbol{\Phi}_{jl,I}$ are given by

$$\frac{\partial \mu_j}{\partial f_j} = 2\pi T \mu_j \left(-\zeta_j + i\sqrt{1-\zeta_j^2} \right)$$

$$\frac{\partial \mu_j}{\partial \zeta_j} = 2\pi T \mu_j \left(-f_j - i\frac{f_j \zeta_j}{\sqrt{1-\zeta_j^2}} \right) \tag{27}$$

$$\frac{\partial \mu_j}{\partial \Phi_{jl,R}} = 0 \ , \quad \frac{\partial \mu_j}{\partial \Phi_{jl,R}} = 0$$

For simplicity the two parameters f_j and ζ_j will be substituted by one parameter ξ_j. The differentiation with respect to ξ_j is therefore differentiation with respect to f_j and ζ_j.

Differentiation of Step 3

Define $\mathbf{0}_{n,m}$ as an $n \times m$ matrix filled with zeros. In the following this matrix will be used whenever a zero matrix having dimensions different from $p \times p$ is needed. The derivatives of Ψ and $\boldsymbol{\mu}$ with respect to ξ_j, $\Phi_{jl,R}$ and $\Phi_{jl,I}$ are then given by

$$\frac{\partial \boldsymbol{\mu}}{\partial \xi_j} = diag\left(\begin{bmatrix} \mathbf{0}_{1,j-1} & \frac{\partial \mu_j}{\partial \xi_j} & \mathbf{0}_{1,np-j} \end{bmatrix} \right) \ , \quad \frac{\partial \boldsymbol{\mu}}{\partial \phi_{jl}} = \mathbf{0}_{np,np}$$

$$\frac{\partial \Psi}{\partial \xi_j} = \begin{bmatrix} \mathbf{0}_{np,j-1} & \begin{bmatrix} \mathbf{0}_{p,1} \\ \frac{\partial \mu_j}{\partial \xi_j}\Phi_j \\ . \\ (n-1)\mu_j^{n-2}\frac{\partial \mu_j}{\partial \xi_j}\Phi_j \end{bmatrix} & \mathbf{0}_{np,np-j} \end{bmatrix}$$

$$\frac{\partial \Psi}{\partial \Phi_{jl,R}} = \begin{bmatrix} \mathbf{0}_{np,j-1} & \begin{bmatrix} \varphi_l \\ \mu_j \varphi_l \\ . \\ \mu_j^{n-1}\varphi_l \end{bmatrix} & \mathbf{0}_{np,np-j} \end{bmatrix} \ , \quad \varphi_l = \begin{bmatrix} \mathbf{0}_{l-1,1} \\ 1 \\ \mathbf{0}_{p-l,1} \end{bmatrix} \tag{28}$$

$$\frac{\partial \Psi}{\partial \Phi_{jl,I}} = \begin{bmatrix} \mathbf{0}_{np,j-1} & \begin{bmatrix} \psi_l \\ \mu_j \psi_l \\ . \\ \mu_j^{n-1}\psi_l \end{bmatrix} & \mathbf{0}_{np,np-j} \end{bmatrix} \ , \quad \psi_l = \begin{bmatrix} \mathbf{0}_{l-1,1} \\ i \\ \mathbf{0}_{p-l,1} \end{bmatrix}$$

Differentiation of Step 4

For simplicity the real and imaginary parts $\Phi_{jl,R}$ and $\Phi_{jl,I}$ of the mode shape coordinates will be replaced by one parameter χ_{jl}. The derivatives of $\boldsymbol{A}$ with respect to ξ_j and χ_{jl} are then given by

$$\frac{\partial \boldsymbol{A}}{\partial \xi_j} = \left(\frac{\partial \Psi}{\partial \xi_j}\boldsymbol{\mu} + \Psi\frac{\partial \boldsymbol{\mu}}{\partial \xi_j} - \boldsymbol{A}\frac{\partial \Psi}{\partial \xi_j} \right) \Psi^{-1}$$

$$\frac{\partial \boldsymbol{A}}{\partial \chi_{jl}} = \left(\frac{\partial \Psi}{\partial \chi_{jl}}\boldsymbol{\mu} - \boldsymbol{A}\frac{\partial \Psi}{\partial \chi_{jl}} \right) \Psi^{-1} \tag{29}$$

Differentiation of step 5

The derivatives of $\boldsymbol{O}(n)$ with respect to ξ_j and χ_{jl} are based the differentiation of increasing integer powers of $\boldsymbol{A}$. The differentiation of $\boldsymbol{A}^m$, $m = 1, 2, \ldots$ is given by

$$\frac{\partial \boldsymbol{A}^m}{\partial \xi_j} = \sum_{j=1}^{m} \boldsymbol{A}^{m-j}\frac{\partial \boldsymbol{A}}{\partial \xi_j}\boldsymbol{A}^{j-1} \ , \quad \frac{\partial \boldsymbol{A}^m}{\partial \chi_{jl}} = \sum_{j=1}^{m} \boldsymbol{A}^{m-j}\frac{\partial \boldsymbol{A}}{\partial \chi_{jl}}\boldsymbol{A}^{j-1} \tag{30}$$

The differentiation of $\boldsymbol{O}(n)$ then follows straightforwardly

$$\frac{\partial \boldsymbol{O}(n)}{\partial \xi_j} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \xi_j} \\ . \\ \boldsymbol{C}\frac{\partial \boldsymbol{A}^{n-1}}{\partial \xi_j} \end{bmatrix} \ , \quad \frac{\partial \boldsymbol{O}(n)}{\partial \chi_{jl}} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{C}\frac{\partial \boldsymbol{A}}{\partial \chi_{jl}} \\ . \\ \boldsymbol{C}\frac{\partial \boldsymbol{A}^{n-1}}{\partial \chi_{jl}} \end{bmatrix} \tag{31}$$

Differentiation of Step 6

Finally, the differentiation of the auto-regressive parameters with respect to ξ_j and χ_{jl} given by

$$\begin{bmatrix} \frac{\partial \boldsymbol{A}_n}{\partial \xi_j} & \frac{\partial \boldsymbol{A}_{n-1}}{\partial \xi_j} & . & . & \frac{\partial \boldsymbol{A}_1}{\partial \xi_j} \end{bmatrix} =$$

$$-\left(\boldsymbol{C}\frac{\partial \boldsymbol{A}^n}{\partial \xi_j} + \begin{bmatrix} \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & . & . & \boldsymbol{A}_1 \end{bmatrix}\frac{\partial \boldsymbol{O}(n)}{\partial \xi_j} \right) \boldsymbol{O}^{-1}(n)$$

$$\begin{bmatrix} \frac{\partial \boldsymbol{A}_n}{\partial \chi_{jl}} & \frac{\partial \boldsymbol{A}_{n-1}}{\partial \chi_{jl}} & . & . & \frac{\partial \boldsymbol{A}_1}{\partial \chi_{jl}} \end{bmatrix} = \tag{32}$$

$$-\left(\boldsymbol{C}\frac{\partial \boldsymbol{A}^n}{\partial \chi_{jl}} + \begin{bmatrix} \boldsymbol{A}_n & \boldsymbol{A}_{n-1} & . & . & \boldsymbol{A}_1 \end{bmatrix}\frac{\partial \boldsymbol{O}(n)}{\partial \chi_{jl}} \right) \boldsymbol{O}^{-1}(n)$$

What remains in order to construct the derivatives $\partial\theta^A/\partial\xi_j$ and $\partial\theta^A/\partial\chi_{jl}$ is to stack the derivatives of the auto-regressive matrix coefficients obtain in (32). The matrix $\boldsymbol{G}(\hat{\kappa}_N)$ can then be constructed by looping over all parameters of $\hat{\kappa}_N$, and the covariance in (26) can be calculated.

3 A SIMULATION STUDY

In order to demonstrate the performance of the two approaches and to compare the computational time needed in each case a simulation study has been performed. The system response of a Gaussian white noise excited 2 DOF linear system has been simulated 500 times. The natural eigenfrequencies and damping ratios of the two modes are presented in table 1.

Mode #	f_j [Hz]	ζ_j [%]
1	1.75	0.63
2	2.66	1.08

Table 1. Modal parameters of the system used in the simulation study.

Gaussian white noise has been added as external disturbance. The level of this disturbance is 10% of the standard deviation of the undisturbed system response. On the basis of each simulation an ARMAV(2,2) model has been calibrated using the prediction error method described in section 1.4. The standard deviations of the modal parameters have then been estimated using the two estimation approaches.

Because of the limited space the results of the mode shape estimates are omitted. In figures 1 and 2, the standard deviations of the first and second modes are shown. The standard deviations obtained from the two approaches are plotted together with the sampled standard deviations. For simulation number i all estimates from 1 to i have been used to calculate the sampled standard deviations. Therefore, the associated curve for the first number of simulations will show some transient behaviour.

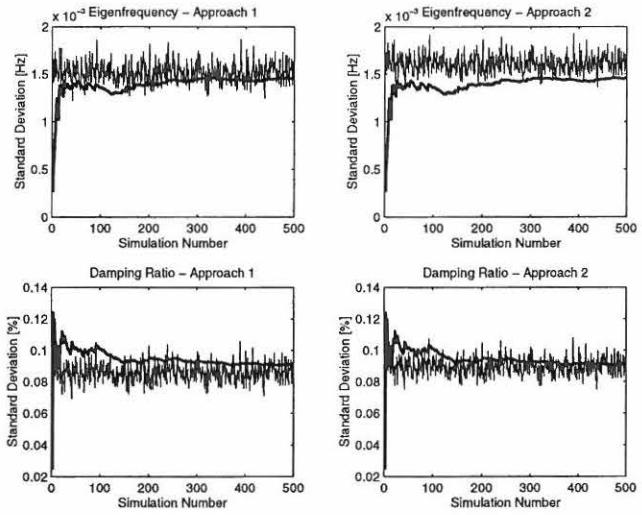

Figure 1. Estimated and sampled standard deviations of the natural eigenfrequency and damping ratio estimates of the first mode. On the left-hand side the comparison is between sampled results and estimated results of approach number 1. The right-hand side shows the sampled results compared with the estimated results of approach number 2.

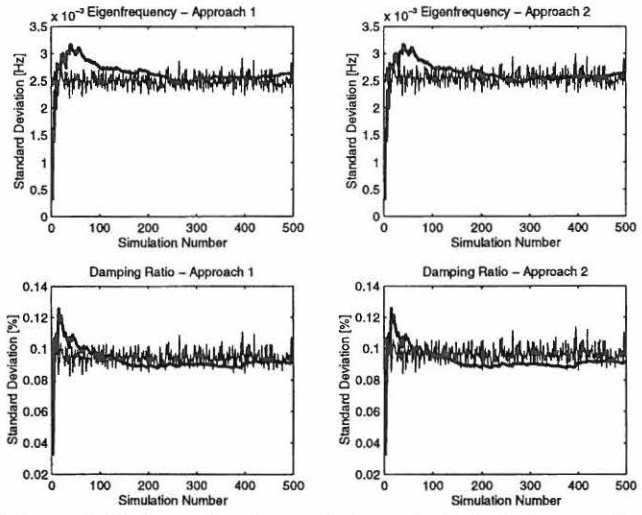

Figure 2. Estimated and sampled standard deviations of the natural eigenfrequency and damping ratio estimates of the second mode.

In this particular case the second approach is 10 times faster than the first approach. As seen the two approaches work almost equally well. A small bias is seen between sampled and estimated standard deviation. This bias is probably caused by the first order approximation and the fact that the model covariance matrix is only an estimate.

4 CONCLUSIONS

The intention of this paper has been to show how modal parameters and especially their estimation errors can be determined. The modal parameter estimation has been based on calibration of ARMAV models to time series data using the prediction error method.

Two approaches for estimation of modal uncertainties have been presented. In each approach, it is the covariance matrix of the modal parameters that has been estimated. This estimation is based on a first order Taylor expansion of the functional relationship between the auto-regressive parameters and the modal parameters.

The first approach involves numerical differentiation of this functional relationship. Implementation of this approach is easy. The second approach involves the construction of analytical differentiation. Implementation of this approach is much more difficult. However, the advantage of the second approach is that an estimate of the covariance matrix is obtained significantly faster than by the first approach.

The performance of the two approaches has been compared by means of a simulation study. This simulation study has shown that the second approach is 10 times faster, and it indicates that the two approaches work almost equally well.

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