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Research and development in the teaching and learning of number systems and arithmetic

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PROCEEDINGS OF

ICME-11 – TOPIC STUDY GROUP 10

RESEARCH AND DEVELOPMENT IN THE TEACHING AND LEARNING OF NUMBER SYSTEMS AND ARITHMETIC

11th INTERNATIONAL CONGRESS ON MATHEMATICAL EDUCATION

July 6-13, 2008
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Editors:

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Preface

The purpose of this ICME-11 Topic Study Group 10 (TSG-10), *Research and Development in the Teaching and Learning of Number Systems and Arithmetic*, is to gather congress participants who are interested in research and development in the teaching and learning of number systems and arithmetic, including operations in the number systems, ratio and proportion, and rational numbers. The focus of the group is broad and includes issues such as the development of number sense in students, the role of contexts and models in teaching and learning about numbers and arithmetic, and the development of teaching/learning units that connect basic arithmetic skills with higher order thinking skills. From an international perspective, we will study and discuss advances in research and practice, new trends, and the state-of-the-art. We also hope that putting together this Proceeding will make it possible to make the congress participants' experience available to people not able to attend ICME-11 or this specific TSG.

We are very proud to have been able to collect ten high quality papers around various issues related the teaching and learning of number systems and arithmetic. These papers come from participants from all over the world and are centred on issues within addition, multiplication, division, fractions, and integers etc. The papers have gone a review process by the Organizational Team of this TSG. Some papers were accepted immediately, some were asked to do some revisions, others, six, were rejected. We regret having to reject those six papers, but within the allocated time for the TGS's we did not have room for more papers. But we would like to thank all who took the time and effort to submit papers for this TSG. Besides the ten accepted papers from congress participants, we also have two invited keynote presentations as well as smaller discussion papers from each of the four members of the Organizational Team of TSG-10.

Our Organizational Team originally consisted of five members. Due to illness, Martha Villalba (Mexico) unfortunately had to leave the group. We are very sorry that we did not get to know her but we hope that we will be able to meet her one day in the future.

The Proceedings of TSG-10 consists of four main parts corresponding to the four sessions that were allocated to the TSG's at ICME-11.

For the first session, we invited two internationally leading scholars in the field to give "state of the art" presentations related to the main issues of this TSG. Here prof. Zalman Usiskin presents *The arithmetic curriculum and the real world* and prof. Darcy Hallett presents *Effects of fraction situations and individual differences: A review of recent research regarding children's understanding of fractions.*

In the second and third sessions, papers around two themes are grouped. The first, *Multiplication, division, fraction* consists of six papers, while the second, *Addition and integers,* consists of four papers. The ten papers could have been shuffled differently, but the team decided that this grouping would give the best possible internal cohesion at each of the sessions.

At the fourth and final session, the chairs and members of the Organizational Team present some personal opinions and reflection on *Research and Development in the Teaching and Learning of Number Systems and Arithmetic* and on the different papers that were presented at the first three sessions.

The chairs and members of the Organizational Team thank all contributors to this Topic Study Group. We also want to thank prof. Marcela Santillán, chair of the IPC for ICME-11, and her team for giving us the opportunity to organise this ICME-11 Topic Study Group and for their assistance to make this enterprise a successful one.

Dirk De Bock (Belgium), chair
Bettina Dahl Søndergaard (Denmark), chair
Bernardo Gómez Alfonso (Spain)
Chun Chor Litwin Cheng (China, Hong Kong SAR)

Monterrey, July 2008

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The Arithmetic Curriculum and the Real World

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The relationships between abstract arithmetic and the real world are dealt with inconsistently in most curricula. Each of the common arithmetic operations is a mathematical model for counting and measure situations found in the real world. These models parallel the theoretical properties of the operations and provide the basis for more sophisticated models found in algebra, geometry, analysis, and statistics. The absence of explicit instruction in these models may explain why many children have difficulty applying arithmetic.

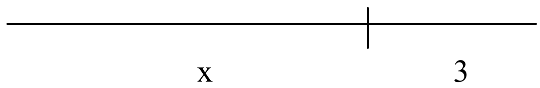
Keywords: arithmetic, modelling, curriculum, applications, mathematics

1 Introduction

I was asked by the organizers to help start the conversation about research in the teaching and learning of number systems and arithmetic by presenting some remarks based on a paper I wrote for the ICMI volume on mathematical modelling and applications and to give an assessment of the state-of-the-art on this issue. I am honoured to have been so asked. The first part of this task – to present the remarks based on a previously-published paper – is easy for me. The second part of this task – the present the state-of-the-art – is difficult. One of the reasons we come to ICME meetings is to learn what is going on. So I hope that if you are doing work related to the subject of my talk or you know of some related work, you will let everyone here know. It might be possible to mention that work in the final version of this paper for the report of TSG 10.

2 Motivation for the research

I began my work with the teaching of applications and modelling thirty years ago with specific attention to school algebra, which in the U.S. is taught in the 9th grade (Usiskin 1979). At that time, contrived problems involving age (Mary is half as old as her father was…), digits, and slightly more realistic but still contrived problems involving distance-rate-time and mixtures were the only things students saw in algebra that in any way connected with applications. I first replaced the contrived problems with problems involving population growth, finances, sports, basic geometry, and other contexts that were quite understandable to my students. But, even with familiar contexts, my students had great difficulty knowing what to do when faced with a new problem. For instance, if a segment of length x cm was placed next to a segment of known length 3 cm, they would guess that the total length was x+3 or 3x or x3 (*not* x^3), not having any idea or any way to check whether their answer was correct. I found it baffling that they needed to memorize the formula for the perimeter of a triangle.



If they were faced with a problem involving distance, rate, and time, they would memorize d = rt (distance = rate X time) but not have any idea why this formula works. Except for the formula C = np (cost = number of items purchased times the price of a single item), they were unlikely to understand any formula and were forced to memorize it. The formula C = np they understood because it generalized the view they had of multiplication as repeated addition.

They would try to latch onto key words – after all, if a problem is in words, what else is there to latch on to? – but if the words were not familiar, they had nothing else to fall back on. And, fundamentally, my probing found out that they could not apply algebra because they could not apply much of the arithmetic on which the algebra was based. For instance, they might be able to use counting to find the distance between -2 and 4 on a number line, but they would not connect this with the amount of increase in a temperature from -2° to 4°, and they would not have any idea that they could have used subtraction to determine the distance. When a situation involved anything other than small whole numbers, they were likely to be helpless. As a consequence, the translation into algebra was to them a matter of memorization rather than generalization from something they understood.

Other than simple situations involving whole numbers, many students are helpless when it comes to relating mathematics and the real world. The meanings of operations most students are taught do not extend beyond whole numbers and sometimes not beyond small whole numbers.

3 Number Systems, Arithmetic, and Models

The title of this TSG, with the language “number systems and arithmetic” was clearly chosen to allow research into almost aspect of the teaching and learning of number ideas. When we think of *number systems*, we think mathematically of integral domains and fields, with properties such as commutativity and associativity and the distributive laws. We think of numeration with decimals and base 10 and perhaps other bases. When we think of *arithmetic*, we think of the basic facts and algorithms associated with addition, subtraction, multiplication and division of whole numbers, fractions, decimals, and percents, and later with some irrational numbers and the real numbers.

Thus we do not usually associate either number systems or arithmetic with knowledge of the *uses of arithmetic*. Nor do we typically associate number systems and arithmetic with *mathematical modelling*, the connection of real-world problem situations with their mathematical counterparts. Conversely, most discussions of mathematical modelling in the curriculum occur with discussions of functions and other mathematical concepts studied at the secondary and tertiary levels, far away from discussions of the learning of arithmetic.

The word “model” is used in many ways in discussions of early mathematics learning. In the context of this paper, a mathematical model of a real situation is a bit of mathematics whose structure is isomorphic or nearly isomorphic to the structure of the situation, so that the mathematics can be employed to answer questions about the situation. For example, if we stack paper cups one on top of the other and we want to know how high the stack will be, an appropriate mathematical model is of the form of the linear equation h = b + an, where h is the height of the stack, b is the height of one cup, n is the number n of cups in the stack and a is the additional height caused by adding one more cup. An inappropriate mathematical model would be $h = bn$, that is to obtain the height by multiplying the height of one cup by the number of cups.

The modelling process is often described as starting with a situation in the real world that one wants to resolve, perhaps simplifying it in order to be studied, translating the situation into a mathematics model of it, working within the model to resolve the situation, translating back into the real world, checking the feasibility of the solution, and if feasible, one is done. I think one reason that mathematical modelling language is not used in discussing the learning of arithmetic is that the real-world situation is so often a counting situation and the mathematical model is not just a rough approximation but precisely isomorphic to the real situation. Thus the emphasis tends to be on precise translation of words rather than on the conceptual structure of a real situation. Thus mathematical models underlie the learning of the uses of arithmetic, but the concepts and language of modelling are often absent, distracted by thinking of every use as a "word problem" or by relating uses to algorithms or numeration. Perhaps this is why many students have difficulty connecting the mathematics they learn with the real world around them, despite the ubiquity of mathematics in that world.

4 Models involving addition

If you have 3 cookies and I have 5 cookies, then together we have 8 cookies. This type of situation is so common that we give it or its generalization to x cookies and y cookies little thought. Throughout the world, students are first introduced to this application of addition by the end of first grade. The model is used to teach students the basic addition facts. Later, because of the number and universality of counting situations like this one, students are asked to memorize answers when x and y are small whole numbers, and to learn algorithms for obtaining answers when x and y are large whole numbers.

In the discussion document for the ICMI 14 study, a useful distinction was made between an application and modelling. In an application of mathematics, we know the mathematics and apply it to the real world. In modelling, we begin with the real world situation and look for some mathematics that we might apply to the situation. In the cookie-counting situation, both directions occur. With very young learners, we use counting to define addition (modelling) as much as we use addition to obtain the total count (application). But the language and the essence of the model is seldom described to young students. It is a fundamental property of addition: If A and B are finite sets with $N(A) = a$ and $N(B) = b$, then $N(A \cup B) = a + b$. It is a property as fundamental as the commutative property or any other property of the operation.

After students apply the mathematical model of addition to answer certain counting problems involving small whole numbers, they are asked to apply the same model to situations such as populations in which the numbers are larger (and the answer cannot be found quickly by counting), to financial situations where the numbers are often written as decimals, and to recipes or probabilities where the numbers are written as fractions.

Students become so accustomed to the model that they apply it where it does not apply, to situations like the following:

(1) Carl has 5 friends and Georges has 6 friends. Carl and Georges decide to give a party together. They invite all their friends. All friends are present. How many friends are there at the party? (Verschaffel et al., 2000, p. 19).

(2) The price of a chair is reduced 20% on sale, and then its sale price is reduced by 10%. What is the total reduction?

(3) A cup of milk is added to a cup of popcorn. How many cups of the mixture will result? (Davis and Hersh, 1981, p. 71)

(4) What will be the temperature of water in a container if you pour 1 jug of water at 80˚ and 1 jug of water at 40˚F into it? (Nesher, 1980, p. 46)

The resolution of the incorrect application is different in each of these situations but is similar to the resolution of more advanced models. To resolve (1), we refine the model to encompass situations in which there is overlap: $N(A \cup B) = N(A) + N(B) - N(A \cap B)$. For (2), though we could use a generalization of this refined model (the reduction is 10% + 20% – 10% X 20%), the situation is more simply described by changing to a multiplication model. Think of a reduction of 20% as a *size change* or *scale factor* of 80%. Then the situation calls for the application of scale factors of 80% and 90%, for a total reduction of 80% X 90%, or 72%, a reduction of 28%. The resolutions of (3) and (4) are more complex and must take into account the chemistry and physics, respectively of the situations.

In geometry, the model is generalized to determine the total length of segments placed end to end, as in calculating perimeter. It is applied to determine the angle measure of some angles formed by the outer rays of two adjacent angles, the area of the union of disjoint planar regions, and the volume of the union of disjoint 3-dimensional solids. In these situations, the property may be called *Angle Addition* or an *Additive Property of Area* or an *Additive Property of Volume*. And then, in the study of combinatorics or probability, the model is typically identified as a *Fundamental Counting Principle*. Mathematics educators identify this model as the *Putting-Together Model for Addition*.

Thus through all of schooling in mathematics, this single model appears, but its appearance is found in various forms and quite different settings. For this reason, to most students, these applications do not share a commonality. The student misses an extraordinarily important point: *Addition is taught, and knowledge of addition is required of all students, because of the ubiquity of important applications of the Putting-Together Model for Addition.*

The Putting-Together Model for Addition does not encompass all of the applications in which addition of numbers is involved. Suppose a temperature of -4°C were to increase by 15°. We find the answer by the addition -4 + 15 = 11. This addition can be interpreted as a putting-together situation only if one stretches the idea of putting together. It is easier to think of this as the same mathematical model applied to a different set of situations, those involving *slides* or *shifts*. In fact, in most textbooks, the geometric idea of slides is used to reinforce or to determine the rules for addition of positive and negative numbers. Students see the geometry as a device or rule to obtain sums and do not realize that, through this process, addition is again a model for a set of real situations. We call it the Slide (or Shift) Model of Addition: If a slide x is followed by a slide y, the result is a slide x+y. This model accounts for applications of complex number and vector addition but its study begins in late primary or early secondary school.

5 Models involving subtraction

Two models for subtraction have long been in the literature: take-away and comparison.

Take-Away Model: If a quantity y is taken away from an original quantity x, the quantity left is x–y.

Comparison Model: The quantity x–y tells how much y is less than the quantity x.

In English, the two most common names for the answer (*"remainder"* for a take-away situation, *"difference"* for a comparison) reflect the different feels that these models have to the user.

These models are first encountered in small whole-number situations but later extended to any positive numbers (and for comparison, to any real numbers) and to the geometry of length, area, and volume. Comparison has its own special cases: change and directed error, and (with the help of absolute value) undirected error and distance on the number line. Thus, as with the addition models, these models appear in different forms and settings, so that the learner does not usually realize the common features.

Some books treat the Putting-Together Model for Addition and the Take-Away Model for Subtraction as a single model: Part-Part-Whole. In the same way, the Slide Model for Addition can be joined with the Comparison Model for Subtraction as a single model: Start-Shift-Finish. This joining of models is analogous to the usual relationship between addition and subtraction in mathematical theory ($a - b = c$ if and only if $a = c + b$), where subtraction is defined in terms of addition and is not treated by itself. The other way of defining subtraction in terms of addition ($a - b = a + -b$), which students encounter when subtracting positive and negative numbers, also is interpretable by models for the operations. A situation in which a temperature of 10° goes down 7° can be viewed as Addition Shift $10 + -7$ or as a Subtraction Shift (a new model) $10 - 7$. Thus there is a structure to the common models of addition and subtraction that complements the mathematical theory of these operations.

Some researchers distinguish variants of a problem that I would view as employing the same mathematical model. For instance, Fuson (2003) identifies 22 types of addition and subtraction problems without going beyond whole numbers. In contrast, some textbooks try to force every application situation of each operation into one model. Some textbook series in the United States have begun to take a middle road, identifying these models of addition, if not on student pages, in the commentary given to teachers.

6 Models involving multiplication

Unlike addition, where the counting of real objects (even if the objects are fingers!) is almost universally present in the learning of addition facts, multiplication tends to be defined theoretically as repeated addition. As a consequence, multiplication "facts" are often memorized by students with little if any real world situation to check them. This is unfortunate, because multiplication models rich and important situations. The lack of connection with real-world situations also results in children missing out on the development of multiplicative reasoning patterns (see, e.g., Harel and Confrey (1994)).

One way of categorizing the models of multiplication is by the *units* of the quantities being multiplied. Multiplication by a scalar covers a class of applications grouped as the Size Change Model of Multiplication: When a quantity x is multiplied by a scalar k, $k \neq 0$, then the product kx is k times the size of the original. Scalar multiplication includes among its applications the "part of" situations resulting from wanting fraction or a percent of a quantity, the "times as many" situations resulting from wanting to enlarge a quantity by a certain factor, and the size change transformations in geometry that result in similar figures. Discounts, taxes, simple interest, scale models, expansions, and contractions all fall under this framework. Repeatedly multiplying by different scalars underlies the multiple discount problem mentioned earlier as an application of putting-together addition. Repeatedly multiplying by the same scalar leads to discrete models of exponentiation as are used in the calculation of compound interest. By viewing multiplication by -1 as changing directions 180°, the geometry can be extended to explain multiplication by negative numbers. This further extends to the view of multiplication by the complex number z as combining a size change of |z| with a rotation of Arg(z).

Multiplication by a quantity with a unit also covers a broad class of applications. One type in this class is the Area Model of Multiplication: The area of a rectangle with length x units and width y units is xy square units. The discrete version is sometimes used to check a multiplication fact: The number of elements in a rectangular array with x rows and y columns is xy. (As with area, in this discrete case the unit of the product is still the product of the units of the factors. For instance, the number of outfits possible with 3 blouses and 4 skirts, then we can view the word "outfit" as an abbreviation for "blouse-skirt". Some books make this point by asking for blouse-skirt combinations.) The volume of a rectangular solid extends this model to three dimensions.

In calculating the area of a rectangle, we may think of its length as acting across its width. By summing many rectangles and taking a limit, the area model generalizes to give the area interpretation of direct integrals in calculus. When one factor is a rate (as in the formula distance = rate X time, or total cost = number of items X unit cost, then the rate acts across the other quantity. This describes the Acting-Across or Rate-Factor Model of Multiplication: When a rate *x* of unit 1 per unit 2 acts across a quantity *y* of unit 2, the total is xy unit 1.

Let me restress the point. We often teach students that they can *picture* multiplication by an array or by area, or *think of* multiplication of fractions by stretching and shrinking. That suggests that we learn applications in order to learn multiplication. Yet *multiplication is taught, and knowledge of multiplication is required of all students, because of the ubiquity of important applications of various models of multiplication.* The many applications for multiplication supply the reason that we ask students to learn multiplication facts and algorithms.

A student equipped with these multiplicative models is far more likely to understand why when y = kx, y varies directly as x, or why the total number of students in a school can be found by multiplying the average number of students per grade by the number of grade. Unfortunately, in the United States, students often know multiplication only as repeated addition, and they consequently find it difficult to find any real world explanation or use of multiplication of fractions, decimals, or positive and negative numbers.

7 Models involving division

The mathematics education literature has long identified two models for division: *partitive* (or *partition*) and *quotitive* (or *measurement*). Sixty years ago, Sutherland (1947) divided these into six different categories. Two of these, rate and ratio, are the categories found in an analysis of operations by Usiskin and Bell (1983) and, with the other models mentioned here, applied in materials for upper elementary and early secondary school students (University of Chicago School Mathematics Project (various years).

Ratio (Partition) Model of Division: When x and y are quantities with the same units, then x/y tells how many y's are in x (or what part of y that x is).

Rate (Measurement) Model of Division: If x and y are quantities with different units, then x/y is the amount of quantity x per quantity y.

In this conception, ratios are scalars, while rates are unitized quantities. In the measurement model, the divisor is most often a rate, as when there are 18 cookies and 2 cookies per child are to be served, and we wish to know how many children can be served, or when one has $18 to spend on items that are $2 each and one wants to know how many items can be bought. But most people mentally change the rate to a simple quantity; 2 cookies per child becomes 2 cookies, and $2 each becomes $2, and then the problem is reduced to a ratio division.

Multiplication and division are theoretically related as addition and subtraction are, so their models can be grouped together. The Rate Model of Division and the Rate Factor Model for Multiplication are related by the definition of division $a \div b = c$ if and only if $a = c \times b$, as are the Ratio Model of Division and the Size Change Model of Multiplication.

8 Arithmetic as the basis for later mathematical modelling

When no language of modelling and models to describe the applications of arithmetic is present, it becomes more difficult for children to apply arithmetic than it ought to be. Given a word problem with certain given numbers, operations are performed on those numbers without a solid base underlying the selection of the operation. If a child does have a generalized conception of a type of a model of an operation, then the child has no guard against linguistic miscues or misuses of the operations (see Verschaffel et al. (2000) and De Bock et al. (2005)).

Models of the operations give a basis for applying mathematics; they are the postulates that connect mathematics to the world of real and fanciful problems. The (applied) models of the operations should be treated as we do the (theoretical) properties of the operations. Carpenter et al. (1996) found that teachers who knew models produced students who performed better on application tasks. Why not teach the students directly?

If models for arithmetic operations are the postulates, then what are the theorems? As in any mathematics, these are statements deducible from the models. A very large number of examples of theorems of this sort exist. The paper-cup example at the beginning of this paper shows how to move from the models for addition and multiplication to explain why situations might lead to linear models of the form $y = mx + b$. The Rate Model for Division and the Comparison Model for Subtraction explain why the formula for slope involves two subtractions and a division. With models for exponentiation in arithmetic, we can derive growth models of exponential functions. And, just as importantly, we can examine why models in some circumstances are descriptive and not necessarily causal, as is often the case when linear regression is used to obtain equations of lines to fit data. And on and on.

It seems worthy of a study of application and modelling in mathematics education to consider the contributions that a discussion of models of arithmetic operations and other primary school content does, could, and should play in the development of the skills and concepts necessary to be a competent user of mathematics. While there may be pitfalls in taking the modeling perspective seriously in early schooling (Verschaffel, 2002), the implications of strong attention to mathematical modelling and applications in primary and early secondary school could be profound.

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Effects of Fraction Situations and Individual Differences: A Review of Recent Research Regarding Children's Understanding of Fractions

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1 Introduction

The goal of this paper is to provide a review of the recent and promising research regarding children's ability to work with fractions. I have chosen to focus on fractions because many researchers have claimed that children have especial difficulty in learning them (Hecht, 1998; Hecht, Close, & Santisi, 2003; Hecht, Vagi, & Torgesen, 2007; Hope & Owens, 1987; Smith, 1995). Streefland (1991) has even stated that fractions are "without doubt the most problematic area in mathematics education" (p. 6).

To illustrate this difficulty, consider this fractions problem. Which of the following numbers is closest to the sum of 7/8 + 12/13: a) 1; b) 2; c) 19; or, d) 21? To answer this questions, a child who understands something about fractions would probably realize that both fractions in the sum to be a little less than one. They would therefore reason that almost one plus almost one would be almost two. But when this question was administered as part of the third National Assessment of Educational Progress (NAEP), only 24% of 13-year-olds correctly answered this question (Bezuck & Cramer, 1989). Perhaps even more surprisingly, 55% of these students chose either 19 or 21. The sum, being equal to a little less than 2, is nowhere near 19 or 21, but it seems that these options were chosen because the numbers in the fractions could be manipulated to produce them (i.e., adding the numerators results in 19, adding the denominators results in 21). Later rounds of the NAEP demonstrate other problems that children have with fractions. Two examples are: 1) only 65% of eighth-graders in 1996, and 73% in 2005, could correctly shade in 1/3 of a rectangle; and, 2) in 2003, only 55% of eighth-graders could correctly solve a word problem that asked how many pieces, each 1/8 of a yard long, could be made from a piece of string 3/4 of a yard long (see Hecht et al., 2007).

There are perhaps many reasons why fractions are more difficult for students, but I propose that there are three particular difficulties that account for the problems that children experience with fractions. First, the quantity of a fraction (i.e., the amount it signifies) is not represented by a single number, but instead by the relationship between two whole numbers (in its simplest form), or the additive combination of the relationship between two whole numbers and a third whole number (at its most complex). Second, the procedures used to compute and manipulate fractions (e.g., the Lowest Common Denominator method to add or subtract fractions) are much more complicated that those used with whole numbers, mostly because of the complexity of how fractions represent quantity. Third, the nature of the relationship between the numerator and denominator can vary across different contexts – sometimes representing a ratio, sometimes a proportion, sometimes a continuous quantity, and sometimes a discrete quantity.

These three complexities of fractions parallel the three topics I will review in this paper: 1) Conceptual knowledge of fractions (represented in part by students' understanding of fractional representation of quantity); 2) Procedural knowledge; and 3) Different fraction situations or contexts. Proceeding in somewhat of a reverse order, I will first consider the last item: the influence fraction situation on fractions learning, reviewing in part some recent research by Mamede (2007) that has yet to be published. I will then consider the first two topics, conceptual knowledge and procedural knowledge of fractions, together, but in two parts. First, I make the argument that both conceptual knowledge and procedural knowledge are needed for children to perform well on fractions problems. Second, I will describe some recent research, by myself and my colleagues, that has investigated individual differences in how children combine conceptual and procedural knowledge when they solve fractions problems.

2 Fractions Situations

Although children do perform differently across problems that vary in problem presentation (e.g., computation versus word problem, see Mack, 1990). fractions situations, as it is defined in this paper, do not refer to differences of this type. Instead, these situations refer to the different meanings of the numerators, denominators and the relationship between them, which means that these situations are unique to rational numbers. Many researchers have developed different ways to classify fractions situation like these (Behr, Lesh, Post, and Silver, 1983; Kieren, 1988; Mack, 2001; Ohlsson, 1988), but I will focus on the recent paradigm of Nunes, Bryant, and their colleagues (Nunes, Bryant, Pretlik, Bell, Evans, & Wade, 2007). These researchers have distinguished between four different fractions situations, which I will now describe in turn.

The first is called *part-whole* situations, where the whole is always seen as one concrete thing that is divided into pieces. For example, the fraction 3/4 would be understood as something like a cake that is divided into 4 equal pieces with 3 of those pieces constituting 3/4 of the cake. In these situations, the whole can be changeable, but the fraction always refers to parts of that whole.

The second context is called a *quotient* situation, where fractions are meant to represent ways of sharing continuous quantities. In this instance, the fraction 3/4 would represent, for example, 3 chocolate bars shared amongst 4 children. The numerator represents the number of wholes to be shared, while the denominator represents the number of people that are sharing. Fractions in this situation, then, are more explicitly a division operation where the numerator is divided by the denominator.

Operator situations, on the other hand, refer to cases when the target to be divided is a discrete quantity like, for example, 24 marbles. If we wanted to know what 3/4 of 24 marbles were, we would need to take the fraction (3/4) and operate on the discrete quantity (24 marbles) using a multiplication and division to generate the result of 18 marbles. In a part-whole situation, we would not be able to generate an answer without first defining what 1/4 of 24 marbles are (which is itself an operation). Likewise, in order to solve this problem using a quotient approach, the 24 marbles would first have to be shared among 4 people, and then the share of 3 of those people would have to be tallied.

All of the preceding fraction situations, while different, can be collectively referred to as extensive quantities, because the quantities involved all depend on the size of the whole (e.g. 3/4 of 36 marbles is more than 3/4 of 24 marbles, and 3/4 of a large cake is more than 3/4 of a small cake). Fractional measures of *intensive quantity*, on the other hand, refer to a quality of something (e.g. density of an object, concentration of an acidic solution) whose value does not change as the total size of the whole changes. The fraction quantifies the relation between the numerator and denominator regardless of overall amount. The density of gold, for example, remains the same (19.3 kg per litre) regardless of whether we are talking about a 1 millilitre gold necklace or a 1 litre bar of gold. Likewise, the concentration of orange juice will be same whether it is in a glass or in a jug.

Although it is interesting in itself that fractions can represent different kinds of situations, those of us in mathematics education are more interested in whether or not these fraction situations are related to children's learning. Will problems framed as one situation be easier to learn than problems framed in another situation? Nunes and her colleagues (2007) investigated this question regarding part-whole and quotient situations. A total of 130 Year 4 and Year 5 British school children were given fraction questions that were either framed as part-whole or quotient situations. Although all the previous instruction experienced by these children had been framed as part-whole problems, children from both years performed significantly better on the quotient problems that the part-whole problems. These results suggest that quotient situations may be easier for children to understand compared to part-whole situations.

A recent series of studies by Mamede (2007) has further examined this question by comparing young children's ability on problems from part-whole, quotient, and operator situations. By studying 6- and 7-year-olds, the intent of these studies were to explore children's intuitive ways of understanding fractions before they received any formal instruction on fractions. Mamede's results demonstrated a consistent performance advantage for those children who were given fractions in quotient situations compared to those in the other two situations. The children who were given quotient problems performed better than children given other types of problems, children who were given all types of problems performed better on the quotient problems, and those students who were trained on quotient problems demonstrated better transfer of understanding to the other situations than those children trained in part-whole and operator situations. These data suggest teaching fractions from a quotient perspective may have some advantages.

While no research to date has compared all four fractions situations, research regarding intensive quantities has demonstrated that, in general, children have more difficulty understanding intensive quantity situations than non-intensive quantity situations (Nunes, Desli, & Bell, 2003).

3 The relative contribution of conceptual and procedural knowledge to fractions understanding

When faced with a fractions problem, what do children do? Do they think about the problem conceptually, using their understanding of fractions to reason through an answer? Or, do they instead execute a procedure – a procedure they have learned will generate the "right answer"? Research in mathematics cognition has a long history of attempting to separate children's conceptual understanding of mathematics from their procedural knowledge used to answer mathematics questions (see Rittle-Johnson & Siegler, 1998; Skemp, 1976). In this literature, conceptual knowledge has been defined as knowledge that is interconnected with other knowledge (Byrnes, 1992; Hiebert & Lefevre, 1986; Kieran, 1993), or knowledge that is rich in relationships with other knowledge. On this account, conceptual knowledge is not knowledge that stands alone but stands in relation to other knowledge. Procedural knowledge, on the other hand, is defined as the use of algorithms, meant to achieve a desired end, that require nothing except the original inputs to execute properly (Byrnes, 1992; Hiebert & Lefevre, 1986). As such, procedural knowledge is knowledge that can be completely separated from meaning and still be successfully executed.

Research has not only examined the influence of conceptual and procedural knowledge in many different domains of mathematical cognition (see Rittle-Johnson & Siegler, 1998 for a review), but also specifically in regards to fractions (Byrnes, 1992; Byrnes & Wasik, 1991; Kerslake, 1986, Peck & Jenks, 1981; Rittle-Johnson, Siegler, & Alibali, 2001). Much of this literature involves a debate on whether type of knowledge is learned first (i.e., do children learn concepts first, or do they learn procedures first). Hecht (1998), however, has generated some interesting research findings that suggest that, regardless of which is learned first, both types of knowledge seem to be important for general fractions learning.

With seventh- and eighth-graders, Hecht (1998) investigated how conceptual and procedural knowledge, as well as the ability to accurately and quickly retrieve answers to simple computations ("math facts"), predict performance on three kinds of fraction problems: 1) computation problems; 2) word problem formulation (i.e., deriving the right formula to solve a word problem); and, 3) fraction size estimation (i.e., estimate the addition of two fractions). Conceptual knowledge was measured with items that probed children's understanding of equivalence (e.g., that fractions with different numbers, like 2/8 and 1/4, are actually equal) and ordering (e.g., being able to tell which fraction, 2/3 or 3/4, is larger). Procedural knowledge was measured using multiple-choice questions that asked students to choose the legal procedure to solve a given problem. Results indicated that, controlling for vocabulary knowledge, both conceptual knowledge and procedural knowledge independently predicted success on the computational problems and the word-problem formulations. Only conceptual knowledge, however, independently predicted success on the fraction-size estimation measure while the math facts was not independently related to any of the problem types.

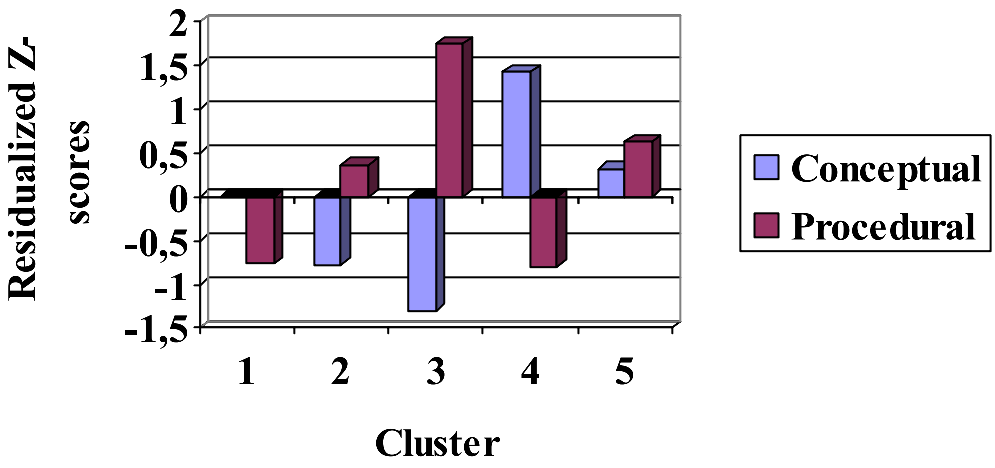
Interestingly, Hecht (1998) has argued that, because only conceptual knowledge independently predicted success across all three problem types, conceptual knowledge is of primary importance when solving fraction problems (see also Hecht et al., 2007). While Hecht's argument regarding the primacy of conceptual knowledge does seem intuitive, it is important to note that procedural knowledge was also independently predictive of both computation problems and word-problem formulations. In the computation problems, procedural knowledge actually accounted for twice the variance compared to conceptual knowledge. The independent relation with word-problem formulations is even more suggestive, because these problems involved only generated the correct formula to solve, not the actual execution of solving the formula. If procedural knowledge is still predictive of these sorts of problems after conceptual knowledge and vocabulary is statistically controlled, that suggests that procedural knowledge is also instrumental to fractions understanding. As such, I propose that both conceptual and procedural knowledge are important in solving fractions problems – a message that is in line with the recent report of the National Mathematics Advisory Panel in the U.S. (U.S. Department of Education, 2008). At present, the research specific to conceptual and procedural understanding of fractions is contradictory (see the next section) and little research has approached this question in the same manner as Hecht (1998). Further research is needed to explore the unique contributions of conceptual and procedural knowledge of fractions understanding.

4 Individual differences in using conceptual and procedural knowledge to solve fractions problems

The last topic in this paper concerns recent research that myself and my colleagues have conducted regarding individual differences in conceptual and procedural knowledge (Hallett, Nunes, & Bryant, under review). As mentioned above, previous research, both in regards to fractions and other kinds of mathematics, has generated conflicting and contradictory results about the exact roles of conceptual and procedural knowledge in mathematical learning. Byrnes and Wasik (1991), for example, found that many of the conceptual aspects of fractions are prerequisites for the procedural ability to perform computations. In contrast, other researchers focused on instances where children seem to approach fraction problems procedurally with little reference to conceptual knowledge (e.g., Kerslake, 1986; Peck & Jencks, 1981). For example, Peck and Jencks (1981) reported that 35% of their sample of sixth-graders were able to correctly carry out procedures to solve fraction problems without possessing a comparable conceptual understanding of fractions. Similarly, Kerslake (1986) observed that many students were unable to explain why they carried out certain procedures, even when they did it correctly. Rittle-Johnson and her colleagues, on the other hand, have argued that conceptual and procedural knowledge continually and incrementally stimulate each other, with neither necessarily preceding the other (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler & Alibali, 2001). Children may first learn a relevant concept, which would then translate into helping to learn a procedure. Or children may instead first learn a procedure, which would then inspire them to understand the conceptual reasons for why the procedure works. Although Rittle-Johnson and her colleagues (2001) have data from an intervention study to support this hypothesis (i.e., an effect for conceptual training on procedural knowledge and an effect of procedural training on conceptual knowledge), their data do not explain why some students (like those reported in Kerslake, 1986) appear to be able to have one type of knowledge without the other.

My colleagues and I have proposed that these contradictory findings in the literature could be explained by individual differences in the ways in which children use conceptual and procedural knowledge (Hallett, Nunes & Bryant, under review). In other words, some students might rely more on their conceptual knowledge, some students might rely more on their procedural knowledge, and some students might rely equally on both. To test this hypothesis, Year 4 and Year 5 children from the U.K. completed a fractions measure containing conceptual and procedural items. From these measures, conceptual and procedural subscale scores were generated for each child, and these scores were then subjected to a cluster analysis. Although we had hypothesized the existence of three clusters (those who relied more on conceptual knowledge, those who relied more on procedural knowledge, and those who relied on both), results indicated five clusters of children that differed in the patterns of conceptual and procedural knowledge. Furthermore, these clusters differed on their performance of the overall fractions measure. Two of the clusters performed poorly (Clusters 1 and 2, see Figure 1), but did so in two different ways: one group demonstrated a lack of conceptual knowledge (without a compensating level of procedural knowledge) and the other demonstrated a lack of procedural knowledge (without a compensating level of conceptual knowledge). Cluster 3 relied more heavily on their procedural knowledge and performed better, but not as well as Cluster 4 that relied more on their conceptual knowledge. Cluster 5, which performed best, relied equally on their conceptual and procedural knowledge. These findings suggest not only that there are individual differences in the ways that children combine conceptual and procedural knowledge, but also that these differences are associated with differences in overall fractions performance.

Figure 1
Cluster Analysis of Year 4 & 5 data



My colleagues and I have extended this research on individual differences to older children (Year 6 and Year 8 students) who presumably had more experience with fractions. To get a better sense of children's procedural and conceptual understanding, each student was individually interviewed in addition to completing written measures. The children were also assessed on their general conceptual and procedural learning (in contrast to that specifically related to fractions) as well as on their dispositional thinking styles (Hallett, Nunes, & Bryant, in prep). Preliminary analyses both reinforce and expand on the findings of the first study. First, conceptual and procedural knowledge specific to fractions understanding are independently predictive of fraction performance even when general conceptual and procedural knowledge are statistically controlled (Hallett, Nunes, & Bryant, in prep). This not only suggests – in line with other studies – that both conceptual and procedural knowledge independently contribute to mathematical performance, but also that students might have strengths and weaknesses specific to fractions learning that is not explained by their general conceptual and procedural learning ability. Second, Year 6 students demonstrate the same cluster pattern, and the same pattern of performance, as that found in the first study (see Figure 2). For Year 8 students, however, the two low-performing clusters are no longer evident and instead only three clusters remain: 1) those who rely more on procedural knowledge; 2) those who rely more on conceptual knowledge; and 3) those who seem to rely on conceptual and procedural knowledge equally (see Figure 3). Performance differences between the Year 8 clusters, however, were very small. It is possible that the patterns of individual differences found in younger children may not persist as children grow older, but further investigation is required.

Figure 2
Cluster Analysis of Year 6 data

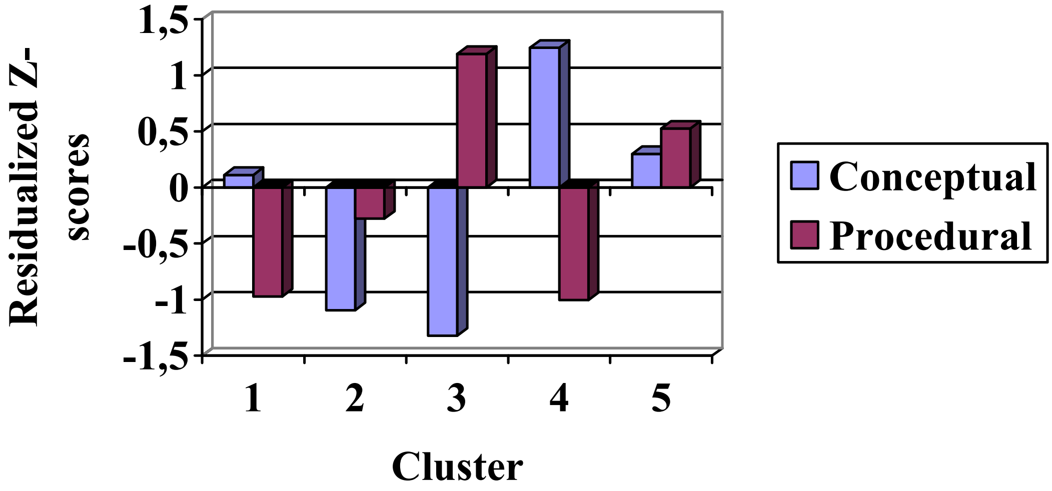
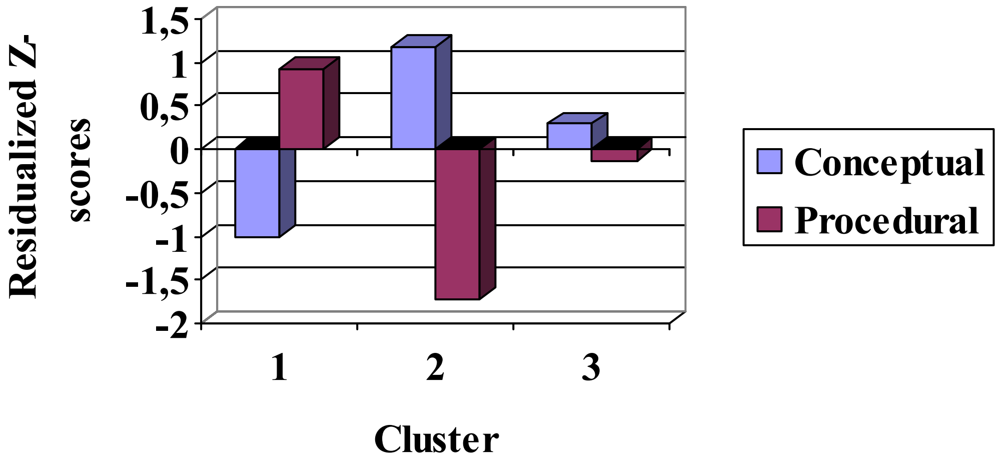


Figure 3
Cluster Analysis of Year 8 data



5 Conclusion

There is still much that we do not know in regards to children's understanding of fractions. The goal of this paper was not to summarize everything we do know, but instead to highlight some key findings about fractions understanding. By reviewing this literature, I was aiming to make three points. First, the situations in which fractions are framed can facilitate understanding, with quotient situations seeming to provide the largest advantage. It may be worthwhile to consider designing curricula to capitalize on this finding. Second, while previous research seems to have focused on the relative primacy of conceptual knowledge versus procedural knowledge, it seems worthwhile to consider that both skills may be necessary to gain competence in utilizing and understanding fractions. Third, recent evidence suggests that different children combine conceptual and procedural knowledge differently when they are learning fractions. If these different profiles can be identified, then teaching methods can be developed to target the child's particular needs. There is still the need for more research, but these research findings have the potential to provide some useful ideas to help design better curricula. In the end, the goal is find a way to better teach children the troublesome knowledge and skills that surround fractions competence.

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PAPER PRESENTATIONS: MULTIPLICATION, DIVISION, FRACTION

Susanne Prediger:
Discontinuities for Mental Models: A Source for Difficulties with the Multiplication of Fractions

Rose Elaine Carbone & Patricia T. Eaton:
Prospective Teachers' Knowledge of Addition and Division of Fractions

Marta Elena Valdemoros:
Planning Fraction Lessons: A Case Study

Wim van Dooren; Dirk De Bock; Marleen Evers & Lieven Verschaffel:
The Role of Number Structures on Pupils' Over-Use of Linearity in Missing-Value Problems

Fátima Mendes & Elvira Ferreira:
Developing Multiplication

Issic K. C. Leung; Cho Paul & Regina M. F. Wong:
Learning Alternate Division Algorithm in Enhancing the concept of Rates and Density

Discontinuities for Mental Models: A Source for Difficulties with the Multiplication of Fractions

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Different theoretical approaches offer different ways of explaining students' well-documented difficulties with arithmetical operations like multiplication of fractions. The article recalls a conceptual framework that integrates approaches focusing on meanings of operations into conceptual change approaches It offers first results from an empirical study on discontinuities and continuities of models for the multiplication of fractions.

Keywords: multiplication, fractions, conceptual change, mental models

1 Different theoretical approaches and an integrating multi-level model for discontinuities with fractions

Many empirical studies have documented enormous difficulties in students' competencies and conceptions in the domain of fractions (and decimals). Whereas *algorithmic competencies* are usually fairly developed, *understanding* is often weaker, as well as the *competencies to solve* word problems or realistic problems including fractions (e.g. Hasemann, 1981; Barash & Klein, 1996; Aksu, 1997).

Different theoretical approaches exist for explaining these difficulties. One common aspect of several approaches is the emphasis on *discontinuities between natural and fractional numbers,* for example the fact that multiplication always makes bigger for natural numbers (apart from 0 and 1), but no more for fractions (e.g. Streefland, 1984; Hartnett & Gelman, 1998). Among different theoretical approaches to explain students' difficulties with these discontinuities, the *conceptual change approach* (Posner et al., 1982) has gained a growing influence in mathematics education research (e.g. Lehtinen, Merenluoto, & Kasanen, 1997; Stafylidou & Vosniadou, 2004; Lehtinen, 2006). On the basis of a constructivist theory of learning and inspired by Piaget's notion of accommodation, the conceptual change approach has emphasized that learning is rarely cumulative in the sense that new knowledge is only added to the prior (as a process of enrichment). Instead, learning often necessitates the discontinuous reconstruction of prior knowledge when confronted with new experiences and challenges. Problems of conceptual change can appear, when learners' prior knowledge is incompatible with new necessary conceptualisations. The key point in the conceptual change approach adopted here is that discrepancies between intended mathematical conceptions and real individual conceptions are not seen as individual deficits but as necessary stages of transition in the process of reconstructing knowledge - in the sense of epistemological obstacles, in Brousseau's terms (1997).

Up to 2006, this discussion on conceptual change was held nearly separately from a second influential theoretical approach that emphasized the importance of underlying mental models (Fischbein et al., 1985, Greer, 1994) or 'Grundvorstellungen' (GVs, see vom Hofe et al.,

2006) for explaining students' difficulties. The notion *mental model* is used here as nearly synonymous to *Grundvorstellung*. It starts from Fischbein's use of model as a "meaningful interpretation of a phenomenon or concept" (Fischbein, 1989, p. 129) which is more specific than the often cited construct *mental model* as used by cognitive scientists like Johnson-Laird (1983). Within the theoretical approaches to which this article refers (Fischbein, 1989, vom Hofe et al., 2006), the formation of mental models is considered to be especially important for mathematical concept acquisition. Mental models constitute the meanings of mathematical concepts based on familiar contexts and experiences. They create mental representations of the concept and they are crucial for the ability to apply a concept to reality by recognizing the respective structure in real life contexts or by modelling a real life situation with the aid of mathematical structures (cf. vom Hofe et al., 2006, p. 2).

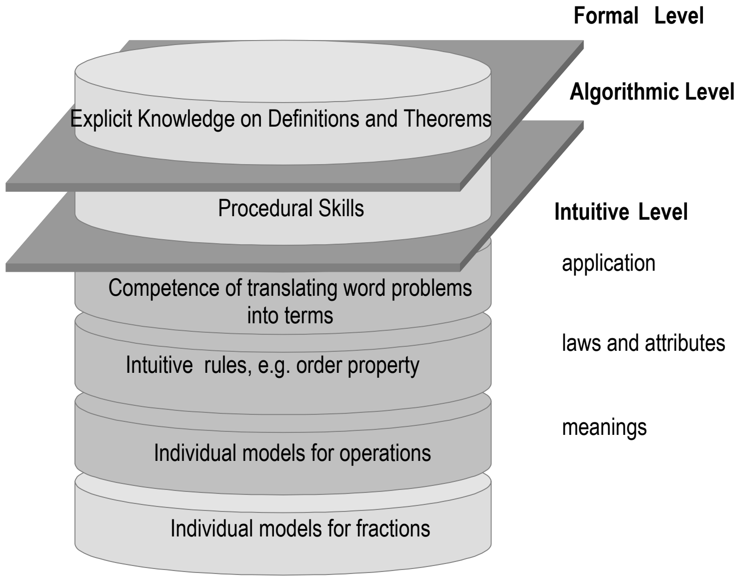


Figure 1: Obstacles can lie deeper – Different levels of students' difficulties

In Prediger (2008), these two so far competing theoretical approaches of conceptual change and mental models were integrated into a multi-level model for knowledge of operations (see Fig. 1). Its main purpose was to provide a conceptual tool for describing the precise locations of students' difficulties with discontinuities, i.e. the epistemological quality of the obstacles hindering students to master the necessary changes in the process of conceptual change.

Following Fischbein et al. (1985), the model differentiates between algorithmic, intuitive and formal understanding. The *formal level* includes the definitions of concepts and of operations, structures, and theorems relevant to a specific content domain. This type of knowledge is formally represented by axioms, definitions, theorems and their proofs. The *algorithmic level* of knowledge is basically procedural in nature and involves students' capability to explain the successive steps included in various, standard procedural operations. Although solving word problems also has procedural aspects, it is assigned to the intuitive level since it necessitates interpretations of mathematical concepts.

Intuitive understanding is characterized as the type of mostly implicit knowledge that we tend to accept directly and confidently as being obvious. On the *intuitive level,* it is worth to distinguish between conceptions about concrete mathematical laws or properties, here called *intuitive rules* (like "multiplication makes bigger") from those about the *meanings* of concepts (like the interpretation "multiplication means repeated addition"). Nearly all studies dealing with conceptual change in the field of fractions have treated intuitive knowledge, but they have mainly focused on the level of intuitive rules. In contrast, they have neglected the level of meanings which consists of mental models. (Remark that the term is used her for domain-specific intuitions, not in a general sense like by Tirosh & Stavy, 1999.)

Natural numbers	→	Fractions
repeated addition (3x5 means 5+5+5, i.e. 3 wands of 5m length in a row)	→	???
area of a rectangle (3x5 is the area of a 3cmx5cm rectangle)	→	area of a rectangle (2/3 x5/4 is the area of a 2/3 cm x 5/4 cm rectangle)
????	→	part-of-interpretation (2/3 x 5/2 means 2/3 of 5/2)
multiplicative comparison (twice as much)	→	multiplicative comparison (half as much)
scaling up (3x5 means 5cm is stretched three times as much)	→	scaling up and down (2/3 x 5/2 means 5/2 cm compressed on 2/3 of it)
combinatorial interpretation (3x5 as number of combining 3 shirts + 5 trousers)	→	????

Figure 2: (Dis-)Continuities of mental models for multiplication in the transition from natural to fractional numbers

The level model allows to re-locate the exact place of the epistemological obstacles in the process of conceptual change from natural to fractional numbers. Most researchers in conceptual change research locate the problem on the level of laws and rules, conceptualizing the transfer of rules from natural numbers to fractions simply as a problem of hasty generalization. In contrast, some researchers (like Fischbein et al., 1985; Prediger, 2008; Greer, 1994) showed the importance of the underlying level of meaning as the more important level to locate discontinuities. Already in 1985, Fischbein et al. showed how many students adhered to the 'repeated addition' as the dominant model for the multiplication of natural numbers. In Prediger (2008), the author pleaded for widening the considerations to *all mental models* for multiplication.

A mathematical (not yet empirical) analysis of mental models as summarized in Figure 2 makes clear that not all mental models for multiplication have to be changed in the transition from natural to fractional numbers. The interpretation as an area of a rectangle or as scaling up can be continued for fractions as well as the multiplicative comparison. In contrast, the basic model 'repeated addition' is not sustainable for fractions, neither the combinatorial interpretation. Vice versa, the basic model of the multiplication of fractions, the part-of-interpretation, has no direct correspondence for the natural numbers (see Fig. 2 and 3 for examples explaining the models).

This mathematical analysis sensitizes for possible locations of obstacles in the process of conceptual change: Not the intuitive rules pose the most urgent problem, but the necessary changes of mental models. *Metaphorically speaking, the discontinuities that possibly generate epistemological obstacles in the transition process can be located in the flashes of Figure 2.*

2 Research questions, test items and research design

2.1 Research questions and test items

So far, the analysis of discontinuities of mental models was only conducted theoretically as a mathematical analysis of meaning. The study presented here tries to show its empirical relevance by treating the following research questions:

1. What mental models do students in Grade 7 and 9 activate?

2. What kind of situations can they describe by a multiplicative term?
3. Is there empirical evidence that those models that have to be changed are more difficult than those which might be continuously transferred from natural numbers to fractions?
4. For which of the models can we find a connection to the intuitive rule "multiplication makes bigger"?

In order to answer these questions, twelve test items were constructed. This paper is focused on those eight items which referred to the intuitive level (see Figure 3, all items were of course given without headline). Item 2 operated on the *level of intuitive rules*, asking in a multiple choice format for the order property of multiplication. Items 5 and 6 operated exploratively on the level of meaning. It was given in an open item format in order not to impose a presupposed mental model but to exploratively gain a great variety of really existing *individual mental models*. Item 7 to 11 referred to the competence of finding multiplicative terms to given word problems. They differed in the necessary mental model that had to be activated.

Selected test items with reference to the intuitive level

Item 2: Order property of multiplication
Which statement is correct (mark with one or more crosses): When I multiply two fractions

- ○ the solution is always bigger than the two fractions
- ○ the solution is always smaller than the two fractions.
- ○ the solution is sometimes bigger, sometimes smaller than the two fractions.

Item 5: Find word problem for an additive equation
When solving word problems, you are supposed to find calculations for given everyday situations.
Here, you are asked to do it vice versa.

Find a word problem that can be solved by means of the equation $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$.

Item 6: Find word problem for a multiplicative equation

Find also a word problem that can be solved by means of the following equation: $\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{12}$.

Item 7: Mathematize a situation with multiplicative comparison

a.) One kilogram tangerine costs € 1.50. Kate wants to buy $\frac{3}{4}$ kg. How can she calculate her price to pay?
(Mark with one or more crosses)
□ $1{,}5 - \frac{3}{4}$ □ $1{,}5 : \frac{3}{4}$ □ $\frac{3}{4}\cdot 1{,}5$
□ none of these, but this:

b.) Justify your answer given in a)

Item 8: Mathematize a situation with repeated addition (natural times fraction)

a.) Every child eats on average $\frac{2}{10}$ kg of mashed potatoes. How can we calculate what 15 children would eat? (Mark with one or more crosses)
□ $15+\frac{2}{10}$ □ $15 : \frac{2}{10}$ □ $\frac{2}{10}\cdot 15$ □ $15\cdot\frac{2}{10}$
□ none of these, but this:

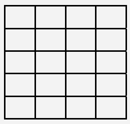
b.) Justify your answer given in a)

Item 9: Mathematize a situation with part-of-whole number (given verbally)

a.) How can we calculate $\frac{2}{3}$ of 36? (Mark with one or more crosses): □ $36-\frac{2}{3}$ □ $36 : \frac{2}{3}$ □ $\frac{2}{3}\cdot 36$
□ none of these, but this:

b.) Justify your answer given in a.)

Item 10: Specify part of a fraction and mathematize



a.) Colourize $\frac{3}{4}$ of the rectangle.
b.) Colourize now $\frac{2}{5}$ of these $\frac{3}{4}$ with another colour.
c.) Give the fraction that describes the part of the rectangle that is double coloured now.
d.) With what calculation could you come to this fraction?

Item 11: Mathematize a situation of scaling down

a.) An African elephant has a body height of 3.60 m.
Anna has a model of the elephant which is scaled down to $\frac{3}{40}$ of the original body height.
Give the height of the model elephant.
Explain your way how you found it.

b.) Can you solve the task in a.) also with one single operation? (if not already done) Which one?

Figure 3: Selected test items

2.2 Design – sample and data analysis

The study was designed as a 60 minutes paper and pencil test, written by 269 students in five Grade 7 classes (age about 12 years) and five Grade 9 classes (age about 14 years) in German grammar schools which comprise the higher achieving 40% of students in Germany.
The students' answers were evaluated quantitatively in a points rationing scheme. Furthermore, the answers to Item 5 and 6 and reasons given in Item 7 to 9 were analysed qualitatively by categorizing the manifested individual conceptions about operations on fractions. In a pre-test, the developed coding scheme achieved an interrater agreement of Cohen's kappa of 0.81 to 0.94.

3 Most important results

3.1 More difficulties for interpreting multiplication than for addition

The comparison of Item 5 and 6 offers well expected results: much more students could find an adequate word problem for a given additive equation than for a multiplicative equation (see Figure 4).

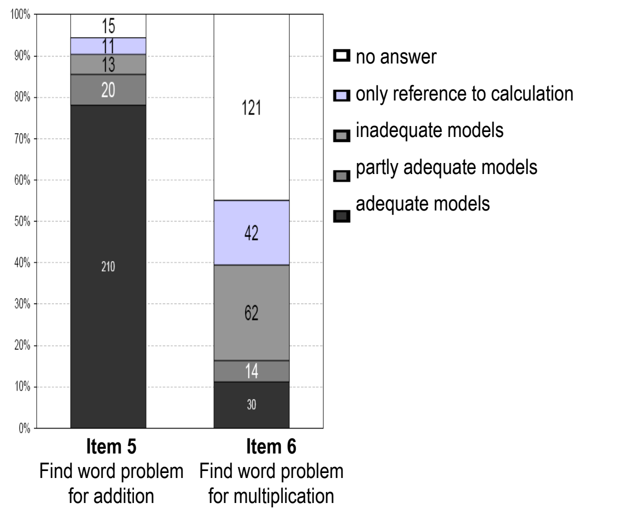


Figure 4: Comparison of occurrences of interpretations for additive and multiplicative equations in Item 5 and 6

Whereas 210 of 269 students (78%) were able to find an adequate model for the addition, only 30 out of 269 (11%) found one for the multiplication, like a word problem concerning "2/3 of 1/4 l milk". In contrast, 15 students gave no answer and 11 referred only to the calculation itself for addition (in sum 10%), but 61% did one of both for multiplication.

In Item 6, a middle group of 14 students gave interpretations which showed partly adequate multiplicative models, but in an incomplete way. A middle group of 20 students gave traces of adequate models for the addition, all of them trying to join parts of different wholes, like in this example:

Lisa completed 2/3 of her English homework and 1/6 of her math homework. Which part has she completed in sum?

Among the 62 wrong answers (23%) with inadequate models for the multiplication, there were 38 answers with a word problem that referred to an additive situation, e.g.

Mr. Miller sells 2/3 of his bread s on one day and ¼ and the next day. He wants to know how much he sold together.

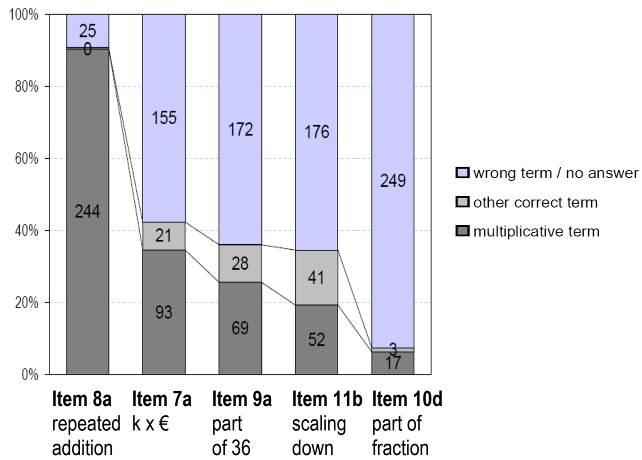


Figure 5: Comparison of chosen terms for different items

3.2 Not all models for multiplication equally difficult

The difficulties of connecting multiplications with situations became equally visible by the five items with the inverse question, in which students should choose terms for mathematizing different given situations.

Figure 5 gives an overview on the decreasing frequency of reached scores for those five items. The far best results were reached for Item 8a, in which 91% of the students could activate the well-known model of repeated addition and chose the multiplicative term $\frac{2}{10}\cdot 15$ or $15\cdot\frac{2}{10}$. (More precisely, 38% chose one, 53% chose both.)

In contrast, in all other items, no more than 93 of 269 students (i.e. 35%) chose the multiplicative term. Only 17 of 269 students (6%) could mathematize the verbally and graphically given $\frac{2}{5}$ of $\frac{3}{4}$ in Item 10d by the multiplication $\frac{2}{5}\cdot\frac{3}{4}$.

3.3 Only a singular connection between mental models and intuitive rule on order property

Due to the low rates of correct answers, it is difficult to determine correlations between different items. Only for Item 9, we find a clear connection to Item 2:

109 students expressed the intuitive rule "When I multiply two fractions, the solution is always bigger than the two fractions." in Item 2, 88% of these 109 (i.e. 96) chose a wrong term for calculating $\frac{2}{3}$ of 36, most preferably (64 answers) the division 36 : $\frac{2}{3}$.

In contrast, among those 116 students who expressed the correct rule "sometimes bigger, sometimes smaller", only 61 chose a wrong term, i.e. only 53%.

Vice versa for those who chose the multiplication $\frac{2}{3}\cdot 36$ correctly: It was chosen by 40 out of 116 students (i.e. 35% of those) with correct order property, and only by 7 of 109, i.e. by 6% of those with false order property (even better was the ratio for the "overgeneralizers" who believe that for fractions, every multiplication makes smaller: 22 of 44, i.e. 50%).

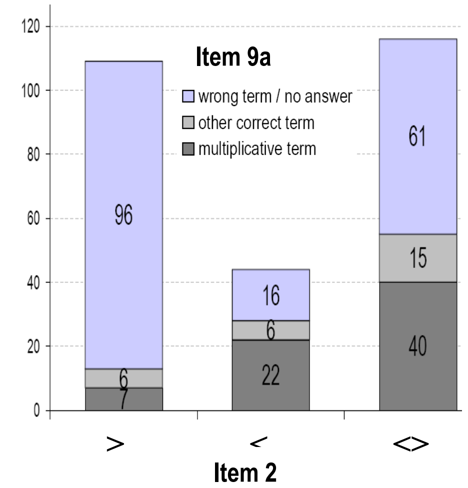


Figure 6: Connection between order property (Item 2) and choice of f operation for part of 36 (Item 9a)

For the determination of a relative part (two third of 36), it seems hence to be helpful to know that multiplication can (sometimes) make smaller.

As the following answer to Item 7b shows, some (singular) answers in other items also refered to the order property:

She would have to calculate 1,5 : ¾ because she wants to buy less than 1kg tangerines.

But this connection could not be found for a statistically significant number of students in Item 7, 8, 10 and 11.

4 Discussion

Why do students have difficulties with the application of multiplication for word problems? Although a quantitative study based on a paper and pencil test can only specify coincidences but no reasons, the results of the study give distinct tendencies which are interesting to discuss.

The first finding "more difficulties for interpreting multiplication than addition" fits perfectly to the explanations given in the here presented *integrated conceptual change approach*: The empirical phenomenon that much more students can formulate a word problem for an additive equation than for a multiplicative equation can be explained by the mathematical fact that there is no epistemological obstacle for addition, that means no conceptual change is necessary in the transition from natural to fractional numbers. In contrast, the large number of students who were not able to find any adequate interpretation for the multiplication in Item 6 can be explained by the epistemological obstacle given by the discontinuity of interpretations for multiplication in the transition from natural to fractional numbers. Fischbein et al. (1985) emphasized already in 1985 that one difficulty lies in the mathematical fact that the most dominant mental model, the repeated addition, cannot be continued for $\frac{2}{3}\cdot\frac{1}{4}=\frac{2}{12}$. This interpretation of the empirical result can be supported by the comparison of difficulties in finding correct terms for differently structured multiplicative situations (second finding "Not all models for multiplication equally difficult" in Figure 5). The word problem asking for repeated addition (natural times fraction) was mathematized significantly better than all other multiplicative situations in which the repeated addition could not be activated. Due to too similar results in Items 7,9,10, and 11, we do not go further in the ranking of different discontinuous models for the multiplication of fractions (see Figure 2). Additionally, we cannot exclude the interpretation that the lowest rates in Item 11 and 10 might also be explained by the fact that there were no pre-given answers as in the multiple choice Items 7 to 9. Even the slight change from decimals to fractions might have contributed to the results.

However, the discontinuity of mental models seem to be more crucial in these cases than the pertinence of the intuitive rule "multiplication makes bigger", since we could only find a statistical connection between Item 2 and Item 9a, but not for Item 7a, 8a, 10d and 11b. Additionally, the low rates of correct answers in the whole test sample might have contributed to the absences of correlations.

As it is not in line with well-known research results, e.g. by Bell et al. (1981), this phenomenon will need further research with a more differentiated look on different mental models, their discontinuities and their connection to intuitive rules on the order property.

5 Outlook: Beyond the example of multiplication of fractions

Although this paper reports on a concrete empirical study on one special operation for a special number set (namely the multiplication of fractions) the author cannot conclude without emphasizing that it should be taken as an example for a wider learning problem and also a wider research program. Although we know relatively much on students' conceptions concerning the numbers, *the interpretations of operations in different situations* is still not enough in view, neither in the view of researchers nor in the view of teachers and text book writers.

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Appendix: Scores of items

Item	Content (in order of difficulty)	Frequency of complete solutions	Average of reached scores	
			absolutely	in %
5	Find word problem for an equation with addition	57%	1,40 of 2	70%
8	Mathematize situation with repeated addition (natural x fraction)	31%	1,35 of 2	67%
2	Order property (does multiplication make bigger?)	43%	1,04 of 2	52%
11	Mathematize situation of scaling down	14%	0,62 of 2	31%
10	Specify part of a fraction and mathematize	4%	1,49 of 5	30%
7	Mathematize situation with multiplicative comparison	9%	0,54 of 2	27%
9	Mathematize situation with part of whole number	7%	0,46 of 2	23%
6	Find word problem for an equation with multiplication	11%	0,30 of 2	15%

Prospective Teachers' Knowledge of Addition and Division of Fractions

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This study reports the initial findings of two collaborating mathematics educators from the United States and Northern Ireland on their prospective elementary teachers' understanding of rational numbers. Prospective elementary teachers were evaluated on their ability to create appropriate real life problems illustrating the addition and division of fractions. The similarity of the misunderstandings that these prospective teachers exhibited offers ways for mathematics educators to inform and improve their teaching. This research also expands international collaborations.

Keywords: rational numbers, prospective elementary teachers, international collaborations

1 Introduction

For at least two decades, concerns about the depth of mathematics understanding of prospective elementary teachers have been discussed in the research of mathematics education (Conference Board of Mathematical Sciences [CBMS], 2000; Ma, 1999; Parker, 1996; Ball, 1990; Silver, 1981). The research suggests that prospective teachers must revisit the mathematics that they have previously learned so that they will be able to effectively teach their future students. The CBMS specifically notes that strengthening rational number knowledge and rational number sense is absolutely essential in the preparation of middle grade mathematics teachers. It further indicates that besides being able to explain procedures, these future teachers need a sufficient depth of understanding to be able to write problems that require specific arithmetic operations. Prospective teachers who obtain this deep understanding of rational numbers will be more prepared to help their future students to develop their own understandings.

2 Methodology

During the spring semester of 2007, 40 United States (US) and Northern Ireland (NI) prospective elementary teachers were evaluated on their understanding of addition and division of fractions. The future teachers were required to demonstrate their knowledge of addition and division of fractions by creating two real life problems that would be appropriate for elementary students. The first problem that was analyzed by the researchers required the prospective teachers to add two fractions whose sum is greater than one. The second problem required the understanding of dividing a mixed number by a fraction. The following problems were completed by 40 prospective teachers from both countries.

1. Write a story problem where students in the elementary grades would add ¾ + ½ to complete the problem.

2. Write a story problem that shows the meaning of $2\frac{1}{2} \div \frac{1}{2}$.

The Northern Ireland subjects were all in the final year of a four year Bachelor of Education degree that prepares them for teaching in elementary (primary) schools. All participants specialize in mathematics and have already spent approximately one quarter of their degree time studying mathematics. They have also spent the equivalent of one quarter of their degree time teaching in elementary schools and have classroom experience in teaching mathematics.

The US subjects are not as advanced in working toward their four year degree in elementary education, but are at various stages in their career. They consist of 2 freshman, 20 sophomores, 7 juniors, 1 senior and 4 post graduates (who were returning to study elementary education). When the data were gathered for the study, the US students were in the final weeks of completing their first required mathematical content knowledge course that included content for Kindergarten through grade 8. A limitation of the study, therefore, is the difference in the background between the US participants and those in Northern Ireland. While the US students were given these problems as part of an 18 question 75-minute exam that assessed their knowledge of fractions, decimals, and percent, the NI participants were not in a testing situation when they responded to the questions, but were simply given 20 minutes to complete four problems regarding fractions, decimals, and percentages. Thus, the time that the NI and US subjects had to complete the problems was approximately the same.

After the subjects completed the problems, their respective professor categorized responses to the two problems as either acceptable or unacceptable. Both principal researchers asked one other researcher from their country to also independently categorize the questions. There was a 100% inter-rater agreement between the two Northern Irish researchers, and 96% inter-rater agreement between two US researchers. The US researchers then shared their responses with the NI research team. As a result of the discussions, there was a final 100% agreement on the acceptable and unacceptable classifications of all problems. A discussion on the emphasis on the referent whole emerged during the collaborations. Within the problems that the prospective teachers created, similar categories of acceptable and unacceptable responses emerged from both countries. The results follow.

3 Results

3.1 Addition of Fractions

In this study, 70% percent (28 out of 40) of the US and Northern Ireland (NI) students provided acceptable responses in designing real-life fraction addition problems for the sum of one half and three-fourths. When the US students completed a pretest with the same problem prior to instruction, only 20% had written acceptable responses. Most of the problems that were classified as unacceptable did not refer to equivalent wholes. During US instruction, the concept of equivalent wholes was emphasized, and as a result, the US students exhibited a higher ability than the NI students to create an appropriate problem in this additive scenario.

The Northern Ireland students had not explicitly discussed referent wholes in class prior to completing this question, thus most of the unacceptable problems that they created did not include this reference.

The major similarity in the responses of the prospective teachers from both countries, whether their responses were classified as acceptable or unacceptable, is that they wrote problems involving food. Both groups of prospective elementary teachers realize that their future elementary students will relate to problems about food. Eighteen problems specifically involved recipes.

A problem that represents a typical acceptable recipe problem follows.

Today we are going to make cookies. We need to add ½ cup of sugar and ¾ cup of brown sugar. Together, how many cups do the two sugars add up to?

Other problems regarding food did elicit several repeated unacceptable responses. A representative example of the salient error is below.

Sarah eats half of a chocolate cake and three quarters of an orange cake. How much cake does she eat altogether?

In similar problems that were deemed unacceptable, students made an underlying assumption that the two chosen objects, in this case the chocolate cake and the orange cake are equivalent in shape and size, and that the answer of one and one quarter cakes would be correct. If the cakes were not equivalent in shape and size, the fractional parts of the cakes could not be added to give the answer of one and one quarter cakes, as Figure 1 below indicates.

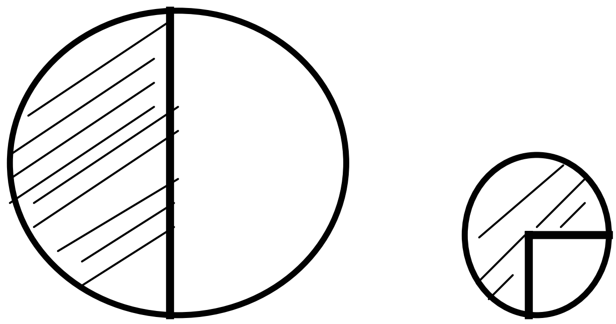

Figure 1: Half of a chocolate cake and three quarters of an orange cake.

An equivalent acceptable response is shown in the example below because it makes reference to both pizzas being equivalent wholes.

You have 2 pizzas that are exactly the same size. You give one pizza to your friend and you decide to see who can eat the most. You eat one half of your pizza. Your friend eats ¾ of his pizza. How much pizza did you both eat all together?

When analyzing and classifying similar problems, the researchers were consistent in requiring that the future teachers clearly noted that the wholes that were being referred to were equivalent, otherwise they would not emphasize to their future students the importance of making reference to the same unit whole.

Another type of unacceptable response in regards to the category of food was not responding to the direction that the problem must require students to add the fractions ½ and ¾. This type of response is demonstrated in the example below.

There are two bars of chocolate with 16 squares in each. Jane eats half of one bar and Peter eats three quarters. How many squares are left to share with Angela?

Here the prospective teacher who designed the question is not requiring the problem solver to carry out the calculation of ½ + ¾. Instead the problem solver can find one half of 16, and then three-quarters of 16, add the results and take the sum from 32 to obtain the answer. It is also not clear in this problem if Peter is eating three quarters of the same bar as Jane or a half of a second bar. The underlying assumption is that it is of the second bar, but the concern again arises that the bars may not be of equivalent sizes thus the pieces of each bar could be different.

Within the category of food the following problem was created.

Goofy had half a cup of chocolate surup (sic) and Donald had ¾ cups of milk. Both wanted chocolate milk and decided to pour both the half cup of chocolate and 3/4 cups milk into a pitcher. How many cups were in the pitcher after the(y) added both the milk and the chocolate surup (sic)?

The researchers categorized this problem as acceptable, but these amounts certainly would not make acceptable chocolate milk. Goofy and Donald are Disney cartoon characters, so the writer may have created this situation to attract the attention of the elementary students. The chemistry of mixing the two substances together may not, in fact give a total measurement of 1 ½ cups of liquid, but the question was posed as how many cups were in the pitcher of both items.

Another category of acceptable problems referred to measuring distances.

While I was running, I kept track of my miles. I ran ½ mile on Monday morning and ¾ mile on Monday night. How far did I run on Monday?

With these types of problems, the referent of measurement being miles is the same. These problems are similar to the first category of acceptable recipe problems that used the cup as the referent unit of measurement. When the prospective teachers referred to units of measurements that are the same, then the fractions could be appropriately added, but pizzas, cakes, candy bars or other foods are not necessarily equivalent units unless specifically noted by the problem writer. The researchers from both countries agree on the importance of emphasizing the referent whole or equivalent unit of measurement so that the two fractions can be appropriately added. Creating the diagrams of the cakes that are not the same size as in Figure 1 can aid students in realizing the need for further clarification of equivalent wholes when creating problems.

The type of problem that exhibited an extreme difficulty in understanding the concept of the addition of two fractions whose sum is greater than one is represented by the example below.

In Mrs. C's class ½ of her students got A's on the test, and ¾ of her student's (sic) got B's. How many student's (sic) got A's and B's.

There were a small number of subjects who wrote this type of problem. These students certainly need to be re-taught the basic idea that the sum of fractional parts of a whole cannot be greater than the whole. The subjects who referred to a second whole, but did not make reference to the equivalence of two wholes, show a greater understanding than the subjects in this particular category.

The main differences in the responses of the NI and US subjects are attributed to the differences in how they were instructed. Since the US participants discussed their results of a pretest where they created a similar addition problem, 82% wrote acceptable responses to the post-test research question. In their instruction, the NI participants did not include discussions of referent wholes and thus did not include reference to equivalent units in their responses. The US students significantly improved from the pre test to the post test as 8 students created acceptable responses prior to instruction (20%), and 28 students created acceptable responses after instruction (82%) The findings show that discussing the referent whole is important in teaching addition of fractions especially when the sum is greater than one. Northern Ireland instructors now plan to include a discussion of the referent whole when teaching addition of fractions.

3.2 Division of Fractions

The similarities and differences that emerged from the students from the two countries in their responses of writing a story problem to show the meaning of $2\frac{1}{2} \div \frac{1}{2}$ were also analyzed.

As with the addition problems, both US and NI prospective teachers wrote the majority of their problems about food. Pizza problems were the most popular type of problem that were presented, although sharing apples for a Healthy Snack Break was included. Some problems even included drawings of cookies, butter, bagels and other foods. A typical acceptable problem in this category is presented below.

If there are two and a half pizzas and each child received half a pizza, how many children share it?

The following pizza problem showed an acceptable insight to the question, but also went a little further in terms of what they asked the pupils to do.

Simon has two and a half pizzas. He wants to share his pizzas with his friends and he wants each person to get half a pizza. How many friends can he share with?

Clearly here the problem solver must complete the prescribed calculation, but then needs to subtract one from the answer to find the number of Simon's friends, to exclude Simon himself.

One subject made a connection from the equivalent unit wholes required in the addition problems and transferred that knowledge to the question he wrote for division.

You have 2 whole pizzas of the same size and exactly ½ of a 3rd pizza that was also the same size. How many halves do you have altogether?

Another salient category of responses to the division problem involved measurement. During US instruction, an example was presented to determine how many ¾ yard bows could be made from 5 yards of ribbon (NCTM, 2000), where the instructor related the problem to the bows that are made for the mum corsages for their campus Homecoming celebrations. Six subjects may have recalled this example as they wrote their division problem about creating bows. For example:

I am making bows. I have 2 ½ yards of material. Each bow uses a ½ yard of material. How many bows can I make?

In the measurement category, four students created similar measurement problems involving running.

Members of a track team ran 2 ½ miles all together. If each person on the team ran ½ of a mile, how many people ran the race?

Several prospective teachers are members of the university track team, and thus related division of fractions to their own interests. The research suggests that making real world connections deepens understanding (Parker, 1996; Wolfe, 2001; Tate, 2003). The frequently used measurement model also uncovered several student misconceptions as noted below.

You are making bows for a wedding. If the amount of ribbon is 2 ½ yards, how many bows can you make with ½ yard of ribbon?

The writer possibly was attempting to ask how many ½ yard ribbons could be made from 2 ½ yards, but did not clearly articulate an accurate division problem that would produce 5 full bows. This writer possibly remembered the bow example discussed in class, but could not present the problem accurately.

Several misconceptions of subtracting a half or dividing in half rather than counting how many halves are included in 2 ½ were demonstrated. The following example shows 2 ½ being divided in half.

Laura was training for a track meet that she has on Saturday. She ran 2 ½ miles around her block, but ended up running half of that. How much did she run?

The only division problem response that was deemed unacceptable for the NI students is provided below.

The group bought two litre bottles of water and a 500ml bottle. When everyone had had a cup there was still half left over. How many mls were left?

It is not clear here when the student states that there was still half left over if they are referring to half a litre or half the total amount of water. If it is the former, there is a possibility that they may have read the question as a subtraction problem. If it was the latter, then this shows a total lack of understanding of this division problem. A cultural difference is noted in this problem, as the US students did not use the metric measurement system for their problems because of the different measuring systems in these countries.

The results of the analysis of division of fractions show that a majority of both groups of prospective teachers have an understanding of division of fractions. Sixty-two percent of US prospective teachers and 83% of Northern Ireland prospective teachers produced acceptable division problems.

4 Conclusions

This study analyzed the depth of understanding that prospective elementary teachers from Northern Ireland and the United States exhibited on rational numbers, specifically addition and division of fractions. As the study was designed for them to create problems regarding the addition of two fractions whose sum is greater than one, the findings show that both groups had similar understandings as well as similar misconceptions. All NI prospective teachers understood that the sum of the two fractions is more than one whole and thus included a second whole in all of their problems, while a small number of US students, even after instruction, continued to write problems where they disregarded the sum being greater than one, and merely posed a problem that asked for the sum of the two fractions, which was not meaningful. The difficulty that arose for the NI students was not making reference to equivalent whole when creating their problems. Due to the results of this research study, Northern Ireland instructors will now include a discussion of the referent whole when teaching addition of fractions. US instructors need to offer additional instruction to the prospective teachers who disregard the sum of two fractions being greater than a whole when creating a real life problem where they add ½ of the whole to ¾ of the same whole.

The Northern Ireland participants overall had a higher percent of acceptable responses than US participants in the area of division, where the US participants had a higher percent in the area of addition. Discussions between the two researchers will continue to determine how the NI instruction of division of fractions is different from the US instruction and how the US instructor can implement the NI instructional strategies. Since the NI participants have had elementary classroom teaching experience, and the US participants have not, perhaps the actual classroom teaching experiences that the NI participants have strengthened their understanding of division of fractions. These types of discussions offer ways for mathematics educators to share ideas to improve their instruction. The researchers are expanding this study to include a group of prospective teachers from South Africa. They also welcome additional international researcher partners.

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Planning Fraction Lessons: A Case Study

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We are doing a case study with three basic education teachers who have joined a master degree focused on a professional strengthening of their teaching experience. In the current research phase we explore how they plan activities for teaching fractions and what kind of difficulties they confront in such planning. In this document we will only make reference to the case of Delia, a fifth grade teacher at an elementary school who has decided to plan her fraction lessons relying on the meaning of measure as the planning's didactic object. This case study has been done from two fundamental methodological instruments: the observation that took place at a master degree seminar (where the activities proposed by Delia for teaching fractions as a measure were presented) and the interviews, where the aforementioned teacher reflected about the obstacles she experienced through the didactic design process.

Keywords: fractions, meaning of measure, planning activities, planning difficulties.

1 Introduction

In a master degree focused on theoretically and practically enriching the educational experience of Mathematics teachers (including instructors of kindergarten, primary and secondary levels), we are in charge of a seminar oriented to teaching and learning fractions. For this matter, we have created a space in which we initiated a case study with three primary and secondary teachers.

The case study will last three years, according to the length of the master degree program. During the first development phase, we explored which were the difficulties on teaching fractions experienced by the three teachers selected. Through the case study's second phase, we are gathering information about how these teachers plan their fraction lessons and which are the obstacles to overcome in that task. The present document corresponds to this last phase and exhibits the case of Delia. Afterwards, in the third phase we will describe and analyze how the teachers apply in the classroom the activities designed during the previous phase, after having joined a master degree in which they had to reflect on their educational experience.

This research's importance dwells in the fact that it provides information about the aspects that make the task of teaching fractions difficult, how these basic education teachers program the instruction, and how reflecting on this may improve their teaching practice.

2 Theoretical Framework

Since this essay's bases are related to different aspects, we will now present each one of them in separate sections.

a) Resuming formal education

Within the wide range of opportunities we can find in adult education, Bishop (2000) states that a very important option is the formal education that takes place at a school, university or higher education institute. The latter is the situation of the subjects being studied in this research, and who have joined a master degree developed at a research center. This is how we gather the features for the profile of "adults that resume higher education studies" (O'Donoghue, 2000) with the purpose of improving their professional activities.

Adult education also draws attention to the exploration of the personal needs of each subject involved, so as to favor subsequent formative interventions that may provide optimal results (Fitz Simons and Godden, 2000). The interests and needs manifested by the adults in formation process are a fundamental starting point for the research done on this phenomena, as well as for the didactic treatments carried out within the formal studies.

Medina Fernández (1997) recognizes that the process of adult formation shows a "critical reflexibility" through which the person's experience is subject to a meticulous analysis that may foster the reorganization of such experience. We take this characteristic as a support to facilitate the reconstruction of the teacher's educational practice that this case study may allow. We base our argument on the fact that the limitations acknowledged explicitly by the person may be overcome in a near future.

In addition, the adult needs to know how to assimilate new knowledge in order to be able to recognize its practical consequences (Cabello Martínez, 1997). Therefore we expect that, through the teacher's own reflection process, we may establish how their conceptions evolve and how this gives place to new modalities of teaching interventions.

b) Fractions in specialized literature

In the master degree seminar on teaching and learning fractions, the six teachers involved worked on reading certain writings that will be immediately presented and that were very important for designing their didactic activities.

Concerning the semantic diversity of fractions, we recovered from Kieren (1980, 1983, 1984, and 1988) and Kieren, Nelson and Smith (1985) the difference among the meanings of **part-whole relationship**, **intuitive quotient**, **measure**, **multiplicative operator** and **ratio**. The concept of **part-whole relationship** is the most primitive and serves as a basis for the rest; Kieren (1980, p. 134) defines the **part-whole relationship** as "some whole is broken up into 'equal' parts, fractional ideas are used to quantify the relationship between the whole and a designated number of parts".

The **intuitive quotient*'s*** construct corresponds to a number in the form a/b, in which the person's interpretation of the numerator and denominator corresponds to the conditions of division and sharing of one or more objects between two or more people.

The sense we give to the fraction as a **measure** is that of a "comparer" (Freudenthal, 1983) between two or more objects so as to determine their size. Kieren (1980, p. 136) states that "the measurements tasks means the assignment of a number to a region (taken here in the general sense of this word; may be 1, 2 or 3 dimensional or have other characteristics). This is usually done through an iteration of the process of counting the number of the whole units usable in 'covering' the region, then equally subdividing a unit to provide the appropriate fit."

We assign the meaning of the relation between two dimensions to the **ratio** (Hart, 1988, Ruiz, 2002, Ruiz and Valdemoros, 2001, 2002 and 2006). About the **ratio** idea Kieren (1980, p. 135) indicates: "The ordered pair notation takes on new significance with respect to ratio relationship – the quantitative comparisons of two qualities. Three tenths (3/10) of a floor surface has a very different meaning than 3/10 which compares the number of girls and boys on a soccer team."

A very important aspect that helps understand the fraction as **multiplicative operator** is recognizing the number that allows going from an initial state to a final state within the problem solving process (Ruiz, 2002, Ruiz and Valdemoros, 2001, 2002 and 2006). Kieren (1980, p. 136) says: "The operator sub-construct portrays rational numbers as mechanisms which maps a set (or region) multiplicatively onto another set. Thus a '2 for 3' operator maps a domain element 12 to a range element 8 and a '2/3' operator maps a region onto a similar region of reduced size."

In an analogue way, an aspect of great importance is Piaget, Inhelder and Szeminska's (1970) contribution on part-whole and part-part relationships as expressions of the structural bases that support fractions; these relationships are, respectively, the pillar of addition and multiplication operations. Here, we emphasis the integration sense of the part in the whole (making reference to the part-whole relationship), as well as the possible consideration of the part as a new whole susceptible to be submitted to division (making reference to the part-part relationship).

We highlight Freudenthal's (1983) observation regarding teaching fractions because he states that the final richness of knowledge depends on the phenomenological diversity with which fractions are taught. Such knowledge will be the product of a great variety of resources used during the teaching process. Particularly, he establishes that teaching fractions cannot be limited to the traditional part-whole relationship, for that way would only bring about proper fractions, this is to say, its scope would be limited. According to Freudenthal's approach, in order to encompass from "fracturers" and "comparers", to "multiplicative operators", we require a non-restrict use of the equivalence, allowing a non-limited fraction production. To achieve this, this researcher proposes not only the use of didactic models of area and length, but also the inclusion of manipulative materials.

Regarding the activities of didactic design, we pay special attention to Streefland (1991), who has remarkably strengthen the process of teaching fractions through programming a course composed of new models supported on real situations and significant manipulative materials. In such course, this researcher recommends the use of variable elements coming from real life situations, for example the atmosphere created at a restaurant where the distribution of the companions at the tables, the way in which food is served, the number of clients, and the chefs' job, altogether, allow an appropriate use of fractions.

c) Teaching fractions in mexican schools

In this section we will present the program and official books used in fifth grade in all mexican schools. As Delia teaches this grade, she has taken into account all this didactic auxiliaries in order to design the appropriate activities for the fraction lessons.

The fifth grade program (Secretaría de Educación Pública, 1993) contemplates the following: introduction to sevenths and ninths as new numbers; application of decimal fractions; use of the fraction as a measure, quotient, ratio and multiplicative operator; use of different resources to establish the equivalence among fractions; recognition of fractions within the number line; introduction of mixed fractions; and solving problems of fraction addition and subtraction with like and unlike denominator.

The students have access to the *Text Book* (Secretaría de Educación Pública, 2002b), which has lessons about fractions that cover the curricular content from the fifth grade program. The aforementioned lessons are presented through situations that call the attention of the students and, at the same time, are related to their reality.

In order to complement and enrich the activities proposed in the child's book, the teacher has an *Activity Book* (Secretaría de Educación Pública, 1994) that presents multiple fraction problems that can be alternated, at the teacher's discretion, with the lessons described above.

Finally, the *Teacher's Book* (Secretaría de Educación Pública, 2002a) suggests the teacher how to use the *Text Book* and the *Activity Book* together and also proposes different didactic strategies to cover each lesson of the child's book.

3 The Case of Delia in the Didactic Design Phase

From the study of three cases already described in the Introduction, this report will only focus on the case of Delia. She is a fifth grade teacher at an elementary school who, at the time of this research's didactic design phase, is 36 years old and has six years of teaching practice.

There were several reasons why we chose Delia from a group of six basic education teachers who joined the master degree. First, she said that teaching fractions was a very difficult practice. Moreover, Delia is a responsible and efficient teacher who is always making an effort to improve her teaching practice and keeps an autocritical attitude towards her performance. Finally, she is an outstanding student in this program.

The previous phase of our research consists in exploring which difficulties in teaching fractions Delia had experienced in her previous professional experience. In such phase we could establish that she had a remarkable dependence on the official teaching books (Secretaría de Educación Pública, 1994, 2002a, and 2002b), which reduced her educational creativity and autonomy. Furthermore, the difficulties she confronted remained linked to the lack of sense in situations involved in the teaching proposals derived of the text book, and thus she tended to develop teaching strategies loaded with algorithms because the predominance of the syntactic procedures and rules allowed her to avoid the didactic treatment of meanings and semantic processes related to fractions.

3.1 The approached research problem

Given that in the current phase of didactic instruments design Delia decided to focus on the treatment of fractions as a measure, we acknowledge that the research problem is **programming activities** for teaching fractions in measurement situations as well as the **design difficulties** that this fifth grade elementary teacher may specifically encounter. Regarding the latter, we assume that the difficulties experienced in the didactic design constitute a particular type of obstacles related to teaching fractions in general.

From the previous statement, we formulated the following research question:

Which are Delia's teaching proposals to deal with the fraction as a measure and what difficulties related to the design has she experienced?

3.2 Methodological instruments

In order to monitor this study, we based its second phase on observation and interviews.

In the seminar for learning and teaching fractions we made an **observation** of the case because Delia and the other participants examined literature specialized on fractions and also exposed progressively and critically the teaching activities programmed for the class. In order to **validate** this case study, we based our research in this observation processes, and as a result generated **triangulations** and contrasts between noteworthy details of this experiences and relevant moments of the interviews. In addition, the information gathered through observation constituted one of the main pillars in this case's analysis.

The two **interviews** were hold individually and designed in a way to allow enrichment of the dialogue. It also gave place to reflection on which bibliographic sources were used by Delia when programming the lessons of fractions as a measure, as well as which design difficulties she confronted while carrying out this task. This information reveals all the aspects that cannot be reconstructed through direct observation.

3.3 The results of the observation

Through observation we could determine the nature of Delia's didactic planning, which consists on dealing with fraction as measure. We could do this through the seminar on learning and teaching fractions, where Delia presented in detail her design and reflected on it. Next we will present the basic aspects of such programming and its analysis afterwards.

The first component of the didactic design was a diagnostic questionnaire, with which Delia decided to explore the student's learning before starting the teaching plan. This questionnaire was composed of ten problems (she created some of them and others were taken from Secretaría de Educación Pública, 1994, 2002b and were adapted afterwards).

Delia planned twelve work sessions and designed them based mainly on Freudenthal (1983), Streefland (1991) and Kieren's (1983, 1984 and 1988) theoretical statements. In this manner, when posing the problematic situations that constitute the twelve sessions, the didactic progression followed up initiated with **estimation** activities of the part-whole relationship, followed by "**direct comparisons**" and finally "**indirect comparisons**" (according to Freudenthal's, 1983, statement, the fraction as a measure is a "**comparer**" that favors "**direct comparisons**" between two objects or "**indirect comparisons**" between two objects, such comparisons are provided by a third object or measurement instrument). Regarding indirect comparisons, at first, the measurement instrument was not conventional (for example, a paper strip, a ribbon or a lace) but soon after it became a conventional measurement unit (meter, kilo, liter, kilogram or time units). In all cases, the fraction came up as a result of the process carried out.

As an example to illustrate the aforementioned, we present one of the first activities of Delia's didactic proposal, which consists in distributing the water contained in a jar, into smaller and different sized containers, so as to favor an **estimation** of the liquid hold in each container, expressed as a fraction of the total amount of water in the jar. Basically, this activity allows the estimation of different fractions from the total amount of liquid, which can be compared and ordered *a posteriori* from a concrete situation in which students may manipulate objects to foster comprehension.

Although Delia's performance in this research's phase was mainly focused on the teaching session design and her proposal will be systematically applied in the future, this did not stop her from trying out the teaching activities programmed with the fifth grade students. Through these preliminary experiences she could improve the design and redirect her course.

In general, if we compare these results with those of the research's previous phase (that in which Delia had proved a strong dependence on official teaching books, Secretaría de Educación Pública, 1994, 2002a, 2002b, thus showing a poor and highly mechanized professional practice) we can recognize a clear progress in her teaching performance, through the appearance of autonomy and creativity signs. Despite these advances in her professional practice, some difficulties still appear when she plans her fraction lessons, but we deal with this in the next section.

3.4 The results of the interviews

In the dialogue developed during the interviews, the most important questions we posed to Delia were: 1) **Why** did she choose **the fraction as a measure** as the approach of the didactic design required in the master degree? 2) **How did she plan** the activities of teaching fractions as a measure in the fifth grade? 3) What **design-related difficulties** did she confront through this planning process?

In the following paragraphs we present a summary of the answers to each question and afterwards we analyze them.

1) As mathematical object of her teaching activities programming, Delia chose the **fraction as a measure** because after revising the child's book (Secretaría de Educación Pública, 2002b) and the books available for fifth grade teacher (Secretaría de Educación Pública, 1994, 2002a) she confirmed that the meaning of measure is the most developed of all.

Regarding the recognition of the main teaching contents, Delia remains subject to the fundamental strategies of the official books used in mexican schools (situation we emphasized in our research's previous phase). If she had referred to the fifth grade national teaching program (Secretaría de Educación Pública, 1993) she would have highlighted that all meanings of fraction are relevant and, from there, she would have exposed other reasons for her election.

2) Before initiating the teaching activities design, Delia searched in literature specialized on fractions (specifically the readings of the seminar) in order to determine the basic notions the students must build on the topic of the fraction as a measure, because as she said: "I had to understand by myself what to design in order to bring the child near to those notions".

About the planning process as such, Delia commented: "my ideal of design is to begin from notions that are common to the students, starting by the estimation so that after, little by little, they can become familiar with conventional measurement units by using fractions". Delia related estimation to the part-whole relationship in situations in which conventional measurement units were still not used. She recently introduced the latter (units of weight, capacity, length and time) after promoting multiple "direct comparisons" and "indirect comparisons" among several objects (according to Freudenthal, 1983).

Delia also pointed out that a very important part of the planning process was to formulate particular objectives that allowed organizing each task's design and that, at the same time, oriented the corresponding construction made by the child. On the other hand, this teacher assigned a function to the design, which consists in explaining the teacher's role as conductor and guide of the teaching process.

Delia emphasized that the teaching activities were designed by her, since she did not take them from any of the authors consulted because she would not be proposing something innovative, instead, she would be limiting herself to repeat what has been proposed by others. This situation is of great importance for us because thanks to this production Delia is making headway in her professional experience, by having more solid practices loaded with sense.

3) One of the main difficulties Delia experienced in the didactic design was setting objectives, because through them she intended to establish the notions the students had to build regarding the fraction as a measure. We could confirm that, after a refinement process, this teacher achieved an appropriate formulation of the final objectives.

According to Delia, another relevant difficulty of planning lessons was to anticipate when and how the teacher should interrogate the children in order to foster reflection and dialogue among them. We think that in a future application of the proposal designed, during the next phase of the research, Delia will have clearer options for interventions in the class.

Delia recognized that an obstacle linked to progression and sequence of activities had to do with finishing any of them and starting the next activity, without leaving anything unfinished or incomplete between them. We think that the fundamental notions this teacher included in her design present a reasonable continuity through the whole process.

The last design-related difficulty Delia recognized was the originality of the planning, according to her own point of view. Concerning this, she said that one tends to design from what others have already programmed, when what we really need is to create original activities and tasks. We interpret her current situation as a remarkable advance, because *Delia decided no reply existing designs* and proved to have done all the planning from her own ideas (a very difficult final result of planning process).

In general, we can say that Delia has overcome most of the difficulties she had previously identified. We suppose that the remaining limitations will be overcome in the near future, when this fifth grade teacher makes a systematic application of her didactic proposal in her class, which will allow her to decide the optimal conditions to develop her design in the classroom.

Nevertheless, what is interesting is that Delia's recognitions may be a remarkable contribution for other teachers, since she provides a critical explanation regarding the location of obstacles when planning fraction lessons.

4 Conclusions

Delia evolved from a period defined by a profound dependence on official educational books and characterized by a lack of personal teaching initiatives, to the current planning phase in which she procured to develop an original design for teaching fraction as a measure in the fifth grade of primary school. Such achievements were possible thanks to a careful revision of literature specialized in fractions and a continual critical reflection on her teaching practice (all of this, in the master degree seminar she joined).

Delia's didactic proposal was supported on an initial process in which the student developed estimations of the part-whole relationship, followed by "direct and indirect comparisons" among several objects that facilitated the introduction of some conventional measurement units (specifically capacity, length, weight and time units), associated to the use of fractions.

Within the task of planning the lessons, Delia recognized several difficulties linked to carrying out the design: achieving an appropriate setting of objectives for the tasks programmed, foreseeing suitable interventions by the teacher in the contemplated teaching process, appropriately following the different didactic activities composing the proposal, and being original for formulating diverse situations that constituted the design. Delia has already overcome most of the difficulties described, but there are still some others remaining that will perhaps disappear in the next stage of application of her teaching proposal for a fifth grade class.

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The Role of Number Structures on Pupils' Over-Use of Linearity in Missing-Value Problems

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This study builds on previous research showing that primary school pupils over-rely on proportional methods when solving non-proportional missing-value word problems. It is hypothesized that when the numbers in word problems form integer ratios, this will stimulate pupils to apply proportional methods, even if this is inappropriate. It is furthermore expected that the effect will diminish from grade 4 to 6 (with pupils' age and proportional reasoning experience). The results confirm both hypotheses.

Keywords: illusion of linearity, ratio and proportion, missing-value problems, number structures

1 Theoretical and empirical background

Contemporary math education curricula consider as an important goal that pupils can model and solve real-world problem situations. Traditionally, mathematical modelling and applied problem solving are taught in primary school through word problems (Verschaffel, Greer, & De Corte, 2000). However, during the last decade, it has been shown that pupils start to perceive word problem solving as a puzzle-like activity with little grounding in the real world. One of the problems is that pupils can often successfully use superficial cues to decide which operations are required to solve word problems in textbooks or tests. Arguably, this does not lead to a disposition to discriminate between problems that can and cannot be modelled and solved by means of (a set of) straightforward arithmetical operations, but rather to a tendency to cope with all problems in a stereotyped and superficial way.

A clear example of such a 'corrupted' modelling process is pupils' tendency to over-use the proportional model. Because of its wide applicability in mathematics and science, proportional reasoning is a major topic in primary and secondary math education. Typically, from 3rd or 4th grade on, pupils are increasingly confronted with missing-value proportionality problems (in which three numbers are given and a fourth is asked). Studies indicate that pupils associate such word problems with the proportionality scheme, even when it does not appropriately model the problem situation (De Bock, Verschaffel, & Janssens, 2002). For example, several studies (see Verschaffel et al., 2000) found that more than 90% of 10-12-year olds answer "170 seconds" to the following item: "*John's best time to run 100 metres is 17 seconds. How long will it take him to run 1 kilometre?*" Another well-documented case relates to 12-16-year old pupils' tendency to give proportional answers to geometry problems like "Farmer Gus needs 8 hours to fertilise a square pasture with sides of 200 metres. How much time will he approximately need to fertilise a square pasture with sides of 600 metres?" (answering "24 hours" in this case) (De Bock et al., 2002; Modestou, Gagatsis, & Pitta-Pantazi, 2004). But also upper secondary and even university students over-use proportionality in various domains like probability (Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003) or calculus (Esteley, Villarreal, & Alagia, 2004).

In a recent study (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005), we attempted to determine when the tendency to routinely apply proportional methods originates, and how it develops with pupils' increasing age and educational experience. For this purpose, we analysed large numbers of 3rd to 8th graders' solutions to various proportional and non-proportional arithmetic problems. An example of a proportional problem used in that study is: "*In the shop, 4 packs of pencils cost 8 euro. The teacher wants to buy 24 packs. How much does she have to pay?*" An example of a non-proportional problem is "*Ellen and Kim are running around a track. They run equally fast, but Ellen started later. When Ellen has run 5 rounds, Kim has run 15 rounds. When Ellen has run 30 rounds, how many has Kim run?*" (proportional answer: $5 \times 3 = 15$ rounds, so $30 \times 3 = 90$ rounds, correct answer: $5 + 10 = 15$ rounds, so $30 + 10 = 40$ rounds). We found that the number of correct answers on the proportional problems considerably increased with age: from 53% correct answers in 3rd grade to 93% in 8th grade. Most learning gains were made between 3rd and 5th grade. But we also found that, as expected, the tendency to *over-use* proportional methods initially developed in parallel with pupils' emerging proportional reasoning skills. In 3rd grade, 30% of all non-proportional problems were answered proportionally, and this increased considerably until 51% in 5th grade (with a decrease thereafter to 22% in 8th grade). We concluded that pupils – at the moment when they acquire proportional reasoning skills as a result of their training in solving 'typical' proportionality problems – tend to overgeneralise proportional methods and learn to apply them on the basis of superficial problem characteristics, like the missing-value formulation of word problems.

2 Proportional reasoning: how numbers affect solutions

Despite the evidence for the over-use of proportional methods in various mathematical domains as documented and analysed by research worldwide, there is one – possibly important – issue that has been largely overlooked in that research so far: the *nature of the numbers* in the non-proportional problems, and the possible impact of these numbers on pupils' tendency to use proportional methods to these problems.

The issue can be clarified by considering the literature on proportional reasoning. A frequently reported error on missing-value proportionality tasks (e.g., Noelting, 1980; Hart, 1984; Karplus, Pulos, & Stage, 1983) is the so-called 'constant difference' or 'additive' strategy. In this strategy, the relationship within the ratios is computed by subtracting one term from a second, and then the difference is applied to the other ratio (instead of considering the multiplicative relationship). For example, "Mixture A has 2 oranges for 6 parts of water. Mixture B tastes the same, and it has 10 oranges. This is 10 – 2 = 8 oranges more, so it needs 6 + 8 = 14 parts of water". The most prominent explanation for this error is that it is a kind of 'fall-back' strategy (especially for less skilled proportional reasoners) to deal with proportionality problems with non-integer ratios, like in the problem "One mixture has 2 oranges to 7 parts of water. Another mixture tastes the same and has 5 oranges. How many parts of water does it have?" (See, e.g., Karplus et al., 1983, who call this the 'fraction avoidance syndrome').

In sum, correct reasoning on proportional (missing-value) tasks sometimes is affected by the nature of the numbers. Particularly less skilled proportional reasoners perform worse if ratios in proportional problems are non-integer. The claim underlying the present study is that this finding also applies to the use of proportional methods to solve *non*-proportional problems. The non-proportional problems in many of the above-mentioned studies (e.g., De Bock et al., 2002; Van

	Item I	Item II
Proportional **(PR)**	**1**	**2**
Non-proportional		
Additive (AD)	**3**	**4**
Constant (CO)	**5**	**6**
Affine (AF)	**7**	**8**

Table 1: Design of test items

Dooren et al., 2003, 2005; Verschaffel et al., 2000) contained 'easy' numbers: Both the internal and the external ratio were integer, so although the problems had no proportional structure, the given numbers somehow invited pupils to conduct proportional calculations. Linchevski, Olivier, Sasman, and Liebenberg (1998) found some indications that such integer ratios could 'trigger' unwarranted proportional reasoning (an error they call the 'proportional multiplication error'), but they did not systematically test this hypothesis. They concluded that "it remains a question for further research to establish whether an approach with non-seductive numbers will *prevent* children from making the multiplication error" (p. 222). In other words, while non-integer ratios cause more errors on proportional problems, they may have an opposite impact in non-proportional problems, as pupils may be less inclined to over-use proportional methods when confronted with non-integer ratios. The goal of the present paper is to test this hypothesis, and in this way, to gain further insight in the determinants of pupils' tendency to over-use proportional methods.

3 Method

508 4th, 5th and 6th graders from 5 randomly chosen Flemish primary schools participated in this study. They received a test containing 8 missing-value word problems presented in random order. The problems were identical to those used by Van Dooren et al. (2005). The design of the test is shown in Table 1 and examples of word problems are given in the left column of Table 2. The test contained one type of *proportional* problems (for which proportional strategies provide the correct answer) and 3 types of *non-proportional* problems (for which another strategy must be applied to find the correct answer). The 3 types of non-proportional problems had different mathematical models underlying them: additive, constant and affine (i.e., a model of the form $f(x) = ax + b$). For each category, 2 items were included.

Central to this study was that the numbers in the word problems were experimentally manipulated, as clarified in Table 2. The manipulation was such that when focussing on the ratios between the numbers, one ends up either with integer (I) ratios or with non-integer (N) ratios. This manipulation led to 4 different versions of each item:

- **II**-version: external ratio (a/b) integer and internal ratio (a/c) integer
- **NI**-version: external ratio (a/b) non-integer but internal ratio (a/c) integer
- **IN**-version: external ratio (a/b) integer but internal ratio (a/c) non-integer
- **NN**-version: external ratio (a/b) non-integer and internal ratio (a/c) non-integer

For example, the II-version of the additive (AD) word problem in Table 2 was:

> Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run 16 rounds, Kim has run 32 rounds. When Ellen has run 48 rounds, how many rounds has Kim run?

A correct reasoning for this II-version is to focus on the (constant) difference between the numbers: *Kim is initially running 16 rounds ahead of Ellen. This remains the same, so when Ellen has 48 rounds, Kim has 48 + 16 = 64 rounds*. When reasoning proportionally here (which is inade

	Example of word problem	Numbers and solutions for each version[a] II	NI	IN	NN
PR	In the shop, ***a*** packs of pencils cost ***b*** euro. The teacher wants to buy ***c*** packs. ***How much*** does she have to pay?	9 27 18 ***C:* 54**	9 24 18 ***C:* 48**	9 27 12 ***C:* 36**	9 24 12 ***C:* 32**
AD	Ellen and Kim are running around a track. They run equally fast but Ellen started later. When Ellen has run ***a*** rounds, Kim has run ***b*** rounds. When Ellen has run ***c*** rounds, ***how many*** has Kim run?	16 32 48 ***C:* 64** ***P:* 96**	16 24 48 ***C:* 56** ***P:* 72**	16 32 36 ***C:* 52** ***P:* 72**	16 24 36 ***C:* 44** ***P:* 54**
CO	A group of ***a*** musicians plays a piece of music in ***b*** minutes. Another group of ***c*** musicians will play the same piece of music. ***How long*** will it take this group to play it?	25 75 50 ***C:* 75** ***P:* 150**	25 40 50 ***C:* 40** ***P:* 80**	25 75 35 ***C:* 75** ***P:* 105**	25 40 35 ***C:* 40** ***P:* 56**
AF	The locomotive of a train is 12[b] m long. If there are ***a*** carriages connected to the locomo-tive, the train is ***b*** m long in total. If there would be ***c*** carriages connected to the locomotive, ***how long*** would the train be?	4 44 8 ***C:* 76** ***P:* 88**	4 42 8 ***C:* 74** ***P:* 84**	4 44 10 ***C:* 92** ***P:* 110**	4 42 10 ***C:* 90** ***P:* 105**

[a] Numbers are schematically represented as ***a*** ***b***
c ***x*** (***C***: correct solution, ***P***: proportional solution)
[b] For the NI and NN-version, this value was 10

Table 2: Examples of word problems and manipulation of numbers in the II-, NI-, IN-, and NN-versions

quate, of course), one needs to focus on the ratios between the numbers: either on the external ratio *a/b* (*initially, Kim has twice as many rounds as Ellen (32/16), so when Ellen has 48 rounds, Kim has 48 × 2 = 96 rounds*), or on the internal ratio *a/c* (*at the end, Ellen has 3 times as many rounds as initially (48/16), so by that time, Kim has 32× 3 = 96 rounds*). For the NN-version of the same word problem, the correct approach is comparably easy (only the constant difference to work with differs). Proportional reasoning, however, is considerably more complex here, because both the internal and external ratio are non-integer: The multiplicative 'jump' from 16 to 24 is far less evident than that from 16 to 32, but for a skilled proportional reasoner, it is still feasible. Reasoning proportionally for the NN-version might be, for example: *initially, Kim has 3/2 times as many rounds as Ellen, so when Ellen has run 36 rounds, Kim has run 36 ×3/2 = 54 rounds*.

The tests were manipulated so that – at a random basis – 2 of the 8 word problems were in the II-version, 2 in the NI-version, 2 in the IN-version and 2 in the NN-version. Pupils' answers to the problems were classified as either *correct* (C, correct answer was given), *proportional error* (P, proportional strategy applied to a non-proportional item) or *other error* (O, another solution procedure was followed).

4 Hypotheses

Due to space restrictions, we limit ourselves to comparing the 'extreme' versions of the proportional and non-proportional items, i.e., the II- and NN-versions with, respectively, *both* (internal and external) ratios integer and *no* ratios integer.

A first set of hypotheses relates to pupils' performances on the *proportional problems*. Based on the literature on proportional reasoning mentioned above, we expect that proportional problems with non-integer ratios (NN-version) will cause more errors (i.e., less correct (C) answers) than proportional problems with integer ratios (II-version) (HYP 1A). Additionally, we anticipate that this effect will be stronger in younger, less experienced proportional reasoners, so we predict that the different performance on the II- and NN-versions will be most pronounced in 4th grade, and that it will gradually diminish through 5th and 6th grade (HYP 1B).

The second set of hypotheses deals with the *non-proportional word problems*. As argued above, we expect that problems with non-integer ratios (NN-version) will elicit less unwarranted proportional (P) answers than problems with integer ratios (II-version) (HYP 2A). We expect that particularly for the additive items (AD), the decrease in P-answers will result in more correct (C) answers – because the 'additive' strategy that pupils often *erroneously* apply to non-integer proportional problems is exactly the *correct* strategy for AD-items –, whereas for the constant (CO) and affine (AF) items, the decrease in P-answers might as well result in more other errors (O-answers) (HYP 2B). Finally, as for the proportional items, we expect that differences in the number of P-answers on the NN- and II-versions of the non-proportional items will be the strongest in the 4th graders, and will gradually diminish through 5th and 6th grade (HYP 2C).

5 Main results

Table 3 shows the percentage of correct answers to the *proportional problems*. As expected (HYP 1A), the NN-versions of the proportional problems elicited less correct answers (56.8%) than the II-versions (82.1%). A repeated measures logistic regression analysis showed that this difference was significant, as there was a main effect of 'number type', $\chi^2(1, N = 508) = 52.51, p < .0001$.

	4th grade	5th grade	6th grade	Total
II	65.2	86.3	96.4	82.1
NN	23.6	63.8	85.5	56.8

Table 3: % correct answers on the proportional problems in the II- and NN-version

The analysis also reveals a 'number type' × 'grade' interaction effect, $\chi^2(2, N = 508) = 166.59, p < .0001$. In line with HYP 1B, the difference between the II- and NN-version was very strong in 4th grade (65.2% correct answers to the II-version and only 23.6% on the NN-version), less strong but still significant in 5th grade (with 86.3% and 63.8% correct answers, respectively), and not significantly different in 6th grade (96.4% and 85.5% correct answers, respectively).

In Table 4 we have split up the results for the 3 different types of *non-proportional problems*.

		4th grade			5th grade			6th grade			Total		
		C	P	O	C	P	O	C	P	O	C	P	O
AD	II	57.3	23.6	19.1	48.8	35.0	16.2	48.1	30.1	21.7	51.6	29.3	19.0
	NN	80.9	0.0	19.1	68.8	12.5	18.8	68.7	25.3	6.0	73.0	12.3	14.7
CO	II	6.9	57.4	35.6	13.6	63.0	23.4	5.7	64.8	30.0	8.6	61.7	29.7
	NN	17.2	8.1	74.7	12.4	38.3	49.4	6.8	61.3	31.8	12.1	36.0	52.0
AF	II	13.8	54.0	32.2	23.5	54.3	22.2	28.4	61.4	10.2	21.9	56.6	21.5
	NN	12.6	12.6	74.7	18.5	38.3	43.2	31.8	52.3	15.9	21.1	34.4	44.5

Table 4: % correct, proportional and other answers
on the non-proportional problems in the II- and NN- version

It shows that the NN-versions elicited considerably less P-answers than the II-versions, and this was true for each type of non-proportional problem. For the additive (AD) problems, the II-versions elicited 29.3% P-answers, and the NN-versions only 12.3%, $\chi^2(1, N = 508) = 23.41$, $p < .0001$. For the constant (CO) items, the II-version elicited 61.7% P-answers vs. 36.0% in the NN-version, $\chi^2(1, N = 508) = 34.03$, $p < .0001$. Finally, for the affine (AF) items, percentages were 56.6% and 34.4%, respectively, $\chi^2(1, N = 508) = 31.54$, $p < .0001$. So **HYP 2A** was confirmed

Table 4 suggests that also **HYP 2B** was confirmed:

- For the AD-items, as expected, the decrease of P-answers resulted in an increased number of C-answers: The II-versions got only 51.6% C-answers whereas the NN-versions got 73.0%, $\chi^2(1, N = 508) = 24.71$, $p < .0001$, while there was no significant difference in the number of O-answers (19.0% and 14.7% respectively).
- For the CO- and AF-items, the decrease in the number of P-answers led to a significantly higher number of O-answers: For the CO-items, there is an increase from 29.7% to 52.0%, $\chi^2(1, N = 508) = 25.99$, $p < .0001$, and for the AF-items, the increase is from 21.5% to 44.5%, $\chi^2(1, N = 508) = 33.82$, $p < .0001$. No significant differences are found in the number of C-answers, neither for the CO-items (8.6% and 12.1%), nor for the AF-items (21.9% and 21.1%).

Finally, **HYP 2C** was confirmed too: The differences in the number of P-answers to the NN- and II-versions were the largest in the 4th graders. In 5th and especially 6th grade, differences were considerably smaller, or even completely gone:

- AD-items: The 'number type' × 'grade' interaction effect for P-answers, $\chi^2(2, N = 508) = 25.19$, $p = .0003$, indicates that 4th graders gave significantly more P-answers to the II-variant (23.6%) than to the NN-variant (0.0%). The difference was still present in 5th grade (35.0% vs. 12.5%), but 6th graders gave almost equal numbers of P-answers to the II- and NN-variant (30.1% vs. 25.3%).
- CO-items: A similar 'number type' × 'grade' interaction effect was found, $\chi^2(2, N = 508) = 40.60$, $p < .0001$: In 4th grade, the II-variant elicited much more P-answers (57.4%) than the NN-variant (8.1%). In 5th grade the difference was smaller but still significant (63.0% vs. 38.3%), but in 6th grade, the difference had disappeared (with 64.8% and 61.3% P-answers, respectively).
- AF-items: Again, a 'number type' × 'grade' interaction effect, $\chi^2(2, N = 508) = 32.83$, $p < .0001$, showing a large difference in P-answers in 4th grade (54.0% on the II-variant vs. 12.6% on the NN-variant), a smaller difference in 5th grade (54.3% vs. 38.3%), and a non-significant difference in 6th grade (61.4% vs. 52.3%).

6 Conclusions and discussion

Earlier studies convincingly indicated that pupils of various ages tend to apply proportional methods to solve various kinds of missing-value word problems, even when this is not appropriate. Remarkably, the problems in these studies always comprised 'easy' numbers (i.e., the internal and external ratio were integer). Some researchers have argued that this may have 'triggered' unwarranted proportional reasoning. The current study explicitly addressed this claim by experimentally manipulating the integer or non-integer character of the ratios in the word problems.

The results on the proportional problems replicated those reported in the proportional reasoning literature: Problems with non-integer ratios elicited less correct answers than variants with integer ratios. Moreover, as expected, this effect was particularly strong in 4th grade, while it became less influential in 5th and especially 6th grade.

With respect to the non-proportional problems, our findings confirmed the hypothesis that pupils are less inclined to over-use proportional methods when the given numbers do not form integer ratios. Also in line with our expectations, the decrease of unwarranted proportional answers resulted in better performances on problems with an additive structure, as the 'additive strategy' – which is often erroneously applied on non-integer proportional problems – is correct for solving this kind of word problems. For constant and affine word problems the decrease in proportional answers did not result in better performances. Instead, pupils started to commit more other errors. Finally, we also found the expected interaction effect: 4th graders were particularly sensitive to the presence of non-integer ratios in non-proportional problems, whereas 5th and especially 6th graders were hardly or not affected by this task characteristic.

Although the scope of the present study was microscopic, it has some important broader theoretical, methodological and practical implications. *Theoretically*, it further documents the variety of superficial cues pupils rely on while doing word problems (Sowder, 1988): Not only problem formulations or key words, but also particular number combinations can be associated with certain solution methods (here, proportional methods). This association moreover interacts with pupils' mathematical knowledge: For more experienced proportional reasoners, a missing-value format seems a 'sufficient condition' to apply proportionality, whereas for less experienced pupils the 'necessary condition' is that the numbers must have an integer multiplicative structure. *Methodologically*, our study warns against the assessment of the over-use of proportionality merely using problems whose numbers have an integer multiplicative structure (like, e.g., in Van Dooren et al., 2005). Nevertheless, this warning only seems to hold for the assessment of younger, less experienced proportional reasoners. *Practically*, our results suggest, that the classroom teaching of proportionality might benefit from explicitly discussing the criteria that pupils use (or do not use) when deciding on the appropriateness of proportional solution methods.

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Developing Multiplication

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The research project "Developing number sense: curricular demands and perspectives"[1] *studies the development of number sense in children from 5 to 11 years old. The project team included classroom teachers and researchers that developed and experimented tasks and task chain that intended to foster number sense.*
This paper focuses on one of the project case studies. This case study analyses the implementation in a 2nd grade class (7-8 years old) of a task chain related with multiplication. We will center the discussion on the strategies used by children in one particular task.

Keywords: number sense, primary education, constructing multiplication, multiplication strategies.

1 Introduction

The project "*Developing number sense: curricular demands and perspectives (DSN)* was developed from January 2005 to December 2007 and its main objective was to study number sense development with children from 5 to 11 years old. Curricular development and teachers' professional practice were the other themes studied in the context of the project.

There is a great consensus that, in today's world, all pupils must understand more than basic skills related with number and operations. They need to have a global understanding, labeled in literature as *number sense* that includes many aspects related with deep understanding of numbers and operations.

The project team discussed the meaning that different authors give to the expression *number sense*. We adopted the one used by McIntosh, Reys & Reys (1992) which considers that number sense comprehends:

- *Knowledge and facility with numbers*, witch includes multiple representations of numbers, recognizing the relative and absolute magnitudes of numbers, composing and decomposing numbers and selecting and using benchmarks.
- *Knowledge and facility with operations*, witch includes the understanding of the effects of operations on numbers, the understanding and the use of the operations properties and their relationships.

[1] This Project provided funds for FCT with reference POCI/CED/59680/2004.

- *Applying knowledge of and facility with numbers and operations to computational settings*, which includes the understanding to make connections between the context of a situation and the computation procedures, requiring knowledge of multiple computational strategies.

The project team worked in two intertwined characteristics: curricular development and educational research. At the curricular level the team developed, experimented in several classrooms and reformulated task and task chains. At the educational research level, the team studied the way children develop number sense in problem solving contexts and the characteristics of the curriculum that promote number sense development (whole numbers, decimals and fractions) (Brocardo, 2006).

During 2007, last year of the Project, the project team analyzed data from the classrooms where the task chains where experimented and prepared several case-studies. With the purpose of promoting a discussion of the theme with other teachers and researchers, a final seminar of the project was also organized.

This paper focus on a case study that analysed a 2nd grade classroom. A more detailed description of this study is in the final project report and still hasn't been published.

2 The task chain – *Forming groups*, *The loft wall* and *Chewing gums*

This chain had 3 tasks and was designed with the objective of developing the understanding on multiplication and the use of different calculus strategies related with multiplication. Each task chain was developed as a hypothetical learning trajectory in the sense used by Simon (1995).

The tasks of this chain are: *Forming groups*, *The loft wall* and *Chewing gums* (see Appendix 1). The chosen contexts for these tasks are familiar to children. They try to facilitate the understanding of the multiplicative structures and the informal use of the multiplication properties. This option follows Treffers & Buys' (2001) ideas about relevant contexts and about the way they see the learning trajectory of multiplication – from solutions specific to a context to a more generalizable solution grounded on models.

2.1 The tasks

The tasks intended to develop a set of ideas and procedures related multiplication. With the task *Forming groups* children could transform informal multiplication procedures in structured multiplication procedures, use the relations halves/doubles and informal division procedures.

The second task *The loft wall*, also facilitated the transformation of informal multiplication procedures in structured multiplication procedures and the use of the relations halves/doubles. In addition, children could use the multiples of doubles, decompositions of 50 and 100 and the distributive property.

The third and last task of this chain, *Chewing gums*, intended to facilitate children to: identify and use benchmarks results, relate the different products of the multiplications tables and use the distributive property.

2.2 The exploration of the task chain in the classroom

The tasks were explored in a 2nd grade class with 19 pupils, located in a small industrial city. The class teacher, Elvira, was a member of the project team.

The lessons were organized in three parts. In the first one, the teacher presented the tasks and clarified the doubts posed by pupils. In the second one, pupils worked alone or in groups on the task. During this phase, the teacher encouraged children to keep track of all the strategies they used. In the third part, the teacher organized an all class discussion. Different strategies and procedures were shared and analyzed. The teacher encouraged children to explain their ways of thinking and understand the explanations of the colleagues.

The teacher decisions had a crucial role in the all class discussion. She selected the order of the pupil's presentations: from the informal to the formal strategies, trying to cover all different procedures the children used. She also encouraged pupils to explain in their own words how they solved the tasks.

The classes were videotaped and later on we transcribed the more relevant episodes. The selection of the episodes to be analyzed was organized, task by task, and following the organization of each lesson. They also illustrated the different strategies used by the pupils and the interactions between children and between the teacher. We also analyzed the children's written work.

In the next section we ilustrate some of the procedures that children used to solve the task *Forming groups*.

3 The procedures that children used to solve the task *Forming groups*

This task (Appendix 1) facilitated the use of strategies related of multiplication. Most of them were informal and not structured. Nevertheless, the different processes used by children show differences in the way of thinking and calculating. In the first part of task, we identified four different strategies:

(i) The understanding and the use of 30 as a group, but without being able to add the several groups of 30, not understanding when to stop – Íris's strategy (Figure 1)

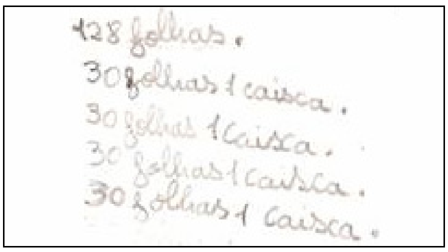

Figure 1 - Íris's strategy

(ii) The use of repeated addition of 30, counting the number of times that the 30 was repeated – Sara's strategy (Figure 2).

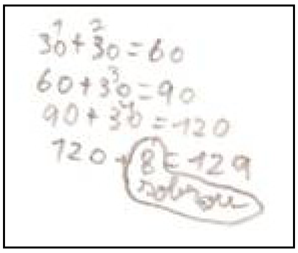

Figure 2 - Sara's strategy

(iii) Counting jumps of 30 and related the counting with the results 30, 60, 90, 120. In this strategy we can see a multiplicative structure based on proportional reasoning – Hugo's strategy (Figure 3).

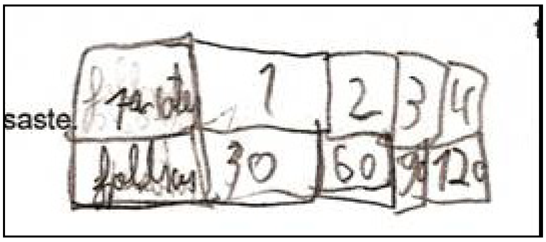

Figure 3 - Hugo's strategy

(iv) The use of division as inverse of multiplication thinking that with 30 sheets we can make 1 group, with 60, 2 groups because 2x30=60, with 90 we can make 3 groups (packages) because 3x30=90 and with 120 we can make 4 packages because 4x30=120 – João Pedro's strategy (Figure 4).

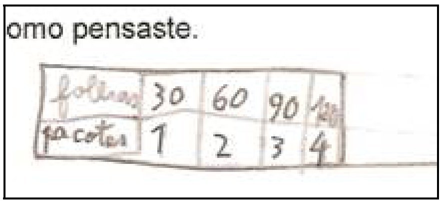

Figure 4 - João Pedro's strategy

The discussion of the others questions was much participated. When the teacher asked *And if we had form packages of 10 sheets?* one pupil answered immediately:

Pupil: 12, because 10x12 =120

After organizing written registrations of the 3rd question – *And if we had form packages of 20 sheets? Gonçalo answered:*

Gonçalo: Because it is the double.

Elvira: The double of what? Yes, there is a double relation. Carolina?

Carolina: Because it is half. Because 6+6=12.

Elvira: And how does it relate with sheets of paper?

Hugo: Because the double of 10 is 20.

Elvira: Can someone explain this better?

Hugo: Because 20 is the double of 10 and 12 is the half of 6. (He reflects and reformulates). Because 12 is the double of 6 e 10 is the double of 20.

Alunos: 10 is the double of? (Laugh)

Elvira: If 20 is the double of 10, what is the relation between 10 and 20?

Alunos: It's half.

Elvira: Others strategies? Gonçalo?

Gonçalo: 5x20=100 and 1x20=20 and 100+20 =120

Theses pupils are able to use the product 12x10 to solve 6x20, using the relation half/double. Gonçalo uses another strategy. He calculates 6x20 using the distributive property – he calculates 5x20 and 1x20 (known facts) adding the partials products.

4 Final Conclusions

During the group discussion and by looking at the various answers on the board, we felt that there was a development on some of the children, who were able to verbalize a reasoning that was slightly more structured and associated with the multiplication procedure. The analysis of the different processes used by the children in the first part of the task, presents some evidence of the fact that some of them seem to understand the effects of the addition and multiplication procedures and their relations. There were two children who even seemed to understand the relation between the multiplication and division procedures, even though in a modestly structured form. As is mentioned by Beishuizen (2003), an entire work based on numbers and their relations further helps pupils with their understanding rather than the premature introduction of the algorithms.

In the second part of the task, although it was only one of the pupils who justified that 120 sheets allow making 12 packages of 10 sheets, almost everybody else seems to have understood that 10x12=12x10=120. From there, they seem to think using more organized procedures, utilizing the doubles and the halves relating directly to the distributive property of the multiplication towards the addition. By justifying their reasoning in this way, there seems to be an indication of some knowledge and dexterity in what concerns the multiplication procedure and its properties. There was also another student, Gonçalo, who used this property.

At least one of the pupils used known products and the 10 factor. It was clear from the discussion, apart from the pupils' seizing of the multiplicative strategies and a growing understanding of the relations between the numbers, that there was a greater relation between the processes used and their verbalization.

Almost every pupil seemed to be aware of the existence of multiple strategies, which was aided by the discussion and the confrontation between the different procedures. On the other hand, from the strategies used by children, there was a focus on the strongest strategies in each situation. By analysing the entire process developed throughout this task, we can say that there is some development in terms of the use by some of the pupils of strategies which are more structured and formal.

We seem to be able to say that, the mentioned chain, and particularly the task illustrated in this paper, contributed for the children's development of their number sense, in aspects concerning the *knowledge and facility with the numbers and the addition and multiplication procedures*, as well as with *applying that knowledge and dexterity in various calculus situations*.

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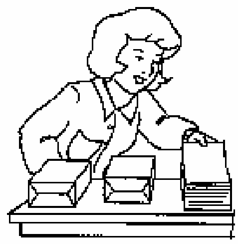
Appendix 1: The task chain – *Forming groups*, *The loft wall* and *Chewing gums*

Task 1 - *Forming groups*



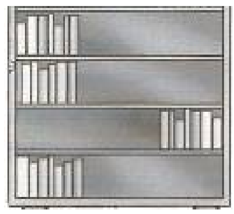
The teacher has 128 sheets. She wants forming groups of 30 sheets.

How many groups can she form? Explain your thinking.

And if she wants forming packages of 10 sheets?

And if she wants forming packages of 20 sheets?

Task 2 – *The loft wall*



Sara's father wants to catch a loft wall with bookcases.

The height of each bookcase is 42 cm.

- Sara's father has been able to heap up 4 bookcases, up to the roof. Which is the height of the wall?

- And if the height of each bookcase is 21 cm? How many bookcases he needs for the same wall?

- Sara's father has experimented to put the bookcases with 21 cm of lenght, side by side, and he can put 9. Which is the length of the wall?

Task 3 - *Chewing gums*

Observe the prices of the ***Chewing gums***

Chewing gums	**Price**
1	20 cents
2	40 cents
4	80 cents
8	160 cents

If you want to buy 5 chewing gums, how much is it?

And how much is 7 chewing gums?

And how much is 10 chewing gums?

Learning Alternate Division Algorithm in Enhancing the concept of Rates and Density

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The traditional long division algorithm assumes that users can apply a guess-and-match type mental process of searching for a maximum that is not greater than the dividend at the initial stage of this algorithm. This optimization process requires heavy cognitive load on metal calculation on applying rules and regulations that does not correlate to life experience of sharing objects. By introducing the special method of learning the concept of division, the Partition of Quotient (POQ), we find that it can enhance the effectiveness of learning the concept of rate in science, in particular, the concept and property of density of an object.

Keywords: division, partition of quotient, rate, density

1 Introduction

How children understanding the concept of rates and ratio, and their difference? Since we need an end value of computational result when looking for either quantity, we inevitably involve in doing procedural division of two quantities. Zweng (1964) introduced the concept of rate to study how well children learned division in contexts that involved the measurement and partition interpretations of the operation. It is helpful as children understand and identify the two types of division. Conversely, our intention is to investigate if learning effectiveness can be enhanced by full understanding of division prior to the learning of the concept of rate and ratio in science. Instead of hard memorizing that division of two quantities with different units is a rate, and is a ratio for those of same units, we start with the interpretation of division traditionally in the following table. That is a modified version of what had been done by Leung et al. (2006).

Traditional Question	Name of Approach	Example	Scientific Interpretation	Example
a ÷ b is interpreted to mean: "From the a objects, how many groups of b	"Measurement" (sometimes referred to as	12 divided by 3 equals 4 because the 12 can be divided into 4	Ratio	In 400cm =4m If 400cm is divided into *lots* (m) of 100cm

objects can we form?"	"quotition")	lots with 3 objects in each.		each, then 4 lots can be formed
a ÷ b is interpreted to mean: "If a number of objects are placed in b equal groups, how many will there be in each group"	"Partition"	12 divided by 3 equals 4 because if 12 objects are dealt to 3 people then each would get 4.	Rate: Amount in one physical quantity occupied in correspondence to 1 single unit of the other quantity	A string of 400cm long is cut into 4 shorter string of equal length, each short string will be of 100cm long

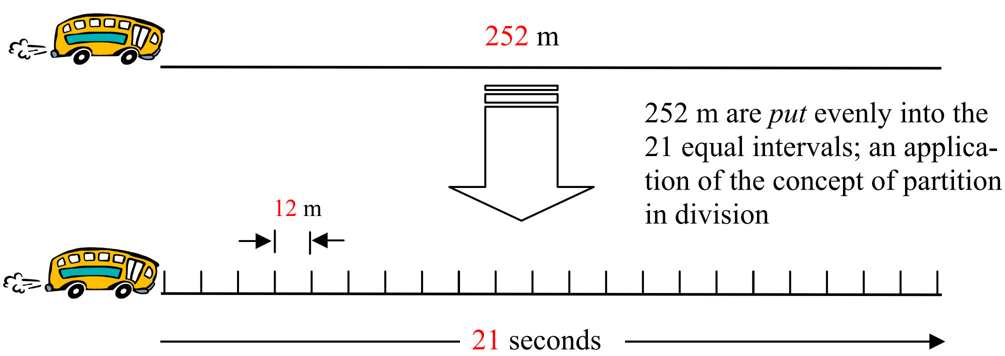
POQ is simply an alternative algorithm different from the traditional long division algorithm on integers. A quotient is made up of one of many additive partitions. In dividing D (dividend) by d (divisor) to get Q (quotient) where all D, Q and d are integers and $D = Q \times d$, we can use the *modified* Distributive Law of division over addition, namely,

$$\begin{aligned} D \div d &= \left(D_1 + D_2 + D_3 + \cdots + D_n\right) \div d \\ &= D_1 \div d + D_2 \div d + D_3 \div d + \cdots + D_n \div d \\ &= q_1 + q_2 + q_3 + \cdots + q_n \\ &= Q \end{aligned}$$

we can deliberately *decompose* this Q as the fundamental partition: 1,1,1…,1, so as to make as many as Q of such 1's. And sum of all of such 1's is corresponding to take away the same number d from D, repeatedly Q times. The intention is to create the "per unit" sense. And each such unit is a partitioned element that will sum up to the whole quotient.

2 Method

Analogously, in an example of physical quantities in definition of speed, this is a rate. we introduce the meaning of speed by telling that a uniformly moving object travels a certain distance within a time interval. A bus travels straight for 252 m in 21 seconds. When working on the average speed of bus within this interval. We are asking how far the bus runs within 1 second. Scientifically, we want to know how many m can be occupied by the moving bus in 1 single second. Projecting this concept to division, we have to divide 252 by 21. In the concept of division, we are placing 252 objects into 21 equal groups (they are equal because of constant time interval), and looking for the number of objects in each group. In the light of the POQ algorithm of division, we are looking for a partition that each of the elements in the partition is identical. And in the calculation process, we can deliberately set up a one-one correspondence of time (in s) interval to one part (12m) of distance interval.



Generalizing this analogue, the speed will then be 12 m per s. The "per s" refers the occupied quantity (distance in m) within 1 single second. Using division concept, the POQ can be deliberately performed 12 times when "dividing 252 by 21", each time contribute an element 1 to a partition. Each time each of the 21 seconds shares 1 such distance (in m) from the 252m. And after 12 rounds of sharing, it can be all shared equally by the 21 seconds. And that is the rate in meters per second (m/s). And the school bus runs with a speed of 12 m/s. The difficulty for students to understand lies in the fact that they experience abstract ingredient when comparing the two horizontal axes. The abstraction here is the one-to-one correspondence of the partitions of two horizontal axes: one is for distance while the other is for time. To a certain extent the abstraction is embedded in the procedure of evaluating rates as we are comparing two quantities with different units. This concept echoes what Hershkowitz et al(2001) have defined the abstraction as "activity of vertically reorganizing previously constructed mathematics into a new mathematical structure"(p. 279, Hassan & Mitchelmore 2006).

Using the same analogue to explain the concept of density to upper primary students, we generalize it into a cluster of many tiny egg-like capsules where each capsule is snap-open-and-close that can be filled with many tiny marbles. The analogue here resembles mass of an object by the quantity of marbles and each capsule represents a unit capacity that possesses a unit volume to contain a certain number of marbles which aggregated to form a sense of mass (weight). Each such capsule is the unit volume of that object (cluster). The single marble occupied in a capsule is an analogue of one single element of the fundamental partition. That is, the mass within a unit volume of that object. And by definition, it is the density of the object. Figures 1 and 2 show the process of two groups of students on putting marbles, arbitrary number of marbles at a time (each marble corresponds to a unit mass) into capsule (unit volume) under the constraints that they must be equal share eventually and share all the marbles without remainder but no restriction on how many times of sharing (picking). Students experienced this concept of sharing marbles is essentially a special way of division. Hypothetically, the cluster of the capsules gives a volume of an object, as a whole, which is partitioned by equal volume of tiny capsules. And each capsule contains an equal number of partitioned marbles representing the mass aggregated to form the whole mass of the object (cluster, see Figure 3).


Figure 1 & 2. Sharing of marbles by capsules of equal volume.

In the lesson, teacher showed a cluster and introduced the following dialogue. Teacher stressed that by putting equal number of marbles into each and every capsule [sharing by POQ method, they just finished the task] is a means to get the density of the cluster. He further reinforced that density was not the total mass in the cluster; rather it was the mass per (shared by a) unit volume. In the current case, the unit volume is most conveniently visualized as the volume of the individual capsule [the make-up of the cluster]. The following dialogue was transcribed:

Figure 3.

Teacher: If the cluster is cut into 2 halves, namely 2 smaller clusters, will its density be changed?
Student: No change.
Teacher: Can you explain?
Student: Density is a rate. You have mentioned that it represents certain amount of stuff [mass] in a certain range [space]. So density is a concept dealing with "average", no matter how many pieces into which you cut the original big cluster, the density of each new cluster so produced is not changed
Teacher: Let me follow his explanation by putting some numbers in the situation. Suppose the original cluster is made up of 1000 marbles and the number of marbles shared in each capsule is 10. Now, if we disassembly the cluster into its make-ups, we will get 1000 marbles and 100 capsules. Make them into 2 heaps; each heap of materials has 500 marbles and 50 capsules. By sharing the 500 marbles equally into the capsules via the POQ algorithm, we will at the end get some fixed number of marbles in each capsule. And this exactly the same number of marbles in the original big cluster. So, the density of the smaller clusters is not changed even they were produced by cutting the big cluster into 2 halves.

One of the two important properties of density of an object is invariance. The above dialogue session supports that a number of students give the correct, feasible solutions since they perceive density as amount of matter per unit volume. That is also invariant. Similarly the teacher raised a cluster that is composed of 28 capsules in another class and asked the students if there was any change in density if the cluster was cut into 2 halves, each smaller cluster consists of 14 capsules. We captured the following:

Teacher: After being turned into 2 smaller clusters, will the small cluster then has the same density as the original big one?
Student: No change.
Teacher: Please explain.
Student: We can take out all the marbles from the 28 eggs [big cluster]. Then divide these marbles into 2 heaps. Share 1 heap of marbles equally into the 14 capsules. We finally will produce a new smaller cluster, in which the make-up capsules [total 14] will have the same number of marbles as the number of marbles as the original, big cluster. Therefore, density of the smaller cluster is not changed.
Teacher: [you are right] Did you remember the secret [which we mentioned before]? So long as each capsule shares the same number of marbles, the density of the new cluster is not changed. … if a butter is cut into 2 halves, each half will have the same density as before.

The above dialogue session supports that the student provided the correct answer since he perceived density adequately as amount of matter per unit volume.

The second property of density is buoyancy in water. To illustrate the floating characteristic of an object in terms of density, we demonstrate the floating power of the capsule even though it contains heavy marbles (they sink by themselves) in it. This activity demonstrates that density of the capsule does not depend on the mass of object alone. We must take the volume into account when we compare the density of object with that of water.

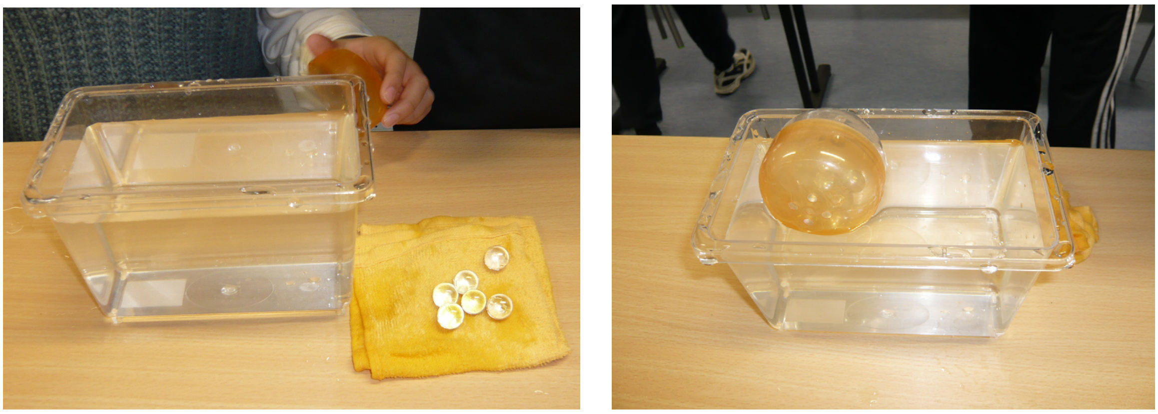


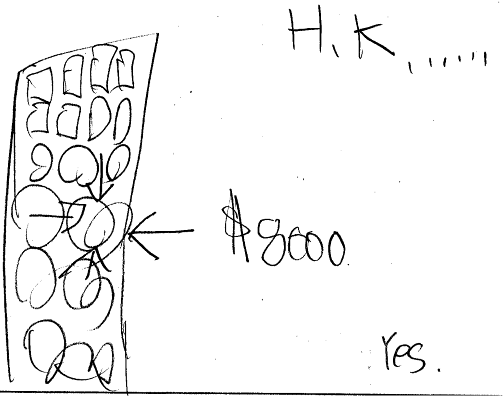
Figure 4 & 5. Demonstration of the buoyancy of capsule containing marbles

The students participated in this study are elementary 6 students. The total number in this cohort is about 230 and they fall into 7 classes. We used school continuous performances in both mathematics and science to determine their initial capability. We treat the average scores of each subject as the pre-test score. From the data, the cohort can be divided into 3 groups: 3 classes in high achievers group; 2 classes in average achievers group and 2 classes in low achievers group. In addition, the cohort's ability in mathematics and science are mirrored. In each group, at least one class belongs to regular (control) class and one class belongs to experimental group. We separately test the effectiveness of their understanding of the concept of density through learning it by POQ (experimental group) against the control group in learning it through traditional introduction of the concept of rate: the division of two quantities with different units. All lessons are a single period lasting 45 minutes in duration; the post-test was finished in 20 minutes and was administered at the end of the 2nd lesson. We expect that the POQ method will also apply to teaching students to identify the difference between rates and ratios.

3 Results and Discussion

We make use of the POQ method of division to point out that the meaning of density is basically a division in a way that a quantity (mass represented by marbles) is shared by each of the element (1 unit volume represented by each capsule)of such fundamental partition. It is an equal sharing a fixed quantity. The concept of sharing is essential ingredient in understanding of the concept of division (Squire & Bryant 2002), which helps to mentally manipulate the concept of equal partitioned distance occupied by a unit time (speed) or partitioned mass occupied by a unit volume (density).

Figure 6.



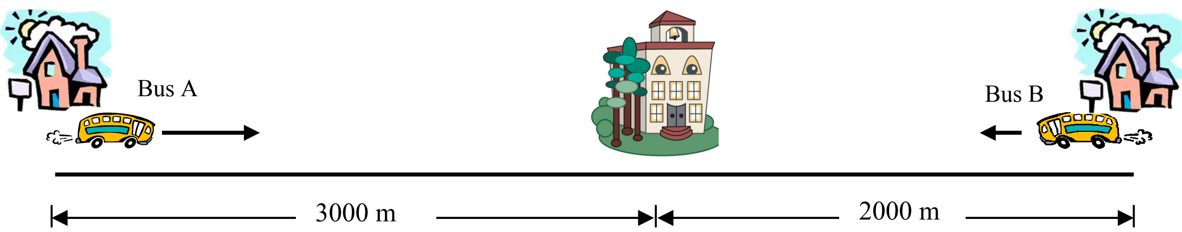
In a test to assess if students can

transfer the concept of the rates learnt from the understanding of speed (linear) and density (cubic) to the concept of rate in area (planar). We ask them if the property flat (apartment) price in \$8000 per square foot a rate. Many students can tell the correct answer that flat rate is a rate. One of them showed us this picture (Figure 6) that each square foot is a partition of the whole floor area where each such small partition is worth \$8000. That is typically a concept of rate.

In another question we ask the students the property about speed

> *Q: Student A takes Bus A to travel to school (the distance between Student A's home and school is 3000 m). Student B takes Bus B to travel to school (the distance between Student B's home and school is 2000 m). Bus A runs at a faster speed than Bus B. Will Student A will arrive earlier at school than Student B does?*

Figure 7.



A student responded with a "No" by illustrating with a picture to show the partition of distance per unit time represented by the symbol "→"at where he wanted to tell that the two buses traveled two partitioned distance of 1500m (by bus A) each and 1000m (by bus B) each respectively. They arrived at the school at the same time.

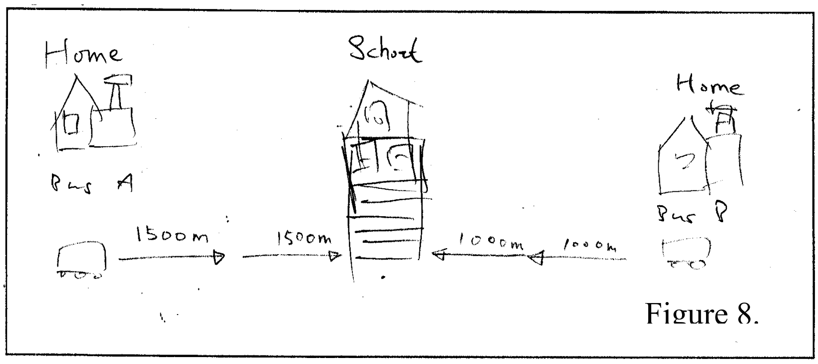


Figure 8.

In a test we asked the students to explain why a big ship can float on water while a big metal block (density larger than water) will sink in contrast. The experimental group students mostly explained that the ship is a lot bigger (in size). The density will be of big different.

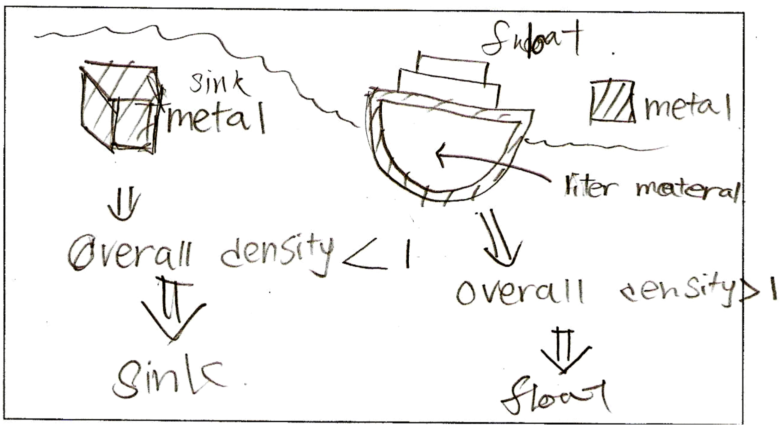
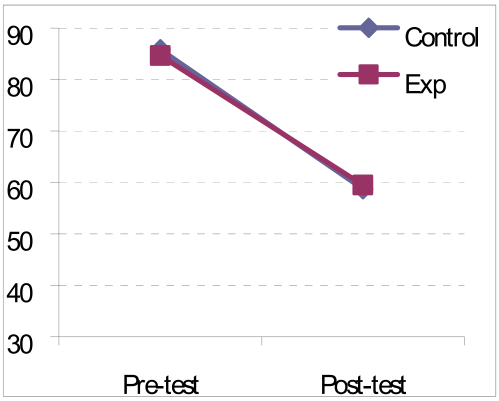


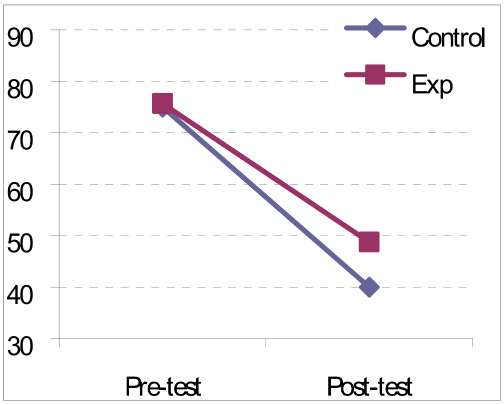
Figure 9.

The illustrative answer (Figure 9) given by one student shows that he knows to compare the density of the ship with water in order to determine if the ship can float or not. He wants to show that hollow trunk of the ship makes its density lesser than 1 (he made mistake too). But unfortunately he fails to indicates that the hollow portion make its volume larger and hence smaller density. However he can basically grasp the concept and property of density.

Quantitatively, we selected the 4 simple questions in the post test to see how well students understood interrelationship among the three quantities, namely, D(density), M(mass) and V(volume) in the definition of density . They were required to find the third quantity if the other two were given. The answer for each question was marked and normalized to compare with the score in the pre test (continuous assessment in school terms). The preliminary results in a post test show that experimental students in the median and low achievers have better understanding of the concept of rates and density while there is no difference between two groups of high achievers (Table 1). Graph 1 shows the qualitative difference between the paired achievers.

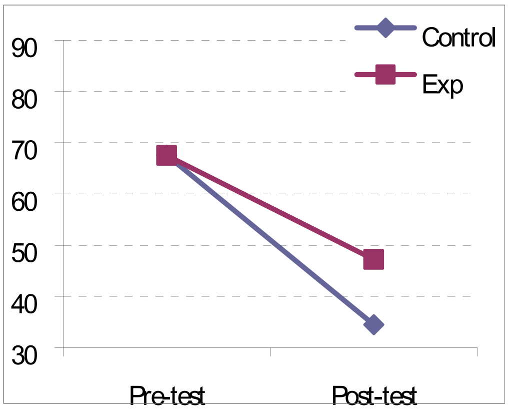
Graph 1. Simple qualitative comparison of students' performance on the pretest and posttest for the three pairs of achievers.





High achievers

Median achievers



Low achievers

Table 1. Students' performance on the pretest and posttest of high, medium and low achievers

				Control Group		Experimental Group	
			Test Items	M	SD	M	SD
High	Control	(n=33)	Pre-test (100 %)	85.8	6.1	84.7	7.1
	Exp	(n=36)	Post-test (100 %)	58.8	22.6	59.5	26.5
Medium	Control	(n=34)	Pre-test (100 %)	75.0	7.4	75.7	7.8
	Exp	(n=32)	Post-test (100 %)	40.0	22.4	48.8	22.3
Low	Control	(n=29)	Pre-test (100 %)	67.6	11.4	67.5	7.8
	Exp	(n=28)	Post-test (100 %)	34.5	27.1	47.3	50.2

4 Conclusion

To distinguish rate from ratio by simply defining it that rate is a division of two quantities with different units is not sufficient. At least it does not tell that the amount of some quantity occupied correspondingly by a unit amount of the other quantity. The partition sense of division, which composed of the concept of equal sharing without remainder, does not exhibit in the traditional way of definition. Students can hardly grasp the meaning of 2 kg of spherical plastic ball, for example, occupied by 0.75cubic meters. They simply mechanically divide these two numbers if the density of the plastic ball is required because the numeric appearance of the 0.75 does not tell the sense of "per cubic unit". Though learners from both approaches need to compute the numerical value of density eventually, the concept of partition is essential for learners for easy transfer of knowledge and concept in identifying a similar physical quantity like flat price per square foot is a rate while 12.5 short-sight students per 100 sample group of students in a city is simply a ratio. Even though the word "per" appears in both descriptions.

We do not intend to say that we should revise the curriculum of teaching the concept of rate and ratio via the new long division algorithm. Rather, introducing the concept of partition of quotient is so helpful that it can play a complementary role in enhancing students' learning effectiveness in this topic. In this piece of pilot study, we believe that the experimental set up can be improved for the sake of collecting valid and reliable data. We do not demonstrate the full statistic analysis of the results, such as inference tests. It will be done in the next step of the investigation.

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PAPER PRESENTATIONS:
ADDITION AND INTEGERS

Using the Jar Model to Improve Students' Understanding of Operations on Integers

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The focus of this paper is to report on a study that assess students' knowledge and understanding of integers before and after the intervention teaching using the 'jar model'. The paper will concentrate on the kind of errors students make in learning integers and how the 'jar model' was supposed to enhance students' understanding instead of memorising rules like 'negative times negative gives positive' etc. Analyses from interviews and performance data of the pre and post-intervention stage revealed that most of the students can understand the jar model and thus improvement can be seen from the result of the post-test.

Keywords: integers, integers operations, jar model, positive and negative numbers

1 Introduction

Students' conceptions of the nature of mathematics and their approaches to studying mathematics have long been matters of interest to mathematics education researchers, largely because both are believed to have an impact on the quality of students' mathematics learning. As is the case in many other parts of the world, educators in Brunei Darussalam are concerned that too many secondary school students pass mathematics examinations without really understanding the subject. Often, students appear to believe that mathematics is a mechanical, rule-bound discipline (Noridah, 1999).

Studies done by Zurina (2003), Khoo (2001), Lim (2000) and Noridah (1999) indicated that the secondary school students in Brunei Darussalam had been taught by methods which emphasize drill and practice with the focus on preparation for the test or examination. Most teachers felt the pressure to move through the mathematics syllabuses as quickly as possible, in order to have the extra time to prepare students for tests and examinations. From the teachers' perspective, this meant that little time is available to attempt to teach for conceptual understanding. For them, teaching for understanding was fine and ideal, but examinations are more crucial. The reason for this is because teachers are usually judged by the examination results and good teachers are usually those who produced good results. On this view, many teachers teach the students for the sake of passing examinations instead of emphasizing understanding of concepts.

Based on one of the researcher's experience of teaching integers in lower secondary government school, it was found that many students were facing difficulties in understanding the topic of integers. Through informal observations and conversations, many secondary mathematics teachers in Brunei Darussalam have expressed their concern over students' poor performances on integers. The teachers indicated that although they recognise that many of their students do not like integers and struggle with integers' questions, they do not know what to do to improve the situation. They tend to say that their students get confused with the signs and operations on integers although they had attempted to explain about it several times. The current number line model that is popular among the teachers is also confusing to the students. The main problems in teaching for understanding of negative numbers and operations are in developing effective strategies for adding, subtracting, multiplying and dividing integers. Some students do not even know how to determine whether one integer is greater than, less than or equal to another integer.

This paper will further describe other studies done to enhance better understanding of integers in other countries, the methodology employed in the research, the findings from the pre-test (the kinds of errors common among the students) and the result of teaching using the jar model on students' understanding of the subject.

2 Literature review

2.1 Research on Teaching and Learning of Integers

Many articles and research papers were found to describe studies in which teachers and students used different strategies in the teaching and learning of integers. Developing effective teaching strategies of integers has been ongoing in many parts of the world. In order to make students understand integers we have to extend their knowledge, help them make logical connections with what they know and use appropriate strategies in learning. Papers have been written on the teaching and learning of integers by Jenny (2002), Dehaene (1997), Hayes (1999), Hart et al (1981), Freudenthal (1973) etc..

Hayes (1996) conducted research on the effectiveness of the most common strategies for negative number concepts and operations at three secondary schools involving students in years seven, eight and nine. The experimental teaching groups used reversible two centimeter square tiles labeled [+1], [–1] and [0]. The major difference in strategy between the experimental and control groups was that the experimental groups started with the tiles. By the end of the topic the experimental group students had also used the number line in context of ordering and 2D point plotting. The outcomes of the study, in terms of student short and long term performance have been compared with those in classes taught by more commonly used strategies. The experimental approach seems to have facilitated better performances for average ability level students. For more able mathematics students, the topic does not appear to be difficult and such students, in both experimental and control groups indicated good levels of general topic mastery. Hayes (1996) found that the use of these tiles led to a significant improvement on a range of test items, including examples requiring the use of brackets, mixed operations and order of operations. It also developed a more confident and secure knowledge of the rules and showed fewer tendencies to confuse sign rules across operations. This study coincides with that of Linchenvski and Williams (1988) involves teaching negative numbers using teaching aids, that is, by using dice with +3, +2, +1, –3, –2 and –1 painted on them. They found that the students soon started to use cancellation and compensation strategies.

Chinese Yin/Yang is one of the examples used by Egan (1997) for teaching and learning of directed number which is quite common in Chinese society. A similar approach has also been observed in a Taiwan book for mathematics educators. Teachers are usually advised to ask students to produce several +1 and –1 figures, so that they can play with the figures to explore the principle of addition and subtraction, then multiplication and division (for the purpose of demonstrating division, some teachers may prefer the use of +4 and –4 instead). This design of teaching and learning activities for directed number is not only effective, but it has its implication on the use of metaphor and students' development of Mythic Understanding (Tang, 2003). The Chinese "Tai ji" (or Yin-yan) symbol consists of two parts: light (yin) and shadow (yan). The light part represents warm and bright sides of the nature while the shadow part represents the cold and dark sides. Thus, the light part can be treated as 'positive' and the shadow part 'negative'. These two parts, when grouped together, have a meaning of balance and harmony. If we use +1 to replace light and –1 to replace shadow (see Fig. ii), the whole diagram now represents the number zero. Teachers can produce several +1 and –1 figures using thick stiff cardboard and use them to explain the principle of addition and subtraction of directed numbers.

Angela (2003) did a study on teaching negative numbers using multi-link cubes with a Year 7 mixed ability group. In her study she found that the students had a physical representation of a negative number but this did not help them to understand what a negative number is. It did not help them when it came to understanding why subtracting a negative number would make the answer bigger. However, the cubes could help students calculate a correct answer but they had no reasoning to help them know that the answer was correct. The multi-link cubes only help students as a counting aid, and once the aid was removed they struggled to answer the questions set. She then furthered her study on a Year 8 group with the top ability group. Though these students had been taught the 'rules of negative numbers' the previous year, they still made common errors by exchanging two negative signs for one negative sign rather than a positive sign. In her study, she chose the context of hot and cold water as she felt that the context of temperature would be something 'real' to all students. Though some students were still confused by the context, it seemed that they were all confident with the temperature idea. Most of the students could answer the questions correctly. This indicated that they had a much more thorough understanding through the use of the temperature context.

It is important for teachers to assess the appropriate ways to teach negative numbers and evaluate the student's understandings. Apart from using mathematical resources to teach, it is also important to teach mathematics with a focus on number sense. These will encourage students to become problem solvers in a wide variety of situations and view mathematics as a discipline in which thinking is important.

2.2 The 'Jar Model'

The method resembling the jar model has been used by Battista (1983) and Paul Griffifth (2002) for teaching integers. In fact, similar models were advocated and can be found at websites such as Homeschool Math (2003) and Learning Math (2002). Basically, this method is not very different from other methods that had been used by some of the researchers mentioned in the literature review (Jenny, 2002; Tang, 2003; Hayes, 1996; Egan, 1997). However, effort was taken to consider students' cultural situation and social context. Please refer to Appendix 1 for the some explanation about the jar model.

3 The Study

The main purpose of this study being undertaken was to investigate the knowledge and understanding of students in Form 1 classes in one government school in Brunei Darussalam on the topic of integers and to investigate if the 'jar model' enhance students' understanding in the topic of integers.

3.1 The Research Questions

The following research questions guide the study:

1. What pre-existing knowledge do the students generally have about integers?
2. To what extent does the strategy used in the intervention enhance the students' performances on operations with integers?

The first research question examined students' prior knowledge including the errors and misconceptions that they hold. These include confusion of rules and instrumental understanding of integers itself.

The second research question examined students' understanding of integers after the teaching of integers and its operations using the 'jar model'.

3.2 Methodology

The present study is an exploratory study which used a multiple perspectives research design. A combination of qualitative and quantitative methods were used to gather data. The sample of the study consisted of Form 1 (grade 7) students in one government secondary school in Brunei. The results of this study presented in this paper were obtained from the analysis of the following data:

1. Document analysis;
2. Analysis of performance data from pencil-and paper pre-test;
3. Interview data analysis – both teachers and students;
4. Analysis of performance data from pencil-and paper post-test;

3.2.1 Pencil and paper test

The Pencil–and–paper test was used to generate pre-test and post-test performance data. The test was administered to all students in July 2006 and September 2006. The Integers test was piloted to test for validity and reliability earlier. There were thirty questions in the Integers Test and they involved questions on each of the four operations of integers (categorised accordingly – positive plus positive, positive minus negative ect.) including the combined operations of integers. Students were categorised into three categories of achievers: high achievers, medium achievers and low achievers.

3.2.2 Interview data Analysis

Class teachers of the classes involved in the study were individually interviewed after the pre-test. The interviews with teachers were audio-taped and analysed. During these interviews, the researcher asked the teachers about the teaching strategies used in their classes, their preferences on which teaching strategies they feel are effective in teaching integers and how they handle students who are still struggling with integers. Each teacher was asked to indicate to what extent most of their students understood the topic integers and on which operations their students had the most problems with.

Twelve students (four high, four medium and four low achievers based on pre-test result), were individually interviewed in July 2006, immediately after the administration of the pre-test. The same twelve students were interviewed again in September 2006 immediately after the post-test. The interviews were tape-recorded. These interviews were conducted to determine the difficulties and the types of errors made by the students. The procedures for interview follows closely the suggestions given by Cohen, Manion and Morrison (2000), to achieve greater validity and to minimise the amount of bias as much as possible.

4 Difficulties in learning integers

The difficulties faced by the students in learning integers are due to the confusion between binary operations of plus and minus and the unary operators which are positive and negative. This confusion is due also to many texts using the same symbols for both plus and positive, and minus and negative. Students always ask 'Why do they have to learn negative numbers and what's the use of negative numbers in our everyday life?' Students often have nothing to relate to, apart from a set of rules governing the combination of negative and positive numbers for the operations. They cannot make sense of the multiplication of a negative number with a negative number and why the product of negative numbers becomes positive.

Teachers also find it easier to teach the rules than to teach for meaning and hope the students' understanding will develop as they operate successfully with the relatively 'simple rules'. Some students find it difficult to establish the rules for themselves; therefore they just rely on remembering them instead of understanding. This can lead to rote learning where students only know how to solve the problems of integers but do not understand why it happens in such a way. Baroody and Ginsburg (1990) described that understanding in mathematics learning involves knowing the concepts and principles related to the procedures being used and making meaningful connections between prior knowledge and the knowledge units being learnt. According to Hart et al (1981) the difficulty is that this stems from the need to work consistently with such rules without recourse to an external, concrete referent and it is this that most secondary school students seem unable to do.

4.1 Some common misunderstandings of integers

Students find integers and operations on integers difficult. The fact that -27 is less than -12 is contrary to the students' experience with (positive) whole numbers. Understanding this requires the students to build mental images and models that allow them to visualize these new comparisons and relationships.

The operation of subtraction, especially subtracting a negative, is difficult for students to make sense of. The idea of subtracting a negative number which gives the same result as adding the opposite of the negative number, is difficult for many students to comprehend. When students have little understanding of subtraction of negative numbers, they may end up just blindly following the rules. Study by Hart et al (1981) found that when students are faced with an expression like $+8 - -6$ many of them use the rule to work out the appropriate sign and then operate with it (in this case adding 8 and 6) ignoring the starting point. This works in some cases but not in others (such as $-2 - -5$, where students would give 7 as the answer). This is exactly the kind of error made by the students investigated in this study. Hayes (1999) found that slight misapplications of the rules, such as applying 'two negatives make a positive' to $-4 + -2$ to get $+6$, are common and are also common among our students.

The pre-teaching interview data suggested that students responded to integers tasks in a totally mechanical way, with little or no understanding of why they did what they did. Often, students did not know which algorithms they needed to use. Some of the students who did know the algorithms could not identify which algorithm should be associated with which type of problem. It seems that students had not learnt the required concepts and skills properly when they were in primary school and in Form 1 (February 2006).

About 35 per cent of the students made errors when adding two negative numbers together to get a positive. An example would be $-2 + (-6) = +8$ as identified by Hayes (1999) earlier. About 40 per cent of the students made the error when adding a negative number with a positive number. For example,$-2 + 6 = -8$ was a common answer, where the students multiplied the negative sign of 2 and the addition operation to get negative and added the numbers. About 41.6 per cent of the students in the study made the error in question $-6 + 2$ giving the answer as -8 and about 47.7 per cent of the students made an error of $-6 + 6$ giving the answer as -12. All the errors were made despite the reteaching using the jar model.

For the subtraction of integers, the students in the study also made the same type of error, as the students in the study done by Hart et al (1981) with an expression like $2 - (-6)$. Many students use the rule to work with the sign, that is, minus and negative become plus and then adding the number. Most of the students ignore the starting sign. It works in this question but not in other questions such as $-2 - (-6)$, where the students would give 8 as the answer. In particular, about 22.1 per cent of the students in the study commited the mistake. About 40.3 per cent of the students made an error of mixing up the rules of addition and subtraction such as $-6 - 2 = -4$, where students took the sign of 6 since it is larger than 2 and subtracted 6 and 2. About 41.6 per cent of the students made the error when subtracting two negative integers together to get a negative result. An example would be $-6 - (-6) = -12$. Students knew that when there were two negatives it will become a positive, that is, $-6 + 6$. Since there is one minus sign the answer is negative. The students were confused with the rule of multiplication, that is positive and negative become negative and added the numbers to get -12. About 34.2 per cent of the students made an error on question $2 - (-6) = -4$. Students took only one of the negative signs instead of changing it to become positive.

For the multiplication of integers, about 36.9 per cent of the students made the error to multiply two negative numbers together to get a negative such as question $-2 \times -2 = -4$. The students had misunderstood some part of the rule, that is, the negative sign after multiplication operation was ignored, so $-2 \times 2 = -4$. The same error was made on the division of integers where about 28.8 per cent of the students made the error of dividing two negative numbers to become a negative such as in question $-6 \div -2 = -3$. The students misunderstood the rule as the same in multiplication, that is, a negative sign after a division operation was ignored.

About 35.6 per cent of the students made the error on question $4 - (-2) + 6$, where students just take one of the negative signs to make $4 -2 + 6 = 8$. This misconception was made by the students in the subtraction of integers. For question $(-4 + 6) \div -2$, about 39.6 per cent of the students made the error of giving the answer as 5. The students got $-4 + 6$ as -10, then divided it by -2. The students seemed to mix up the rules of addition and multiplication. About

38.9 per cent of the students made the error on question $4 \times (-2) - (-6)$ by giving the answer as 2. The students made the error of $4 \times (-2) = 8$. Then $8 - (-6) = 2$ where the students only took one of the negative signs thinking that they were just the same. The same error was made by the students on the multiplication and subtraction of integers. For question $-4 \times (2 - 6) \div 8$, about 45.6 per cent of the students made the error by working out $-4 \times -4 \div 8 = -2$. Students made the same error in the multiplication of integers where multiplying two negatives become negative.

5 Results

Performances of the Form 1 students on the pre- and post-teaching test were compared using quantitative procedures. The "test performance" vantage point of Form 1 indicated that the pre-test mean score was 16.50 (out of possible 30) on the integers test. Analyses of the post-teaching data revealed that the mean score of Form 1 students on the integers test was higher than at the pre-teaching stage. The post-test mean score of 21.26 revealed that the students' performances were significantly enhanced. Using the paired t-test, the researcher confirmed that the students' achievements in the five classes are all significantly different. Teaching effects and history was confounded here as well. The post test was given one month after the pre-test. The students were possibly encouraged and motivated to study harder for the test which could have reflected in the improved performances.

Table 1 also shows that all classes scored significantly higher after the intervention using the jar mode.

Class	Post- Test		Pre-Test		Number of Students	t-test	Sig.
	Mean	SD	Mean	SD			
1A	24.92	4.09	20.16	4.60	25	5.170	.000
1B	24.00	4.20	17.58	4.12	36	9.262	.000
1C	21.00	4.44	16.26	4.21	23	4.519	.000
1D	18.25	5.24	15.72	4.79	32	3.009	.005
1E	18.58	4.23	13.48	3.19	33	6.901	.000
Overall	21.26	5.21	16.50	4.67	149	12.46	.000

*$p < .05$

Table 1: Pre and Post-Test Mean Total Score, Standard Deviation, and t-test Results for Students from each of the five classes involved in the study

As for the qualitative data, most of the 12 students that was interviewed seemed to have a slightly better grasp at the post-teaching stage than at the pre-teaching stage of which rules needed to be used to answer questions. The interviewee also tended to make fewer skills manipulation errors than at the pre-teaching stage. However, at the post-teaching stage some interviewees did not have a firm grasp of the jar model concept especially on multiplication and division of integers using the jar model. Post-teaching interviews revealed that the high achiever students managed to answer almost all the 11 questions asked during the interview compared with the pre-teaching interview. Some interviewees were still confused when to remove or add the positive/negative chips from the jar.

Post-teaching interviews had also shown that most of the medium and low achieving students struggled to remember steps on the jar model that had been taught to them. They could not remember when to remove or add the positive and negative chips from the jar especially the subtraction, multiplication and division of integers using the jar model. That was probably because the students may have had to try to learn too many separate skills by rote. This is similar to data reported in Noridah's (1999), Lim's (2000), Khoo's (2001), Zurina's (2003) and Sarina's (2004) dissertations which under examination pressure, most Bruneian students in secondary schools could not remember which skills should be associated with which problems.

6 Discussion and Conclusion

The jar method seemed to generate a better understanding of operations on integers. However, this model can still be improved because students seemed to be confused with some aspects of the model. The model fails to explain situations when a negative number is multiplied by a negative number and when a positive number is divided by a negative number. In cases like this some other explanation need to be given other model need to be combined with the jar model. However, from the study carried out, we can confidently say that the jar model is less confusing than the number line model and created better understanding in the students compared to the rules and analogies that teachers are fond of using before.

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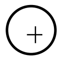
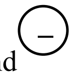
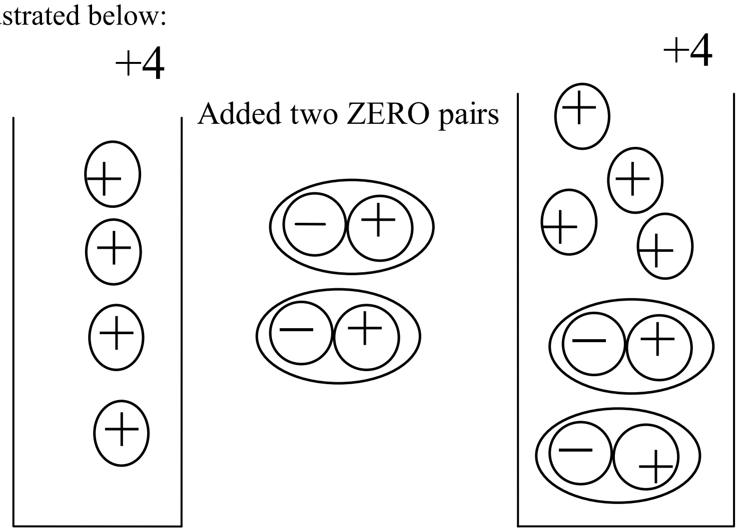
Appendix 1: The Jar Model

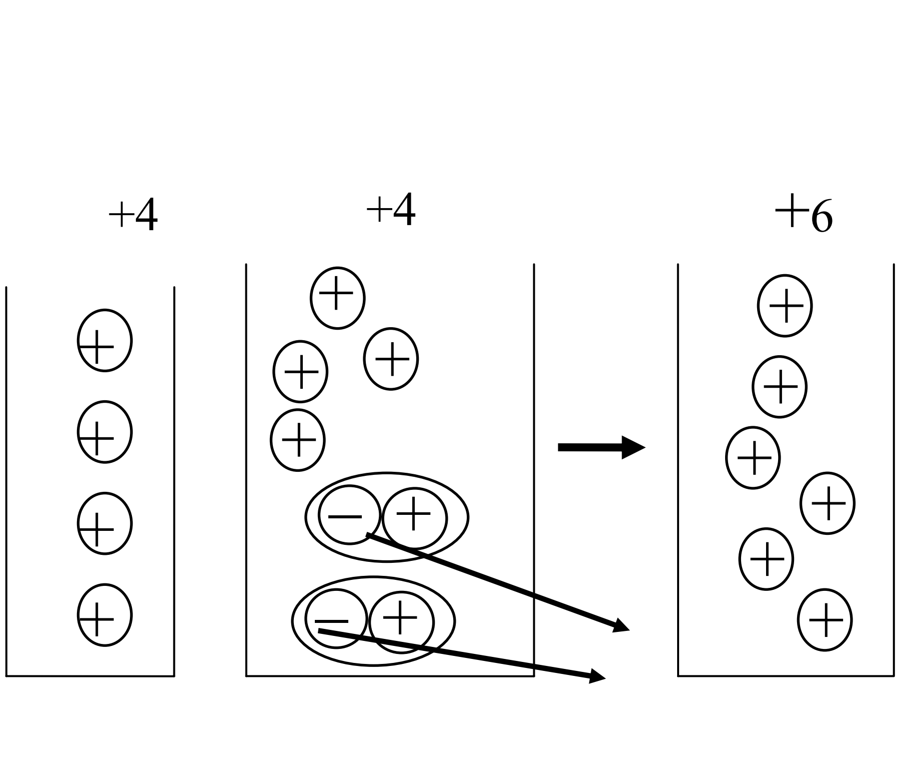
The jar model is the model used to teach the students in this study when intervention was implemented. The jar model used positive and negative counters,  and , for students to work with. The same idea had been used by Battista (1983) for teaching integers. Paul Griffith (2002) also advocated a model resembling the jar model. In fact similar models can be found mentioned at many websites (for example, Homeschool Math and Learning Math).

Example: +4 – (–2). You want to remove 2 negative chips from the jar. But there are only 4 positive chips in the jar. To remove 2 negative chips from the jar, we have to add 2 positive/negative pairs into the jar. It is illustrated below:



Now you can take away 2 negative chips and you are left with +6 chips.

$\therefore\ +4 - (-2) = +6$



A New Algorithm for Column Addition

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We show a modern version of an old "dot algorithm" for column addition of whole numbers and decimals. This new "dot" algorithm is at least as efficient as the "standard" written algorithm currently taught in schools, but has 2 advantages: It is easier to use, especially for adding more than two numbers; and it is not "mechanical". Users can develop their own strategies based on patterns of digits in 1 column, making computation faster and easier. This makes the new algorithm more challenging and interesting. Theoretical underpinnings of the algorithm, historical data, and comments of people who have already learned it will be given.

Keywords: algorithm, addition, whole numbers, decimals

1 Introduction

Children in elementary school are at some point supposed to master the "standard" algorithm for adding a list of whole numbers (or even decimals). According to Carpenter et al. (1998), 99% of children start using a standard algorithm for some problems in addition by the end of third grade. And learning the standard algorithm is one of the focal points in NCTM's "Curricular Focal Points" (2006) for the second grade. But even when the skill requirements are very low, some students never completely master the algorithm, and others soon forget the procedure.

By the "standard" algorithm, we mean a right-to-left column algorithm in which each column is processed in top-down fashion, and the "carry" is written at the top of the next column to the left.

The use of this algorithm in the schools in the USA is illustrated by the most complex addition exercise presented in a typical arithmetic textbook in different years:

Year	Author	
1810	J. Joyce	sixteen 5-digit numbers
1901	E. E. White	nine 6-digit numbers
1934	L. J. Brueckner et al.	nine 3-digit numbers
1990	S. Hake & J. Saxon (4th edition)	seven 3-digit numbers

The fault that many children do not master written addition seems to lie in the standard algorithm, which was designed six hundred years ago for merchants, bankers, and other professionals who needed an algorithm that provided not only a result but also a permanent business record. The standard algorithm does not fulfil modern educational or practical needs. (See Van de Walle, 2005).

In this paper we present a new algorithm, which is a modification of an old “dot” algorithm (See: J. Gough 1798, I. A. Clark 1846) by showing some generic examples using whole numbers. This new algorithm is as efficient as the standard one but is better aligned with current pedagogy. We compare the new algorithm to the standard algorithm using the following criteria:

1. The content of long-term memory. (This is a set of facts that need to be recalled.)
2. Modularity. (This is a partition of an algorithm into subparts that can be executed separately.)
3. Properties of individual modules. (This includes the number of operations and their complexity, the load on working memory, and the number of times working memory is updated.)
4. Flexibility. (This entails choices that a person makes during the process of computation.)

The new algorithm is better according to each of the four criteria.

2 Addition algorithm

One-digit numbers:	1 2 3 4 5 6 7 8 9
Their complements to 10:	9 8 7 6 5 4 3 2 1

When we add a column of numbers, instead of adding the next digit, we may subtract its complement and mark this digit to indicate that we need to carry 1 to the next column. Later we count the number of marks and write it at the top of the next column to the left as a carry.

Example 1

$$\begin{array}{rrrl} & 2 & & \leftarrow \text{carry} \\ & 1 & 2 & \\ & 2 & 7 & \\ & 4 & 8\bullet & \\ & 6\bullet & 1 & \\ + & 7\bullet & 3\bullet & \\ \hline 2 & 2 & 1 & \end{array}$$

Remark:

An algorithm can be flexible (the technical term is “non-deterministic”). This means that the algorithm doesn’t specify all the steps, but provides the user with well-defined options.

Historical remark:

Writing dots next to a digit when the sum exceeds ten was used in the past. The only differences are that at that time the addition of a column of digits was carried out bottom-up and not top down, and that the number carried (here the number of dots in a column) was not written down but remembered.

Here is an example from *Practical Arithmetick* by John Gough (1798).

$$\begin{array}{llll} 7\bullet & 4\bullet & 3\bullet & 6 \\ 2 & 1 & 7 & 9\bullet \\ 5\bullet & 0 & 8\bullet & 7\bullet \\ 6 & 8 & 5\bullet & 3 \end{array}$$

$$\begin{array}{r}
2{\bullet}4{\bullet}\;6\;\;4{\bullet} \\
+\;\;7\;\;2\;\;8{\bullet}\;7 \\ \hline
3\;\;1\;\;3\;\;0\;\;6
\end{array}$$

The next example is from a business arithmetic by I. A. Clark (1846).

$$\begin{array}{r}
6{\bullet}4{\bullet} \\
7\;6\;\; \\
9{\bullet}8{\bullet} \\
5{\bullet}2\;\; \\
3\;3{\bullet} \\
+\,2\;7\;\; \\ \hline
3\;5\;0\;\;
\end{array}$$

In this algorithm the user is allowed to add the numbers in one column in any order. But in order to avoid errors the user has to mark or cross out any digit that has been used.

We look again at Example 1 above.

Example 1 again

$$\begin{array}{r}
\cancel{2}\;\;\;\; \\
\cancel{1}\;\;\cancel{2}\;\; \\
\cancel{2}\;\;\cancel{7}{\bullet} \\
4\;\;\cancel{8}{\bullet} \\
\cancel{6}{\bullet}\cancel{1}\;\; \\
+\;\cancel{7}{\bullet}\;\cancel{3}\;\; \\ \hline
2\;\;2\;\;1\;\;
\end{array}$$

2.1 Modules, strategies, and macros

The modules of an algorithm are the parts of it that can be processed independently. In this algorithm, processing any list of numbers in one column that add to zero is a module.

In example 1 we had four modules.

Processing each of the following pairs:

- 2 and 8 in the right column,
- 3 and 7 in the right column,
- 4 and 6 in the left column, and
- 1, 2, and 7 in the left column,

is a module.

$$\begin{array}{r}
1\;\;2\;\; \\
2\;\;7^{\bullet} \\
4\;\;8^{\bullet} \\
6^{\bullet}\;1\;\; \\
+7^{\bullet}\;3\;\; \\ \hline
2\;\;2\;\;1\;\;
\end{array}$$

You may take a break after executing a module and resume your computation any time later, so even very long computations do not require a long attention span.

The table below shows the frequencies of occurrence of modules.

Length of the sequence of digits 1 through 9 (no zeroes) to be added:	Percentage (rounded to .01%) of sequences of this length which contain modules:

2	11.11%
3	39.64%
4	79.52%
5	98.33%
6	99.94%
7	100.00%
8	100.00%
9	100.00%

Every sequence of length 10 contains a module.

(Computed by Michael Main, University of Colorado, Boulder.)

Example 2

Typical long column written as a line:		7	3	9	6	2	7	7	9	1	2	2	5	7	6	8
Modules:	(1)	7	3													
	(2)								9	1						
	(3)											2				8
	(4)				6	2					2					
	(5)			9			7	7						7		
Not in a module:													5		6	

2.2 Strategies

A strategy in a flexible algorithm consists of additional rules which put restrictions on available options. Some strategies lead to good performance, and some may lead to poor performance. Using only the top-down order of addition of digits in a column is an example of an inefficient strategy. But different users may have different preferences, so the concept of a "good strategy" is subjective. In this algorithm the number of different strategies (good and bad) is practically unlimited.

2.3 Macros

Macro-operations (macros) are groups of operations that a user executes as one unit. Consider the following example:

$$\begin{array}{r} 1 \\ 1 \\ 1 \\ 1 \\ \underline{+\,6} \end{array}$$

One person may compute it as follows: 1, 2, 3, 4, minus 4, 0. This person uses 3 additions and 1 subtraction (cancellation). Another person may see at a glance that this group of digits forms a module, cross out the 1's and mark 6. The main difference between experts and novices executing a flexible algorithm is in their use of macros.

Remark:

Using macros in order to speed up addition is not new. See for example their treatment in a business arithmetic book (Sutton & Lennes) published in 1938.

3 Comparison of the new algorithm to the standard addition algorithm

3.1 Content of long-term memory

In order to use either algorithm (standard or new), children need to master some addition "facts". They have to be able to recall these facts rather effortlessly and not to rely on external help such as counting on their fingers. The standard algorithm requires mastery of 81 facts belonging to 45 families. For example, 6 + 7 = 13 and 7 + 6 = 13 are two facts in the same family.

Remark:

A family of addition and subtraction facts is described by one equality, a + b = c, which contains the following facts (described also by equalities):

a + b = c, b + a = c, c – a = b, and c – b = a.

The new algorithm also requires 81 addition and subtraction facts, but they belong to only 25 families, because they are restricted to pairs of numbers whose sum is smaller than or equal to ten. For example, the following four facts:

2 + 6 = 8, 6 + 2 = 8, 8 – 2 = 6, and 8 – 6 = 2,

belong to the same family. So users of the new algorithm need to know fewer equalities (25 instead of 45), but they need to use them more flexibly.

3.2 Modularity

The standard algorithm requires that the computation of a whole column be done in one pass without interruption, because partial sums have to be remembered. So its difficulty increases when more numbers are added.

The new algorithm requires only that individual modules be computed without interruption. The most common modules contain only two digits, and modules containing more than five digits are very rare, so the difficulty doesn't increase when more numbers are added. The only increase in difficulty is due to counting the marks. When we add 5 numbers, we need only to count up to 4 marks per column. When we add 20 numbers, we may need to count almost 20 marks.

3.3 Properties of modules

In the standard algorithm, adding digits in one column (containing n non-zero digits) requires n – 1 additions of one-digit numbers to one- or two-digit numbers (n – 2 of them being held in memory). The memory load counted in bits is the logarithm to the base two of the number of possible items to be remembered. When we add n numbers, the maximal memory load is $\log_2(n) + \log_2(10)$. So when we add from 2, 3, …, 10 numbers, the memory load increases from 4.3 to 6.6 bits. When we process a module containing m digits, we make m – 2 additions and subtractions of one-digit numbers and one cancellation (for example 5 – 5 is a cancellation). The memory load during the processing of one module is at most $\log_2(10) \approx 3.3$. So when we add n numbers, the maximum memory load is $\max(\log_2(n), \log_2(10))$, where $\log_2(n)$ is due to counting marks. Therefore when we add 2, 3, …,10 numbers, the memory load is at most 3.3 bits. We say that working memory is updated when we add to or subtract from the number that is held in memory and we have to remember the result. In the standard algorithm, the addition of n non-zero digits in a column requires n – 2 memory updates. Processing a module containing m numbers requires m – 3 updates.

We illustrate these comparisons with the following example.

Standard algorithm:	Think:	New algorithm:	Module numbers:		
5	5	$5^\bullet$			3
7	12	$7^\bullet$	1		
4	16	4			3
3	19	3	1		
4	23	4		2	
1	24	1			3
6	30	$6^\bullet$	2		
+ 8	38	+ 8			
3 8		3 8			

	Standard algorithm	New algorithm
Additions and subtractions:	7	1
Cancellations:	0	3
Memory updates:	6	0

4 Flexibility

The standard algorithm is rigid. It prescribes every step of the computation. Therefore after it is mastered it can be performed automatically without any thought. This was a very important and positive feature for accountants and other human computers, who spent hours and hours doing sums. But at present, computers, and not humans, should do all mindless calculations.

The new algorithm requires planning and reflection in finding and choosing which modules to process. The same task can be done in many different ways, some of them better than others. This opens the door to comparisons and discussions. Because modules are very small units, and computation can be interrupted after each module, discussions can be carried out even during computation.

5 Experimental data

When we implement an algorithm on a computer, we can theoretically predict its performance. When we teach children or adults an algorithm, we cannot theoretically predict how well they will be able to use it.

We taught this algorithm to 19 students (3 practicing teachers and 16 future teachers) taking a class in elementary mathematics at New Mexico State University in fall semester 2007. They practiced it for approximately 15 minutes per week for fourteen weeks.

In an anonymous questionnaire given at the end of the course, 18 students said that they liked the new algorithm and one said that she didn't. The main reasons for liking it are illustrated by the following comments.

"It is easy and it is challenging …"

"So much easier and interesting."

"Makes adding fun and fast!"

"I think it allows you to add easier."

We also observed that 15 out of the 19 students who learned this algorithm started using it in other tasks that required adding several whole numbers or decimals.

We plan to continue to collect experimental data from both adults and children.

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Experimental Teaching of Arithmetic by Using Computers

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Paper deal with using computer on the mathematical lessons. It is focused on the teaching of the integers by simple program based on the Theory of Constructivism. We provide an experiment in 3 classes and compare their results. There were no significant difference between them, but we find important result – pupils, which use educational software based on the theory of constructivism have better results in context tasks than the pupils, which do not use this kind of software.

Keywords: teaching integers, educational software, constructivism,

1 Introduction

Teaching of the arithmetic of the integers looks easy, there are some basic rules for computing: commutative law, distributive law, sign laws (i.e.: $a.(-b) = -a.b$, $-a.(-b) = a.b$). But to really teach someone to compute with this numbers in age of 12 is a bit difficult. There are many of researches pointed on the problem of integers (Hejny, Slavíčková…), there were some recommendations for new way of teaching this theme, but we can not see any results in the real life. Teachers still use the method who can find in the textbooks.

2 Theoretical frameworks

We choose the Theory of Constructivism as a base theory for our research. We decide to use software, which is based on this theory (it means that this software is not for training existing knowledge). In general point of view the *Constructivism is a set of assumptions about the nature of human learning that guide constructivist learning theories and methods of education. Constructivism values developmentally appropriate teacher-supported learning that is initiated and directed by the student*. (Wikipedia, 2007). The main idea of the Constructivism is that peoples' knowledge is not give, but it is constructed. We can construct our knowledge in the situation with new impulses by reflecting on our experiences. The idea of Constructivism is not new. *In 300 BC, Socrates (470-399BC) engaged his learners by asking questions (know as the Socratic or dialectic method). He often insisted that he really knew nothing, but his questioning skills allowed others to learn by self-generated understanding*. (Clark, 2000).

There are many historical figures that influence the constructivism – Jean Piaget, John Dewey, Ernst von Glasersfeld, Lev Vygotsky, Jerome Bruner and others. There are many types of constructivism – social, radical, physical, evolutionary, post-modern, information-processing constructivism. Ernest (1995) points out that *there are as many varieties of constructivism as there are researchers*. Psychologist Ernst von Glasersfeld whose thinking has been profoundly influenced by the theories of Piaget, is typically associated with radical constructivism - radical *because it breaks conventions and develops a theory of knowledge in which knowledge does not reflect an objective, ontological reality but exclusively an ordering and organization of a world constituted by our experience* (von Glasersfeld, 1984)

Recent mathematics curriculum documents, such as the NCTM Standards (2000), as well as researchers in mathematics education, value mathematical investigation based on the pedagogical belief that students learn best when they are given the opportunity to actively construct personal understandings of mathematical concepts and relationships. (Mousoulides & Philippou, 2005)

There is a lot of researcher all over the world (Uhlířová – Czech Republic, Andersen M. – Denmark, Mousoulides N., Philippou G. – Cyprus, Hegedus S. – USA, Sanne A. - Norway and others). The results are very similar, almost the same: better and deeper understanding. When the pupils were asked why they do thing so, the answers were*: 'because you can easily get series of graphs', 'you do not stuck in technical details','it is easy to see examples' or 'you do not have to remember a lot of techniques but may concentrate on the ideas'* (Andersen M., 2005). The effort is the same – find good tool – software which can be use on the mathematics lessons to make pupils/students knowledge more durable, deeper and to motivate students to explore, formulate hypotheses and verify them (for example – graph of the sinus function: $y = a.\sin(bx + c) + d$ what happened, if we change parameter *a*, and what if we change other parameters?)

The situation in Slovakia is going to be good, but there are still not enough teachers who use computer in their lessons, there is not enough technical support for teaching with computer. The software for primary school is mostly in English, so preparing micro worlds and small programs for smaller children is (in my opinion) good way how to start use the computer on the primary schools and prepare pupils better for secondary school and what is more important – prepare them for real life.

3 Methodology of research

We use a method of real experiment to verify if the teaching arithmetic of integers by using of the computer is (more) effective. It means that we choose 3 classes on lower secondary school to teach them and compare their results. The criterion for choosing the groups was that the classes should be at the statistical same level of knowledge in mathematics. There was not a significant difference between chosen classes before the experiment.

We planned the lessons for each class according to curriculum and the school plan for mathematics on 6th grade. We decide that 1st group will not use computer at all, 2nd group will use computer only for practising and 3rd group will use computer in whole educational process. The plan of lessons shows Table 1.

Table 1

<table>
<tr><th>No. of lesson</th><th>1st group</th><th>2nd group</th><th>3rd group</th></tr>
<tr><td>1.</td><td colspan="2">In the classroom: addition and subtraction of integers - rules, examples, games</td><td>In the computer room: addition and subtraction of integers</td></tr>
<tr><td>2.</td><td>Classroom - practicing addition and subtraction of the integers</td><td>Computer room - practising of addition and subtraction of the integers using software</td><td>Classroom - practicing addition and subtraction of the integers</td></tr>
</table>

<table>
<tr><td>3.</td><td colspan="3">Classroom - practicing "longer" exercises in addition and subtraction of the integers</td></tr>
<tr><td>4.</td><td colspan="2">Classroom: multiplication of the integers - explanation, rules, practicing</td><td>Computer room: multiplication of the integers</td></tr>
<tr><td>5.</td><td>Classroom: multiplication of the integers - practicing</td><td>Computer room: multiplication of the integers, practicing</td><td>Computer room: multiplication of the integers, practicing</td></tr>
<tr><td>6.</td><td colspan="3">Classroom - practicing of the addition, subtraction and multiplication</td></tr>
<tr><td>7.</td><td colspan="2">Classroom: division of the integers - explanation, rules, analogy and practicing</td><td>Computer room: division of the integers</td></tr>
<tr><td>8.</td><td>Classroom: practicing</td><td>Computer room: division of the integers - practicing</td><td>Computer room: division of the integers - practicing</td></tr>
<tr><td>9.</td><td colspan="3">Classroom: practicing, exceptions, exercises for multiplication and division</td></tr>
<tr><td>10.</td><td>Classroom: practicing of all arithmetical operation</td><td>Computer room: practicing all arithmetical operation</td><td>Computer room: practicing all arithmetical operation</td></tr>
<tr><td>11.</td><td colspan="3">Classroom: context tasks for using all of the operation</td></tr>
<tr><td>12.</td><td colspan="3">Classroom: combination of all arithmetical operation
(exercises like: $(-25+18):19+(-8+6).5=$)</td></tr>
<tr><td>13.</td><td>Classroom: repetition</td><td>Computer room: repetition, making context exercises</td><td>Computer room: repetition, making context exercises</td></tr>
<tr><td>14.</td><td colspan="3">Classroom: context exercises, mathematical crossword and puzzles</td></tr>
<tr><td>15.</td><td colspan="3">Classroom: context exercises, mathematical crossword and puzzles</td></tr>
<tr><td>16.</td><td colspan="3">Classroom: repetition</td></tr>
<tr><td>17.</td><td colspan="3">The final test</td></tr>
</table>

We used also didactical games (definition in Vankúš, 2006) on the lessons in classroom. The games were based on the team work of all pupils, teacher (me) was just a coordinator. The pupils were divided into two groups – sellers and customer. They should make ordering, borrowing, selling and counting (it depends on the group) and check if they are in debt or they profit. After the 17^{th} lesson we compare results in two ways:

1. Classrooms between each other – using F-test
2. Each classroom separately – comparison after 11 month after the experiment, to know durability of their knowledge.

The software we use for experiment was developed as a part of doctoral thesis and is free of use. There are to main parts of this software – Teaching (the constructivist part) and Exercise (only practising existing knowledge)

4 Descriptions of the Lesson Series

We would like to deeply describe the lessons with computer in 3^{rd} group. The other two approaches are common and no important for this paper.

4.1 Lessons one, two and three

We start with part Teaching and the Addition and Subtraction. The pupils saw the environment on the Figure 1.

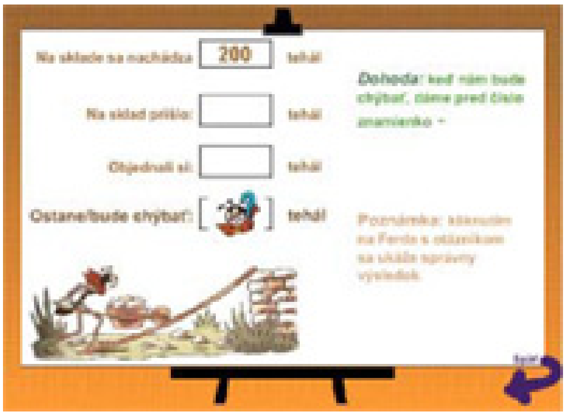

Figure 1

The environment is very easy to understand and it is also very easy to make some operation. At the beginning we gave them 5 minutes to “play with the program” and wrote some notes about computing of the program. After this time we asked them for their notes (without commentary) and then started with coordinated activity.

We started asked question: “Do you know how to have on the store 100 packs? Do it!” Then we continue: “There is delivery of 200 packs on the store. How many packs do you have now?”, “There are first customers and they ordered 150 packs. How many packs do you have now?” We continue with questions like before until we gave them question, when the number of packs on the storage was negative. In this moment we made a deal with them: If we don’t have packs, we will mark it with sign “-“. And then we start to give them the tasks which result was mostly negative number. After this activity (computer as the thing, that knows answer and later as the control thing), we start to formulated rules for counting with positive and negative numbers. The 2nd and 3rd lessons were in class room. We use output from the previous lesson to start 2nd lesson. We also use school books to solve simple exercises (addition or subtraction of two numbers), on the 3rd lesson we also compute longer exercise (addition and subtraction 3 and more numbers).

4.2 Lesson four

On the fourth lesson we start with multiplying of integers. We use software again. In this time we use part Teaching and Multiply positive and negative number. The pupils saw new environment, but it was again very easy. (Figure 2)

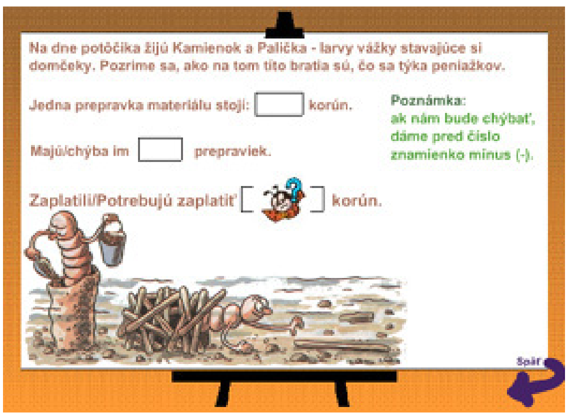

Figure 2

The activity was similar as in the first lesson – pupils start to play with program, then the teacher start to asked them question like: “What happened if 200 bags should borrow 10 crones?” After the activity we start to formulate the rules for multiplying numbers with different sign. Verifying the rules was by using the program again: Teacher wrote on the blackboard some situation and the pupils start to solve it without computer and after that they “ask computer” for correct solution.

4.3 Lesson five and six

On this lesson we stay in computer room again. We start with multiplication of two negative numbers (Part Teaching, Multiply two negative numbers). The environment for the pupils is on the Figure 3. There is a sequence of the multiplying positive and negative number with the answer, the sequence has an increasing character. The role of pupil is easy – guess what the solution of the next multiplying is.

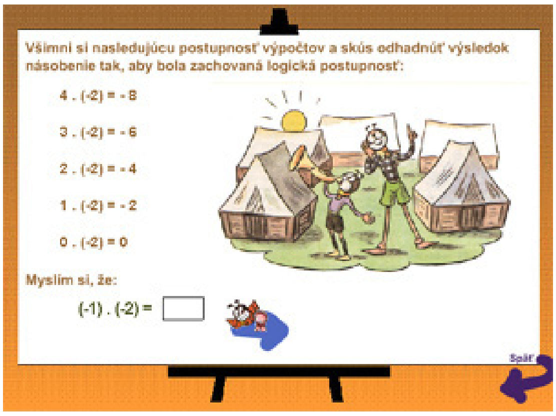

Figure 3

The pupils worked without coordinator. The role of teacher was only taken care of the pupils. After 10 minutes teacher start to ask them question, what they wrote down and why. The answers were correct. So the pupil can write down to their notebooks the other rule for multiplying. On the 6th lessons we were in classroom and compute exercises from school book where we practising addition, subtraction and multiplication.

4.4 Lesson seven

The teaching of the division of two integers were very easy, not only because of the computer, but also because of the pupils reaction after few minutes working with program: “It is same as for multiplying!”. So in this lesson we work with 3 environments: Teaching, Division positive and negative number, Division two negative numbers and Exercise, Division. Teacher ask pupil still question “why?” to make sure that the pupils really understand what are they do (that it is not only routine without thinking)

4.5 Lesson eight and nine

We planed here to practicing division of integers, but we could practice all arithmetical operation in the part Exercise. The pupils were very good in counting that exercises. After the finishing counting and solving exercises in the part Exercise the pupils got reaction of the program to their solution and recommendation for entrenching their knowledge, for example "Well done", "Keep training"… So the lesson was very nice – pupils start to talk the teacher, that the computer say them that they are clever and it was more that if teacher or someone else talk this to them – computer is very clever and make no mistakes, computer is the best in mathematic and this "genius person" talk them, that they are good. On 9th lesson, similar as on 6th lesson, we stayed in the classroom and used school book to practising all arithmetical operation with integers.

4.6 Lessons ten, eleven and twelve

In this lesson we again practicing and repetitive all arithmetical operation. This lesson was combining – computer and blackboard. Pupils solve exercises first in the computer and then on the blackboard (other one, not the same). At the beginning: teacher stayed in front of the blackboard and ask pupils for the exercise for the multiplying, pupils start with the program and tell the teacher, then teacher on the blackboard and the pupils with computer solve that. Then first pupil is come to the blackboard and other pupils tell him the exercise, then other pupils is going to the blackboard and the other pupils tell him the exercise. The 11th lesson we solve context tasks from real life. Some of these tasks were from school book; some of them were prepared by the teacher. Most of the pupils had no problem with finding solution even more complicated exercises. On the 12th lesson we compute combine exercises from school book. Some of the exercises were too difficult for the pupils. It could be because of many rules for computing, which were not practising and entrench on the lessons before.

4.7 Lesson thirteenth

We think that this one was one of the most important lessons. In this lesson pupils run on part from the part Exercise and start to develop a context task which can be solved by the exercises (Figure 4)

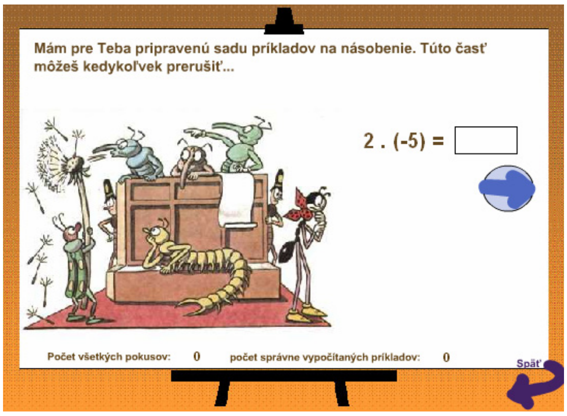

Figure 4

The context tasks were easy: "I borrow 2 time 5 crones. How many crowns I owe?" We do not find more complicated context task in the solution of the pupils. Some of them just write: "dept 5, person 2, how many together?"

4.8 Lessons fourteen, fifteen and sixteen

On these lessons without computer we solve problems which lead to the finding of commutative law, distributive law and solving longer tasks (for example -3 + (+12) + (-29) - (-17) = …..). These lessons were connected to the lessons with computer. We asked pupils questions in the program context, it means, if in the program was Ferdo bug, in the question was Ferdo and has a similar problem like in the program (packs on the storage, borrowing …)

4.9 Lesson seventeen

On the last lesson we gave the pupils test from arithmetic of integers. They could spend all the 45 minutes for solving exercises. There were 6 tasks on the test, 4 only for computing (type Solve!) and 2 were with context from real life.

5 Results

After research we gave the pupils a test from arithmetic of the integers and we start with analysis of the result. We can briefly show what we find.

5.1 comparison classes between each other

For comparing differences between classes we used the F-test, the statistical test for comparing 2 and more groups. The results are in the Table 2.

Table 2

	Sum of squares of deviation	Degrees of freedom	Mean squares	F-test criteria	Probability
Variability between classes	0,0416	2	0,0207	0,35859	**67,8%**
Variability within class	3,53682724	61	0,0579		
Total variability	2,4106	63			

We can say, that there is no significant difference between classes (the knowledge is equivalent in the each group).

5.2 comparisons inside classes

We compare the result after the experiment and 11 month after experiment. We gave the pupils very similar test like after the experiment and we compare the results from these two tests separately for each group and each pupil. The results are in the table 3.

Table 3

	Level of difference	Result
1st group	86,95%	Not significant difference
2nd group	26,29%	Not significant difference
3rd group	26,11%	Not significant difference

We can see that there is also no significant difference between knowledge after experiment and after 11 month after experiment. However there is difference between the 1st group and the two other. Differences in the 2nd and the 3rd group are very small on the other hand difference in the 1st group is considerable. We can not say unambiguous conclusion. There is necessary to make bigger experiment with more groups. Now we can say that there is possibility of more durable knowledge if we use the computer in the proper way.

5.3 Important facts from the experiment

During the experiment and the evaluating of the tests we find out:

- The 3rd group (computer all the time) was the best in the solving tasks from real life (Figure 5, tasks 5 and 6 were context tasks from real life)
- The 3rd group has better results from mathematics than from other themes (we can not say this about other two groups) – this difference is not statistical significant, but it is significant for the pupils – better mark from mathematics
- The environment on the lessons with computer was more friendly than in common lesson – pupils was not so shy as usual they were
- The reaction of the other teacher for this lessons was positive, they would like to teach this way, but unfortunately, they do not have time in the schedule to do it

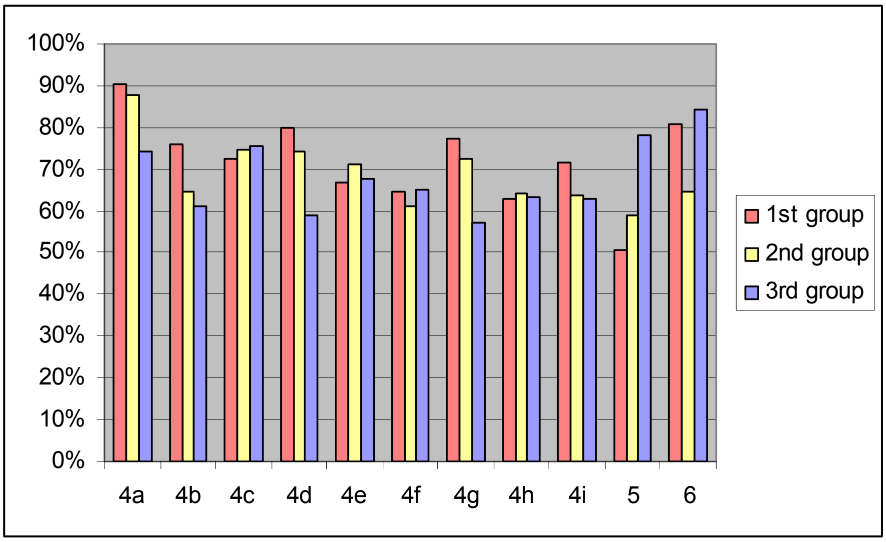

Figure 5

6 Conclusion

In this paper we briefly describe experiment focused on efficacy of using computers and software based on the Theory of Constructivism on mathematics lessons. The results are not significant and positive. The reasons of this failing should be small experimental group (60 pupils), troubles with computer room at the beginning of the experiment (software was not installed, pupils did not have a permission for using computers, etc) and the "foreign element" of teacher (researcher did not teach at that classes whole school year). To find out more significant results it is necessary to have bigger experimental group and eliminate problems with computer room and foreign element. We think that it takes a time to find the optimal way on how to use computer in the lower secondary school. We showed that it is possible if there is a time (in schedule of the computer room), enough courage and optimism. The results could be interesting not only for teachers but also for pupils.

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A Framework for Characterizing the Development of Arithmetical Thinking[2]

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Based on a previous study and the Van Hiele Model about levels of geometrical thinking development, I propose a framework for characterizing the development of arithmetical thinking. The framework is based on the profile of students' reasoning and explanations of arithmetical activities. Data were collected from 190 questionnaires. The quantitative analysis of the results of the questionnaires included calculations of the relative frequencies of levels of arithmetical thinking in the population surveyed. What is outlined in the present paper may provide a possible tool to be used by mathematics teachers.

Keywords: development of arithmetical thinking, arithmetical concepts, mathematics instruction.

Modern researchers have suggested theories which aim to explain and to predict the cognitive development of students during the course of their learning of mathematics: Piaget's *Stage-Theory*, Dubinsky's *APOS Theory*, Van Hiele's *Theory of geometrical development*, the *SOLO model* and so forth (Pegg & Tall, 2005). Most of the theories speak about the stages of cognitive development while learning mathematics referring to the students' age. Van Hiele's Theory of geometrical development, however, claims to relate to the levels of development of thinking *not* related to age. This approach allows the use of such a theory for entire body of students: from those in school until pre-service mathematical teachers.

Van Hiele concludes that "the transition from one level to the next is not a natural process: it takes place under the influence of a teaching-learning process" (Van Hiele, 1986). We can see this transition by means of students' language when working with arithmetic. Furthermore, we can identify the level of development of students' arithmetical thinking by means of their arithmetical reasoning and explanations.

1 Theoretical Considerations

Since Wirszup (1976) introduced the Van Hiele theory to American mathematics educators, numerous researchers have tried to identify and to validate this theory. Some researchers have tried to determine whether the Van Hiele theory describes the development of geometric thinking (Burger & Shaughnessy, 1986; Fuys, Geddes & Tischler, 1988; Gutierrez, Jaime & Fortuny, 1991; Gutierrez, Jaime, Burger & Shaughnessy, 1991; Mayberry, 1983; Senk, 1989; Usiskin, 1982; Wilson, 1990). Other researchers have tried to determine and to analyze the properties of the levels of this theory (Crowley, 1990; De Villiers, 1987). Still more research-

[2] This article is based on the author's doctoral dissertation, completed in 2007 at the Ben-Gurion University of the Negev (Israel) under the direction of Professor Shlomo Vinner.

ers have developed strategies for the implementation of this theory in order to impact classroom instruction (Ben-Chaim, Lappan & Houang, 1988; Clements, Battista & Sarama, 2001; Gutierrez, 1992). Moreover, some researchers have used adaptations of this theory of the development of geometrical thinking for other domains in mathematics. Examples of such studies are by Gutierrez (1992, 1993, 1996) who speaks about 3-dimensional geometry and by Isoda (1996) who discusses the development of the language students use when speaking about function.

In this paper I use an adaptation of Van Hiele's theory to describe the levels of development of *arithmetical thinking*. Van Hiele's theory and following studies related to the development of mathematical thinking have enabled the formulation of this framework and to support it by study.

2 The Method

The process of the development of this framework contained several stages. First, five levels of development of arithmetical thinking were formulated a priori based on my experience in teaching math. Then the types of answers that conformed to the different levels were determined. Following this, I designed a questionnaire that contained 20 items. Each item consisted of a multiple-choice question and a request for an explanation for the chosen answer. This questionnaire was validated by means of opinions from experts, a pilot study and a Guttman scalogram analysis. Participating in this study were students specializing in teaching mathematics in the elementary school and studying in four academic teacher-training colleges. Data were collected from 190 questionnaires. The quantitative analysis of the results of the questionnaires included calculations of the relative frequencies of levels of arithmetical thinking in the population surveyed. After this, an analysis was made of the explanations, which the students provided for each question in the questionnaire. Though original Van Hiele theory refers to five levels, the present discussion refers only to four levels, since the research population did not include representatives of the fifth level.

The framework below is the result of the analysis of the students' reasoning and explanations on the arithmetical activities.

3 Framework for Characterizing the Development of Arithmetical Thinking

The basic assumption of this framework is that a development from one level of arithmetical thinking to the subsequent level is parallel to the development of arithmetical language; it is therefore possible to identify by means of the arithmetical language the students' level of thinking.

3.1 A Key principle for the identification of the levels of mathematical thinking

What defines the students' level of thinking is not the task, but the students' explanations for their solutions. Below is an example of one of the items that was presented to the students in this study.

The answer in the exercise shown below lacks the decimal point: $5.5 \times 3.2 = 176$
Select the correct answer: a. 1.76 b. 0.176 c. 17.6 d. 176 e. none of these.
Explain your answer.

We can classify students' explanation into three main groups:

- Students that multiplied 5.5 by 3.2 (some students used algorithm for multiplying decimal numbers; others used algorithm for common fractions).
- Students that multiplied and then spoke about estimating the product.
- Students that explained their answers only by means of estimation.

When we observe and analyze these groups of explanations, we can see a significant difference between students' level of abstraction; level of explanation (or proof in the advanced tasks); level of experience with new principles or new concepts and language the student understood and used. Searching for common properties in the students' explanations enabled the formulation of the key principle for identification of levels of arithmetical thinking:

> *If a student was found to be at the level j, then his/her answers to tasks have the specific characteristics T_j . Student who was found to be at level j could not give answers from the profile T_k ($j \neq k$) on majority of arithmetical tasks .That means that the specific level is not determined by means of the student's ability to solve specific problem; it's determined by means of the profile of the student's answers.*

As a result of analyzing the explanations and reasoning provided by students, four types of responses were identified:

Type T_1 - An uncontrolled response to a preliminary stimulus to perform an action.

Type T_2 - A correct global concept of methods for writing numbers (the decimal system, the method for representing fractions etc.) while displaying difficulty or inability to provide general explanations.

Type T_3 - An ability to distinguish between arithmetical relations and connections together with an ability to explain via examples.

Type T_4 - An ability to understand the logical structure of arithmetical sentences and an ability to explain and perform a logical analysis of data.

The pilot study showed that there was no significant difference between the verbal and written reasoning of students. In this research I decided to focus on written reasoning. A further study could include personal interviews to clarify whether there is difference.

3.2 The levels of development of arithmetical thinking

As a result of an analysis of the explanations provided by the students to the questions, the inherent characteristics of four levels in the development of arithmetical thinking were described.

Level 1. Students at this level can recognize different kinds of numbers and know how to accomplish arithmetical operations with these numbers. These students do not recognize yet the properties of numbers and properties of the arithmetical operations. They are characterized by an inability to calculate effectively; in general, their calculation ability is low. Their explanations are poor; their mathematical language is very weak.

These students are characterized as students that rely on instrumental understanding of arithmetic.

Level 2. Students at this level are learning the systems for writing different kinds of numbers (the decimal system, the system for writing rational numbers and so on). They can compare numbers from the same group and from different groups, all this – when specific numbers are given. Students on this level recognize the properties of numbers and the properties of the arithmetical operations, but they do not know how to connect between operations and their properties. These students are characterized by a superficial understanding of arithmetical operations as a result of a lack of ability to connect between the different properties of the operations. They can explain their claims (or given claims) by means of specific examples. In addition, they are characterized by the gap between ability of performance as opposed to difficulty in verbalization their thinking. The arithmetical terminology of these students is partial and deficient; they have a partial ability to arrive at generalization.

Level 3. Students at this level understand the existence of reciprocity between arithmetical operations. They can connect the properties of numbers with properties of arithmetical operations, but they will do it only if asked. These students are able to reason informally using general examples, partial algebraic tools and so forth. Furthermore, students at this level are able to follow deductive reasoning and even do a little deductive reasoning themselves.

Level 4. The students at this level understand the logic required to establish their mathematical conclusions. They can analyze an arithmetical claim, identify what is given and for which domain of numbers this claim is correct. These students are able to recognize the logical connections between data and they can present the formal proof (sometimes this proof integrates part of a general example and part of a deductive proof; sometimes it is not complete). At this level, the students begin to use central concepts in the building of mathematical theory: claim, theorem, proof and others. However, the meaning of these concepts is not entirely obvious to these students.

3.3 Types of students' answers as a tool to determine their level of arithmetical thinking: some examples.

The following examples will hopefully illustrate the differences between the students' answers to various tasks and the correlation between their reasoning and their level of arithmetical thinking.

Example 1.

> Among the following triplets of numbers, choose one from whose numbers it is impossible to construct a correct arithmetical exercise:
> **a.** *3; 3; 9.* **b**. *1; 14; 15.* **c**. *2; 4; 12* **d.** *100; 10; 10* **e**. *there is no such triplet.*
> Explain your answer.

The majority of students at the first level found the correct answer and claimed that "it is impossible to write an exercise" with the triplet **c.** Their ways of reasoning were very similar to each other, e.g.:

- $2:12 \neq 4$; $12-2 \neq 4$; $4:2 \neq 12$; $12:4 \neq 2$; $4-2 \neq 12$; $2+4 \neq 12$; $4 \times 2 \neq 12$
  All my attempts to use other arithmetical operations did not lead to a correct exercise.

As can be seen, a student using this explanation has tried to use all the operations: addition, subtraction, multiplication, division. One cannot discern any attempt to analyze the situation or to look more deeply into the task: just simple and immediate action. The student builds a lot of exercises, some of which are completely useless, such as those whose result will obviously be less than one, while all the given numbers are greater than one. There is little reasoning and no method behind it: it is only an exercise in writing a number of operations and computing their results. Thus, students' mathematical behavior at the first level may be characterized by immediate performance of the task without looking deeper into it. Their reasoning is just the report of the performance procedure. Hence, we can discern the answer presented in the example as one of T_1 type, and the students' results of arithmetical development test actually indicated the first level.

The students who are at the second level also succeeded in finding the correct answer, but their reasoning was somewhat different. Here is an example typical of this level:

- $2 \cdot 4 \neq 12$, $2+4 \neq 12$ *, therefore, it is not possible to find an exercise. All my attempts with other operations also did not lead to a correct exercise.*

This answer indicates that this student is aware of the fact that there are essentially two basic arithmetical situations: that of multiplication and that of addition. This means that if it is impossible to find an addition exercise with one of the triplets, there is no use looking for a one that will work with subtraction. Nevertheless, the student reports that he or she did attempt to perform other operations, regardless of the fact that the operations presented in the beginning of the answer are sufficient to arrive at the correct conclusion. One can state that the student possesses some general understanding of natural numbers and the operations on them, including, for example, the relation between subtraction and addition, but he or she still prefers to check it "to be on the safe side".

In order to emphasize the importance of reference to a student's profile of answers, I shall quote the answer of a student who is at the fourth level: "*It is possible only if one of the numbers appear more that once, e.g.* $4^2 - 4 = 12$, *or* $2^4 + 4 = 12$." One may suppose that this student knows the general meaning of "the solution to a problem": finds all the possible answers or prove that there is none. He or she has checked the essential operations and found out that there is no way that these operations will suffice to provide an example with the given numbers, but kept trying to look for less obvious and less routine solutions. This indicates flexibility of thinking; presence of number sense; ability to see through the logical structure of the task and ability to perform a deep analysis of possible solutions. Students on these higher levels did not try to write down exercises.

Example 2.

Refer to the following addition exercise: 296 + 884 + 258 + 704 + 1116.
The best order of computation is (choose the answer that is correct, in your opinion):
a. (296+1116) + (884+704) + 258; *b. (884+1116) + (704+296) + 258 ;*
c. (704+1116) + (884+296) + 258; *d. (704+1116) + (884+258) + 296;*
e. (258+296+1116) + (884+704).

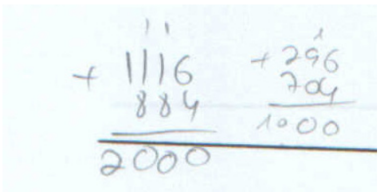
Explain your answer.

The majority of the students succeeded in this task, which is actually a comparison between various ways of adding. The essential variations were in their ways of reasoning, e.g:

- *"The best thing to do is to follow the order of the summands: if we add up the first two and then another two, the sum will be shorter and thus easier, in my opinion"* (a first-level student).
- *"In **e** there are three addition operations and in other sums there are four"*(a first-level student).

The first argument points to the search for a "safe" order of addition. The second one indicates that a student asserts that if there are three summands in the parenthesis, then one is cancelled (i.e. one can perform three operations instead of four), hence this way is preferable. One may assert that this student does not actually understand the meaning of addition as a binary operation. The common feature of these two answers is the emphasis on the procedural aspect, with no reference to efficiency or to laws of addition etc.

Some other students found the most efficient way to add (which is c), but in their reasoning, they confined themselves to the description of the procedure, for example:



This is actually a fully performed written solution, whereas the purpose of the task was to test the student's ability to use arithmetical laws to arrive at the most efficient way to add, which in this case could be purely mental.

Another type of second level students' reasoning was short operational-type answer, with no "extra" words, e.g.:

- "*Add up to tens";*

Word-saving phrasing in this case brings about non-sufficient accuracy, since in this case one should actually obtain full thousands; otherwise, tens alone can equally lead to the answer b, which is less efficient than **c**, since the next step would involve addition with two borrowings. From all the examples of reasoning of students at the second level, one can see that they prefer verbal descriptions of operations that they performed.

Some students argued that "the numbers in the parentheses add up to tens, and thus it is easier to proceed, for example: $704 + 296 = 1000$, $884 + 1116 = 2000$". One can observe that such a student discerns the relation between numbers where addition is concerned, and is able to find efficient ways to find their sums. This is a typical argument of the T_3 type.

4 Discussion and conclusions

The theory of levels of arithmetical thinking enables us to discern students who are essentially different from each other as far as arithmetical thinking goes. These differences are first of all in mathematical language and its usage, in their ways of reasoning and justifying assertions and in the choice of tools they use for judgment and decision making. One of the most important tools to determine one's level of mathematical thinking is the profile of his or her answers to arithmetical tasks. It is important to emphasize that *the specific level is not appointed by* *means of the student's ability to solve a specific problem; it is appointed by means of the profile of the student's answers.*

The level of thinking is a complex concept, which includes, among other things, the ability to achieve the level of abstraction with respect to the matter being learned; the level of argumentation and justification; experience with principles and new concepts, etc. Ascending in levels involves the expansion and enrichment of mathematical language. This is one of the reasons why two people at different levels may not understand each other: they actually speak different languages. The teacher who is to teach arithmetic needs a simple and efficient tool, which would help him or her easily determine what is the level of a specific student and how one must proceed in teaching this student. What is outlined in the present paper may provide a tool for this purpose to be used by mathematics teachers.

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Appendix: Comparison between the levels of Thinking: in Geometry vs. in Arithmetic

	Geometrical Thinking Descriptions of the levels were obtained from the writings of Van Hiele (1986, 1999):	**Arithmetical Thinking**
Level 1	The visual level. At the visual level of thinking, figures are judged by their appearance. A figure is perceived as a whole, recognizable by its visible form, but properties of a figure are not perceived.	Students at this level can recognize different kinds of numbers and know how to accomplish arithmetical operations with these numbers. These students do not recognize yet the properties of numbers and properties of the arithmetical operations. They are characterized by an inability to calculate effectively; in general, their calculation ability is low. Their explanations are poor; their mathematical language is very weak. These students are characterized as students that rely on instrumental understanding of arithmetic.
Level 2	The descriptive level. On this level figures are the bearers of their properties. A figure is no longer judged because "it looks like one" but rather because it has certain properties. However, at the descriptive level, properties are not yet logically ordered, so a triangle with equal sides is not necessarily one with equal angles.	Students at this level are learning the systems for writing different kinds of numbers. They can compare numbers from the same group and from different groups, all this – when specific numbers are given. Students on this level recognize the properties of numbers and the properties of the arithmetical operations, but they do not know how to connect between operations and their properties. These students are characterized by a superficial understanding of arithmetical operations as a result of a lack of ability to connect between the different properties of the operations. They can explain their claims (or given claims) by means of specific examples. The arithmetical terminology of these students is partial and deficient; they have a partial ability to arrive at generalization.
Level 3	The informal deduction level. On this level properties are logically ordered. They are deduced from one another; one property precedes or follows from another property.	Students at this level understand the existence of reciprocity between arithmetical operations. They can connect the properties of numbers with properties of arithmetical operations, but they will do it only if asked. These students are able to reason informally using general examples, partial algebraic tools and so forth. Furthermore, students at this level are able to follow deductive reasoning and even do a little deductive reasoning themselves.
Level 4	The formal deduction level. At this level, deduction is meaningful. The student can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions.	The students at this level understand the logic required to establish their mathematical conclusions. They can analyze an arithmetical claim, identify what is given and for which domain of numbers this claim is correct. These students are able to recognize the logical connections between data and they can present the formal proof (sometimes this proof integrates part of a general example and part of a deductive proof; sometimes it is not complete). At this level, the students begin to use central concepts in the building of mathematical theory: claim, theorem others. However, the meaning of these concepts is not entirely obvious to these students.

DISCUSSION PAPERS

Dirk De Bock:
Operations in the number systems: Towards a modelling perspective

Bettina Dahl Søndergaard:
A Brick in the Wall of Mathematics Education Research in Number Systems and Arithmetic

Bernardo Gómez Alfonso:
Models, Main Problem in TSG10

Chun Chor Litwin Cheng:
Concepts Acquisition in Addition and Place Value

Operations in the Number Systems: Towards a Modelling Perspective

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Elementary mathematics education often focuses one-sidedly on the technically correct and fluent execution of basic operations like addition, multiplication, and direct and inverse proportionality. As a drawback, children tend to perform these operations also beyond their proper range of applicability. In this discussion paper we first provide some research-based illustrations of this phenomenon. Second, we formulate some recommendations for the improvement of educational practice by bringing the modelling perspective more to the forefront of mathematics education.

Keywords: addition, multiplication, misconception, illusion of linearity, modelling perspective

1 "Adders" in proportional situations

From the seventies on math education researchers started to study pupils' "misconceptions" about and primitive strategies for solving proportional tasks (Hart, 1981; Karplus, Pulos, & Stage, 1983; Lin, 1991). These researchers not only identified several faulty applications of "correct strategies" (e.g. incorrect uses of the rule of three), but also strategies that are completely inappropriate for solving proportional problems. The best-known example is the constant difference or additive strategy. In this strategy, the relationship within ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio. An example is the Mr. Short paperclip task (Karplus, Karplus, Formisano, & Paulsen, 1975), adapted by Lin (1991): *"Given a picture to show Mr. Short's height is 6 paperclips. When we measure Mr. Short and Mr. Tall with matchsticks: Mr. Short's height is 4 matchsticks and Mr. Tall's height is 6 matchsticks. How many paperclips are needed for Mr. Tall's height?"* In a large-scale comparative study with 2257 English and 1599 Taiwan 13–15-year old students, the erroneous "additive" answer (8 paperclips) for this task was given by, respectively, 47% and 24% of the students (Lin, 1991).

Additive errors have been widely observed, from childhood through adulthood, in pupils with different cultural backgrounds and for different types of problems. Often, it is observed that this erroneous strategy is used as a fall-back strategy by less skilled proportional reasoners when confronted with non-integer ratios: a child may use a correct strategy on integer ratios and then the constant difference strategy on non-integer ratios. Although, 25 Taiwan "adders" out of a group of 33 students that was individually interviewed while solving a geometrical enlargement problem (in which two similar figures were given and the interviewer asked for an unknown length in one of these figures) tended to use an additive *apart from the complexity of the number structure* of the problem (Lin, 1991), which suggests that these students really believed that this type of problem is an additive one. One of the interview protocols seems to confirm this assumption: *"...Same shape just means the length of a bigger diagram is increased, so I add..."* (p. 7).

2 "Multipliers" in additive situations

As a part of an introduction to teaching ratio and proportion ideas, Cramer, Post, and Currier (1993) confronted 33 preservice elementary school teachers with the following additive problem: *"Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?"* Thirty-two student teachers solved this problem by setting up and solving a proportion: $9/3 = x/15$; $3x = 135$; $x = 45$ instead of using the additive structure in the problem (i.e., Sue always has run six rounds more than Julie). The authors argued *"we cannot define a proportional reasoner simply as one who knows how to set up and solve a proportion"* (p. 160) and diagnosed that textbooks do not sufficiently emphasize the ability to discriminate linear and non-linear situations. Undoubtedly, the students in this study possessed all necessary mathematical tools to solve this problem correctly, but were misled because the problem is stated in a missing-value format. There is as such nothing wrong with this formulation, but it strongly cues proportional schemes and procedures because most proportional reasoning tasks students encounter in their school careers are stated in a missing-value format (whereas additive problems are rarely, if ever, stated in a missing-value format) (De Bock, Verschaffel, & Janssens, 2002).

In a recent study on the overgeneralization of the linear model in students' solving of arithmetic word problems, Van Dooren, De Bock, Hessels, Janssens, and Verschaffel (2005), used (a slightly adapted version of) the 'runners' problem in a paper-and-pencil test and analysed a large numbers of 3rd to 6th graders' solutions of that problem. They found that the number of incorrect proportional answers on this additive problem *increased* by grade (resp. 5%, 17%, 36%, and 50% in grades 3, 4, 5, and 6). The data suggest that 3rd graders (who were until then most frequently confronted with additive situations) more easily noticed the additive model underlying this situation than the 5th and 6th graders (who had been more often confronted with multiplicative/proportional situations in the recent past). The tendency to overrely on proportional methods appears to develop in parallel with the ability to solve proportional word problems.

3 "Proportional reasoners" in a non-proportional geometrical context

An extensively investigated case of students' misuse of proportionality relates to the effect of an enlargement or reduction of a geometrical figure on its area or volume (for an overview of this research, see De Bock, Van Dooren, Janssens, & Verschaffel, 2007; a specific study in this domain is reported in this volume by Van Dooren, De Bock, Evers, & Verschaffel). While an enlargement or reduction of any geometrical figure by a linear factor k, multiplies lengths by k, areas by k^2, and volumes by k^3, pupils strongly tend to see the relations between length and area or between length and volume as linear and thus apply the factor k to determine enlarged or reduced areas and volumes. De Bock et al. (2007) gave tests with proportional and non-proportional problems about the lengths, perimeters, areas, and volumes of different types of figures to 12–16-year old students. An example of a non-proportional problem is: *"Farmer Carl needs approximately 8 hours to fertilize a square pasture with a side of 200 m. How many hours would he need to fertilize a square pasture with a side of 600 m?"* More than 90% of the 12-year olds and more than 80% of the 16-year olds gave a proportional answer (here: "24 hours") to this type of non-proportional problems. Even with considerable support (e.g., the request to make a drawing or the provision of a ready-made drawing on plain or squared paper), very few students made the shift to correct, non-proportional responses. The only experimental manipulation that had significant impact was rephrasing the missing-value problems within a so-called comparison format (e.g., for the earlier mentioned item: *"Today, farmer Carl fertilized a square pasture. Tomorrow, he has to fertilize a square pasture* with a side being three times as big. How much more time *would he approximately need to fertilize this pasture?"*). In this case, the number of correct answers increased from 23% for missing-value problems to 41% for comparison problems.

A subsequent interview study pointed at three underlying causes. A first cause is that students selected and used a proportional method in an intuitive way (in the sense of Fischbein, 1987). Students opted immediately for it, were strongly convinced about its correctness, but found it very difficult to justify their choice. A second cause are particular deficiencies in students' geometrical knowledge (e.g., the misconception that the concept of area only applies to regular figures, or that a similarly enlarged figure is not necessarily enlarged to the same extent in all dimensions), which often prohibited them from discovering the incorrectness of their proportional errors. A third cause lies in students' inadequate beliefs and attitudes towards solving mathematical school word problems (e.g., the belief that the first solution is always the best), and their low self-monitoring while problem-solving.

In a last empirical study on geometry problems, non-linear problems were offered as meaningful, authentic performance tasks with concrete materials instead of a traditional, school-like word problem. The study showed that this manipulation was very beneficial, as linear reasoning almost disappeared, but the correct solution of this kind of task did not affect students' performances on school-like word problems afterwards.

Peter Bryant (2007) provided an interesting comment on this type of research results: "Quite often students apply an *inappropriate* solution to a mathematical problem, i.e. a solution that is not right for this problem but would be for another one. There are two possible explanations for this kind of mistake. One is that the student is incapable of producing the right solution because the solution is too difficult for the student. This is how Piaget and many others (Karplus, Noelting, Wilkening) explained the fact that many students wrongly apply *additive* solutions to *proportional* problems. They attribute this mistake to the relative difficulty of multiplicative reasoning. This is a one-way account. The second possible explanation for inappropriate solutions is that the student could in principle provide the right solution, but can't always work out what is the appropriate solution for the problem at hand. This would be true if students also applied *proportional* reasoning to *additive* problems, i.e. if the confusion goes *both* ways. This would be a two-way confusion. The confusion between additive and proportional reasoning goes *both* ways. Therefore, it is not just a matter of multiplicative reasoning being more difficult and developing later than additive reasoning, as Piaget and others claimed. It is also a matter of children not understanding *(or not trying to understand)* additive and multiplicative problems well enough to know which problem."

4 A minimal response

Students' inappropriate application of as such "correct" operations and strategies, as evidenced by many research in our field (cf. supra), is a major problem and calls for some modifications of current instructional practice. In our view, already at the elementary level, it is an important goal to improve the quality of problems and the way teachers handle these problems, as that have been suggested already over many years and that include the following (see Verschaffel, Greer, & De Corte, 2002, p. 270–271):

- Break up the expectation that any word problem can be solved by adding, subtracting, multiplying, or dividing, or by a simple combination thereof.
- Eliminate the flaws in textbooks that allow superficial solution strategies to be undeservedly successful. If a specific context or presentational structure (e.g. the "missing-value" format) tends to elicit a specific problem-solving routine, this behaviour can be questioned by confronting students with mathematically different problems within the same context and/or presentational structure, and with mathematically identical problems presented in a different context and/or presentational structure. This kind of "discrimination training using counter-examples" (Greer, 2006) is especially recommendable for the context or presentational structure in which students encountered a certain mathematical model for the first time.
- Vary problems so that it cannot be assumed that all data included in the problem, and only those data, are required for solution. As in real-life, students must learn to select the data that are needed to solve a problem, and eventually look for 'missing' data. In this respect, one could think about project-like tasks embedded in a rich context in which several data are given or can be calculated, measured or estimated for different problem-solving purposes.
- Weed out word problems in which the situation, the numbers, and/or the question do not correspond to real life or for which the mathematical model that students are expected to find and apply does not fit (well) with the situation evoked by the problem statement. Students implicitly learn from such tasks to put reality between brackets in the mathematics classroom!
- Legitimize forms of answer other than exact numerical answers, e.g. estimations, commentaries, drawings, graphs, etc. This recommendation fits with the plea of many math educators to emphasise the process rather than the result of a problem-solving activity. It can also stimulate students to increase their repertoire of problem-solving strategies and approaches.
- Include alternative forms of tasks such as classification tasks and 'problem posing' tasks. Alternative tasks can move the attention from individually calculating numerical answers towards classroom discussions on the link between problem situations and arithmetical operations.

5 A more comprehensive approach

Although many of the above-mentioned recommendations can be realised by users of current curricula and textbooks, they actually fit within a more general reform approach, namely to teach mathematics from a genuine modelling perspective. A genuine modelling perspective implies that all phases of the modelling process are considered equally important and receive ample attention. Many authors have proposed descriptions of this process, but, essentially, they all involve the following stages (see Verschaffel, Greer, De Corte, 2000):

1. understanding the phenomenon under investigation, leading to a model of the relevant elements, relations and conditions that are embedded in the situation (situation model),

2. constructing a mathematical model of the relevant elements, relations and conditions available in the situation model,

3. working through the mathematical model using disciplinary methods in order to derive some mathematical results,

4. interpreting the outcome of the computational work to arrive at a solution to the real-world problem situation that gave rise to the mathematical model,

5. evaluating the model by checking if the interpreted mathematical outcome is appropriate and reasonable for the original problem situation and

6. communicating the solution of the original real-world problem.

Already while teaching basic concepts like addition, subtraction, multiplication, division, and direct and inverse proportionality, immediate stress should be laid on these concepts' capacity of modelling some situations (at some level of precision) and their inadequacy of modelling others (Usiskin in this volume). In fact, although mathematical modelling is generally associated with courses at the tertiary or, to an increasing extent, secondary level of instruction, an early exposure to essential modelling ideas by re-conceptualysing the basic arithmetic operations and other primary school content as modelling exercises, can provide a solid base for competently applying mathematics at the primary school level and for further extensions of these mathematical tools found in algebra, geometry, calculus, and statistics at the secondary and tertiary level. Prototypically clean situations can be used to develop students' ability to easily recognize mathematical models and to fluently apply related procedures (or strategies), but at regular times, they should be alternated with exercises in relating more authentic real-world situations to these mathematical models and in reflecting on this relation as a corrective to an oversimplistic view of the world that many supposed applications of mathematics tend to establish. According to Mukhopadhyay and Greer (2001), it is important and also feasible to start applying the modelling perspective successfully in mathematics education of all students already from a (very) young age on and with a diversity of learners.

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A Brick in the Wall of Mathematics Education Research in Number Systems and Arithmetic

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This paper reflects on the state of the research and development of teaching and learning of number systems and arithmetic. This includes a discussion of the descriptive/ explanatory and normative dimensions of mathematics education research in relation to the ten papers in the proceeding. I also refer to an example of a normative statement about standard algorithms in Danish primary education. I conclude that we have reached far but we are not finished yet.

Keywords: nature of research, normative dimension, algorithms, Denmark

1 Introduction

This year we attend the 11th quadrennial International Congress in Mathematics Education (ICME-11). This congress is probably the largest and most prestigious international mathematics education conference and it gathers several thousands participants from all over the world. ICME is held under the auspices of the International Commission on Mathematical Instruction (ICMI) established at the International Congress of Mathematicians in 1908 but it is planned and organised by separate committees working independently of ICMI. The aim of ICME is to present the current states and trends in mathematics education research and in the practice of mathematics teaching at all levels (ICME-11 website).

Different parts of the programme have been delegated out to various organising teams invited by the International Programme Committee (IPC) for ICME-11. One such part is the Topic Study Groups (TSG) whose purpose is to gather participants interested in a specific area. Our group is TSG-10 and our specific topic is *Research and development in the teaching and learning of number systems and arithmetic, including operations in the number systems, ratio and proportion, rational numbers*. We therefore sent out a call for papers stating that any current issue related to this theme may be considered in discussion. We wanted from an international perspective to study and discuss advances in research and practice, new trends, and the state-of-the-art. As the congress participants, or reader, might have noticed, we have ten papers from all over the world reporting various research, and we also have two more general state-of-the-art papers written by invitation. Then the question is, did we succeed in the endeavour set out by us and ICME?

2 How far are we by now?

We know from Niss (1999, p. 1) that the field of mathematics education research is around four decades old. So how much do we now know about the teaching and learning of number systems and arithmetic? In an ICMI Study from 1998 entitled: *Mathematics Education as a Research Domain: A Search for Identity*, Adda referred to a speech by Freudenthal at ICME-4 in 1980 where he presented the major problems of mathematics education at that time. She writes: "Looking at Freudenthal's list of thirteen problems, the thing that strikes one the most is that not only are none of them solved yet today, but they are still of major interest and, in addition, that they have produced important new problems. The first problem was: *Why can Jennifer not do arithmetic?*" (Adda, 1998, p. 49). Hence, teaching and learning arithmetic was a major problem in 1980 and it was not yet solved in 1998 – but is it solved today?

How does knowledge of mathematics education in fact grow? I discussed this in more details in Dahl (2006) as well as pointed at some problems with adding to the body of knowledge in our field. I wrote that we "need a series of 'State of the Art' articles pulling together present research in an attempt to create/discover a meta theory" (p. 68). I also quoted Mewborn: "Moving toward predictive frameworks is not going to come from doing more studies alone; it will come from thoughtful analysis of a large collection of existing studies" (2005, p. 5). This was *inter alia* discussed based on work done at ICME-10 in 2004 at a Discussion Group (DG-10) entitled: *Different perspectives, positions, and approaches in mathematics education research*. The DG-10 report states that it is difficult to accumulate knowledge in mathematics education research due to various research approaches which sometimes appear as fashion waves. The diversity might be useful, if it provides a more complete picture but it also results in fragmentation. There is not a common knowledge base on which to refute the claims made (English & Sierpinska 2004). This might to some extent explain the state of research in mathematics education. Does our TSG also suffer from this? Turning to Prediger's paper (this volume) we read: "two so far competing theoretical approaches of conceptual change and mental models [that she] … integrated into a multi-level model for knowledge of operations" (2008, p. 30). This is an example of such creation/discovery of a meta theory mentioned above. It indicates that we are moving forward – and we need to do a lot more like this.

3 Dimensions of mathematics education research in TSG-10

Niss (1999, pp. 5-6) states that the field of educational research in mathematics has a dual nature. One dimension is descriptive/explanatory the other is normative focusing on 'what *ought to be* the case?' and *why*. Niss argues that the normative dimension "is unavoidable in the same way as it is unavoidable to operate with the notion of 'good health' and 'sound treatment' in much medical research" (1999, p. 6). In the case of TSG-10, the nominative dimension would be answers to questions such as 'how does good teaching and learning of number systems and arithmetic look'? But are both dimensions present in TSG-10? In Sections 3.1 and 3.2, all references are to papers in this volume, hence they do not appear in the References unless they are also referred to elsewhere.

3.1 The descriptive/explanatory dimension

This is clearly the most common dimension in the ten papers since six papers belongs to this category. The first example is Prediger (2008) who refers to different theoretical frameworks to explain pupils' difficulties in competencies and conceptions in fractions and decimals. She reports a study of 269 pupils to show what mental models grades 7 and 9 pupils activate. Valdemoros (2008) reports a case study of a fifth grade teacher who joined a master's degree programme. The study explores the difficulties she encounters planning original design for teaching fraction and it describes her development. Dooren et al. (2008) describe primary school pupils' over-relying on proportional methods while solving non-proportional missing-value word problems. This particularly occurs when word problems form integer rations, but the effect diminishes from grade 4 to 6. Mendes & Ferreira (2008) describe the development of numbers in children aged 5-11, particularly the implementation in a grade 2 class of a task chain related to multiplication. Badarudin & Khalid (2008) assess student' knowledge and understanding of integers before and after an intervention using the 'jar model'. Their analysis showed that the jar model created better understanding compared to the rules and analogies the teachers usually used. Baggett & Ehrenfeucht (2008) describe a study showing that a modern version of the old 'dot algorithm' for column addition of whole numbers and decimals is at least as efficient as the standard algorithm usually taught in school. It is easier to use and not mechanical – users can develop their own strategies.

3.2 Mixed descriptive/explanatory and normative dimensions

None of the papers were completely normative, but four of them had normative elements based on results from a description/explanatory study. One of such papers is that by Carbone & Eaton (2008) who refer to research suggesting that prospective teachers must revisit the mathematics they have previously learnt to be able to teach effectively – for instance is their knowledge on rational numbers essential in preparing middle school mathematics teachers. A study on some US and Northern Ireland prospective elementary teachers' understanding of addition and division of fractions, lead to some normative recommendations for teacher preparation. Another example is Leung et al. (2008) who investigate how the Partition of Quotient (POQ) method of learning the concept of division enhances the effectiveness of learning the concept of rate in science. They conclude: "We do not want to say that we should revise the curriculum of teaching the concept of rate and ratio via the new long division algorithm. Rather, introducing the concept of partition of quotient is so helpful that it can play a complementary role in enhancing students' learning effectiveness in this topic" (p. 80). Slavicková (2008) reports a study comparing three grade 6 classes being taught integers using a specific computer programme. It did not show any significant differences but nevertheless showed that pupils who had used the education software had better results on context tasks than the others. We see that the author aims at suggesting normative claims in the following quote: "We think that it takes time to find the optimal way on how to use computer in the lower secondary school" (p. 111). Finally, Guberman (2008) proposes a framework to characterize the development of arithmetical thinking. The framework is *inter alia* based on Van Hiele's model of geometrical thinking. The conclusion states: "The teacher who is to teach arithmetic needs a simple and efficient tool, which would help him or her easily determine what is the level of a specific student and how one must proceed in teaching this student. What is outlined in the present paper may provide a tool for this purpose to be used by mathematics teachers" (p. 119).

3.3 Why so few normative statements?

The title is in no way to indicate that the descriptive/explanatory dimension is of lesser value! However, if Niss is right, and we need both dimensions, we do need to ask ourselves how, and if, we do it yet. One could therefore ask why not more papers use a mixed dimension such as the papers in Section 3.2? One answer could be lack of space, since we were very strict on page limit, but overall, could some of the papers support a normative element? In case of the papers in the proceeding I will leave it to the readers to decide and instead add some thoughts about when we can infer a normative statement from a descriptive/explanatory study.

The space here is short, but obviously first of all, the study must have *internal validity* to even have a sound description/ analysis. But, secondly, do the research results also need to have *external validity (generalisability)*? I believe so. If a description/explanation does not have any explanatory power beyond the researched case itself; how can one make a normative statement based on this? Unless of course the normative statement is solely aimed at the already studied case, in which it become rather uninteresting. Usually researchers agree that quantitative research can produce general statements, but what about qualitative studies? Generalisability in the sense of producing laws that apply universally is in fact not a useful standard for qualitative research (Guba & Lincoln, 1989, p. 61). Instead Schofield suggests to replace the notion of generalisability by ‘fittingness’: “the degree to which the situation matches other situations in which we are interested” (1990, p. 207). Hence, also qualitative studies can produce generalisable results to similar settings if studies are detailed enough described in order for others to see if it “fits” their situation.

A third factor needs to be taken into consideration. Balacheff et al. argues: “Any result is relative to a *problematique*, to a theoretical framework on which it is directly or indirectly based, and to the methodology through which it was obtained” (1998, p. 7). Hence, also the theoretical framework and methodology need to be discussed. Niss writes: “For normative issues to be subject of research it is necessary to reveal and explain the values implicated as honestly and clearly as possible, and to make them subject to scrutiny; and to undertake an analysis, as objective and neutral as possible, of the logical, philosophical, and material relations between the elements involved” (1999, p. 6). This however implies that we *have* shared views of how to do such an analysis and how to be neutral and objective. As seen above, such consensus does not seem to exist in our field. This is perhaps even more so due to postmodernism where there is “no ‘meta-narrative’ of rationality to which we can appeal and which will bring a certain unity to this diversity” (Pring, 2000, p. 110). In order to make valid normative statements, we need to move beyond postmodernism and we need to reveal, explain and discuss our values. It is not going to be easy, which the following example might illustrate.

3.4 Standard algorithms in Danish primary education – an example

In Denmark schooling of grades 1-9 (10) takes place in a compulsory system. The Ministry of Education lays down the official guidelines stated in the document *Common Goals* (Danish Ministry of Education, 2003). These consists of binding national end-goals after completing compulsory education as well as step-goals for grades 1-3, 4-6, and 7-9, respectively. Hence, these documents contain normative claims about how teaching (and learning) *should* take place. For grades 1-3 in mathematics, it is stated that the pupils’ intuitive understanding of mathematics should gradually be developed to mathematical concepts. Particularly regarding numbers and algebra, it is stated that the pupil *must have* the opportunity to, on the basis of his own understanding, develop methods to do addition and subtraction.[3] For grades 4-6, it is stated that while working with the natural numbers, the pupils still develop their own calculation methods. Algorithms are introduced *if* it makes it simpler for the pupil.[4]

Some Danish student teachers find it difficult to implement the idea of pupils developing their own algorithms while withholding the standard algorithm - and what is learn in education programmes is quickly de-learnt once confronted with reality (personal communications). It seems that Kilpatrick’s question can still be posed: “Why is it that so many intelligent, well-trained, well-intentioned teachers put such a premium on developing students’ skill in the routines of arithmetic and algebra despite decades of advice to the contrary from so-called experts? What is it that teachers know that others do not?” (1988).

[3] Author translation from Danish: ” Den enkelte elev skal have mulighed for på baggrund af egen forståelse at udvikle metoder til antalsbestemmelse ved addition og subtraktion.”

[4] Author translation from Danish: ”I arbejdet med de naturlige tal udvikler eleverne fortsat beregningsmetoder. Regneopstillinger indføres, hvis det for eleven er en forenkling af arbejdet.”

Do some of our papers have something to add to this dilemma? Indeed – many of them writes about algorithms. E.g. Baggett & Ehrenfeucht write that "learning the standard algorithm is one of the focal points in NCTM[5]'s 'Curricular Focal Points' (2006) for the second grade" (2008, p. 97). The NCTM-document itself writes: "develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems (NCTM, 2006, p. 14)". This is grade 2, but the same phrase is also used regarding multiplying, grade 4 (p. 16) and dividing, grade 5 (p. 17). Hence, the standard algorithm certainly has a place in NCTM's recommendations. This is different from the Danish *Common Goals* which explicitly states that these should only be introduced if it is a help.

In the Netherlands, Treffers et al. (2001, p. 147) distinguish between 'algorithm calculations' (traditional algorithms) and 'column calculation' using a 'splitting strategy' where interim results are calculated. They write that since 1985 due to realistic mathematics education and the wider use of calculators there is less emphasis on algorithm calculations and instead more on "mental arithmetic, estimation, and the appropriate use of the calculator; this is an international trend. It is also clear that the position of algorithm calculation has not yet been clearly defined" (2001, p. 149). They have a balanced view of the pros and cons of the two types of calculations: "the previously ascertained developments in the Netherlands can lead to a more subtle distinction than simply for or against. A few facts: Learning the calculation algorithm requires at least one hundred class hours. … an early introduction to algorithm calculation and an extensive sequel form a major obstacle to the development of mental arithmetic with handy, varied calculations; it also hampers estimation. … Column calculation promotes mental arithmetic and estimation partly due to the calculation structure from large to small … Column calculation links up naturally with the informal approaches used by children … Children can learn the algorithm-based addition procedure … in about five lessons after they have become familiar with column addition. This algorithm calculation skill can also be used with column multiplication. The same applied to algorithm-based calculation in subtraction and column division" (p. 149). What can we conclude? It is certainly more complex that the Danish *Common Goals* seems to suggest, and Kilpatrick's teachers' might be more inclined to agree that perhaps that it is not the *usual* standard algorithm that should be taught in primary school; but other algorithms. But all of these decisions rest on a variety of values about what it means to learn arithmetic – and what the purpose of schooling is.

4 Conclusions

Balacheff et al. (1998, p. 8) suggest that a category of results in mathematics education is "demolishers of illusions" which are results that undermine the beliefs and assumptions. There are many such beliefs in teachers, politicians, and researchers, and perhaps research can demolish some of them. But to do this, we first need to agree about what constitutes good research and develop a common knowledge base. Naturally we cannot hope with ten papers and two invited keynote presentations to have completed the task or covered the whole range of research and development in the area of TSG-10. I asked above, if the problem of teaching and learning arithmetic has been solved now? The answer is no, but hopefully this TSG, and this proceeding is a step forward – a brick in the wall.

[5] NCTM: National Council of Teachers of Mathematics, USA

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Models, Main Problem in TSG10[6]

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The models are present in several of the contributions from the TSG10, who are concerned about the difficulties presented in teaching and learning fractions. Some of the difficulties are related to the generalisation of multiplying and dividing operations. In this study, the alternative historical approaches to tackle this generalisation are analysed. In one of them, connections are sought between models of operations with natural numbers and those with fractions, in order to facilitate the conceptual change, and in the other this conceptual change is avoided.

Keywords: models, difficulties, fractions, division

1 Important issues in TSG10

1.1 The main lines

In any discipline, the concept that researchers have of their discipline as well as their professional interest and work is what determines the scope of their study. In TSG-10 and in Mathematical Education in general, this scope of study may include three main lines of investigation, which involve a wide variety of problems related to different topics. These lines are:

1. The study of the theoretical foundations of the teaching and learning of mathematics.
2. The study, development, implementation and evaluation of the knowledge involved in mathematics classroom practice.
3. The analysis and concretion of knowledge and practice that supports plans for professionally qualifying and improving mathematics teachers.

1.2 The main topics

In the TSG-10, these three main lines involve the following key mathematical topics: number systems and arithmetic, including operations in the number systems, ratio and proportion, and rational numbers**.** Although this is not the time nor place to focus the debate on what "arithmetic" really includes or should include, it is worth pointing out that the relation of key topics is given by the internal logic of mathematics, as opposed to what one see in the Handbook of the NCTM that have recently appeared (Lester, 2007), where the research logic aimed at building and understanding mathematical concepts is adopted. Namely: Whole number concepts (Single-digit computation, multi-digit computation, estimation and number sense, word

[6] This work was supported in part by a grant from the Spanish MEC.. Ref.: SEJ2005-06697/EDUC.

problems, the structure of the whole-number system, and rational number concepts (fractions and decimals, ratio and proportions) (see Lester, 2007).

The reason for pointing this out is to highlight two ideas. One is that the fundamental nucleus of the research effort of the TSG-10, ICME 11, is the mathematical content with respect to objects that must be taught and learnt.[7] And the other is the direction taken by the reflection that follows.

1.3 The main problems

Almost thirty years ago, in a Plenary Meeting of the ICME 4, Freudenthal, one of those who helped to re-launch the International Congress on Mathematical Education in 1969, stated what in his opinion were the main problems in mathematics education.

Allow me to start with the most down to earth problem I can think of. Among the major ones it is the most urgent. There is even the problem of how to formulate it correctly and unmistakably. Let us try a pre-formulation. It runs (thus*): Why can Johnny not do arithmetic?*

Freudenthal (1981, p, 134) was referring to how to approach the question as to why many children do not learn arithmetic as they are expected to. Since then, the research carried out has shown that the problem is so complex that the solution to general problems of teaching and learning mathematics is a long way off. This has led the concerns of a good many researchers to still be focussed on understanding and approaching didactic problems, and on developing trustworthy criteria to evaluate their eventual advances and relevance, etc. It is difficult for there to be plenty of results directly applicable to the classroom (Hitt, 2001, p. 166).

1.4 Issues in TSG-10

The different papers submitted by participants, reviewed and accepted by the TSG-10 organizational team, consider the three lines of research and the key topics that have been indicated in the previous points. For the oral presentation of these, they have been divided into two sessions; one of these includes issues, which could refer to the wide domain of research known as the conceptual multiplicative field; in this way a variety of key topics can be tied in, from the multiplication of natural numbers to fractions (thanks to the proportional nature of rational numbers).

The papers provide information about aspects of individual performance on, and understanding of, these key topics from different angles (discontinuities of models, linearity, number sense, posing problems) and different perspectives (children's, prospective teachers, and in-service teachers). In several of these, models and modelling are implicitly or explicitly present, related to the limited competence and conceptions in the domain of fractions

2 Models and the division of fractions

2.1 Models and modeling

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[7] Seeking to identify and characterise the learning problems they bring with them and the mathematical discourse that is best adapted to the students in each educational circumstance.

Although the different meanings of the terms ‘models” and “modelling” have been widely dealt with in the literature (for example, Gravemeijer, Lehrer, & Vereschaffel, 2002), in what follows the term “model” is used in the sense that is compatible with what is called ‘mathematical modelling”. And mathematical modelling is used to refer to the process of building mathematical objects that symbolically reproduce essential characteristics of a phenomenon or situation from the real world that one intends to study. The purpose of these mathematical models of reality is to explain, deduce or predict results, draw conclusions, or answer questions about the real phenomenon or situation that they model. This is what is done, for example, with the derivative, when it is taken as a mathematical model of instantaneous velocity.

2.2 The reversible nature of modelling in elemental arithmetic. Two sides of a coin.

In the teaching of elemental arithmetic, the modelling process can be seen as the reverse of what usually appears in mathematics in general. What is first provided is a physical phenomenon situation that acts as a model for studying the mathematical object, such that instead of modelling the phenomenon or situation by means of the mathematical object, the mathematical object is modelled by means of the phenomenon or situation[8]. Thus, for example, temperature is taken as a phenomenon to model integers, a pizza or a cake is taken as a situation to model fractions, and certain word problems are taken as models of elemental mathematics operations. When the teacher considers that the child, aided by the physical model, has built up a “good enough” conception of the mathematical object, he or she then inverts the terms again and proposes different questions that can be modelled by this mathematical object, thus showing that it has «applications» that «are useful for solving real problems». As the phenomena or situations are closer to the child’s daily experience, they produce significant interpretations about the arithmetic notions that one wants to teach. However, the limitations of teaching means that only one or several of the modelling phenomena or situations are chosen, not all the possible ones. As a consequence, a restriction is produced in the semantic field and a conceptual limitation.

2.3 Difficulties with the multiplication and division of fractions

Traditional teaching in the multiplication and division of fractions emphasizes drill and practice with the focus on algorithms, on numerical examples, not on word problems (Only one side of a coin). When teachers propose resolving different word problems, the children’s activity is reduced to the selection and execution of the operation to be modelled. But, to date, several studies have shown children’s difficulty in the selection of an operation for solving a big variety of fraction, multiplication and division problems. The efforts aimed at understanding how children attempt to solve multiplicative word problems have shown that children’s ability to solve these problems is influenced by a large variety of factors interacting in multiple ways. Among these factors it is worth noting the following:

a) The presence of certain key words in the problem text, the association between the situation described in the problem and some of the primitive models of operations, the type, size and structure of numbers embedded in the problem text and their relation with the result of this calculation.

[8] *We learn applications in order to learn multiplication* (Usiskin, this issue).

b) The way in which teachers conceive and treat problems in the classroom. Their very poor preparation to plan their fraction lessons; limitations of the models which they use to represent the concept of division, and their "remarkable dependence on the official teaching books, which reduced her educational creativity and autonomy" (Valdemoros, this issues)

In the different theoretical approaches that exist for explaining difficulties in students' competences and conceptions in the domain of fractions, one common aspect of several approaches is the emphasis on discontinuities between natural and fractional numbers. To explain students' difficulties with these discontinuities, one theoretical approach is the conceptual change approach and another is the theoretical approach which emphasizes the importance of underlying mental models (Prediger, in this issue). With multiplication and division operations, on changing the numeric field the operations are no longer the same, though their name does not change. Using the terminology of "levels" used by Predinger, it may be said that on the formal level the definition changes, on the algorithmic level the rules change, and on the intuitive level the meanings change. Predinger, attempts to reconcile (or integrate) both theoretical focuses by relating conceptual change with the change in some mental models of multiplication.

not all mental models for multiplication have to be changed in the transition from natural to fractional numbers. The interpretation as an area of a rectangle or as scaling up can be continued for fractions as well as the multiplicative comparison. In contrast, the basic 'repeated addition' model is not sustainable for fractions, nor the combinatorial interpretation. Vice versa: the basic model of the multiplication of a fraction, the part-of-interpretation, has no direct correspondence for natural numbers (Predinger, this issue).

The same could be said of the division of fractions, where it does not work in the partition model, whereas the measurement model works well enough in both directions: writing a mathematical expression for a problem and problem posing for a mathematical expression (see Carbone & Eaton, this issue, related to a prospective teacher posing problems that shows the meaning of $2\frac{1}{2} \div \frac{1}{2}$).

2.4 The historical approach in old textbooks

How should children learn? Or, how do people learn?

This is the question with which Freudenthal (ob. cit. p, 137) drew up his second major problem. To which he answered that

The way to answer it would be: by observing learning processes, analysing them and reporting paradigms.

And that, amongst the learners,

the biggest one, mankind, is also a learner. Observing its learning process is what we call history

To observe history one must turn to textbooks of the past. There, one may track the process of generalizing the division of whole numbers to the division of fractions.

1. In the first arithmetic texts printed, which followed Arabic traditions, the definition of multiplication and division operations were introduced through the proportional model.

Multiplication

To multiply one number by itself or by another is to find from two given numbers a third number which contains one of these numbers as many times as there are units in the other (Treviso Arithmetic, 1478/1989, p. 67).

Division, with two options: $\frac{\boldsymbol{D}}{\boldsymbol{d}} \div \frac{\boldsymbol{c}}{\boldsymbol{1}}$ (1) and $\frac{\boldsymbol{D}}{\boldsymbol{c}} \div \frac{\boldsymbol{d}}{\boldsymbol{1}}$ (2)

1. *I say that division is the operation of finding, from two given numbers, a third number which is contained as many times in the greater number as unity is contained in the lesser number* (Treviso Arithmetic, 1989, p. 85).

2. *Dividing one number by another means looking for another third number which is to be found with unity in such a proportion as the number that we divide with the divisor* (Pérez de Moya, 1562/1998, p. 197).

Influential authors like Lacroix (1825, p. 47) accompanied these definitions with commentaries aimed at explaining the conceptual change, in terms that intended to extend "to all cases" the definition that they had previously adopted of multiplying or dividing (repeated addition or partition). In these commentaries the authors began to show the discontinuity of the model, beyond natural numbers, and later they made an effort to create connections between the previous definition and the new one, making us see that the first one is a specific case of the second one.

Multiplication

The doctrine of fractions enables us to generalize the definition of multiplication given in article 21. When the multiplicand is a whole number, it shows how many times the multiplicand is to be repeated; but the term multiplication, extended to fractional expressions, does not always imply augmentation, as in the case of whole numbers. To comprehend in one statement every possible case, it may be said that to multiply one number by another is to form a number by means of the first, in the same manner as the second is formed, by means of unity. In reality, when it is necessary to multiply by 2, by 3, etc. the product consists of twice, three times, etc. the multiplicand, in the same way as the multiplier consists of two, three, etc. units; and to multiply any number by a fraction, 1/5 for example, is to take the fifth part of it, because the multiplier 1/5 being the fifth part of unity, shows that product ought to be the fifth part of the multiplicand. Also, to multiply any number by 4/5 is to take out of this number or the multiplicand a part which shall be four fifths of it, or equal to four times one fifth.

Division

The word 'contain', in its strict sense, is not more proper in the different cases presented by division, than the word 'repeat' in those presented by multiplication; for it cannot be said that the dividend contains the divisor, when it is less than the latter; the expressions is generally used, but only by analogy and extension.

To generalize division, the dividend must be considered as having the same relation to the quotient, that the divisor has to unity, because the divisor and quotient are the two factors of the dividend. This consideration is applicable to every case that division can present. When, for instance, the divisor is 5, the dividend is equal to 5 times the quotient, and consequently, the latter is the fifth part of the dividend. If the divisor is a fraction, ½ for instance, the dividend cannot be but half of the quotient, or the latter must be double that of the former.

2. In more recent textbooks, some authors like Rey Pastor and Puig Adam (1935, p. 210), introduced a conception of the operations of multiplying and dividing fractions that did not implement the conceptual change.

The multiplication of fractions was presented, on the basis of an interpretation of fractions as an operator, as a double operation of multiplying and dividing by a natural number:

The so-called problems of multiplication of fractions are, strictly speaking, problems of combined multiplication and division.

The division of fractions was presented as an inverse multiplication operation.

Showing the operation $\frac{2}{9} \div \frac{3}{7}$ *means: Is there a fraction which when multiplied by* $\frac{3}{7}$ *gives* $\frac{2}{9}$ *?*

2.5 Key implications for teaching and learning

In the process of generalizing the division of whole numbers to the division of fractions, according to the textbooks there are two options; in one, attention is paid to the conceptual change, in the other it is ignored.

1. When opting for conceptual change, in which the multiplication and division of fractions is conceived through the proportional model, the quaternary relationship is made to appear (described by Vergnaud 1983) which allows direct operations of multiplying or dividing to be identified as "missing-value proportional problems (in which three numbers are given, one of them is unity, and a fourth is asked for).

In this way, word problems of direct operations that involve fractions can be solved using a general method: the rule of three, which implicitly involves the idea of linearity[9]. However, the rule of three is often learnt routinely, giving priority to procedural knowledge (the rules that prescribe how to organise and operate the data), above conceptual knowledge, which is what is needed to exercise control of "linearity" and to limit pupils' tendency to over–use the proportional model. This is a tendency that increases the more pupils acquire linear reasoning skills through practising and solving typical linear problems (Van Dooren, De Bock & Verschaffel, 2006, p. 120) and which is stimulated by various factors, among them the numerical structure of the data involved in the problem (Van Dooren, De Bock. Evers & Verschaffel (this issue).

2. When choosing to drop the conceptual change, the focus is on the approach based on interpretations associated with fractions, and models with natural numbers.

If we look at textbooks we can see that the application of this approach to problem-solving is based on 'analytical' methods (reducing a problem to a simpler case that one already knows how to solve) that are based on a change of unity, through unitary fractions or through reduction to a common denominator. This can be seen, for example, in the way of solving the following problems taken from the text by Rey Pastor & Puig Adam (ob. cit., pgs. 209-211):

Example. 1. *Each metre of cloth costs 3/5 euros. How much do 7/4 m cost?*

Example 2. $\frac{3}{7}$ *of pizza weighs* $\frac{2}{9}$ *kilos. How much does the pizza weigh?*

[9] This idea can be treated from different perspectives: through proportionality, through first-degree equations, and through linear function. Equations and functions are not formally a part of arithmetic. De Bork (this issue) uses the term linearity through the perspective of arithmetic, so he refers it to the 'proportional model'.

Example 3. *If each pizza weighs* $\frac{3}{7}$ *kg, what portion of pizza will I have with* $\frac{2}{9}$ *kg?*

These three problems are direct operation problems, but in this operation it is not easily recognisable.

To explain that they correspond to the multiplication $\frac{3}{5}\cdot\frac{7}{4}$ and the division $\frac{2}{9}\div\frac{3}{7}$, the author of the text resorts to 'analysis' of the problem statement to reduce the problem to a "recognisable" form.

Example 1

We will indicate the operation thus: $\frac{3}{5}\cdot\frac{7}{4}$*; and we will say: If each m costs* $\frac{3}{5}$ *euros, a fourth of a m, that is to say* $\frac{1}{4}$ *m, will cost* $\frac{3}{5}\div 4=\frac{3}{5\cdot 4}$ *euros, and* $\frac{7}{4}$ *will cost seven times more, that is to say* $\frac{3\cdot 7}{5\cdot 4}=\frac{21}{20}$ *euros.*

Here, the author connects with the interpretation of fractions as "operators".

Example 2

If 3 sevenths of pizza weigh $\frac{2}{9}$ *kilos,*

1 seventh of pizza will weigh $\frac{2}{9}\div 3=\frac{2}{9\cdot 3}$

and the whole pizza = 7 sevenths of pizza, will weigh $\frac{2}{9\cdot 3}\times 7=\frac{2\cdot 7}{9\cdot 3}$

One notices in this last paragraph that on taking "the sevenths" as a new unity, the problem has been reduced to a known multiplication model with natural numbers: "if 1 unity weighs x, how much will 7 units cost?".

Example 3

Reducing the weights to the same proportional part of kgs, we are posing the question in this different way: If each pizza weighs $\frac{3\cdot 9}{7\cdot 9}$ *kilograms, how much will I have for* $\frac{2\cdot 7}{9\cdot 7}$*?*

Hence, taking $\frac{1}{7\cdot 9}$ *kg for a new unit, the pizza weights 3·9 units, so with 2·7 units I will have a portion of pizza equal to* $\frac{2\cdot 7}{3\cdot 7}$

This abstract quotient has the same expression as before; but now it represents a fraction of cake, whereas before it was of a kilo.

Just as before, taking the common denominator as a new unity, the problem is reduced to a known division (measurement) problem with natural numbers.

3 Conclusions

Research has suggested that teachers and textbook authors should revise the notions of multiplication and division of fractions. With respect to this matter, the immediate question is how to approach the process of generalising the multiplication and division of natural numbers to fractions:

a) To help students through the approach of models, it will be necessary to help them create the connections hidden among the natural number models and fraction models.

b) However, if multiplication and division of fractions are notions resistant to models, as happens with multiplying negative numbers, then it will be necessary to drop the model approach or else resort to an 'analysis' of the problem in order to reduce it to a simpler one that the student already knows how to solve. Otherwise, there is a third way which would direct the conceptual change towards a more formal mathematical conception.

In any case, in order to answer the proposed question with a sound base, it seems there is still the need for much more research to be done.

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Concepts Acquisition in Addition and Place Value

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The concept of place value and addition are presented in several of the contributions in TSG10, and the learning difficulties and possible concepts formation in learning addition are addressed. Many researches showed that children's learning experience in addition is not as easy is we thought. Most of the time, we teach addition according to textbook material and many textbooks are algorithm-teaching based. Algorithm teaching resulted in systematic error and lack of focus on the development of place value, which hindered children to understand algorithm in multi-digits addition. Also, the representation used in carrying in addition does not relate to the concept of place value or addition. In this paper, the process of addition learning and formation of place value is analysed.

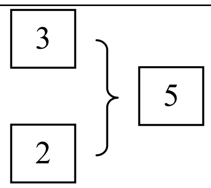
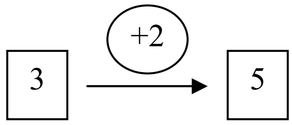
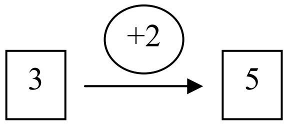
Key words: Addition, Counting, Place Value, Number operation

1 Issues in TSG 10 and learning of addition

The focus of TSG 10 is on relating the theoretical framework of learning arithmetic and the research results from practical teaching. The concept acquisition of addition and arithmetic is a long researched area. Piaget (1965) proposed that educators should not be too concerned with investigating how the child learns addition and subtraction tables as this kind of learning are frequently verbal. Carpenter and Moser (1983) summarised the results of word problem with addition and subtraction into six categories: (1) Join (addition), (2) Separate (subtraction), (3) Combine (subtraction), (4) Combine (addition), (5) Compare (subtraction) and (6) Join missing addend (subtraction). Similarly, Fuson (1992) concluded that the ability of addition come from the following stages, (1) count all from 1, (2) count on from arbitrary number, (3) count on from larger number, (4) count on from either number, (5) regrouping.

Nunes and Bryant (1996) proposed that context and situation is important for learning addition. Nunes regarded the context as (1) "change situation", (2) "part-whole situation" and (3) "comparison". The following are three most common contexts in addition.

Context explanation 3 □ 2 □ 5 in three different context:		
Context 1: A has three candies. B has two candies. They have 5 candies.	Context 2: A has three candies, he got two more. He finally has 5 candies.	Context 3: A has three candies; B has two candies more than A. B has 5 candies.

		
part-part-whole (Carpenter & Moser, 1983 Union (Brown, 1991)	Joining (Carpenter and Moser, 1983 Adding On, (Brown, 1991)	Compare (Carpenter and Moser, 1983)

2 Developing Concept of Place Value and Representation

Concept of place values is the ability to count with different base and represent a number with different base. For example, 25 can be expressed as two "10" and one "5" or five "5". Though student may be able to decompose 25 into "20 + 5", a process to build up the concept of place value, the result does not guarantee that student can have concept of place value.

Brown (1981) found that students could not answer question such as "The meter of a car showed that it has run 6299 miles, after running another mile, what number will appear on the meter?" Only 24% of the children can answer the question, knowing that 1 after 6299 is 6300.

Kamii (1985, 1994) maintained that given a suitable environment; children can re-invent the algorithm of addition and subtraction through regrouping and combine. This is also how children learn the concept of place value through their exploration. Children need not taught to combine and group into 10, they can perform such knowledge when they calculate 165 + 99 and obtain the answer 165 + 99 = (160 + 90) + (9 + 1) + 4 = 274 as they use a lot of their own regrouping.

To sum up, the development of place value depends very much on the following abilities (1) count on from any number, (2) regrouping of numbers and (3) using arbitrary unit (especially base 10) as base.

(1) Repeated Counting on (serial of addition) of single digit number

Children understand place value when they add up the numbers through count on.

For example, the following sequence provide children the chance to sum over 10, 20, and 30: "4 + 7 + 6 + 8 + 9 = 11 + 6 + 8 + 9 = 17 + 8 + 9 = 25 + 9 = 34".

(2) Base and place value

Formation of place value needs experience in operation of different base value. The following two contexts helps to build up the concept of place value. Adding with different base (dollar coin and 10c coins, hours and minutes)

	Dollars	**10c**
	4	8
+	8	7
	12	15

	Hours	**Minutes**
	19	50
+	1	15
	20	65

13	5

21	05

(3) Regrouping, splitting number into sum of different base

Regrouping is an important concept in addition (Nunes et al., 1993, Carroll, 1996).

Example: Splitting a four digits number	
2345 = 1000 + 1345.	2345 = 1000 +1000 + 345.

3 Teaching of addition and procedural learning

Liping Ma (1999) found that many teachers teach according to textbook and representation in addition did not reflect children's thinking process. Many mathematics textbooks add from the right digits. In fact, many children use both directions for adding up numbers. Adding from the left even reflect more of the children thinking (which is 14 + 3), a thinking process similar to count on with from large number, while adding start from the right did not correspondence to this thinking process.

$$\begin{array}{rr} 1 & 4 \\ + & 3 \\ \hline 1 & 7 \end{array} \qquad\qquad \begin{array}{rr} 1 & 4 \\ + & 3 \\ \hline 1 & 7 \end{array}$$

From left $\longrightarrow$ $\longleftarrow$ From right

(analogy: Count on) (procedure)

The following representation seems to be universal in many mathematics textbooks, where the carrying is marked with a small 1 (denote "10"). This is an example of procedural learning and the hidden process of operation of the 10. However, the algorithm teaching did not reflect the process of operation in place value.

Universal representation 1 (Addition)	Universal representation 2 (Addition)
$\begin{array}{rr} 2 & 5 \\ + \ 3_{1} & 7 \\ \hline 6 & 2 \\ \hline \end{array}$	$\begin{array}{rr} \scriptstyle 1 & \\ 2 & 5 \\ + \ 3 & 7 \\ \hline 6 & 2 \\ \hline \end{array}$

4 Learning of addition and place value

The process and representation of addition should reflect the thinking process of children. The following four possible processes reflect the understanding of place value, using count on or regrouping.

Representation 1	Representation 2	Representation 3	Representation 4
$\begin{array}{rrr} & 2 & 5 \\ + & 3 & 7 \\ \hline & 1 & 2 \\ + & 5 & 0 \\ \hline & 6 & 2 \end{array}$	$\begin{array}{rrr} & 2 & 5 \\ + & 3 & 7 \\ \hline & 5 & 0 \\ & 1 & 2 \\ \hline & 6 & 2 \end{array}$	$\begin{array}{rrr} & 2 & 5 \\ + & 3 & 7 \\ \hline & 5 & 5 \\ & & 7 \\ \hline & 6 & 2 \end{array}$	$\begin{array}{rrr} & 2 & 5 \\ + & 3 & 7 \\ \hline & 3 & 2 \\ & 3 & 0 \\ \hline & 6 & 2 \end{array}$
Adding "1" and "10" as unit	Adding "1" and "10" as unit	Count on from 25 and separate 37 into 30+7	Count on from 32 and separate 25 into 20+5
All process above need the concept of place value			

The above process reflects the correspondence between the addition process and the grouping of numbers, and most importantly, the thinking process. Hence representation of the process of addition is important for conceptual learning.

In order that learning is meaningful, different version of representation reflecting operation reflecting can help understanding of the concept of addition.

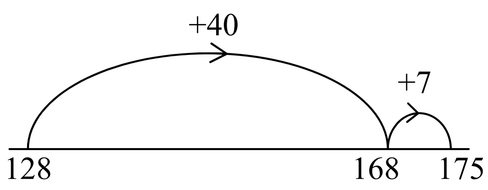
(1) Using the 10-grids table for count on process (13 + 19 = 13 + 10 + 9 = 32).

~~31~~	32	33	34	35	36	37	38	39	40
21	22	~~23~~	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	~~29~~	30
11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10

13 + 10 = 23, 23 + 9 = 32

(2) Using count on with number line (128 + 47 = 128 + 40 + 7)

Start from 128 and add 40, then add 7 (or add 7, then add 40)



$$\begin{array}{rrrr} & 1 & 2 & 8 \\ + & & 4 & 7 \\ \hline & 1 & 6 & 8 \\ & & & 7 \\ \hline & 1 & 7 & 5 \end{array}$$

(3) Using count on, 47 + 36 = 47 + 30 + 6, 294 + 148 = 294+100 + 40 + 8.

$$\begin{array}{rrr} & 4 & 7 \\ + & 3 & 6 \\ \hline & 7 & 7 \end{array} \qquad \begin{array}{rrrr} & 2 & 9 & 4 \\ + & 1 & 4 & 8 \\ \hline & 3 & 9 & 4 \end{array}$$

$$\begin{array}{r} +\quad 6 \\ \hline 83 \end{array} \qquad\qquad \begin{array}{r} +\quad 40 \\ \hline 434 \\ +\quad 8 \\ \hline 442 \end{array}$$

(4) Using combination and regrouping of 10 (or 100) to indicate place value

$$\begin{array}{r} 47 \\ +\ 36 \\ \hline 77 \\ +\quad 6 \\ \hline 83 \end{array} \qquad\qquad \begin{array}{rl} 294 & \\ +\ 148 & \\ \hline 300 & \quad 200 + 100 = 300 \\ 130 & \quad 90 + 40 = 130 \\ 12 & \quad 4 + 8 = 12 \\ \hline 442 & \quad 300 + 130 + 12 = 442 \end{array}$$

(5) Using variation of representation to indicate place value

The above example of 294 + 148 can correspond to the following representation, other than a vertical representation.

$$\begin{array}{rcrcrl} 200 & + & 90 & + & 4 & \\ 100 & + & 40 & + & 8 & \\ \hline 300 & + & 130 & + & 12 & = 442 \end{array}$$

5 Conclusions

The teaching of addition by algorithm teaching require memory of procedure, which increases the cognitive load and memory load of children in their learning. Algorithm teaching has little connection for further exploration of concepts and investigation of addition. The teaching of addition should base on contextual situation and the representation of the calculation process should reflect the thinking process of the children. That is, having a correspondence to the operation of objects.

In most cases, failure in performing addition is due to the lack of concepts in place value. The count on activities contributes to the development of concepts in place value and the sense of numbers when adding the sum. From research literature on how children re-invent arithmetic, we know that repeated count on activities, regrouping with arbitrary unit are important steps for place value development and successful acquisition of concepts in addition. We need the extension of concept, not extension of algorithm teaching.

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