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Nielsen, Morten; Sikic, Hrvoje

Publication date:
2010

Document Version
Early version, also known as pre-print

AALBORG UNIVERSITY

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by

Morten Nielsen and Hrvoje Šikić

R-2010-15 November 2010

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

MAXIMAL FUNCTIONS, PRODUCT CONDITION AND ITS ECCENTRICITY

MORTEN NIELSEN AND HRVOJE ŠIKIĆ*

Abstract. We characterize Muckenhoupt A_p weights in the product case on $\mathbb{R}^N$ in terms of a graded family of A_p conditions defined by rectangles with a lower bound on eccentricity. The connection to maximal functions and geometric coverings is also studied.

1. Introduction

In this paper we deal with some of the problems that arise in extending basic facts about the Hardy-Littlewood maximal function in one dimension to higher dimensions. Recall, see C. Fefferman [2], that the typical multiplier operator does not exhibit the same properties in both cases. For a detailed account of various problems we refer to S.-Y. Chang and R. Fefferman [1].

We focus here on the base of sets used to define maximal functions. For $f \in L^1_{\text{loc}}(\mathbb{R})$ one forms the means $\frac{1}{|B|}\int_B |f|\,dx$, using essentially one base $\mathcal{B} = \{B\}$ of open sets, namely the open intervals containing the point of interest. The fundamental result states that the corresponding maximal operator $M_{\mathcal{B}}$ is bounded on the L^p spaces, $1 < p < \infty$, with weight w if and only if w satisfies the Muckenhoupt A_p condition.

As is well known, when the dimension is greater than one, the base $\mathcal{B}$ that is used to define both the maximal function and the corresponding A_p condition, must be restricted to obtain boundedness of the corresponding maximal operator. In order to resolve this, B. Jawerth introduces a condition $(C)_p$ in [4] and proves a theorem (Theorem 3.4 in [4]) that plays the main role in our paper as well. Although the theorem is valid, there is a part of the proof that requires a correction. One of the contributions of our paper is to revisit some of Jawerth's ideas and straighten and clarify some proofs in [4].

There is, however, another interesting point that influenced our research in this matter. We recently observed (see our article [7]) that there is a connection between the A_p condition, which is a notion in harmonic analysis, and the Schauder basis property, which comes from functional analysis. The study of

2000 *Mathematics Subject Classification.* 42B25.

Key words and phrases. Maximal function, product condition, Muckenhoupt weight.

*The second named author is supported by the MZOS grant 037-0372790-2799 of the Republic of Croatia.

wavelets, Gabor systems, and other reproducing function systems leads naturally to the study of systems of translates, i.e., to shift invariant spaces. It turns out that properly ordered systems of translates form a Schauder basis if and only if an associated weight satisfies the A_p condition, see [7] for the scalar valued case, [6] for matrix valued weights, and K. Moen [5] for other recent developments. Let us observe that the appearance of the A_p condition in the study of shift invariant spaces is not an isolated event, but rather a well placed condition in the hierarchy of basis type conditions (see [3] for a recent systematic presentation).

Let us point out that the study of the A_p condition for non-scalar valued weights is well on its way (see A. Volberg [9] and references therein). However, the issue that we would like to address here is of interest even in the case of scalar valued weights. When one extends the base of open intervals to the higher dimensional case, one faces numerous candidates for the choice of base $\mathcal{B}$. For example, one could take the family of all squares (or, eqivalently, balls), or, as another example, the family of all rectangles. Recall that it is well known that the classes just selected do not induce equivalent A_p conditions. Furthermore, not only are the two choices above the most useful ones, but the rectangle A_p condition, which is also referred to as the product A_p condition, is also the proper one for the multivariate Schauder basis property (see [5, 6] for precise statements).

We propose to fill the gap between the two families by producing a monotone collection of bases that builds up to the full product condition. As pointed out to us (in a private communication) by K. Moen, one can develop abstract theorems solely based on the monotonicity property. We opt, however, for a particular choice of the collection of bases, since such classes are of independent interest and they also form the most natural choice to connect the non-product with the product condition.

We introduce the notion of eccentricity in order to successfully create a natural grading of all rectangles. Then associated Muckenhoupt A_p classes for each eccentricity are introduced. In general, a weight in the standard Muckenhoupt class A_p will be contained in each of the eccentricity A_p classes, but the corresponding A_p constants may not be uniformly bounded as the eccentricity varies. As one would intuitively hope for, the weights for which the eccentricity A_p constants are uniformly bounded are exactly the weights in the product A_p class. The proof of this theorem turned out to be more demanding than one may expect. In the following sections we present our results, while we leave the proofs for the last section.

Acknowledgements. We would like to thank Professor Guido L. Weiss for careful and detailed discussions about this article. We would also like to thank Professors Kabe Moen and Edward N. Wilson for several remarks and valuable suggestions to improve the presentation of this article.

2. Notation and Results

For fixed $d_1, \dots, d_k \in \mathbb{N}$, $N := \sum_j d_j$, we consider the product space

$$\mathcal{P} := \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \cdots \times \mathbb{R}^{d_k} \approx \mathbb{R}^N.$$

A rectangle in $\mathcal{P}$ is a product

$$R = B_1 \times B_2 \times \cdots \times B_k,$$

where B_j is an Euclidean ball in $\mathbb{R}^{d_j}$. We denote by $\mathcal{R}$ the family of all such rectangles in $\mathcal{P}$. The eccentricity of $R := B_1 \times \cdots \times B_k \in \mathcal{R}$ is defined to be

$$e(R) := \frac{\min_i |B_i|}{\max_j |B_j|},$$

with $|B_j|$ the Lebesgue measure of B_j in $\mathbb{R}^{d_j}$. For $0 < \delta \le 1$, we define the restricted class

$$\mathcal{R}^\delta := \{R \in \mathcal{R} : e(R) \ge \delta\}.$$

Notice that $\mathcal{R}^\delta \subseteq \mathcal{R}^\eta$ for $0 < \eta \le \delta$, and clearly,

$$\mathcal{R} = \bigcup_{\delta > 0} \mathcal{R}^\delta.$$

For notational convenience, we denote $\mathcal{R}^0 := \mathcal{R}$. For $\delta \in [0,1]$, and $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, we define the maximal function

$$M_\delta f(x) := \sup_{R \in \mathcal{R}^\delta} \frac{1}{|R|} \int_R |f(y)|\, dy. \tag{2.1}$$

It is easy to verify that

$$Mf(x) := M_0 f(x) = \sup_{\delta > 0} M_\delta f(x) = \lim_{\delta \to 0^+} M_\delta f(x).$$

The Muckenhoupt class $A_p(\mathcal{R}^\delta)$, $1 < p < \infty$, is defined to be the family of locally integrable weights $w : \mathbb{R}^N \to \mathbb{R}_+$ satisfying $[w]_{A_p(\mathcal{R}^\delta)} < \infty$, with

$$[w]_{A_p(\mathcal{R}^\delta)} := \sup_{R \in \mathcal{R}^\delta} \frac{1}{|R|} \int_R w(x) dx \cdot \left[\frac{1}{|R|} \int_R w(x)^{-p'/p} dx \right]^{p/p'},$$

where p' is the dual exponent to p, i.e., $1/p + 1/p' = 1$. It is easy to check that $[w]_{A_p(\mathcal{R}^\delta)} \le [w]_{A_p(\mathcal{R}^\eta)}$ whenever $0 \le \eta \le \delta$.

We notice that for a fixed $N \ge 3$, there are several ways to decompose N as a sum of integers; each choice gives rise to a unique class of rectangles. The "finest" decomposition $d_1 = \cdots = d_N := 1$ yields the largest class of rectangles, which consequently produces the smallest A_p-class.

The Muckenhoupt A_p condition is closely related to geometric covering properties. Following B. Jawerth [4], we introduce the covering property which plays the crucial role in the proof of our main theorem. Let $w : \mathbb{R}^N \to \mathbb{R}_+$ be a locally integrable weight. For a Borel set $\Omega \subseteq \mathbb{R}^N$, we let $w(\Omega) := \int_\Omega w(x)dx$.

We say that w satisfies condition S_δ with constant c provided that for any finite double sequence $\{R_j^k\}_{(j,k)\in H} \subset \mathcal{R}^\delta$, $H \subset \mathbb{Z} \times \mathbb{Z}$, there exists a double sequence of pairwise disjoint sets $\{E_j^k\}_{(j,k)\in H}$ such that

$$E_j^k \subseteq R_j^k \text{ for } (j,k) \in H \tag{2.2}$$

$$\sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} R_j^k\Big) \leq c \sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} E_j^k\Big) \tag{2.3}$$

$$\Big\| \sum_{j,k\in\mathbb{Z}} 2^{k(p-1)} \frac{w(E_j^k)}{|R_j^k|} \chi_{R_j^k} \Big\|_{L_{p'}(w^{-1/(p-1)})} \leq c \Big(\sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} R_j^k\Big)\Big)^{1/p'}. \tag{2.4}$$

We wish to make the point that Jawerth's first two conditions above are actually always satisfied for a locally integrable weight. This is the content of the following lemma. Hence, Jawerth's condition (2.4) essentially defines the class S_δ.

Lemma 2.1. *Let $1 < p < \infty$, and let $w : \mathbb{R}^N \to \mathbb{R}_+$ be a locally integrable weight. For any finite double sequence of measurable sets $\{R_j^k\}_{(j,k)\in H}$ there exists a double sequence of pairwise disjoint sets $\{E_j^k\}_{(j,k)\in H}$ satisfying $E_j^k \subseteq R_j^k$ for $(j,k) \in H$ and*

$$\sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} R_j^k\Big) \leq 2 \sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} E_j^k\Big). \tag{2.5}$$

Let us now state our main result.

Theorem 2.2. *Let $w : \mathbb{R}^N \to \mathbb{R}_+$ be a locally integrable weight. Then for $1 < p < \infty$ the following conditions are equivalent.*

i) $w \in A_p(\mathcal{R})$

ii) $\sup_{\delta>0} [w]_{A_p(\mathcal{R}^\delta)} < \infty$

iii) *w satisfies condition S_δ with constant independent of δ*

iv) *There exists a constant $C := C(p,w)$ such that for any $\delta > 0$,*

$$\|M_\delta f\|_{L_p(w)} \leq C\|f\|_{L_p(w)}$$

v) *There exists a constant $C := C(p,w)$ such that*

$$\|Mf\|_{L_p(w)} \leq C\|f\|_{L_p(w)}$$

We notice that any weight in the standard A_p class on $\mathbb{R}^N$, defined using Euclidean balls in $\mathbb{R}^N$, is contained in each class $A_p(\mathcal{R}^\delta)$, $\delta > 0$. This follows easily from the fact that any rectangle with eccentricity at most δ is contained in an Euclidean ball of comparable measure. Theorem 2.2 thus shows that the weights in the full product class $A_p(\mathcal{R})$ are exactly the weights from the standard A_p-class on $\mathbb{R}^N$ that are uniformly in $A_p(\mathcal{R}^\delta)$ for $0 < \delta \leq 1$.

In fact, the proof actually shows that for any locally integrable weight $w : \mathbb{R}^N \to \mathbb{R}_+$, the quantities

- $[w]_{A_p(\mathcal{R}^\delta)}$
- The S_δ constant for w
- $\sup_{\|f\|_{L_p(w)}\leq 1} \|M_\delta f\|_{L_p(w)}$,

are equivalent independent of δ. From this point of view, it is tempting to introduce a "smoothness scale" on weights in $A_p(\mathbb{R}^N)$ by classifying weight functions by a growth condition such as $[w]_{A_p(\mathcal{R}^\delta)} = O(\delta^{-s})$ as $\delta \to 0^+$, for $s \geq 0$. On such a scale, $s = 0$ corresponds to the product class $A_p(\mathcal{R})$. This grading of weights could potentially give a better understanding of weights that fail to be in the product class $A_p(\mathcal{R})$. However, we leave this issue open for further study.

3. Proofs

In this final section, we give the proofs of Lemma 2.1 and Theorem 2.2. We prove Lemma 2.1 first.

Proof of Lemma 2.1. We construct the sets $\{E_j^k\}_{(j,k)\in H}$ inductively. Put $N = \max\{k : (j,k) \in H\}$, and let $\Omega_k := \bigcup_j R_j^k$. Using standard techniques, we first pick pairwise disjoint measurable sets $\{E_j^N\}_j$ such that $E_j^N \subseteq R_j^N$ and $\bigcup_j E_j^N = \Omega_N$. Then we pick pairwise disjoint measurable sets $\{E_j^{N-1}\}_j$ such that $E_j^{N-1} \subseteq R_j^{N-1}$ and $\bigcup_j E_j^{N-1} = \Omega_{N-1}\backslash\Omega_N$. Let $\ell \geq 2$, and suppose $\{E_j^{N-\ell+1}\}_j$ has been properly defined. We then pick pairwise disjoint measurable sets $\{E_j^{N-\ell}\}_j$ such that $E_j^{N-\ell} \subseteq R_j^{N-\ell}$ and $\bigcup_j E_j^{N-\ell} = \Omega_{N-\ell}\backslash B_\ell$, where $B_\ell := \bigcup_{s=N-\ell+1}^N \Omega_s$. We continue until the sets in $\{R_j^k\}_{(j,k)\in H}$ have been exhausted.

For notational convenience put $B_0 := \emptyset$, and notice that $\{(\Omega_{N-\ell}\backslash B_\ell)\}_{\ell\geq 0}$ forms a pairwise disjoint partition of $\bigcup_j \Omega_j$. We have, by the additivity of any measure,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{kp} w(\Omega_k) &= \sum_{\ell=0}^\infty \sum_{k\in\mathbb{Z}} 2^{kp} w\big((\Omega_{N-\ell}\backslash B_\ell) \cap \Omega_k\big) \\ &\leq \sum_{\ell=0}^\infty \sum_{k\leq N-\ell} 2^{kp} w\big(\Omega_{N-\ell}\backslash B_\ell\big)\end{aligned}$$

$$\leq \frac{2^p - 1}{2^p} \sum_{\ell=0}^{\infty} 2^{p(N-\ell)} w\Big(\bigcup_j E_j^{N-\ell}\Big)$$
$$\leq 2 \sum_{\ell=0}^{\infty} 2^{p(N-\ell)} w\Big(\bigcup_j E_j^{N-\ell}\Big),$$

where we also used the fact that $\sum_{k \leq M-1} 2^{kp} = 2^{Mp}/(2^p - 1)$. □

In order to prove Theorem 2.2, we shall focus on the equivalence between (iii) and (iv). The remaining steps are standard (for some of the remaining steps see also the results in K. Moen [5]). Let us point out that the equivalence between i) and ii), as well as the equivalence between iv) and v), is valid for any such nested collection of bases.

We mention that our proof follows Jawerth [4, Theorem 3.4], and one of our contributions is to correct some issues in the proof of [4, Theorem 3.4] by using the result from Lemma 2.1.

Proof of Theorem 2.2, iii) ⇔ iv): Let $\{R_j^k\}_{(j,k)\in H} \subset \mathcal{R}^\delta$ be any finite collection. Let $\{E_j^k\}_{(j,k)\in H}$ be the corresponding family of pairwise disjoint sets given by Lemma 2.1. We define a linear operator by

$$Lf(x) := \sum_{j,k} \chi_{E_j^k}(x) \frac{1}{|R_j^k|} \int_{R_j^k} f(y)\, dy.$$

Clearly, $|Lf(x)| \leq M_\delta f(x)$ so $L : L_p(w) \to L_p(w)$ is bounded with at most the same norm as M_δ. A straightforward calculation shows that the adjoint of L is given by

$$L^* g(x) = \sum_{j,k} \chi_{R_j^k}(x) \frac{1}{|R_j^k|} \int_{E_j^k} g(y)\, dy.$$

It follows that

$$\|L^* g\|_{L_{p'}(w^{-1/(p-1)})} \leq C \|g\|_{L_{p'}(w^{-1/(p-1)})}.$$

We put $g = \sum_{j,k} 2^{k(p-1)} w(\cdot) \chi_{E_j^k}(\cdot)$, and notice that

$$\|L^* g\|_{L_{p'}(w^{-1/(p-1)})} = \Big\| \sum_{j,k\in\mathbb{Z}} 2^{k(p-1)} \frac{w(E_j^k)}{|R_j^k|} \chi_{R_j^k} \Big\|_{L_{p'}(w^{-1/(p-1)})},$$

while

$$\|g\|_{L_{p'}(w^{-1/(p-1)})} = \Big(\sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} E_j^k\Big)\Big)^{1/p'} \leq \Big(\sum_{k\in\mathbb{Z}} 2^{kp} w\Big(\bigcup_{j\in\mathbb{Z}} R_j^k\Big)\Big)^{1/p'}.$$

The above estimates show that condition (2.4) is satisfied.

Conversely, for $|k| \leq S$ with S large we choose compact sets

$$K_k \subseteq \{x \in \mathbb{R}^N : 2^k < M_\delta f(x) \leq 2^{k+1}\}.$$

For each k we choose a finite cover $\{R_j^k\}_j$ such that $K_k \subseteq \bigcup_j R_j^k$ and

$$\frac{1}{|R_j^k|} \int_{R_j^k} |f(y)|\, dy \geq 2^k.$$

Now, we have

$$\begin{aligned}
\int_{\bigcup_k K_k} |M_\delta f(x)|^p w(x)\, dx &\leq 2 \sum_k 2^{kp} w\Big(\bigcup_j R_j^k\Big) \\
&\leq 4 \sum_k \sum_j 2^{kp} w(E_j^k) \\
&\leq 4 \sum_k \sum_j 2^{k(p-1)} w(E_j^k) \left(\frac{1}{|R_j^k|} \int_{R_j^k} |f(y)|\, dy\right) \\
&= 4 \int_{\mathbb{R}^N} \sum_k \sum_j 2^{k(p-1)} \frac{w(E_j^k)}{|R_j^k|} \chi_{R_j^k}(y) |f(y)|\, dy.
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \sum_k \sum_j 2^{k(p-1)} \frac{w(E_j^k)}{|R_j^k|} \chi_{R_j^k}(y) |f(y)|\, dy \\
&\qquad \leq \left\| \sum_{j,k \in \mathbb{Z}} 2^{k(p-1)} \frac{w(E_j^k)}{|R_j^k|} \chi_{R_j^k} \right\|_{L_{p'}(w^{-1/(p-1)})} \cdot \|f\|_{L_p(w)} \\
&\qquad \leq C \left(\sum_{k \in \mathbb{Z}} 2^{kp} w\Big(\bigcup_{j \in \mathbb{Z}} R_j^k\Big)\right)^{1/p'} \cdot \|f\|_{L_p(w)}.
\end{aligned}$$

Hence,

$$\left(\sum_{k \in \mathbb{Z}} 2^{kp} w\Big(\bigcup_{j \in \mathbb{Z}} R_j^k\Big)\right)^{1/p} \leq C \|f\|_{L_p(w)},$$

so

$$\left(\int_{\bigcup_k K_k} |M_\delta f(x)|^p w(x)\, dx \right)^{1/p} \leq 2^{1/p} \left[\sum_k 2^{kp} w\Big(\bigcup_j R_j^k\Big)\right]^{1/p} \leq 2^{1/p} C \|f\|_{L_p(w)}.$$

By a limiting argument, it follows directly that

$$\|M_\delta f\|_{L_p(w)} \leq 2^{1/p} C \|f\|_{L_p(w)}.$$

□

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Department of Mathematical Sciences, Aalborg University, Frederik Bajersvej 7G, DK - 9220 Aalborg East, Denmark

E-mail address: mnielsen@math.aau.dk

Department of Mathematics, University of Zagreb, Bijenička 30, HR-10000, Zagreb, Croatia

E-mail address: hsikic@math.hr