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SEDIMENT TRANSPORT

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Preface

Flow and sediment transport are important in relation to several engineering topics, e.g. erosion around structures, backfilling of dredged channels and nearshore morphological change.

The purpose of the present book is to describe both the basic hydrodynamics and the basic sediment transport mechanism. The reader's background should be a basic course in wave theory and fluid mechanics.

Chapter 1 deals with fundamentals in fluid mechanics with emphasis on bed shear stress by currents, while Chapter 3 focuses on bed shear stress by waves. They are both written with a view to sediment transport.

Sediment transport in rivers, cross-shore and longshore are dealt with in Chapters 2, 4 and 5, respectively.

It is not the intention of the book to give a broad review of the literature on this very wide topic. The book tries to *pick up* information which is of engineering importance.

An obstacle to the study of sedimentation is the scale effect in model tests. Whenever small-scale tests, large-scale tests and field investigations are available, it is always the result from field investigations which is referred to.

Contents

1 Steady uniform flow in open channels

This chapter is written with a view to sediment transport. The main outcome is the current friction coefficient.

The coordinate system applied in this chapter is shown in Fig.1.

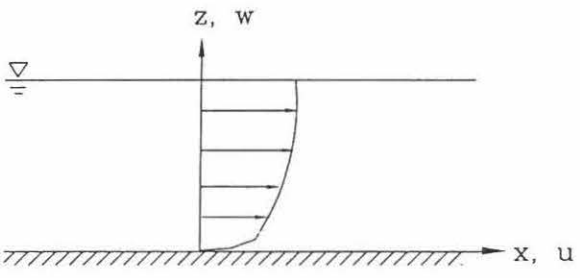


Fig.1. Coordinate system for the flow in open channels.

1.1 Types of flow

Description of various types of flow are given in the following.

Laminar versus turbulent

Laminar flow occurs at relatively low fluid velocity. The flow is visualized as layers which slide smoothly over each other without macroscopic mixing of fluid particles. The shear stress in laminar flow is given by Newton's law of viscosity

$$\tau_\nu = \rho \, \nu \, \frac{du}{dz} \tag{1}$$

where ρ is density of water and ν kinematic viscosity ($\nu = 10^{-6}\ m^2/s$ at 20^0C).

Most flows in nature are turbulent. Turbulence is generated by instability in the flow, which trigger vortices. However, a thin layer exists near the boundary where the fluid motion is still laminar.

A typical phenomenon of turbulent flow is the fluctuation of velocity

$$U = u + u' \qquad W = w + w' \tag{2}$$

where U W instantaneous velocity, in x and z directions respectively

u w time-averaged velocity, in x and z directions respectively

u' w' instantaneous velocity fluctuation, in x and z directions respectively

Turbulent flow is often given as the mean flow, described by u and w.

In turbulent flow the water particles move in very irregular paths, causing an exchange of momentum from one portion of fluid to another, and hence, the turbulent shear stress (Reynolds stress). The turbulent shear stress, given by time-averaging of the Navier-Stokes equation, is

$$\tau_t = -\rho \, \overline{u' \, w'} \tag{3}$$

Note that $\overline{u' \, w'}$ is always negative. In turbulent flow both viscosity and turbulence contribute to shear stress. The total shear stress is

$$\tau = \tau_\nu + \tau_t = \rho \, \nu \, \frac{\mathrm{d}u}{\mathrm{d}z} + (-\rho \, \overline{u' \, w'}) \tag{4}$$

Steady versus unsteady

A flow is steady when the flow properties (e.g. density, velocity, pressure etc.) at any point are constant with respect to time. However, these properties may vary from point to point. In mathematical language,

$$\frac{\partial \, (\text{any flow property})}{\partial t} = 0 \tag{5}$$

In the case of turbulent flow, steady flow means that the statistical parameters (mean and standard deviation) of the flow do not change with respect to time.

If the flow is not steady, it is unsteady.

Uniform versus non-uniform

A flow is uniform when the flow velocity does not change along the flow direction, cf. Fig.2. Otherwise it is non-uniform flow.

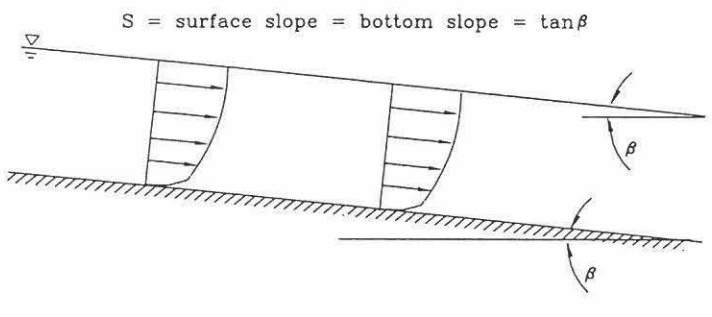


Fig.2. Steady uniform flow in a open channel.

Boundary layer flow

Prandtl developed the concept of the boundary layer. It provides an important link between ideal-fluid flow and real-fluid flow.

Here is the original description. *For fluids having small viscosity, the effect of internal friction in the flow is appreciable only in a thin layer surrounding the flow boundaries.* However, we will demonstrate that the boundary layer fulfil the whole flow in open channels.

The boundary layer thickness (δ) is defined as the distance from the boundary surface to the point where $u = 0.995\ U$. The boundary layer development can be expressed as

$$\text{laminar flow} \quad \frac{\delta}{x} = 5\left(\frac{U\,x}{\nu}\right)^{-0.5} \quad \text{when } Re_x = \frac{U\,x}{\nu} < 5\times 10^5$$

$$\text{turbulent flow} \quad \frac{\delta}{x} = 0.4\left(\frac{U\,x}{\nu}\right)^{-0.2} \quad \text{when } Re_x = \frac{U\,x}{\nu} > 5\times 10^5$$

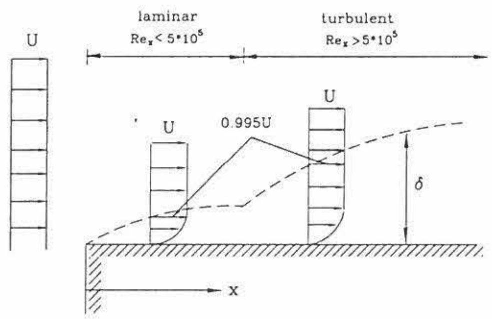


Fig.3. Development of the boundary layer.

Example 1	Development of the boundary layer flow.
Given	flow velocity $U = 1m/s$ and water depth $h = 10m$
Wanted	1) x value where the boundary layer flow starts to fulfil the whole depth
	2) type of the boundary layer flow
Solution	Based on the expression for turbulent boundary layer flow

$$x|_{\delta=h} = \left(\frac{(U/\nu)^{0.25}}{0.4^{0.8}}\,h^{1.25}\right) = \left(\frac{(1/10^{-6})^{0.25}}{0.4^{0.8}}\,10^{1.25}\right) = 1171\ m$$

$$Re_x = \frac{U\,x}{\nu} = \frac{1\times 1171}{10^{-6}} = 1.171\times 10^9 > 5\times 10^5 \quad \text{turbulent}$$

Comment — The example demonstrates that the flow in open channels is always a turbulent boundary layer flow.

1.2 Prandtl's mixing length theory

Prandtl introduced the mixing length concept in order to calculate the turbulent shear stress. He assumed that a fluid parcel travels over a length ℓ before its momentum is transferred.

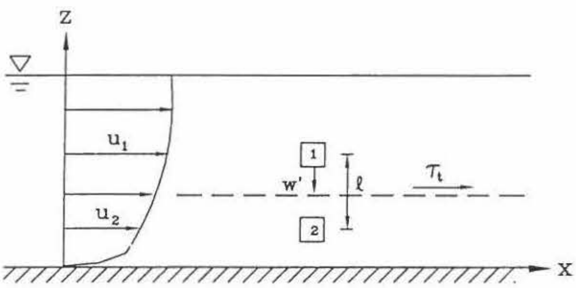


Fig.4. Prandtl's mixing length theory.

Fig.4 shows the time-averaged velocity profile. The fluid parcel, locating in layer 1 and having the velocity u_1, moves to layer 2 due to eddy motion. There is no momentum transfer during movement, i.e. the velocity of the fluid parcel is still u_1 when it just arrives at layer 2, and decreases to u_2 some time later by the momentum exchange with other fluid in layer 2. This action will speed up the fluid in layer 2, which can be seen as a turbulent shear stress τ_t acting on layer 2 trying to accelerate layer 2. cf. Fig.4

The horizontal instantaneous velocity fluctuation of the fluid parcel in layer 2 is

$$u' = u_1 - u_2 = \ell \frac{du}{dz} \tag{6}$$

Assuming the vertical instantaneous velocity fluctuation having the same magnitude

$$w' = -\ell \frac{du}{dz} \tag{7}$$

where negative sign is due to the downward movement of the fluid parcel, the turbulent shear stress now becomes

$$\tau_t = -\rho\, u'\, w' = \rho\, \ell^2 \left(\frac{du}{dz}\right)^2$$

If we define kinematic eddy viscosity

$$\varepsilon = \ell^2 \frac{du}{dz} \tag{8}$$

the turbulent shear stress can be expressed in a way similar to viscous shear stress

$$\tau_t = \rho\, \varepsilon \frac{du}{dz} \tag{9}$$

1.3 Fluid shear stress and friction velocity

Fluid shear stress

The forces on a fluid element with unit width is shown in Fig.5. Because the flow is uniform (no acceleration), the force equilibrium in x-direction reads

$$\tau_z \; \Delta x \; = \; \rho \; g \; (h - z) \; \Delta x \; \sin \beta$$

For small slope we have $\sin \beta \approx \tan \beta = S$. Therefore

$$\tau_z \; = \; \rho \; g \; (h - z) \; S$$

The bottom shear stress is

$$\tau_b \; = \; \tau_{z=0} \; = \; \rho \; g \; h \; S \tag{10}$$

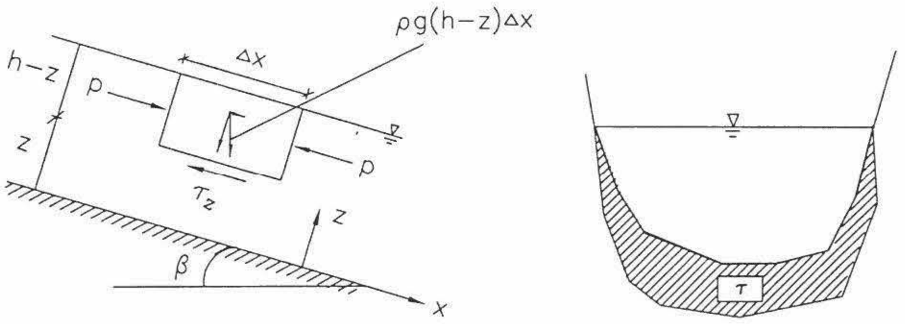


Fig.5. Fluid force and bottom shear stress.

Bottom shear stress

In the case of arbitrary cross section, the shear stress acting on the boundary changes along the wetted perimeter, cf. Fig.5. Then the bottom shear stress means actually the average of the shear stress along the wetted perimeter. The force equilibrium reads

$$\tau_b \; O \; \Delta x \; = \; \rho \; g \; A \; \Delta x \; \sin \beta$$

where O is the wetted perimeter and A the area of the cross section. By applying the hydraulic radius ($R = A/O$) we get

$$\tau_b \; = \; \rho \; g \; R \; S \tag{11}$$

In the case of wide and shallow channel, R is approximately equal to h, eq (11) is identical to eq (10).

Friction velocity

The bottom shear stress is often represented by friction velocity, defined by

$$u_* = \sqrt{\frac{\tau_b}{\rho}} \tag{12}$$

The term *friction velocity* comes from the fact that $\sqrt{\tau_b/\rho}$ has the same unit as velocity and it has something to do with friction force.

Inserting eq (11) into eq (12), we get

$$u_* = \sqrt{g\,R\,S} \tag{13}$$

Viscous shear stress versus turbulent shear stress

Eq (10) states that the shear stress in flow increases linearly with water depth, cf. Fig.6.

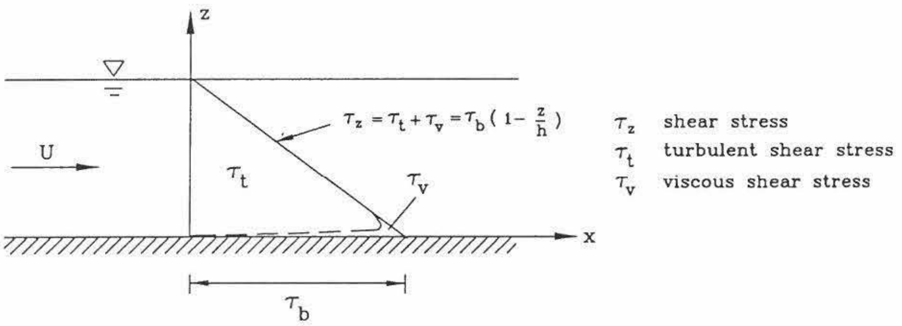


Fig.6. Shear stress components and distribution.

As the shear stress is consisted of viscosity and turbulence, we have

$$\tau_z = \tau_\nu + \tau_t = \rho\,g\,(h-z)\,S \tag{14}$$

On the bottom surface, there is no turbulence ($u = w = 0, u' = w' = 0$), the turbulent shear stress

$$\tau_t = -\rho\,\overline{u'\,w'} = 0$$

Therefore, in a very thin layer above the bottom, viscous shear stress is dominant, and hence the flow is laminar. This thin layer is called viscous sublayer. Above the viscous sublayer, i.e. in the major part of flow, the turbulent shear stress dominates, cf. fig.6.

The measurement shows the shear stress in the viscous sublayer is constant and equal to the bottom shear stress, not increasing linearly with depth as indicated by Fig.6.

1.4 Classification of flow layer

Scientific classification

Fig.7 shows the classification of flow layers. Starting from the bottom we have

1) Viscous sublayer: a thin layer just above the bottom. In this layer there is almost no turbulence. Measurement shows that the viscous shear stress in this layer is constant. The flow is laminar. Above this layer the flow is turbulent.

2) Transition layer: also called buffer layer. viscosity and turbulence are equally important.

3) Turbulent logarithmic layer: viscous shear stress can be neglected in this layer. Based on measurement, it is assumed that the turbulent shear stress is constant and equal to bottom shear stress. It is in this layer where Prandtl introduced the mixing length concept and derived the logarithmic velocity profile.

4) Turbulent outer layer: velocities are almost constant because of the presence of large eddies which produce strong mixing of the flow.

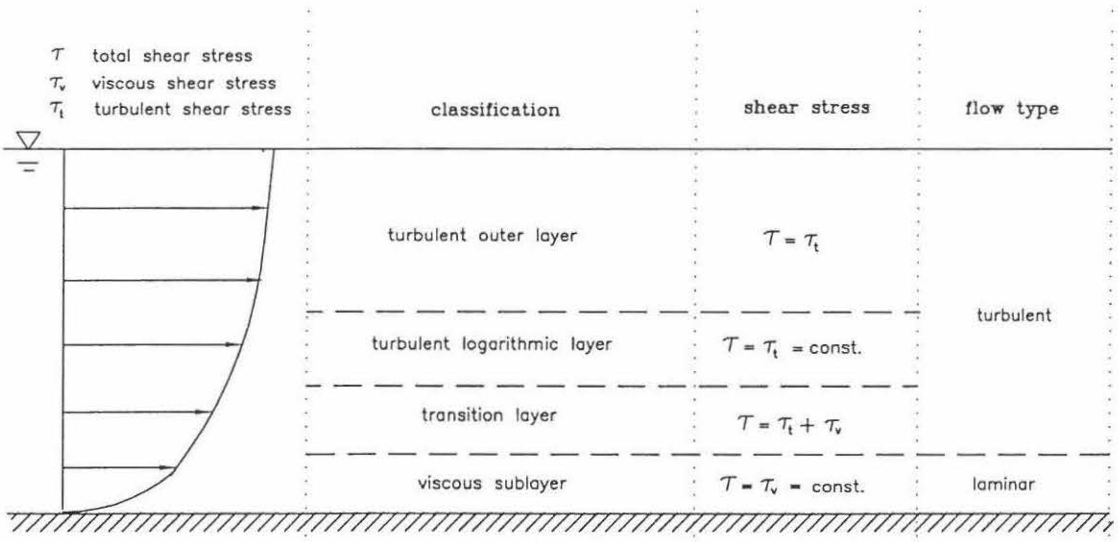


Fig.7. Scientific classification of flow region (Layer thickness is not to scale, turbulent outer layer accounts for 80% - 90% of the region).

Engineering classification

In the turbulent logarithmic layer the measurements show that the turbulent shear stress is constant and equal to the bottom shear stress. By assuming that the mixing length is proportional to the distance to the bottom ($\ell = \kappa z$), Prandtl obtained the logarithmic velocity profile.

Various expressions have been proposed for the velocity distribution in the transitional layer and the turbulent outer layer. None of them are widely accepted. However, By the modification of the mixing length assumption, cf. next section, the logarithmic velocity profile applies also to the transitional layer and the turbulent outer layer. Measurement and computed velocities show reasonable agreement.

Therefore in engineering point of view, a turbulent layer with the logarithmic velocity profile covers the transitional layer, the turbulent logarithmic layer and the turbulent outer layer, cf. Fig.8.

As to the viscous sublayer. The effect of the bottom (or wall) roughness on the velocity distribution was first investigated for pipe flow by Nikurase. He introduced the concept of equivalent grain roughness k_s (Nikurase roughness, bed roughness). Based on experimental data, it was found

1) Hydraulically smooth flow for $\frac{u_* k_s}{\nu} \leq 5$

   Bed roughness is much smaller than the thickness of viscous sublayer. Therefore, the bed roughness will not affect the velocity distribution.

2) Hydraulically rough flow for $\frac{u_* k_s}{\nu} \geq 70$

   Bed roughness is so large that it produces eddies close to the bottom. A viscous sublayer does not exist and the flow velocity is not dependent on viscosity.

3) Hydraulically transitional flow for $5 \leq \frac{u_* k_s}{\nu} \leq 70$

   The velocity distribution is affected by bed roughness and viscosity.

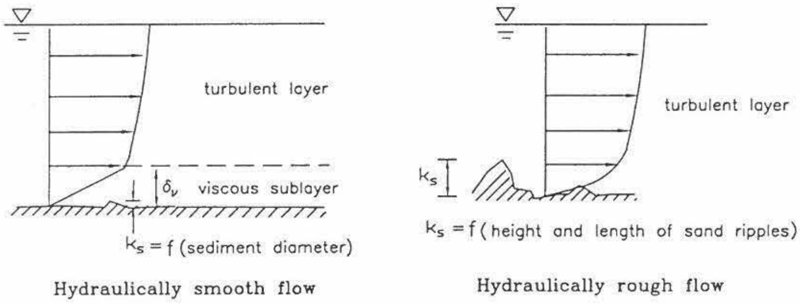


Fig.8. Engineering classification of flow region (Layer thickness is not to scale).

1.5 Velocity distribution

Turbulent layer

In the turbulent layer the total shear stress contains only the turbulent shear stress. The total shear stress increases linearly with depth (eq (10) or Fig.6), i.e.

$$\tau_t(z) = \tau_b \left(1 - \frac{z}{h}\right)$$

By prandtl's mixing length theory

$$\tau_t = \rho\, \ell^2 \left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2$$

and assuming the mixing length

$$\ell = \kappa\, z \left(1 - \frac{z}{h}\right)^{0.5}$$

where Von Karman constant $\kappa = 0.4$ [1], we get

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \frac{\sqrt{\tau_b/\rho}}{\kappa\, z} = \frac{u_*}{\kappa\, z} \quad .$$

Integration of the equation gives the famous logarithmic velocity profile

$$u\,(z) = \frac{u_*}{\kappa} \ln\left(\frac{z}{z_0}\right) \tag{15}$$

where the integration constant z_0 is the elevation corresponding to zero velocity ($u_{z=z_0} = 0$), given by Nikurase by the study of the pipe flows.

$$Z_0 = \begin{cases} 0.11\, \frac{\nu}{u_*} & \text{Hydraulically smooth flow} \quad \frac{u_*\, k_s}{\nu} \le 5 \\ 0.033\, k_s & \text{Hydraulically rough flow} \quad \frac{u_*\, k_s}{\nu} \ge 70 \\ 0.11\, \frac{\nu}{u_*} + 0.033\, k_s & \text{Hydraulically transition flow} \quad 5 < \frac{u_*\, k_s}{\nu} < 70 \end{cases} \tag{16}$$

It is interesting to note that the friction velocity u_*, which, by definition, has nothing to do with velocity, is the flow velocity at the elevation $z = z_0 e^{\kappa}$, i.e.

$$u_{z=z_0 e^{\kappa}} = u_*$$

In the study of sediment transport, it is important to know that the friction velocity is the fluid velocity very close to the bottom, cf. Fig.9.

[1] $\kappa = 0.4$ is obtained experimentally in pipe flow

Viscous sublayer

In the case of hydraulically smooth flow there is a viscous sublayer. Viscous shear stress is constant in this layer and equal to the bottom shear stress, i.e.

$$\tau_\nu = \rho\,\nu\,\frac{\mathrm{d}u}{\mathrm{d}z} = \tau_b$$

Integrating and applying $u|_{z=0} = 0$ gives

$$u\,(z) = \frac{\frac{\tau_b}{\rho}}{\nu}\,z = \frac{u_*^2}{\nu}\,z \qquad (17)$$

Thus, there is a linear velocity distribution in the viscous sublayer.

The linear velocity distribution intersect with the logarithmic velocity distribution at the elevation $z = 11.6\nu/u_*$, yielding a theoretical viscous sublayer thickness

$$\delta_\nu = 11.6\,\frac{\nu}{u_*}$$

The velocity profile is illustrated in Fig.9, with the detailed description of the fluid velocity near the bottom.

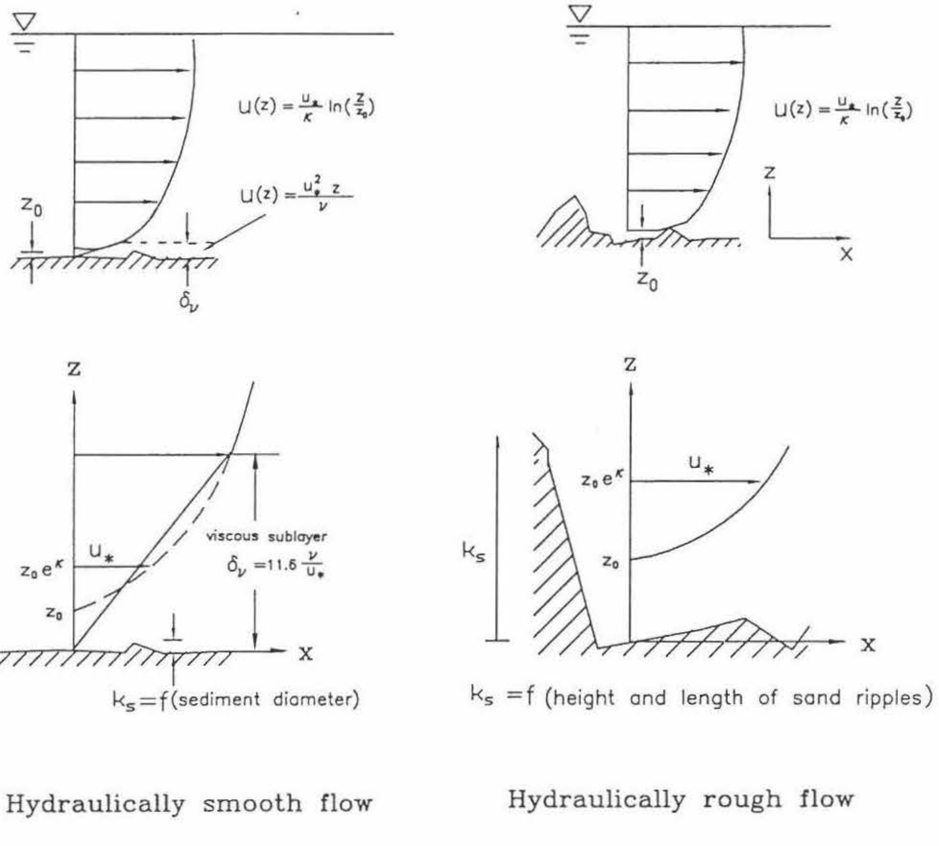


Fig.9. Illustration of the velocity profile in hydraulically smooth and rough flows.

Bed roughness

The bed roughness k_s is also called the equivalent Nikurase grain roughness, because it was originally introduced by Nikurase in his pipe flow experiments, where grains are glued to the smooth wall of the pipes.

The only situation where we can directly obtain the bed roughness is a flat bed consisting of uniform spheres, where k_s = diameter of sphere.

But in nature the bed is composed of grains with different size. Moreover, the bed is not flat, various bed forms, e.g. sand ripples or dunes, will appear depending on grain size and current. In that case the bed roughness can be obtained indirectly by the velocity measurement, as demonstrated by the following example.

Example 2 — Find the bed roughness from velocity measurement.

Given — Flume tests with water depth $h = 1m$, the measured velocities at the elevation of 0.1, 0.2, 0.4 and 0.6 m are 0.53, 0.58, 0.64 and 0.67 m/s, respectively.

Wanted — bed roughness k_s

Solution — The fitting of the measured velocities to the logarithmic velocity profile by the least square method gives $u_* = 0.03$ m/s and $z_0 = 0.0000825$ m. Hence,

$$k_s = z_0/0.033 = 0.0025m \qquad \tau_b = \rho\, u_*^2 = 0.9 \text{ N/m}^2$$

we can confirm that it is hydraulically rough flow by $u_* k_s/\nu = 75 > 70$

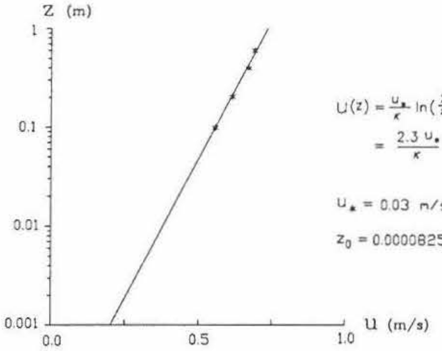


Fig.10. Fitting of the measured velocities to logarithmic velocity profile.

Comment — The logarithmic velocity profile suggests that the maximum velocity occurs at the flow surface. However, the measurements reveal that the maximum velocity occurs some distance under the flow surface due to the surface shear from the air. Moreover, the logarithmic velocity is basically developed for the logarithmic turbulent layer which is close to the bottom. Therefore, the velocity measurement in connection with the determination of τ_b and k_s is preferred to take place at the elevation $k_s < z < 0.2\,h$

The following k_s values have been suggested based on flume tests

Concrete bottom	$k_s = 0.001 - 0.01\ m$
Flat sand bed	$k_s = (1 - 10) \times d_{50}$
Bed with sand ripples	$k_s = (0.5 - 1) \times$ (height of sand ripple)

1.6 Chézy coefficient

Chézy proposed an empirical formula for the average velocity of steady uniform channel flow

$$U = C \sqrt{R\,S} \tag{18}$$

where R Hydraulic radius, i.e. area of cross section divided by wetted parameter

S Bed slope

C Empirical coefficient called Chézy coefficient. C was originally thought to be constant. Various formulas for C have been proposed

Here we will see that C can be theoretically determined by averaging the logarithmic velocity profile.

Recalling that the friction velocity is (eq (13))

$$u_* = \sqrt{g\,R\,S}$$

and applying it into eq (18), we get the expression of C

$$C = \frac{U}{u_*}\sqrt{g} \tag{19}$$

Averaging the logarithmic velocity profile gives

$$\begin{aligned} U &= \frac{1}{h}\int_{z_0}^{h} u(z)\,\mathrm{d}z = \frac{u_*}{\kappa\,h}\int_{z_0}^{h} \ln\left(\frac{z}{z_0}\right)\,\mathrm{d}z \\ &= \frac{u_*}{\kappa}\left(\ln\left(\frac{h}{z_0}\right) - 1 + \frac{z_0}{h}\right) \approx \frac{u_*}{\kappa}\ln\left(\frac{h}{z_0\,e}\right) \end{aligned} \tag{20}$$

Inserting the above equation into eq (19) gives

$$C = \frac{\sqrt{g}}{\kappa}\ln\left(\frac{h}{z_0\,e}\right) \tag{21}$$

$$\approx \begin{cases} 18\,\log\left(\frac{12\,h}{3.3\,\nu/u_*}\right) & \text{Hydraulically smooth flow} \quad \frac{u_*\,k_s}{\nu} \le 5 \\ 18\,\log\left(\frac{12\,h}{k_s}\right) & \text{Hydraulically rough flow} \quad \frac{u_*\,k_s}{\nu} \ge 70 \end{cases} \tag{22}$$

where the expression for z_0 has been used and Ln has been converted to Log. Moreover the inclusion of $g = 9.8\ m/s^2$ means that C has the unit $\sqrt{m}/s$.

Example 3 Chézy coefficient and bottom shear stress.

Given A project is to be located at the water depth h=5 m in Kattegat strait. The measured tidal current velocity is U=1 m/s (Havnecon a/s). The sediment size is d_{90}=0.15 mm (Danish Geotechnic Institute). It is estimated that the height of sand ripples is app. 10 cm.

Wanted 1) Bottom shear stress τ_b, when there are sand ripples with height of app. 10 cm on the bed

2) Bottom shear stress τ_b, if the flow is hydraulically smooth.

Solution 1) When there are sand ripples on the bed

Bed roughness $k_s \approx 0.75 \times \text{(height of sand ripple)} = 0.075\ m$

Chézy coefficient $C = 18 \log\left(\frac{12\ h}{k_s}\right) = 18 \log\left(\frac{12\times5}{0.075}\right) = 52.3\ \sqrt{m}/s$

Friction velocity $u_* = \frac{U}{C}\sqrt{g} = \frac{1}{52.3}\sqrt{9.8} = 0.06\ m/s$

Bed shear stress $\tau_b = \rho\ u_*^2 = 1000 \times 0.06^2 = 3.6\ N/m^2$

The flow is hydraulically rough as

$$\frac{u_*\ k_s}{\nu} = \frac{0.06 \times 0.075}{10^{-6}} = 4500 > 70$$

2) If we assume that the flow is hydraulically smooth (not the case in reality), we have

$$C = 18 \log\left(\frac{12\ h}{3.3\ \nu/u_*}\right)$$

By inserting $u_* = U\ \sqrt{g}/C$ into the above equation, we get

$$C = 18 \log\left(\frac{11.4\ U\ h}{\nu\ C}\right) = 18 \log\left(\frac{11.4 \times 1 \times 5}{10^{-6}\ C}\right) \quad \sqrt{m}/s$$

The solution of the equation is $C = 103\ \sqrt{m}/s$

Friction velocity $u_* = \frac{U}{C}\sqrt{g} = \frac{1}{103}\sqrt{9.8} = 0.03\ m/s$

Bed shear stress $\tau_b = \rho\ u_*^2 = 1000 \times 0.03^2 = 0.9\ N/m^2$

Comment The example shows that if we know only the average velocity, which is often the case, it is easier working on C.

For turbulent flow over ripple bed, The bottom shear stress obtained in the above example is consisted of skin friction shear stress τ_b' and form pressure of ripples τ_b''. It is τ_b' which drives grains as bed-load transport. More details will be given in the next chapter.

1.7 Drag coefficient, lift coefficient and friction coefficient

Drag and lift coefficients

A real fluid moving past a body will exert a drag force on the body, cf. Fig.11.

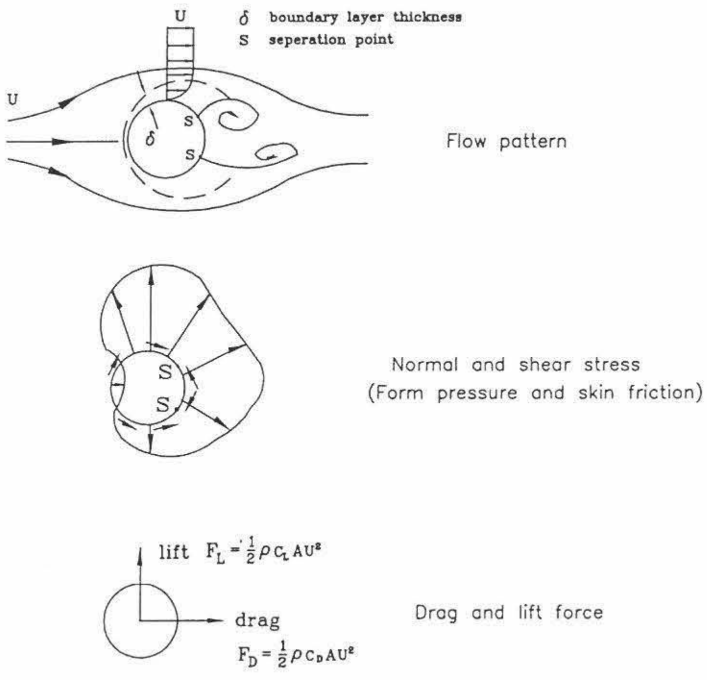


Fig.11. Drag force and lift force.

Drag force is consisted of friction drag and form drag, the former comes from the projection of skin friction force in the flow direction, and the latter from the projection of the form pressure force in the flow direction. The total drag is written as

$$F_D = \frac{1}{2}\,\rho\, C_D\, A\, U^2 \tag{23}$$

The lift force is written in the same way

$$F_L = \frac{1}{2}\,\rho\, C_L\, A\, U^2 \tag{24}$$

where A — Projected area of the body to the plane perpendicular to the flow direction.

C_D, C_L — Drag and lift coefficients, depend on the shape and surface roughness of the body and the Reynolds number. They are usually determined by experiments

Friction coefficient

Fig.12 illustrates fluid forces acting on a grain resting on the bed. The drag force

$$F_D = \frac{1}{2}\,\rho\,C_D\,A\,(\alpha\,U)^2$$

where α is included because we do not know the fluid velocity past the grain, but we can reasonably assume that it is the function of the average velocity and other parameters.

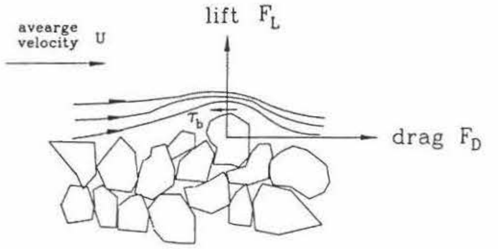


Fig.12. Fluid forces acting on a grain resting on the bed.

We can also say that the grain exerts a resistant force F_D on the flow. If A' is the projected area of the grain to the horizontal plane, the bottom shear stress is

$$\tau_b = \frac{F_D}{A'} = \frac{1}{2}\,\rho\,\left(C_D\,\alpha^2\,\frac{A}{A'}\right)U^2 = \frac{1}{2}\,\rho\,f\,U^2 \tag{25}$$

where f is the friction coefficient of the bed, which is a dimensionless parameter. By applying the Chézy coefficient we get

$$f = \frac{2\,g}{C^2} = \begin{cases} \dfrac{0.06}{\left(\log(\frac{12\,h}{3.3\,\nu/u_*})\right)^2} & \text{Hydraulically smooth flow} \quad \frac{u_*\,k_s}{\nu} \le 5 \\[2ex] \dfrac{0.06}{\left(\log(\frac{12\,h}{k_s})\right)^2} & \text{Hydraulically rough flow} \quad \frac{u_*\,k_s}{\nu} \ge 70 \end{cases} \tag{26}$$

Example 4 Calculate the bed friction coefficient in Example 3.

Solution The bed friction coefficient is

$$f = \frac{0.06}{\left(\log\left(\frac{12\,h}{k_s}\right)\right)^2} = \frac{0.06}{\left(\log\left(\frac{12\times5}{0.075}\right)\right)^2} = 0.0071$$

Therefore, the bed shear stress is

$$\tau_b = \frac{1}{2}\,\rho\,f\,U^2 = \frac{1}{2}\,1000\times0.0071\times1^2 = 3.6\;N/m^2$$

Comment f is preferred over C because f is non-dimensional.

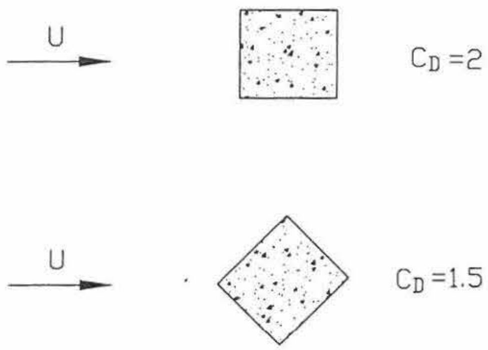
Darcy-Weisbach friction coefficient obtained in pipe flow is $f_{DW} = \frac{8\,g}{c^2}$

1.8 Exercise

1) A bridge across a river is supported by piers with a square cross section (length=width=B=1 m). The water depth is h=10 m. The velocity distribution in the river can be expressed by

$$u(z) = \frac{6\,U_*}{\kappa} \left(\frac{z}{h}\right)^{1/6}$$

where the friction velocity is $U_* = 0.05$ m/s.
The square piers can be placed in two ways with different drag coefficients, see the figure.



   1) Which placement of the pier gives minimum fluid force ?

   2) Calculate this minimum force.

2) A project is to be located at the water depth h=5 m in Kattegat strait. The measured tidal current velocity is U=1.5 m/s (Havnecon a/s). The sediment size is d_{90}=0.15 mm (Danish Geotechnic Institute). It is estimated that the height of sand ripples is app. 10 cm.

   1) Chézy coefficient C.

   2) friction coefficient f.

   3) bottom shear stress τ_b.

2 Sediment transport in open channels

2.1 Sediment properties

Density

The density of natural sediments is $\rho_s = 2650\ kg/m^3$. Therefore, the relative density is $s = \rho_s/\rho = 2.65$.

Size and shape of a grain

Generally grains are triaxial ellipsoids, having a long diameter d_a, intermediate diameter d_b and short diameter d_c. Corel shape factor gives the most useful description of the shape of a grain

$$S_{Corel} = \frac{d_c}{\sqrt{d_a\ d_b}} \tag{1}$$

For natural grains typically $S_{Corel} = 0.7$.

The diameter of a grain can be presented as

- d_s Sieve diameter, obtained by sieve analysis
- d_n Nominal diameter, which is the diameter of the sphere having the same volume and weight as the grain. $d_n \approx d_b$ and d_n is slightly larger than d_s.
- d_f Fall diameter, which is the diameter of the smooth sphere having the same fall velocity in still water at 24^0C as the grain. Fall diameter is the best description of the grain size, because it takes into account the grain shape.

Grain size distribution

The most useful and convenient method for the analysis of the grain size distribution is the sieve analysis, cf Fig.1. The median diameter of the sample is d_{50}, i.e. 50% of the grains by weight pass through.

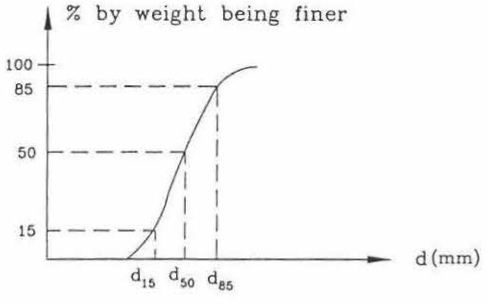


Fig.1. Grain size distribution.

Settling velocity

When a grain falls down in still water, it obtains a constant velocity when the upward fluid drag force on the grain is equal to the downward submerged weight of the grain. This constant velocity is defined as the settling velocity (fall velocity) of the grain.

Considering now the settling of a sphere with diameter d, cf. Fig.2

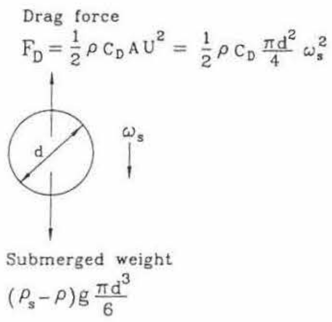


Fig.2. Settling of a sphere in still water.

The force balance between the drag force and the submerged weight gives

$$\frac{1}{2} \rho C_D \frac{\pi d^2}{4} \omega_s^2 = (\rho_s - \rho) g \frac{\pi d^3}{6}$$

Therefore the settling velocity of the sphere is

$$\omega_s = \sqrt{\frac{4 (s - 1) g d}{3 C_D}} \tag{2}$$

The drag coefficient of a sphere depends on the Reynolds number ($Re = \omega_s d/\nu$)

Laminar ($Re < 0.5$)	$C_D = \frac{24}{Re}$ (by theory)	$\omega_s = \frac{1}{18} (s-1) \frac{g d^2}{\nu}$
Turbulent ($Re > 10^3$)	$C_D \approx 0.4$ (by experiment)	$\omega_s = \sqrt{3 (s-1) g d}$

As to natural grains, Fredsøe et al.(1992) gives the empirical expression for the drag coefficient

$$C_D = 1.4 + \frac{36}{Re} \tag{3}$$

Inserting the equation into eq (2) and solving for ω_s give

$$\omega_s = \frac{\sqrt{\left(\frac{36 \nu}{d_n}\right)^2 + 7.5 (s-1) g d_n} - \frac{36 \nu}{d_n}}{2.8} \tag{4}$$

2.2 Threshold of sediment

Let us consider the steady flow over the bed composed of cohesionless grains. The forces acting on the grain is shown in Fig.3.

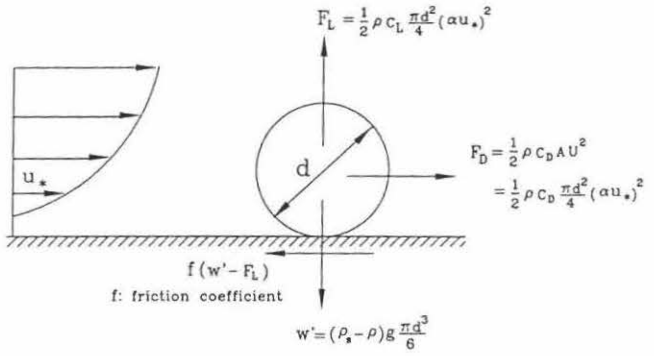


Fig.3. Forces acting on a grain resting on the bed.

The driving force is the flow drag force on the grain

$$F_D = \frac{1}{2}\,\rho\,C_D\,\frac{\pi\,d^2}{4}\,(\alpha\,u_*)^2$$

where the friction velocity u_* is the flow velocity close to the bed. α is a coefficient, used to modify u_* so that αu_* forms the characteristic flow velocity past the grain. The stabilizing force can be modelled as the friction force acting on the grain.

If $u_{*,c}$, critical friction velocity, denotes the situation where the grain is about to move, then the drag force is equal to the friction force, i.e.

$$\frac{1}{2}\,\rho\,C_D\,\frac{\pi\,d^2}{4}\,(\alpha\,u_{*,c})^2 = f\,\left((\rho_s - \rho)\,g\,\frac{\pi\,d^3}{6} - \frac{1}{2}\,\rho\,C_L\,\frac{\pi\,d^2}{4}\,(\alpha\,u_{*,c})^2\right)$$

which can be re-arranged into

$$\frac{u_{*,c}^2}{(s-1)\,g\,d} = \frac{f}{\alpha^2 C_D + f\,\alpha^2 C_L}\,\frac{4}{3\,\alpha^2}$$

Shields parameter is defined as

$$\theta = \frac{u_*^2}{(s-1)\,g\,d} \tag{5}$$

We say that sediment starts to move if

	$u_* > u_{*,c}$	critical friction velocity $u_{*,c}$
or	$\tau_b > \tau_{b,c}$	critical bottom shear stress $\tau_{b,c} = \rho\,u_{*,c}$
or	$\theta > \theta_c$	critical Shields parameter $\theta_c = \frac{u_{*,c}^2}{(s-1)\,g\,d}$

Fig.4 shows Shields experimental results which relate θ_c to the grain Reynolds number defined as

$$Pe = \frac{u_* d_n}{\nu} \tag{6}$$

The figure has 3 distinct zones corresponding to 3 flow situations

1) Hydraulically smooth flow for $Pe = \frac{u_* d_n}{\nu} \leq 2$

   d_n is much smaller than the thickness of viscous sublayer. Grains are embedded in the viscous sublayer and hence, θ_c is independent of the grain diameter. By experiments it is found $\theta_c = 0.1/Pe$

2) Hydraulically rough flow for $Pe \geq 500$

   The viscous sublayer does not exist and hence, θ_c is independent of the fluid viscosity. θ_c has a constant value of 0.06.

3) Hydraulically transitional flow for $2 \leq Pe \leq 500$

   Grain size is the same order as the thickness of the viscous sublayer. There is a minimum value of θ_c of 0.032 corresponding to $Pe = 10$.

Note that the flow classification is similar to that of the Nikurase pipe flow (Fig.8 of Chapter 1), where the bed roughness k_s is applied in stead of d_n.

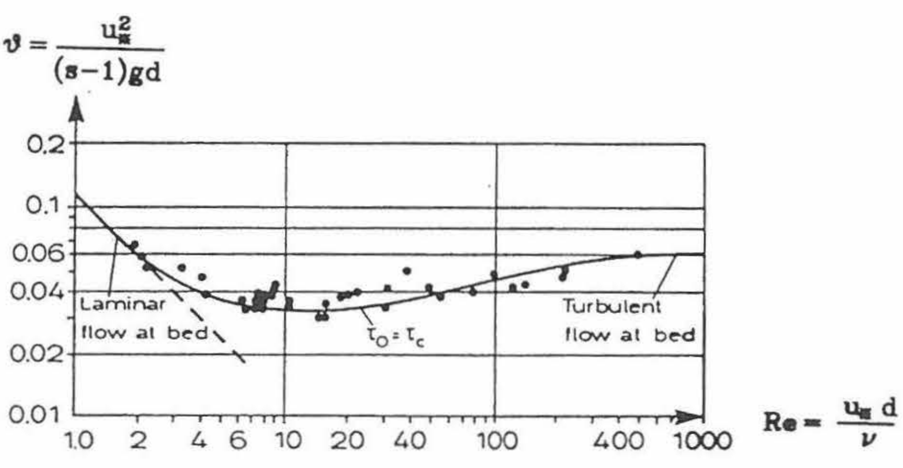


Fig.4. The Shields diagram giving θ_c as a function of Pe (uniform and cohesionless grain).[1]

[1]The critical Shields parameter of sand in air is $0.01 \leq \theta_c \leq 0.02$.

It is not convenient to apply the Shields diagram because the friction velocity u_* appears in both axes. Madsen et al. (1976) converted the Shields diagram into the diagram showing the relation between the critical Shields parameter θ_c and the so-called sediment-fluid parameter S_*

$$S_* = \frac{d \sqrt{(s-1)\, g\, d}}{4\, \nu} \tag{7}$$

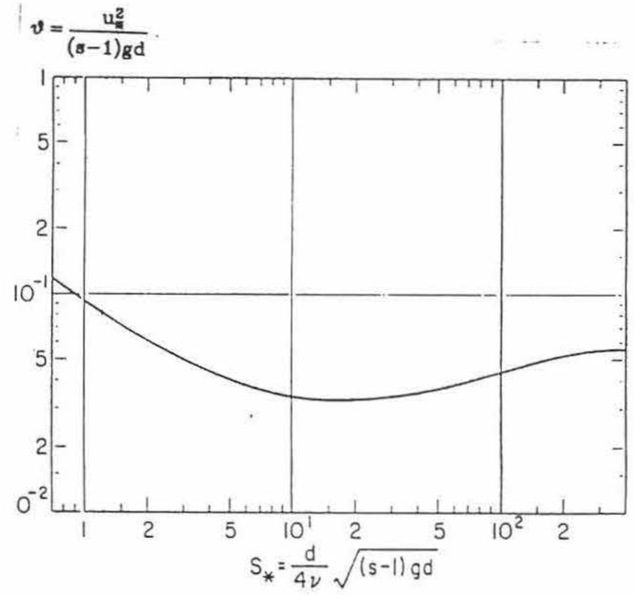


Fig.5. The Shields diagram giving θ_c as a function of S_*.

Example 1 Threshold of sediment

Given Sediment is quartz sand with $\rho_s = 2650\ kg/m^3$ and $d = 0.2mm$.
Fluid is sea water with $\rho = 1025\ kg/m^3$ and $\nu = 10^{-6}\ m^2/s$.

Wanted Critical shear stress $\tau_{b,c}$

Solution The relative density is $s = \rho_s/\rho = 2.59$
The sediment-fluid parameter is

$$S_* = \frac{d \sqrt{(s-1)\, g\, d}}{4\, \nu} = \frac{0.0002 \sqrt{(2.59-1) \times 9.8 \times 0.0002}}{4 \times 10^{-6}} = 2.79$$

From Fig.5 it is found $\theta_c = 0.052$, therefore,

$$u_{*,c} = \sqrt{\theta_c\, (s-1)\, g\, d} = 0.0127\ m/s$$

$$\tau_{b,c} = \rho\, u_{*,c}^2 = 0.165\ N/m^2$$

2.3 Bedforms, bed roughness and effective shear stress

Bedforms

Once sediment starts to move, various bedforms occur. In laboratory flumes the sequence of bedforms with increasing flow intensity is

Flat bed ⇒ Ripples ⇒ Dunes ⇒ High stage flat bed ⇒ Antidunes

Ripples — Ripples are formed at relatively weak flow intensity and are linked with fine materials, with d_{50} less than 0.7 mm.
The size of ripples is mainly controlled by grain size. By observations the typical height and length of ripples are

$$H_r \approx 100d_{50} \qquad L_r \approx 1000d_{50}$$

At low flow intensity the ripples have a fairly regular form with an upstream slope 6^0 and downstream slope 32^0. With the increase of flow intensity, ripples become three dimensional.

Dunes — The shape of dunes is very similar to that of ripples, but it is much larger. The size of dunes is mainly controlled by flow depth. Dunes are linked with coarse grains, with d_{50} bigger than 0.6 mm.
With the increase of flow intensity, dunes grow up, and the water depth at the crest of dunes becomes smaller. It means a fairly high velocity at the crest, dunes will be washed-out and the high stage flat bed is formed.

Antidunes — When Froude number exceeds unity antidunes occur. The wave height on the water surface is the same order as the antidune height. The surface wave is unstable and can grow and break in an upstream direction, which moves the antidunes upstream.

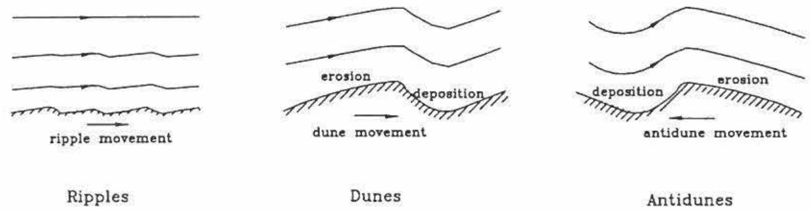


Fig.6. Illustration of flow over ripples, dunes and antidunes, and their movement.

If we know the average velocity of the current, water depth and sediment size, the bed forms can be predicted by empirical diagrams, e.g. the one by Znamenskaya (1969), cf. Fig.7, where the sediment size is represented by the fall velocity of the sediment (ω_s). The ripples speed (c) is also given so that the figure can be used to estimate the bed-load transport.

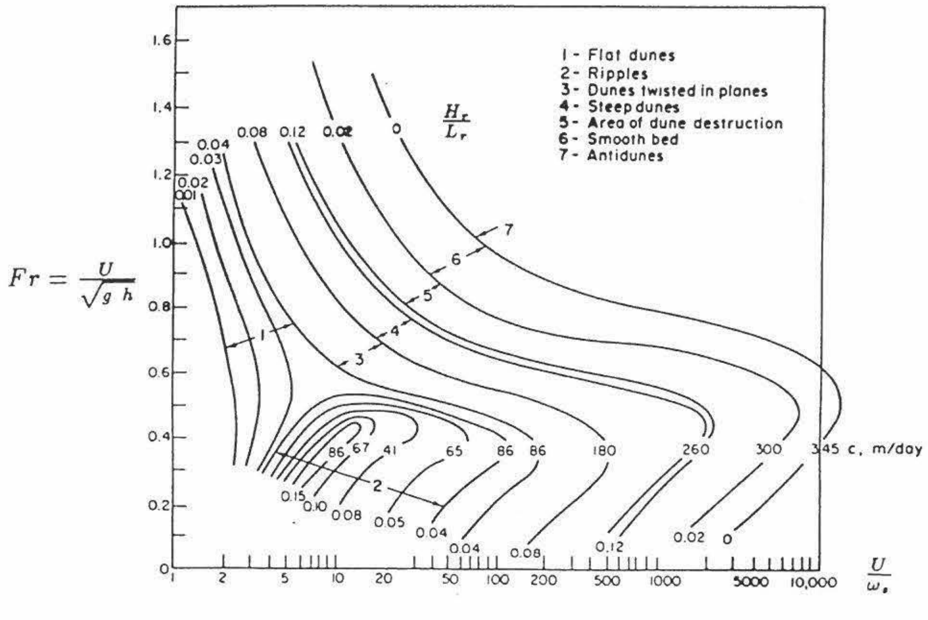


Fig.7. Bed forms given by Znamenskaya (taken from Raudkivi, 1976).

<u>*Bed roughness*</u>

The bed roughness k_s is also called the equivalent Nikurase grain roughness, because it was originally introduced by Nikurase in his pipe flow experiments, where grains are glued to the smooth wall of the pipes.

The only situation where we can directly obtain the bed roughness is a flat bed consisting of uniform spheres, where k_s = diameter of sphere.

Generally the bed roughness can be obtained indirectly by the velocity measurement, as demonstrated by Example 2 in Section 1.5.

The large collection of bed roughness values, obtained by velocity measurement and fitting, covering various flow regions with different sediment size, shows

$$k_s \approx \begin{cases} (1 - 10)d_{50} & \text{flat bed} \\ 100d_{50} = H_r & \text{rippled bed} \end{cases} \tag{8}$$

Effective shear stress

In the presence of ripples, the resistance to the flow consists of two parts, one originating from the skin friction, another due to the form pressure of the ripples, i.e.

$$\tau_b \;=\; \tau_b' \;+\; \tau_b'' \tag{9}$$

where τ_b' is also called effective shear stress, because it is τ_b' which is acting on single sediment.

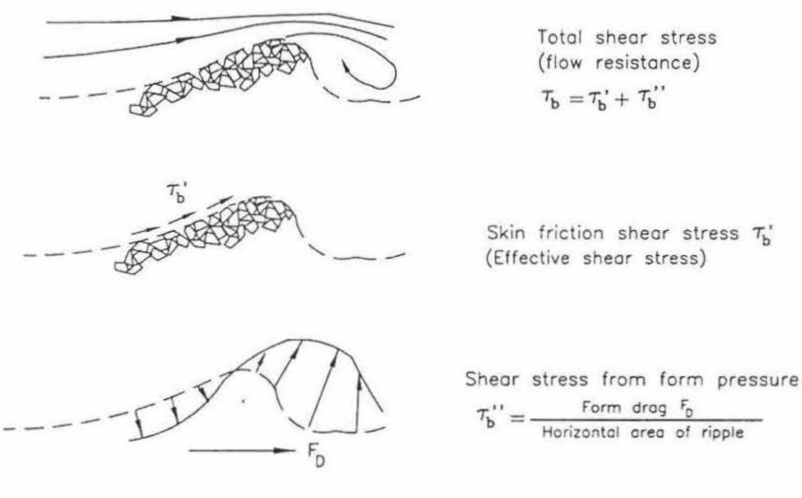


Fig.8. The resistance to flow over a rippled bed.

In the case of flat bed, $\tau_b'' = 0$, and the bed roughness is usually taken as $2.5d_{50}$, the effective shear stress is[2]

$$\tau_b' \;=\; \tau_b \;=\; \frac{1}{2}\,\rho\, f\, U^2 \;=\; \frac{1}{2}\,\rho\,\left(\frac{0.06}{\left(\,\log\left(\frac{12\,h}{2.5\,d_{50}}\right)\,\right)^2}\right)\,U^2 \tag{10}$$

where h is water depth and U current average velocity. In the case of a rippled bed, τ_b' is the same as above, but the total stress is larger due to form pressure.

$$\tau_b \;=\; \frac{1}{2}\,\rho\,\left(\frac{0.06}{\left(\,\log\left(\frac{12\,h}{H_r}\right)\,\right)^2}\right)\,U^2 \tag{11}$$

where the bed roughness is assumed equal to the height of ripples (H_r).

The distinction between τ_b and τ_b' explains the phenomenon that with the appearance of rippled bed, and hence the increase of τ_b, the bed-load transport does not increase.

[2]Assume the flow is always hydraulically rough.

2.4 Transport modes

There are three sediment transport modes

Wash load — very fine particles which are transported by the water, but these particles do not exist on the bed. Therefore the knowledge of bed material composition does not permit any prediction of wash load transport. Hence, wash load will not be considered in this book.

Bed-load — the part of the total load which has more or less continuous contact with the bed. Thus the bed load must be determined in relation to the effective shear stress which acts directly on the grain surface.

Suspended load — the part of the total load which is moving without continuous contact with the bed as the result of the agitation of the fluid turbulence. The appearance of ripples will increase the bed shear stress (flow resistance). On the other hand, more grains will be suspended due to the flow separation on the lee side of the ripples, cf. Fig.8. Thus the suspended load is related to the total bed shear stress.

The basic idea of splitting the total sediment load into bed-load and suspended load is that, as described above, two different mechanisms are effective during the transport.

As to the boundary between the bed-load and the suspended load, argument is still going on. Einstein (1950) suggests the boundary to be some grain diameters, typically $2d_{50}$, above the bed. But this is not realistic when the bed is rippled, which is almost always the case. Therefore Bijker (1971) proposed that the bed-load transport takes place inside a layer with a thickness being equal to the bed roughness (height of ripples).

The SI unit for sediment transport is $\frac{m^3}{m*s}$, readed *cubic meter of sediment per meter width per second.*

Moreover, only cohesionless sediment will be treated in this book.

2.5 Bed-load transport formulae

Bed-load transport q_B is often expressed in the dimensionless form

$$\Phi_B = \frac{q_B}{d\sqrt{(s-1)\,g\,d}} \tag{12}$$

Kalinske-Frijlink formula

Kalinske-Frijlink (1952) formula is a curve fitted to all data available at that time

$$q_B = 2\,d_{50}\,\sqrt{\frac{\tau_b}{\rho}}\,exp\left(\frac{-0.27\,(s-1)\,d_{50}\,\rho\,g}{\tau_b'}\right) \tag{13}$$

where τ_b and τ_b' are bottom shear stress and effective shear stress, respectively.

Meyer-Peter formula

The fitting of large amount of experimental data by Meyer-Peter (1948) gives

$$\Phi_B = 8\,(\theta' - \theta_c)^{1.5} \tag{14}$$

where θ' effective Shields parameter $\theta' = \frac{\tau_b'/\rho}{(s-1)\,g\,d}$

τ_b' effective shear stress

θ_c critical Shields parameter

Einstein-Brown formula

The principle of Einstein's analysis is as follows: the number of deposited grains in a unit area depends on the number of grains in motion and the probability that the hydrodynamic forces permit the grains to deposit. The number of eroded grains in the same unit area depends on the number of grains in that area and the probability that the hydrodynamic forces are strong enough to move them. For equilibrium conditions the number of grains deposited must be equal to the number of grains eroded, which, together with experimental data fitting, gives

$$\Phi_B = 40\,K\,(\theta')^3 \tag{15}$$

$$K = \sqrt{\frac{2}{3} + \frac{36\nu^2}{(s-1)\,g\,d_{50}^3}} - \sqrt{\frac{36\nu^2}{(s-1)\,g\,d_{50}^3}}$$

Bagnold formula

Bagnold proposed a formula based on the work done by current. The formula has the same form as the modified Meyer-Peter formula.

Example 2 Bed-load transport

Given A river with

sea water	$\rho = 1025\ kg/m^3$	$\nu = 10^{-6}\ m^2/s$
flow	$U = 1\ m/s$	h=2 m
sediment	$\rho_s = 2650\ kg/m^3$	$d_{50} = 0.2mm.$

Wanted Bed-load transport q_B

Solution 1) critical Shields parameter

The relative density is $s = \rho_s/\rho = 2.59$
The sediment-fluid parameter is

$$S_* = \frac{d_{50}\sqrt{(s-1)\ g\ d_{50}}}{4\ \nu} = \frac{0.0002\sqrt{(2.59-1)\times 9.8\times 0.0002}}{4\times 10^{-6}} = 2.79$$

From Fig.5 it is found the critical Shields parameter is $\theta_c = 0.052$.

2) Effective Shields parameter

The effective shear stress is

$$\tau_b' = \frac{1}{2}\rho\left(\frac{0.06}{\left(\log\left(\frac{12\ h}{2.5\ d_{50}}\right)\right)^2}\right) U^2 = 1.40\ N/m^2$$

The effective Shields parameter is $\theta' = \frac{\tau_b'/\rho}{(s-1)\ g\ d_{50}} = 0.44$

As we do not have information on ripple height, we take $H_r = 100\ d_{50} = 0.02\ m$, the bottom shear stress is

$$\tau_b = \frac{1}{2}\rho\left(\frac{0.06}{\left(\log\left(\frac{12\ h}{H_r}\right)\right)^2}\right) U^2 = 3.24\ N/m^2$$

The coefficient in the Einstein-Brown formula is

$$K = \sqrt{\frac{2}{3} + \frac{36\nu^2}{(s-1)\ g\ d_{50}^3}} - \sqrt{\frac{36\nu^2}{(s-1)\ g\ d_{50}^3}} = 0.44$$

3) Calculate q_B by formulae

formula	Kalinske-Frijlink	Meyer-Peter	Einstein-Brown
$q_B\ (m^3/(m*s))$	0.0000121	0.0000215	0.0000167

Comment The total bed-load transport in the river depends on the width of the river.

When the accuracy of sediment transport formulae is concerned, experts say that if a formula gives the correct order of magnitude, it is a good formula.

It is not surprising that the formulae give more or less the same result, because all formulae include parameters to be determined by the fitting of experimental results.

2.6 Suspended load

Sediment concentration in a steady current

Consider a steady flow in a open channel. The sediment is kept in suspension by turbulent fluctuations. Sediment concentration c has the unit m^3/m^3, i.e. the volume of sediments in 1 cubic meter water. The classical approach to calculate the vertical distribution of suspended sediment is to apply Prandtl's mixing length theory, cf. Fig.9

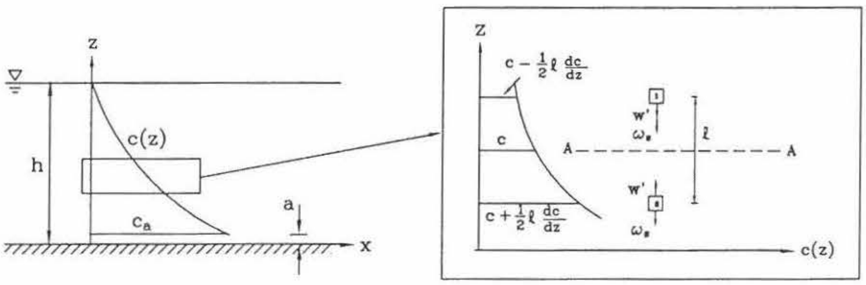


Fig.9. Suspended sediment in steady turbulent flow.

Consider a uniform sand with a settling velocity ω_s. In a unit time, through a unit area on the horizontal plan A-A, the volume of sediment travelling upward and downward are

$$q_u = (w' - \omega_s)\left(c - \frac{1}{2}\ell\frac{dc}{dz}\right)$$

$$q_d = (w' + \omega_s)\left(c + \frac{1}{2}\ell\frac{dc}{dz}\right)$$

In a steady situation, q_u and q_d must be equal to each other, which gives

$$c\,\omega_s + \frac{1}{2}\,w'\,\ell\,\frac{dc}{dz} = 0 \qquad (16)$$

By assuming that

$$\frac{1}{2}\,w'\,\ell = \kappa\,u_*\,z\left(1 - \frac{z}{h}\right)$$

where $\kappa = 0.4$ and u_* is the friction velocity, we get

$$c\,\omega_s + \kappa\,u_*\,z\left(1 - \frac{z}{h}\right)\frac{dc}{dz} = 0 \qquad (17)$$

which is integrated with the integration constant given by $c|_{z=a} = c_a$

$$c(z) = c_a\left(\frac{h-z}{z}\,\frac{a}{h-a}\right)^{\left(\frac{\omega_s}{\kappa\,u_*}\right)} \qquad (18)$$

Reference elevation and reference sediment concentration

a and c_a in eq (18) are called reference elevation and reference sediment concentration, respectively.

The reference elevation a is the boundary between the bed load and the suspended load. Bijker (1992) suggests that a is taken as the bed roughness k_s and relates c_a to the bed-load transport q_B.

It is assumed that bed-load transport takes place in the bed-load layer from $z = 0$ to $z = a = k_s$, and in the bed-load layer there is a constant sediment concentration c_a. Fig.10 shows the velocity profile applied by Bijker. He argues that in hydraulically rough flow there is still a viscous sublayer, which starts from $z = 0$ to $z = z_0\, e$ where the linear velocity distribution is tangent with the logarithmic velocity distribution. Note the thickness of the viscous sublayer is much smaller than that in hydraulically smooth flow (Fig.9 in Chapter 1).

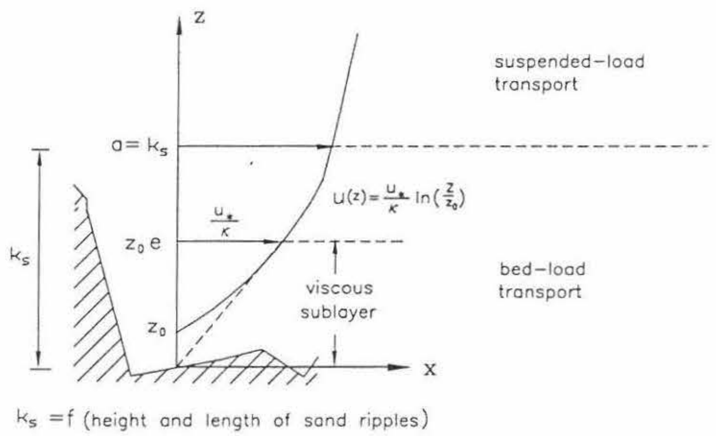


Fig.10. Viscous sublayer in hydraulically rough flow.

By the logarithmic velocity profile we get

$$u\big|_{z=z_0 e} = u_*/\kappa$$

The averaged velocity in the bed-load layer is

$$U_b = \frac{1}{k_s}\left(\frac{1}{2}\,\frac{u_*}{\kappa}\,Z_0\,e + \int_{z_0 e}^{k_s}\frac{u_*}{\kappa}\,\ln\left(\frac{z}{z_0}\right)\,\mathrm{d}z\right) \approx 6.34\,u_*$$

therefore, the bed-load transport is

$$q_B = U_b\,k_s\,c_a$$

hence we obtain the reference sediment concentration

$$c_a = \frac{q_B}{U_b\,k_s} = \frac{q_B}{6.34\,u_*\,k_s} \qquad (19)$$

Suspended sediment transport

Now we know the vertical distribution of both the suspended sediment concentration and the fluid velocity, cf. Fig.11.

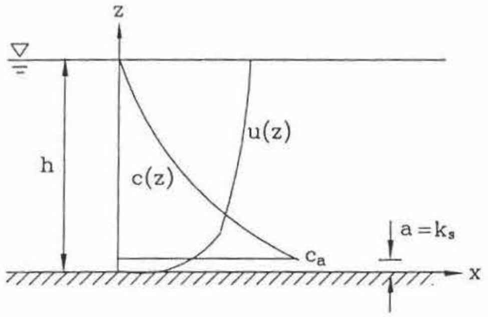


Fig.11. Illustration of vertical distribution of c and u.

the suspended sediment transport can be calculated as

$$
\begin{aligned}
q_s &= \int_a^h u(z)\, c(z)\, \mathrm{d}z \\
&= \int_a^h \left(\frac{u_*}{\kappa} \ln\left(\frac{z}{z_0}\right) c_a \left(\frac{h-z}{z}\,\frac{a}{h-a}\right)^{\left(\frac{\omega_s}{\kappa\, u_*}\right)} \right) \mathrm{d}z \\
&= 11.6\, u_*\, c_a\, a \left(I_1 \ln\left(\frac{h}{0.033\, k_s}\right) + I_2 \right)
\end{aligned}
$$

where I_1 and I_2 are Einstein integrals given by

$$
I_1 = 0.216 \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \mathrm{d}B
$$

$$
I_2 = 0.216 \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \ln B\, \mathrm{d}B
$$

where $A = \frac{k_s}{h}$ $\quad B = \frac{z}{h}$ $\quad z_* = \frac{\omega_s}{\kappa\, u_*}$

By applying Bijker's recommendation on a and c_a, we get

$$
q_s = 1.83\, q_B \left(I_1 \ln\left(\frac{h}{0.033\, k_s}\right) + I_2 \right) \tag{20}
$$

Example 3 Suspended sediment transport

Given As in Example 2, i.e.)

sea water	$\rho = 1025\ kg/m^3$	$\nu = 10^{-6}\ m^2/s$
flow	$U = 1\ m/s$	h=2 m
sediment	$\rho_s = 2650\ kg/m^3$	$d_{50} = 0.2mm$.

Wanted Suspended sediment transport q_S

Solution In Example 2 by Meyer-Peter formula we get $q_B = 0.0000215\ \frac{m^3}{m*s}$, and by assuming $H_r = k_s = 100\ d_{50} = 0.02\ m$, we obtain $\tau_b = 3.24\ N/m^2$

The fall velocity

$$\omega_s = \frac{\sqrt{\left(\frac{36\ \nu}{d_{50}}\right)^2 + 7.5\ (s-1)\ g\ d_{50}} - \frac{36\ \nu}{d_{50}}}{2.8} = 0.02\ m/s$$

The friction velocity $u_* = \sqrt{\frac{\tau_b}{\rho}} = 0.056\ m/s$

Therefore $A = \frac{H_r}{h} = 0.01 \qquad z_* = \frac{\omega_s}{\kappa\ u_*} = 0.89$

and we get the Einstein integrals by numerical integration

$$I_1 = 0.216\ \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \mathrm{d}B = 1.00$$

$$I_2 = 0.216\ \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \ln B\ \mathrm{d}B = -2.50$$

The suspended sediment transport is

$$\begin{aligned} q_S &= 1.83\ q_B \left(I_1\ \ln\left(\frac{h}{0.033\ k_s}\right) + I_2\right) \\ &= 1.83 \times 0.0000215 \left(1.00 \times \ln\left(\frac{2}{0.033 \times 0.02}\right) - 2.50\right) \\ &= 0.000217\ \frac{m^3}{m*s} \end{aligned}$$

The ratio between the bed-load and suspended transport is

$$Q = \frac{q_S}{q_B} = 1.83 \left(I_1\ \ln\left(\frac{h}{0.033\ k_s}\right) + I_2\right) = 10$$

2.7 Total sediment transport

There are numerous formulae, cf. Raudkivi (1976). Two of them are

Bijker $$q_T = q_B + q_S = q_B \left(1 + 1.83 \left(I_1 \ln\left(\frac{h}{0.033\ k_s}\right) + I_2\right)\right)$$

Engelund $$q_T = 0.05\ U^2 \sqrt{\frac{d_{50}}{(s-1)\ g}} \left(\frac{\tau_b}{(\rho_s-\rho)\ g\ d_{50}}\right)^{1.5}$$

Example 4 Total sediment transport

Given As in Examples 2 & 3, i.e.)

sea water	$\rho = 1025\ kg/m^3$	$\nu = 10^{-6}\ m^2/s$
flow	$U = 1\ m/s$	h=2 m
sediment	$\rho_s = 2650\ kg/m^3$	$d_{50} = 0.2mm.$

Wanted Total sediment transport q_T

Solution In Examples 2 & 3 we obtained

$$\tau_b = 3.24\ N/m^2$$

$$q_B = 0.0000215\ \frac{m^3}{m*s}$$

$$q_S = 0.000217\ \frac{m^3}{m*s}$$

The total sediment transport

Bijker $$q_T = q_B + q_S = 0.0000215 + 0.000217$$

$$= 0.000239\ \frac{m^3}{m*s}$$

Engelund $$q_T = 0.05\ U^2 \sqrt{\frac{d_{50}}{(s-1)\ g}} \left(\frac{\tau_b}{(\rho_s-\rho)\ g\ d_{50}}\right)^{1.5}$$

$$= 0.05 \times 1^2 \sqrt{\frac{0.0002}{(2.59-1)\times 9.81}} \left(\frac{3.24}{(2650-1025)\times 9.81\times 0.0002}\right)^{1.5}$$

$$= 0.000179\ \frac{m^3}{m*s}$$

2.8 Exercise

1) A river with

sea water	$\rho = 1025\ kg/m^3$	$\nu = 10^{-6}\ m^2/s$
flow	$U = 1.5\ m/s$	h=3 m
sediment	$\rho_s = 2650\ kg/m^3$	$d_{50} = 0.2mm.$

1) threshold of sediment

2) bed roughness

3) bed-load transport

4) suspended-load transport

5) total transport

3 Wave boundary layer

This chapter is written with a view to coastal sediment transport. The main outcome is the bottom shear stress of sea bed.

3.1 Concept of wave boundary layer

Linear wave theory gives the amplitude of water particle oscillation on the bottom

$$A = \frac{H}{2} \frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} \tag{1}$$

and the horizontal velocity of water particle

$$u_b = U_m \sin(\omega t) \tag{2}$$

where U_m is the maximum horizontal velocity

$$U_m = \frac{\pi\,H}{T} \frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = A\,\omega \tag{3}$$

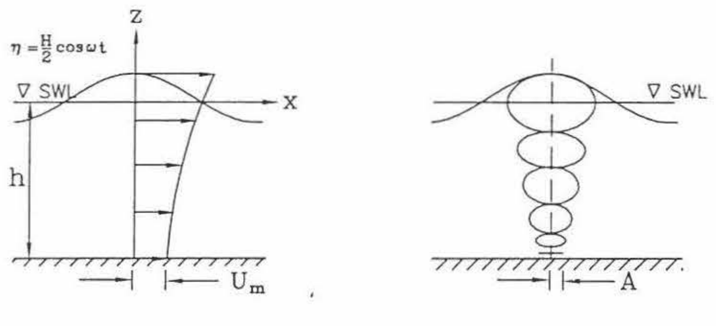


Fig.1. Horizontal velocity profile and water particle orbit by linear wave theory.

Linear wave theory assumes that the fluid is ideal (no viscosity), so there is water particle movement on the bottom, which is not the case in reality. Unfortunately, for the study of sediment transport, the flow pattern close to the bottom is of great interest. To overcome this contradiction, the concept of wave boundary layer is introduced.

Prandtl developed the concept of fluid boundary layer in general: *For fluids having small viscosity, the effect of internal friction in the flow is appreciable only in a thin layer surrounding the flow boundaries.* Under wave action, this thin layer is called wave boundary layer.

3.2 Laminar wave boundary layer on smooth bed

First we will get some impression of wave boundary layer by looking at the laminar wave boundary layer on smooth bed, which can be described theoretically.

Oscillating water tunnel

Lundgren and Sørensen (1956) invented the oscillating tunnel to model the wave boundary layer, cf. Fig.2. Note that the piston movement is the same as the water particle on the sea bed given by linear wave theory.

The flow in the test section is horizontal and uniform. The thickness of the boundary layer changes with time, but remains very thin due to oscillation. Outside the boundary layer, the flow is undisturbed.

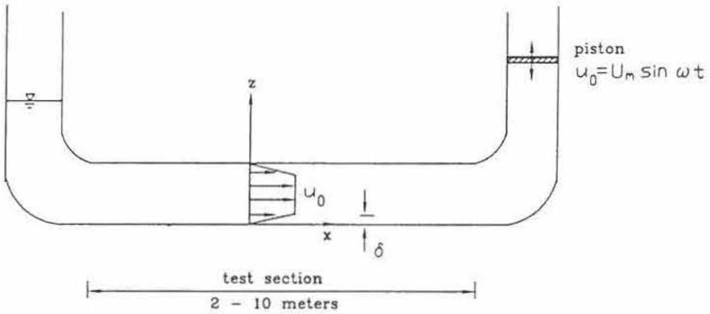


Fig.2. Oscillating water tunnel.

Formulation of equation of motion

We start from the Navier-Stokes equation in the horizontal direction

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial z} + \frac{\partial \tau}{\partial z} \tag{4}$$

The flow in the test section is horizontal ($w = 0$) and uniform ($\frac{\partial u}{\partial x} = 0$)

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \frac{\partial \tau}{\partial z} \tag{5}$$

Outside the boundary layer we have $u = u_0$ and $\tau = 0$, therefore

$$\rho \frac{\partial u_0}{\partial t} = -\frac{\partial p}{\partial z} \tag{6}$$

which, minus eq (5), gives

$$\rho \frac{\partial (u - u_0)}{\partial t} = \frac{\partial \tau}{\partial z} \tag{7}$$

Because the flow is laminar, shear stress can be expressed by Newton's law of viscosity

$$\tau = \rho \, \nu \frac{\partial u}{\partial z} \tag{8}$$

we get the equation of motion

$$\frac{\partial (u - u_0)}{\partial t} = \nu \, \frac{\partial^2 u}{\partial^2 z} \tag{9}$$

The boundary condition is

$$u|_{z=0} = 0 \qquad u|_{z=\infty} = U_m \, \sin(\omega t) \tag{10}$$

Velocity profile and bottom shear stress

The solution of eq (9) is

$$u = U_m \, \sin(\omega t) - U_m \, \exp\left(-\frac{z}{\sqrt{2\nu/\omega}}\right) \, \sin\left(\omega t - \frac{z}{\sqrt{2\nu/\omega}}\right) \tag{11}$$

The second term is a dampened wave, which decays quickly away from the bed, cf. Fig.3.

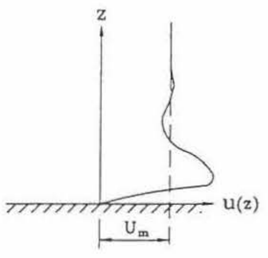


Fig.3. The local velocity amplitude oscillating around U_m.

The bottom shear stress is given by

$$\tau_b = \rho \, \nu \frac{\partial u}{\partial z}\Big|_{z=0} = \frac{\rho \, \nu \, U_m}{\sqrt{2\nu/\omega}} \left[\sin(\omega t) + \cos(\omega t)\right]$$

$$= \tau_{b,max} \, \sin(\omega t + 45^0)$$

$$\text{where} \qquad \tau_{b,max} = \frac{\rho \, \nu \, U_m}{\sqrt{\nu/\omega}}$$

As the free stream velocity is $u_0 = U_m \sin(\omega t)$, we can see that the phase shift between τ_b and u_0 is 45^0, i.e. $\tau_{b,max}$ will appear ahead of U_m by 45^0.

3.3 Wave boundary layer thickness

With the inclusion of the wave boundary layer, the velocity profile corresponding to wave crest is shown in Fig.4.

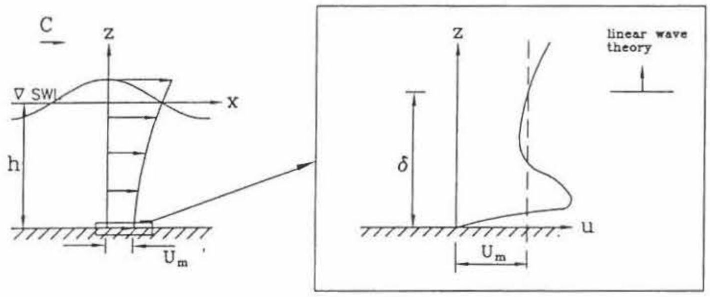


Fig.4. wave boundary layer (laminar flow on smooth bed).

The boundary layer thickness depends on how we define the top of the boundary layer. Jonsson (1966) defined the top of the boundary layer as the minimum elevation where the velocity amplitude is equal to U_m, which, by eq (11), gives

$$\delta_j = \frac{\pi}{2}\sqrt{\frac{2\,\nu}{\omega}} \tag{12}$$

Sleath (1987) defined the top of the boundary layer as the elevation where the velocity amplitude is 95% of U_m, i.e. 5% relative difference, which gives the boundary layer thickness

$$\delta_{0.05} = 3\sqrt{\frac{2\,\nu}{\omega}} \tag{13}$$

In reality, the flow type is turbulent flow on rippled bed, the boundary layer thickness is affected by bed roughness, cf. Fig.5. Sleath (1987) gives the empirical formula

$$\frac{\delta_{0.05}}{k_s} = 0.26\left(\frac{A}{k_s}\right)^{0.70} \tag{14}$$

where k_s is the bed roughness and A the amplitude of water particle oscillation on the bottom.

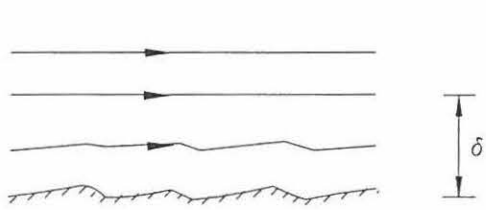


Fig.5. wave boundary layer thickness over rippled bed.

Example 1 Wave boundary layer thickness

Given Wave height H=2 m, wave length L= 80 m
Water depth h=5 m, sea bed ripple height H_r=15 cm

Wanted Wave boundary layer thickness $\delta_{0.05}$

Solution By linear wave theory the amplitude of the water particle on the bottom

$$A = \frac{H}{2} \frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = 2.48 \ m$$

The bed roughness is taken as the ripple height $k_s = H_r$ =0.15 m, then

$$\delta_{0.05} = 0.26 \left(\frac{A}{k_s}\right)^{0.70} k_s = 0.28 \ m$$

Comment Boundary layer thickness at one location varies with time. $\delta_{0.05} = 0.28\,m$ is the one when wave crest passes the location (maximum boundary layer thickness).

In Section 1.1 we have shown that the boundary layer fulfils the whole flow depth in channel flow. However, the wave boundary layer will remains thin due to the oscillation of water particles. Let us imagine a progressive wave with a period of 8 seconds. First water particles close to the bottom move forward, wave boundary layer is developing, but the development is stopped after 4 seconds, because the water particles stop and start to move backward, and a new boundary layer starts to develop.

3.4 Wave friction coefficient

Definition of wave friction coefficient

The current friction coefficient f_c is defined as

$$\tau_b = \frac{1}{2} \rho f_c U^2 \tag{15}$$

As to sea bed, the bottom shear stress varies with time. Jonsson (1966) defined the wave friction coefficient f_w as

$$\tau_{b,max} = \frac{1}{2} \rho f_w U_m^2 \tag{16}$$

where U_m is the maximum horizontal velocity of water particle on sea bed, given by the Airy wave theory. f_w is a fictional coefficient because $\tau_{b,max}$ and U_m do not occur at the same time.

Wave friction velocity is defined as

$$u_{*,w} = \sqrt{\frac{\tau_{b,max}}{\rho}} = \sqrt{\frac{f_w}{2}} U_m \tag{17}$$

f_w: Laminar boundary layer and smooth bed

The theoretical expression of $\tau_{b,max}$ for laminar boundary layer on smooth bed is

$$\tau_{b,max} = \frac{\rho \nu U_m}{\sqrt{\nu/\omega}}$$

By comparison with eq (16), we get

$$f_w = \frac{2}{U_m} \frac{\nu}{\sqrt{\frac{2\nu}{\omega}}} = \frac{2}{A\omega} \frac{\nu}{\sqrt{\frac{2\nu}{\omega}}} = 2\left(\frac{\nu}{A^2 \omega}\right)^{0.5}$$

By observation it is found that laminar boundary layer on smooth bed corresponds to $A^2\omega/\nu < 3 \times 10^5$

f_w: Turbulent boundary layer and smooth bed

Justesen (1988) suggests

$$f_w = 0.024 \left(\frac{\nu}{A^2 \omega}\right)^{-0.123} \qquad \text{for } 10^6 < \frac{A^2 \omega}{\nu} < 10^8 \tag{18}$$

f_w: Turbulent boundary layer and rough bed

In reality the flow is always hydraulically turbulent over rough bed. Jonsson (1966) gives an implicit empirical formula for f_w, which is approximated by Swart (1974) in an explicit form

$$f_w = exp\left(5.213\left(\frac{k_s}{A}\right)^{0.194} - 5.977\right) \tag{19}$$

Nielsen (1992) means that Swart formula tends to overpredict f_w for small k_s/A, cf. Fig.6. The new fitting gives

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) \tag{20}$$

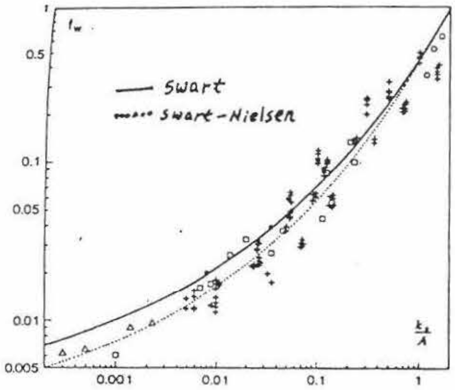


Fig.6. Observed wave friction coefficient.

Example 2 Wave friction coefficient f_w

Given Wave height H=2 m, wave length L= 80 m
Water depth h=5 m, sea bed ripple height H_r=15 cm

Wanted Wave friction coefficient f_w

Solution By linear wave theory the amplitude of the water particle on the bottom

$$A = \frac{H}{2}\frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = 2.48\ m$$

The bed roughness is taken as the ripple height $k_s = H_r$ =0.15 m, then

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) = 0.050$$

Comment Current friction coefficient is

$$f_c = \frac{0.06}{\left(\log\left(\frac{12\,h}{k_s}\right)\right)^2} = 0.00886$$

3.5 Mechanism of sediment transport in coastal regions

The mechanism of coastal sediment transport is: *Wave stirs up sediment and current transports the sediment.*

We have shown that wave boundary layer remains thin and current boundary layer fulfils the whole flow depth. In the coexistence of wave and current, even if the current velocity is much larger than the wave-induced velocity near the water surface, the wave-induced velocity will dominate the situation close to the bottom, cf. Fig.7.

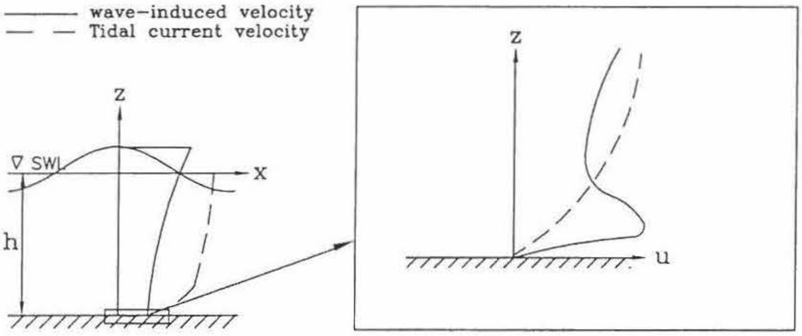


Fig.7. Comparison of current and wave velocity profiles.

However, because of the *one step forwards and one step backwards* nature of water particle movement due to waves, current will usually be the main transporter of the sediments stirred up by the waves, except for wave breaking zones, where a longshore current is produced due to wave breaking.

Example 3	Bed shear stress by currents and waves respectively
Given	As in Example 2, i.e. Wave height H=2 m, wave length L= 80 m Water depth h=5 m, sea bed ripple height H_r=15 cm Current velocity U= 1 m/s
Wanted	Bottom shear stress by current and wave respectively.
Solution	In Example 2 we have obtained $f_w = 0.050$, $f_c = 0.00886$ and $A = 3.48\ m$ By linear wave theory we get $T \approx 12\ s$ and $U_m = A\omega = 1.3\ m/s$

current $\tau_c = \frac{1}{2}\,\rho\, f_c\, U^2 \;=\; 4.43\ \ N/m^2$

wave $\tau_{w,max} = \frac{1}{2}\,\rho\, f_w\, U_m^2 \;=\; 42.3\ \ N/m^2$

3.6 Boundary layer of irregular waves

The preceding analysis of wave boundary layer mechanics is based on the assumption of one periodic wave. In reality, wind waves are formed of the superposition of many periodic waves with different frequency and amplitude.

Madsen et al. (1988) presented the wave boundary layer model described by the spectrum of the near-bottom orbit velocity of water particles. The main effort is devoted to find *a representative periodic wave* which gives a boundary layer mechanics close to that of irregular waves. It is found that the representative periodic wave has the wave height equal to the root-mean-square of irregular wave heights H_{rms} ($= H_s/\sqrt{2}$), and the wave period equal to the significant wave period T_s.

3.7 Boundary layer of wave and current: Fredsøe's model

For the sediment transport, we need to know the bed shear stress and the current velocity profile under combined waves and currents. The analysis in this section is based on Fredsøe (1981).

With wave alone, the wave boundary layer thickness is δ, and the flow is divided into two zones, outside the wave boundary layer ($z > \delta$) where the flow is frictionless, and inside the wave boundary layer ($z < \delta$). With the superposition of a weak current, turbulence is produced outside the wave boundary layer by the current. Inside the wave boundary layer both the wave and the current contribute to turbulence. But the current is so weak that the wave boundary layer thickness is app. the same.

First we will consider the case where waves and currents are propagating in the same direction.

Mean bottom shear stress

With wave alone, the maximum bed shear stress and the wave friction velocity are

$$\tau_{w,max} = \frac{1}{2}\,\rho\, f_w\, U_m^2 \qquad u_{*,w} = \sqrt{\frac{\tau_{b,max}}{\rho}} = \sqrt{\frac{f_w}{2}}\, U_m$$

where U_m is the maximum horizontal velocity on bed given by the linear wave theory. Because the wave boundary layer is very thin, U_m can be taken as the velocity on the top of the boundary. The instantaneous bottom velocity and bottom shear stress are

$$u_w = U_m\,\sin(\omega t) \qquad \tau_w = \frac{1}{2}\,\rho\, f_w\, u_w^2$$

Now the current is superimposed, the current velocity on the top of the boundary is U_δ, the combined instantaneous flow velocity on the top of the boundary is

$$u = u_w + U_\delta = U_m\,\sin(\omega t) + U_\delta$$

The combined bed shear stress is

$$\tau_{wc} = \frac{1}{2} \rho f_w u^2$$

The mean bed shear stress is

$$\overline{\tau_{wc}} = \frac{1}{T} \int_0^T \tau_{wc} \, dt \approx \frac{2}{\pi} \rho f_w U_m U_\delta$$

If we know the wave (H, T, h) and the current (average velocity U), then δ, f_w and U_m can be calculated from wave alone, but U_δ is unknown at the moment because the velocity profile has been distorted by waves.

Velocity profile outside the wave boundary layer

Without the wave, the current velocity profile is

$$u(z) = \frac{u_{*,c}}{\kappa} \ln\left(\frac{z}{0.033 \, k_s}\right)$$

where $u_{*,c}$ is the current friction velocity, k_s bed roughness, Van Karmen constant $\kappa = 0.4$.

With the wave, Grant and Madsen (1979) suggest the velocity profile

$$u(z) = \frac{u_{*,wc}}{\kappa} \ln\left(\frac{z}{0.033 \, k_w}\right) \tag{21}$$

where the combined wave-current friction velocity is

$$u_{*,wc} = \sqrt{\frac{\overline{\tau_{wc}}}{\rho}} \approx \sqrt{\frac{2}{\pi} f_w U_m U_\delta} \tag{22}$$

k_w can be interpreted as the bed roughness under the combined wave and current flow. Inserting $u|_{z=\delta} = U_\delta$ into eq (21), we get

$$k_w = 30 \, \delta \exp\left(-\frac{\kappa U_\delta}{u_{*,wc}}\right) \tag{23}$$

The average velocity of the current is

$$U = \frac{1}{h} \int_\delta^h u(z) \, dz = \frac{u_{*,wc}}{\kappa h} \int_\delta^h \ln\left(\frac{z}{k_w}\right) dz$$

$$\approx u_{*,wc} \left(6.2 + \frac{1}{\kappa} \ln\left(\frac{h}{k_w}\right)\right) \tag{24}$$

Combining eqs (22), (23) and (24) gives

$$U_\delta = C - \sqrt{C^2 - U^2}$$

$$\text{where } C = U + \tfrac{1}{\pi} f_w U_m \left(6.2 + \tfrac{1}{\kappa} \ln\left(\tfrac{h}{30 \, \delta}\right)\right)^2$$

Velocity profile inside the wave boundary layer

Inside the wave boundary layer, turbulence comes from the wave and the current. The combined eddy viscosity is

$$\varepsilon_{wc} \;=\; \varepsilon_c \;+\; \varepsilon_w \;=\; \kappa \left(u_{*,wc} \;+\; u_{*,w}\right) z \tag{25}$$

Therefore the mean bottom shear stress is

$$\overline{\tau_{wc}} \;=\; \rho \; \varepsilon_{wc} \frac{\mathrm{d}u}{\mathrm{d}z} \tag{26}$$

which can be rewritten into

$$\frac{\mathrm{d}u}{\mathrm{d}z} \;=\; \frac{u_{*,wc}^2}{u_{*,wc} \;+\; u_{*,w}} \; \frac{1}{z} \tag{27}$$

The integration of the above equation gives

$$u \;=\; \frac{1}{\kappa} \; \frac{u_{*,wc}^2}{u_{*,wc} \;+\; u_{*,w}} \; \ln\left(\frac{z}{z_0}\right) \tag{28}$$

where the integration constant z_0 is the elevation corresponding to zero velocity $(u|_{z=z_0} = 0)$. Nikurase gives $z_0 = 0.033 \; k_s$.

Fig.8 gives an example of the velocity profile with and without waves.

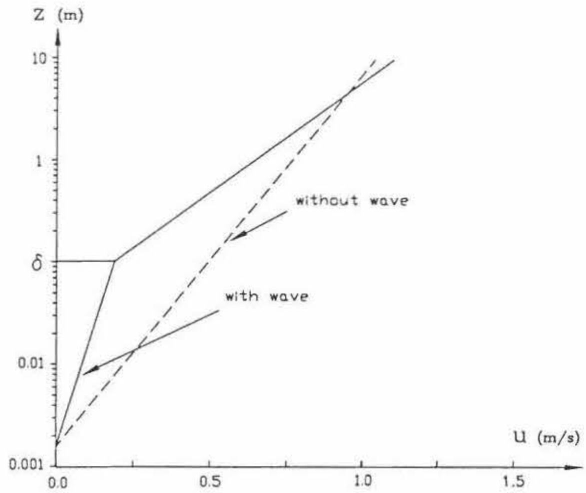


Fig.8. Velocity profile with and without wave for the same water discharge.

Example 4 Bed shear stress and velocity profile in combined wave and current

Given Average current velocity U=1 m/s
Wave height H=2 m, wave period T= 8 s
Water depth h=10 m, sea bed ripple height H_r=5 cm
Wave and current propagate in the same direction

Wanted Bottom shear stress and velocity profile

Solution By linear wave theory, we get

$$L \approx 71\ m, \quad A = \frac{H}{2}\ \frac{1}{\sinh\left(\frac{2\ \pi\ h}{L}\right)} = 0.98\ m, \quad U_m = A\omega = A\ \frac{2\ \pi}{T}\ =\ 0.77\ m/s$$

The bed roughness is taken as the ripple height $k_s = H_r$ =0.05 m, then

$$\delta\ =\ 0.26\ \left(\frac{A}{k_s}\right)^{0.70}\ k_s\ =\ 0.10\ m$$

$$f_w\ =\ exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2}\ -\ 6.3\right)\ =\ 0.038$$

$$u_{*,w}\ =\ \sqrt{\frac{\tau_{b,max}}{\rho}}\ =\ \sqrt{\frac{f_w}{2}}\ U_m\ =\ 0.11\ m/s$$

$$C\ =\ U\ +\ \frac{1}{\pi}\ f_w\ U_m\ \left(6.2\ +\ \frac{1}{\kappa}\ln\left(\frac{h}{30\ \delta}\right)\right)^2\ =\ 1.79\ m/s$$

$$U_\delta\ = C\ -\ \sqrt{C^2\ -\ U^2}\ =\ 0.31\ m/s$$

Mean bottom shear stress $\overline{\tau_{wc}}\ =\ \frac{2}{\pi}\ \rho\ f_w\ U_m\ U_\delta\ =\ 5.68\ N/m^2$

$$u_{*,wc}\ =\ \sqrt{\frac{\overline{\tau_{wc}}}{\rho}}\ =\ 0.075\ m/s$$

$$k_w\ =\ 30\ \delta\ \exp\left(-\frac{\kappa\ U_\delta}{u_{*,wc}}\right)\ =\ 0.59\ m$$

Outside the boundary layer $u\ (z)\ =\ \frac{u_{*,wc}}{\kappa}\ \ln\left(\frac{z}{0.033\ k_w}\right)\ =\ 0.19\ \ln\left(\frac{z}{0.019}\right)$

Inside the boundary layer $u\ (z)\ =\ \frac{1}{\kappa}\ \frac{u_{*,wc}^2}{u_{*,wc}\ +\ u_{*,w}}\ \ln\left(\frac{z}{0.033\ k_s}\right)\ =\ 0.076\ \ln\left(\frac{z}{0.0017}\right)$

Comment Current alone, we have

$$f_c\ =\ \frac{0.06}{\left(\ \log\left(\frac{12\ h}{k_s}\right)\ \right)^2}\ =\ 0.0053$$

$$u_{*,c}\ =\ \sqrt{\frac{\tau_c}{\rho}}\ =\ \sqrt{\frac{f_c}{2}}\ U\ =\ 0.051\ m/s$$

$$u\ (z)\ =\ \frac{u_{*,c}}{\kappa}\ \ln\left(\frac{z}{0.033\ k_s}\right)\ =\ 0.13\ \ln\left(\frac{z}{0.0017}\right)$$

Comparison of velocity profile is shown in Fig.8.

Wave and current forms an angle β

Fig.9 shows the horizontal velocity vector on the top of the wave boundary layer ($z = \delta$).

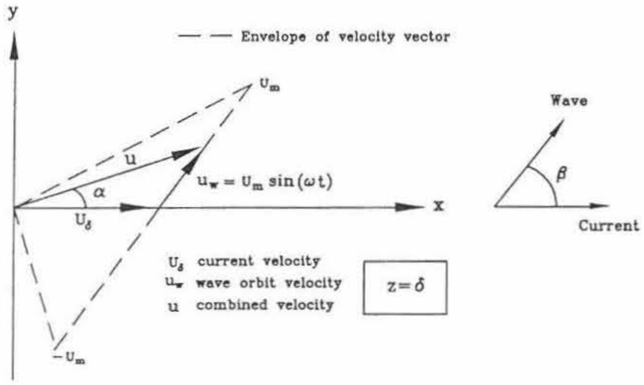


Fig.9. Instantaneous velocity on the top of the wave boundary layer.

The combined velocity u is

$$u = \sqrt{U_\delta^2 + u_w^2 + 2\, U_\delta\, u_w \cos\beta} \tag{29}$$

The bottom shear stress is

$$\frac{1}{2}\, \rho\, f_w\, u^2$$

which acts in the same direction as u, cf. Fig.9. The bottom shear stress in the current direction is

$$\tau_{wc} = \frac{1}{2}\, \rho\, f_w\, u^2 \cos\alpha = \frac{1}{2}\, \rho\, f_w\, u^2\, \frac{U_\delta + u_w \cos\beta}{u}$$

The mean bed shear stress in the current direction is

$$\overline{\tau_{wc}} = \frac{1}{T} \int_0^T \tau_{wc}\, \mathrm{d}t = \frac{2}{\pi}\, \rho\, f_w\, U_m\, U_\delta \frac{1 + \cos^2\beta}{2} \tag{30}$$

3.8 Boundary layer of wave and current: Bijker's model

Opposite to Fredsøe, Bijker (1971) assumes that the wave is so weak that it will not affect the thickness of the current viscous sublayer.

First we consider the current alone. Bijker assumes that there is a viscous sublayer, starting from $z = 0$ to $z = z_0\, e$ where the linear velocity distribution is tangent with the logarithmic velocity distribution, cf. Fig.10.

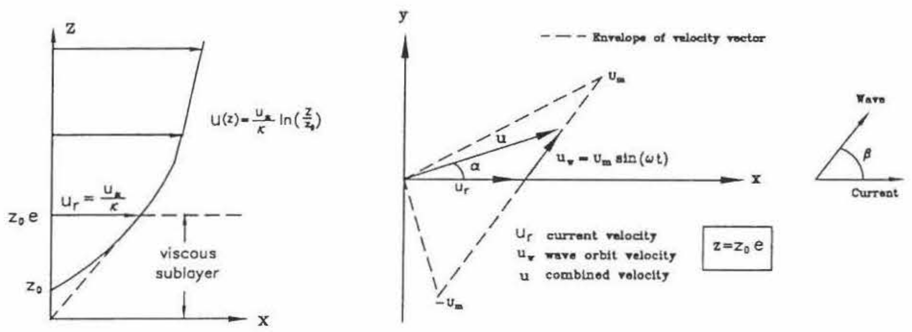


Fig.10. Viscous sublayer in hydraulically rough flow.

By the logarithmic velocity profile we get

$$U_r \;=\; u\big|_{z=z_0\,e} \;=\; u_*/\kappa$$

and the bottom shear stress is

$$\tau_c \;=\; \rho\; u_{*,c}^2 \;=\; \rho\; \kappa^2\; U_r^2$$

Now the wave is superimposed, cf. Fig.10, the combined instantaneous flow velocity on the top of the viscous sublayer is

$$u \;=\; \sqrt{U_r^2 \;+\; u_w^2 \;+\; 2\; U_r\; u_w\; \cos\beta}$$

The combined bed shear stress is

$$\tau_{wc} \;=\; \rho\; \kappa^2\; u^2$$

The mean bed shear stress is

$$\overline{\tau_{wc}} \;=\; \frac{1}{T}\int_0^T \tau_{wc}\; \mathrm{d}t \;\approx\; \tau_c \;+\; \frac{1}{2}\; \tau_{w,max} \tag{31}$$

Bijker's mean bed shear stress is in the direction of the combined velocity u.

Example 5 Bed shear stress in combined wave and current by Bijker's model

Given As in Example 4, i.e.

Average current velocity U=1 m/s
Wave height H=2 m, wave period T= 8 s
Water depth h=10 m, sea bed ripple height H_r=5 cm
Wave and current propagate in the same direction

Wanted Bottom shear stress

Solution By linear wave theory, we get

$$L \approx 71\ m, \qquad A = \frac{H}{2}\,\frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = 0.98\ m, \qquad U_m = A\omega = A\,\frac{2\,\pi}{T} = 0.77\ m/s$$

The bed roughness is taken as the ripple height $k_s = H_r$ =0.05 m, then

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) = 0.038$$

$$\tau_{w,max} = \frac{1}{2}\,\rho\,f_w\,U_m^2 = 11.3\ N/m^2$$

$$f_c = \frac{0.06}{\left(\log\left(\frac{12\,h}{k_s}\right)\right)^2} = 0.0053$$

$$\tau_c = \frac{1}{2}\,\rho\,f_c\,U^2 = 2.65\ N/m^2$$

$$\overline{\tau_{wc}} = \tau_c + \frac{1}{2}\,\tau_{w,max} = 8.3\ N/m^2$$

Comment Fredsøe's model gives (Example 4)

$$\overline{\tau_{wc}} = 5.68\ N/m^2$$

3.9 Exercise

1) Wave height H=3 m, wave length L= 80 m
   Water depth h=6 m, sea bed ripple height H_r=15 cm

   1) Calculate wave boundary layer thickness $\delta_{0.05}$

   2) Calculate wave friction factor f_w

   3) Calculate max. wave-induced bottom shear stress

2) Average current velocity U=1.5 m/s
   Wave height H=3 m, wave period T= 8 s
   Water depth h=8 m, sea bed ripple height H_r=5 cm
   Wave and current propagate in the same direction

   1) bottom shear stress, current alone

   2) bottom shear stress, wave alone

   3) bottom shear stress, current+wave

4 Cross-shore sediment transport and beach profile

4.1 Sediment size and its sorting on beaches

Sediments in coastal regions may be composed of any materials that are available in significant quantity and is of a suitable grain size to remain on the beach.

Grain size is sometimes given as the ϕ scale. It relates to the diameter d in millimeter by $d = 1/2^{\phi}$.

Grain size in beach varies from more than 1 meter for boulders to less than 0.1 mm for very fine sands. Generally grain size ranges from 0.1 to 2 mm.

There are three dominant factors controlling the mean grain size of beach sediment: the sediment source, the wave energy level and the beach slope.

The sorting of sediments along a beach profile produces cross-shore variations in sediment grain size. As shown in Fig.1, the mean grain size reflects the wave energy loss. Note that the incoming waves first break over the offshore bar, without much energy dissipation by turbulence. The waves then reform and break for a second time, plunging at the base of the beach face where they expend most of their energy.

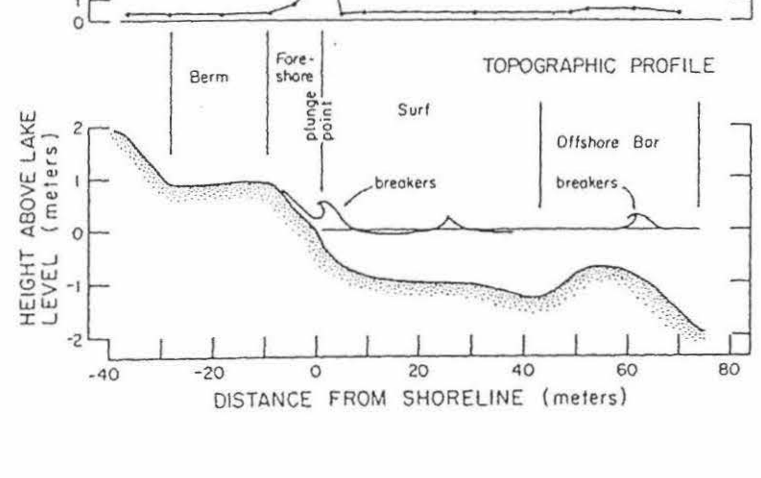


Fig.1. Grain size across the Lake Michigan beach (scanned from Fox et al. 1966).

4.2 Threshold of sediment under wave actions

We have learned from channel flow that sediments start to move if the Shields parameter is bigger than the critical Shields parameter. The Shields parameter is defined as

$$\theta \;=\; \frac{u_*^2}{(s-1)\;g\;d} \;=\; \frac{\tau}{\rho\;(s-1)\;g\;d} \tag{1}$$

and the critical Shields parameter is given in Fig.2.

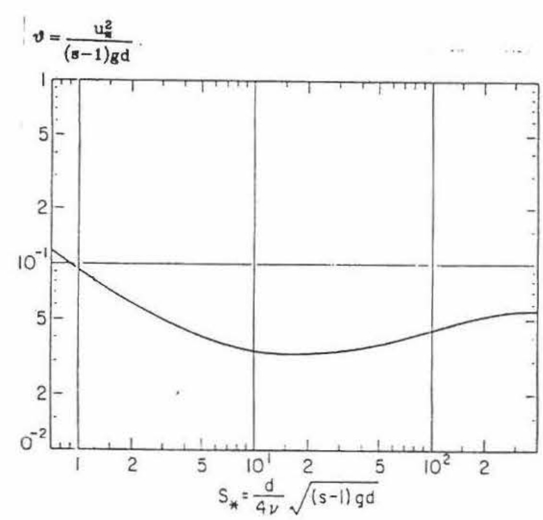


Fig.2. The Shields diagram giving θ_c as a function of S_*.

Laboratory results have shown that the Shields diagram for currents can be used directly for wave, with the current-induced bottom shear stress replaced by Jonsson's definition of wave-induced bottom shear stress

$$\tau_{w,max} \;=\; \frac{1}{2}\;\rho\;f_w\;U_m^2 \tag{2}$$

where U_m is the maximum horizontal velocity of water particle on sea bed, given by the linear wave theory,

$$U_m \;=\; \frac{\pi\;H}{T}\;\frac{1}{\sinh\left(\frac{2\;\pi\;h}{L}\right)} \tag{3}$$

and f_w is wave friction coefficient,

$$f_w \;=\; exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} \;-\; 6.3\right) \tag{4}$$

where k_s is the bed roughness and A the amplitude of oscillation of water particle on sea bed, see Chapter 3.

Example 1	Threshold of sediment under wave
Given	Sediment is quartz sand with $\rho_s = 2650\ kg/m^3$ and $d = 0.2mm$. Fluid is sea water with $\rho = 1025\ kg/m^3$ and $\nu = 10^{-6}\ m^2/s$. Wave height H=4 m, wave period T=7.2 s Water depth h=50 m, bed ripple height H_r=1 cm
Wanted	Will sediments start to move ?
Solution	The relative density is $s = \rho_s/\rho = 2.59$ The sediment-fluid parameter is

$$S_* = \frac{d\sqrt{(s-1)\,g\,d}}{4\,\nu} = \frac{0.0002\sqrt{(2.59-1)\times 9.8\times 0.0002}}{4\times 10^{-6}} = 2.79$$

From Fig.2 it is found the critical Shields parameter $\theta_c = 0.052$

By linear wave theory

$$L = 80\ m$$

$$A = \frac{H}{2}\,\frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = 0.078\ \ m$$

$$U_m = A\omega = A\,\frac{2\pi}{T} = 0.068\ m/s$$

The bed roughness is taken as the ripple height $k_s = H_r$ =0.01 m, then the wave friction factor is

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) = 0.078$$

The wave-induced maximum bottom shear stress is

$$\tau_{w,max} = \frac{1}{2}\,\rho\,f_w\,U_m^2 = 0.18\ \ N/m^2$$

The Shields parameter

$$\theta = \frac{\tau_{w,max}}{\rho\,(s-1)\,g\,d} = 0.056$$

Because $\theta > \theta_c$, sediment movement takes place.

4.3 Depth of closure

Example 1 has shown that the sediments start to move at the water depth of 50 meters. However, because of the one step forwards and one step backwards nature of water particle movement, the net movement of sediment in one wave period is zero.

We conceptualize that a beach profile responds to wave action between two limits, one limit on the landward side where the wave run-up ends, the other limit in relatively deep water where the waves can no longer produce a measurable change in depth. This latter limit is called the depth of closure.

Obviously the depth closure is not the location where sediment ceases to move (threshold depth), but that location of minimum depth where the profile surveys before and after a storm overlap each other.

Laboratory measurement has shown that the net movement of sediment starts when $\theta > 2 \times \theta_c$. In example 1 it can be calculated that the depth of closure is app. 20 meters, cf. Fig.3.

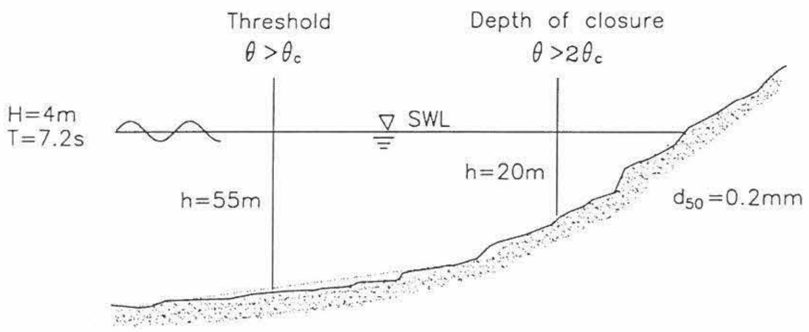


Fig.3. Depth of threshold and depth of closure.

The depth of closure is time dependent. We expect that average depth of closure for the summer is smaller than that in winter. In engineering project, the depth of closure is best determined through repeated accurate profile survey. If such data are not available, an analytic method introduced by Hallermeier (1983) can be used to estimate the limiting depth

$$\frac{h_c}{H_{s,0}} = 2.28 - 10.9 \frac{H_{s,0}}{L_{s,0}} \tag{5}$$

where $H_{s,0}$ is the deep-water significant wave height exceeded 12 hours per year and $L_{s,0}$ deep-water wave length corresponding to significant wave period.

4.4 Bed form and bed roughness

As in steady flow, a sequence of bed forms occurs above the threshold of sediment movement as the magnitude of orbital amplitude increases:

Flat bed ⇒ rolling grain ripples ⇒ vortex ripples ⇒ High stage flat bed

Rolling grain ripples	At the threshold of motion, grains start to move over the surface by are not lifted.
Vortex ripples	It occurs when $\theta > 2 \times \theta_c$.
High stage flat bed	Based on the laboratory data, Komar et al (1975) proposed $\theta \; > \; 0.413 \; d$, where d is in cm.

Wave-induced ripples can be distinguished from current-induced ripples by their shape. Wave-induced ripples have symmetrical profiles due to the oscillation of water particle, cf. Fig.4.

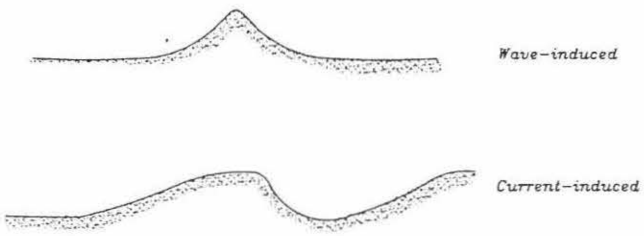


Fig.4. Wave-induced and current-induced ripples.

Wave generated ripple length is (Brøker 1985)

$$L_r \;=\; 2 \times A \tag{6}$$

where A is the amplitude of water particle oscillation on the bottom, given by linear wave theory.

The ripple steepness is (Nielsen 1979)

$$\frac{H_r}{L_r} \;=\; 0.182 \;-\; 0.24\,(\theta')^{3/2} \tag{7}$$

where θ' is the effective Shields parameter due to skin friction, i.e. calculated by setting the bed roughness to $2.5 \times d_{50}$.

The bed roughness (equivalent Nikurase bed roughness) is

$$k \;=\; (1-4)\; H_r \tag{8}$$

Example 2 Sea bed roughness

Given Sediment is quartz sand with $\rho_s = 2650\ kg/m^3$ and $d = 0.2mm$.
Fluid is sea water with $\rho = 1025\ kg/m^3$ and $\nu = 10^{-6}\ m^2/s$.
Wave height H=4 m, wave period T=7.2 s
Water depth h=50 m

Wanted Sea bed roughness k_s

Solution By linear wave theory

$$L = 80\ m$$

$$A = \frac{H}{2}\,\frac{1}{\sinh\left(\frac{2\,\pi\,h}{L}\right)} = 0.078\ m$$

$$U_m = A\omega = A\,\frac{2\pi}{T} = 0.068\ m/s$$

The wave friction factor corresponding to $k_s = 2.5\ d = 0.0005\ m$

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) = 0.014$$

The effective bottom shear stress is

$$\tau' = \frac{1}{2}\,\rho\,f_w\,U_m^2 = 0.033\ N/m^2$$

The relative density is $s = \rho_s/\rho = 2.59$
The effective Shields parameter

$$\theta' = \frac{\tau'}{\rho\,(s-1)\,g\,d} = 0.0041$$

the wave generated ripple length is

$$L_r = 2 \times A = 0.156\ m$$

The ripple steepness is

$$\frac{H_r}{L_r} = 0.182 - 0.24\,(\theta')^{3/2} = 0.182$$

The ripple height is

$$H_r = 0.182\ L_r = 0.028\ m$$

The sea bed roughness is

$$k_r = (1-4)\ H_r = 0.028 - 0.11\ m$$

4.5 Beach classification

The morphology of a beach profile depends on grain size and physical process of waves, currents and tides. The pattern of beach profile response to wave intensity is illustrated schematically in Fig.5.

In the summer, lower wave heights move sand shoreward along the beach profile and deposit on the beach face, often to form a wide beach (*summer profile or berm-type*).

In the winter, longshore bars are formed as a result of a strong offshore transport in the surf zone and a weak onshore transport outside the surf zone (*winter profile or bar-type*).

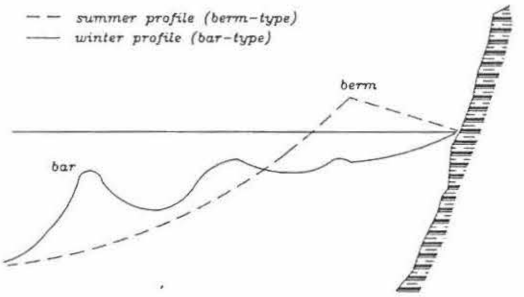


Fig.5. Illustration of summer profile and winter profile.

With respect to type of wave breaking, and hence energy transformation on beach, beaches can be classified as *dissipative, intermediate and reflective*, corresponding to spilling breaker, plunging breaker and surging breaker, respectively, cf. Fig.6.

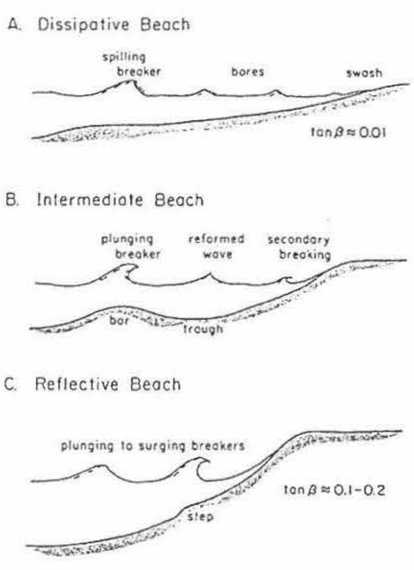


Fig.6. Type of wave breaking.

4.6 Berms and longshore bars

Berms

Berms are formed by onshore movement of sediments. The onshore movement of sediments takes place in two situations. In the summer, lower wave heights move sand shoreward along the beach profile and deposit on the beach face, often to form a wide beach (berm). At the end of a storm in the winter, when the wave decreases in height while maintaining a long period, the sediment transport direction reverses from offshore to onshore, and the material builds up on the foreshore to form a berm. Successive storms are, therefore, usually separated by an interval of onshore transport and berm formation, which means that storms would not have a cumulative erosion impact on the coast owing to berm formation inbetween (storm recovery process).

For the large wave tank experiment with regular waves, Larson et al.(1989) found a fairly clear relation between the berm height and deep-water waves

$$\frac{Z_R}{H_0} = 1.47\ \zeta_0^{0.79} \tag{9}$$

where $\zeta_0 = \tan\beta(H_0/L_0)^{-1/2}$ is the deep-water surf similarity parameter and $\tan\beta$ the initial beach profile slope.

The beach face slope is normally very linear. Kriebel et al (1991) reanalyzed the field data of Sunamura and obtained the following expression for the beach face slope

$$m_f = 0.15\left(\frac{\omega_s\, T}{H_b}\right)^{1/2} \tag{10}$$

where ω_s is the fall velocity of sediment, T wave period and H_b wave height at breaking point.

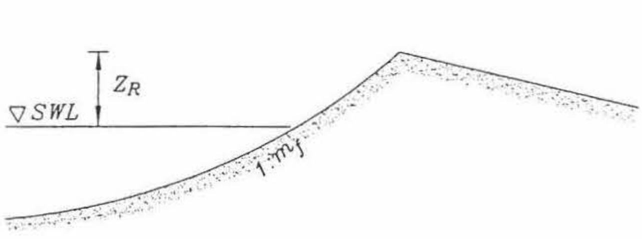


Fig.7. Illustration of a berm.

Longshore bars

A longshore bar is formed around the wave breaking point as a result of a strong offshore transport in the surf zone and a weak onshore transport outside the surf zone, cf. Fig.8

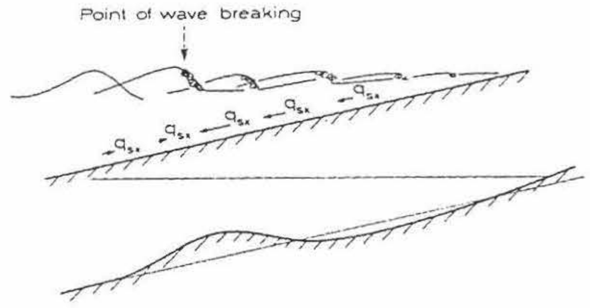


Fig.8. Illustration of formation of a longshore bar.

A break-point bar forms rapidly at first, and thereafter its volume increases more slowly as the profile shape approaches equilibrium. At the same time the longshore bar is moving seaward, in association with the seaward moving of wave breaking point. The distance of the bar crest from the shoreline X_B in large tank experiments with monochromatic waves (Kraus 1992) is

$$\frac{X_B}{L_0} = A_B \left(1 - \exp\left(-\alpha \frac{t}{T}\right)\right) \tag{11}$$

where A_B is the ultimate relative distance

$$A_B = 11000 \left(\frac{H_b}{g\,T^2}\right)^2 \tag{12}$$

and the decay coefficient

$$\alpha = 3.5 \times 10^{-8} \left(\frac{H_b}{g\,T^2}\right)^{-1.5} \tag{13}$$

and L_0 is deep-water wave length, H_b wave height at breaking point, T wave period, g gravitational acceleration and t time.

The growth of a bar is ultimately limited by the maximum slope the sand grains can maintain under the action of gravity and fluid motion. In the field, random waves and varying water level exert a smoothing effect on the profile. The maximum slope in the field is less than 10 degrees while in laboratories it can reach up to 25 degrees with monochromatic waves. In the field and in the laboratory, the shoreward bar slope is almost always steeper than the seaward slope.

At equilibrium the geometry of a longshore bar is described by (Kraus 1992)

$$h_c = 0.66\ H_b \qquad (14)$$

$$\frac{h_t}{h_c} = 2.5\ \left(\frac{H_0}{L_0}\right)^{0.092} \qquad (15)$$

$$Z_B = 0.122\ \left(\frac{H_0}{\omega_s\ T}\right)^{0.59}\ \left(\frac{H_0}{L_0}\right)^{0.73} \qquad (16)$$

where ω_s is the fall velocity of sediment.

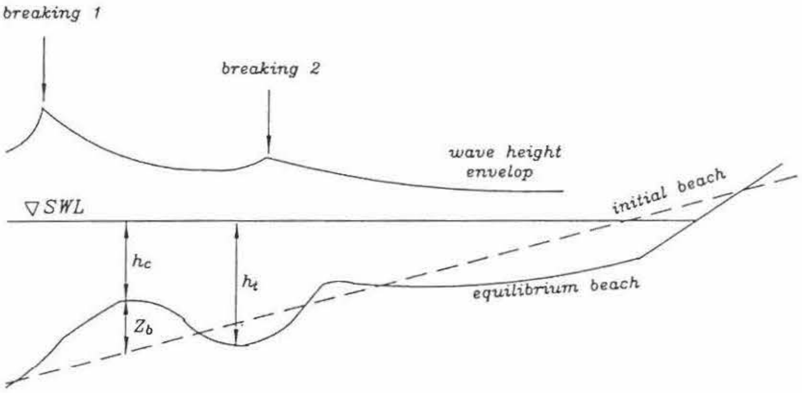


Fig.9. Illustration of a longshore bar.

4.7 Equilibrium beach profile ($x^{2/3}$)

If we become more abstract in characterizing the beach profile, the berms and bars may be considered as small perturbations and hence omitted. Bruun (1954) and Dean (1977) have shown that many beach profiles exhibit a concave shape such that the depth varies as the two-thirds power of the distance from the shoreline, i.e.

$$h = A\ x^{2/3} \qquad (17)$$

where A is the profile shape parameter, which controls the steepness of the profile. The fitting of the field data gives (Moore 1982 and Kraus 1992)

$$A = 0.41\ (d_{50})^{0.94} \qquad d_{50} < 0.4$$

$$A = 0.23\ (d_{50})^{0.32} \qquad 0.4 \leq d_{50} < 10$$

$$A = 0.23\ (d_{50})^{0.28} \qquad 10 \leq d_{50} < 40$$

$$A = 0.46\ (d_{50})^{0.11} \qquad 40 \leq d_{50}$$

4.8 Erosion and accretion predictors

In this section, we describe some of the simple techniques that have been found capable of predicting whether a beach will erode or accrete by *cross-shore transport processes.*

It is well known that winter storm waves and hurricanes tend to remove material from the beach face and deposit it offshore as a bar, whereas summer swell and swell generated during a decay of a storm tend to build the berm and widen the beach.

There have been numerous studies of erosion and accretion predictors with small-scale wave tank tests, large-scale wave tank tests and field investigations (Larson et al. 1989).

Fig.10 gives the results of field investigations. The dashed line were developed by assigning a 10% variability in deep-water significant wave height H_s, significant wave period T_s and fall velocity of grains ω_s (H_0, T, w in the figure). Originally the following simple predictor is proposed

$$\frac{H_{s,0}}{\omega_s\ T_s} \begin{cases} < 3.2 & \text{accretion} \\ \geq 3.2 & \text{erosion} \end{cases} \tag{18}$$

Later on the criterion is replaced by the diagonal line.

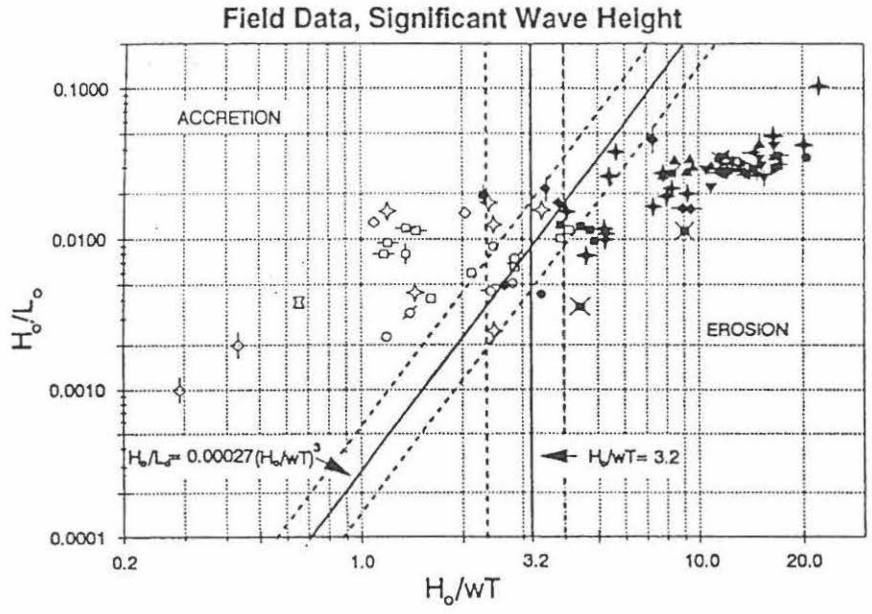


Fig.10. Erosion and accretion predictor by field data (after Kraus 1992).

4.9 Shoreline retreat due to sea level rise

The sea level rise is best recorded by tide gauges. Long-term tide gauge records from a specific coastal site give the relative change of sea level S, the sum of the global sea level change and local land level change.

On average the global sea level rise is about 2 mm/year. A local land level change can be positive (subsidence of the land) or negative (uplift of the land)

The first and best-known of the models that relate shoreline retreat to an increase of sea level is that proposed by Bruun (1962, 1988)

$$R = \frac{L_*}{h_c} S \tag{19}$$

where R is the shoreline retreat rate, S sea level rise, h_c depth of closure (depth to which nearshore sedimentation exists), cf. Fig.11.

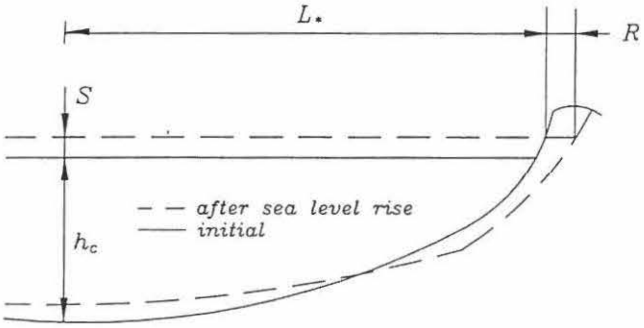


Fig.11. Shoreline retreat due to sea level rise.

Eq (19) can also be expressed as

$$R = m_s S \tag{20}$$

where m_s is the average slope of the beach profile. With respect to the fact that an average beach slope is 1:50 to 1:100 for many coastal sites, eq (20) yields R=50S to 100S, which is often used as *rule of thumb* to calculate the expected shoreline retreat rate R from a sea level rise S.

4.10 Exercise

1) Oil and gas are transported from the North Sea by pipeline. When the water depth is lower than the depth of closure, a trench needs to be dredged so that the pipeline can be buried.
In a beach along the Danish west coast, the deep-water significant wave height distribution is given by extreme wave height analysis

$$\text{Gumbel} \quad F(H_s) = e^{-e^{-\left(\frac{H_s - B}{A}\right)}} \tag{21}$$

with A=0.5 and B=1.7;
Estimate the depth from which the pipeline should be buried.

5 Longshore sediment transport

When waves approach the coast at an oblique angle, a longshore current will be generated. Waves and currents may transport considerable amounts of sediment along the coast. Longshore sediment transport will often be the dominant factor in the sediment budget, and hence in the erosion or accretion of beach, cf. Fig.1.

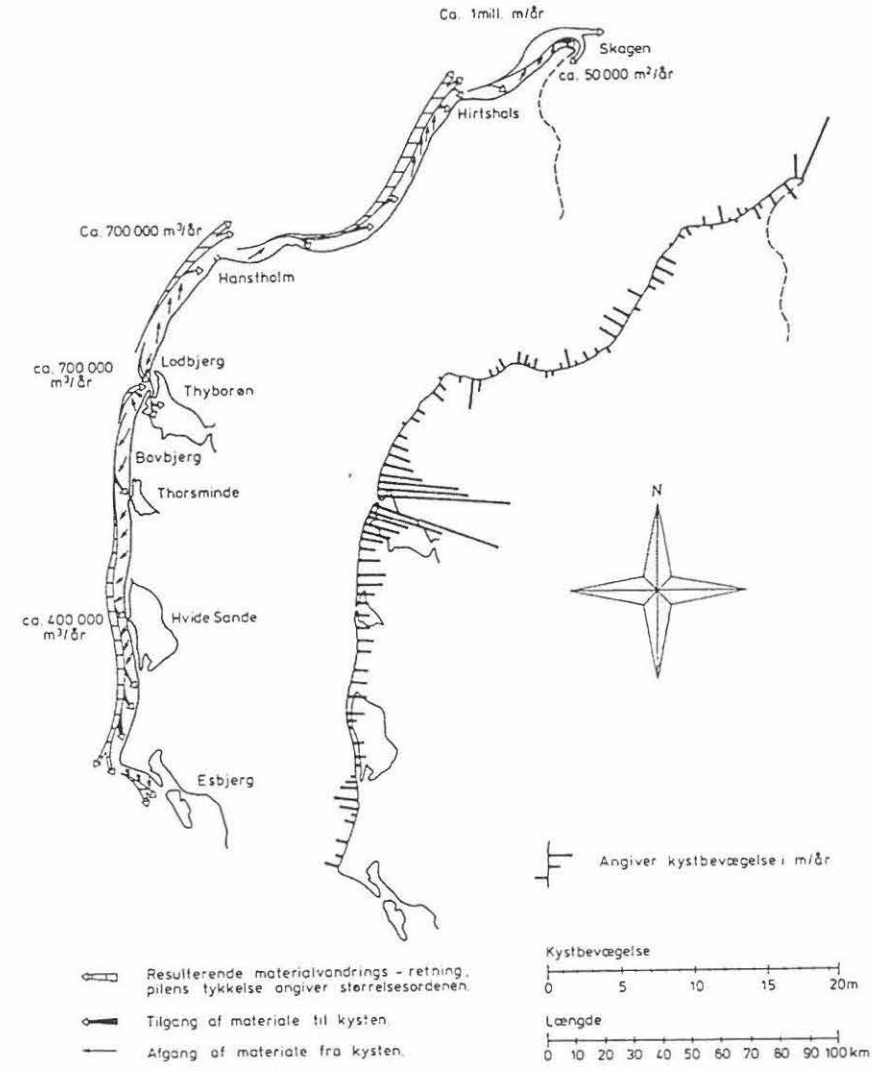


Fig.1. Longshore sediment transport and coastal line movement along the west coast in Jutland, Denmark, Jubilæumsskriftet: Vandbygningsvæsenet 1868-1968 (scanned from Burcharth 1984).

5.1 CERC-formula

One of the oldest and still most useful formula for calculating the longshore sediment transport is known as the CERC-formula, presented in the Shore Protection Manual (CERC, 1984).

The CERC-formula states that the longshore sediment transport is proportional to the longshore wave energy flux.

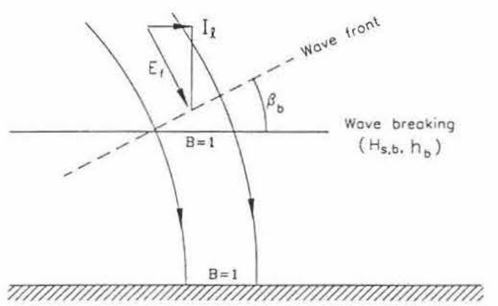


Fig.2. Longshore wave energy flux within a unit width of beach.

Let Q_ℓ be the solid volume sediment transport (m^3/s). The submerged weight of the transported sediment is $(\rho_s - \rho)\ g\ Q_\ell \quad (N/s)$.

By linear wave theory, the wave energy flux at the breaking point is

$$E_{f,b} \;=\; E_b\ c_{g,b} \;=\; \frac{1}{8}\ \rho\ g\ H_{s,b}^2\ \frac{c_b}{2}\ \left(1 \;+\; \frac{\frac{4\ \pi\ h_b}{L_b}}{\sinh\left(\frac{4\ \pi\ h_b}{L_b}\right)}\right) \qquad \text{Joule/(m}\times\text{s) = N/s}$$

where the subscript b denotes the parameters at the breaking point. Over a unit width of beach, cf. Fig.2, the incoming wave energy at the breaking point is

$$E_{f,b}\ B\ \cos\beta_b \;=\; E_f\ \cos\beta_b$$

The longshore energy flux is

$$I_\ell \;=\; E_{f,b}\ \cos\beta_b\ \sin\beta_b$$

The CERC-formula reads

$$(\rho_s - \rho)\ g\ Q_\ell \;=\; K_c\ I_\ell \qquad (1)$$

The fitting to the field measurement gives $K_c = 0.41$.

Example 1 Application of the CERC-formula

Given A trench was dredged for the oil pipeline. Before the lay-down of the pipeline, there was a storm with $H_{s,b} = 3$ m, $T_p = 8$ s, $\beta_b = 45^0$. The storm lasted 2 days.

Wanted Estimate the backfilling in the trench.

Solution The water depth where wave breaking takes place is estimated to be

$$h_b = \frac{H_{s,b}}{0.55} = 5.5 \ m$$

By the linear wave theory

$$L_b = 55 \ m \qquad c_b = \frac{L_b}{T_p} = 6.9 \ m/s$$

The longshore energy flux is

$$I_\ell = E_{f,b} \cos\beta_b \sin\beta_b$$

$$= \frac{1}{8} \rho \, g \, H_{s,b}^2 \frac{c_b}{2} \left(1 + \frac{\frac{4 \pi h_b}{L_b}}{\sinh\left(\frac{4 \pi h_b}{L_b}\right)} \right) \cos\beta_b \sin\beta_b$$

$$= 33825 \ N/s$$

The longshore sediment transport is

$$Q_\ell = \frac{K_c \, I_\ell}{(\rho_s - \rho) \, g} = 0.86 \ m^3/s$$

The sediment which might deposit in the dredged trench is

$$Q_\ell \, (2 \times 24 \times 60 \times 60) = 148608 \ m^3$$

Comment Note that Q_ℓ is the solid volume of deposited sediment. If the porosity of the sand is 0.3, the total volume to be re-dredged would be

$$\left(1 + \frac{0.3}{0.7}\right) \times 148608 = 212297 \ m^3$$

The CERC-formula has only the characteristics of the incoming waves as input. This is not realistic, as the sediment transport must be expected to depend on the sediment size and beach profile. Kamphuis (1990) have made en extensive analysis of field and laboratory data and proposed a longshore sediment transport formula:

$$Q_\ell = 1.28 \, \frac{\tan\alpha \, H_{s,b}^{3.5}}{d_{50}} \sin(2 \, \beta_b)$$

where $\tan\alpha$ is the beach slope. The formula is dimensional, $H_{s,b}$ and d_{50} in meter, Q_ℓ in kg/s.

5.2 Bijker's method: Wave + current

Bijker (1971) presented a method for the calculation of sediment transport in the combined wave and current. The method has been extended to the calculation of longshore sediment transport, where the longshore current is produced by wave breaking.

In the chapter on wave boundary layer theory it has been shown that sediment transport mechanism in coastal regions is: *wave stirs up sediment and current transports the sediment.*

We have many formulae for calculating sediment transport by currents. These formulae contain the current-induced bed shear stress in several places. Bijker divides the formulae into *stirring up part* and *transporting part.* Under the combination of wave and current, the wave action will only contribute to the stirring up part. This contribution is expressed by replacing the current bed shear stress in the stirring up part by the wave-current bed shear stress. In the transporting part the bed shear stress remains current-induced.

<u>Bed shear stress by wave and current</u>

In the combined wave and current, the mean bed shear stress by Bijker is given in Chapter 3, i.e.

$$\tau_{wc} = \tau_c + \frac{1}{2}\,\tau_{w,max} \tag{2}$$

where τ_c is the bed shear stress by current alone, and $\tau_{w,max}$ is the maximum bed shear stress by wave alone,

$$\tau_c = \frac{1}{2}\,\rho\,\left(\frac{0.06}{\left(\log\left(\frac{12\,h}{k_s}\right)\right)^2}\right)\,U^2$$

$$\tau_{w,max} = \frac{1}{2}\,\rho\,f_w\,U_m^2$$

$$f_w = exp\left(5.5\left(\frac{k_s}{A}\right)^{0.2} - 6.3\right)$$

where h water depth

k_s bed roughness

U average velocity of current

A amplitude of the water particle on the bottom

U_m maximum horizontal velocity of the water particle on the bottom

Bed-load by wave and current

Kalinske-Frijlink (1952) formula for bed-load transport by current reads

$$q_B = \underbrace{2\, d_{50} \sqrt{\frac{\tau_c}{\rho}}}_{\text{transporting}} \; \underbrace{exp\left(\frac{-0.27\ (s-1)\ d_{50}\ \rho\ g}{\mu_r\ \tau_c} \right)}_{\text{stirring up}}$$

Under combined wave and current, the current-induced bed shear stress in the stirring up part is replaced by τ_{wc},

$$q_B = \underbrace{2\, d_{50} \sqrt{\frac{\tau_c}{\rho}}}_{\text{transporting}} \; \underbrace{exp\left(\frac{-0.27\ (s-1)\ d_{50}\ \rho\ g}{\mu_r\ \tau_{wc}} \right)}_{\text{stirring up}} \tag{3}$$

suspended load by wave and current

Einstein-Bijker formula for suspended sediment transport under current alone reads

$$q_S = 1.83\ q_B \left(I_1\ \ln\left(\frac{h}{0.033\ k_s}\right) + I_2 \right) \tag{4}$$

where I_1 and I_2 are Einstein integrals given by

$$I_1 = 0.216\ \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \mathrm{d}B$$

$$I_2 = 0.216\ \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \ln B\ \mathrm{d}B$$

$$\text{where} \quad A = \frac{k_s}{h} \qquad B = \frac{z}{h} \qquad z_* = \frac{\omega_s}{\kappa\ u_{*,c}}$$

where $u_{*,c}$ is the current friction velocity.

Under combined wave and current, $u_{*,c}$ is replaced by $u_{*,wc}$, wave-current friction velocity, and q_B is the bed-load under combined wave and current.

Example 2 Application of Bijker's method

Given In a coastal region

sea water	$\rho = 1025\ kg/m^3$	$\nu = 10^{-6}\ m^2/s$
flow	$U = 1\ m/s$	$h = 2\ m$
wave	$H = 0.5\ m$	$T = 8\ s\ (L = 35\ m)$
sediment	$\rho_s = 2650\ kg/m^3$	$d_{50} = 0.2\ mm.$
bed roughness	$k_s = 2\ cm$	

Wanted Sediment transport under current alone and under combined wave and current.

Solution We consider first current alone.

The effective bottom shear stress is

$$\tau_c' = \frac{1}{2}\,\rho\,\left(\frac{0.06}{\left(\log\left(\frac{12\ h}{2.5\ d_{50}}\right)\right)^2}\right)\,U^2 = 1.40\ N/m^2$$

The total bottom shear stress is

$$\tau_c = \frac{1}{2}\,\rho\,\left(\frac{0.06}{\left(\log\left(\frac{12\ h}{k_s}\right)\right)^2}\right)\,U^2 = 3.24\ N/m^2$$

The ripple factor is

$$\mu_r = \frac{\tau_c'}{\tau_c} = 0.43$$

The bed-load transport is

$$q_B = 2\ d_{50}\,\sqrt{\frac{\tau_c}{\rho}}\ exp\left(\frac{-0.27\ (s-1)\ d_{50}\ \rho\ g}{\mu_r\ \tau_c}\right) = 0.0000121\ \frac{m^3}{m \times s}$$

The relative density of the sediment $s = \rho_s/\rho = 2.59$.
The fall velocity of the sediment

$$\omega_s = \frac{\sqrt{\left(\frac{36\ \nu}{d_{50}}\right)^2 + 7.5\ (s-1)\ g\ d_{50}} - \frac{36\ \nu}{d_{50}}}{2.8} = 0.02\ m/s$$

The friction velocity $u_{*,c} = \sqrt{\frac{\tau_c}{\rho}} = 0.056\ m/s$

Therefore $A = \frac{k_s}{h} = 0.01$ $z_* = \frac{\omega_s}{\kappa\ u_{*,c}} = 0.89$

and we get the Einstein integrals by numerical integration

$$I_1 = 0.216 \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \mathrm{d}B = 1.00$$

$$I_2 = 0.216 \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \ln B \, \mathrm{d}B = -2.50$$

The suspended sediment transport is

$$\begin{aligned} q_S &= 1.83 \, q_B \left(I_1 \ln\left(\frac{h}{0.033 \, k_s}\right) + I_2 \right) \\ &= 1.83 \times 0.0000215 \left(1.00 \times \ln\left(\frac{2}{0.033 \times 0.02}\right) - 2.50 \right) \\ &= 0.000217 \frac{m^3}{m * s} \end{aligned}$$

The total sediment transport under current alone is

$$q_T = q_B + q_S = 0.000239 \frac{m^3}{m \times s}$$

Now we consider combined wave and current.

By linear wave theory the amplitude of the water particle on the bottom

$$A = \frac{H}{2} \frac{1}{\sinh\left(\frac{2 \pi h}{L}\right)} = 0.68 \ m$$

The maximum horizontal velocity of water particle on the bottom is

$$U_m = A \, \omega = A \frac{2 \pi}{T} = 0.53 \ m/s$$

The wave friction coefficient is

$$f_w = exp\left(5.5 \left(\frac{k_s}{A}\right)^{0.2} - 6.3\right) = 0.028$$

The maximum bottom shear stress by wave is

$$\tau_{w,max} = \frac{1}{2} \rho \, f_w \, U_m^2 = 4 \ N/m^2$$

The mean bottom shear stress under combined wave and current

$$\tau_{wc} = \tau_c + \frac{1}{2} \tau_{w,max} = 5.24 \ N/m^2$$

The bed-load transport is

$$q_B = 2\, d_{50} \sqrt{\frac{\tau_c}{\rho}}\, exp\left(\frac{-0.27\,(s-1)\, d_{50}\, \rho\, g}{\mu_r\, \tau_{wc}} \right) = 0.0000153 \, \frac{m^3}{m \times s}$$

The friction velocity $u_{*,wc} = \sqrt{\frac{\tau_{wc}}{\rho}} = 0.071\ m/s$

Therefore $A = \frac{k_s}{h} = 0.01$ $z_* = \frac{\omega_s}{\kappa\, u_{*,wc}} = 0.70$

and we get the Einstein integrals by numerical integration

$$I_1 = 0.216\, \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \mathrm{d}B = 1.83$$

$$I_2 = 0.216\, \frac{A^{(z_*-1)}}{(1-A)^{z_*}} \int_A^1 \left(\frac{1-B}{B}\right)^{z_*} \ln B\, \mathrm{d}B = -3.73$$

The suspended sediment transport is

$$\begin{aligned} q_S &= 1.83\, q_B \left(I_1\, \ln\left(\frac{h}{0.033\, k_s}\right) + I_2 \right) \\ &= 1.83 \times 0.0000153 \left(1.83 \times \ln\left(\frac{2}{0.033 \times 0.02}\right) - 3.75 \right) \\ &= 0.000306 \, \frac{m^3}{m * s} \end{aligned}$$

The total sediment transport under combined wave and current is

$$q_T = q_B + q_S = 0.000321 \, \frac{m^3}{m \times s}$$

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