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Nonlinear periodic response analysis of mooring cables using harmonic balance ² method

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6 Abstract

3

Mooring cables are critical components of ocean renewable energy systems including offshore floating wind turbines and wave energy converters. Mooring cable dynamics is strongly nonlinear resulting from the geometric effect, 8 hydrodynamic loads and probably seabed interactions. Time-domain methods are commonly used for numerical sim-9 ulation. This study formulates a nonlinear frequency domain multi-harmonic balance method for efficient analysis of 10 a mooring cable subjected to periodic fairlead motions. The periodic responses are of particular interest to investi-11 gate the mooring effect on the platform. In the formulation, the governing equations of the three-dimensional cable 12 motions are spatially discretized using the finite difference method; the nonlinear ordinary differential equations are 13 subsequently transformed into frequency domain by expanding both the structural responses and the nonlinear nodal 14 forces using truncated Fourier series, leading to a set of nonlinear algebraic equations of the Fourier coefficients. The 15 equations are eventually solved using Newton's method where the alternating frequency/time domain method is used 16 to handle the nonlinearity effect. The presented method is then compared to a time-domain method by numerical 17 studies of a mooring cable. The results show that the method is of comparable accuracy as the time-domain method 18 while it is generally more efficient. The proposed method shows promising results even when the cable tension be-19 comes non-positive for a period of time during the cable motion, which is a known ill-posed problem for time-domain 20

²¹ methods.

22 Keywords: Mooring cables; nonlinear dynamics; harmonic balance method; periodic response; alternating

²³ frequency/time domain technique.

24 1. Introduction

Offshore winds and waves are promising renewable energy sources and are receiving intensive research attention 25 recently. Modeling mooring systems is one of the challenging tasks in simulation and design of such floating offshore 26 structures [1, 2]. Several comparison studies have already shown the importance of mooring cable dynamics on 27 28 floating wind turbines [3–7]. In the last decade, a number of cable models have been explored, validated or coupled with the multi-body dynamics of floating offshore wind turbines and wave energy devices for numerical simulation, 29 including the finite element model [8, 9], the multi-body dynamics model [10], the lumped mass models [11, 12] and 30 the finite difference model [13–17]. A review of the available models and simulation tools of mooring cables can be 31 found in [18, 19]. Presently, mathematical modeling of mooring cables is still a topic area, e.g. a high-order spectral 32 method has been developed by [20, 21] and modeling cables using bar elements in an open-source library has been 33 conducted in [22]. 34

³⁵ Despite a large number of models available for dynamic analyses of the mooring cables, the understanding of the ³⁶ mooring cable dynamics is still limited. This is due to the complex nonlinearity arising from the geometric effect, ³⁷ hydrodynamic loads and the seabed contact. Besides, for nonlinear analysis, hundreds of degrees of freedom of one

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Nomenclature

()	variables are constant or dependent of cable static	d	cable diameter
	solution	е	error
$b_{n-1/2}$	vectorized Fourier coefficients of $\hat{\mathbf{f}}_{n-1/2}$	EA	cable axial stiffness
$\mathbf{c}_k^{()}, \mathbf{s}_k^{()}$	Fourier coefficients of \mathbf{y}_n or $\mathbf{f}_{n-1/2}$, indicated by the superscripts	F_{dt}, F_{dn}	, F_{db} drag forces per unit length in the local coordinate
β	structural damping coefficient	$F_{\rm v}$ $F_{\rm v}$	F _z fairlead forces
$h_{n-1/2}, h$	$\mathbf{h}_1, \mathbf{h}_N$ residual vector of the resulting nonlinear al-	h 1	static/initial cable depth and radius
	gebraic equations corresponding to intermediate	i i	indexes
	and boundary nodes	l, j k	index of harmonics in Fourier series
$\mathbf{I}_{()}$	identity matrix with dimension indicated by the	к I о	unstretched cable length
$\sigma(t)$	subscript	L ₀	cable mass per unit length
$\mathbf{q}(t)$	vector containing the sine and cosine series $\begin{bmatrix} z & z \\ z \end{bmatrix}^T$	<i>m</i>	added mass per unit length
$\mathbf{y}(s,t)$	$= \begin{bmatrix} \hat{\varepsilon} \ u \ v \ w \ \phi \ \theta \end{bmatrix}, \text{ vector of nodal variables}$	m _a N	added mass per unit length
\mathbf{Z}_n	vectorized Fourier coefficients of \mathbf{y}_n	IN	
$\mathcal{F}, \mathcal{F}'$	inverse FF1 and FF1 operators	n	cable node index
$\Delta \mathbf{z}_n$	time derivatives	N _c	number of Fourier coefficients for each unknown
	cable segment length between node $n-1$ and node	N_h	number of harmonics retained in truncated Fourier
Δs_{n-1}	n	N	
γ	relaxation factor	IN _t	number of time instances used for discretization
$\hat{f}_{n-1/2}$	vectorized $\tilde{\mathbf{f}}_{n-1/2}$ samples in one oscillation period	N	number of rodal variables
$\hat{\mathbf{y}}_n$	vectorized \mathbf{y}_n samples in one oscillation period	IV _y	are length according to of the unstratched cable
M , K , f	mass and stiffness matrices, and force vector after	S	arc length coordinate of the unstretched cable
	moving all nonlinear terms to the force vector	t	
$\mathbf{Q}(\omega)$	Fourier series sampled at discrete time points	u(s,t), v	v(s,t), w(s,t) cable velocity in tangential, normal
ω, T_f	characteristic angular frequency and period of the		and DI-normal directions in the moving La-
	forced fairlead motion	U(t) V(t)	(t) $W(t)$ forced fairled velocities at time t and
8	Kronecker product operator	U(l), V(l)	node N in vertical horizontal and out-of-plane di-
$\phi, heta$	angles		rections of the fixed reference frame
ρ_w	the density of water	U V	W current velocities in the vertical and horizontal
Θ, Θ_k	partial differential operator in frequency domain	$U_c, V_c,$	and out-of-plane directions of the fixed reference
$\tilde{()}$	and its block element		coordinate system
()	an integer to account for subharmonics in Fourier	$u_{r}(s,t)$.	$v_{r}(s, t), w_{r}(s, t)$ relative velocities of the cable with
υ	series	, (0, .),	respect to fluid current in the moving Lagrangian
$\varepsilon(s,t)$	cable strain		reference frame
C_{dt} . C_{du}	C_{dh} drag coefficients in tangential. normal and bi-	w_0	submerged cable weight per unit length
- ui , ~ un	normal directions of the moving Lagrangian coor-	We	effective cable submerged weight per unit length
	dinate system	-	considering seabed effect

cable need to be considered for accuracy and hence the computational demand is another difficulty. Characterizing

³⁹ the cable dynamics is important for improving the computational efficiency for the coupled analyses, e.g. by model

⁴⁰ reduction and also for the interpretation of the coupled analysis results. In this context, this study focuses on the

research gap of nonlinear responses of a mooring cable subjected to periodic fairlead excitations, which represent an important subset of the cable dynamics and also are important for understanding the nonlinear mooring loads on the

structures in the steady state. For understanding the dynamic behavior of submerged cables, linearization methods

⁴⁴ [23–25] including linearization based frequency domain methods have been used to study towed cable dynamics [26].

⁴⁵ The second-order nonlinear dynamics of catenary pipelines/cables have been studied using a perturbation technique

⁴⁶ based on the finite difference model [27]. However, those methods can only give approximate solutions of the cable
 ⁴⁷ responses. In this study, the nonlinear periodic motion is proposed to be solved efficiently and accurately using a
 ⁴⁸ multi-Harmonic Balance (multi-HB) method.

The harmonic balance method may date back to [28-30] and it has been widely used as an efficient method 49 for computing periodic and steady-state responses of nonlinear systems and, hence to gain insight into system non-50 linear characteristics. Furthermore, the introduction of the Fast Fourier Transform (FFT) and the Alternating Fre-51 quency/Time (AFT) technique [31-34] enables the use of multiple (high-order) harmonics and accurate consideration 52 of strong nonlinearity such as friction. With the AFT technique, it has been shown that the Jacobian matrix of the 53 nonlinear algebraic equations resulting from multi-HB analysis can be formulated analytically, even for stiff systems 54 with friction interfaces, which guarantees the computational efficiency. Currently, the multi-HB method is capable 55 of studying the stability and nonlinear normal modes of large nonlinear systems as described in [35, 36]. It has been 56 applied to aerospace structures [37], flexible structures with local nonlinear attachments [38–40], stay cables [41], and 57 nonlinear mechanical systems [42], to name but a few. The single-term harmonic balance has been used for linearizing 58 mooring dynamics by [24, 43, 44]. The multi-HB method, however, has not been applied to submerged cables with 59 hydrodynamic effects so far. 60

This paper is structured as follows. After this introduction, Section 2 presents the nonlinear hydrodynamics of mooring cables along with a finite difference method for spatial discretization. Section 3 formulates the multi-HB method for mooring cables together with the AFT technique. Numerical studies are presented in Section 4 to demonstrate the effectiveness and advantages of the method by comparison with a time-domain method. A brief conclusion is provided in Section 5.

66 2. Nonlinear hydrodynamics of mooring cables

The mooring cable under consideration has uniform properties and circular or annular cross-section with the outter diameter *d*, mass per unit length *m*, and submerged weight per unit length w_0 when unstretched. A linear strain and tension relationship is considered with *EA* denoting the axial stiffness. The unstretched cable length is denoted by L_0 . The density of water is denoted by ρ_w . The initial cable depth and radius are denoted by *h* and *l* respectively, as shown in Fig. 1





Figure 1: Submerged mooring cable and the coordinate system for describing its motion: (a) front view; (b) top view.

72 2.1. Governing Equations

The cable model derived in [13, 14] is used here. The bending and torsional stiffnesses are ignored because they 73 are quite small for mooring cables and hence have limited effects on the cable responses. The coordinate systems 74 for describing the three-dimensional mooring configuration and motion are shown in Fig. 1. The origin of the fixed 75 reference frame (X, Y, Z) is located at the cable anchorage on the seabed with X - Y plane as the vertical plane defined 76 by the anchor and the initial cable top end location and the X-axis is pointing upwards. A moving Lagrangian reference 77 frame (x, y, z) is attached to the cable at an arc length s of the unstretched cable measuring from the seabed anchor. 78 The x-axis is aligned with the local tangent direction. The angle between x-axis and X-axis in X - Y plane is denoted 79 by ϕ and the angle between x-axis and Y-axis in Y - Z plane is denoted by θ . The cable curvatures in the X - Y plane 80 and the Y – Z plane are then defined by $\partial \phi / \partial s$ and $\partial \theta / \partial s$ respectively. For the case concerned here where the cable is 81 axisymmetric and the bending and torsion are ignored, the two angles (ϕ, θ) along with s are found to be sufficient to 82 define the cable configuration [13]. The normal and binormal directions (y, z) of the local reference frame are defined 83 correspondingly after the transformation, using the two angles, to align X-axis to the local tangent direction. 84

⁸⁵ Considering a steady current velocity with three components in the fixed reference frame, denoted by U_c , V_c and ⁸⁶ W_c respectively, from the balance of forces in the Lagrangian reference frame together with the compatibility relations, ⁸⁷ the partial differential equations (PDEs) governing the cable motion are given as [13]

$$0 = EA\frac{\partial\varepsilon}{\partial s} - m\frac{\partial u}{\partial t} + mv\cos\theta\frac{\partial\phi}{\partial t} - mw\frac{\partial\theta}{\partial t} + EA\beta\frac{\partial\varepsilon}{\partial t} - w_0\cos\phi\cos\theta + F_{dt}$$
(1)

$$0 = EA\varepsilon\cos\theta\frac{\partial\phi}{\partial s} - (m+m_a)\frac{\partial v}{\partial t} - m(u\cos\theta + w\sin\theta)\frac{\partial\phi}{\partial t} - C_m\rho_w\frac{\pi d^2}{4}(U_c\cos\phi + V_c\sin\phi)\frac{\partial\phi}{\partial t} + w_0\sin\phi + F_{dn}$$
(2)

$$0 = -EA\varepsilon\frac{\partial\theta}{\partial s} - (m + m_a)\frac{\partial w}{\partial t} + mv\sin\theta\frac{\partial\phi}{\partial t} - C_m\rho_w\frac{\pi d^2}{4}(U_c\sin\phi\sin\theta - V_c\cos\phi\sin\theta)\frac{\partial\phi}{\partial t} + mu\frac{\partial\theta}{\partial t}$$
(3)

$$+ C_m \rho_w \frac{\pi d^2}{4} (U_c \cos\phi\cos\theta + V_c \sin\phi\cos\theta - W_c \sin\theta) \frac{\partial\theta}{\partial t} - w_0 \cos\phi\sin\theta + F_{db}$$

$$0 = \frac{\partial u}{\partial s} - \frac{\partial \varepsilon}{\partial t} + w \frac{\partial \theta}{\partial s} - v \frac{\partial \phi}{\partial s} \cos \theta$$
(4)

$$0 = \frac{\partial v}{\partial s} - (1 + \varepsilon)\cos\theta\frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial s}\cos\theta(u + w\tan\theta)$$
(5)

$$0 = \frac{\partial w}{\partial s} + (1 + \varepsilon)\frac{\partial \theta}{\partial t} - v\frac{\partial \phi}{\partial s}\sin\theta - \frac{\partial \theta}{\partial s}u$$
(6)

where $\varepsilon(s, t) = \text{cable strain}; u(s, t), v(s, t)$ and w(s, t) are the tangential, normal, and bi-normal components of cable velocity. The buoyancy effect is included in calculation of the submerged weight per unit length w_0 ; the Froude-Krylov force, hydrodynamic mass and drag forces are considered using modified Morison's formula [45]. The added mass is calculated by $m_a = C_a \rho_w \pi d^2/4$ with the added mass coefficient denoted by C_a . The inertia coefficient is given as $C_m = 1 + C_a$. A structural damping term is added into Eq. (1) as $\beta EA\partial\varepsilon/\partial t$ assuming proportional damping [12]. The cable velocities relative to the fluid in the Lagrangian reference frame are denoted as $u_r(s, t), v_r(s, t)$ and $w_r(s, t)$, i.e.

$$u_r = u - (U_c \cos\phi\cos\theta + V_c \sin\phi\cos\theta - W_c \sin\theta)$$
(7)

$$v_r = v - (-U_c \sin \phi + V_c \cos \phi) \tag{8}$$

$$w_r = w - (U_c \cos\phi \sin\theta + V_c \sin\phi \sin\theta + W_c \cos\theta)$$
(9)

⁹⁵ The hydrodynamic drag forces are given as

$$F_{dt} = -\frac{1}{2}\rho_w d\pi C_{dt} |u_r| u_r \sqrt{1+\varepsilon}$$
⁽¹⁰⁾

$$F_{dn} = -\frac{1}{2}\rho_w dC_{dn} v_r \sqrt{v_r^2 + w_r^2} \sqrt{1 + \varepsilon}$$
⁽¹¹⁾

$$F_{db} = -\frac{1}{2}\rho_w dC_{db} w_r \sqrt{v_r^2 + w_r^2} \sqrt{1 + \varepsilon}$$
(12)

- where C_{dt} , C_{dn} , and C_{db} are the drag coefficients in tangential, normal and binormal directions.
- ⁹⁷ 2.2. Incremental form of the governing equations
- ⁹⁸ For dynamic analysis, the static solution is assumed to be known, which fulfills the static equations that

$$0 = EA \frac{d\bar{\varepsilon}}{ds} - w_0 \cos \bar{\phi} \cos \bar{\theta} + \bar{F}_{dt}$$
(13)

$$0 = EA\bar{\varepsilon}\cos\bar{\theta}\frac{d\phi}{ds} + w_0\sin\bar{\phi} + \bar{F}_{dn}$$
(14)

$$0 = -EA\bar{\varepsilon}\frac{\mathrm{d}\theta}{\mathrm{d}s} - w_0 \cos\bar{\phi}\sin\bar{\theta} + \bar{F}_{db}$$
(15)

- ⁹⁹ where $\bar{\varepsilon}, \bar{\phi}$ and $\bar{\theta}$ are the static solutions, and $\bar{F}_{dt}, \bar{F}_{dn}$ and \bar{F}_{db} are hydaulic drag forces when the cable is at rest. The
- ¹⁰⁰ preceding equations can be rewritten as

$$\frac{d\bar{\varepsilon}}{ds} = \frac{1}{EA} \left(w_0 \cos \bar{\phi} \cos \bar{\theta} - \bar{F}_{dt} \right) \tag{16}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = -\frac{1}{EA\bar{\varepsilon}\cos\bar{\theta}}\left(w_0\sin\bar{\phi} + \bar{F}_{dn}\right) \tag{17}$$

$$\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}s} = -\frac{1}{EA\bar{\varepsilon}} \left(w_0 \cos\bar{\phi}\sin\bar{\theta} - \bar{F}_{db} \right) \tag{18}$$

¹⁰¹ Correspondingly, the relative velocity of the static cable with respect to water is defined as

 $\begin{aligned} \bar{u}_r &= -U_c \cos \bar{\phi} \cos \bar{\theta} - V_c \sin \bar{\phi} \cos \bar{\theta} + W_c \sin \bar{\theta} \\ \bar{v}_r &= U_c \sin \bar{\phi} - V_c \cos \bar{\phi} \\ \bar{w}_r &= -U_c \cos \bar{\phi} \sin \bar{\theta} - V_c \sin \bar{\phi} \sin \bar{\theta} - W_c \cos \bar{\theta} \end{aligned}$

¹⁰² The cable state can be expressed as the summation of its static and dynamic components as

$$\varepsilon(s,t) = \bar{\varepsilon}(s) + \tilde{\varepsilon}(s,t), \phi(s,t) = \bar{\phi}(s) + \tilde{\phi}(s,t), \theta(s,t) = \bar{\theta}(s) + \tilde{\theta}(s,t)$$
(19)

¹⁰³ and hence the governing equations are rewritten in an incremental form as

$$0 = EA \frac{\partial(\bar{\varepsilon} + \tilde{\varepsilon})}{\partial s} - m \frac{\partial u}{\partial t} + mv \cos\theta \frac{\partial \tilde{\phi}}{\partial t} - mw \frac{\partial \tilde{\theta}}{\partial t} + EA\beta \frac{\partial \tilde{\varepsilon}}{\partial t} - w_0 \cos\phi \cos\theta + F_{dt}$$
(20)
$$\frac{\partial(\bar{\phi} + \tilde{\phi})}{\partial t} - \frac{\partial v}{\partial t} = 0$$

$$0 = EA(\bar{\varepsilon} + \tilde{\varepsilon})\cos\theta\frac{\partial(\phi + \phi)}{\partial s} - (m + m_a)\frac{\partial v}{\partial t} - m\left[u\cos\theta + w\sin\theta\right]\frac{\partial\tilde{\phi}}{\partial t} - C_m\rho_w\frac{\pi d^2}{4}\left(U_c\cos\phi + V_c\sin\phi\right)\frac{\partial\tilde{\phi}}{\partial t} + w_0\sin\phi + F_{dn}$$
(21)

$$0 = -EA\varepsilon \frac{\partial(\bar{\theta} + \tilde{\theta})}{\partial s} - (m + m_a)\frac{\partial w}{\partial t} + mv\sin\theta\frac{\partial\tilde{\phi}}{\partial t} - C_m\rho_w\frac{\pi d^2}{4} (U_c\sin\phi\sin\theta - V_c\cos\phi\sin\theta)\frac{\partial\tilde{\phi}}{\partial t}$$
(22)

$$+ mu\frac{\partial\theta}{\partial t} + C_m\rho_w\frac{d^2}{4} \left(U_c\cos\phi\cos\theta + V_c\sin\phi\cos\theta - W_c\sin\theta\right)\frac{\partial\theta}{\partial t} - w_0\cos\phi\sin\theta + F_{db}$$
$$\frac{\partial\mu}{\partial t} = \frac{\partial(\bar{\theta} + \tilde{\theta})}{\partial(\bar{\theta} + \tilde{\theta})} + \frac{\partial(\bar{\theta} + \tilde{\theta})}{\partial(\bar{\theta} + \tilde{\theta})} = 0$$

$$0 = \frac{\partial u}{\partial s} - \frac{\partial \varepsilon}{\partial t} + w \frac{\partial(\theta + \theta)}{\partial s} - v \frac{\partial(\phi + \phi)}{\partial s} \cos(\bar{\theta} + \bar{\theta})$$
(23)

$$0 = \frac{\partial v}{\partial s} - (1 + \bar{\varepsilon} + \tilde{\varepsilon})\cos(\theta + \tilde{\theta})\frac{\partial \phi}{\partial t} + \frac{\partial(\phi + \phi)}{\partial s}\cos(\bar{\theta} + \tilde{\theta})\left[u + w\tan(\bar{\theta} + \tilde{\theta})\right]$$
(24)

$$0 = \frac{\partial w}{\partial s} + (1 + \bar{\varepsilon} + \tilde{\varepsilon})\frac{\partial \tilde{\theta}}{\partial t} - v\frac{\partial(\bar{\phi} + \tilde{\phi})}{\partial s}\sin(\bar{\theta} + \tilde{\theta}) - \frac{\partial(\bar{\theta} + \tilde{\theta})}{\partial s}u$$
(25)

104 2.3. Equations in matrix form

$$\mathbf{M}(\mathbf{y})\frac{\partial \mathbf{y}}{\partial t} + \mathbf{K}(\mathbf{y})\frac{\partial \mathbf{y}}{\partial s} + \mathbf{f}(\mathbf{y}) = 0$$
(26)

where the nodal state vector is $\mathbf{y}(s,t) = [\tilde{\varepsilon}, u, v, w, \tilde{\phi}, \tilde{\theta}]^{\top}$, **M** and **K** are mass and stiffness matrices, and **f** is the nodal force vector. They are given as below

$$\mathbf{K} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & EA\varepsilon\cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & -EA\varepsilon \\ 0 & 1 & 0 & 0 & -v\cos\theta & w \\ 0 & 0 & 1 & 0 & u\cos\theta + w\sin\theta & 0 \\ 0 & 0 & 0 & 1 & -v\sin\theta & -u \end{bmatrix}$$
(27)

$$\mathbf{M} = \begin{bmatrix} EA\beta & -m & 0 & 0 & mv\cos\theta & -mw \\ 0 & 0 & -(m+m_a) & 0 & M_{2,5} & 0 \\ 0 & 0 & 0 & -(m+m_a) & M_{3,5} & M_{3,6} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{5,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\varepsilon \end{bmatrix}$$
(28)

109 where

108

$$M_{2,5} = -m(w\sin\theta + u\cos\theta) - C_m\rho_w \frac{\pi d^2}{4} (U_c\cos\phi + V_c\sin\phi)$$
$$M_{3,5} = mv\sin\theta - C_m\rho_w \frac{\pi d^2}{4} (U_c\sin\phi\sin\theta - V_c\cos\phi\sin\theta)$$
$$M_{3,6} = mu + C_m\rho_w \frac{\pi d^2}{4} (U_c\cos\phi\cos\theta + V_c\sin\phi\cos\theta - W_c\sin\theta)$$
$$M_{5,5} = -(1+\varepsilon)\cos\theta$$

and the force vector $\mathbf{f} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}^{\top}$

$$f_1 = -w_0 \cos\phi \cos\theta + F_{dt} + EA \frac{d\bar{\varepsilon}}{ds}$$
(29)

$$f_2 = w_0 \sin \phi + F_{dn} - EA\varepsilon \cos \theta \frac{\mathrm{d}\phi}{\mathrm{d}s}$$
(30)

$$f_3 = -w_0 \cos\phi \sin\theta + F_{db} - EA\varepsilon \frac{d\theta}{ds}$$
(31)

$$f_4 = -v\cos\theta \frac{\mathrm{d}\bar{\phi}}{\mathrm{d}s} + w\frac{\mathrm{d}\theta}{\mathrm{d}s}$$
(32)

$$f_5 = (u\cos\theta + w\sin\theta)\frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{33}$$

$$f_6 = -v\sin\theta \frac{\mathrm{d}\bar{\phi}}{\mathrm{d}s} + u\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}s} \tag{34}$$

Noting that \mathbf{M}, \mathbf{K} and \mathbf{f} depend on \mathbf{y} , for the convenience of formulating the harmonic balance analysis, the governing equations are rewritten by collecting all the \mathbf{y} -dependent terms in the nodal force vector such that

$$\bar{\mathbf{M}}\frac{\partial \mathbf{y}}{\partial t} + \bar{\mathbf{K}}\frac{\partial \mathbf{y}}{\partial s} + \tilde{\mathbf{M}}(\mathbf{y})\frac{\partial \mathbf{y}}{\partial t} + \tilde{\mathbf{K}}(\mathbf{y})\frac{\partial \mathbf{y}}{\partial s} + \mathbf{f}(\mathbf{y}) = \mathbf{0}$$
(35)

The mass and stiffness matrices, and force vector thus become

$$\bar{\mathbf{M}} = \begin{bmatrix} EA\beta & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & -m - m_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -m - m_a & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(1 + \bar{\varepsilon})\cos\bar{\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 + \bar{\varepsilon} \end{bmatrix}$$
(36)

114

$$\tilde{\mathbf{K}} = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & EA\bar{\varepsilon}\cos\bar{\theta} & 0 \\ 0 & 0 & 0 & 0 & 0 & -EA\bar{\varepsilon} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
(37)

115 and

As in [13–16], Eq. (26) along with boundary conditions can be discritized in both time and space using the finite difference method and then solved using the relaxation method [46]. For formulating the multi-HB analysis, the equation is only spatially discretized using the finite difference method.

119 2.4. Spatial discretization

For spatial discretization, the spatial derivatives Eq. (26) are replaced by the central differences [13, 16]. Let the cable be discretized into N - 1 segments with N nodal points in total along the cable length. The first node is at the

seabed origin and the *N*th node is at the cable top end. Hence, a set of N - 1 matrix equations (one equation per half grid) can be obtained as

$$\begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \left\{ \begin{array}{c} \dot{\mathbf{y}}_{n-1} \\ \dot{\mathbf{y}}_n \end{array} \right\} + \frac{1}{\Delta s_{n-1}} \begin{bmatrix} -\bar{\mathbf{K}}_{n-1} - \bar{\mathbf{K}}_n & \bar{\mathbf{K}}_{n-1} + \bar{\mathbf{K}}_n \end{bmatrix} \left\{ \begin{array}{c} \mathbf{y}_{n-1} \\ \mathbf{y}_n \end{array} \right\} + \begin{bmatrix} \tilde{\mathbf{M}}_{n-1} & \tilde{\mathbf{M}}_n \end{bmatrix} \left\{ \begin{array}{c} \dot{\mathbf{y}}_{n-1} \\ \dot{\mathbf{y}}_n \end{array} \right\} + \frac{1}{\Delta s_{n-1}} \begin{bmatrix} -\bar{\mathbf{K}}_{n-1} - \bar{\mathbf{K}}_n & \bar{\mathbf{K}}_{n-1} + \bar{\mathbf{K}}_n \end{bmatrix} \left\{ \begin{array}{c} \mathbf{y}_{n-1} \\ \mathbf{y}_n \end{array} \right\} + \mathbf{f}_{n-1} + \mathbf{f}_n = \mathbf{0}$$

$$(39)$$

124 which is further written as

$$\begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \left\{ \begin{array}{c} \dot{\mathbf{y}}_{n-1} \\ \dot{\mathbf{y}}_n \end{array} \right\} + \begin{bmatrix} -\bar{\mathbf{K}}_{n-1/2} & \bar{\mathbf{K}}_{n-1/2} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{y}_{n-1} \\ \mathbf{y}_n \end{array} \right\} + \tilde{\mathbf{f}}_{n-1/2} = \mathbf{0}$$
(40)

125 with

$$\bar{\mathbf{K}}_{n-1/2} = \left(\bar{\mathbf{K}}_{n-1} + \bar{\mathbf{K}}_n\right) / \Delta s_{n-1},
\tilde{\mathbf{f}}_{n-1/2} = \left[\tilde{\mathbf{M}}_{n-1} \ \tilde{\mathbf{M}}_n\right] \left\{ \begin{array}{c} \dot{\mathbf{y}}_{n-1} \\ \dot{\mathbf{y}}_n \end{array} \right\} + \frac{1}{\Delta s_{n-1}} \left[-\tilde{\mathbf{K}}_{n-1} - \tilde{\mathbf{K}}_n \ \tilde{\mathbf{K}}_{n-1} + \tilde{\mathbf{K}}_n \right] \left\{ \begin{array}{c} \mathbf{y}_{n-1} \\ \mathbf{y}_n \end{array} \right\} + \mathbf{f}_{n-1} + \mathbf{f}_n$$
(41)

126 2.5. Boundary conditions

¹²⁷ The cable is often fixed at the seabed anchor such that the velocity at the first node is constantly zero

$$u_1 = 0, v_1 = 0, w_1 = 0 \tag{42}$$

¹²⁸ On the other hand, the fairlead is subjected to excitations resulting from platform motion. Let the excitation velocity

¹²⁹ be represented by its three components in the fixed cable coordinate system, i.e. U(t), V(t) and W(t) respectively. ¹³⁰ Hence, the boundary equations at the fairlead node are given at time *t* as

$$0 = u_N \cos \phi_N \cos \theta_N - v_N \sin \phi_N + w_N \cos \phi_N \sin \theta_N - U(t)$$
(43)

 $0 = u_N \sin \phi_N \cos \theta_N + v_N \cos \phi_N + w_N \sin \phi_N \sin \theta_N - V(t)$ (44)

$$0 = -u_N \sin \theta_N + w_N \cos \theta_N - W(t) \tag{45}$$

Correspondingly the cable tension at the fairlead has three components in the fixed cable coordinate system, as illustrated in Fig. 1, given as

$$F_X(t) = EA\varepsilon_N \cos\phi_N \cos\theta_N \tag{46}$$

$$F_Y(t) = EA\varepsilon_N \sin\phi_N \cos\theta_N \tag{47}$$

$$F_Z(t) = -EA\varepsilon_N \sin\theta_N \tag{48}$$

¹³³ In cases where the fairlead force is known, the preceding three equations are the boundary conditions at the fairlead. ¹³⁴ This is usually the case for static analysis.

The mooring cable usually lies partly grounded on the seabed to avoid the lift force to the anchor. For considering 135 the cable-seabed contact effect, the method proposed in [15] is adopted herein. Flat seabed is considered and it is 136 modeled as elastic spring with stiffness k_{sb} which provides a vertical support force when the cable is grounded. This 137 can be easily accounted for by modifying the effective submerged cable weight. In other words, the effective weight 138 per unit length at node *n* is given as $w_e^n = w_0 + k_{sb}X(s_n)$ and $0 \le w_e^n \le w_0$ for static problem. In solving the PDEs using 139 iterative method, to consider the seabed effect, w_0 in Eqs. (29-34) is replaced by w_e^n which is evaluated based on the 140 cable nodal position obtained in the previous iteration step. After solving the equations in the Lagrangian coordinate 141 system, the cable nodal displacement and position can be integrated node by node from the seabed anchor using s, ϕ 142

143 and θ [13, 15].

144 **3.** Multi-harmonic balance analysis

The nonlinear ordinary equation (39) can also be solved in time domain by replacing the time derivatives using finite differences [16]. However, the time domain method may be subjected to numerical stability issues and to obtain the steady-state responses and a long time simulation may be required to arrive the steady state. This section therefore formulates the multi-HB method for efficiently solving the cable responses when it is subjected to periodic fairlead excitations.

150 3.1. Governing equations in frequency domain

¹⁵¹ Considering the cable subjected to a periodic excitation with a period T_f and the corresponding characteristic ¹⁵² angular frequency $\omega = 2\pi/T_f$ at its fairlead, the periodic cable response \mathbf{y}_n is pursued herein and hence the nodal ¹⁵³ force \mathbf{f}_n is also periodic. Therefore, they can be approximated using truncated Fourier series as follows,

$$\mathbf{y}_n(t) \approx \frac{\mathbf{c}_0^{y_n}}{\sqrt{2}} + \sum_{k=1}^{N_h} \left(\mathbf{s}_k^{y_n} \sin \frac{k\omega t}{\upsilon} + \mathbf{c}_k^{y_n} \cos \frac{k\omega t}{\upsilon} \right)$$
(49)

154

$$\tilde{\mathbf{f}}_{n-1/2}(t) \approx \frac{\mathbf{c}_0^{f_{n-1/2}}}{\sqrt{2}} + \sum_{k=1}^{N_h} \left(\mathbf{s}_k^{f_{n-1/2}} \sin \frac{k\omega t}{\upsilon} + \mathbf{c}_k^{f_{n-1/2}} \cos \frac{k\omega t}{\upsilon} \right)$$
(50)

¹⁵⁵ in which the index *k* represents the *k*th harmonic component and N_h = the number of harmonics retained. Noting ¹⁵⁶ that generally the constant terms need to be retained since the presence of the current may induce constant drift of the ¹⁵⁷ solution. The total number of coefficient for each degree of freedom is denoted by $N_c = 2N_h + 1$ for general cases and ¹⁵⁸ $N_c = 2N_h$ if the constant term is omitted. The integer v accounts for subharmonics of the excitation frequency ω . The ¹⁵⁹ coefficients $\mathbf{c}_k^{(.)}$ and $\mathbf{s}_k^{(.)}$ can be reshaped to $N_cN_y \times 1$ vectors as

$$\mathbf{z}_{n} = \begin{bmatrix} c_{0}^{y_{n,1}} & s_{1}^{y_{n,1}} & c_{1}^{y_{n,1}} & \cdots & s_{N_{h}}^{y_{n,1}} & c_{N_{h}}^{y_{n,1}} & \cdots & s_{N_{h}}^{y_{n,N_{y}}} & c_{N_{h}}^{y_{n,N_{y}}} \end{bmatrix}^{\mathsf{T}}$$
(51)

160

$$\mathbf{b}_{n-1/2} = \begin{bmatrix} c_0^{f_{n-1/2,1}} & s_1^{f_{n-1/2,1}} & c_1^{f_{n-1/2,1}} & \cdots & s_{N_h}^{f_{n-1/2,1}} & c_{N_h}^{f_{n-1/2,1}} & \cdots & s_{N_h}^{f_{n-1/2,N_y}} & c_{N_h}^{f_{n-1/2,N_y}} \end{bmatrix}^{\mathsf{T}}$$
(52)

where N_y denotes the number of variables at each node, i.e. $N_y = 6$ here. Note that the coefficient arrangements here are different from [47] for the convenience of using relaxation method in the subsequent solving procedure [46]. Then the nodal response and force can be recast into a compact form as

$$\mathbf{y}_{n}(t) = \left[\mathbf{I}_{N_{y}} \otimes \mathbf{q}(t)\right] \mathbf{z}_{n}$$
(53)

$$\mathbf{f}_{n-1/2}(t) = \left| \mathbf{I}_{N_y} \otimes \mathbf{q}(t) \right| \mathbf{b}_{n-1/2}$$
(54)

where \otimes stands for an operation on two matrices which gives another matrix that is formed by multiplying the second matrix by each element of the first matrix (known as Kronecker product of two matrices, see Appendix A for an example). The matrix \mathbf{I}_{N_y} is an identity matrix of size $N_y \times N_y$, and $\mathbf{q}(t)$ is a row vector containing the sine and cosine series

$$\mathbf{q}(t) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sin\frac{\omega t}{\upsilon} & \cos\frac{\omega t}{\upsilon} & \cdots & \sin\frac{k\omega t}{\upsilon} & \cos\frac{k\omega t}{\upsilon} & \cdots & \sin\frac{N_h \omega t}{\upsilon} & \cos\frac{N_h \omega t}{\upsilon} \end{bmatrix}$$
(55)

¹⁶⁸ From Eq. (53), one obtains

$$\dot{\mathbf{y}}_{n}(t) = \left[\mathbf{I}_{N_{y}} \otimes \dot{\mathbf{q}}(t)\right] \mathbf{z}_{n} = \left\{\mathbf{I}_{N_{y}} \otimes \left[\mathbf{q}(t)\boldsymbol{\Theta}\right]\right\} \mathbf{z}_{n}$$
(56)

¹⁶⁹ in which the matrix $\boldsymbol{\Theta}$ is given as

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{\Theta}_{1} & & & \\ & \ddots & & \\ & & \boldsymbol{\Theta}_{k} & \\ & & \ddots & \\ & & & \boldsymbol{\Theta}_{N_{h}} \end{bmatrix}$$
(57)

170 with block entries that

$$\mathbf{\Theta}_{k} = \begin{bmatrix} 0 & -k\omega/\nu \\ k\omega/\nu & 0 \end{bmatrix}$$
(58)

¹⁷¹ Substituting expressions (53,56) into Eq. (39), one obtains

$$\begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \{ \mathbf{I}_{N_y} \otimes [\mathbf{q}(t)\boldsymbol{\Theta}] \} \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \begin{bmatrix} -\bar{\mathbf{K}}_{n-1/2} & \bar{\mathbf{K}}_{n-1/2} \end{bmatrix} \mathbf{I}_{N_y} \otimes \mathbf{q}(t) \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \begin{bmatrix} \mathbf{I}_{N_y} \otimes \mathbf{q}(t) \end{bmatrix} \mathbf{b}_{n-1/2} = \mathbf{0}_{N_y \times 1}$$
(59)

where $\mathbf{0}_{N_y \times 1}$ represents a vector of size N_y containing zeros. The preceding equation can be further simplified as

$$\left\{ \begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \otimes \begin{bmatrix} \mathbf{q}(t) \Theta \end{bmatrix} \right\} \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \left\{ \begin{bmatrix} -\bar{\mathbf{K}}_{n-1/2} & \bar{\mathbf{K}}_{n-1/2} \end{bmatrix} \otimes \mathbf{q}(t) \right\} \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \begin{bmatrix} \mathbf{I}_{N_y} \otimes \mathbf{q}(t) \end{bmatrix} \mathbf{b}_{n-1/2} = \mathbf{0}_{N_y \times 1} \tag{60}$$

To eliminate the time dependency of the preceding equation, a Galerkin procedure projects the preceding equation on the orthogonal trigonometric basis of $\mathbf{q}(t)$, namely

$$\left\{ \begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \otimes \begin{bmatrix} \frac{2}{T_f} \int_0^{T_f} \mathbf{q}^{\mathsf{T}}(t) \mathbf{q}(t) \mathrm{d}t \Theta \end{bmatrix} + \begin{bmatrix} -\bar{\mathbf{K}}_{n-1/2} & \bar{\mathbf{K}}_{n-1/2} \end{bmatrix} \otimes \begin{bmatrix} \frac{2}{T_f} \int_0^{T_f} \mathbf{q}^{\mathsf{T}}(t) \mathbf{q}(t) \mathrm{d}t \end{bmatrix} \right\} \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \left\{ \mathbf{I}_{N_y} \otimes \begin{bmatrix} \frac{2}{T_f} \int_0^{T_f} \mathbf{q}^{\mathsf{T}}(t) \mathbf{q}(t) \mathrm{d}t \end{bmatrix} \right\} \mathbf{b}_{n-1/2} = \mathbf{0}_{N_y \times 1}$$
(61)

175 Note that

$$\frac{2}{T_f} \int_0^{T_f} \mathbf{q}^\top(t) \mathbf{q}(t) \mathrm{d}t = \mathbf{I}_{N_c}$$

¹⁷⁶ The governing equations are eventually expressed in frequency domain as

$$\left\{ \begin{bmatrix} \bar{\mathbf{M}}_{n-1} & \bar{\mathbf{M}}_n \end{bmatrix} \otimes \boldsymbol{\Theta} + \begin{bmatrix} -\bar{\mathbf{K}}_{n-1/2} & \bar{\mathbf{K}}_{n-1/2} \end{bmatrix} \otimes \mathbf{I}_{N_c} \right\} \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \mathbf{b}_{n-1/2} = \mathbf{0}_{N_c N_y \times 1}$$
(62)

177 The the left-hand side of the preceding equation is defined as the residual, i.e.

$$\mathbf{h}_{n-1/2} = \left[\bar{\mathbf{M}}_{n-1} \otimes \boldsymbol{\Theta} - \bar{\mathbf{K}}_{n-1/2} \otimes \mathbf{I}_{N_c} \ \bar{\mathbf{M}}_n \otimes \boldsymbol{\Theta} + \bar{\mathbf{K}}_{n-1/2} \otimes \mathbf{I}_{N_c} \right] \left\{ \begin{array}{c} \mathbf{z}_{n-1} \\ \mathbf{z}_n \end{array} \right\} + \mathbf{b}_{n-1/2}$$
(63)

for each intermediate node $1 < n \le N$. Similarly, for the boundary nodes, it reads

$$\mathbf{h}_1 = \mathbf{b}_1, \ \mathbf{h}_N = \mathbf{b}_N \tag{64}$$

For implementation of multi-HB method, it is crucial to determine \mathbf{b}_n and also the Jacobian matrix for gradient based correction of the solution. The AFT method is applied [32], as detailed in the following subsection.

181 3.2. AFT technique for handling nonlinearity

¹⁸² The expressions for the nonlinear nodal forces as expressed in Eq. (29-34) are difficult to be analytically trans-

formed into frequency domain to obtain the coefficients in the Fourier series. The AFT technique offers a convenient

184 procedure as

$$\mathbf{z}_{n}, \mathbf{z}_{n-1} \xrightarrow{\mathcal{F}^{-}} \mathbf{y}_{n}, \mathbf{y}_{n-1}, \dot{\mathbf{y}}_{n}, \dot{\mathbf{y}}_{n-1} \to \tilde{\mathbf{f}}_{n-1/2} \xrightarrow{\mathcal{F}^{+}} \mathbf{b}_{n-1/2}$$
(65)

where \mathcal{F}^- denotes the inverse FFT operator and correspondingly the FFT operator is denoted by \mathcal{F}^+ . In other words, in each iteration step, the nonlinear nodal force is obtained by evaluating Eq. (29-34) in time domain, using time series of the nodal state which are transformed from \mathbf{z}_n and \mathbf{z}_{n-1} using inverse FFT, and further the nodal force time series are transformed into frequency domain for \mathbf{b}_n and \mathbf{b}_{n-1} via FFT. Let the time period be discretized by N_t equally distributed sampling points in the FFT. One can define vectors $\hat{\mathbf{y}}$ and $\hat{\mathbf{f}}$ containing the concatenated $N_t \cdot N_y$ time samples of the nodal states and the forces, respectively. For the *n*th node, one thus obtains

$$\hat{\mathbf{y}}_n = \begin{bmatrix} y_{n,1}(t_1) & \cdots & y_{n,1}(t_{N_t}) & \cdots & y_{n,N_y}(t_1) & \cdots & y_{n,N_y}(t_{N_t}) \end{bmatrix}^{\top}$$
(66)

$$\hat{\mathbf{f}}_{n-1/2} = \begin{bmatrix} \tilde{f}_{n-1/2,1}(t_1) & \cdots & \tilde{f}_{n-1/2,1}(t_{N_t}) & \cdots & \tilde{f}_{n-1/2,N_y}(t_1) & \cdots & \tilde{f}_{n-1/2,N_y}(t_{N_t}) \end{bmatrix}^{\mathsf{T}}$$
(67)

¹⁹² The inverse FFT can then be written as a linear operation

$$\hat{\mathbf{y}}_n = \mathcal{F}^- \mathbf{z}_n, \hat{\mathbf{f}}_{n-1/2} = \mathcal{F}^- \mathbf{b}_{n-1/2}$$
(68)

¹⁹³ with the sparse operator

$$\mathcal{F}^{-} = \mathbf{I}_{N_{y}} \otimes \mathbf{Q}(\omega) \tag{69}$$

where $\mathbf{Q}(\omega)$ is the matrix of the time samples of trigonometrical functions

$$\mathbf{Q}(\omega) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \sin\frac{\omega t_1}{\upsilon} & \cos\frac{\omega t_1}{\upsilon} & \cdots & \sin\frac{N_h \omega t_1}{\upsilon} & \cos\frac{N_h \omega t_1}{\upsilon} \\ \frac{1}{\sqrt{2}} & \sin\frac{\omega t_2}{\upsilon} & \cos\frac{\omega t_2}{\upsilon} & \cdots & \sin\frac{N_h \omega t_2}{\upsilon} & \cos\frac{N_h \omega t_2}{\upsilon} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{2}} & \sin\frac{\omega t_{N_t}}{\upsilon} & \cos\frac{\omega t_{N_t}}{\upsilon} & \cdots & \sin\frac{N_h \omega t_{N_t}}{\upsilon} & \cos\frac{N_h \omega t_{N_t}}{\upsilon} \end{bmatrix}$$
(70)

¹⁹⁵ Similarly, the FFT to obtain the Fourier coefficients is written as

$$\mathbf{z}_n = \mathcal{F}^+ \hat{\mathbf{y}}_n, \mathbf{b}_{n-1/2} = \mathcal{F}^+ \hat{\mathbf{f}}_{n-1/2}$$
(71)

where the FFT operator is computed by $\mathcal{F}^+ = (\mathcal{F}^-)^\top [\mathcal{F}^-(\mathcal{F}^-)^\top]^{-1}$.

¹⁹⁷ The Jacobian matrix of the residual function (63) with respect to \mathbf{z}_{n-1} and \mathbf{z}_n can be obtained as

$$\frac{\partial \mathbf{h}_{n-1/2}}{\partial \mathbf{z}_{n-1}} = \left(\bar{\mathbf{M}}_{n-1} \otimes \boldsymbol{\Theta} - \bar{\mathbf{K}}_{n-1/2} \otimes \mathbf{I}_{N_c}\right) + \frac{\partial \mathbf{b}_{n-1/2}}{\partial \mathbf{z}_{n-1}}$$
(72)

198

$$\frac{\partial \mathbf{h}_{n-1/2}}{\partial \mathbf{z}_n} = \left(\bar{\mathbf{M}}_n \otimes \boldsymbol{\Theta} + \bar{\mathbf{K}}_{n-1/2} \otimes \mathbf{I}_{N_c}\right) + \frac{\partial \mathbf{b}_{n-1/2}}{\partial \mathbf{z}_n}$$
(73)

¹⁹⁹ The difficulty in evaluating the Jacobian matrix lies in the computation of $\partial \mathbf{b}_{n-1/2}/\partial \mathbf{z}_n$. This also requires the AFT ²⁰⁰ technique. Noting that $\mathbf{\tilde{f}}_{n-1/2}$ is a function of both \mathbf{y}_n and $\dot{\mathbf{y}}_n$ so that the Jacobian matrix computation needs to be ²⁰¹ written as

$$\frac{\partial \mathbf{b}_{n-1/2}}{\partial \mathbf{z}_n} = \frac{\partial \mathbf{b}_{n-1/2}}{\partial \hat{\mathbf{f}}_{n-1/2}} \frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \mathbf{z}_n} = \mathcal{F}^+ \frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \mathbf{z}_n} \\
= \mathcal{F}^+ \left(\frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \hat{\mathbf{y}}_n} \frac{\partial \hat{\mathbf{y}}_n}{\partial \mathbf{z}_n} + \frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \dot{\mathbf{y}}_n} \frac{\partial \dot{\mathbf{y}}_n}{\partial \mathbf{z}_n} \right) = \mathcal{F}^+ \frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \hat{\mathbf{y}}_n} \mathcal{F}^- + \mathcal{F}^+ \frac{\partial \hat{\mathbf{f}}_{n-1/2}}{\partial \dot{\mathbf{y}}_n} \left\{ \mathbf{I}_{N_y} \otimes \left[\mathbf{Q}(\omega) \mathbf{\Theta} \right] \right\} \tag{74}$$

Similar procedure is applicable for computing $\partial \mathbf{b}_{n-1/2}/\partial \mathbf{z}_{n-1}$. The same method is also applied for handling the 202 boundary nodal equations (64) but it is noteworthy that the FFT operator for boundary equations is of size $N_c \times 3N_t$ 203 here. From Eqs. (29-34) $\partial \tilde{\mathbf{f}}_{n-1/2}/\partial \mathbf{y}_n$ and $\partial \tilde{\mathbf{f}}_{n-1/2}/\partial \dot{\mathbf{y}}_n$ can be derived analytically (see Appendix B), which are then 204 evaluated at the sampled time instances in a period and further rearranged to obtain $\partial \mathbf{f}_{n-1/2}/\partial \hat{\mathbf{y}}_n$ and $\partial \mathbf{f}_{n-1/2}/\partial \hat{\mathbf{y}}_n$. 205 Once the residual and Jacobian matrix are available, Newton's method can be used for iteration to solve the equation. 206 In addition, for this two-point boundary valued problem spatially discretized using the first-order finite difference, 207 only the two neigboring nodes are coupled and hence the problem can be solved from the fairlead node by node 208 without assembling the global mass, stiffness matrices and the force vector. In iteration for solving the equations, the 209 coefficient vector \mathbf{z}_n for all the nodes are updated by 210

$$\mathbf{z}_n^{i+1} = \mathbf{z}_n^i + \gamma \Delta \mathbf{z}_n^i \tag{75}$$

- where $\Delta \mathbf{z}_n^i$ is the state increment, γ is the relaxation factor which is in the range of 0 and 1 to slow the update, and *i* is the iteration step index. The adjustment method of the relaxation factor proposed in [16] is applied here.
- the iteration step index. The adjustment method of the relaxation factor proposed in [16] is appli-The error is defined as [16]

$$e^{i} = \frac{1}{N_{c}N} \sum_{n=1}^{N} \sum_{j=1}^{N_{c}} \left| \Delta z_{n,j}^{i} \right|$$
(76)

214 3.3. Solving procedure

Providing the cable and environmental parameters and the cable static solution, the presented method can be used
 to conduct periodic responses analysis. Given the forced motion frequency, the FFT and inverse FFT operators can be
 prepared for use. The solving procedure is summarized as below.

- i Evaluate $\bar{\mathbf{M}}_n$ and $\bar{\mathbf{K}}_n$ for $1 \le n \le N$ from the static solution using Eqs. (36,37);
- ²¹⁹ ii Initialize \mathbf{z}_n for all nodes;
- iii Evaluate $\hat{\mathbf{y}}_n$ and $\dot{\mathbf{y}}_n$ using the inverse FFT, Eq. (68), and rearrange the vector to obtain $\mathbf{y}_n(t_j)$ and $\dot{\mathbf{y}}_n(t_j)$ for all nodes and time instances;
- iv Evaluate $\tilde{\mathbf{M}}_n$, $\tilde{\mathbf{K}}_n$ and \mathbf{f}_n for all nodes using Eqs. (38,29–34);
- v Evaluate $\tilde{\mathbf{f}}_{n-1/2}$, $\partial \tilde{\mathbf{f}}_{n-1/2}/\partial \mathbf{y}_n$, $\partial \tilde{\mathbf{f}}_{n-1/2}/\partial \mathbf{y}_{n-1}$, $\tilde{\mathbf{f}}_{n-1/2}/\partial \dot{\mathbf{y}}_n$ and $\tilde{\mathbf{f}}_{n-1/2}/\partial \dot{\mathbf{y}}_{n-1}$ using Eq. (40) and for boundary nodes using Eqs. (42-45);
- vi Rearrange $\mathbf{\tilde{f}}_{n-1/2}$ for all nodes and time instances to obtain $\mathbf{\hat{f}}_{n-1/2}$ and rearrange $\mathbf{\tilde{f}}_{n-1/2}/\partial \mathbf{y}_n$ and $\mathbf{\tilde{f}}_{n-1/2}/\partial \mathbf{y}_{n-1}$ to obtain $\mathbf{\hat{f}}_{n-1/2}/\partial \mathbf{\hat{y}}_{n-1}$, $\mathbf{\hat{f}}_{n-1/2}/\partial \mathbf{\hat{y}}_{n-1}$ and $\mathbf{\hat{f}}_{n-1/2}/\partial \mathbf{\hat{y}}_{n-1}$;
- vii Obtain $\mathbf{h}_{n-1/2}$ using Eq. (63) and \mathbf{h}_1 and \mathbf{h}_N using Eq. (64) and $\partial \mathbf{h}_{n-1/2}/\partial \mathbf{z}_n$ and $\partial \mathbf{h}_{n-1/2}/\partial \mathbf{z}_n$ using Eqs. (72,73) with the FFT and inverse FFT operators;
- viii Solve $\Delta \mathbf{z}_n$ and evaluate the error and update \mathbf{z}_n using Eq. (75);
- ix Check the error using Eq. (76) and repeat steps 3-8 before convergence.
- x Stop if convergence is achieved or the maximum number of iterations is reached.
- ²³² The method is implemented in C++ with Eigen library [48] for handling linear algebra, matrix and vector operations.

233 4. Application and discussion

In this section, a typical mooring cable is analyzed using the presented method and the results are compared with corresponding time-domain analysis results. The open-source mooring system simulation program developed by the authors, named OpenMOOR, which has been verified and applied in [7, 49], is used for the time-domain analysis. The generalized- α method is used for time stepping [50].

238 4.1. Description of the simulated mooring cable

The mooring cable of the OC3 Hywind platform for the spar-type floating offshore wind turbine is used [51, 52]. The cable properties are listed in Table 1 along with the hydrodynamic coefficients which are adopted following [5]. In the following numerical analyses, to focus on the multi-HB method, the cable is considered to be in still water and the seabed interaction is ignored here. In this case, the constant terms in the Fourier expansion, i.e. Eqs. (49,50), are omitted. Hence $N_c = 2N_h$ and correspondingly the first column of the matrix $\mathbf{Q}(\omega)$ in Eq. (70) is also eliminated.

The cable static profile is shown in Fig. 2 which is solved using a shooting procedure [15]. The cable is divided into 49 segments with 50 nodes. For a fair comparison of the computation efficiency, in using the time-domain method,

the static solution is used as the starting point. In harmonic balance analysis, the static solution is used along with zero dynamic responses as initial guess, i.e. $\mathbf{z}_n = \mathbf{0}$.



Figure 2: Static profile of the simulated cable (circles indicate the node position).

Table 1. Cable properties and environmental parameters					
Parameter	Symbol	Unit	Value		
Diameter	d	m	0.09		
Unstretched length	L_0	m	902.2		
Mass per unit length	m	kg/m	77.7066		
Submerged weight per unit length	w_0	N/m	698.094		
Elastic stiffness	EA	Ν	3.84243E8		
Static cable depth	h	m	250		
Static cable radius	l	m	848.67		
Added mass coefficient	C_a	-	1.0		
Drag coefficient	C_{dt}, C_{dn}, C_{db}	-	0, 1.6, 1.6		

Table 1: Cable properties and environmental parameter

248 4.2. Convergence of the multi-HB method

The convergence of the presented method is first studied. The cable is forced with harmonic motions at the 249 fairlead in the Y direction (surge motion). The frequency is considered to be 0.05 Hz when the non-linearity effect 250 can be clearly seen in the following results. In using the multi-HB method, the integer v = 1 is adopted since no sub-251 harmonic responses have been observed in this case; the initial relaxation factor is set to be 1.0, and the convergence 252 tolerance is considered to be 10^{-10} . The first case considers the amplitude of the fairlead displacement to be 5.0 m and 253 in a second case, the amplitude is increased to 9.0 m. The error evolution with respect to the iteration step is plotted in 254 Fig. 3 and the obtained cable tensions, $EA\varepsilon$, at the fairlead are plotted in Fig. 4. Results solved using different values 255 of the harmonic balance parameters are presented for comparison. 256

It can be seen from Fig. 3 that for all the cases, the computation is able to achieve a fast convergence within 257 ten steps. Generally, more iterations are required when higher-order harmonics are included and a larger number of 258 time points are used. More importantly, even for the difficult case when cable tension becomes non-positive during 259 the cable motion, which is known as an ill-posed problem for perfectly flexible cables [53], convergence can still 260 be achieved with the multi-HB method, as shown in Fig. 4 for the case when the forced motion has an amplitude 261 of 9.0 m and the fairlead tension is zero around t = 15 s. This is because the Jacobian matrix of the multi-HB 262 method is constructed from all the time instances in one period and therefore it is nonsingular even if the Jacobian 263 matrix of the time-domain equation is singular for some time instances. In Fig. 4, by comparing the results obtained 264 using 9 harmonics and more, it can be concluded that the solution is converged with 9 harmonics in these two cases. 265 Actually, with 3 harmonics the method can already achieve a quite high accuracy as compared to those obtained using 266 9 harmonics. 267



Figure 3: Error evolution in the multi-HB analysis (left) forced fairlead motion in *Y* direction of amplitude 5.0 m; (right) forced fairlead motion in *Y* direction of amplitude 9.0 m.



Figure 4: Fairlead tensions solved using the multi-HB method (left) forced fairlead motion in *Y* direction of amplitude 5.0 m; (right) forced fairlead motion in *Y* direction of amplitude 9.0 m.

268 4.3. Comparison of time- and frequency-domain methods

To verify the presented method and to show its advantages, simulations are carried out with the comparison to 269 the time-domain method. As shown in the preceding section, $N_h = 9$ and $N_t = 32$ are sufficient for the convergence 270 of the multi-HB analysis and they are thus adopted in the following analysis. In using the time-domain method, the 271 convergence tolerance is also set to be 10^{-10} . Time domain analyses commonly start with the static solution with 272 zero displacements and zero velocity, and hence some time is required to dissipate the transient responses so that 273 the cable motion can reach the steady state. In the analysis, the displacement is ramped to the target value in half 274 of a period and therefore at least two periods are required. The time needed for the transient responses to dying out 275 depends on the system damping. For mooring cables, due to the hydrodynamic drag effect, several periods may be 276 sufficient. As shown in Fig. 5, with the transverse drag coefficient of 1.6 in Table 1, the solution reaches the steady 277 state in 3 periods, while decreasing the drag coefficient to 0.1, at least four periods are required. The accuracy and 278 computational efficiency also depend on the time step, a test shows that a time step of 0.1 s is the minimum step to 279 prevent the numerical drift of the cable displacement at the fairlead which is a known issue of this finite difference 280 cable modal [16]. In the following, the time step is considered to be 0.1 s and the time-domain analysis is performed 281 for three periods of the forced motion. The cable responses in the third period are assumed to be the steady responses 282 to be compared with the harmonic balance analysis results. 283

Four cases are studied for comparison. In the first three cases, a forced motion of a displacement amplitude of 5.0 m and frequency of 0.05 Hz is considered respectively in *Y*, *X*, and *Z* directions. In the last case, the forced motion is considered in both *Y* and *Z* directions: the motion in the *Y* direction has a displacement amplitude of 2.5 m and

frequency of 0.1 Hz and the motion in the Z direction is of a displacement amplitude of 5.0 m and a frequency of 287 0.05 Hz. The computation times to obtain the steady-state response using the time-domain method and the multi-HB 288 method are compared in Table 2. Note that the value listed for the time-domain method is the time taken to complete 289 a three-period simulation. The computations were performed on a 16-core Windows desktop (Intel i7-8700 CPU @ 290 3.20 GHz). It is seen that the multi-HB method is much efficient and the computational time is almost equivalent to 291 the time needed to run the time-domain method for one period. Additionally, for such a nonlinear system, the time 292 required for the system to reach the steady state is not known beforehand and for systems with less damping, more 293 time is required as shown in Fig. 5. 294



Figure 5: Fairlead tension time history solved using the time domain method (left) with hydrodynamic drag coefficients $C_{dt} = 0$ and $C_{dn} = C_{db} = 1.6$; (right) with hydrodynamic drag coefficients $C_{dt} = 0$ and $C_{dn} = C_{db} = 0.1$.

Table 2: Computational efficiency comparison						
Case no.	Time-domain computation (s)	Multi-HB computation (s)	Description			
1	6.03	1.40	motion in Y direction			
2	6.07	1.40	motion in X direction			
3	6.47	2.43	motion in Z direction			
4	6.74	3.01	motion in both Y and Z direction			

The cable responses solved using the time-domain method and the multi-HB are compared in Figs. 6–9 where the fairlead tension and the nodal solution corresponding to the 25th node (with s = 441.11 m) are plotted. In most of the graphics, results obtained using the two methods are found to be pretty consistent. Relatively observable differences are seen in the fairlead tension and the nodal strain of Fig. 8 when the forced motion is in the out-of-plane direction. This is because the overall tension/strain variation is small since the forced motion is in the out-of-plane. Besides, the super-harmonic responses are clearly captured in the cable tension, as also reported in the numerical study using time domain methods by [54].

The numerical results clearly demonstrate that in the analyzed cases, the multi-HB method has achieved comparable accuracy as the time-domain analysis while it is more efficient. In addition, as seen from the solving procedure given in Section 3.3, the AFT technique requires the FFT and inverse FFT operations for all nodes which can be done in parallel for further improving the computational efficiency. To summarize, the multi-HB method is found to be advantageous for periodic analyses of mooring cables.

307 5. Conclusion

This study has proposed and formulated a multi-HB method for a three-dimensional mooring cable under periodic fairlead motion. The governing equations of the cable are first represented in an incremental form and then spatially discretized using the finite difference method. The nodal equations are transformed into the frequency domain by



Figure 6: Comparison of the fairlead tension and nodal solution corresponding to the 25th node with s = 441.11 m when the forced motion is in Y direction. Lines correspond to the time-domain analysis results and symbols indicate the multi-HB analysis results.

Fourier expansion. The AFT technique is applied to handle the geometrical and hydrodynamic non-linearity accurately. The presented method is implemented and compared with a time domain method based on numerical studies of a typical mooring cable. The following conclusions can be drawn:

i The multi-HB method together with the AFT technique is promising to solve periodic mooring cable motion. It can handle the geometric and hydrodynamic nonlinearity and the case when cable tension becomes zero.

ii The multi-HB method is accurate and more efficient as compared to the time-domain method for analyzing the
 periodic cable responses.

iii The method is able to capture the super-harmonic cable responses and is promising for further parametric analyses
 of mooring cables.

Future studies will focus on local and global stability analysis of mooring cables with nonlinear hydrodynamics based on the presented method.

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Figure 7: Comparison of the fairlead tension and nodal solution corresponding to the 25th node with s = 441.11 m when the forced motion is in X direction. Lines correspond to the time-domain analysis results and symbols indicate the multi-HB analysis results.

327 Appendix A. Kronecker product

The Kronecker product of two matrices is calculated by

$$\mathbf{A} \otimes \mathbf{E} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix} \otimes \mathbf{E} = \begin{bmatrix} a_{11}\mathbf{E} & a_{12}\mathbf{E} & \cdots & a_{1q}\mathbf{E} \\ a_{21}\mathbf{E} & a_{22}\mathbf{E} & \cdots & a_{2q}\mathbf{E} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}\mathbf{E} & a_{p2}\mathbf{E} & \cdots & a_{pq}\mathbf{E} \end{bmatrix}$$

where $a_{(\cdot,\cdot)} \in \mathbb{R}$ is an element of matrix \mathbf{A} $(p \times q)$.

329 Appendix B. Matrix differentiation

In the derivation of $\partial \tilde{\mathbf{f}}_{n-1/2}/\partial \mathbf{y}$, the following matrix differentiation expression is used

$$\frac{\partial \mathbf{M}(\mathbf{y})\mathbf{x}}{\partial \mathbf{y}} = \begin{vmatrix} \vdots \\ \mathbf{x}^{\top} \frac{\partial \mathbf{m}_{p}^{\top}}{\partial \mathbf{y}} \\ \vdots \end{vmatrix}$$
(B.1)

where **x** is a vector and \mathbf{m}_p denotes the *p*th row of the **M** matrix.



Figure 8: Comparison of the fairlead tension and nodal solution corresponding to the 25th node with s = 441.11 m when the forced motion is in Z direction. Lines correspond to the time-domain analysis results and symbols indicate the multi-HB analysis results.

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Figure 9: Comparison of the fairlead tension and nodal solution corresponding to the 25th node with s = 441.11 m when the forced motion is in both Y and Z directions. Lines correspond to the time-domain analysis results and symbols indicate the multi-HB analysis results.

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