



**AALBORG UNIVERSITY**  
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## **Papers**

*Volume 7: 2004-2008*

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*Publication date:*  
2008

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*  
Thoft-Christensen, P. (2008). *Papers: Volume 7: 2004-2008*. Department of Civil Engineering, Aalborg University.

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# CHAPTER 124

## SYSTEM RELIABILITY <sup>1</sup>

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### 1. INTRODUCTION

A fully satisfactory estimate of the reliability of a structure is based on a systems approach. In some situations it may be sufficient to estimate the reliability of the individual structural members of a structural system. This is the case for statically determinate structures where failure in any member will result in failure of the total system. However, failure of a single element in a structural system will generally not result in failure of the total system, because the remaining elements may be able to sustain the external load by redistribution of the internal load effects. This is typically the case of statically indeterminate (redundant) structures, where failure of the structural system always requires that more than one element fail. A structural system will usually have a great number of failure modes, and the most significant failure modes must be taken into account in an estimate of the reliability of the structure.

From an application point of view, reliability of structural systems is a relatively new area. However, extensive research has been conducted in the last decades and a number of effective methods are developed. Some of these methods have a limited scope and some are more general. One may argue that this area is still in a phase of development and therefore not yet sufficiently clarified for practical application. However, a number of real practical applications are made with success. This presentation does not try to cover all aspects of structural systems reliability. No attempt is made to include all methods that can be used in estimating the reliability of structural systems. Only the  $\beta$ -unzipping method is described in detail here, since the author has extensive experience from using that method.

This section is to some degree based on the books by Thoft-Christensen & Baker [1] and Thoft-Christensen & Murotsu [2].

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<sup>1</sup> Chapter 15 in “Engineering Design Reliability Handbook” (Eds. E. Nikolaidis, D.M, Ghiocel & S. Singhal), CRC Press, INC., 2004, pp. 15.1-15.45.

## 2. MODELLING OF STRUCTURAL SYSTEMS

### 2.1 Introduction

A real structural system is so complex that direct exact calculation of the probability of failure is completely impossible. The number of possible different failure modes is so large that they cannot all be taken into account, and even if they could all be included in the analysis, exact probabilities of failure cannot be calculated. It is therefore necessary to idealize the structure so that the estimate of the reliability becomes manageable. Not only the structure itself, but also the loading must be idealized. Because of these idealizations it is important to bear in mind that the estimates of e.g. probabilities of failure are related to the idealized system (the model) and not directly to the structural system. The main objective of a structural reliability design is to be able to design a structure so that the probability of failure is minimized in some sense. Therefore, the model must be chosen carefully so that the most important failure modes for the real structures are reflected in the model.

It is assumed that the total reliability of the structural system can be estimated by considering a finite number of failure modes and combining them in complex reliability systems. The majority of structural failures are caused by human errors. Human errors are usually defined as serious mistakes in design, analysis, construction, maintenance, or use of the structure, and they cannot be included in the reliability modeling presented in this chapter. However, it should be stated that the probabilities of failure calculated by the methods presented in this book are much smaller than those observed in practice due to gross errors. The failures included in structural reliability theory are caused by random fluctuations in the basic variables such as extremely low strength capacities or extremely high loads.

Only truss and frame structures are considered, although the methods used can be extended to a broader class of structures. Two-dimensional (plane) as well as three-dimensional (spatial) structures are treated. It is assumed that the structures consist of a finite number of bars and beams and that these structural elements are connected by a finite number of joints. In the model of the structural system, the failure elements are connected to the structural elements (bars, beams, and joints), see section 2.3. For each of the structural elements a number of different failure modes exist. Each failure mode results in element failure, but systems failure will in general only occur when a number of simultaneous element failures occur. A more precise definition of systems failure is one of the main objects of this chapter. It will be assumed that the reliability of a structural system can be estimated on the basis of a series system modeling, where the elements are failure modes. The failure modes are modeled by parallel systems.

### 2.2 Fundamental Systems

Consider a statically determinate (non-redundant) structure with  $n$  structural elements and assume that each structural element has only one failure element. The total number of failure elements is therefore also  $n$ ; see figure 1. For such a structure the total structural system fails as soon as any structural element fails. This is symbolized by a series system.

For a statically indeterminate (redundant) structure, failure in a single structural element will not always result in failure of the total system. The reason for this is that



Figure 1. Series system with  $n$  elements.

the remaining structural elements will be able to sustain the external loading by redistribution

of the load effects. For statically indeterminate structures total failure will usually require that failure takes place in more than one structural element. It is necessary to define what is understood by total failure of a structural system. This problem will be

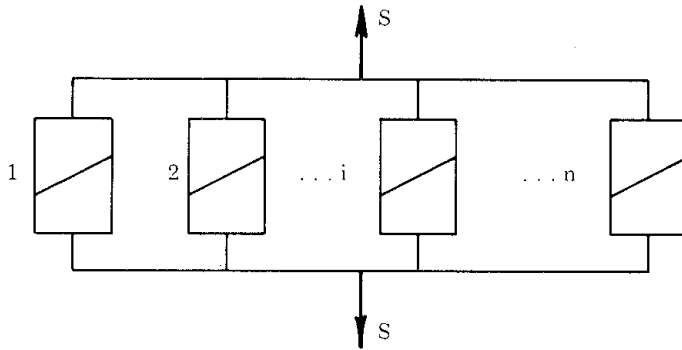


Figure 2. Parallel system with  $n$  elements.

addressed in more detail, but formation of a mechanism is the most frequently used definition. If this definition is used here, failure in a set of failure elements forming a mechanism is called a failure mode. Formation of a failure mode will therefore require simultaneous failure in a number of failure elements. This is symbolized by a parallel system; see figure 2.

In this section it is assumed that all basic variables (load variables and strength variables) are normally distributed. All geometrical quantities and elasticity coefficients are assumed deterministic. This assumption significantly facilitates the estimation of the failure probability, but, in general, basic variables cannot be modelled by normally distributed variables at a satisfactory degree of accuracy. To overcome this problem a number of different transformation methods have been suggested. The most well-known method was suggested by Rackwitz & Fiessler [3]. One drawback to these methods is that they increase the computational work considerably due to the fact that they are iterative methods. A simpler (but also less accurate) method called the multiplication factor has been proposed by Thoft-Christensen [4]. The multiplication factor method does not increase the computational work. Without loss of generality it is assumed that all basic variables are standardized, i.e. the mean value is 0 and the variance 1.

The  $\beta$ -unzipping method is only used for trussed and framed structures but it can easily be modified to other classes of structures. The structure is considered at a fixed point in time, so that only static behavior has been addressed. It is assumed that failure in a structural element (section) is either pure tension/compression or failure in bending. Combined failure criteria have also been used in connection with the  $\beta$ -unzipping method, but only little experience has been obtained till now.

Let the vector  $\bar{X} = (X_1, \dots, X_n)$  be the vector of the standardized normally distributed basic variables with the joint probability density function  $\varphi_n$  and let failure of a failure element be determined by a failure function  $f: \omega \rightarrow R$ , where  $\omega$  is the  $n$ -dimensional basic variable space. Let the failure function  $f$  be defined in such a way that the space  $\omega$  is divided into a failure region  $\omega_f = \{\bar{x}: f(\bar{x}) \leq 0\}$  and a safe region  $\omega_s = \{\bar{x}: f(\bar{x}) > 0\}$  by the failure surface (limit state)  $\partial\omega = \{\bar{x}: f(\bar{x}) = 0\}$  where the vector  $\bar{x}$  is a realization of the random vector  $\bar{X}$ . Then the probability of failure  $P_f$  for the failure element in question is given by

$$P_f = P(f(\bar{X}) \leq 0) = \int_{\omega_f} \varphi_n(\bar{x}) d\bar{x} \quad (1)$$

If the function  $f$  is linearized at the so-called design point at the distance  $\beta$  to the

origin of the coordinate system, then an approximate value for  $P_f$  is given by

$$P_f \approx P(\alpha_1 X_1 + \dots + \alpha_n X_n + \beta \leq 0) = P(\alpha_1 X_1 + \dots + \alpha_n X_n \leq -\beta) = \Phi(-\beta) \quad (2)$$

where  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$  is the vector of directional cosines of the linearized failure surface.  $\beta$  is the Hasofer-Lind reliability index.  $\Phi$  is the standardized normal distribution function. The random variable

$$M = \alpha_1 X_1 + \dots + \alpha_n X_n + \beta \quad (3)$$

is the *linearized safety margin* for the failure element.

Next consider a *series system* with  $k$  elements. An estimate of the failure probability  $P_f^s$  of this series system can be obtained on the basis of the linearized safety margin of the form (3) for the  $k$  elements

$$\begin{aligned} P_f^s &= P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{X} + \beta_i \leq 0)\right) = P\left(\bigcup_{i=1}^k (\bar{\alpha}_i \bar{X} \leq -\beta_i)\right) = 1 - P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{X} > -\beta_i)\right) \\ &= 1 - P\left(\bigcap_{i=1}^k (-\bar{\alpha}_i \bar{X} < \beta_i)\right) = 1 - \Phi_k(\bar{\beta}; \bar{\rho}) \end{aligned} \quad (4)$$

where  $\bar{\alpha}_i$  and  $\beta_i$  are the directional cosines and the reliability index for failure element  $i$ ,  $i = 1, \dots, k$  and where  $\bar{\beta} = (\beta_1, \dots, \beta_k)$ .  $\bar{\rho} = \{\rho_{ij}\}$  is the correlation coefficient matrix given by  $\rho_{ij} = \bar{\alpha}_i^T \bar{\alpha}_j$  for all  $i \neq j$ .  $\Phi_k$  is the standardized  $k$ -dimensional normal distribution function. Series systems will be treated in more detail in section 5.3.

For a *parallel system* with  $k$  elements an estimate of the failure probability  $P_f^p$  can be obtained in the following way

$$P_f^p = P\left(\bigcap_{i=1}^k (\bar{\alpha}_i \bar{X} + \beta_i \leq 0)\right) = P\left(\bigcap_{i=1}^k (-\bar{\alpha}_i \bar{X} < -\beta_i)\right) = \Phi_k(\bar{\beta}; \bar{\rho}) \quad (5)$$

where the same notations as above are used. Parallel systems will be treated in more detail in section 5.4.

It is important to note the approximation behind (4) and (5), namely the linearization of the general non-linear failure surfaces at the distinct design points for the failure elements. The main problem in connection with application of (4) and (5) is numerical calculation of the  $n$ -dimensional normal distribution function  $\Phi_n$  for  $n \geq 3$ . This problem will be addressed later in this chapter where a number of methods to get approximate values for  $\Phi_n$  are mentioned.

### 2.3 Modeling of systems at level $N$

Clearly, the definition of failure modes for a structural system is of great importance in estimating the reliability of the structural system. In this section failure modes are classified in a systematic way convenient for the subsequent reliability estimate. A very simple estimate of the reliability of a structural system is based on failure of a single failure element, namely the failure element with the lowest reliability index (highest failure probability) of all failure elements. Failure elements are structural elements or cross-sections where failure can take place. The number of failure elements will usually be considerably higher than the number of structural elements. Such a reliability analysis is in fact not a system reliability analysis, but from a classification point of view it is convenient to call it *system reliability analysis at level 0*. Let a structure

consist of  $n$  failure elements and let the reliability index (see e.g. Thoft-Christensen & Baker [1]) for failure element  $i$  be  $\beta_i$  then the system reliability index  $\beta_s^0$  at level 0 is

$$\beta_s^0 = \min_{i=1, \dots, n} \beta_i \quad (6)$$

Clearly, such an estimate of the system reliability is too optimistic. A more satisfactory estimate is obtained by taking into account the possibility of failure of any failure element by modeling the structural system as a series system with the failure elements as elements of the system (see

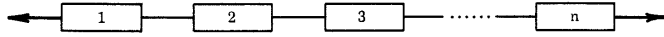


Figure 3. System modeling at level 1.

figure 3). The probability of failure for this series system is then estimated on the basis of the reliability indices  $\beta_i, i=1, 2, \dots, n$ , and the correlation between the safety margins for the failure elements. This reliability analysis is called *system reliability analysis at level 1*. In general it is only necessary to include some of the failure elements in the series system (namely those with the smallest  $\beta$ -indices) to get a good estimate of the system failure probability  $P_f^1$  and the corresponding generalized reliability index  $\beta_s^1$ , where

$$\beta_s^1 = -\Phi^{-1}(P_f^1) \quad (7)$$

and where  $\Phi$  is the standardized normal distribution function. The failure elements included in the reliability analysis are called *critical failure elements*.

The modeling of the system at level 1 is natural for a statically determinate structure, but failure in a single failure element in a structural system will not always result in failure of the total system, because the remaining elements may be able to sustain the external loads due to redistribution of the load effects. This situation is characteristic of statically indeterminate structures.

For such structures *system reliability analysis at level 2* or higher levels may be reasonable. At level 2 the systems reliability is estimated on the basis of a series system

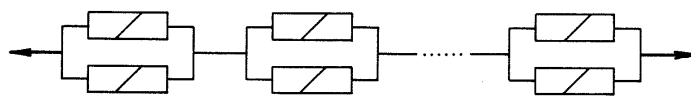


Figure 4. System modelling at level 2.

where the elements are parallel systems each with two failure elements - so-called *critical pairs of failure elements* (see figure 4). These critical pairs

of failure elements are obtained by modifying the structure by assuming in turn failure in the critical failure elements and adding fictitious loads corresponding to the load-carrying capacity of the elements in failure. If e.g. element  $i$  is a critical failure element, then the structure is modified by assuming failure in element  $i$  and the load-carrying capacity of the failure element is added as fictitious loads if the element is ductile. If the failure element is brittle, no fictitious loads are added. The modified structure is then analyzed elastically and new  $\beta$ -values are calculated for all the remaining failure elements. Failure elements with low  $\beta$ -values are then combined with failure element  $i$  so that a number of critical pairs of failure elements are defined.

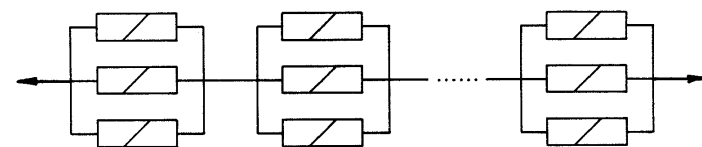


Figure 5. System modeling at level 3.

Analyzing modified structures where failure is assumed in critical pairs of failure elements now continues the procedure sketched above. In this way

*critical triples of failure elements* are identified and a *reliability analysis at level 3* can be made on the basis of a series system, where the elements are parallel systems each with three failure elements (see figure 5). By continuing in the same way, reliability estimates at levels 4, 5, etc. can be performed, but in general analysis beyond level 3 is of minor interest.

## 2.4 Modelling of systems at mechanism level

Many recent investigations in structural system theory concern structures, which can be modelled as elastic-plastic structures. In such cases failure of the structure is usually defined as formation of a mechanism. When this failure definition is used it is of great importance to be able to identify the most significant failure modes because the total number of mechanisms is usually much too high to be included in the reliability analysis. The  $\beta$ -unzipping method can be used for this purpose simply by continuing the procedure described above until a mechanism has been found. However, this will be very expensive due to the great number of reanalyses needed. It turns out to be much better to base the unzipping on reliability indices for fundamental mechanisms and linear combination of fundamental mechanisms.

When system failure is defined as formation of a mechanism the probability of failure of the structural system is estimated by modeling the structural system as a series system with the significant mechanisms as elements (see figure 6). Reliability analysis based on the mechanism failure definition is called *system reliability analysis at mechanism level*.

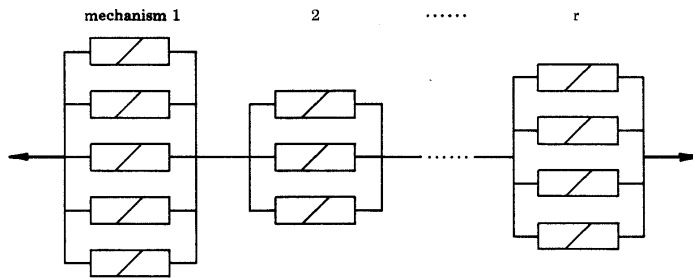


Figure 6. System modelling at mechanism level.

For real structures a mechanism will often involve a relatively large number of yield hinges and the deflections at the moment of formation of a mechanism can usually not be neglected. Therefore, the failure definition must be combined with some kind of deflection failure definition.

## 2.5 Formal representation of systems

This section gives a brief introduction to a new promising area within the reliability theory of structural systems called *mathematical theory of system reliability*. Only in the last decade this method has been applied to structural systems, but it has been applied successfully within other reliability areas. However, a lot of research in this area is being conducted and it can be expected that this mathematical theory will be useful also for structural systems in the future. The presentation here corresponds to some extent to the presentation given by Kaufmann, Grouchko & Cruon [5].

Consider a structural system  $S$  with  $n$  failure elements  $E_1, \dots, E_n$ . Each failure element  $E_i$ ,  $i=1, \dots, n$ , is assumed to be either in a “state of failure” or in a “state of non-failure”. Therefore, a so-called Boolean variable (indicator function)  $e_i$  defined by

$$e_i = \begin{cases} 1 & \text{if the failure element is in a non-failure state} \\ 0 & \text{if the failure element is in a failure state} \end{cases} \quad (8)$$

is associated with each failure element  $E_i$ ,  $i = 1, \dots, n$ .

The state of the system  $S$  is therefore determined by the *element state vector*

$$\bar{e} = (e_1, \dots, e_n) \quad (9)$$

The system  $S$  will also be assumed to be either in a “state of failure” or in a “state of non-failure”. Therefore, a Boolean variable  $s$ , defined by

$$s = \begin{cases} 1 & \text{if the system is in a non-failure state} \\ 0 & \text{if the system is in a failure state} \end{cases} \quad (10)$$

is associated with the system  $S$ . Since the state of the system  $S$  is determined solely by the vector  $\bar{e}$  there is a function called the *systems structure function*,  $\varphi: \bar{e} \rightarrow s$ , i.e.

$$s = \varphi(\bar{e}) \quad (11)$$

*Example 1.*

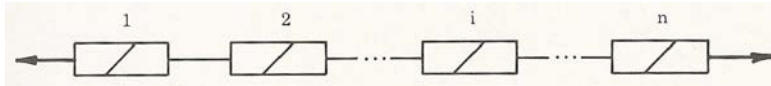


Figure 7. Series system with  $n$  elements.

Consider a *series system* with  $n$  elements as shown in figure 7. This series system is in a safe (non-failure) state if and only if all elements are in a non-failure state. Therefore, the structural function  $s_S$  for a series system is given by

$$s_S = \varphi_S(\bar{e}) = \prod_{i=1}^n e_i \quad (12)$$

where  $\bar{e}$  is given by (9). Note that  $s_S$  can also be written

$$s_S = \min(e_1, e_2, \dots, e_n) \quad (13)$$

*Example 2.*

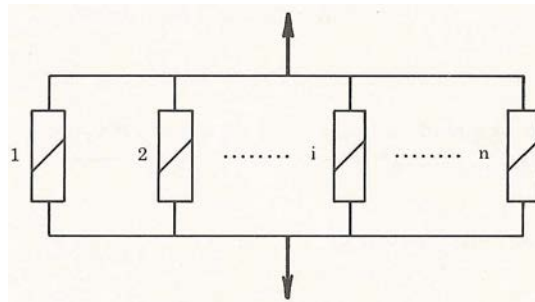


Figure 8. Parallel system with  $n$  elements.

Consider a *parallel system* with  $n$  elements, as shown in figure 8. This parallel system is in a safe state (non-failure state) only if at least one of its elements is in a non-failure state. Therefore, the structural function  $s_P$  for a parallel system is given by

$$s_P = \varphi_P(\bar{e}) = 1 - \prod_{i=1}^n (1 - e_i) \quad (14)$$

where  $\bar{e}$  is given by (9). Note that  $s_P$  can also be written

$$s_P = \max(e_1, e_2, \dots, e_n) \quad (15)$$



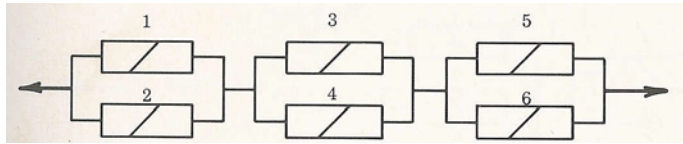


Figure 9. A level 2 systems modeling.

It is easy to combine the structural functions of the system  $s_S$  and  $s_P$  shown in (12) and (14) so that the structural function for a more complicated system can be

obtained. As an example, consider the system used in system modeling at level 2 and shown in figure 9. It is a series system with three elements and each of these elements is parallel systems with two failure elements. With the numbering shown in figure 9 the structural function becomes

$$s_S = s_{P_1} s_{P_2} s_{P_3} = [1 - (1 - e_1)(1 - e_2)][1 - (1 - e_3)(1 - e_4)][1 - (1 - e_5)(1 - e_6)] \quad (16)$$

where  $s_{P_i}$ ,  $i = 1, 2, 3$ , is the structural function for the parallel system  $i$  and where  $e^i$ ,  $i = 1, \dots, 6$ , is the Boolean variable for element  $i$ .

Consider a structure  $S$  with a set of  $n$  failure elements  $E = \{E_1, \dots, E_n\}$  and let the structural function of the system  $s$  be given by

$$s = \varphi(\bar{e}) = \varphi(e_1, \dots, e_n) \quad (17)$$

where  $\bar{e} = (e_1, \dots, e_n)$  is the element state vector. A subset,  $A = \{E_i | i \in I\}$ ,  $I \subset \{1, 2, \dots, n\}$ , of  $E$  is called a path set (or a link set) if

$$\left. \begin{array}{l} e_i = 1, i \in I \\ e_i = 0, i \notin I \end{array} \right\} \Rightarrow s = 1 \quad (18)$$

According to the definition a subset  $A \subset E$  is a path set if the structure is in a non-failure state and all elements in  $E \setminus A$  are in a failure state.

*Example 3.*

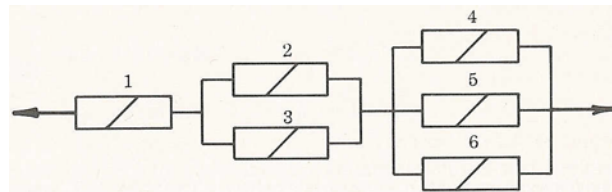


Figure 10.

Consider the system shown in figure 10. Clearly the following subsets of  $E = \{E_1, \dots, E_6\}$  are all path sets:  $A_1 = \{E_1, E_2, E_4\}$ ,  $A_2 = \{E_1, E_2, E_5\}$ ,  $A_3 = \{E_1, E_2, E_6\}$ ,  $A_4 = \{E_1, E_3, E_4\}$ ,  $A_5 = \{E_1, E_3, E_5\}$ , and  $A_6 = \{E_1, E_3, E_6\}$ . The path set  $A_1$  is illustrated in figure 11.

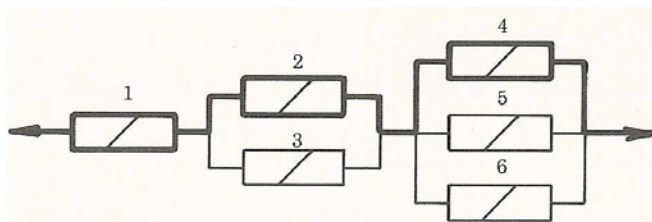


Figure 11. Path set  $A_1$ .

If a path set  $A \subset E$  has the property that a subset of  $A$ , which is also a path set, does not exist then  $A$  is called a *minimal path set*. In other words, a path set is a minimal path set if failure of any failure element in  $A$  results in system failure.

Another useful concept is the *cut set* concept. Consider again a structure  $S$  defined by (17) and let  $A \subset E$  be defined by

$$A = \{E_i | i \in I\}, I \subset \{1, 2, \dots, n\} \quad (19)$$

$A$  is then called a cut set, if

$$\left. \begin{array}{l} e_i = 0, i \in I \\ e_i = 1, i \notin I \end{array} \right\} \Rightarrow s = 0 \quad (20)$$

According to the definition (20) a subset  $A \subset E$  is a cut set if the structure is in a failure state when all failure elements in  $A$  are in a failure state and all elements in  $E \setminus A$  are in a non-failure state.

*Example 4.*

Consider again the structure shown in figure 10. Clearly, the following subsets of  $E = \{E_1, \dots, E_6\}$  are all cut sets:  $A_1 = \{E_1\}$ ,  $A_2 = \{E_2, E_3\}$ , and  $A_3 = \{E_4, E_5, E_6\}$ . The cut set  $A_2$  is illustrated in figure 12.

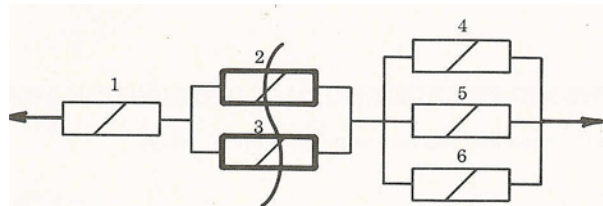


Figure 12. Cut set  $A_2$ .

If a cut set  $A \subset E$  has the property that a subset of  $A$ , which is also a cut set, then  $A$  is called a *minimal cut set*. In other words, a cut set  $A$  is a minimal cut set if non-failure of any failure element in  $A$  results in system non-failure.

It is interesting to note that one can easily prove that any  $n$ -tuple  $(e_1, \dots, e_n)$  with  $e_i = 0$  or  $1$ ,  $i = 1, \dots, n$ , corresponds to either a path set or a cut set.

For many structural systems it is convenient to describe the state of the system by the state of the failure elements on the basis of the system function as described in this section. The next step is then to estimate the reliability of the system when the reliabilities of the failure elements are known. The reliability  $R_i$  of failure element  $E_i$  is given by

$$R_i = P(e_i = 1) = 1 \times P(e_i = 1) + 0 \times P(e_i = 0) = E[e_i] \quad (21)$$

where  $e_i$ , the Boolean variable for failure element  $E_i$ , is considered a random variable and where  $E[e_i]$  is the expected value of  $e_i$ .

Similarly, the reliability  $R_S$  of the system  $S$  is

$$R_s = P(s=1) = 1 \times P(s=1) + 0 \times P(s=0) = E[s] = E[\varphi(\bar{e})] \quad (22)$$

where the element state vector  $\bar{e}$  is considered a random vector and where  $\varphi$  is the structural function of the system.

Unfortunately an estimate of  $E[\varphi(\bar{e})]$  is only simple when the failure elements are uncorrelated and when the system is simple, e.g. a series system. In civil engineering failure elements will often be correlated. Therefore, the presentation above is only useful for very simple structures.

### 3. RELIABILITY ASSESSMENT OF SERIES SYSTEMS

#### 3.1 Introduction

To illustrate the problems involved in estimating the reliability of systems, consider a structural element or structural system with two potential failure modes defined by safety margins  $M_1 = f_1(X_1, X_2)$  and  $M_2 = f_2(X_1, X_2)$ , where  $X_1$  and  $X_2$  are standardized normally distributed basic variables. The corresponding failure surface and reliability indices  $\beta_1$  and  $\beta_2$  are shown in figure 13.

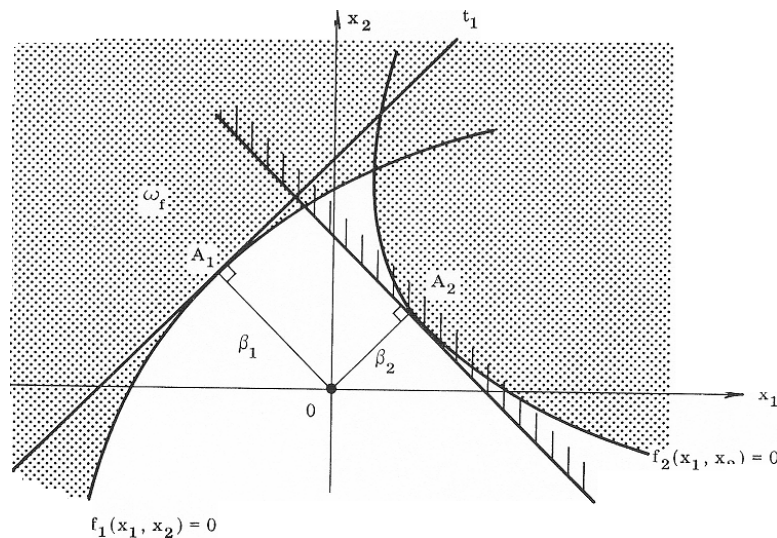


Figure 13. Illustration of series system with two failure elements.

Realizations  $(x_1, x_2)$  in the dotted area  $\omega_f$  will result in failure, and the probability of failure  $P_f$  is equal to

$$P_f = \int_{\omega_f} \varphi_{X_1, X_2}(x_1, x_2; 0) d\bar{x} \quad (23)$$

where  $\varphi_{X_1, X_2}$  is the bivariate normal density function for the random vector  $\bar{X} = (X_1, X_2)$ . Let  $\beta_2 < \beta_1$  as shown in figure 13, and assume that the reliability index  $\beta$  for the considered structural element or structural system is equal to the shortest distance from the origin 0 to the failure surface, i.e.  $\beta = \beta_2$ . Estimating the probability of failure by the formula

$$P_f \approx \Phi(-\beta) = \Phi(-\beta_2) \quad (24)$$

will then correspond to integrating over the hatched area (to the right of the tangent  $t_2$ ). Clearly the approximation (2.2) will in many cases be very different from the exact  $P_f$  calculated by (2.1). It is therefore of great interest to find a better approximation of  $P_f$  and then define a reliability index  $\beta$  by

$$\beta = -\Phi^{-1}(P_f) \quad (25)$$

Let the two failure modes be defined by the safety margins  $M_1 = f_1(X_1, X_2)$  and  $M_2 = f_2(X_1, X_2)$  and let  $F_i = \{M_i \leq 0\}$ ,  $i = 1, 2$ . Then the probability of failure  $P_f$  of the structural system is

$$P_f = P(F_1 \cup F_2) \quad (26)$$

corresponding to evaluating the probability of failure of a series system with two elements. An approximation of  $P_f$  can be obtained by assuming that the safety margins  $M_1$  and  $M_2$  are linearized at their respective design points  $A_1$  and  $A_2$

$$M_1 = a_1 X_1 + a_2 X_2 + \beta_1 \quad (27)$$

$$M_2 = b_1 X_1 + b_2 X_2 + \beta_2 \quad (28)$$

where  $\beta_1$  and  $\beta_2$  are the corresponding reliability indices when  $\bar{a} = (a_1, a_2)$  and  $\bar{b} = (b_1, b_2)$  are chosen as unit vectors. Then an approximation of  $P_f$  is

$$\begin{aligned} P_f &\approx P(\{\bar{a}^T \bar{X} + \beta_1 \leq 0\} \cup \{\bar{b}^T \bar{X} + \beta_2 \leq 0\}) = P(\{\bar{a}^T \bar{X} \leq -\beta_1\} \cup \{\bar{b}^T \bar{X} \leq -\beta_2\}) \\ &= 1 - P(\{\bar{a}^T \bar{X} > -\beta_1\} \cap \{\bar{b}^T \bar{X} > -\beta_2\}) = 1 - P(\{-\bar{a}^T \bar{X} < \beta_1\} \cap \{-\bar{b}^T \bar{X} < \beta_2\}) \quad (29) \\ &= 1 - \Phi_2(\beta_1, \beta_2; \rho) \end{aligned}$$

where  $\bar{X} = (X_1, X_2)$  is an independent standard normal vector, and  $\rho$  is the correlation coefficient given by

$$\rho = \bar{a}^T \bar{b} = a_1 b_1 + a_2 b_2 \quad (30)$$

$\Phi_2$  is the bivariate normal distribution function defined by

$$\Phi_2(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \varphi_2(t_1, t_2; \rho) dt_1 dt_2 \quad (31)$$

where the bivariate normal density function with zero mean  $\varphi_2$  is given by

$$\varphi_2(t_1, t_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(t_1^2 + t_2^2 - 2\rho t_1 t_2)\right) \quad (32)$$

A formal reliability index  $\beta$  for the system can then be defined by

$$\beta = -\Phi^{-1}(P_f) \approx -\Phi^{-1}(1 - \Phi_2(\beta_1, \beta_2; \rho)) \quad (33)$$

The reliability of a structural system may be estimated based on modeling by a series system where the elements are parallel systems. It is therefore of great importance to have accurate methods by which the reliability of series systems can be evaluated. In this section it will be shown how the approximation of  $P_f$  by (29) can easily be extended to a series system with  $n$  elements. Further, some bounding and approximate methods are presented.

### 3.2 Assessment of the probability of failure of series systems

Consider a series system with  $n$  elements as shown in figure 3 and let the safety margin for element  $i$  be given by

$$M_i = g_i(\bar{X}) \quad , \quad i = 1, 2, \dots, n \quad (34)$$

where  $\bar{X} = (X_1, \dots, X_k)$  are the basic variables and where  $g_i$ ,  $i = 1, 2, \dots, n$  are non-linear functions.

The probability of failure  $P_{f_i}$  of *element*  $i$  can then be estimated in the following way. Assume that there is a transformation  $\bar{Z} = \bar{T}(\bar{X})$  by which the basic variables  $\bar{X} = (X_1, \dots, X_k)$  are transformed into independent standard normal variables  $\bar{Z} = (Z_1, \dots, Z_k)$  so that

$$P_{f,i} = P(M_i \leq 0) = P(f_i(\bar{X}) \leq 0) = P(f_i(\bar{T}^{-1}(\bar{Z})) \leq 0) = P(h_i(\bar{Z}) \leq 0) \quad (35)$$

where  $h_i$  is defined by (35). An approximation of  $P_{f,i}$  can then be obtained by linearization of  $h_i$  at the design point

$$P_{f,i} = P(h_i(\bar{Z}) \leq 0) \approx P(\bar{\alpha}_i^T \bar{Z} + \beta_i \leq 0) \quad (36)$$

where  $\bar{\alpha}_i$  is the unit normal vector in the design point and  $\beta_i$  the Hasofer-Lind reliability index.

The approximation (36) can be written

$$P_{f,i} \approx P(\bar{\alpha}_i^T \bar{Z} + \beta_i \leq 0) = P(\bar{\alpha}_i^T \bar{Z} \leq -\beta_i) = \Phi(-\beta_i) \quad (37)$$

where  $\Phi$  is the standard normal distribution function.

Return to the *series system* shown in figure 3. An approximation of the probability of failure  $P_{f_s}$  of this system can then be obtained by using the same transformation  $\bar{T}$  as for the single elements and by linearization of

$$h_i(\bar{Z}) = g_i(\bar{T}^{-1}(\bar{Z})) \quad , \quad i = 1, 2, \dots, n \quad (38)$$

at the design points for each element. Then (see e.g. Hohenbichler & Rackwitz [6])

$$\begin{aligned} P_{f_s} &= P\left(\bigcup_{i=1}^n \{M_i \leq 0\}\right) = P\left(\bigcup_{i=1}^n \{f_i(\bar{X}) \leq 0\}\right) = P\left(\bigcup_{i=1}^n \{f_i(\bar{T}^{-1}(\bar{Z})) \leq 0\}\right) \\ &= P\left(\bigcup_{i=1}^n \{h_i(\bar{Z}) \leq 0\}\right) \approx P\left(\bigcup_{i=1}^n \{\bar{\alpha}_i^T \bar{Z} + \beta_i \leq 0\}\right) = P\left(\bigcup_{i=1}^n \{\bar{\alpha}_i^T \bar{Z} \leq -\beta_i\}\right) \\ &= 1 - P\left(\bigcap_{i=1}^n \{\bar{\alpha}_i^T \bar{Z} \leq -\beta_i\}\right) = 1 - P\left(\bigcap_{i=1}^n \{-\bar{\alpha}_i^T \bar{Z} < \beta_i\}\right) = 1 - \Phi_n(\bar{\beta}; \bar{\rho}) \end{aligned} \quad (39)$$

where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  and where  $\bar{\rho} = [\rho_{ij}]$  is the correlation matrix for the linearized safety margins, i.e.  $\rho_{ij} = \bar{\alpha}_i^T \bar{\alpha}_j$ .  $\Phi_n$  is the  $n$ -dimensional standardized normal distribution function.

By (39) the calculation of the probability of failure of a series system with linear and normally distributed safety margins is reduced to calculation of a value of  $\Phi_n$ .

### 3.3 Reliability bounds for series systems

It has been emphasized several times in this chapter that numerical calculation of the multi-normal distribution function  $\Phi_n$  is extremely time consuming or perhaps even impossible for values of  $n$  greater than, say four. Therefore, approximate techniques or bounding techniques must be used. In this section the so-called *simple bounds* and *Ditlevsen bounds* are derived.

#### 3.3.1 Simple bounds

First the simple bounds will be derived. For this purpose it is convenient to use the Boolean variables introduced in section 2.5. Consider a series system  $S$  with  $n$  failure elements  $E_1, \dots, E_i, E_{i+1}, \dots, E_n$ . For each failure element  $E_i$ ,  $i = 1, \dots, n$ , a Boolean variable  $e_i$  is defined by (see (8))

$$e_i = \begin{cases} 1 & \text{if the failure element is in a non-failure state} \\ 0 & \text{if the failure element is in a failure state} \end{cases} \quad (40)$$

Then the probability of failure  $P_{fs}$  of the series system is

$$\begin{aligned} P_{fs} &= 1 - P\left(\bigcap_{i=1}^n e_i = 1\right) = 1 - P(e_1 = 1) \frac{P(e_1 = 1 \cap e_2 = 1)}{P(e_1 = 1)} \dots \frac{P(e_1 = 1 \cap \dots \cap e_n = 1)}{P(e_1 = 1 \cap \dots \cap e_{n-1} = 1)} \\ &\leq 1 - \prod_{i=1}^n P(e_i = 1) = 1 - \prod_{i=1}^n (1 - P(e_i = 0)) \end{aligned} \quad (41)$$

if

$$P(e_1 = 1 \cap e_2 = 1) \geq P(e_1 = 1)P(e_2 = 1) \quad (42)$$

etc., or in general,

$$P\left(\bigcap_{j=1}^{i+1} e_j = 1\right) \geq P\left(\bigcap_{j=1}^i e_j = 1\right)P(e_{i+1} = 1) \quad (43)$$

for all  $1 \leq i \leq n-1$ . It can be shown that the condition (43) is satisfied when the safety margins for  $E_i$ ,  $i = 1, \dots, n$ , are normally distributed and positively correlated. When (43) is satisfied an upper bound of  $P_{fs}$  is given by (41). A simple lower bound is clearly the maximum probability of failure of any failure element  $E_i$ ,  $i = 1, \dots, n$ . Therefore, the following simple bounds exist when (43) is satisfied

$$\max_{i=1}^n P(e_i = 0) \leq P_{fs} \leq 1 - \prod_{i=1}^n (1 - P(e_i = 0)) \quad (44)$$

The lower bound in (44) is equal to the exact value of  $P_{fs}$  if there is full dependence between all elements ( $\rho_{ij} = 1$  for all  $i$  and  $j$ ) and the upper bound in (44) corresponds to no dependence between any pair of elements ( $\rho_{ij} = 0$ ,  $i \neq j$ ).

#### Example 5

Consider the structural system shown in figure 14 loaded by a single concentrated load  $p$ . Assume that system failure is failure in compression in element 1 or in element 2. Let the load-carrying capacity in the elements 1 and 2 be  $1.5 \times n_F$  and  $n_F$ , respectively, and assume that  $p$  and  $n_F$  are realizations of independent normally distributed random variables  $P$  and  $N_F$  with

$$\mu_p = 4 \text{ kN} \quad \sigma_p = 0.8 \text{ kN}$$

$$\mu_{N_F} = 4 \text{ kN} \quad \sigma_{N_F} = 0.4 \text{ kN}$$

Safety margins for elements 1 and 2 are

$$M_1 = \frac{3}{2}N_F - \frac{\sqrt{2}}{2}P, \quad M_2 = N_F - \frac{\sqrt{2}}{2}P$$

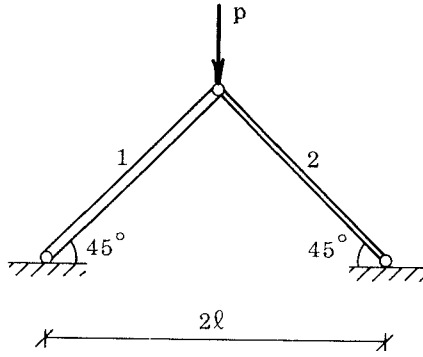


Figure 14.

or expressed by standardized random variables  $X_1 = (N_F - 4)/0.4$  and  $X_2 = (P - 4)/0.8$ , where the coefficients of  $X_1$  and  $X_2$  are chosen so that they are components of unit vectors. The reliability indices are  $\beta_1 = 3.85$  and  $\beta_2 = 1.69$  and the correlation coefficient between the safety margins is

$$\rho = 0.728 \times 0.577 + 0.686 \times 0.816 = 0.98.$$

Therefore the probability of failure of the system is

$$P_{fs} = 1 - \Phi_2(3.85, 1.69; 0.98)$$

The probabilities of failure of the failure elements  $E_i$ ,  $i = 1, 2$ , are

$$P_{f_1} = P(e_1 = 0) = \Phi(-3.85) = 0.00006, \quad P_{f_2} = P(e_2 = 0) = \Phi(-1.65) = 0.04947$$

Bounds for the probability of failure  $P_{fs}$  are then according to (44)

$$0.04947 \leq P_{fs} \leq 1 - (1 - 0.00006)(1 - 0.04947) = 0.04953$$

For this series system  $\rho = 0.98$ . Therefore, the lower bound can be expected to be close to  $P_{fs}$ .

### 3.3.2 Ditlevsen bounds

For small probabilities of element failure the upper bound in (44) is very close to the sum of the probabilities of failure of the single elements. Therefore, when the probability of failure of one failure element is predominant in relation to the other failure element then the probability of failure of series systems is approximately equal to the predominant probability of failure and the gap between the upper and lower bounds in (44) is narrow. However, when the probabilities of failure of the failure elements are of the same order then the simple bounds (44) are of very little use and there is a need for narrower bounds.

Consider again the above-mentioned series system  $S$  shown in figure 3.4 and define the Boolean variable  $s$  by (37). Then it follows from (12) that

$$\begin{aligned} s &= e_1 \times e_2 \times \cdots \times e_n = e_1 \times e_2 \times \cdots \times e_{n-1} - e_1 \times e_2 \times \cdots \times e_{n-1} (1 - e_n) \\ &= e_1 - e_1(1 - e_2) - e_1 \times e_2(1 - e_3) - \cdots - e_1 \times e_2 \times \cdots \times e_{n-1}(1 - e_n) \end{aligned} \quad (45)$$

Hence, in accordance with (22)

$$\begin{aligned}
P_{fs} &= 1 - R_S = 1 - E[s] \\
&= E[1 - e_1] + E[e_1(1 - e_2)] + E[e_1e_2(1 - e_3)] + \cdots + E[e_1e_2 \cdots e_{n-1}(1 - e_n)]
\end{aligned} \tag{46}$$

It is easy to see that

$$1 - ((1 - e_1)(1 - e_2) + \cdots + (1 - e_n)) \leq e_1e_2 \cdots e_n \leq e_i \quad \text{for } i = 1, 2, \dots, n \tag{47}$$

It is then seen from (46)

$$P_{fs} \leq \sum_{i=1}^n P(e_i = 0) - \sum_{i=2}^n \max_{j < i} P(e_i = 0 \cap e_j = 0) \tag{48}$$

and

$$P_{fs} \geq P(e_1 = 0) + \sum_{i=2}^n \max P(e_i = 0) - \sum_{j=1}^{i-1} P(e_i = 0 \cap e_j = 0, 0) \tag{49}$$

These bounds have been suggested in a slightly different form by Kounais [7]. In structural reliability they are called *Ditlevsen bounds* [8]. The numbering of the failure elements may influence the bounds (48) and (49). However, experience suggests that it is a good choice to arrange the failure elements so that  $P(e_1 = 0) \geq P(e_2 = 0) \geq \cdots \geq P(e_n = 0)$  i.e. according to decreasing probability of element failure.

The gap between the Ditlevsen bounds (48) and (49) is usually much smaller than the gap between the simple bounds (44). However, the bounds (48) and (49) require calculation of the joint probabilities  $P(e_i = 0 \cap e_j = 0)$  and these calculations are not trivial. Usually a numerical technique must be used.

### 3.4 Series systems with equally correlated elements

The  $n$ -dimensional standardized normal distribution function  $\Phi_n(\bar{x}; \bar{\rho})$  can be easily evaluated when  $\rho_{ij} = \rho > 0$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, n$ ,  $i \neq j$ , i.e. when

$$\bar{\rho} = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} \tag{50}$$

By the correlation matrix (50) it has been shown by Dunnett & Sobel [9] that

$$\Phi_n(\bar{x}; \bar{\rho}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^n \Phi\left(\frac{x_i - \sqrt{\rho}t}{\sqrt{1 - \rho}}\right) dt \tag{51}$$

Equation (50) can be generalized to the case where  $\rho_{ij} = \lambda_i \lambda_j$ ,  $i \neq j$ ,  $|\lambda_i| \leq 1$ ,  $|\lambda_j| \leq 1$ , i.e.

$$\bar{\rho} = \begin{bmatrix} 1 & \lambda_1 \lambda_2 & \cdots & \lambda_1 \lambda_n \\ \lambda_2 \lambda_1 & 1 & \cdots & \lambda_2 \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n \lambda_1 & \lambda_n \lambda_2 & \cdots & 1 \end{bmatrix} \tag{52}$$

For such correlation matrices Dunnett & Sobel [9] have shown that



$$\Phi_n(\bar{x}; \bar{\rho}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^n \Phi\left(\frac{x_i - \lambda_i t}{\sqrt{1 - \lambda_i^2}}\right) dt \quad (53)$$

For series systems with equally correlated failure elements the probability of failure  $P_{fs}$  can then be written (see (39))

$$P_{fs} = 1 - \Phi_n(\bar{\beta}; \bar{\rho}) = 1 - \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^n \Phi\left(\frac{x_i - \sqrt{\rho}t}{\sqrt{1 - \rho}}\right) dt \quad (54)$$

where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  are the reliability indices for the single failure elements and  $\rho$  is the common correlation coefficient between any pair of safety margins  $M_i$  and  $M_j$ ,  $i \neq j$ .

A further specialization is the case where all failure elements have the same reliability index  $\beta_e$ , i.e.  $\beta_i = \beta_e$  for  $i = 1, \dots, n$ . Then

$$P_{fs} = 1 - \int_{-\infty}^{\infty} \varphi(t) \left[ \Phi\left(\frac{\beta_e - \sqrt{\rho}t}{\sqrt{1 - \rho}}\right) \right]^n dt \quad (55)$$

*Example 6.*

Consider a series system with  $n = 10$  failure elements, common element reliability index  $\beta_e$  and common correlation coefficient  $\rho$ . The probability of failure  $P_{fs}$  of this series system as a function of  $\rho$  is illustrated in figure 15 for  $\rho = 2.50$  and  $3.00$ . Note that, as expected, the probability of failure  $P_{fs}$  decreases with  $\rho$ .

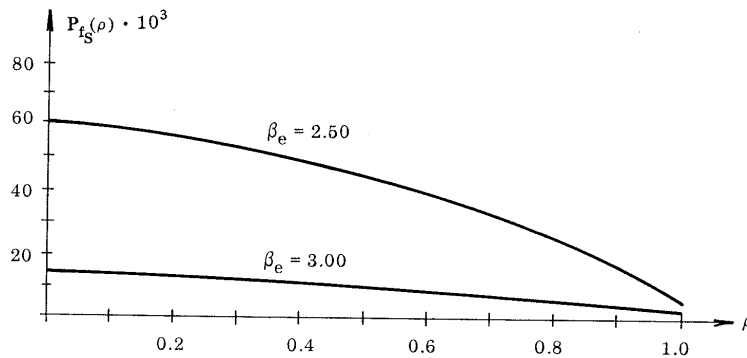


Figure 15.

To summarize for a series system with equally correlated failure elements where the safety margins are linear and normally distributed, the probability of failure  $P_{fs}$  can be calculated by (54). A formal reliability index  $\beta_s$  for the series system can then be calculated by

$$P_{fs} = \Phi(-\beta_s) \Leftrightarrow \beta_s = -\Phi^{-1}(P_{fs}) \quad (56)$$

### 3.5 Series Systems with Unequally Correlated Elements

It has been investigated by Thoft-Christensen & Sørensen [10] whether an equivalent correlation coefficient can be used in (55) with satisfactory accuracy when the failure elements are unequally correlated (or (54) if the reliability indices are not equal).

Define the *average correlation coefficient*  $\bar{\rho}$  by

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \rho_{ij} \quad (57)$$

$\bar{\rho}$  is the average of all  $\rho_{ij}, i \neq j$ . Using  $\bar{\rho}$  corresponds to the approximation

$$\Phi_n(\bar{\beta}; \bar{\rho}) \approx \Phi_n(\bar{\beta}; [\bar{\rho}]) \quad (58)$$

where the correlation matrix  $[\bar{\rho}]$  is given by

$$[\bar{\rho}] = \begin{bmatrix} 1 & \bar{\rho} & \cdots & \bar{\rho} \\ \bar{\rho} & 1 & \cdots & \bar{\rho} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\rho} & \bar{\rho} & \cdots & 1 \end{bmatrix} \quad (59)$$

Thoft-Christensen & Sørensen [3.7] have shown by extensive simulation that in many situations

$$\Phi_n(\bar{\beta}; [\bar{\rho}]) \leq \Phi_n(\bar{\beta}; \bar{\rho}) \quad (60)$$

so that an estimate of the probability of failure  $P_{fs}$  using the average correlation coefficient will in such cases be conservative. Ditlevsen [11] has investigated this more closely by a Taylor expansion for the special case  $\beta_i = \beta_e, i = 1, \dots, n$  with the conclusion that (60) holds for most cases, when  $\beta_e > 3, n < 100, \bar{\rho} < 0.4$ .

Thoft-Christensen & Sørensen [10] have shown that a better approximation can be obtained by

$$\Phi_n(\bar{\beta}; \bar{\rho}) \approx \Phi_n(\bar{\beta}; [\bar{\rho}]) + \Phi_2(\beta_e, \beta_e; \rho_{\max}) - \Phi_2(\beta_e, \beta_e; \bar{\rho}) \quad (61)$$

where  $\bar{\beta} = (\beta_e, \dots, \beta_e)$  and where

$$\rho_{\max} = \max_{\substack{i,j=1 \\ i \neq j}}^n \rho_{ij} \quad (62)$$

By this equation the probability of failure  $P_{fs}$  is approximated by

$$P_{fs} \approx P_{fs}([\rho_{i,j}]) = P_{fs}([\bar{\rho}]) + P_{fs}(n=2, \rho = \rho_{\max}) - P_{fs}(n=2, \rho = \bar{\rho}) \quad (63)$$

**Example 7.** Consider a series system with 5 failure elements and common reliability index  $\beta_e = 3.50$  and the correlation matrix

$$\bar{\rho} = \begin{bmatrix} 1 & 0.8 & 0.6 & 0 & 0 \\ 0.8 & 1 & 0.4 & 0 & 0 \\ 0.6 & 0.4 & 1 & 0.1 & 0.2 \\ 0 & 0 & 0.1 & 1 & 0.7 \\ 0 & 0 & 0.2 & 0.7 & 1 \end{bmatrix}$$

Using (58) with  $\bar{\rho} = 0.28$  then gives  $P_{fs} \approx 0.00115$ . It can be shown that the Ditlevsen bounds are  $0.00107 \leq P_{fs} \leq 0.00107$ . The approximation (63) gives  $P_{fs} \approx 0.00111$ .

## 4. RELIABILITY ASSESSMENT OF PARALLEL SYSTEMS

### 4.1 Introduction

As mentioned earlier, the reliability of a structural system may be modelled by a series system of parallel systems. Each parallel system corresponds to a failure mode and this modelling is called system modelling at level  $N$ ,  $N = 1, 2, \dots$  if all parallel systems have the same number  $N$  of failure elements. In section 5 it is shown how the most significant failure modes (parallel systems) can be identified by the  $\beta$ -unzipping method. After identification of significant (critical) failure modes (parallel systems) the next step is an estimate of the probability of failure  $P_{fp}$  for each parallel system and the correlation between the parallel systems. The final step is the estimate of the probability of failure  $P_f$  of the series system of parallel systems by the methods discussed in section 3.

Consider a parallel system with only two failure elements and let the safety margins be  $M_1 = f_1(X_1, X_2)$  and  $M_2 = f_2(X_1, X_2)$ , where  $X_1$  and  $X_2$  are independent standard normally distributed random variables. If  $F_i = \{M_i \leq 0\}$ ,  $i = 1, 2$ , then the probability of failure  $P_{fp}$  of the parallel system is

$$P_{fp} = P(F_1 \cap F_2) \quad (64)$$

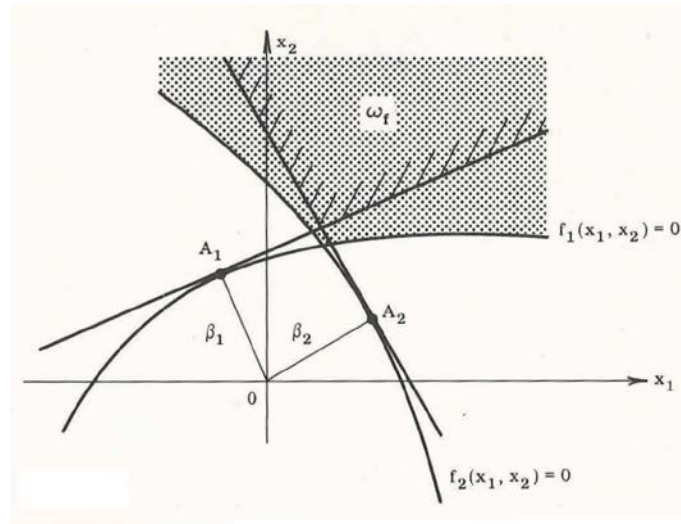


Figure 16.

An approximation of  $P_{fp}$  can be obtained by assuming that the safety margins  $M_1$  and  $M_2$  are linearized at their respective design points  $A_1$  and  $A_2$  (see figure 16)

$$M_1 = a_1 X_1 + a_2 X_2 + \beta_1 \quad (65)$$

$$M_2 = b_1 X_1 + b_2 X_2 + \beta_2 \quad (66)$$

where  $\beta_1$  and  $\beta_2$  are the corresponding reliability indices when  $\bar{a} = (a_1, a_2)$  and  $\bar{b} = (b_1, b_2)$  are chosen as unit vectors. Then an approximation of  $P_{fp}$  is

$$\begin{aligned} P_{fp} &\approx P(\{\bar{a}^T \bar{X} + \beta_1 \leq 0\} \cap \{\bar{b}^T \bar{X} + \beta_2 \leq 0\}) \\ &= P(\{\bar{a}^T \bar{X} \leq -\beta_1\} \cap \{\bar{b}^T \bar{X} \leq -\beta_2\}) = \Phi_2(-\beta_1, -\beta_2; \rho) \end{aligned} \quad (67)$$

where  $\bar{X} = (X_1, X_2)$  and where  $\rho$  is the correlation coefficient given by  $\rho = \bar{a}^T \bar{b} = a_1 b_1 + a_2 b_2$ .  $\Phi_2$  is the bivariate normal distribution function defined by (32). A formal reliability index  $\beta_p$  for the parallel system can then be defined by

$$\beta_p = -\Phi^{-1}(P_{fp}) \approx -\Phi^{-1}(\Phi_2(-\beta_1, -\beta_2; \rho)) \quad (68)$$

The formula (67) which gives an approximate value for the probability of failure of a parallel system with two failure elements will be generalized in section 4.2 to the general case where the parallel system has  $n$  failure elements and where the number of basic variables is  $k$ .

#### 4.2 Assessment of the probability of failure of parallel systems

Consider a parallel system with  $n$  elements as shown in figure 17 and let the safety margin for element  $i$  be given by

$$M_i = g_i(\bar{X}) \quad , \quad i = 1, \dots, n \quad (69)$$

where  $X = (X_1, \dots, X_k)$  are basic variables and where  $g_i$ ,  $i = 1, 2, \dots, n$  are non-linear functions.

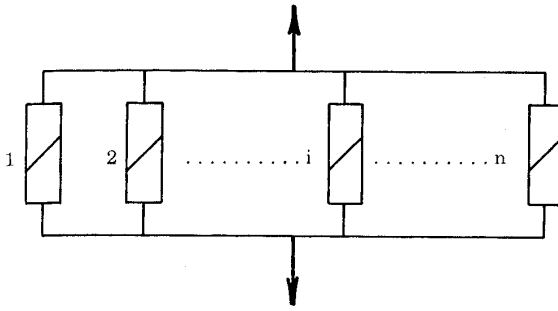


Figure 17. Parallel system with  $n$  elements.

The probability of failure  $P_{fi}$  of element  $i$  can then be estimated in the similar way as shown on page 20. An approximation of the probability of failure  $P_{fp}$  of the parallel system in figure 17 can then be obtained by using the same transformation as for the single elements and by linearization of

$$h_i(\bar{Z}) = g_i(\bar{T}^{-1}(\bar{Z})) \quad , \quad i = 1, \dots, n \quad (70)$$

at the design points for each element. Then (see e.g. Hohenbichler & Rackwitz [6])

$$\begin{aligned} P_{fp} &= P\left(\bigcap_{i=1}^n \{M_i \leq 0\}\right) = P\left(\bigcap_{i=1}^n \{g_i(\bar{X}) \leq 0\}\right) = P\left(\bigcap_{i=1}^n \{g_i(\bar{T}^{-1}(\bar{Z})) \leq 0\}\right) \\ &= P\left(\bigcap_{i=1}^n \{h_i(\bar{Z}) \leq 0\}\right) \approx P\left(\bigcap_{i=1}^n \{\bar{\alpha}_i^T \bar{Z} + \beta_i \leq 0\}\right) = P\left(\bigcap_{i=1}^n \{\bar{\alpha}_i^T \bar{Z} \leq -\beta_i\}\right) \\ &= \Phi_n(-\bar{\beta}; \bar{\rho}) \end{aligned} \quad (71)$$

where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$ , and where  $\bar{\rho} = [\rho_{ij}]$  is the correlation matrix for the linearized safety margins, i.e.  $\rho_{ij} = \bar{\alpha}_i^T \bar{\alpha}_j$ .  $\Phi_n$  is the  $n$ -dimensional standardized normal distribution function.

By (71) the calculation of the probability of failure of a parallel system with linear and normally distributed safety margins is reduced to calculation of a value of  $\Phi_n$ .

#### 4.3 Reliability bounds for parallel systems

In general, numerical calculation of the multi-normal distribution function  $\Phi_n$  is very

time consuming. Therefore, approximate techniques or bounding techniques must be used.

Simple bounds for the probability of failure  $P_{fp}$  analogous to the simple bounds for series systems (44) can easily be derived for parallel systems. Consider a parallel system  $P$  with  $n$  failure elements  $E_1, \dots, E_n$ . For each failure element  $E_i$ ,  $i = 1, \dots, n$ , a Boolean variable  $e_i$  is defined by (see (8))

$$e_i = \begin{cases} 1 & \text{if the failure element is in a non-failure state} \\ 0 & \text{if the failure element is in a failure state} \end{cases} \quad (72)$$

Then the probability of failure  $P_{fp}$  of the parallel system is

$$P_{fp} = P\left(\bigcap_{i=1}^n e_i = 0\right) = P(e_1 = 0) \frac{P(e_1 = 0 \cap e_2 = 0)}{P(e_1 = 0)} \dots \frac{P(e_1 = 0 \cap \dots \cap e_n = 0)}{P(e_1 = 0 \cap \dots \cap e_{n-1} = 0)} \geq \prod_{i=1}^n P(e_i = 0) \quad (73)$$

if

$$P(e_1 = 0 \cap e_2 = 0) \geq P(e_1 = 0)P(e_2 = 0) \quad (74)$$

etc., or in general

$$P\left(\bigcap_{j=1}^{i+1} e_j = 1\right) \geq P\left(\bigcap_{j=1}^i e_j = 1\right)P(e_{i+1} = 1) \quad (75)$$

for all  $1 \leq i \leq n-1$ . A simple upper bound is clearly the maximum probability of failure of the minimum probability of failure of any failure element  $E_i$ ,  $i = 1, \dots, n$ . Therefore, the following simple bounds exist when (75) is satisfied:

$$\prod_{i=1}^n P(e_i = 0) \leq P_{fp} \leq \min_{i=1}^n P(e_i = 0) \quad (76)$$

The lower bound in (76) is equal to the exact value of  $P_{fp}$  if there is no dependence between any pair of elements ( $\rho_{ij} = 0$ ,  $i \neq j$ ), and the upper bound in (76) corresponds to full dependence between all elements ( $\rho_{ij} = 1$  for all  $i$  and  $j$ ).

The simple bounds (76) will in most cases be so wide that they are of very little use. A better upper bound of  $P_{fp}$  has been suggested by Murotsu et al. [12]

$$P_{fp} \leq \min_{i,j=1}^n [P(e_i = 0) \cap P(e_j = 0)] \quad (77)$$

The derivation of (77) is very simple. For the case  $n = 3$ , (77) is illustrated in figure 18.

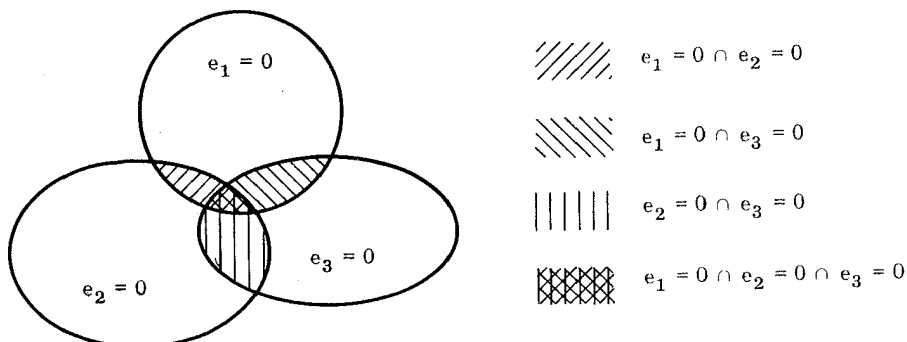


Figure 18.

*Example 8.*

Consider a parallel system with the reliability indices  $\bar{\beta} = (3.57, 3.41, 4.24, 5.48)$  and the correlation matrix:

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.62 & 0.91 & 0.62 \\ 0.62 & 1.00 & 0.58 & 0.58 \\ 0.91 & 0.58 & 1.00 & 0.55 \\ 0.62 & 0.58 & 0.55 & 1.00 \end{bmatrix}$$

Further, assume that the safety margins for the 4 elements are linear and normally distributed. The probability of failure of the parallel system then is  $P_{fp} = \Phi_4(-3.57, -3.41, -4.24, -5.48; \bar{\rho})$ .

The simple bounds (76) are

$$(1.79 \times 10^{-4}) \times (3.24 \times 10^{-4}) \times (0.11 \times 10^{-4}) \times (0.21 \times 10^{-7}) \leq P_{fp} \leq 0.21 \times 10^{-7}$$

or  $0 \leq P_{fp} \leq 0.21 \times 10^{-7}$ . The corresponding bounds of the formal reliability index  $\beta_p$  are  $5.48 \leq \beta_p \leq \infty$ .

With the ordering of the four elements shown above the probabilities of the intersections of  $e_1$  and  $e_2$  can be shown as

$$\left[ P(e_i = 0 \cap e_j = 0) \right] = \begin{bmatrix} - & 1.71 \times 10^{-5} & 9.60 \times 10^{-6} & 9.93 \times 10^{-9} \\ 1.71 \times 10^{-5} & - & 1.76 \times 10^{-6} & 9.27 \times 10^{-9} \\ 9.60 \times 10^{-6} & 1.76 \times 10^{-6} & - & 1.90 \times 10^{-9} \\ 9.93 \times 10^{-9} & 9.27 \times 10^{-9} & 1.90 \times 10^{-9} & - \end{bmatrix}$$

Therefore, from (77):  $\beta_p \geq -\Phi^{-1}(1.90 \times 10^{-9}) = 5.89$ .

#### 4.4 Equivalent linear safety margins for parallel systems

In section 4.2 it is shown how the probability of failure of a parallel system can be evaluated in a simple way when the safety margin for each failure element is *linear* and normally distributed. Consider a parallel system with  $n$  such failure elements. Then the probability of failure  $P_{fp}$  of the parallel system is (see (71)):

$$P_{fp} = \Phi_n(-\bar{\beta}; \bar{\rho}) \quad (78)$$

where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  is a vector whose components are the reliability indices of the failure elements, and where  $\bar{\rho}$  is the correlation matrix for the *linear* and normally distributed safety margins of the failure elements.

When the reliability of a structural system is modelled by a series system of parallel systems (failure modes) the reliability is evaluated by the following steps:

- evaluate the probability of failure of each parallel system by equation (78)
- evaluate the correlation between the parallel systems
- evaluate the probability of failure of the series system by equation (39)

Evaluation of the correlation between a pair of parallel systems can easily be performed if the safety margins for the parallel systems are linear. However, in general this will clearly not be the case. It is therefore natural to investigate the possibility of introducing an *equivalent linear safety margin* for each parallel system. In this section

an equivalent linear safety margin suggested by Gollwitzer & Rackwitz [13] will be described.

Consider a parallel system with  $n$  elements as shown in figure 17 and let the safety margin for element  $i$ ,  $i = 1, 2, \dots, n$  be linear

$$M_i = \alpha_{i1}Z_1 + \dots + \alpha_{ik}Z_k + \beta_i = \sum_{j=1}^k \alpha_{ij}Z_j + \beta_i \quad (79)$$

where the basic variables  $Z_i$ ,  $i = 1, \dots, k$  are independent standard normally distributed variables where  $\bar{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{ik})$  is a unit vector, and where  $\beta_i$  is the Hasofer-Lind reliability index. A formal (generalized) reliability index  $\beta_p$  for the parallel system is then given by

$$\beta_p = -\Phi^{-1}\left(\Phi_n(-\bar{\beta}; \bar{\rho})\right) \quad (80)$$

where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  and  $\bar{\rho} = [\rho_{ij}] = [\bar{\alpha}_i^T \bar{\alpha}_j]$ .

The equivalent linear safety margin  $M^e$  is then defined in such a way that the corresponding reliability index  $\beta^e$  is equal to  $\beta_p$  and so that it has the same sensitivity as the parallel system against changes in the basic variables  $Z_i$ ,  $i = 1, 2, \dots, k$ .

Let the vector  $\bar{Z}$  of basic variables be increased by a (small) vector  $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$ . Then the corresponding reliability index  $\beta_p(\bar{\varepsilon})$  for the parallel system is

$$\begin{aligned} \beta_p(\bar{\varepsilon}) &= -\Phi^{-1}\left(P\left(\bigcap_{i=1}^n \left\{\sum_{j=1}^k \alpha_{ij}(Z_j + \varepsilon_j) + \beta_i \leq 0\right\}\right)\right) \\ &= -\Phi^{-1}\left(\Phi_n(-\bar{\beta} - \bar{\alpha} \bar{\varepsilon}; \bar{\rho})\right) \end{aligned} \quad (81)$$

where  $\bar{\alpha} = [\alpha_{ij}]$ .

Let the equivalent linear safety margin  $M^e$  be given by

$$M^e = \alpha_1^e Z_1 + \dots + \alpha_k^e Z_k + \beta^e = \sum_{j=1}^k \alpha_j^e Z_j + \beta^e \quad (82)$$

where  $\bar{\alpha}^e = (\alpha_1^e, \dots, \alpha_k^e)$  is a unit vector and where  $\beta^e = \beta_p$ . By the same increase  $\bar{\varepsilon}$  of the basic variables the reliability index  $\beta^e(\bar{\varepsilon})$  is

$$\beta_p(\bar{\varepsilon}) = -\Phi^{-1}\left(\Phi(-\beta^e - \bar{\alpha}^e \bar{\varepsilon})\right) = \beta^e + \alpha_1^e \varepsilon_1 + \dots + \alpha_k^e \varepsilon_k \quad (83)$$

It is seen from (83) and by putting  $\beta_p(\bar{0}) = \beta^e(\bar{0})$  that

$$\alpha_i^e = \frac{\left. \frac{\partial \beta_p}{\partial \varepsilon_i} \right|_{\bar{\varepsilon}=\bar{0}}}{\sqrt{\sum_{j=1}^n \left( \left. \frac{\partial \beta_p}{\partial \varepsilon_j} \right|_{\bar{\varepsilon}=\bar{0}} \right)^2}}, \quad i = 1, \dots, k \quad (84)$$

An approximate value of  $\alpha_i^e$ ,  $i = 1, \dots, k$  can easily be obtained by numerical differentiation as shown in example 9.

*Example 9.*

Consider a parallel system with two failure elements and let the safety margin for the failure elements be

$$M_1 = 0.8 Z_1 - 0.6 Z_2 + 3.0 \text{ and } M_2 = 0.1 Z_1 - 0.995 Z_2 + 3.5$$

where  $Z_1$  and  $Z_2$  are independent standard normally distributed variables.

The correlation  $\rho$  between the safety margins is

$$\rho = 0.8 \times 0.1 + 0.6 \times 0.9950 = 0.68 .$$

Then the reliability index  $\beta_p$  of the parallel system is

$$\beta_p = -\Phi^{-1}(\Phi_2(-3.0, -3.5; 0.68)) = 3.83 .$$

To obtain the equivalent linear safety margin  $Z_1$  and  $Z_2$  are in turn given an increment  $\varepsilon_i = 0.1, i = 1, 2$ . With  $\bar{\varepsilon} = (0.1, 0)$  one gets

$$-\bar{\beta} - \bar{\alpha}\bar{\varepsilon} = \begin{bmatrix} -3.0 \\ -3.5 \end{bmatrix} - \begin{bmatrix} 0.8 & -0.6 \\ 0.1 & -0.995 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3.08 \\ -3.51 \end{bmatrix}$$

and by (81)  $\beta_p = -\Phi^{-1}(\Phi_2(-3.08, -3.51; 0.68)) = 3.87$ .

Therefore,

$$\left. \frac{\partial \beta_p}{\partial \varepsilon_1} \right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{3.87 - 3.83}{0.1} = 0.40 .$$

Likewise with  $\bar{\varepsilon} = (0, 0.1)$

$$P_{fp} = \int_{-\infty}^{\infty} \varphi(t) \left[ \Phi \left( \frac{-\beta_e - \sqrt{\rho}t}{\sqrt{1-\rho}} \right) \right]^n dt$$

and  $\beta_p(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.94, -3.40; 0.68)) = 3.74$ .

Therefore,

$$\left. \frac{\partial \beta_p}{\partial \varepsilon_2} \right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{3.74 - 3.83}{0.1} = -0.90 .$$

By normalizing  $\bar{\alpha}^e = (\alpha_1^e, \alpha_2^e) = (0.4061, -0.9138)$  and the equivalent safety margin is

$$M^e = 0.4061 Z_1 - 0.9138 Z_2 + 3.83$$

The safety margins  $M_1, M_2$ , and  $M^e$  are shown in figure 19.

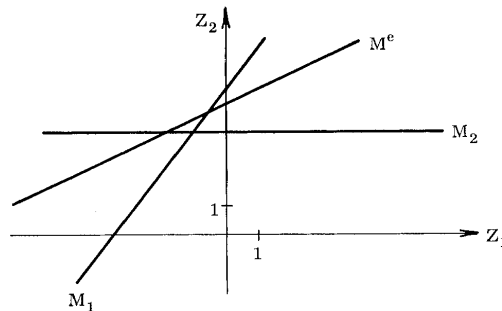


Figure 19



#### 4.5 Parallel systems with equally correlated elements

It is shown in section 4.2 that the probability of failure  $P_{f_p}$  of a parallel system with  $n$  failure elements is equal to  $P_{f_p} \approx \Phi_n(-\bar{\beta}; \bar{\rho})$  when the safety margins are linear and normally distributed.

Dunnett & Sobel [9] have shown for the special case where the failure elements are equally correlated with the correlation coefficient  $\rho$ , i.e.

$$\bar{\rho} = \begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix} \quad (85)$$

that (see (51))

$$\Phi_n(\bar{x}; \bar{\rho}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^n \Phi\left(\frac{x_i - \sqrt{\rho}t}{\sqrt{1-\rho}}\right) dt \quad (86)$$

If all failure elements have the same reliability index  $\beta_e$ , then

$$P_{f_p} = \int_{-\infty}^{\infty} \varphi(t) \left[ \Phi\left(\frac{-\beta_e - \sqrt{\rho}t}{\sqrt{1-\rho}}\right) \right]^n dt \quad (87)$$

A formal reliability index  $\beta_p$  for the parallel system can then be calculated by  $\beta_p = -\Phi^{-1}(P_{f_p})$ .

*Example 10.*

Consider a parallel system with  $n = 10$  failure elements and common element reliability index  $\beta_e$  and common correlation coefficient  $\rho$ . The formal reliability index  $\beta_p$  for this parallel system as a function of  $\rho$  is illustrated in figure 20 for  $\beta_e = 2.50$  and  $3.50$ . Note that, as expected, the reliability index  $\beta_p$  decreases with  $\rho$ .

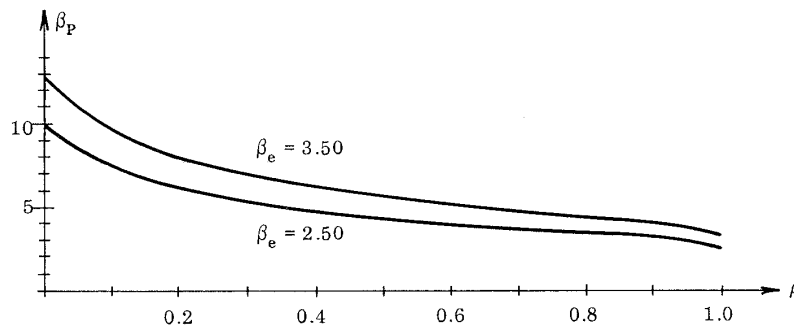


Figure 20.

The strength of a *fibre bundle* with  $n$  ductile fibres may be modelled by a parallel system as shown in figure 21. The strength  $R$  of the fibre bundle is

$$R = \sum_{i=1}^n R_i \quad (88)$$

where the random variable  $R_i$  is the strength of fibre  $i$ ,  $i = 1, 2, \dots, n$ . Let  $R_i$  be identically and normally distributed  $N(\mu, \sigma)$  with common correlation coefficient  $\rho$ . The strength  $R$  is then normally distributed  $N(\mu_R, \sigma_R)$  where  $\mu_R = n\mu$  and  $\sigma_R^2 = n\sigma^2 + n(n-1)\rho\sigma^2$ .

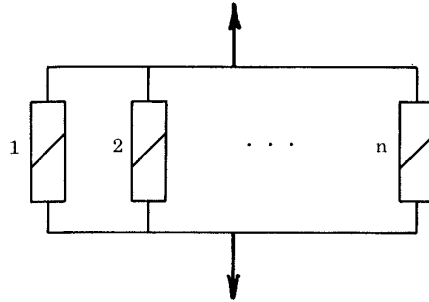


Figure 21. Modelling of a fibre bundle.

Assume that the fibre bundle is loaded by a deterministic and time independent load  $S = nS_e$ , where  $S = \text{constant}$  is the load of fibre  $i$ ,  $i = 1, 2, \dots, n$ . The reliability indices of the fibres are the same for all fibres and equal to

$$\beta_e = \frac{\mu - S_e}{\sigma} \quad (89)$$

Therefore,  $S = nS_e = n(\mu - \beta_e\sigma)$  and the reliability index  $\beta_F$  for the fibre bundle (the parallel system) is

$$\beta_F = \frac{\mu_R - S}{\sigma_R} = \frac{n\mu - n(\mu - \beta_e\sigma)}{(n\sigma^2 + n(n-1)\sigma^2\rho)^{1/2}} = \beta_e \sqrt{\frac{n}{1 + \rho(n-1)}} \quad (90)$$

Equation (90) has been derived by Grigoriu & Turkstra [14]. In section 4.6 it will be shown that (90) can easily be modified so that the assumption of the common correlation coefficient  $\rho$  can be removed.

#### 4.6 Parallel systems with unequally correlated elements

It is shown earlier that an approximation of the probability of failure of a parallel system with  $n$  failure elements with normally distributed safety margins is  $P_{f_p} \approx \Phi_n(-\bar{\beta}; \bar{\rho})$  where  $\bar{\beta} = (\beta_1, \dots, \beta_n)$  is the reliability indices of the failure elements and  $\bar{\rho} = [\rho_{ij}]$  is the correlation matrix. In section 4.5 it is shown that the reliability index  $\beta_F$  for a *fibre bundle* with  $n$  ductile fibres can easily be calculated if the following assumptions are fulfilled:

1. the load  $S$  of the fibre bundle is deterministic and constant with time,
2. the strength of the fibres is identically normally distributed  $N(\mu, \sigma)$ ,
3. the fibres have a common reliability index  $\beta_e$
4. common correlation coefficient  $\rho$  between the strengths of any pair of fibres.

Under these assumptions it was shown that (see (90))

$$\beta_F = \beta_e \sqrt{\frac{n}{1 + \rho(n-1)}} \quad (91)$$

Now the assumption 4 above will be relaxed. Let the correlation coefficient between fibre  $i$  and fibre  $j$  be denoted  $\rho_{ij}$ . The reliability index  $\beta_F$  for such a fibre bundle with unequally correlated fibres can then be calculated similarly as used in deriving (91) in section 4.5 (see Thoft-Christensen & Sørensen [15])

$$\beta_F = \frac{\mu_R - S}{\sigma_R} = (n\mu - (n\mu - n\beta_e\sigma))(n\sigma^2 + \sigma^2 \sum_{i,j=1, i \neq j}^n \rho_{ij})^{-1/2} = \beta_e \sqrt{\frac{n}{1 + \bar{\rho}(n-1)}} \quad (92)$$

where

$$\bar{\rho} = \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^n \rho_{ij} \quad (93)$$

By comparing (91) and (92) it is seen that for systems with unequal correlation coefficients, the reliability index  $\beta_{F_p}$  can be calculated by the simple expression (91) by inserting for  $\rho$  the average correlation coefficient  $\bar{\rho}$  defined by (93).  $\bar{\rho}$  is the average of all  $\rho_{ij}, i \neq j$ .

## 5. IDENTIFICATION OF CRITICAL FAILURE MECHANISMS

### 5.1 Introduction

A number of different methods to identify critical failure modes have been suggested (see e.g. Ferregut-Avila [16], Moses [17], Gorman [18], Ma & Ang [19], Klingmüller [20], Murotsu et al. [21] and Kappler [22]). In this section the  $\beta$ -unzipping method [4], [23] - [25] is used.

The  $\beta$ -unzipping method is a method by which the reliability of structures can be estimated at a number of different levels. The aim has been to develop a method which is at the same time simple to use and reasonably accurate. The  $\beta$ -unzipping method is quite general in the sense that it can be used for two-dimensional and three-dimensional framed and trussed structures, for structures with ductile or brittle elements and also in relation to a number of different failure mode definitions.

Estimate of the reliability of a structural system on the basis of failure of a single structural element (namely the element with the lowest reliability index of all elements) is called system reliability at *level 0*. At level 0 the reliability of a structural system is equal to the reliability of this single element. Therefore, such a reliability analysis is in fact not a system reliability analysis but an element reliability analysis. At level 0 each element is considered isolated from the other elements and the interaction between the elements is not taken into account in estimating the reliability. Let a structure consist of  $n$  failure elements (i.e. elements or points where failure can take place) and let the reliability index for failure element  $i$  be denoted  $\beta_i$ . Then at level 0 the system reliability index  $\beta_0$  is simply given as  $\beta_0 = \min \beta_i$ .

### 5.2 Assessment of system reliability at level 1

At *level 1* the system reliability is defined as the reliability of a series system with  $n$  elements - the  $n$  failure elements, see figure 1. Therefore, the first step is to calculate  $\beta$ -values for all failure elements and then use equation (39). As mentioned earlier, equation (39) can in general not be used directly. However, upper and lower bounds exist for series systems as shown in section 3.3.

Usually for a structure with  $n$  failure elements, the estimate of the probability of failure of the series system with  $n$  elements can be calculated with sufficient accuracy by only including some of the failure elements, namely those with the smallest reliability indices. One way of selecting is to include only failure elements with  $\beta$  - values in an interval  $[\beta_{\min}, \beta_{\min} + \Delta\beta_1]$ , where  $\beta_{\min}$  is the smallest reliability index of all failure element indices and where  $\Delta\beta_1$  is a prescribed positive number. The failure elements chosen to be included in the system reliability analysis at level 1 are called *critical failure elements*. If two or more critical failure elements are perfectly correlated, then only one of them is included in the series system of critical failure elements.

### 5.3 Assessment of system reliability at level 2

At level 2 the system reliability is estimated as the reliability of a series system where the elements are parallel systems each with 2 failure elements (see figure 4) - so-called *critical pairs of failure elements*. Let the structure be modeled by  $n$  failure elements and let the number of critical failure elements at level 1 be  $n_1$ . Let the critical failure element  $l$  have the lowest reliability index  $\beta$  of all critical failure elements. Failure is then assumed in failure element  $l$  and the structure is modified by removing the corresponding failure element and adding a pair of so-called fictitious loads  $F_l$  (axial forces or moments). If the removed failure element is brittle, no fictitious loads are added. However, if the removed failure element  $l$  is ductile the fictitious load  $F_l$  is a stochastic load given by  $F_l = \gamma_l R_l$ , where  $R_l$  is the load-carrying capacity of failure element  $l$  and where  $0 < \gamma_l \leq 1$ .

The modified structure with the loads  $P_1, \dots, P_k$  and the fictitious load  $F_l$  (axial force or moment) is then reanalyzed and influence coefficients  $a_{ij}$  with respect to  $P_1, \dots, P_k$  and  $a'_{il}$  with respect to  $F_l$  are calculated. The load effect (force or moment) in the remaining failure elements is then described by a stochastic variable. The load effect in failure element  $i$  is called  $S_{i|l}$  (load effect in failure element  $i$  given failure in failure element  $l$ ) and

$$S_{i|l} = \sum_{j=1}^k a_{ij} P_j + a'_{il} F_l \quad (94)$$

The corresponding safety margin  $M_{i|l}$  then is

$$M_{i|l} = \min(R_i^+ - S_{i|l}, R_i^- + S_{i|l}) \quad (95)$$

where  $R_i^+$  and  $R_i^-$  are the stochastic variables describing the (yield) strength capacity in “tension” and “compression” for failure element  $i$ . In the following  $M_{i|l}$  will be approximated by either  $R_i^+ - S_{i|l}$  or  $R_i^- + S_{i|l}$  depending on the corresponding reliability indices. The reliability index for failure element  $i$ , given failure in failure element  $l$ , is

$$\beta_{i|l} = \mu_{M_{i|l}} / \sigma_{M_{i|l}} \quad (96)$$

In this way new reliability indices are calculated for all failure elements (except the one where failure is assumed) and the smallest  $\beta$  -value is called  $\beta_{\min}$ . The failure elements with  $\beta$  -values in the interval  $[\beta_{\min}, \beta_{\min} + \Delta\beta_2]$ , where  $\Delta\beta_2$  is a prescribed

positive number, are then in turn combined with failure element  $l$  to form a number of parallel systems.

The next step is then to evaluate the probability of failure for each critical pair of failure elements. Consider a parallel system with failure elements  $l$  and  $r$ . During the reliability analysis at level 1 the safety margin  $M_l$  for failure element  $l$  is determined and the safety margin  $M_{r|l}$  for failure element  $r$  has the form (11). From these safety margins the reliability indices  $\beta_1 = \beta_l$  and  $\beta_2 = \beta_{r|l}$  and the correlation coefficient  $\rho = \rho_{l,r|l}$  can easily be calculated. The probability of failure for the parallel system then is

$$P_f = \Phi_2(-\beta_1, -\beta_2; \rho) \quad (97)$$

The same procedure is then in turn used for all critical failure elements and further critical pairs of failure elements are identified. In this way the total series system used in the reliability analysis at level 2 is determined (see figure 4). The next step is then to estimate the probability of failure for each critical pair of failure elements (see (97)) and also to determine a safety margin for each critical pair of failure elements. When this is done generalized reliability indices for all parallel systems in figure 4 and correlation coefficients between any pair of parallel systems are calculated. Finally, the probability of failure  $P_f$  for the series system (figure 2) is estimated. The so-called equivalent linear safety margin introduced in section 4.4 is used as approximations for safety margins for the parallel systems.

An important property by the  $\beta$ -unzipping method is the possibility of using the method when brittle failure elements occur in the structure. When failure occurs in a brittle failure element then the  $\beta$ -unzipping method is used in exactly the same way as presented above, the only difference being that no fictitious loads are introduced. If e.g. brittle failure occurs in a tensile bar in a trussed structure then the bar is simply removed without adding fictitious tensile loads. Likewise, if brittle failure occurs in bending, then a yield hinge is introduced, but no (yielding) fictitious bending moments are added.

#### 5.4 Assessment of system reliability at level $N > 2$

The method presented above can easily be generalized to higher levels  $N > 2$ . At level 3 the estimate of the system reliability is based on so-called *critical triples of failure elements*, i.e. a set of three failure elements. The critical triples of failure elements are identified by the  $\beta$ -unzipping method and each triple forms a parallel system with three failure elements. These parallel systems are then elements in a series system (see figure 5). Finally, the estimate of the reliability of the structural system at level 3 is defined as the reliability of this series system.

Assume that the critical pair of failure elements ( $l, m$ ) has the lowest reliability index  $\beta_{l,m}$  of all critical pairs of failure elements. Failure is then assumed in the failure elements  $l$  and  $m$  adding for each of them a pair of fictitious loads  $F_l$  and  $F_m$  (axial forces or moments).

The modified structure with the loads  $P_1, \dots, P_k$  and the fictitious loads  $F_l$  and  $F_m$  are then reanalyzed and influence coefficients with respect to  $P_1, \dots, P_k$  and  $F_l$  and  $F_m$  are calculated. The load effect in each of the remaining failure elements is then described by a stochastic variable  $S_{i|l,m}$  (load effect in failure element  $i$  given failure in failure elements  $l$  and  $m$ ) and

$$S_{i|l,m} = \sum_{j=1}^k a_{ij} P_j + a'_{il} F_l + a'_{im} F_m \quad (98)$$

The corresponding safety margin  $M_{i|l,m}$  then is

$$M_{i|l,m} = \min(R_i^+ - S_{i|l,m}, R_i^- + S_{i|l,m}) \quad (99)$$

where  $R_i^+$  and  $R_i^-$  are the stochastic variables describing the load-carrying capacity in “tension” and “compression” for failure element  $i$ . In the following  $M_{i|l,m}$  will be approximated by either  $R_i^- + S_{i|l,m}$  or  $R_i^+ - S_{i|l,m}$  depending on the corresponding reliability indices. The reliability index for failure element  $i$ , given failure in failure elements  $l$  and  $m$ , is then given by

$$\beta_{i|l,m} = \mu_{M_{i|l,m}} / \sigma_{M_{i|l,m}} \quad (100)$$

In this way new reliability indices are calculated for all failure elements (except  $Q$  and  $m$ ) and the smallest  $\beta$ -value is called  $\beta_{\min}$ . These failure elements with  $\beta$ -values in the interval  $[\beta_{\min}, \beta_{\min} + \Delta\beta_3]$ , where  $\Delta\beta_3$  is a prescribed positive number, are then in turn combined with failure elements  $l$  and  $m$  to form a number of parallel systems.

The next step is then to evaluate the probability of failure for each of the critical triples of failure elements. Consider the parallel system with failure elements  $l$ ,  $m$ , and  $r$ . During the reliability analysis at level 1 the safety margin  $M_l$  for failure element  $l$  is determined and during the reliability analysis at level 2 the safety margin  $M_{m|l}$  for the failure element  $m$  is determined. The safety margin  $M_{r|l,m}$  for safety element  $r$  has the form (15). From these safety margins the reliability indices  $\beta_1 = \beta_l$ ,  $\beta_2 = \beta_{m|l}$  and  $\beta_3 = \beta_{r|l,m}$  and the correlation matrix  $\bar{\rho}$  can easily be calculated. The probability of failure for the parallel system then is

$$P_f = \Phi_2(-\beta_1, -\beta_2, -\beta_3; \bar{\rho}) \quad (101)$$

An equivalent safety margin  $M_{i,j,k}$  can be determined by the procedure mentioned above. When the equivalent safety margins are determined for all critical triples of failure elements the correlation between them two and two can easily be calculated. The final step is then to arrange all the critical triples as elements in a series system (see figure 5) and estimate the probability of failure  $P_f$  and the generalized reliability index  $\beta_s$  for the series system.

The  $\beta$ -unzipping method can be used in exactly the same way as described in the preceding text to estimate the system reliability at levels  $N > 3$ . However, a definition of failure modes based on a fixed number of failure elements greater than 3 will hardly be of practical interest.

### 5.5 Assessment of system reliability at mechanism level

The application of the  $\beta$ -unzipping method presented above can also be used when failure is defined as formation of a mechanism. However, it is much more efficient to use the  $\beta$ -unzipping method in connection with *fundamental mechanisms*. Experience has shown that such a procedure is less computer time consuming than unzipping based on failure elements.

If unzipping is based on failure elements, then formation of a mechanism can be unveiled by the fact that the corresponding stiffness matrix is singular. Therefore, the unzipping is simply continued until the determinant of the stiffness matrix is zero. By this procedure a number of mechanisms with different numbers of failure elements will be identified. The number of failure elements in a mechanism will often be quite high so that several re-analyses of the structure are necessary.

As emphasized above it is more efficient to use the  $\beta$ -unzipping method in connection with fundamental mechanisms. Consider an elasto-plastic structure and let the number of potential failure elements (e.g. yield hinges) be  $n$ . It is then known from the theory of plasticity that the number of fundamental mechanisms is  $m = n - r$ , where  $r$  is the degree of redundancy. All other mechanisms can then be formed by linear combinations of the fundamental mechanisms. Some of the fundamental mechanisms are so-called joint mechanisms. They are important in the formation of new mechanisms by linear combinations of fundamental mechanisms, but they are not real failure mechanisms. Real failure mechanisms are by definition mechanisms, which are not joint mechanisms.

Let the number of loads be  $k$ . The safety margin for fundamental mechanism  $i$  can then be written

$$M_i = \sum_{j=1}^n |a_{ij}| R_j - \sum_{j=1}^k b_{ij} P_j \quad (102)$$

where  $a_{ij}$  and  $b_{ij}$  are the influence coefficients.  $R_j$  is the yield strength of failure element  $j$  and  $P_j$  is load number  $j$ .  $a_{ij}$  is the rotation of yield hinge  $j$  corresponding to the yield mechanism  $i$  and  $b_{ij}$  is the corresponding displacement of load  $j$ . The numerical value of  $a_{ij}$  is used in the first summation at the right-hand side of (102) to make sure that all terms in this summation are non-negative.

The total number of mechanisms for a structure is usually too high to include all possible mechanisms in the estimate of the system reliability. It is also unnecessary to include all mechanisms because the majority of them will in general have a relatively small probability of occurrence. Only the most critical or most significant failure modes should be included. The problem is then how the most significant mechanisms (failure modes) can be identified. In this section it is shown how the  $\beta$ -unzipping method can be used for this purpose. It is not possible to prove that the  $\beta$ -unzipping method identifies all significant mechanisms, but experience with structures where all mechanisms can be taken into account seems to confirm that the  $\beta$ -unzipping method gives reasonably good results. Note that since some mechanisms are excluded the estimate of the probability of failure by the  $\beta$ -unzipping method is a lower bound for the correct probability of failure. The corresponding generalized reliability index determined by the  $\beta$ -unzipping method is therefore an upper bound of the correct generalized reliability index. However, the difference between these two indices is usually negligible.

The first step is to identify all fundamental mechanisms and calculate the corresponding reliability indices. Fundamental mechanisms can be automatically generated by a method suggested by Watwood [26], but when the structure is not too complicated the fundamental mechanisms can be identified manually.

The next step is then to select a number of fundamental mechanisms as starting points for the unzipping. By the  $\beta$ -unzipping method this is done on the basis of the reliability index  $\beta_{\min}$  for the real fundamental mechanism that has the smallest

reliability index and on the basis of a preselected constant  $\varepsilon_1$  (e.g.  $\varepsilon_1 = 0.50$ ). Only real fundamental mechanisms with  $\beta$ -indices in the interval  $[\beta_{\min}; \beta_{\min} + \varepsilon_1]$  are used as starting mechanisms in the  $\beta$ -unzipping method. Let  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_f$  be an ordered set of reliability indices for  $f$  real fundamental mechanisms 1, 2, ...,  $f$ , selected by this simple procedure.

The  $f$  fundamental mechanisms selected as described above are now in turn combined linearly with all  $m$  (real and joint) mechanisms to form new mechanisms. First the fundamental mechanism 1 is combined with the fundamental mechanisms 2, 3, ...,  $m$  and reliability indices ( $\beta_{1,2}, \dots, \beta_{1,m}$ ) for the new mechanisms are calculated. The smallest reliability index is determined, and the new mechanisms with reliability indices within a distance  $\varepsilon_2$  from the smallest reliability index are selected for further investigation. The same procedure is then used on the basis of the fundamental mechanisms 2, ...,  $f$  and a failure tree as the one shown in figure 22 is constructed.

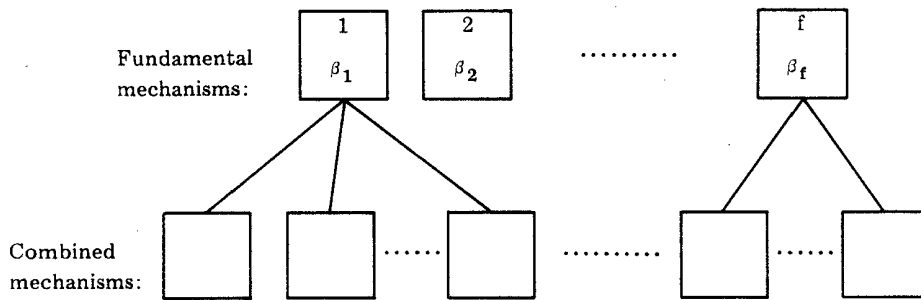


Figure 22. Construction of new mechanisms.

Let the safety margins  $M_i$  and  $M_j$  of two fundamental mechanisms  $i$  and  $j$  combined as described above (see (102)) be

$$M_i = \sum_{r=1}^n |a_{ir}| R_r - \sum_{s=1}^k b_{is} P_s \tag{103}$$

$$M_j = \sum_{r=1}^n |a_{jr}| R_r - \sum_{s=1}^k b_{js} P_s \tag{104}$$

The combined mechanism  $i \pm j$  then has the safety margin

$$M_{i \pm j} = \sum_{r=1}^n |a_{ir} \pm a_{jr}| R_r - \sum_{s=1}^k (b_{is} \pm b_{js}) P_s \tag{105}$$

where + or - is chosen dependent on which sign will result in the smallest reliability index. From the linear safety margin (105) the reliability index  $\beta_{i \pm j}$  for the combined mechanism can easily be calculated.

More mechanisms can be identified on the basis of the combined mechanisms in the second row of the failure tree in figure 22 by adding or subtracting fundamental mechanisms. Note that in some cases it is necessary to improve the technique by modifying (105), namely when a new mechanism requires not only a combination with  $1 \times$  but a combination with  $k \times$  a new fundamental mechanism. The modified version is

$$M_{i \pm k} = \sum_{r=1}^n |a_{ir} \pm k a_{jr}| R_r - \sum_{s=1}^k (b_{is} \pm k b_{js}) P_s \tag{106}$$



where  $k$  is chosen equal to e.g.  $-1, +1, -2, +2, -3$  or  $+3$  dependent on which value of  $k$  will result in the smallest reliability index. By (106) it is easy to calculate the reliability index  $\beta_{i \pm k}$  for the combined mechanism  $i + kj$ .

By repeating this simple procedure the failure tree for the structure in question can be constructed. The maximum number of rows in the failure tree must be chosen and can typically be  $m + 2$ , where  $m$  is the number of fundamental mechanisms. A satisfactory estimate of the system reliability index can usually be obtained by using the same  $\varepsilon_2$ -value for all rows in the failure tree.

During the identification of new mechanisms it will often occur that a mechanism already identified will turn up again. If this is the case, then the corresponding branch of the failure tree is terminated just one step earlier, so that the same mechanism does not occur more than once in the failure tree.

The final step is the application of the  $\beta$ -unzipping method in evaluating the reliability of an elasto-plastic structure at mechanism level is to select the significant mechanisms from the mechanisms identified in the failure tree. This selection can, in accordance with the selection criteria used in making the failure tree, e.g. be made by first identifying the smallest  $\beta$ -value  $\beta_{\min}$  of all mechanisms in the failure tree and then selecting a constant  $\varepsilon_3$ . The significant mechanisms are then by definition those with  $\beta$ -values in the interval  $[\beta_{\min}; \beta_{\min} + \varepsilon_3]$ . The probability of failure of the structure is then estimated by modeling the structural system as a series system with the significant mechanisms as elements (see figure 6).

## 5.6 Examples

### 5.6.1 Two-storey braced frame with ductile elements

Consider the two-storey braced frame in figure 23. The geometry and the loading are shown in the figure. This example is taken from [27] where all detailed calculations are shown. The area  $A$  and the moment of inertia  $I$  for each structural member are shown in the figure. In the same figure the expected values of the yield moment  $M$  and the tensile strength capacity  $R$  for all structural members are also stated. The compression strength capacity of one structural member is assumed to be one half of the tensile strength

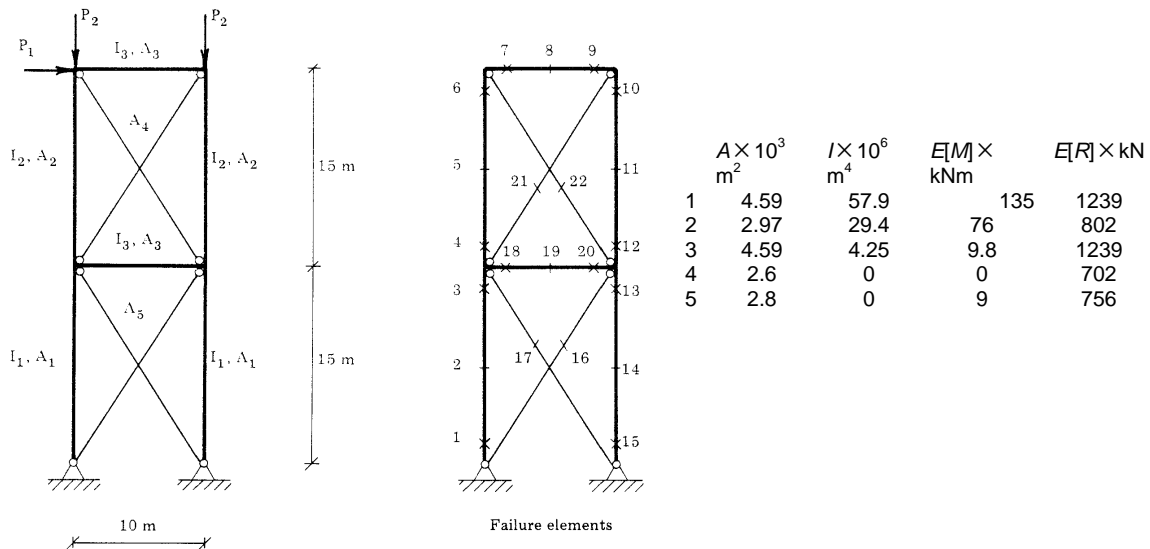


Figure 23. Two-storey brace frame.

capacity. The expected values of the loading are  $E[P_1] = 100$  kN and  $E[P_2] = 350$  kN. For the sake of simplicity the coefficient of variation for any load or strength is assumed to be  $V[\cdot] = 0.1$ . All elements are assumed to be perfectly ductile and made of a material having the same modulus of elasticity  $E = 0.21 \times 10^9$  kN/m<sup>2</sup>.

The failure elements are shown in figure 23.  $\beta_s^1$  indicates a potential yield hinge and  $\uparrow$  indicates failure in tension/compression. The total number of failure elements is 22, namely  $2 \times 6 = 12$  yield hinges in 6 beams and 10 tension/compression possibilities of failure in the 10 structural elements. The following pairs of failure elements (1, 3), (4, 6), (7, 9), (10, 12), (13, 15) and (18, 20) are assumed fully correlated. All other pairs of failure elements are uncorrelated. Further, the loads  $P_1$  and  $P_2$  are uncorrelated.

The  $\beta$ -values for all failure elements are shown in table 1. Failure element 14 has the lowest reliability index  $\beta_{14} = 1.80$  of all failure elements. Therefore, at level 0 the system reliability index is  $\beta_s^0 = 1.80$ .

Failure element	2	3	4	5	6	7	8	9	10	11
$\beta$ -value	8.13	9.96	9.92	4.97	9.98	9.84	9.79	9.86	9.98	1.81
Failure element	12	13	14	16	17	18	19	20	21	22
$\beta$ -value	9.94	9.97	1.80	9.16	4.67	9.91	8.91	9.91	8.04	3.34

Table 1. Reliability indices at level 0 (and level 1).

Let  $\Delta\beta_1 = 0$ . It then follows from table 1 that the critical failure elements are 14, 11, 22, and 17. The corresponding correlation matrix (between the safety margins in the same order) is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.24 & 0.20 & 0.17 \\ 0.24 & 1.00 & 0.21 & 0.16 \\ 0.20 & 0.21 & 1.00 & 0.14 \\ 0.17 & 0.16 & 0.14 & 1.00 \end{bmatrix} \quad (107)$$

The Ditlevsen bounds for the system probability of failure  $P_f^1$  (see figure 24) gives

$$0.06843 \leq P_f^1 \leq 0.06849$$

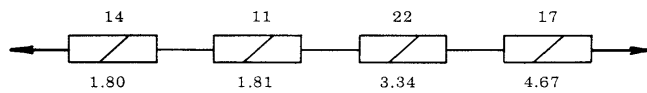


Figure 24. Series system used in estimating the system reliability at level 1.

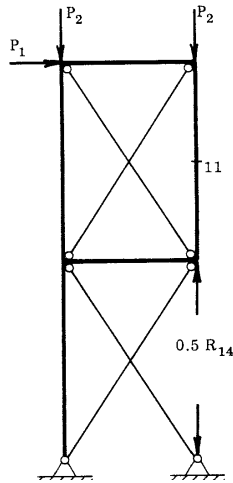
Therefore, a (good) estimate for the system reliability index at level 1 is  $\beta_s^1 = 1.49$ . It follows from (107) that the coefficients of correlation are rather small.

Therefore, the simple upper bound in (44) can be expected to give a good approximation. One gets  $\beta_s^1 = 1.48$ . A third estimate can be obtained by (54) and (57).

The result is  $\beta_s^1 = 1.49$ .

At level 2 it is initially assumed that the ductile failure element 14 fails (in compression) and a fictitious load equal to  $0.5 \times R_{14}$  is added (see figure 25). This modified structure is then analysed elastically and new reliability indices are calculated for all the remaining failure elements (see figure 25). Failure element 11 has the lowest

$\beta$ -value 1.87. With  $\Delta\beta_2 = 1.00$  failure element 11 is the only failure element with a  $\beta$ -value in the interval  $[1.87, 1.87 + \Delta\beta_2]$ . Therefore, in this case only one critical pair of failure elements is obtained by initiating the unzipping with failure element 14.



Failure element	2	3	4	5	6	7	8	9	10	11
$\beta$ -value	9.67	10.02	9.99	5.01	9.99	9.92	9.97	9.93	9.99	1.87
Failure element	12	13	16	17	18	19	20	21	22	
$\beta$ -value	10.00	10.03	5.48	3.41	10.24	8.87	10.25	9.08	4.46	

Figure 25. Modified structure when failure takes place in failure element 14.

Based on the safety margin  $M_{14}$  for failure element 14 and the safety margin

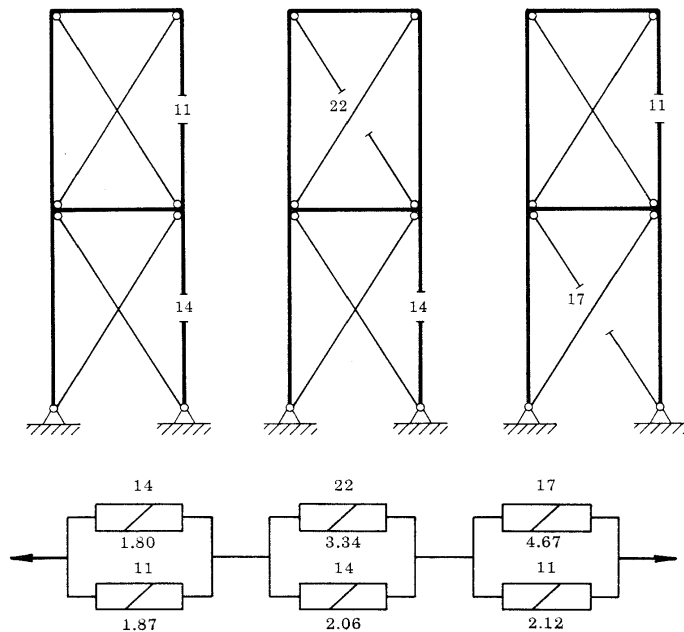


Figure 26. Failure modes used to estimate the reliability at level 2.

$M_{11,14}$  for failure element 11, given failure in failure element 14, the correlation coefficient can be calculated as  $\rho = 0.28$ .

Therefore, the probability of failure for this parallel system is  $P_f = \Phi_2(-1.80, -1.87; 0.28)$

$= 0.00347$ , and the corresponding generalized

index  $\beta_{14,11} = 2.70$ . The same procedure can be performed with the three other critical failure elements 11, 22 and 17.

The corresponding failure modes are shown in figure 26.

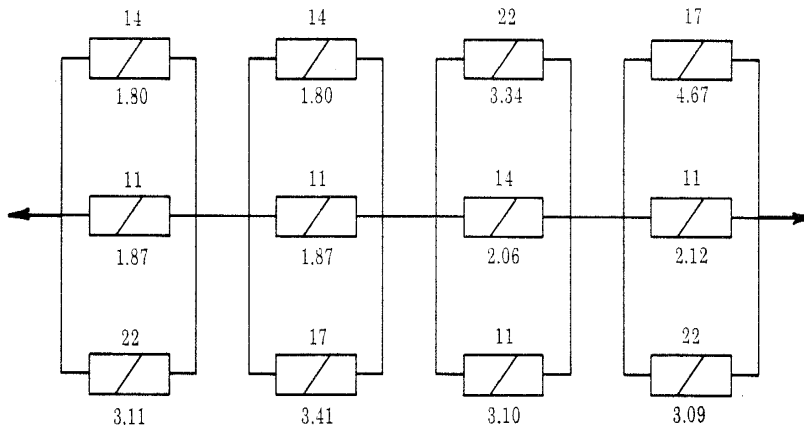
Generalized reliability indices and approximate equivalent safety margins for each parallel system in figure 26 are calculated. Then the

correlation matrix  $\bar{\rho}$  can be calculated

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.56 & 0.45 \\ 0.56 & 1.00 & 0.26 \\ 0.45 & 0.26 & 1.00 \end{bmatrix}$$

The Ditlevsen bounds for the probability of failure of the series system in figure 26 are  $0.3488 \times 10^{-2} \leq P_f^2 \leq 0.3488 \times 10^{-2}$ . Therefore, an estimate of the system

reliability at level 2 is  $\beta_s^2 = 2.70$ . At level 3 (with  $\Delta\beta_3 = 1.00$ ), four critical triples of



failure elements are identified (see figure 27) and an estimate of the system reliability at level 3 is  $\beta_s^3 = 3.30$ .

It is of interest to note that the estimates of the system reliability index at levels 1, 2, and 3 are very different:  $\beta_s^1 = 1.49$ ,  $\beta_s^2 = 2.70$ , and  $\beta_s^3 = 3.30$ .

Figure 27. Reliability modeling at level 3 of the two-storied braced frame.

### 5.6.3 Two-storey braced frame with ductile and brittle elements

Consider the same structure as in section 5.5.1, but now the failure elements 2, 5, 11, and 14 are assumed brittle. All other data are unchanged. By a linear elastic analysis the same reliability indices for all (brittle and ductile) failure elements as in table 1 are calculated. Therefore, the critical failure elements are 14 and 11, and the estimate of the system reliability at level 1 is unchanged, i.e.  $\beta_s^1 = 1.49$ .

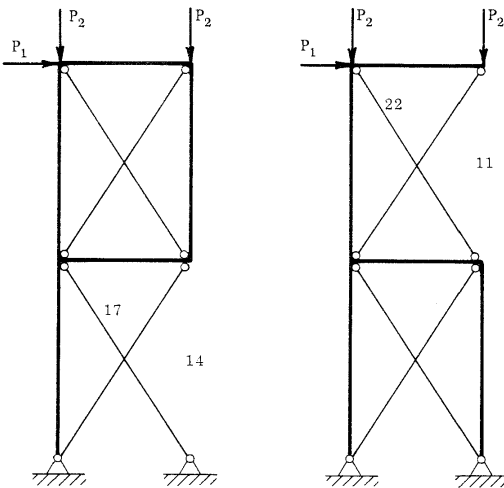


Figure 28. Modified structures.

The next step is to assume brittle failure in failure element 14 and remove the corresponding part of the structure without adding fictitious loads (see figure 28, left). The modified structure is then linear elastically analyzed and reliability indices are calculated for all remaining failure elements. Failure element 17 now has the lowest reliability index, namely the negative value  $\beta_{17|14} = -6.01$ . This very low negative value indicates that failure takes place in failure element 17 instantly after failure in failure element 14. The failure mode identified in this way is a mechanism and it is the only one when  $\Delta\beta_2 = 1.00$ . It can be mentioned that

$\beta_{16|14} = -3.74$  so that failure element 16 also fails instantly after failure element 14.

Again, by assuming brittle failure in failure element 11 (see figure 28, right) only one critical pair of failure elements is identified, namely the pair of failure elements 11 and 22, where  $\beta_{22|11} = -4.33$ . This failure mode is not a mechanism. The series system used in calculating an estimate of the system reliability at level 2 is shown in figure 29. Due to the small reliability indices the strength variables 17 and 22 do not affect the safety margins for the two parallel systems in figure 29 significantly. Therefore, the reliability index at level 2 is unchanged from level 1, namely  $\beta_s^2 = 1.49$ .

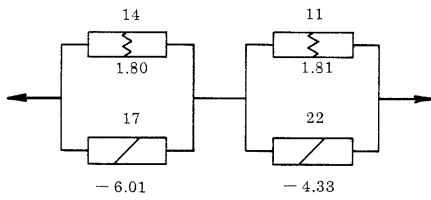


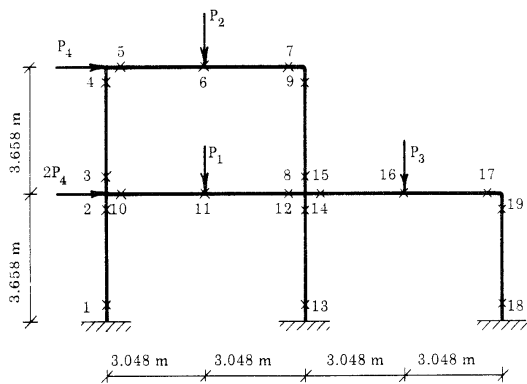
Figure 29. Modelling at level 2.

As expected this value is much lower than the value 2.70 (see section 5.5.1) calculated for the structure with only ductile failure elements. This fact stresses the importance of the reliability modeling of the structure.

It is of interest to note that the  $\beta$ -unzipping method was capable of disclosing that the structure cannot survive failure in failure element 14. Therefore, when brittle failure occurs it is often reasonable to define failure of the structure as failure of just one failure element. This is equivalent to estimating the reliability of the structure at level 1.

### 5.6.3 Elastic-plastic framed structure

In this example it is shown how the system reliability at mechanism level can be estimated in an efficient way. Consider the simple framed structure in figure 30 with corresponding expected values and coefficients of variation for the basic variables.



Variables	Expected values	Coefficients of variation
$P_1$	169 kN	0.15
$P_2$	89 kN	0.25
$P_3$	116 kN	0.25
$P_4$	31 kN	0.25
$R_1, R_2, R_{13}, R_{14}, R_{18}, R_{19}$	95 kNm	0.15
$R_3, R_4, R_8, R_9$	95 kNm	0.15
$R_5, R_6, R_7$	122 kNm	0.15
$R_{10}, R_{11}, R_{12}$	204 kNm	0.15
$R_{15}, R_{16}, R_{17}$	163 kNm	0.15

Figure 30. Geometry, loading and potential yield hinges

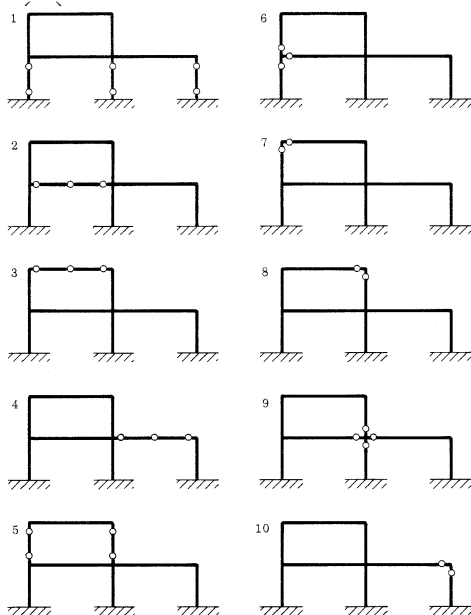


Figure 31. Set of fundamental mechanisms.

The load variables are  $P_i, i = 1, \dots, 4$  and the yield moments are  $R_i, i = 1, \dots, 19$ . Yield moments in the same line are considered fully correlated and the yield moments in different lines are mutually independent. The number of potential yield hinges is  $n = 19$  and the degree of redundancy is  $r = 9$ . Therefore, the number of fundamental mechanisms is  $n - r = 10$ .

One possible set of fundamental mechanisms is shown in figure 31. The safety margins  $M_i$  for the fundamental mechanisms can be written

$$M_i = \sum_{j=1}^{19} a_{ij} R_j - \sum_{i=1}^4 b_{ij} P_j, i = 1, \dots, 10 \quad (108)$$

where the influence coefficients  $a_{ij}$  and  $b_{ij}$  are determined by considering the mechanisms in the deformed state.

i	1	2	3	4	5	6	7	8	9	10
$\beta_i$	1.91	2.08	2.17	2.26	4.19	10.75	9.36	9.36	12.65	9.12

Table 2. Reliability indices for the fundamental mechanisms.

between the yield moments. The result is shown in table 2.

With  $\varepsilon_1 = 0.50$  the fundamental mechanisms 1, 2, 3, and 4 are selected as starting mechanisms in the  $\beta$ -unzipping and combined in turn with the remaining fundamental mechanisms. As an example, consider the combination 1 + 6 of mechanisms 1 and 6. The linear safety margin  $M_{1+6}$  is obtained from the linear safety margins  $M_1$  and  $M_6$  by addition taking into account the signs of the coefficients. The corresponding reliability index is  $\beta_{1+6} = 3.74$ .

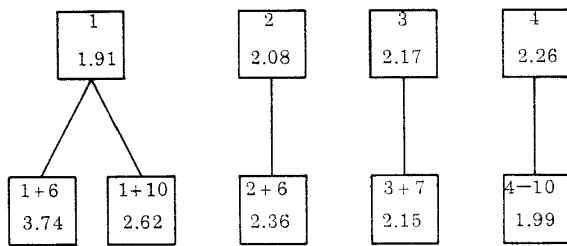


Figure 32. The first two rows in the failure tree.

The reliability indices  $\beta_i$ ,  $i = 1, \dots, 10$  for the 10 fundamental mechanisms can be calculated from the safety margins taking into account the correlation

With  $\varepsilon_2 = 1.20$  the following new mechanisms 1 + 6, 1 + 10, 2 + 6, 3 + 7, and 4 - 10 are identified by this procedure. The failure tree at this stage is shown in figure 16. It contains 4 + 5 = 9 mechanisms. The reliability indices and the

fundamental mechanisms involved are shown in the same figure.

This procedure is now continued as explained earlier by adding or subtracting fundamental mechanisms. If the procedure is continued 8 times (up to 10 fundamental mechanisms in one mechanism) and if the significant mechanisms are selected by  $\varepsilon_3 = 0.31$ , then the system modeling at mechanism level will be a series system where the elements are 12 parallel systems. These 12 parallel systems (significant mechanisms) and corresponding reliability indices are shown in table 3. The correlation matrix is

$$\bar{\rho} = \begin{bmatrix} 1.00 & 0.65 & 0.89 & 0.44 & 0.04 & 0.91 & 0.59 & 0.97 & 0.87 & 0.81 & 0.36 & 0.92 \\ 0.65 & 1.00 & 0.55 & 0.00 & 0.09 & 0.67 & 0.00 & 0.63 & 0.54 & 0.58 & 0.00 & 0.71 \\ 0.89 & 0.55 & 1.00 & 0.35 & 0.45 & 0.83 & 0.61 & 0.88 & 0.98 & 0.95 & 0.31 & 0.83 \\ 0.44 & 0.00 & 0.35 & 1.00 & 0.00 & 0.03 & 0.00 & 0.45 & 0.37 & 0.04 & 0.89 & 0.08 \\ 0.04 & 0.09 & 0.45 & 0.00 & 1.00 & 0.05 & 0.00 & 0.04 & 0.44 & 0.48 & 0.00 & 0.05 \\ 0.91 & 0.67 & 0.83 & 0.03 & 0.05 & 1.00 & 0.73 & 0.87 & 0.80 & 0.88 & 0.00 & 0.98 \\ 0.59 & 0.00 & 0.61 & 0.00 & 0.00 & 0.73 & 1.00 & 0.58 & 0.60 & 0.65 & 0.00 & 0.64 \\ 0.97 & 0.63 & 0.88 & 0.45 & 0.04 & 0.87 & 0.58 & 1.00 & 0.90 & 0.77 & 0.48 & 0.87 \\ 0.87 & 0.54 & 0.98 & 0.37 & 0.44 & 0.80 & 0.60 & 0.90 & 1.00 & 0.90 & 0.41 & 0.78 \\ 0.81 & 0.58 & 0.95 & 0.04 & 0.48 & 0.88 & 0.65 & 0.77 & 0.90 & 1.00 & 0.00 & 0.88 \\ 0.36 & 0.00 & 0.31 & 0.89 & 0.00 & 0.00 & 0.00 & 0.48 & 0.41 & 0.00 & 1.00 & 0.00 \\ 0.92 & 0.71 & 0.83 & 0.08 & 0.05 & 0.98 & 0.64 & 0.87 & 0.78 & 0.88 & 0.00 & 1.00 \end{bmatrix}$$

No.	Significant mechanisms	$\beta$
1	1 + 6 + 2 + 5 + 7 + 3 - 8	1.88
2	1	1.91
3	1 + 6 + 2 + 5 + 7 + 3 + 4 - 8 - 10	1.94
4	3 + 7 - 8	1.98
5	4 - 10	1.99
6	1 + 6 + 2	1.99
7	2	2.08
8	1 + 6 + 2 + 5 + 7 + 3 + 8	2.09
9	1 + 6 + 2 + 5 + 7 + 3 + 8 + 9 + 4 - 10	2.11
10	1 + 6 + 2 + 5 + 9 + 4 - 10	2.17
11	3	2.17
12	1 + 6 + 2 + 5	2.18

Table 3.

The probability of failure  $P_f$  for the series system with the 12 significant mechanisms as elements can then be estimated by the usual techniques. The Ditlevsen bounds are  $0.08646 \leq P_f \leq 0.1277$ . If the average value of the lower and upper bounds is used, the estimate of the reliability index  $\beta_s$  at mechanism level is  $\beta_s = 1.25$ . Hohenbichler [28] has derived an approximate method to calculate estimates for the system probability of failure  $P_f$  (and the corresponding reliability index  $\beta_s$ ). The estimate of  $\beta_s$  is  $\beta_s = 1.21$ . It can finally be noted that Monte Carlo simulation gives  $\beta_s = 1.20$ .

## 6. ILLUSTRATIVE EXAMPLES

### 6.1 Reliability of a tubular joint

The single tubular beams in a steel jacket structure will in general at least have three important failure elements, namely failure in yielding under combined bending moments and axial tension/compression at points close to the ends of the beam and failure in buckling (instability). A tubular joint will likewise have a number of potential

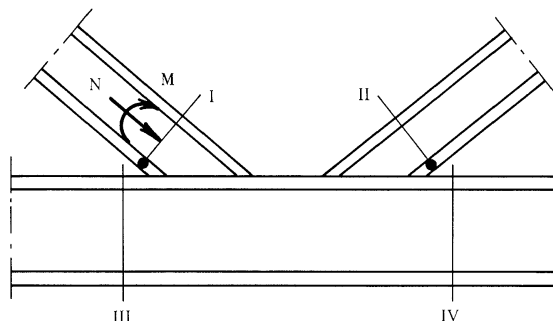


Figure 33. Tubular K-joint. I - IV are the critical sections. • indicates hot spots.

failure elements. Consider the tubular K-joint shown in figure 33.

This tubular joint is analyzed in more detail in a paper by Thoft-Christensen & Sørensen [29]. It is assumed that the joint has four critical sections as indicated in figure 1 by I, II, III and IV. The load effects in each critical section are an axial force  $N$  and a bending moment  $M$ . The structure is assumed to be linear elastic. For this K-joint 12 failure elements are considered, namely

- Failure in yielding in the four critical sections I, II, III and IV ,
- Punching failure in the braces (cross-sections I and II) ,
- Buckling failure of the four tubular members (cross-sections I, II, III and IV),
- Fatigue failure in the critical sections I and II (hot spots are indicated in figure 33).

For these failure elements safety margins have been formulated by Thoft-Christensen & Sørensen [29]. For failure in yielding the safety margins are of the form

$$M^Y = Z^Y - \left( \left| \frac{M}{M_F} \right| - \cos \left( \frac{\pi}{2} \frac{N}{N_F} \right) \right) \quad (109)$$

where  $M_F$  and  $N_F$  are the yield capacities in pure bending and pure axial loading and where  $Z^Y$  is a model uncertainty variable. For punching failure the following safety margin is used

$$M^P = Z^P - \left( \left| \frac{N}{Z^N N_U} \right| + \left( \frac{|M|}{Z^M M_U} \right)^{1.2} \right) \quad (110)$$

$N_U$  and  $M_U$  are ultimate punching capacities in pure axial loading and pure bending, respectively. In this case 3 model uncertainty variables  $Z^P$ ,  $Z^N$ , and  $Z^M$  are included. The following safety margin is used for buckling failure

$$M^B = Z^B - \left( \frac{N}{N_B} + \frac{M}{M_B} \right) \quad (111)$$

where  $N_B$  and  $M_B$  are functions of the geometry and the yield stress and where  $Z^B$  is a model uncertainty variable. Finally, with regard to fatigue failure the safety margin for each of the two hot points, shown in figure 33, has the form

$$M^F = Z^F - \left( Z^L \right)^m K^{-1} (\max(t/32, 1)^{M_1}) g \quad (112)$$

where  $m (= 3)$  and  $K$  are constants in the  $S$ - $N$  relation used in Miner's rule.  $K$  is modeled as a random variable.  $t$  is the wall thickness,  $M_1$  is a random variable and  $g$  is a constant.  $Z^F$  and  $Z^L$  are model uncertainty variables.

For this particular K-joint in a plane model of a tubular steel jacket structure analyzed by Thoft-Christensen & Sørensen [29] only four failure elements are significant, namely (referring to figure 1) fatigue in cross-section II ( $\beta = 3.62$ ), punching in the same cross-section II ( $\beta = 4.10$ ), buckling in cross-section II ( $\beta = 4.58$ ) and fatigue in cross-section I ( $\beta = 4.69$ ).

The correlation coefficient matrix of the linearized safety margins of the 4 significant failure elements is

$$\bar{\bar{\rho}} = \begin{bmatrix} 1 & 0 & 0 & 0.86 \\ 0 & 1 & 0.82 & 0 \\ 0 & 0.82 & 1 & 0 \\ 0.86 & 0 & 0 & 1 \end{bmatrix}$$

and the probability of failure of the joint

$$P_f \approx 1 - \Phi_4(\bar{\beta}, \bar{\bar{\rho}}) = 1.718 \times 10^{-4}$$

The corresponding reliability index for the joint is

$$\beta = -\Phi^{-1}(P_f) = 3.58$$

It is important to note that in this estimate of the reliability of the K-joint interaction between the different significant failure elements in e.g. cross-section II is not taken into account. Each failure element (failure mode) is considered independent of the other although such interaction will influence the reliability of the joint.

## 6.2 Reliability-based optimal design of a steel-jacket offshore structure

In Thoft-Christensen & Murotsu [2], an extensive number of references to reliability-based optimal design can be found. In this section the optimal design problem is briefly stated and illustrated with an example taken from Thoft-Christensen & Sørensen [30]. In reliability-based optimization the objective function is often chosen as the weight  $F$



of the structure. The constraints can either be related to the reliability of the single elements or to the reliability of the structural system. In the last-mentioned case the optimization problem for a structure with  $h$  elements may be written

$$\begin{aligned} \min \quad & F(\bar{y}) = \sum_{i=1}^h \varphi_i l_i A_i(\bar{y}) \\ \text{s.t.} \quad & \beta^s(\bar{y}) \geq \beta_0^s \\ & y_i^l \leq y_i \leq y_i^u \quad , \quad i = 1, \dots, n \end{aligned} \quad (113)$$

where  $A_i$ ,  $l_i$  and  $\varphi_i$  are the cross-sectional area, the length, and the density of element no.  $i$ .  $\bar{y} = (y_1, \dots, y_n)$  are the design variables.  $\beta_0^s$  is the target systems reliability index.  $y_i^l$  and  $y_i^u$  are lower and upper bounds for the design variable  $y_i$ ,  $i=1, \dots, n$ .

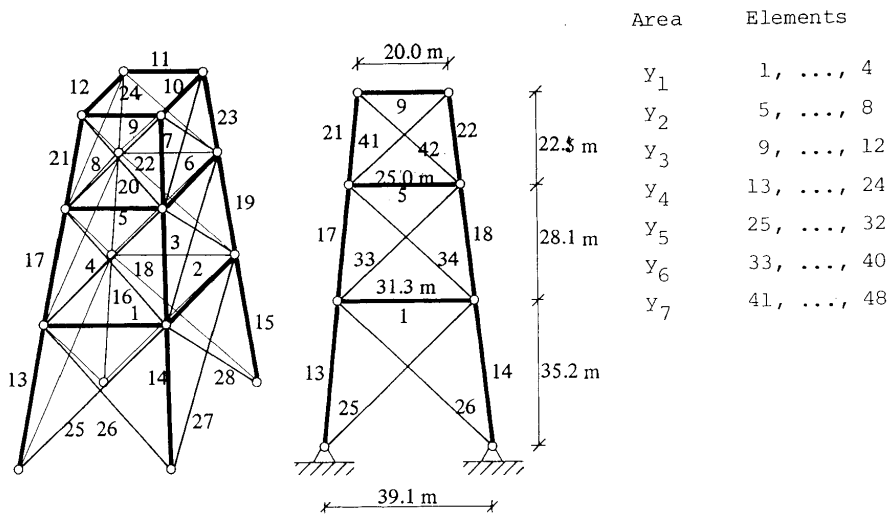


Figure 34. Space truss tower with design variables.

Consider the three-dimensional truss model of a steel-jacket offshore structure shown in figure 34. The load and the geometry are described in detail in Thoft-Christensen & Sørensen [30] and in Sørensen, Thoft-Christensen & Sigurdsson [31]. The load is modeled by two random variables and the yield capacities of the 48 truss elements are modeled as random variables with expected values  $270 \times 10^6 \text{ Nm}^{-2}$  and coefficients of variation equal to 0.15. The correlation structure of the normally distributed variables is described in [31]. The design variables  $y_i$ ,  $i = 1, \dots, 7$  are the cross-sectional areas ( $\text{m}^2$ ) of the seven groups of structural elements (see figure 41).

The optimization problem is

$$\begin{aligned} \min \quad & F(\bar{y}) = 125y_1 + 100y_2 + 80y_3 + 384y_4 + 399y_5 + 319y_6 + 255y_7 \quad (\text{m}^3) \\ \text{s.t.} \quad & \beta^s(\bar{y}) \geq \beta_0^s = 3.00 \\ & 0 = y_i^l \leq y_i \leq y_i^u = 1 \quad , \quad i = 1, \dots, 7 \end{aligned} \quad (114)$$

$\beta^s$  is the systems reliability index at level 1. The solution is  $\bar{y} = (0.01, 0.001, 0.073, 0.575, 0.010, 0.009, 0.011) \text{ m}^2$  and  $F(\bar{y}) = 215 \text{ m}^3$ . The iteration history for the weight function  $F(\bar{y})$  and the systems reliability index  $\beta^s$  is shown in figure 35.

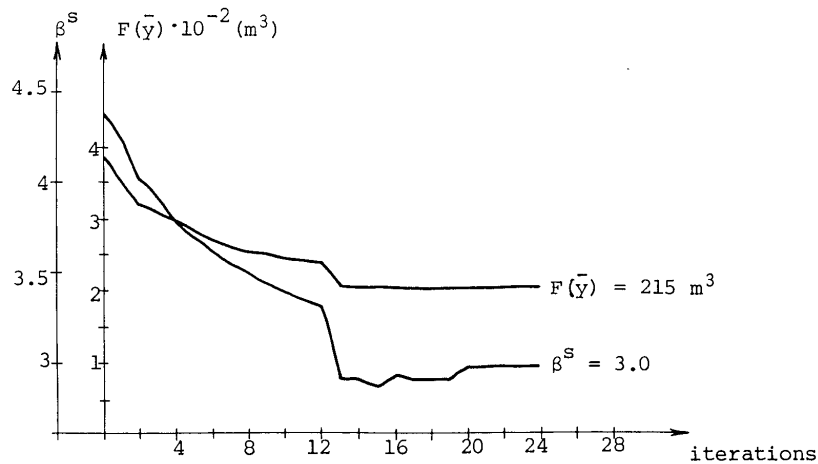


Figure 35. Iteration history.

### 6.3 Reliability-based optimal maintenance strategies

An interesting application of optimization methods is related to deriving optimal strategies for inspection and repair of structural systems. In a paper by Thoft-Christensen & Sørensen [32] such a strategy was presented with the intention of minimizing the cost of inspection and repair of a structure in its lifetime  $T$  under the constraint that the structure has an acceptable reliability. In a paper by Sørensen & Thoft-Christensen [33] this work was extended by including not only inspection costs and repair costs but also the production (initial) cost of the structure in the objective function.

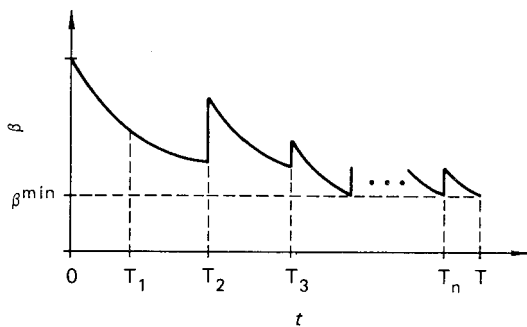


Figure 36. Maintenance strategy.

the inspection times and  $\beta^{\min}$  is the minimum acceptable reliability of the structure in its lifetime.

Let  $t_i = T_i - T_{i-1}$ ,  $i = 1, \dots, n$  and let the quantity of inspection at time  $T_i$  be  $q_i$ ,  $i = 1, \dots, n$ . Then, with a given number of inspection times  $n$ , the design variables are  $t_i$ ,  $i = 1, \dots, n$  and  $q_i$ ,  $i = 1, \dots, n$ . As an illustration consider the maintenance strategy shown in Figure 36. At time  $T_1$  the solution of the optimization problem is  $q_i = 0$ , i.e. no inspection takes place at that time. Inspection takes place at time  $T_2$ , and according to the result of the inspection it is decided whether or not repair should be performed. If repair is performed the reliability is improved. If no repair is performed the reliability is also improved because then updating of the strengths of the structural elements takes place. Therefore, the variation of the reliability of the structure with time will be as shown in Figure 36. The shape of the curves between inspection times will depend on

The purpose of a simple optimal strategy for inspection and repair of civil engineering structures is to minimize the expenses of inspection and repair of a given structure so that the structure in its expected service life has an acceptable reliability. The strategy is illustrated in figure 36, where  $T$  is the lifetime of the structure and  $\beta$  is a measure of the reliability of the structure. The reliability  $\beta$  is assumed to be a non-increasing function with time  $t$ .  $T_i$ ,  $i = 1, 2, \dots, n$  are

the relevant types of deterioration, for example whether corrosion or fatigue is considered.

A very brief de-scription of the application of this strategy is given here based on [33]. The design variables are cross-sectional parameters  $z_1, \dots, z_m$ , inspection qualities  $q_1, \dots, q_N$ , and time between inspections  $t_1, \dots, t_N$  where  $m$  is the number of cross-sections to be designed and  $N$  the number of inspections (and repairs). The optimization for a structural system modeled by  $s$  failure elements can then be formulated in the following way

$$\min_{\bar{q}=(q_1, \dots, q_N), \bar{t}=(t_1, \dots, t_N)} C(\bar{z}, \bar{q}, \bar{t}) = C_I(\bar{z}) + \sum_{i=1}^N \sum_{j=1}^s C_{IN,j}(q_i) e^{-rT_i} + \sum_{i=1}^N \sum_{j=1}^s C_{R,j}(\bar{z}) E[R_{ij}(\bar{z}, \bar{q}, \bar{t})] e^{-rT_i} \quad (114)$$

$$s.t. \beta^s(T_i) \geq \beta^{\min}, \quad i = 1, \dots, N, N+1$$

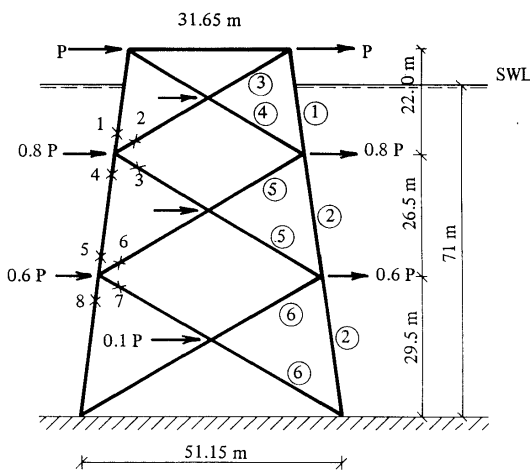


Figure 37. Plan model of a steel jacket platform.

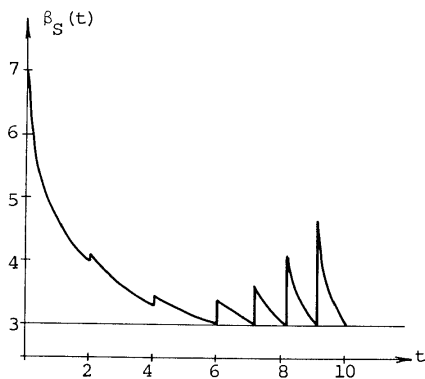


Figure 38.  $\beta^s$  as a function of the time  $t$  for optimal design and inspection variables and for  $N = 6$ .

following optimal solution is determined:

$$\begin{aligned} \bar{t} &= (2, 2, 2, 1.19, 0.994, 0.925) \text{ years} \\ \bar{q} &= (0.1, 0.133, 0.261, 0.332, 0.385, 0.423) \\ \bar{z} &= (68.9, 62.7, 30.0, 30.0, 50.4, 32.0) \text{ mm} \end{aligned}$$

as shown in figure 38.

where the trivial bounds on the design variables are omitted.  $C_I$  is the initial cost of the structure,  $C_{IN,j}(q_i)$  the cost of an inspection of element  $j$  with the inspection quality  $q_i$ ,  $C_{R,j}$  is the cost of an repair of element  $j$ .  $r$  is the discount rate, and  $E[R_{ij}(\bar{z}, \bar{q}, \bar{t})]$  is the expected number of repairs at the time  $T_i$  in element  $j$ .  $\beta^s(T_i)$  is the systems reliability index at level 1 at the inspection time  $T_i$  and  $\beta^{\min}$  the lowest acceptable systems reliability index.

Consider the plane model of a steel jacket platform shown in figure 37 (see [33], where all details are described). Due to symmetry only the 8 fatigue failure elements indicated by  $\times$  in figure 9 are considered. Design variables are the tubular thicknesses of the 6 groups of elements indicated by  $\circ$  in figure 37.

Using  $\beta^{\min} = 3.00$ ,  $T = 10$  years,  $r = 0$  and  $N = 6$  the

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