Spectral edge regularity of magnetic Hamiltonians

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Abstract

We analyse the spectral edge regularity of a large class of magnetic Hamiltonians when the perturbation is generated by a globally bounded magnetic field. We can prove Lipschitz regularity of spectral edges if the magnetic field perturbation is either constant or slowly variable. We also recover an older result by G. Nenciu who proved Lipschitz regularity up to a logarithmic factor for general globally bounded magnetic field perturbations.

1. Introduction and main results

This is the second paper of the authors on the spectral regularity with respect to perturbations induced by Peierls-type magnetic flux phases. We assume that the flux is generated by a globally bounded magnetic field whose intensity is proportional with $\epsilon \in \mathbb{R}$. In a previous paper [**13**], the regularity of the Hausdorff distance between the perturbed and unperturbed spectra was investigated. In the current paper, we analyse the regularity of spectral edges when ϵ varies.

It is well known that the magnetic perturbation induced by a non-decaying magnetic field is a singular perturbation and the spectral stability is not obvious. The first proof of spectral stability of nearest-neighbour Harper operators with constant magnetic fields can be found in [**15**], while in [**10**], it is shown that the gap boundaries/spectral edges are $\frac{1}{3}$-Hölder continuous in ϵ. Later results [**4**, **16**, **18**, **19**] show that Hausdorff distance between spectra goes like $|\epsilon - \epsilon_0|^{1/2}$. This result is optimal in the sense that it is known that gaps can appear/close down precisely like $|\epsilon - \epsilon_0|^{1/2}$ if ϵ_0 generates a rational flux or if the lattice is triangular, see [**5**, **7**, **17**, **19**].

The first proof of Lipschitz continuity of gap edges for Harper-like operators with constant magnetic fields was given by Bellissard [**6**] (later on Kotani [**23**] extended his method to more general regular lattices and dimensions larger than 2).

In the continuous case of Schrödinger operators with bounded magnetic fields, the stability of gaps was first proved in [**3**, **28**]. Then in [**8**] the Hölder exponent of gap edges was shown to be at least $\frac{2}{3}$, while [**11**] provided a new proof of the results of [**6**] and extended them to continuous two-dimensional Schrödinger operators perturbed by weak *constant* magnetic fields. We note that purely magnetic Schrödinger operators of Iwatsuka type (see [**14**] and references therein) have magnetic bands whose width is proportional with the total variation of the magnetic field. An interesting open problem would be to see whether such a behaviour remains true when the magnetic field is slightly perturbed around a non-zero constant value, and this perturbation is not a function of just one variable.

A general discrete problem was formulated by Nenciu [**30**], where he worked with more general real and antisymmetric phases obeying a certain area condition (see (1.7)). These phases appear very naturally in different physical problems, see [**9**, **12**, **21**, **24**–**26**, **29**, **31**, **32**].

Using a completely different method of proof, Nenciu showed in [**30**] that the gap edges are Lipschitz up to a logarithmic factor. His method uses a regularity property of almost mid-convex functions, and works for arbitrary bounded magnetic fields, not necessarily constant.

In the current paper, we significantly improve our previous results in [**11**, **13**]. In particular, we recover the results of [**30**] and, moreover, we can prove Lipschitz regularity of spectral edges if the magnetic field perturbation is either constant or slowly variable. We also obtain results in the case in which the off-diagonal localization of the unperturbed kernels is weak.

The structure of the paper is as follows. This section continues with a detailed description of the physical motivation, then we state the main results in Theorem 1.1 and Corollary 1.2, where we also discuss which class of magnetic Hamiltonians/ΨDO's are covered. The last two sections contain the proofs.

1.1. *Physical motivation coming from magnetic Bloch systems*

In order to better motivate our paper, let us argue why the problem of spectrum location for magnetic Bloch systems described by magnetic Schrödinger operators can be reduced to the study of integral operators perturbed by magnetic phases.

We use the notation $\langle \mathbf{x} \rangle := (1 + |\mathbf{x}|^2)^{1/2}$ for any $\mathbf{x} \in \mathbb{R}^d$. Let V be a bounded, $\mathbb{Z}^d$-periodic scalar potential. Assume that the Hamiltonian $H_0 = -\Delta + V$ has an isolated spectral island σ_0 which consists of the range of a finite number of Bloch bands. Up to the addition of a constant, we may assume that $\min(\sigma_0) < 0 < \max(\sigma_0)$. The orthogonal projector P_0 corresponding to σ_0 can be written as a Riesz integral, which implies that the band Hamiltonian reads as:

$$T_0 := H_0 P_0 = \frac{i}{2\pi} \int_{\mathcal{C}} z(H_0 - z)^{-1} \, dz,$$

where $\mathcal{C}$ is a simple, positively oriented contour which surrounds σ_0 and no other part of the spectrum of H_0. The resolvent $(H_0 - z)^{-1}$ has an integral kernel $(H_0 - z)^{-1}(\mathbf{x}, \mathbf{x}')$, which is continuous outside the diagonal, behaves like $\ln |\mathbf{x} - \mathbf{x}'|$ at the diagonal when $d = 2$, and like $|\mathbf{x} - \mathbf{x}'|^{-d+2}$ when $d \geqslant 3$, and decays like $e^{-\alpha(z)|\mathbf{x} - \mathbf{x}'|}$ when $|\mathbf{x} - \mathbf{x}'| > 1$ with some $\alpha(z) > 0$. Using the formula:

$$(H_0 - z)^{-1} = \sum_{j=1}^{n} (z - i)^{j-1}(H_0 - i)^{-j} + (z - i)^n (H_0 - z)^{-1}(H_0 - i)^{-n}, \quad n \geqslant 1,$$

we obtain

$$T_0 = \frac{i}{2\pi} \int_{\mathcal{C}} z(z - i)^n (H_0 - z)^{-1}(H_0 - i)^{-n} \, dz \quad \forall n \geqslant 1.$$

Choosing n large enough (depending on d), one can prove that T_0 has a jointly continuous integral kernel $K_0(\mathbf{x}, \mathbf{x}')$, and there exist $\alpha > 0$ and $C < \infty$ such that

$$|K_0(\mathbf{x}, \mathbf{x}')| \leqslant C \, e^{-\alpha |\mathbf{x} - \mathbf{x}'|} \quad \forall \mathbf{x}, \mathbf{x}' \in \mathbb{R}^d. \tag{1.1}$$

From now on, C will denote a generic numerical constant. Now let us introduce the magnetic perturbation. For us, a stationary magnetic field will be described by a closed 2-form B on $\mathbb{R}^d$ (that is, $dB = 0$) with bounded and smooth components $B_{jk}(\mathbf{x}) = -B_{kj}(\mathbf{x})$. Given B, we can always consider the 'transverse magnetic vector potential' A with the components

$$A_m(\mathbf{x}) := \sum_{k=1}^{d} \int_0^1 dt \, t B_{km}(t\mathbf{x}) x_k \tag{1.2}$$

which obeys $\sum_{j=1}^d x_j A_j(\mathbf{x}) = 0$ and $B = dA$, that is, $B_{jk}(\mathbf{x}) = \partial_j A_k(\mathbf{x}) - \partial_k A_j(\mathbf{x})$.

A quantity we are interested in is the flux of this 2-form through triangles (here $\langle \mathbf{x}, \mathbf{y}, \mathbf{x}' \rangle$ denotes the triangle with vertices $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}'$ in $\mathbb{R}^d$):

$$\Phi^B(\mathbf{x}, \mathbf{y}, \mathbf{x}') := \int_{\langle \mathbf{x}, \mathbf{y}, \mathbf{x}' \rangle} B = \sum_{j,k} (y_j - x_j)(x_k' - y_k)$$
$$\times \int_0^1 dt \int_0^t ds B_{jk}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}) + s(\mathbf{x}' - \mathbf{y})). \tag{1.3}$$

We define

$$\phi^A(\mathbf{y}, \mathbf{z}) := -\int_{[\mathbf{y}, \mathbf{z}]} A = -\Phi^B(0, \mathbf{y}, \mathbf{z})$$
$$= -\sum_{j<k} (y_j z_k - y_k z_j) \int_0^1 dt \int_0^t ds B_{jk}(t\mathbf{y} + s(\mathbf{z} - \mathbf{y})). \tag{1.4}$$

In the case of a constant unit magnetic field, we have $B_{jk} = -B_{kj} = 1$ if $j < k$ and $B_{jj} = 0$. Thus (1.4) gives

$$\phi^A(\mathbf{x}, \mathbf{y}) = -\frac{1}{2} \sum_{j<k} (x_j y_k - x_k y_j). \tag{1.5}$$

It is easy to see that ϕ^A is antisymmetric and, due to the Stokes theorem, we have

$$\phi^A(\mathbf{x}, \mathbf{y}) + \phi^A(\mathbf{y}, \mathbf{x}') - \phi^A(\mathbf{x}, \mathbf{x}') = -\Phi^B(\mathbf{x}, \mathbf{y}, \mathbf{x}'). \tag{1.6}$$

Thus if the components of B obey $\|B_{jk}\|_\infty \leqslant \text{const}$, then we see that we have constructed a 2-point function $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:

$$\phi(\mathbf{x}, \mathbf{y}) = -\phi(\mathbf{y}, \mathbf{x}), \quad \text{and} \quad |\phi(\mathbf{x}, \mathbf{y}) + \phi(\mathbf{y}, \mathbf{z}) + \phi(\mathbf{z}, \mathbf{x})| \leqslant \text{const} \, |\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle|. \tag{1.7}$$

If $\epsilon \geqslant 0$, then consider the magnetic Schrödinger operator $H_\epsilon := (-i\nabla - \epsilon A)^2 + V$. For simplicity, assume that $d = 2$. Fix a compact set $\mathcal{C} \subset \rho(H_0)$. Then (see Section 4 in [11]) there exist $\epsilon_0 > 0$, $\alpha < \infty$ and $C < \infty$ such that for every $0 \leqslant \epsilon \leqslant \epsilon_0$ we have that $\mathcal{C} \subset \rho(H_\epsilon)$ and

$$\sup_{z \in \mathcal{C}} |(H_\epsilon - z)^{-1}(\mathbf{x}, \mathbf{x}') - e^{i\epsilon \phi^A(\mathbf{x}, \mathbf{x}')}(H_0 - z)^{-1}(\mathbf{x}, \mathbf{x}')|$$
$$\leqslant C\epsilon \, e^{-\alpha|\mathbf{x} - \mathbf{x}'|} \quad \forall \mathbf{x} \neq \mathbf{x}' \in \mathbb{R}^2. \tag{1.8}$$

This estimate also implies that H_ϵ has an isolated spectral island σ_ϵ close to σ_0 when ϵ is small enough. Moreover, one can prove (see [8, 13]) that the Hausdorff distance between σ_ϵ and σ_0 is of order $\sqrt{\epsilon}$.

The main physical question to be addressed in this paper is the following: can one show that $\max(\sigma_\epsilon) - \max(\sigma_0)$ and $\min(\sigma_\epsilon) - \min(\sigma_0)$ can go to zero faster than $\sqrt{\epsilon}$?

Let us show how we can reduce this question to the study of integral operators perturbed by magnetic phases. Let P_ϵ be the spectral projection corresponding to σ_ϵ and $H_\epsilon P_\epsilon$ the corresponding band operator. Applying the Riesz integral formula (remember that $H_0 P_0 = T_0$), we obtain

$$|H_\epsilon P_\epsilon(\mathbf{x}, \mathbf{x}') - e^{i\epsilon \phi^A(\mathbf{x}, \mathbf{x}')} K_0(\mathbf{x}, \mathbf{x}')| \leqslant C\epsilon \, e^{-\alpha|\mathbf{x} - \mathbf{x}'|}. \tag{1.9}$$

Denote by T_ϵ the bounded self-adjoint operator with the kernel

$$K_{\epsilon A}(\mathbf{x}, \mathbf{x}') := e^{i\epsilon \phi^A(\mathbf{x}, \mathbf{x}')} K_0(\mathbf{x}, \mathbf{x}').$$

Up to a Schur-type estimate, (1.9) implies

$$\|H_\epsilon P_\epsilon - T_\epsilon\| \leqslant C\epsilon. \tag{1.10}$$

In general, if A and B are two bounded and self-adjoint operators and u is a norm-one vector, then we have

$$\langle u, Au \rangle \leqslant \langle u, Bu \rangle + \|A - B\| \leqslant \max \sigma(B) + \|A - B\|$$

which leads to

$$\max \sigma(A) \leqslant \max \sigma(B) + \|A - B\|, \quad |\max \sigma(A) - \max \sigma(B)| \leqslant \|A - B\|, \qquad (1.11)$$

hence (1.10) leads to

$$|\max \sigma(H_\epsilon P_\epsilon) - \max \sigma(T_\epsilon)| \leqslant C\epsilon. \qquad (1.12)$$

The spectrum of $H_\epsilon P_\epsilon$ is the set $\{0\} \cup \sigma_\epsilon$. Since we assumed that $\min(\sigma_0) < 0 < \max(\sigma_0)$ and $\sigma_0 = [\min(\sigma_0), \max(\sigma_0)]$, we have that $\{0\} \cup \sigma_0 = \sigma(T_0) = \sigma_0$; in [13] he has proved that the Hausdorff distance between $\sigma(T_\epsilon)$ and $\sigma(T_0)$ goes at least like $\sqrt{\epsilon}$, hence

$$|\max \sigma(T_\epsilon) - \max \sigma(T_0)| \leqslant C\sqrt{\epsilon}. \qquad (1.13)$$

Together with (1.12), this implies that $\max \sigma(H_\epsilon P_\epsilon)$ converges to $\max(\sigma_0) > 0$, which means that $\max \sigma(H_\epsilon P_\epsilon) = \max(\sigma_\epsilon)$ for ϵ small enough. Then (1.12) shows that $\max(\sigma_\epsilon) - \max \sigma(T_\epsilon) \sim \epsilon$, hence the only remaining question is whether we can improve the exponent of ϵ in (1.13).

The conclusion is that the ϵ-regularity of the edges of σ_ϵ is given by the ϵ-regularity of the spectral edges of T_ϵ.

1.2. *The main theorem*

DEFINITION 1.   We say that a linear bounded operator $T_K \in \mathbb{B}(L^2(\mathbb{R}^d))$ has an *off-diagonal polynomial decay of order* $\alpha \geqslant 0$ if it is defined by an integral kernel K such that

$$\|T_K\|_\alpha := \max \left\{ \sup_{\mathbf{x} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(\mathbf{x}, \mathbf{y})| \langle \mathbf{x} - \mathbf{y} \rangle^\alpha \, d\mathbf{y}, \ \sup_{\mathbf{y} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(\mathbf{x}, \mathbf{y})| \langle \mathbf{x} - \mathbf{y} \rangle^\alpha \, d\mathbf{x} \right\} < \infty.$$

We denote by $\mathscr{C}^\alpha$ the complex linear space of these operators with the norm $\| \cdot \|_\alpha$, including by definition the identity operator Id.

REMARK 1.   Throughout this paper, A indicates the transverse magnetic gauge (1.2) associated to a magnetic field B whose components B_{jk} are continuous and bounded on $\mathbb{R}^d$, together with their partial derivatives of all orders (no decay conditions are imposed on the field).

A different situation is when the magnetic field perturbation comes from a slowly varying vector potential $A_\epsilon(\mathbf{x}) := a(\epsilon \mathbf{x})$, where the partial derivatives of all orders of each a_j are globally bounded and continuous (note that a can grow linearly in $\mathbf{x}$). In this case, the magnetic field perturbation is of the form $\epsilon B_\epsilon(\mathbf{x})$ with $B_\epsilon(\mathbf{x}) := (da)(\epsilon \mathbf{x})$. Then we define

$$\phi^{A_\epsilon}(\mathbf{x}, \mathbf{x}') := -\epsilon \Phi^{B_\epsilon}(0, \mathbf{x}, \mathbf{x}'), \quad K_{A_\epsilon}(\mathbf{x}, \mathbf{y}) := e^{i\phi^{A_\epsilon}(\mathbf{x}, \mathbf{y})} K(\mathbf{x}, \mathbf{y}). \qquad (1.14)$$

For every operator $T_K \in \mathscr{C}^\alpha \subset \mathbb{B}(L^2(\mathbb{R}^d))$, we can define a family of bounded linear operators $T_{K_A} \in \mathscr{C}^\alpha$ whose kernels are given by

$$K_A(\mathbf{x}, \mathbf{y}) := e^{i\phi^A(\mathbf{x}, \mathbf{y})} K(\mathbf{x}, \mathbf{y}).$$

If one uses a different vector potential A' such that $dA' = dA = B$, then Stokes theorem ensures that

$$\phi^A(\mathbf{x}, \mathbf{y}) = -\int_{[\mathbf{x}, \mathbf{y}]} A' + \int_{[0, \mathbf{y}]} A' - \int_{[0, \mathbf{x}]} A',$$

which shows that the operator with kernel $e^{-i\int_{[\mathbf{x}, \mathbf{y}]} A'} K(\mathbf{x}, \mathbf{y})$ is unitarily equivalent with T_{K_A}.

Let $T_K \in \mathscr{C}^\alpha$ be a self-adjoint operator. Since both $\phi^{\epsilon A}$ and ϕ^{A_ϵ} are antisymmetric, both $T_{K_{\epsilon A}}$ and $T_{K_{A_\epsilon}}$ belong to $\mathscr{C}^\alpha$ and are self-adjoint.

THEOREM 1.1.   *Denote by $\mathcal{E}(\epsilon) = \sup \sigma(T_{K_{\epsilon A}})$ and by $\mathcal{E}_\epsilon := \sup \sigma(T_{K_{A_\epsilon}})$. The following statements hold true uniformly in $|\epsilon| \leqslant \frac{1}{2}$:*

(i) *if $1 \leqslant \alpha < 2$, then there exists a numerical constant $C_\alpha > 0$ with $\lim_{\alpha \nearrow 2} C_\alpha = \infty$, such that*

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leqslant C_\alpha \|T_K\|_\alpha |\epsilon|^{\alpha/2}; \tag{1.15}$$

(ii) *if $\alpha \geqslant 2$, then there exists a numerical constant $C > 0$ such that*

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leqslant C \|T_K\|_2 |\epsilon| \ln(1/|\epsilon|); \tag{1.16}$$

(iii) *let $\alpha \geqslant 2$ and assume that either B is a constant magnetic field, or the magnetic field perturbation comes from a slowly varying vector potential A_ϵ. Then there exists a numerical constant $C > 0$ such that*

$$\max\{|\mathcal{E}(\epsilon) - \mathcal{E}(0)|, |\mathcal{E}_\epsilon - \mathcal{E}_0|\} \leqslant C \|T_K\|_2 |\epsilon|. \tag{1.17}$$

REMARK 2.   Because $\inf(\sigma(T_{K_A})) = -\sup(\sigma(T_{(-K)_A}))$, the theorem also holds true if $\mathcal{E}(\epsilon) = \inf \sigma(T_{K_{\epsilon A}})$ and $\mathcal{E}_\epsilon = \inf \sigma(T_{K_{A_\epsilon}})$.

REMARK 3.   We have the general identity

$$\|T_{K_A}\| = \max\{-\inf(\sigma(T_{K_A})), \sup(\sigma(T_{K_A}))\}.$$

If $F(\epsilon) := \max\{f_1(\epsilon), f_2(\epsilon)\}$, then let us show that $|F(\epsilon) - F(0)| \leqslant \max_n |f_n(\epsilon) - f_n(0)|$. Assume without loss that $F(\epsilon) \geqslant F(0)$. If $F(\epsilon) = f_j(\epsilon)$ and $F(0) = f_j(0)$, then the inequality is trivial. If $F(\epsilon) = f_j(\epsilon)$ and $F(0) = f_k(0)$ with $j \neq k$, then we have

$$|F(\epsilon) - F(0)| = F(\epsilon) - F(0) = f_j(\epsilon) - f_k(0) \leqslant f_j(\epsilon) - f_j(0)$$
$$\leqslant \max_{n=1,2} |f_n(\epsilon) - f_n(0)|.$$

Thus the theorem also holds true if $\mathcal{E}(\epsilon) = \|T_{K_{\epsilon A}}\|$ and $\mathcal{E}_\epsilon = \|T_{K_{A_\epsilon}}\|$.

REMARK 4.   Since $\phi^{\epsilon A} = \epsilon \phi^A = (\epsilon - \epsilon_0)\phi^A + \epsilon_0 \phi^A$, we can absorb $e^{i\epsilon_0 \phi^A(\mathbf{x}, \mathbf{y})}$ into the kernel K without changing its off-diagonal decay properties. Thus the above results for $\mathcal{E}(\epsilon)$ can be easily extended near any $\epsilon_0 \in \mathbb{R}$, with $|\epsilon|$ replaced by $|\epsilon - \epsilon_0|$. The dependence on ϵ of ϕ^{A_ϵ} is nonlinear, and it seems that the results on $\mathcal{E}_\epsilon$ cannot be extended. In fact, if ϵ is large, then we can no longer talk about a slowly varying magnetic field.

1.3. *Application to magnetic pseudodifferential operators*

Let us briefly present the setting behind them. Denote by $\mathcal{X} := \mathbb{R}^d$ the *configuration space* of a physical system, by $\mathcal{X}^* \cong \mathbb{R}^d$ its dual (*the space of momenta*), by $\langle \cdot, \cdot \rangle : \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ the duality bilinear form, and by $\Xi := \mathcal{X} \times \mathcal{X}^*$ the *phase space* with the canonical symplectic form $\sigma((x, \xi), (y, \eta)) := \langle \xi, y \rangle - \langle \eta, x \rangle$. Let us recall from [**25, 26**] that to any *classical Hamiltonian* described by a real smooth function $h : \Xi \to \mathbb{R}$ (with polynomial growth together with all its derivatives) and to any bounded smooth magnetic field described by a closed 2-form having components $B_{jk} \in BC^\infty(\mathcal{X})$ (that is, smooth and uniformly bounded with all their partial derivatives of all orders), we associate a *quantum Hamiltonian* defined by the following action

on test functions (as oscillating integrals):

$$(\mathfrak{Op}^A(h)f)(\mathbf{x}) := (2\pi)^{-d} \int_{\mathcal{X}} \int_{\mathcal{X}^*} e^{i\langle \xi, \mathbf{x}-\mathbf{y}\rangle} \Lambda^A(\mathbf{x},\mathbf{y}) h\left(\frac{\mathbf{x}+\mathbf{y}}{2}, \xi\right) f(\mathbf{y})\, d\mathbf{y}\, d\xi, \qquad (1.18)$$

where A is a vector potential such that $dA = B$, $\Lambda^A(\mathbf{x},\mathbf{y}) := e^{i\phi^A(\mathbf{x},\mathbf{y})}$, $f \in \mathscr{S}(\mathcal{X})$, and $\mathbf{x} \in \mathcal{X}$. Let us remind ourselves here that the quantum Hamiltonian depends on the choice of the vector potential A, but different choices lead to unitarily equivalent operators. Choosing a vector potential of class $C^\infty_{\mathsf{pol}}(\mathcal{X})$ (that is, its components are smooth and have a polynomial growth at infinity together with all their derivatives) is always possible and, with such a choice, Proposition 3.5 in [26] states that the application $\mathfrak{Op}^A$ defines a bijection from the tempered distributions on Ξ to the continuous operators from $\mathscr{S}(\mathcal{X})$ to $\mathscr{S}'(\mathcal{X})$. Thus the composition of operators (when possible) induces a composition law on $\mathscr{S}'(\Xi)$ that we call *the magnetic Moyal product*, and is explicitly given by the following formula (for a pair of test functions ϕ and ψ from $\mathscr{S}(\Xi)$) that only depends on the magnetic field and not on the vector potential

$$(\phi \sharp^B \psi)(X) = \pi^{-2d} \int_\Xi \int_\Xi e^{-2i\sigma(X-Y,X-Z)}\, e^{-i\int_{\mathcal{T}(\mathbf{x},\mathbf{y},\mathbf{z})} B} \phi(Y)\psi(Z)\, dY\, dZ,$$

where we have introduced the notation of the form $X = (\mathbf{x}, \xi)$ for the points of Ξ, and we have denoted by $\mathcal{T}(\mathbf{x},\mathbf{y},\mathbf{z})$ the triangle with vertices $\mathbf{x}-\mathbf{y}+\mathbf{z}$, $\mathbf{y}-\mathbf{z}+\mathbf{x}$, and $\mathbf{z}-\mathbf{x}+\mathbf{y}$.

Before stating the second main result of our paper, we need some more notation. Let $S^m_\rho(\Xi)$ denote the set of classical Hörmander symbols $F(\mathbf{x}, \xi)$ such that

$$\sup_{(\mathbf{x},\xi)\in\Xi} \langle \xi \rangle^{\rho|\beta|-m} |D_\mathbf{x}^\alpha D_\xi^\beta F(\mathbf{x},\xi)| < \infty.$$

An interesting class of symbols, related to Onsager–Peierls effective Hamiltonians, are symbols from $S^0_0(\Xi)$ that do not depend on the first variable $\mathbf{x} \in \mathcal{X}$, and are periodic in the second variable $\xi \in \mathcal{X}^*$ with respect to a lattice $\Gamma_* \subset \mathcal{X}^*$. We denote by $S^0_{\Gamma_*}$ this class of symbols.

Here is the second main result of our paper.

COROLLARY 1.2. *Let F be either a symbol in $S^t_1(\Xi)$ with $t < 0$, or a symbol in $S^0_{\Gamma_*}$. Let $\mathcal{E}(\epsilon)$ denote either $\sup \sigma(\mathfrak{Op}^{\epsilon A}(F))$, $\inf \sigma(\mathfrak{Op}^{\epsilon A}(F))$, or $\|\mathfrak{Op}^{\epsilon A}(F)\|$. Let $\mathcal{E}_\epsilon$ denote the same quantities defined with A_ϵ instead of ϵA.*

(i) *There exists a constant $C < \infty$ such that:*

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leqslant C|\epsilon| \ln(1/|\epsilon|), \quad |\epsilon| \leqslant \tfrac{1}{2}.$$

(ii) *If the magnetic field is either constant or slowly variable, then the logarithmic factor is absent*:

$$\max\{|\mathcal{E}(\epsilon) - \mathcal{E}(0)|, |\mathcal{E}_\epsilon - \mathcal{E}_0|\} \leqslant C|\epsilon|, \quad |\epsilon| \leqslant \tfrac{1}{2}.$$

REMARK 5. Let us recall from [21] that if h is a real elliptic symbol (of Hörmander type), of strictly positive order m, then the corresponding magnetic ΨDO can be extended to a lower semibounded self-adjoint operator denoted by H_A, acting in $L^2(\mathcal{X})$ with domain *a magnetic Sobolev space* (as defined also in [21]). If we work with a Schrödinger symbol $h(\mathbf{x}, \xi) = \xi^2 + V(\mathbf{x})$, then $H_A = (-i\nabla - A(x))^2 + V(x)$. Moreover, let us recall that Proposition 6.5 from [22] implies the existence for any $\mathfrak{z} \in \rho(H_A)$ of a symbol $r_B(h, \mathfrak{z}) \in S^{-m}_1(\mathcal{X})$ such that $(H_A - \mathfrak{z})^{-1} = \mathfrak{Op}^A(r_B(h, \mathfrak{z}))$. In [2, 13], we proved that the spectrum of $H_{\epsilon A}$ varies continuously (as a subset of $\mathbb{R}$) with the parameter ϵ.

REMARK 6. If h is as before and $\Phi \in C_0^\infty(\mathbb{R})$, then by using the Dynkin–Helffer–Sjöstrand formula, it was proved in [**22**, Proposition 6.7] that there exists a symbol $\Phi_B[h] \in S_1^{-\infty}(\mathcal{X})$ such that the operator $\Phi(H_A)$ defined by functional calculus with self-adjoint operators is in fact of the form $\Phi(H_A) = \mathfrak{Op}^A(\Phi_B[h])$. The results in [**22**] imply that

$$\|\Phi(H_{\epsilon A}) - \mathfrak{Op}^{\epsilon A}(\Phi_0[h])\| \leqslant C|\epsilon|, \quad \Phi_0[h] \in S_1^{-\infty}(\mathcal{X}),$$

which shows that the eventual non-Lipschitz behaviour in the spectrum of $\Phi(H_{\epsilon A})$ can only come from the phase factor. We observe that $\mathfrak{Op}^{\epsilon A}(\Phi_0[h])$ is covered by Corollary 1.2.

REMARK 7. With h as before, suppose that the Weyl quantized operator $H := \mathfrak{Op}(h)$ has a bounded and isolated spectral island σ_0. Then one can find a function $\Phi \in C_0^\infty(\mathbb{R})$ such that $\Phi(t) = t$ on σ_0 and the support of Φ is disjoint from the rest of $\sigma(H)$. Thus $\sigma(\Phi(H)) = \sigma_0 \cup \{0\}$. Up to a translation in energy we may suppose that $0 \in \sigma_0$. It follows (using Remark 6 and mimicking the argument in the introduction) that $H_{\epsilon A}$ will still have an isolated spectral island σ_ϵ near σ_0 if ϵ is small enough, and its edges behave as in Corollary 1.2.

2. Proof of Theorem 1.1

Let us fix a non-zero and non-negative symmetric function $f \in C_0^\infty(\mathbb{R}^d)$. If $\delta > 0$ and $\mathbf{x} \in \mathbb{R}^d$, then we denote by $f_\delta(\mathbf{x}) := f(\delta\mathbf{x})$ and by $\widetilde{f}_{(\delta)} := f_\delta * f_\delta$. We have

$$0 \leqslant \widetilde{f}_{(\delta)}(\mathbf{x}) \leqslant \|f_\delta\|_2^2 = \int_{\mathbb{R}^d} d\mathbf{y}\, f_\delta(-\mathbf{y}) f_\delta(\mathbf{y}) = \widetilde{f}_{(\delta)}(0) = \delta^{-d}\|f\|_2^2.$$

For any kernel K such that $T_K \in \mathscr{C}^\alpha$ for some $\alpha \geqslant 0$, we define

$$K_{(\delta)}(\mathbf{x}, \mathbf{y}) := K(\mathbf{x}, \mathbf{y}) \widetilde{f}_{(\delta)}(\mathbf{x} - \mathbf{y})\|f_\delta\|_2^{-2}. \tag{2.1}$$

Since $|K_{(\delta)}(\mathbf{x}, \mathbf{y})| \leqslant |K(\mathbf{x}, \mathbf{y})|$, we have $\|T_{K_{(\delta)}}\| \leqslant \|T_{K_{(\delta)}}\|_\alpha \leqslant \|T_K\|_\alpha$.

Denote by $\widetilde{F}_{(\delta)}(\mathbf{x}, \mathbf{y}) := \widetilde{f}_{(\delta)}(\mathbf{x} - \mathbf{y})$ and $F_\delta(\mathbf{x}, \mathbf{y}) := f_\delta(\mathbf{x} - \mathbf{y})$ the kernels associated to the obvious convolution operators. We now write two simple but very important identities. The first one is

$$\|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} |F_\delta(\mathbf{y}, \mathbf{x})|^2 \, d\mathbf{y} = \|f_\delta\|_2^{-2} \widetilde{F}_{(\delta)}(\mathbf{x}, \mathbf{x}) = \|f_\delta\|_2^{-2} \widetilde{f}_{(\delta)}(0) = 1 \quad \forall \mathbf{x} \in \mathbb{R}^d. \tag{2.2}$$

The second one is (we use (1.6))

$$
\begin{aligned}
e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{x}')}\widetilde{F}_{(\delta)}(\mathbf{x}, \mathbf{x}') &= \int_{\mathbb{R}^d} e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} [e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{y})} F_\delta(\mathbf{x}, \mathbf{y})][e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x}')} F_\delta(\mathbf{y}, \mathbf{x}')] \, d\mathbf{y} \\
&= \int_{\mathbb{R}^d} \overline{[e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x})} F_\delta(\mathbf{y}, \mathbf{x})]}[e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x}')} F_\delta(\mathbf{y}, \mathbf{x}')] \, d\mathbf{y} \\
&\quad + \int_{\mathbb{R}^d} [e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1][e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{y})} F_\delta(\mathbf{x}, \mathbf{y})][e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x}')} F_\delta(\mathbf{y}, \mathbf{x}')] \, d\mathbf{y} \\
&= \int_{\mathbb{R}^d} \overline{[e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x})} F_\delta(\mathbf{y}, \mathbf{x})]}[e^{i\phi^{\epsilon A}(\mathbf{y}, \mathbf{x}')} F_\delta(\mathbf{y}, \mathbf{x}')] \, d\mathbf{y} \\
&\quad + e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{x}')} \int_{\mathbb{R}^d} [e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1] F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \, d\mathbf{y} \\
&\quad + e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{x}')} \int_{\mathbb{R}^d} |e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1|^2 F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \, d\mathbf{y}. \tag{2.3}
\end{aligned}
$$

If we multiply the left-hand side of (2.3) with $\|f_\delta\|_2^{-2} K(\mathbf{x}, \mathbf{x}')$ we obtain $e^{i\phi^{\epsilon A}(\mathbf{x}, \mathbf{x}')} K_{(\delta)}(\mathbf{x}, \mathbf{x}')$, see (2.1). Then we can compute the quadratic form of $T_{K_{(\delta), \epsilon A}}$ on some $u \in L^2(\mathbb{R}^d)$ with $\|u\|_{L^2(\mathbb{R}^d)} = 1$:

$$
\begin{aligned}
\langle u, T_{K_{(\delta), \epsilon A}} u \rangle = {}& \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \langle [e^{i\phi^{\epsilon A}(\mathbf{y}, \cdot)} F_\delta(\mathbf{y}, \cdot) u(\cdot)], \ T_K [e^{i\phi^{\epsilon A}(\mathbf{y}, \cdot)} F_\delta(\mathbf{y}, \cdot) u(\cdot)] \rangle \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' [e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1] \\
& \times K_{\epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}') \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' |e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1|^2 \\
& \times K_{\epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}') \\
\leqslant {}& \mathcal{E}(0) \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \|e^{i\phi^{\epsilon A}(\mathbf{y}, \cdot)} F_\delta(\mathbf{y}, \cdot) u(\cdot)\|_{L^2(\mathbb{R}^d)}^2 \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' [e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1] \\
& \times K_{\epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}') \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' |e^{i\Phi^{\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1|^2 \\
& \times K_{\epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}').
\end{aligned}
\tag{2.4}
$$

Using (2.2) in the last inequality, we obtain

$$
\begin{aligned}
\langle u, T_{K_{(\delta), \pm \epsilon A}} u \rangle \leqslant {}& \mathcal{E}(0) + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' [e^{i\Phi^{\pm \epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1] \\
& \times K_{\pm \epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}') \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' |e^{i\Phi^{\pm \epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1|^2 \\
& \times K_{\pm \epsilon A}(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}').
\end{aligned}
\tag{2.5}
$$

Another simple but very important observation which we want to underline here, is that T_K and $T_{K_{\pm \epsilon A}}$ have the same Schur α-norms, see Definition 1. Moreover, we have the obvious identities:

$$
K(\mathbf{x}, \mathbf{x}') = e^{i\phi^{\pm \epsilon A}(\mathbf{x}, \mathbf{x}')} (e^{i\phi^{\mp \epsilon A}(\mathbf{x}, \mathbf{x}')} K(\mathbf{x}, \mathbf{x}')), \quad T_K = T_{(K_{\mp \epsilon A})_{\pm \epsilon A}}.
\tag{2.6}
$$

Thus changing K with $K_{\mp \epsilon A}$ in (2.5), we obtain

$$
\begin{aligned}
\langle u, T_{K_{(\delta)}} u \rangle \leqslant {}& \mathcal{E}(\mp \epsilon) + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' [e^{i\Phi^{\pm \epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1] \\
& \times K(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}') \\
& + 4 \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}' |\sin(\Phi^{|\epsilon| B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')/2)|^2 \\
& \times K(\mathbf{x}, \mathbf{x}') F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \overline{u(\mathbf{x})} u(\mathbf{x}').
\end{aligned}
\tag{2.7}
$$

This implies

$$
\begin{aligned}
\langle u, T_{K_{(\delta)}} u \rangle \leqslant\ & \frac{1}{2}\mathcal{E}(\epsilon) + \frac{1}{2}\mathcal{E}(-\epsilon) \\
& + \|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}'(\cos(\Phi^{|\epsilon|B}(\mathbf{x},\mathbf{y},\mathbf{x}')) - 1) \\
& \times K(\mathbf{x},\mathbf{x}') F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')\overline{u(\mathbf{x})}u(\mathbf{x}') \\
& + 4\|f_\delta\|_2^{-2} \int_{\mathbb{R}^d} d\mathbf{y} \int_{\mathbb{R}^d} d\mathbf{x} \int_{\mathbb{R}^d} d\mathbf{x}'|\sin(\Phi^{|\epsilon|B}(\mathbf{x},\mathbf{y},\mathbf{x}')/2)|^2 \\
& \times K(\mathbf{x},\mathbf{x}') F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')\overline{u(\mathbf{x})}u(\mathbf{x}'). \tag{2.8}
\end{aligned}
$$

Taking the supremum with respect to $u \in L^2(\mathbb{R}^d)$ with $\|u\|_{L^2(\mathbb{R}^d)} = 1$, we obtain

$$
\begin{aligned}
\sup(\sigma(T_{K_{(\delta)}})) \leqslant\ & \frac{1}{2}\mathcal{E}(\epsilon) + \frac{1}{2}\mathcal{E}(-\epsilon) + \|f_\delta\|_2^{-2} \\
& \times \Bigg(\sup_{\|u\|=1} \int_{\mathbb{R}^{3d}} d\mathbf{y}\, d\mathbf{x}\, d\mathbf{x}'|\cos(\Phi^{|\epsilon|B}(\mathbf{x},\mathbf{y},\mathbf{x}')) - 1| \\
& \times |K(\mathbf{x},\mathbf{x}')|F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')|u(\mathbf{x})||u(\mathbf{x}')| \\
& + 4 \sup_{\|u\|=1} \int_{\mathbb{R}^{3d}} d\mathbf{y}\, d\mathbf{x}\, d\mathbf{x}'|\sin(\Phi^{|\epsilon|B}(\mathbf{x},\mathbf{y},\mathbf{x}')/2)|^2 \\
& \times |K(\mathbf{x},\mathbf{x}')|F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')|u(\mathbf{x})||u(\mathbf{x}')| \Bigg). \tag{2.9}
\end{aligned}
$$

Using (1.3) and (1.7), we obtain that for all $\mathbf{x},\mathbf{y},\mathbf{x}' \in \mathbb{R}^d$:

$$
|\Phi^{\epsilon B}(\mathbf{x},\mathbf{y},\mathbf{x}')| \leqslant C|\epsilon||\langle \mathbf{x},\mathbf{y},\mathbf{x}'\rangle| \leqslant \frac{C|\epsilon|}{2}|\mathbf{x}-\mathbf{x}'||\mathbf{x}-\mathbf{y}|^{1/2}|\mathbf{y}-\mathbf{x}'|^{1/2}. \tag{2.10}
$$

Using (2.10) and elementary properties of sin and cos, we can bound the last two terms from the right-hand side of (2.9) by

$$
C\|f_\delta\|_2^{-2} \sup_{\|u\|=1} \int_{\mathbb{R}^{3d}} |\Phi^{\epsilon B}(\mathbf{x},\mathbf{y},\mathbf{x}')|^\alpha |K(\mathbf{x},\mathbf{x}')|F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')|u(\mathbf{x})|\,|u(\mathbf{x}')|\,d\mathbf{y}\,d\mathbf{x}\,d\mathbf{x}'
$$

$$
\leqslant C|\epsilon|^\alpha \|T_K\|_\alpha \|f_\delta\|_2^{-2} \left(\int_{\mathbb{R}^d} |\mathbf{y}|^\alpha f_\delta(\mathbf{y})^2 d\mathbf{y} \right) \leqslant C|\epsilon|^\alpha \delta^{-\alpha}\|T_K\|_\alpha, \quad 1 \leqslant \alpha < 2,
$$

where the last inequality is due to the fact that $|\mathbf{y}| \leqslant C\delta^{-1}$ on the support of f_δ. Moreover,

$$
C\|f_\delta\|_2^{-2} \sup_{\|u\|=1} \int_{\mathbb{R}^{3d}} |\Phi^{\epsilon B}(\mathbf{x},\mathbf{y},\mathbf{x}')|^2 |K(\mathbf{x},\mathbf{x}')|F_\delta(\mathbf{x},\mathbf{y}) F_\delta(\mathbf{y},\mathbf{x}')|u(\mathbf{x})||u(\mathbf{x}')|\,d\mathbf{y}\,d\mathbf{x}\,d\mathbf{x}'
$$

$$
\leqslant C|\epsilon|^2 \|T_K\|_2 \|f_\delta\|_2^{-2} \left(\int_{\mathbb{R}^d} |\mathbf{y}|^2 f_\delta(\mathbf{y})^2\, d\mathbf{y} \right) \leqslant C|\epsilon|^2 \delta^{-2}\|T_K\|_2, \quad \alpha \geqslant 2.
$$

Using this in (2.9), we obtain

$$
\begin{aligned}
\sup(\sigma(T_{K_{(\delta)}})) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C|\epsilon|^\alpha \delta^{-\alpha}\|T_K\|_\alpha, \quad 1 \leqslant \alpha < 2, \\
\sup(\sigma(T_{K_{(\delta)}})) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C|\epsilon|^2 \delta^{-2}\|T_K\|_2, \quad \alpha \geqslant 2.
\end{aligned} \tag{2.11}
$$

Applying (1.11) with $A = T_K$ and $B = T_{K_{(\delta)}}$, and using (2.11) we have

$$
\begin{aligned}
\mathcal{E}(0) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C|\epsilon|^\alpha \delta^{-\alpha}\|T_K\|_\alpha + \|T_{K_{(\delta)}} - T_K\|, \quad 1 \leqslant \alpha < 2, \\
\mathcal{E}(0) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C|\epsilon|^2 \delta^{-2}\|T_K\|_2 + \|T_{K_{(\delta)}} - T_K\|, \quad \alpha \geqslant 2.
\end{aligned} \tag{2.12}
$$

Because $T_{K_{(\delta)}} - T_K = T_{K_{(\delta)} - K}$, using the Schur–Holmgren bound we have

$$\|T_{K_{(\delta)}} - T_K\| \leqslant \|f_\delta\|_2^{-2} \sup_{\mathbf{x} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\widetilde{f}_{(\delta)}(\mathbf{x} - \mathbf{x}') - \widetilde{f}_{(\delta)}(0)||K(\mathbf{x}, \mathbf{x}')| \, d\mathbf{x}'. \tag{2.13}$$

The following bound is valid for all $\mathbf{x} \in \mathbb{R}^d$:

$$|\widetilde{f}_{(\delta)}(\mathbf{x}) - \widetilde{f}_{(\delta)}(0)| \leqslant \delta|\mathbf{x}| \int_0^1 \int_{\mathbb{R}^d} |\nabla f(\delta t \mathbf{x} - \delta \mathbf{y})| \, |f(\delta \mathbf{y})| \, d\mathbf{y} \, dt \leqslant C|\mathbf{x}|\delta^{1-d}.$$

Moreover, because $\widetilde{f}_{(\delta)}$ has a maximum at $\mathbf{x} = 0$, by expanding up to the second-order around $\mathbf{x} = 0$, we have

$$|\widetilde{f}_{(\delta)}(\mathbf{x}) - \widetilde{f}_{(\delta)}(0)| \leqslant \delta^2|\mathbf{x}|^2 \int_0^1 dt \int_0^t ds \int_{\mathbb{R}^d} \sqrt{\sum_{1 \leqslant j,k \leqslant d} |\partial_{jk}^2 f(\delta s \mathbf{x} - \delta \mathbf{y})|^2} |f(\delta \mathbf{y})| \, d\mathbf{y}$$

which leads to

$$|\widetilde{f}_{(\delta)}(\mathbf{x}) - \widetilde{f}_{(\delta)}(0)| \leqslant C|\mathbf{x}|^2 \delta^{2-d}. \tag{2.14}$$

If $1 \leqslant \alpha \leqslant 2$, then we can combine the last two inequalities and get

$$|\widetilde{f}_{(\delta)}(\mathbf{x}) - \widetilde{f}_{(\delta)}(0)| \leqslant C|\mathbf{x}|^\alpha \delta^{\alpha-d}. \tag{2.15}$$

Introducing (2.15) in (2.13), we obtain

$$\|T_{K_{(\delta)}} - T_K\| \leqslant C\delta^d \sup_{\mathbf{x} \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathbf{x} - \mathbf{x}'|^\alpha \delta^{\alpha-d} |K(\mathbf{x}, \mathbf{x}')| \, d\mathbf{x}'$$
$$\leqslant C\delta^\alpha \|T_K\|_\alpha, \quad 1 \leqslant \alpha \leqslant 2. \tag{2.16}$$

If $T_K \in \mathscr{C}_\alpha$ with $\alpha \geqslant 2$, then we introduce (2.14) into (2.13) and obtain

$$\|T_{K_{(\delta)}} - T_K\| \leqslant C\delta^2 \|T_K\|_2, \quad \alpha \geqslant 2. \tag{2.17}$$

Introducing the last two estimates in (2.12), we obtain

$$\mathcal{E}(0) \leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C\|T_K\|_\alpha (\delta^\alpha + |\epsilon|^\alpha \delta^{-\alpha}), \quad 1 \leqslant \alpha < 2,$$

or

$$\mathcal{E}(0) \leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C\|T_K\|_2 (\delta^2 + |\epsilon|^2 \delta^{-2}), \quad \alpha \geqslant 2.$$

Until now, δ and ϵ have been independent. Keeping ϵ fixed and minimizing the right-hand side with respect to δ imposes the condition $\delta = |\epsilon|^{1/2}$ in both cases. Thus we obtain

$$\begin{aligned} \mathcal{E}(0) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C\|T_K\|_\alpha |\epsilon|^{\alpha/2}, \quad 1 \leqslant \alpha < 2, \\ \mathcal{E}(0) &\leqslant \tfrac{1}{2}\mathcal{E}(\epsilon) + \tfrac{1}{2}\mathcal{E}(-\epsilon) + C\|T_K\|_2 |\epsilon|, \quad \alpha \geqslant 2. \end{aligned} \tag{2.18}$$

As we commented in Remark 4, the above estimates can be obtained near any ϵ_0 by redefining K. Thus we have just proved that the map $\mathbb{R} \ni x \mapsto \mathcal{E}(x) \in \mathbb{R}$ is a bounded, almost mid-convex function which obeys

$$\begin{aligned} \mathcal{E}\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\mathcal{E}(a) + \frac{1}{2}\mathcal{E}(b) + C\|T_K\|_{2\beta} \left|\frac{b-a}{2}\right|^\beta, \quad 1/2 \leqslant \beta := \alpha/2 < 1, \\ \mathcal{E}\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{2}\mathcal{E}(a) + \frac{1}{2}\mathcal{E}(b) + C\|T_K\|_2 \left|\frac{b-a}{2}\right|, \quad \alpha \geqslant 2. \end{aligned} \tag{2.19}$$

Regularity of bounded and almost mid-convex functions. Now we shall prove that (2.19) implies (1.15), essentially following Nenciu [30]. Assume that $1/2 \leqslant \beta < 1$ is fixed and denote by $M := C\|T_K\|_{2\beta}$. Thus we have

$$\mathcal{E}\left(\frac{a+b}{2}\right) \leqslant \frac{1}{2}\mathcal{E}(a) + \frac{1}{2}\mathcal{E}(b) + M \left|\frac{b-a}{2}\right|^\beta \quad \forall a, b \in \mathbb{R}.$$

The strategy is to construct a constant $C_\beta > 0$ such that for every $x \in \mathbb{R}$ and $0 < \eta < 1/2$ to have

$$- C_\beta \eta^\beta \leqslant \mathcal{E}(x + \eta) - \mathcal{E}(x) \leqslant C_\beta \eta^\beta. \tag{2.20}$$

One can easily prove by induction the following two inequalities (assume that $a < b$ and $n \geqslant 1$):

$$\mathcal{E}((2^{-1} + \cdots + 2^{-n})a + 2^{-n}b) \leqslant (2^{-1} + \cdots + 2^{-n})\mathcal{E}(a) + 2^{-n}\mathcal{E}(b)$$
$$+ M\left(\frac{b-a}{2^n}\right)^\beta \left(1 + \frac{1}{2^{1-\beta}} + \cdots + \frac{1}{(2^{1-\beta})^{n-1}}\right) \tag{2.21}$$

and

$$\mathcal{E}(2^{-n}a + (2^{-1} + \cdots + 2^{-n})b) \leqslant 2^{-n}\mathcal{E}(a) + (2^{-1} + \cdots + 2^{-n})\mathcal{E}(b)$$
$$+ M\left(\frac{b-a}{2^n}\right)^\beta \left(1 + \frac{1}{2^{1-\beta}} + \cdots + \frac{1}{(2^{1-\beta})^{n-1}}\right). \tag{2.22}$$

Given $\eta \in (0, 1/2)$, we define $N_\eta := \lfloor \ln(1/\eta)/\ln(2) \rfloor + 1$. We have $N_\eta - 1 \leqslant \ln(1/\eta)/\ln(2) < N_\eta$, that is, $1 < \eta 2^{N_\eta} \leqslant 2$. Replace $a = x$, $b = x + \eta 2^{N_\eta}$, and $n = N_\eta$ in (2.21). We have $(2^{-1} + \cdots + 2^{-N_\eta})a + 2^{-N_\eta}b = x + \eta$ and

$$\mathcal{E}(x + \eta) - \mathcal{E}(x) \leqslant \eta \frac{\mathcal{E}(b) - \mathcal{E}(a)}{b - a} + M\eta^\beta \frac{1}{1 - 2^{\beta-1}}.$$

Since $\mathcal{E}$ is bounded and we always have $1 < b - a \leqslant 2$, the right-hand side is of order η^β. Thus the right-hand side of (2.20) is proved.

For the other inequality in (2.20), we replace $a = x + \eta - \eta 2^{N_\eta}$, $b = x + \eta$ and $n = N_\eta$ in (2.22). We have $2^{-N_\eta}a + (2^{-1} + \cdots + 2^{-N_\eta})b = x$ and

$$\mathcal{E}(x) - \mathcal{E}(x + \eta) \leqslant -\eta \frac{\mathcal{E}(b) - \mathcal{E}(a)}{b - a} + M\eta^\beta \frac{1}{1 - 2^{\beta-1}}.$$

Now we have to change sign and note again that $1 < b - a \leqslant 2$, uniformly in η. Thus (2.20) is proved, and so is the first part of our Theorem 1.1.

Concerning the second part, that is, the estimate (1.16), we see that both (2.21) and (2.22) hold true even if $\beta = 1$. In this case, we can no longer use the geometric series and we get an extra N_η. This is the reason for having the logarithmic factor in (1.16). We give no further details.

Constant magnetic field. Now let us separately treat the case in which the perturbation comes from a constant magnetic field and $\alpha \geqslant 2$. In this case, we shall see that one can directly prove a Lipschitz regularity for $\mathcal{E}$, without the logarithmic factor, and without using the trick based on almost mid-convex functions.

Going back to the inequality (2.5), we see that we can isolate the $\mathbf{y}$ integral in the second term on the right-hand side. This integral is

$$\int_{\mathbb{R}^d} F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}')(e^{i\Phi^{\pm\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}')} - 1) \, d\mathbf{y}.$$

Now let us show that the first-order term in ϵ is just zero

$$\int_{\mathbb{R}^d} F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \Phi^{\pm\epsilon B}(\mathbf{x}, \mathbf{y}, \mathbf{x}') \, d\mathbf{y} = 0.$$

Let us start by noticing that the above integral is proportional with

$$\sum_{j,k} B_{jk} \int_{\mathbb{R}^d} (x_k - y_k) f_\delta(\mathbf{x} - \mathbf{y})(y_j - x_j') f_\delta(\mathbf{y}' - \mathbf{x}') \, d\mathbf{y}.$$

Denote by $\mathcal{F}$ and $\mathcal{F}^-$ the Fourier transform and its inverse. Then

$$(x_k - y_k) f_\delta(\mathbf{x} - \mathbf{y}) \sim [\mathcal{F}^-(\partial_k \mathcal{F} f_\delta)](\mathbf{x} - \mathbf{y}), \quad (y_j - x'_j) f_\delta(\mathbf{y} - \mathbf{x}')$$
$$\sim [\mathcal{F}^-(\partial_j \mathcal{F} f_\delta)](\mathbf{y} - \mathbf{x}').$$

Hence

$$\int_{\mathbb{R}^d} (x_k - y_k) f_\delta(\mathbf{x} - \mathbf{y})(y_j - x'_j) f_\delta(\mathbf{y}' - \mathbf{x}') \, d\mathbf{y}$$
$$\sim \{[\mathcal{F}^-(\partial_k \mathcal{F} f_\delta)] * [\mathcal{F}^-(\partial_j \mathcal{F} f_\delta)]\}(\mathbf{x} - \mathbf{x}') \sim \{\mathcal{F}^-[(\partial_k \mathcal{F} f_\delta)(\partial_j \mathcal{F} f_\delta)]\}(\mathbf{x} - \mathbf{x}').$$

But the product $(\partial_k \mathcal{F} f_\delta)(\partial_j \mathcal{F} f_\delta)$ is symmetric in k and j while B_{jk} is antisymmetric, hence the sum gives zero.

Thus we see that in this case, the linear term disappears without having to appeal to the arithmetic mean, as we did in (2.9). By performing the same type of analysis as before in order to deal with the quadratic terms, we obtain in particular that

$$|\mathcal{E}(\epsilon) - \mathcal{E}(0)| \leqslant C\|T\|_2 (\delta^2 + |\epsilon|^2/\delta^2).$$

Choose $\delta = |\epsilon|^{1/2}$ and the proof is over.

Slowly varying magnetic field. In this case, the antisymmetric form entering the flux formula (1.3) is of the form $B(\epsilon \cdot)$, while the total magnetic field perturbation is $\epsilon B(\epsilon \cdot)$. Also, $\phi^{\pm \epsilon A}$ has to be replaced with $\phi^{\pm A_\epsilon}$. Again, the only obstacle in getting the Lipschitz behaviour is the linear term as before. Let us show that we have the bound

$$\left| \int_{\mathbb{R}^d} F_\delta(\mathbf{x}, \mathbf{y}) F_\delta(\mathbf{y}, \mathbf{x}') \Phi^{\pm \epsilon B_\epsilon}(\mathbf{x}, \mathbf{y}, \mathbf{x}') \, d\mathbf{y} \right| \leqslant C|\mathbf{x} - \mathbf{x}'| \, |\epsilon|^2/\delta^2. \tag{2.23}$$

Indeed, using (1.3) and Taylor's formula, we have

$$\left| \Phi^{\epsilon B_\epsilon}(\mathbf{x}, \mathbf{y}, \mathbf{x}') - \frac{\epsilon}{2} \sum_{j,k} B_{jk}(\epsilon \mathbf{x})(y_j - x_j)(x'_k - y_k) \right|$$
$$\leqslant C|\epsilon|^2 |\langle \mathbf{x}, \mathbf{y}, \mathbf{x}' \rangle|(|\mathbf{y} - \mathbf{x}| + |\mathbf{y} - \mathbf{x}'|).$$

The contribution coming from $\epsilon \sum_{j,k} B_{jk}(\epsilon \mathbf{x})(y_j - x_j)(x'_k - y_k)$ is zero, as in the constant case. The right-hand side can be bounded by

$$C|\epsilon|^2 |\mathbf{x} - \mathbf{x}'||\mathbf{x} - \mathbf{y}|(|\mathbf{y} - \mathbf{x}| + |\mathbf{y} - \mathbf{x}'|)$$

term having a polynomial growth which introduced in the integral will generate a diverging factor δ^{-2}. Note that $|\mathbf{x} - \mathbf{x}'|$ can be coupled later on with $K(\mathbf{x}, \mathbf{x}')$. Having proved (2.23), the estimate we get in the end is

$$|\mathcal{E}_\epsilon - \mathcal{E}_0| \leqslant C\|T\|_1 |\epsilon|^2/\delta^2 + C\|T\|_2 (\delta^2 + |\epsilon|^2/\delta^2)$$

which gives the Lipschitz regularity by again taking $\delta = |\epsilon|^{1/2}$. The proof is over.

3. *Proof of Corollary* 1.2

It was proved in [**26**] that the magnetic quantization associated to the vector potential A is a topological vector space isomorphism $\mathscr{S}'(\Xi) \to \mathbb{B}(\mathscr{S}(\mathcal{X}); \mathscr{S}'(\mathcal{X}^*))$. We have also given the explicit form of this isomorphism by constructing the distribution kernel associated to a symbol. More precisely, let us denote by $S_W : \mathcal{X}^2 \to \mathcal{X}^2$ the linear isomorphism $S_W(\mathbf{x}, \mathbf{y}) := ((\mathbf{x} + \mathbf{y})/2, \mathbf{x} - \mathbf{y})$, by $S_W^* : \mathscr{S}'(\mathcal{X}^2) \to \mathscr{S}'(\mathcal{X}^2)$ its transposed map $S_W^*(F) := F \circ S_W$ and by $\mathcal{F} : \mathscr{S}'(\mathcal{X}) \to \mathscr{S}'(\mathcal{X}^*)$ the Fourier transform (normed in order to give a unitary map $L^2(\mathcal{X}) \to L^2(\mathcal{X}^*)$); we shall denote its inverse by $\mathcal{F}^-$. Then the map $\mathfrak{K}_W := S_W^* \circ (\mathbb{1} \otimes \mathcal{F}^-) : \mathscr{S}'(\Xi) \to \mathscr{S}'(\mathcal{X}^2)$ is a bijection associated to any 'symbol' on Ξ an 'integral kernel' on $\mathcal{X}$.

We denote by $T_K : \mathscr{S}(\mathcal{X}) \to \mathscr{S}'(\mathcal{X})$ the operator associated to the integral kernel $K \in \mathscr{S}'(\mathcal{X}^2)$, that is,

$$\langle v, T_K u \rangle := K(v \otimes u) \quad \forall v, u \in \mathscr{S}(\mathcal{X}),$$

or formally

$$(T_K u)(\mathbf{x}) := \int_{\mathcal{X}} K(\mathbf{x}, \mathbf{z}) u(\mathbf{z}) \, d\mathbf{z}.$$

Then we have the equality

$$\mathfrak{Op}(F) = T_{S_W^* \circ (\mathbb{1} \otimes \mathcal{F}^-)F} \quad \forall F \in \mathscr{S}'(\Xi).$$

We make the important observation that the magnetic quantization can be expressed as

$$\mathfrak{Op}^A(F) = T_{e^{i\phi^A} S_W^* \circ (\mathbb{1} \otimes \mathcal{F}^-)F},$$

where $\phi^A(\mathbf{x}, \mathbf{y}) = -\int_{[\mathbf{x}, \mathbf{y}]} A$. If $K := S_W^* \circ (\mathbb{1} \otimes \mathcal{F}^-)F$, then the 'magnetic integral kernel' of [11, 29, 30] is

$$K_A(\mathbf{x}, \mathbf{y}) = e^{i\phi^A(\mathbf{x}, \mathbf{y})} K(\mathbf{x}, \mathbf{y}).$$

Thus we have an explicit way of transferring results and formulas between the two representations, working with the one which is more suitable for a given problem. The magnetic pseudodifferential calculus developed in [21, 22, 26, 27] is an equivalent formulation of the calculus with magnetic integral kernels proposed in [9, 11, 29–32], the equivalence being realized through the application taking a symbol into the distribution kernel associated to the pseudodifferential operator the symbol generates.

3.1. Decaying symbols

Using Proposition 1.3.3 from [1] and its variant given in [27], we see that for any symbol F of the type $S_1^t(\Xi)$ with $t < 0$, its partial inverse Fourier transform $(\mathbb{1} \otimes \mathcal{F}^-)F$ is a function for which there exists a constant C such that

$$\sup_{\mathbf{x} \in \mathbb{R}^d} |[(\mathbb{1} \otimes \mathcal{F}^-)F](\mathbf{x}, \mathbf{x}')| \leqslant C|\mathbf{x}'|^{-d-t}, \quad \mathbf{x}' \neq 0,$$

and has rapid decrease in the second variable (thus in $\mathbf{x} - \mathbf{y}$ for the kernel). Thus through our identification discussed above, it defines an integral operator with a kernel of class $\mathscr{C}^N$ for any $N \in \mathbb{N}$ (see Definition 1). Thus for this class of symbols, the Corollary is an immediate consequence of Theorem 1.1.

3.2. Periodic symbols

For any $\lambda \in S_{\Gamma_*}^0$, we denote by $\tilde{\lambda} := (\mathbb{1} \otimes \mathcal{F}^-)\lambda$ and, taking into account the theorem in [20] concerning the Fourier transform of periodic distributions and denoting by $\Gamma \subset \mathcal{X}$ the dual lattice of Γ_*, we obtain

$$(\mathfrak{Op}^A(\lambda)u)(\mathbf{x}) = \sum_{\gamma'' \in \Gamma} \Lambda^A(\mathbf{x}, \mathbf{x} - \gamma'') \tilde{\lambda}(\gamma'') u(\mathbf{x} - \gamma'') \tag{3.1}$$

and the operator $\mathfrak{Op}^A(\lambda)$ has the distribution kernel

$$K_\lambda^A(\mathbf{x}, \mathbf{y}) := \sum_{\gamma'' \in \Gamma} e^{i\phi^A(\mathbf{x}, \mathbf{y})} \tilde{\lambda}(\gamma'') \delta(\mathbf{x} - \mathbf{y} - \gamma'') \tag{3.2}$$

with $\tilde{\lambda}(\gamma)$ having rapid decay with respect to $\gamma \in \Gamma$.

There exists a d-dimensional parallelepiped Ω such that every $\mathbf{x} \in \mathbb{R}^d$ can be uniquely represented as $\gamma + \underline{x}$, with $\gamma \in \Gamma$ and $\underline{x} \in \Omega$. We can see $\mathfrak{Op}^{\epsilon A}(\lambda)$ as an operator in

$$l^2(\Gamma; L^2(\Omega)) \sim l^2(\Gamma) \otimes L^2(\Omega), \quad \mathfrak{Op}^A(\lambda) = \{T_{\gamma\gamma'}\}_{\gamma,\gamma' \in \Gamma}, \quad T_{\gamma\gamma'} \in \mathbb{B}(L^2(\Omega)),$$

where the operator $T_{\gamma\gamma'}$ has the distribution kernel

$$T_{\gamma\gamma'}(\underline{x}, \underline{x}') := K_\lambda^{\epsilon A}(\gamma + \underline{x}, \gamma' + \underline{x}') = e^{i\phi^{\epsilon A}(\underline{x}+\gamma, \underline{x}'+\gamma')}\tilde{\lambda}(\gamma - \gamma')\delta(\underline{x} - \underline{x}').$$

We see that $T_{\gamma\gamma'}$ is a multiplication operator

$$L^2(\Omega) \ni f \longmapsto [T_{\gamma\gamma'}f](\underline{x}) = e^{i\phi^{\epsilon A}(\underline{x}+\gamma, \underline{x}+\gamma')}\tilde{\lambda}(\gamma - \gamma')f(\underline{x}) \in L^2(\Omega).$$

Consider the unitary operator

$$U_\epsilon : l^2(\Gamma) \otimes L^2(\Omega) \longmapsto l^2(\Gamma) \otimes L^2(\Omega), \quad [U_\epsilon\Psi]_\gamma(\underline{x}) := e^{i\phi^{\epsilon A}(\gamma, \underline{x}+\gamma)}\Psi_\gamma(\underline{x}).$$

The operator

$$T^\epsilon := U_\epsilon \mathfrak{Op}^{\epsilon A}(\lambda)U_\epsilon^*, \quad [T_{\gamma\gamma'}^\epsilon f](\underline{x})$$
$$= e^{i\phi^{\epsilon A}(\gamma, \underline{x}+\gamma)}e^{i\phi^{\epsilon A}(\underline{x}+\gamma, \underline{x}+\gamma')}e^{i\phi^{\epsilon A}(\underline{x}+\gamma', \gamma')}\tilde{\lambda}(\gamma - \gamma')f(\underline{x})$$

will have the same spectrum as $\mathfrak{Op}^{\epsilon A}(\lambda)$. Define the operator

$$\widetilde{T}^\epsilon = \{\widetilde{T}_{\gamma,\gamma'}^\epsilon\}_{\gamma\gamma' \in \Gamma}, \quad \widetilde{T}_{\gamma\gamma'}^\epsilon = e^{i\phi^{\epsilon A}(\gamma,\gamma')}\tilde{\lambda}(\gamma - \gamma')\mathbb{1}.$$

The following estimate

$$|\phi^{\epsilon A}(\gamma, \underline{x} + \gamma) + \phi^{\epsilon A}(\underline{x} + \gamma, \underline{x} + \gamma') + \phi^{\epsilon A}(\underline{x} + \gamma', \gamma') - \phi^{\epsilon A}(\gamma, \gamma')|$$
$$\leqslant |\epsilon|C(|\langle\gamma, \underline{x} + \gamma, \underline{x} + \gamma'\rangle| + |\langle\gamma, \underline{x} + \gamma', \gamma'\rangle|)$$

is a consequence of (1.6) applied twice. We observe that since Ω is bounded, the areas of both triangles are bounded from above by $|\gamma - \gamma'|$. Using a Schur–Holmgren-type bound, we obtain that

$$\|T^\epsilon - \widetilde{T}^\epsilon\| \leqslant C|\epsilon| \sum_{\gamma \in \Gamma} |\tilde{\lambda}(\gamma)| \, |\gamma|,$$

which shows that the spectrum of $\mathfrak{Op}^{\epsilon A}(\lambda)$ is at an $|\epsilon|$-Hausdorff distance from the spectrum of $\widetilde{T}^\epsilon$. Hence the spectral edges of $\mathfrak{Op}^{\epsilon A}(\lambda)$ have the same regularity as those of $\widetilde{T}^\epsilon$. The operator $\widetilde{T}^\epsilon$ is independent of the $\underline{x}$ variable, and we have

$$\widetilde{T}^\epsilon = \tilde{t}^\epsilon \otimes \mathbb{1}, \quad \tilde{t}^\epsilon = \{\tilde{t}_{\gamma\gamma'}^\epsilon\}_{\gamma,\gamma' \in \Gamma} \in \mathbb{B}(l^2(\Gamma)), \quad \tilde{t}_{\gamma\gamma'}^\epsilon = e^{i\phi^{\epsilon A}(\gamma,\gamma')}\tilde{\lambda}(\gamma - \gamma').$$

Hence it is enough to study the spectral edges of the discrete operator $\tilde{t}^\epsilon$ acting on $l^2(\Gamma)$, which is exactly of the form previously considered in [**11**, **30**]. Although the Lipschitz behaviour up to the logarithmic factor is essentially proved in [**30**], let us show how one can modify the proof of our Theorem 1.1 in order to cover the discrete case.

First of all, the space $\mathscr{C}^\alpha$ introduced in Definition 1 will now consist of operators $t \in \mathbb{B}(l^2(\Gamma))$ for which

$$\|t\|_\alpha = \max\left\{\sup_{\gamma \in \Gamma} \sum_{\gamma'} |t_{\gamma\gamma'}|\langle\gamma - \gamma'\rangle^\alpha, \ \sup_{\gamma' \in \Gamma} \sum_{\gamma \in \Gamma} |t_{\gamma\gamma'}|\langle\gamma - \gamma'\rangle^\alpha \, d\mathbf{x}\right\} < \infty.$$

The proof of Theorem 1.1 remains true if the following replacements take place:

(i) $x \in \mathbb{R}^d$ and $x' \in \mathbb{R}^d$ are replaced by $\gamma \in \Gamma$ and $\gamma' \in \Gamma$;
(ii) the corresponding integrals over $\mathbb{R}^d$ with respect to dx and dx' are replaced by sums over Γ;
(iii) $L^2(\mathbb{R}^d)$ is replaced by $l^2(\Gamma)$.

Note the important fact that the integration with respect to $\mathbf{y}$ must not be replaced with a discrete sum.

We conclude that the spectral edges (and the norm) of $\widetilde{t}^{\epsilon}$ (hence $\mathfrak{Op}^{\epsilon A}(\lambda)$) obey the estimates announced in Corollary 1.2, where the constants are proportional with the quantity $\sum_{\gamma \in \Gamma} \langle \gamma \rangle^2 |\tilde{\lambda}(\gamma)|$. The proof is over.

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