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COMPACTLY SUPPORTED CURVELET TYPE SYSTEMS

KENNETH N. RASMUSSEN AND MORTEN NIELSEN

Abstract. In this article we study a flexible method for constructing curvelet type frames. These curvelet type systems have the same sparse representation properties as curvelets for appropriate classes of smooth functions, and the flexibility of the method allows us to construct curvelet type systems with a prescribed nature such as compact support in direct space. The method consists of using the machinery of almost diagonal matrices to show that a system of curvelet molecules which is sufficiently close to curvelets constitutes a frame for curvelet type spaces. Such a system of curvelet molecules is then constructed using finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay.

1. Introduction

Second generation curvelets were introduced by Candès and Donoho, who also proved that curvelets give an essentially optimal sparse representation of images (functions) that are C^2 except for discontinuities along piecewise C^2-curves [4]. It follows that efficient compression of such images can be archived by thresholding their curvelet expansions. Curvelets form a multiscale system with effective support that follows a parabolic scaling relation *width* $\approx$ *length*^{2}. Moreover, they also provide an essentially optimal sparse representation of Fourier integral operators [2] and an optimal sparse and well organized solution operator for a wide class of linear hyperbolic differential equations [3]. However, curvelets are band-limited, and contrary to wavelets it is an open question whether compactly supported curvelet type systems exist.

In this article we study a flexible method for generating curvelet type systems with the same sparse representation properties as curvelets (when sparseness is measured in curvelet type sequence spaces). The method uses a perturbation principle which was first introduced in [10], further generalized in [8] and refined for frames in [9]. The constructed curvelet type system consists of finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay. This allows us to the flexibility to construct curvelet type systems with a prescribed nature (see Section 6) such as compact support in direct space. For the sake of convenience the construction will only

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be done in $\mathbb{R}^2$, but it can easily be extended to $\mathbb{R}^d$. The main results can be found in Sections 4 and 5.

The curvelet type sequence spaces we use are associated with curvelet type spaces $G^s_{p,q}$ which were introduced in [1]. Here $G^s_{p,q}$ was constructed by applying a curvelet type splitting of the frequency space to a general construction of decomposition spaces; thereby obtaining a natural family of smoothness spaces for which curvelets constitute frames (see Section 2). Originally, this construction of decomposition spaces based on structured coverings of the frequency space was introduced by Feichtinger and Gröbner [6] and Feichtinger [5]. For example, the classical Triebel-Lizorkin and Besov spaces correspond to dyadic coverings of the frequency space (see [12]).

The outline of the article is as follows. In Section 2 we define second generation curvelets and curvelet type spaces. Furthermore, we introduce curvelet molecules which will be the building blocks for our compactly supported curvelet type frames. Next, in Section 3 we use the properties of curvelet molecules to show that the "change of frame coefficient" matrix is almost diagonal if the curvelet molecules have sufficient regularity. With the machinery of almost diagonal matrices, we can then in Section 4 show that curvelet molecules which are close enough to curvelets constitute frames for the curvelet type spaces. Finally, in Section 5 we construct these curvelet molecules from finite linear combinations of shifts and dilates of a single function with sufficient smoothness and decay. We conclude the paper with a short discussion in Section 6 of the possible functions which can used to construct the curvelet molecules.

2. Second generation curvelets

We begin this section with a brief definition of curvelets and curvelet molecules which will later be used to construct curvelet type frames. Furthermore, we define the curvelet type spaces for which curvelets constitute frames. For a much more detailed discussion of the curvelet construction, we refer the reader to [3,4], and for decomposition spaces, of which the curvelet type spaces are a subclass, we refer to [1,6].

Let ν be an even $C^\infty(\mathbb{R})$ window that is supported on $[-\pi,\pi]$ such that its 2π-periodic extension obeys $|\nu(\theta)|^2+|\nu(\theta-\pi)|^2=1$, for $\theta\in[0,2\pi)$. Define $\nu_{j,l}(\theta):=\nu(2^{\lfloor j/2\rfloor}\theta-\pi l)$ for $j\geq 2$ and $l=0,1,\dots,2^{\lfloor j/2\rfloor}-1$. Next, with the angular window in place, let $w\in C_c^\infty(\mathbb{R})$ obey

$$|w_0(t)|^2+\sum_{j\geq 2}|w(2^{-j}t)|^2=1,\ t\in\mathbb{R},$$

with $w_0\in C_c^\infty(\mathbb{R})$ supported in a neighborhood of the origin. We then define

$$\phi_{j,l}(\xi):=w(2^{-j}|\xi|)(\nu_{j,l}(\theta)+\nu_{j,l}(\theta+\pi)),\ \xi=|\xi|(\cos\theta,\sin\theta)\in\mathbb{R}^2. \tag{2.1}$$

Notice that the support of $w(2^{-j}|\xi|)\nu_{j,0}(\theta)$ is contained in a rectangle $R_j = I_{1j} \times I_{2j}$ given by

$$I_{1j} := \{\xi_1, t_j \leq \xi_1 \leq t_j + L_j\}, \qquad I_{2j} := \{\xi_2, 2|\xi_2| \leq l_j\},$$

where t_j is defined by minimizing L_j, $L_j := \delta_1 \pi 2^j$ and $l_j := \delta_2 2\pi 2^{j/2}$ (δ_1 depends weakly on j, see [4, Section 2.2]). With $\tilde{I}_{1j} := \pm I_{1j}$ and $\tilde{R}_j = \tilde{I}_{1j} \times I_{2j}$ the system

$$e_{j,k}(\xi) := \frac{2^{-3j/4}}{2\pi\sqrt{\delta_1\delta_2}} e^{i\frac{(k_1+1/2)2^{-j}\xi_1}{\delta_1}} e^{i\frac{k_2 2^{-j/2}\xi_2}{\delta_2}},\ k \in \mathbb{Z}^2,$$

is an orthonormal basis for $L_2(\tilde{R}_j)$. Finally, we define

$$\hat{\eta}_\mu(\xi) := \phi_{j,l}(\xi) e_{j,k}(R_{\theta_\mu}^\top \xi),\ \mu = (j,l,k), \tag{2.2}$$

where R_{θ_μ} is rotation by the angle $\theta_\mu := \pi 2^{-\lfloor j/2 \rfloor} l$, and as coarse-scale elements we define $\hat{\eta}_{1,0,k}(\xi) := \delta_0^{-1}\phi_{1,0}(\xi)e^{ik\cdot\xi/\delta_0}$, where $\phi_{1,0}(\xi) := \omega_0(|\xi|)$ and $\delta_0 > 0$ is sufficiently small. The system $\{\eta_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ is called *curvelets*, $\mathcal{J} := \{(j,l) | j \geq 1, l = 0, 1, \ldots, 2^{\lfloor j/2 \rfloor} - 1\}$. It can be shown that curvelets constitute a tight frame for $L_2(\mathbb{R}^2)$ (see [4, Section 2.2]).

To later construct curvelet type frames, we need a system of functions which share the essential properties of curvelets. As we shall see, curvelet molecules, which were introduced in [3] and used there to study hyperbolic differential equations, have all the properties we need. For $\kappa \in \mathbb{N}_0^2$, we define $|\kappa| := \kappa_1 + \kappa_2$, and for suitably differentiable functions we define $f^{(\kappa)} := \frac{\partial^{|\kappa|} f}{\partial_{\xi_1}^{\kappa_1} \partial_{\xi_2}^{\kappa_2}}$.

Definition 2.1.
A family of functions $\{\psi_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ is said to be a *family of curvelet molecules with regularity* R, $R \in \mathbb{N}$, if for $j \geq 2$ they may be expressed as

$$\psi_\mu(x) = 2^{\frac{3j}{4}} a_\mu(D_{2^{-j}} R_{\theta_\mu} x - (k_1/\delta_1, k_2/\delta_2)),$$

where $D_{2^{-j}}\xi = (2^{-j}\xi_1, 2^{-j/2}\xi_2)$, $\delta_1, \delta_2 > 0$ and all functions a_μ satisfy the following:

- For $|\kappa| \leq R$ there exists constants $C > 0$ such that

$$|a_\mu^{(\kappa)}(x)| \leq C(1 + |x|)^{-2R}. \tag{2.3}$$

- There exists constants $C > 0$ such that

$$|\hat{a}_\mu(\xi)| \leq C \min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^R. \tag{2.4}$$

Here the constants may be chosen independent of μ, and the coarse-scale molecules, $j = 1$, must satisfy an obvious modification. ◇

It can be shown that curvelets constitute a family of curvelet molecules with regularity R for any $R \in \mathbb{N}$.

To define the curvelet type spaces which together with the associated sequence spaces will characterize the sparse representation properties of curvelets we need a suitable partition of unity.

Definition 2.2.
Let $Q_{j,l} := \text{supp}(\phi_{j,l})$ for $(j,l) \in \mathcal{J}$, where $\phi_{j,l}$ was defined (2.1). in A bounded admissible partition of unity (BAPU) is a family of functions $\{\varphi_{j,l}\}_{(j,l)\in\mathcal{J}} \subset \mathcal{S} := \mathcal{S}(\mathbb{R}^2)$ satisfying:

- $\text{supp}(\varphi_{j,l}) \subseteq Q_{j,l}$, $(j,l) \in \mathcal{J}$.
- $\sum_{(j,l)\in\mathcal{J}} \varphi_{j,l}(\xi) = 1$, $\xi \in \mathbb{R}^2$.
- $\sup_{(j,l)\in\mathcal{J}} |Q_{j,l}|^{1/p-1} \|\mathcal{F}^{-1}\varphi\|_{L_p(\mathbb{R}^2)} < \infty$, $p \in (0,1]$.

◇

We are now ready to define curvelet-type spaces. We let $\hat{f}(\xi) := \mathcal{F}(f)(\xi) := (2\pi)^{-1} \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi}\, \mathrm{d}x, f \in L_1(\mathbb{R}^2)$, and by duality extend it uniquely from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$.

Definition 2.3.
Let $\{\varphi_{j,l}\}_{(j,l)\in\mathcal{J}}$ be a BAPU and $\varphi_{j,l}(D)f := \mathcal{F}^{-1}(\varphi_{j,l}\mathcal{F}f)$. For $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$, we define $G^s_{p,q} := G^s_{p,q}(\mathbb{R}^2)$ as the set of distributions $f \in \mathcal{S}' := \mathcal{S}'(\mathbb{R}^2)$ satisfying

$$\|f\|_{G^s_{p,q}} := \Big(\sum_{(j,l)\in\mathcal{J}} \|2^{js}\varphi_{j,l}(D)f\|^q_{L_p} \Big)^{1/q} < \infty.$$

◇

It can be shown that $G^s_{p,q}$ is a quasi-Banach space (Banach space for $p, q \geq 1$), and $\mathcal{S}$ is dense in $G^s_{p,q}$ (see [1] and [6]).

We also need the sequence spaces associated with the curvelet-type spaces. For the sake of convenience, we write $\|f_k\|$ instead of $\|\{f_k\}_{k\in K}\|$ when the index set is clear from the context.

Definition 2.4.
For $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$, we define the sequence space $g^s_{p,q}$ as the set of sequences $\{s_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \subset \mathbb{C}$ satisfying

$$\|s_\mu\|_{g^s_{p,q}} := \Big\| 2^{j\left(s+\frac{3}{2}(\frac{1}{2}-\frac{1}{p})\right)} \Big(\sum_{k\in\mathbb{Z}^2} |s_\mu|^p \Big)^{1/p} \Big\|_{l_q} < \infty,$$

where the l_p-norm is replaced with the l_∞-norm if $p = \infty$. ◇

Notice that the sequence spaces l_q are special cases of $g^s_{p,q}$ as we have $g_{q,q}^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} = l_q$.

Next, we introduce frames for $G^s_{p,q}$ and use the notation $F \asymp G$ when there exists two constants $0 < C_1 \le C_2 < \infty$, depending only on "allowable" parameters, such that $C_1 F \le G \le C_2 F$.

Definition 2.5.
We say that a family of functions $\{\psi_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ in the dual of $G^s_{p,q}$ is a frame for $G^s_{p,q}$ if for all $f \in G^s_{p,q}$ we have

$$\|f\|_{G^s_{p,q}} \asymp \|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}}.$$

The following is called the frame expansion of $\{\psi_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ when it exists,

$$f = \sum_{\mu \in \mathcal{J}\times\mathbb{Z}^2} \langle f, S^{-1}\psi_\mu\rangle \psi_\mu \tag{2.5}$$

in the sense of $\mathcal{S}'$, where S is the frame operator $Sf = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle f, \psi_\mu\rangle\psi_\mu$, $f \in G^s_{p,q}$. ◇

From [1, Lemma 4 and Section 7.3] we have that curvelets (2.2) constitute a frame for the curvelet type spaces with a frame operator S that is equal to the identity, $S = I$:

Proposition 2.6.
Assume that $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \le \infty$. For any finite sequence $\{s_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \subset \mathbb{C}$, we have

$$\Big\| \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} s_\mu \eta_\mu \Big\|_{G^s_{p,q}} \le C \|s_\mu\|_{g^s_{p,q}}.$$

Furthermore, $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ with frame operator $S = I$,

$$\|f\|_{G^s_{p,q}} \asymp \|\langle f, \eta_\mu\rangle\|_{g^s_{p,q}}, \; f \in G^s_{p,q}.$$

□

Notice that frame expansions for two frames $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ and $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ have the same sparseness when measured in the associated sequence space $g^s_{p,q}$ if $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ and $\{S^{-1}\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ also constitute frames for $G^s_{p,q}$,

$$\|\langle f, S^{-1}\psi_\mu\rangle\|_{g^s_{p,q}} \asymp \|f\|_{G^s_{p,q}(\mathbb{R}^2)} \asymp \|\langle f, S^{-1}\eta_\mu\rangle\|_{g^s_{p,q}}.$$

Hence, to get a curvelet type system $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ with the same sparse representation properties as curvelets $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$, it suffices to prove that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ constitutes a frame for $G^s_{p,q}$.

3. Almost diagonal matrices

To generate curvelet type frames in the following sections we introduce the machinery of almost diagonal matrices in this section. Almost diagonal matrices where used in [7] on Besov spaces, and here we find an associated notion of almost diagonal matrices on $g^s_{p,q}$. The goal is to find a definition so that the composition of two almost diagonal matrices gives a new almost diagonal matrix and almost diagonal matrices are bounded on $g^s_{p,q}$.

To help us define almost diagonal matrices we use a slight variation of the pseudodistance introduced in [11] which was constructed in [3]. For this we need the center of η_μ in direct space, $x_\mu := R_{\theta_\mu}(k_1 2^{-j}/\delta_1, k_2 2^{-j/2}\delta_2)$, and the "direction" of η_μ, $\rho_\mu := (\cos\theta_\mu, \sin\theta_\mu)$.

Definition 3.1.
Given a pair of indices $\mu = (j,k,l)$ and $\mu' = (j',k',l')$, we define the *dyadic-parabolic pseudodistance* as

$$\omega(\mu,\mu') := 2^{|j-j'|}(1+\min(2^j, 2^{j'})d(\mu,\mu')),$$

where

$$d(\mu,\mu') := |\theta_\mu - \theta_{\mu'}|^2 + |x_\mu - x_{\mu'}|^2 + |\langle \rho_\mu, x_\mu - x_{\mu'}\rangle|.$$

◇

The dyadic-parabolic distance was studied in detail in [3], and from there we can deduce the following properties:

- For $\delta > 0$ there exists $C > 0$ such that

$$\sum_{k\in\mathbb{Z}^2} \omega(\mu,\mu')^{-\frac{3}{2}-\delta} \le C. \tag{3.1}$$

- For $\delta > 0$ there exists $C > 0$ such that

$$\sum_{(j,l)\in\mathcal{J}} \omega(\mu,\mu')^{-\frac{1}{2}-\delta} \le C. \tag{3.2}$$

- For $N \ge 2$ and $\delta > 0$ there exists $C > 0$ such that

$$\sum_{\mu''\in\mathcal{J}\times\mathbb{Z}^2} \omega(\mu,\mu'')^{-N-\delta}\omega(\mu'',\mu')^{-N-\delta} \le C\omega(\mu,\mu')^{-N-\frac{\delta}{2}}. \tag{3.3}$$

- Let $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ and $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ be two families of curvelet molecules with regularity $4R$, $R\in\mathbb{N}$. Then there exists $C > 0$ such that

$$|\langle \psi_\mu, \eta_{\mu'}\rangle| \le C\omega(\mu,\mu')^{-R}. \tag{3.4}$$

These properties lead us to the following definition of almost diagonal matrices on $g^s_{p,q}$.

Definition 3.2.
Assume that $s \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$. Let $r := \min(1, p, q)$ and $t := s + \frac{3}{2}(\frac{1}{2} - \frac{1}{p})$. A matrix $\mathbf{A} = \{a_{\mu\mu'}\}_{\mu,\mu' \in \mathcal{J}\times\mathbb{Z}^2}$ is called *almost diagonal on* $g^s_{p,q}$ if there exists $C, \delta > 0$ such that

$$|a_{\mu\mu'}| \leq C 2^{(j'-j)t} \omega(\mu, \mu')^{-\frac{2}{r}-\delta}.$$

◇

Remark 3.3.
Note that by using (3.3), we get that the composition of two almost diagonal matrices on $g^s_{p,q}$ gives a new almost diagonal matrix on $g^s_{p,q}$. ◦

We are now ready to show the most important property of almost diagonal matrices; they act boundedly on the curvelet type spaces.

Proposition 3.4.
If $\mathbf{A}$ is almost diagonal on $g^s_{p,q}$, then $\mathbf{A}$ is bounded on $g^s_{p,q}$.

Proof:
We only prove the result for $p < \infty$ as the result for $p = \infty$ follows in a similar way with l_p replaced by l_∞. Let $\omega_0(\mu, \mu') := \omega(j, l, 0, j', l', 0)$, $\{s_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2} \in g^s_{p,q}$, and assume for now that $p \geq 1$. We begin with looking at the l_p-norm of $\|\mathbf{A}s\|_{g^s_{p,q}}$. By using Minkowski's inequality, Hölder's inequality and (3.1) we get

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}^2} |(\mathbf{A}s)_\mu|^p\Big)^{1/p} \\
&\quad\leq C\bigg(\sum_{k\in\mathbb{Z}^2}\bigg(\sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\bigg)^p\bigg)^{1/p} \\
&\quad\leq C \sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \bigg(\sum_{k\in\mathbb{Z}^2}\bigg(\sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\bigg)^p\bigg)^{1/p} \\
&\quad\leq C \sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \\
&\qquad\qquad \times \bigg(\sum_{k\in\mathbb{Z}^2}\sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|^p\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\Big(\sum_{k'\in\mathbb{Z}^2}\omega(\mu,\mu')^{-\frac{3}{2r}-\frac{\delta}{2}}\Big)^{p-1}\bigg)^{1/p} \\
&\quad\leq C \sum_{(j',l')\in\mathcal{J}} 2^{(j'-j)t}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \Big(\sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|^p\Big)^{1/p}.
\end{aligned}$$

We then have

$$\|\mathbf{A}s\|_{g^s_{p,q}} \leq C\Bigg(\sum_{(j,l)\in\mathcal{J}} \Bigg(\sum_{(j',l')\in\mathcal{J}} 2^{j't}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \Big(\sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|^p\Big)^{1/p}\Bigg)^q\Bigg)^{1/q}.$$

For $q \geq 1$ we use Hölder's inequality and (3.2) to get

$$\begin{aligned}\|\mathbf{A}s\|_{g^s_{p,q}} \leq & C\Bigg(\sum_{(j,l)\in\mathcal{J}} \sum_{(j',l')\in\mathcal{J}} 2^{j'qt}\omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}} \\ & \times \Big(\sum_{k'\in\mathbb{Z}^2} |s_{\mu'}|^p\Big)^{q/p} \Big(\sum_{(j',l')\in\mathcal{J}} \omega_0(\mu,\mu')^{-\frac{1}{2r}-\frac{\delta}{2}}\Big)^{q-1}\Bigg)^{1/q} \\ \leq & C\|s\|_{g^s_{p,q}}.\end{aligned}$$

For $q < 1$ the result follows by a direct estimate. The case $p < 1$ remains, and here we first observe that

$$\tilde{\mathbf{A}} := \{\tilde{a}_{\mu\mu'}\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2} = \big\{|a_{\mu\mu'}|^p 2^{(j'-j)(t-tp)}\big\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$$

is almost diagonal on $g^s_{1,\frac{q}{p}}$. Furthermore, if we let
$v := \{v_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} := \{|s_\mu|^p 2^{-j(t-tp)}\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ we have

$$\|v\|^{1/p}_{g^s_{1,\frac{q}{p}}} = \Bigg(\sum_{(j,l)\in\mathcal{J}} \Bigg(\sum_{k\in\mathbb{Z}^2} 2^{jtp}|s_\mu|^p\Bigg)^{q/p}\Bigg)^{1/q} = \|s\|_{g^s_{p,q}}.$$

Before we can put these two observations into use, we need that

$$|(\mathbf{A}s)_\mu|^p \leq \sum_{(j',l')\in\mathcal{J}} \sum_{k'\in\mathbb{Z}^2} |a_{\mu\mu'}|^p|s_{\mu'}|^p = 2^{j(t-tp)} \sum_{(j',l')\in\mathcal{J}} \sum_{k'\in\mathbb{Z}^2} \tilde{a}_{\mu\mu'}v_\mu.$$

We then have

$$\|\mathbf{A}s\|_{g^s_{p,q}} \leq \|\tilde{\mathbf{A}}v\|^{1/q}_{g^s_{1,\frac{q}{p}}} \leq C\|v\|^{1/q}_{g^s_{1,\frac{q}{p}}} = C\|v\|_{g^s_{p,q}}.$$

■

4. Curvelet type frames

In this section we study a family of curvelet molecules $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ which is a small perturbation of curvelets $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$.The goal is first to show that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is close enough to $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$, then $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for

$G^s_{p,q}$. Next to get a frame expansion, we show that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a frame, where S is the frame operator

$$Sf = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle f, \psi_\mu\rangle \psi_\mu.$$

The results are inspired by [9] where perturbations of frames were studied in Triebel-Lizorkin and Besov spaces.

Let $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ be a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ independent of ε for some $\varepsilon, \delta > 0$. Then $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$, and motivated by $\{\eta_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ being a tight frame for $L_2(\mathbb{R}^2)$, we formally define $\langle f, \psi_{\mu'}\rangle$ as

$$\langle f, \psi_{\mu'}\rangle := \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle \eta_\mu, \psi_{\mu'}\rangle \langle f, \eta_\mu\rangle, \; f \in G^s_{p,q}. \tag{4.1}$$

It follows from (3.4) and Proposition 3.4 that $\langle \cdot, \psi_{\mu'}\rangle$ is a bounded linear functional on $G^s_{p,q}$; in fact we have

$$\begin{aligned}\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} |\langle \eta_\mu, \psi_{\mu'}\rangle||\langle f, \eta_\mu\rangle| &\le \Big\| \Big\{ \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} |\langle \eta_\mu, \psi_{\mu'}\rangle||\langle f, \eta_\mu\rangle| \Big\}_{\mu'\in\mathcal{J}\times\mathbb{Z}^2} \Big\|_{g^s_{p,q}} \\ &\le C\|\langle f, \eta_\mu\rangle\|_{g^s_{p,q}} \le C\|f\|_{G^s_{p,q}}.\end{aligned} \tag{4.2}$$

Furthermore, $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a norming family for $G^s_{p,q}$ as it satisfies $\|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}} \le C\|f\|_{G^s_{p,q}}$. This can be used to show that S is a bounded operator on $G^s_{p,q}$, and for small enough ε this will be the key to showing that $\{\psi_\mu\}$ is a frame for $G^s_{p,q}$.

Theorem 4.1.
There exists $\varepsilon_0, C_1, C_2 > 0$ such that if $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ independent of ε for some $\varepsilon \le \varepsilon_0$ and $\delta > 0$, then we have

$$C_1\|f\|_{G^s_{p,q}} \le \|\langle f, \psi_\mu\rangle\|_{g^s_{p,q}} \le C_2\|f\|_{G^s_{p,q}} \; f \in G^s_{p,q}, \tag{4.3}$$

where we used the notation from Definition 3.2.

Proof:
That $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a norming family gives the upper bound, thus we only need to establish the lower bound. For this we use that $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a norming family so we have

$$\|\langle f, \eta_\mu - \psi_\mu\rangle\|_{g^s_{p,q}} \le C\varepsilon\|f\|_{G^s_{p,q}}.$$

It then follows that

$$\|f\|_{G^s_{p,q}} \le C\|\langle f, \eta_\mu\rangle\|_{g^s_{p,q}}$$

$$\leq C(\|\langle f,\psi_\mu\rangle\|_{g^s_\mu} + \|\langle f,\eta_\mu - \psi_\mu\rangle\|_{g^s_{p,q}})$$
$$\leq C(\|\langle f,\psi_\mu\rangle\|_{g^s_{p,q}} + \varepsilon\|f\|_{G^s_{p,q}}).$$

By choosing $\varepsilon < 1/C$ we get the lower bound.

■

As one might guess from Theorem 4.1, the boundedness of the matrix $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ on $g^s_{p,q}$ is the key to showing that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is also a frame for $G^s_{p,q}$. For the sake of convenience, we use the notation from Definition 3.2 in the following.

Proposition 4.2.
There exists $\varepsilon_0 > 0$ such that if $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ independent of ε for some $\varepsilon \leq \varepsilon_0$ and $\delta > 0$, and $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^0_{22} = L_2(\mathbb{R}^2)$, then $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu\,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

Proof:
The fact that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $L_2(\mathbb{R}^2)$ ensures that S^{-1} is a bounded operator on $L_2(\mathbb{R}^2)$. We first show that S^{-1} is bounded on $G^s_{p,q}$. This will follow from showing that

$$\|(I-S)f\|_{G^s_{p,q}} \leq C\varepsilon\|f\|_{G^s_{p,q}}, f\in G^s_{p,q}, \tag{4.4}$$

choosing ε small enough and using the Neumann series. Assume for a moment that
$\mathcal{D} := \{d_{\mu'\mu}\}_{\mu',\mu\in\mathcal{J}\times\mathbb{Z}^2} := \{\langle(I-S)\eta_\mu, \eta_{\mu'}\rangle\}_{\mu',\mu\in\mathcal{J}\times\mathbb{Z}^2}$ satisfies

$$\|\mathcal{D}s\|_{g^s_{p,q}} \leq C\varepsilon\|s\|_{g^s_{p,q}}. \tag{4.5}$$

By using that S is self-adjoint, we then have

$$\|(I-S)f\|_{G^s_{p,q}} \leq C\|\{\langle(I-S)f,\eta_{\mu'}\rangle\}\|_{g^s_{p,q}} = C\|\mathcal{D}\{\langle f,\eta_\mu\rangle\}\|_{g^s_{p,q}}$$
$$\leq C\varepsilon\|\{\langle f,\eta_\mu\rangle\}\|_{g^s_{p,q}} \leq C\varepsilon\|f\|_{G^s_{p,q}}.$$

So to show (4.4) it suffices to prove (4.5). Note that

$$\langle(I-S)\eta_\mu,\eta_{\mu'}\rangle = \sum_{\mu''\in\mathcal{J}\times\mathbb{Z}^2}\langle\eta_\mu,\eta_{\mu''}\rangle\langle\eta_{\mu''},\eta_{\mu'}\rangle - \sum_{\mu''\in\mathcal{J}\times\mathbb{Z}^2}\langle\eta_\mu,\psi_{\mu''}\rangle\langle\psi_{\mu''},\eta_{\mu'}\rangle$$
$$= \sum_{\mu''\in\mathcal{J}\times\mathbb{Z}^2}\langle\eta_\mu,\eta_{\mu''}\rangle\langle\eta_{\mu''}-\psi_{\mu''},\eta_{\mu'}\rangle + \sum_{\mu''\in\mathcal{J}\times\mathbb{Z}^2}\langle\eta_\mu,\eta_{\mu''}-\psi_{\mu''}\rangle\langle\psi_{\mu''},\eta_{\mu'}\rangle.$$

By setting

$$\mathcal{D}_1 := \{d_{1(\mu')(\mu'')}\} := \{\langle\eta_{\mu''}-\psi_{\mu''},\eta_{\mu'}\rangle\},$$
$$\mathcal{D}_2 := \{d_{2(\mu'')(\mu)}\} := \{\langle\eta_\mu,\eta_{\mu''}\rangle\},$$

$$\mathcal{D}_3 := \{d_{3(\mu')(\mu'')}\} := \{\langle \psi_{\mu''}, \eta_{\mu'}\rangle\},$$
$$\mathcal{D}_4 := \{d_{4(\mu'')(\mu)}\} := \{\langle \eta_\mu, \eta_{\mu''} - \psi_{\mu''}\rangle\},$$

we have the decomposition

$$\mathcal{D} = \mathcal{D}_1\mathcal{D}_2 + \mathcal{D}_3\mathcal{D}_4.$$

Since $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ independent of ε, we have from (3.4) that $\varepsilon^{-1}\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \varepsilon^{-1}\mathcal{D}_4$ are almost diagonal on $g^s_{p,q}$. Next, we use Remark 3.3, and by Proposition 3.4,

$$\|\mathcal{D}s\|_{g^s_{p,q}} \le C\varepsilon\|s\|_{g^s_{p,q}}.$$

Consequently, (4.4) holds, and for sufficiently small ε the operator S^{-1} is bounded on $G^s_{p,q}$. Finally, let $s := \{s_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \in g^s_{p,q}$ and $h =: \sum_\mu s_\mu\eta_\mu$. By using (2.6) we have that $h \in G^s_{p,q}$, and as $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $L_2(\mathbb{R}^2)$, we have that S^{-1} is self-adjoint which gives

$$\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle s_\mu = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle S^{-1}\eta_\mu, \psi_{\mu'}\rangle s_\mu = \langle S^{-1}h, \psi_{\mu'}\rangle.$$

If we combine this with $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ being a norming family (4.2), we get

$$\begin{aligned}\Big\| \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle s_\mu \Big\|_{g^s_{p,q}} &= \|\langle S^{-1}h, \psi_{\mu'}\rangle\|_{g^s_{p,q}} \le C\|S^{-1}h\|_{G^s_{p,q}} \\ &\le C\|h\|_{G^s_{p,q}} \le C\|s\|_{g^s_{p,q}}\end{aligned}$$

which proves that $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

■

That $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ now follows as a consequence of $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ being bounded on $g^s_{p,q}$. We state the following results without proofs as they follow directly in the same way as in the Besov space case. The proofs can be found in [9]. First, we have the frame expansion.

Lemma 4.3.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t| + \frac{2}{r} + \delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. If $\{\langle \eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$, then for $f \in G^s_{p,q}$ we have

$$f = \sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2} \langle f, S^{-1}\psi_\mu\rangle\psi_\mu$$

in the sense of $\mathcal{S}'$.

□

Next, we have that $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$

Theorem 4.4.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t|+\frac{2}{r}+\delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. Then $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a frame for $G^s_{p,q}$ if and only if $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$.

□

It follows from Proposition 4.2, Lemma 4.3 and Theorem 4.4 that if $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules which is close enough to curvelets, then the representation $\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, S^{-1}\psi_\mu\rangle\psi_\mu$, $f\in G^s_{p,q}$, has the same sparse representation properties as curvelets when measured in $g^s_{p,q}$. As a final result we also have a frame expansion with $\{S^{-1}\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$.

Lemma 4.5.
Assume that $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4\lceil |t|+\frac{2}{r}+\delta\rceil$ and a frame for $L_2(\mathbb{R}^2)$. If the transpose of $\{\langle\eta_\mu, S^{-1}\psi_{\mu'}\rangle\}_{\mu,\mu'\in\mathcal{J}\times\mathbb{Z}^2}$ is bounded on $g^s_{p,q}$, then for $f\in G^s_{p,q}$ we have

$$f=\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f,\psi_\mu\rangle S^{-1}\psi_\mu$$

in the sense of $\mathcal{S}'$.

□

All that remains now is to construct a flexible family of curvelet molecules which is close enough to curvelets in the sense of Proposition 4.2.

5. Construction of curvelet type systems

In this section we construct a flexible curvelet type systems. We do this by showing that finite linear combinations of shifts and dilates of a function g with sufficient smoothness and decay can constitute a system $\{\psi_\mu\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ such that $\{\varepsilon^{-1}(\eta_\mu-\psi_\mu)\}_{\mu\in\mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules with regularity $4(|t|+\frac{2}{r}+\delta)$ independent of $\varepsilon>0$. From the previous section, we then have that the representation $\sum_{\mu\in\mathcal{J}\times\mathbb{Z}^2}\langle f, S^{-1}\psi_\mu\rangle\psi_\mu$, $f\in G^s_{p,q}$, has the same sparse representation properties as curvelets when measured in $g^s_{p,q}$. Notice that by starting out with a compactly supported function g, we get a compactly supported curvelet type system.

First we take $g\in C^{M+1}(\mathbb{R}^2)$, $\hat{g}(0)\neq 0$, which for fixed $N'>2, M>0$ satisfies

$$|g^{(\kappa)}(x)|\le C(1+|x|)^{-N'},\ |\kappa|\le M+1, \tag{5.1}$$

Next, for $m \geq 1$ we define $g_m(x) := C_g m^2 g(mx)$, where $C_g =: \hat{g}(0)^{-1}$. It then follows that

$$|g_m^{(\kappa)}(x)| \leq C m^{2+|\kappa|}(1+m|x|)^{-N'}, \; |\kappa| \leq M+1,$$

$$\int_{\mathbb{R}^2} g_m(x)\,\mathrm{d}x = 1. \tag{5.2}$$

We recall that curvelets (2.2) are a family of curvelet molecules for any regularity $R \in \mathbb{N}$. From the definition of a family of curvelet molecules (Definition 2.1), we have that for $j \geq 2$ curvelet molecules can be expressed as

$$\eta_\mu(x) = 2^{\frac{3j}{4}} a_\mu(D_{2^{-j}} R_{\theta_\mu} x - (k_1/\delta_1, k_2/\delta_2)),$$

where a_μ must satisfy (2.3) and (2.4). For the coarse scale, $j = 1$, similar requirements exist. So to construct a family of curvelet molecules $\{\psi_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ such that $\{\varepsilon^{-1}(\eta_\mu - \psi_\mu)\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ is a family of curvelet molecules, we need to construct a family of functions $\{b_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ such that $\varepsilon^{-1}(a_\mu - b_\mu)$ satisfy (2.3) and (2.4). We define $\{\psi_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ as

$$\psi_\mu(x) := 2^{\frac{3j}{4}} b_\mu(D_{2^{-j}} R_{\theta_\mu} x - (k_1/\delta_1, k_2/\delta_2))$$

for $j \geq 2$, and similar for the coarse scale elements, $j = 1$. To construct $\{b_\mu\}_{\mu \in \mathcal{J}\times\mathbb{Z}^2}$ we use the following set of finite linear combinations,

$$\Theta_{K,m} := \{b_\mu : b_\mu(\cdot) = \sum_{i=1}^{K} c_i g_m(\cdot + d_i), c_i \in \mathbb{R}, d_i \in \mathbb{R}^2\}.$$

Proposition 5.1.
Let $N' > N > 2$, $M > 0$ and $j > 0$. If $g \in C^{M+1}(\mathbb{R}^2)$, $\hat{g}(0) \neq 0$, fulfills (5.1) and $a_\mu \in L_2(\mathbb{R}^2) \cap C^{M+1}(\mathbb{R}^2)$ fulfills

$$|a_\mu^{(\kappa)}(x)| \leq C(1+|x|)^{-N'}, \; |\kappa| \leq M+1$$

$$|\hat{a}_\mu(\xi)| \leq C \min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^{M+1},$$

then for any $\varepsilon > 0$ there exists $K, m \geq 1$ (m independent of j) and $b_\mu \in \Theta_{K,m}$ such that

$$|a_\mu^{(\kappa)}(x) - b_\mu^{(\kappa)}(x)| \leq \varepsilon(1+|x|)^{-N}, \; |\kappa| \leq M \tag{5.3}$$

$$|\hat{a}_\mu(\xi) - \hat{b}_\mu(\xi)| \leq \varepsilon \min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^{M}. \tag{5.4}$$

Proof:
Let $\varepsilon > 0$ and κ, $|\kappa| \leq M$, be given. We construct the approximation of a_μ in direct space in three steps. First by a convolution operator $\omega_m = a_\mu * g_m$, then

by $\theta_{q,m}$ which is the integral in ω_m taken over a dyadic cube Q, and finally by a discretization over smaller dyadic cubes $b_{l,q,m}$. From (5.2) we have

$$a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x) = \int_{\mathbb{R}^2} \left(a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)\right) g_m(y)\,\mathrm{d}y. \tag{5.5}$$

Define $U := m^{\lambda/2N}$, where $\lambda := \min(1, N' - N)$. For $|x| \le U$, we use the mean value theorem to get

$$|a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)| \le C \min(1, |y|).$$

Inserting this in (5.5) we have

$$\begin{aligned} |a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x)| &\le C \int_{\mathbb{R}^2} \frac{\min(1,|y|) m^2}{(1+m|y|)^{N'}}\,\mathrm{d}y \\ &\le C m^{-\lambda} \le \frac{C m^{-\lambda/2}}{U^N} \le \frac{C m^{-\lambda/2}}{(1+|x|)^N}. \end{aligned} \tag{5.6}$$

For $|x| > U$, we split the integral over $\Omega := \{y : |y| \le |x|/2\}$ and Ω^c. If $y \in \Omega$, then $|x-y| \ge |x|/2$, and we have

$$\begin{aligned} \int_\Omega |a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)||g_m(y)|\,\mathrm{d}y &\le C(1+|x|)^{-N'} \\ &\le \frac{C}{(1+U)^\lambda (1+|x|)^N} \le \frac{C m^{-\lambda^2/2N}}{(1+|x|)^N}. \end{aligned} \tag{5.7}$$

Integrating over Ω^c with $|x| > U$ gives

$$\begin{aligned} &\int_{\Omega^c} |a_\mu^{(\kappa)}(x) - a_\mu^{(\kappa)}(x-y)||g_m(y)|\,\mathrm{d}y \\ &\qquad \le \frac{C}{(1+|x|)^{N'}} + \int_{\Omega^c} \frac{C m^2}{(1+|x-y|)^{N'}(1+m|y|)^{N'}}\,\mathrm{d}y \\ &\qquad \le \frac{C}{(1+|x|)^{N'}} + \frac{C m^{-\lambda}}{(1+|x|)^N} \le \frac{C(m^{-\lambda^2/2N} + m^{-\lambda})}{(1+|x|)^N}. \end{aligned} \tag{5.8}$$

So by choosing m sufficiently large in (5.6)-(5.8), we get

$$|a_\mu^{(\kappa)}(x) - \omega_m^{(\kappa)}(x)| \le \frac{\varepsilon}{3}(1+|x|)^{-N}. \tag{5.9}$$

For the next step we fix m and choose $q \in \mathbb{N}$. Let Q denote the dyadic cube with sidelength 2^{q+1}, sides parallel with the axes and centered at the origin. We then approximate ω_m with $\theta_{q,m}$ defined as

$$\theta_{q,m}(\cdot) = \int_Q a_\mu(y) g_m(\cdot - y)\,\mathrm{d}y.$$

In which case we have

$$\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x) = \int_{Q^c} a_\mu(y) g_m^{(\kappa)}(x-y)\,\mathrm{d}y,$$

and it follows that,

$$|\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x)| \le \int_{|y|\ge 2^q} \frac{Cm^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|_B)^{N'}}\,\mathrm{d}y := L.$$

We first estimate the integral for $|x| \le 2^{q-1}$ which gives $|y| > |x|$ and $|x-y| \ge 2^{q-1}$. Hence we obtain

$$L \le \frac{Cm^{2+|\kappa|}}{(1+|x|)^{N'}} \int_{|u|\ge 2^{q-1}} \frac{1}{(1+m|u|)^{N'}}\,\mathrm{d}u \le \frac{Cm^{|\kappa|-\lambda}2^{-\lambda q}}{(1+|x|)^{N'}}. \tag{5.10}$$

For $|x| > 2^{q-1}$, we split the integral over $\Omega := \{y : |y| \ge 2^q\} \cap \{y : |y| \le |x|/2\}$ and $\Omega' := \{y : |y| \ge 2^q\} \cap \{y : |y| > |x|/2\}$. If $y \in \Omega$, then $|x-y| \ge |x|/2$, and we get

$$\begin{aligned}\int_\Omega \frac{m^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|)^{N'}}\,\mathrm{d}y &\le \frac{Cm^{2+|\kappa|}}{(1+m|x|)^{N'}} \int_{|y|\ge 2^q} \frac{1}{(1+|y|)^{N'}}\,\mathrm{d}y \\ &\le \frac{Cm^{|\kappa|-\lambda}2^{-\lambda q}}{(1+|x|)^N}. \end{aligned}\tag{5.11}$$

Similar for Ω' we have

$$\begin{aligned}\int_{\Omega'} \frac{m^{2+|\kappa|}}{(1+|y|)^{N'}(1+m|x-y|)^{N'}}\,\mathrm{d}y &\le \frac{C}{(1+|x|)^{N'}} \int_{\mathbb{R}^2} \frac{m^{2+|\kappa|}}{(1+m|x-y|)^{N'}}\,\mathrm{d}y \\ &\le \frac{Cm^{|\kappa|}}{(1+|x|)^{N'}} \le \frac{m^{|\kappa|}2^{-\lambda q}}{(1+|x|)^N}. \end{aligned}\tag{5.12}$$

By choosing q sufficiently large in (5.10)-(5.12), we obtain

$$|\omega_m^{(\kappa)}(x) - \theta_{q,m}^{(\kappa)}(x)| \le \frac{\varepsilon}{3}(1+|x|)^{-N}. \tag{5.13}$$

For the final step we fix q, choose $l \in \mathbb{N}$ and approximate $\theta_{q,m}$ by a discretization

$$b_{l,q,m}(\cdot) = \sum_{I \in H_{l,q}} |I| a_\mu(x_I) g_m(\cdot - x_I),$$

where x_I is the center of the dyadic cube I and $H_{l,q}$ is the set of dyadic cubes with sidelength 2^{-l} which together give Q. Note that $b_{l,q,m} \in \Theta_{K,m}$, $K = 2^{q+l+1}$. We introduce $F(\cdot) := a_\mu(\cdot) g_m^{(\kappa)}(x - \cdot)$ which gives

$$|\theta_{q,m}^{(\kappa)}(x) - b_{l,q,m}^{(\kappa)}(x)| \le \sum_{I \in H_{l,q}} \int_I |a_\mu(y) g_m^{(\kappa)}(x-y) - a_\mu(x_I) g_m^{(\kappa)}(x - x_I)|\,\mathrm{d}y$$

$$\leq \sum_{I\in H_{l,q}} \int_I |F(y) - F(x_I)|\,\mathrm{d}y.$$

By using the mean value theorem, we then get

$$|\theta^{(\kappa)}_{q,m}(x) - b^{(\kappa)}_{l,q,m}(x)| \leq \sum_{I\in H_{l,q}} \int_I |y - x_I| \max_{\substack{z\in l(x_I,y)\\ |\kappa'|\leq 1}} |F^{(\kappa')}(z)|\,\mathrm{d}y$$

$$\leq C2^{2q-l} \max_{\substack{|z|\leq 2^{q+1}\\ |\kappa'|\leq|\kappa|+1}} |g^{(\kappa')}_m(x-z)|, \tag{5.14}$$

where $l(x_I, y)$ is the line-segment between x_I and y. If $|x| \leq 2^{q+2}$ and $|\kappa'| \leq |\kappa| + 1$, then we have

$$|g^{(\kappa')}_m(x-z)| \leq Cm^{3+|\kappa|} \leq \frac{Cm^{3+|\kappa|}2^{qN}}{(1+|x|)^N}. \tag{5.15}$$

For $|x| > 2^{q+2}$ and $|z| \leq 2^{q+1}$, we have $|x - z| \geq |x|/2$, and hence for $|\kappa'| \leq |\kappa| + 1$, it follows that

$$|g^{(\kappa')}_m(x-z)| \leq \frac{Cm^{3+|\kappa|}}{(1+m|x|)^{N'}} \leq \frac{Cm^{3+|\kappa|}}{(1+|x|)^{N'}}. \tag{5.16}$$

By choosing l sufficiently large, we obtain by combining (5.14)-(5.16) that

$$|\theta^{(\kappa)}_{q,m}(x) - b^{(\kappa)}_{l,q,m}(x)| \leq \frac{\varepsilon}{3}(1+|x|)^{-N}. \tag{5.17}$$

Finally by combining (5.9), (5.13) and (5.17), we get

$$|a^{(\kappa)}_\mu(x) - b^{(\kappa)}_{l,q,m}(x)| \leq \varepsilon(1+|x|)^{-N}. \tag{5.18}$$

To approximate a_μ in frequency space we use three steps similar to the approximation in direct space. Note that $b_{l,q,m}$ still fulfills (5.18) if we choose l, q, m even larger. First we use $\hat{\omega}_m$ to approximate $\hat{a}_\mu$ in which case we have

$$|\hat{a}_\mu(\xi) - \hat{\omega}_m(\xi)| = |\hat{a}_\mu(\xi)^{\frac{M}{1+M}}\hat{a}_\mu(\xi)^{\frac{1}{1+M}}(1 - C_g\hat{g}(\xi/m))|$$

$$\leq C\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M(1+|\xi|)^{-1}|1 - C_g\hat{g}(\xi/m)|.$$

By choosing $\xi_g > 0$ such that $C(1+\xi_g)^{-1}|1 - C_g\hat{g}(\xi/m)| \leq \varepsilon/3$ and m such that $C|1 - C_g\hat{g}(\xi/m)| \leq \varepsilon/3$ for $|\xi| < \xi_g$, we get

$$|\hat{a}_\mu(\xi) - \hat{\omega_m}(\xi)| \leq \frac{\varepsilon}{3}\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M. \tag{5.19}$$

Next, we fix m, choose q and limit the Fourier integral of a_μ to Q from the approximation in direct space,

$$\theta'_{q,m}(\xi) = \hat{g}_m(\xi)\int_Q a_\mu(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

This gives

$$|\hat{\omega}_m(\xi) - \theta'_{q,m}(\xi)| \le |\hat{g}_m(\xi)|\int_{|x|>2^q} |a_\mu(x)e^{ix\cdot\xi}|\,\mathrm{d}x \le C2^{-\lambda q}. \tag{5.20}$$

In the last step, we fix q and approximate $\theta'_{q,m}$ by $\hat{b}_{l,q,m}$. We introduce $G(x) := a_\mu(x)e^{ix\cdot\xi}$ which gives

$$\begin{aligned}|\theta'_{q,m}(\xi) - \hat{b}_{l,q,m}(\xi)| \le& |\hat{g}_m(\xi)|\Big|\int_Q a_\mu(x)e^{ix\cdot\xi}\,\mathrm{d}x - \sum_{I\in H_{l,q}} |I|a_\mu(x_I)e^{ix_I\cdot\xi}\Big| \\ \le& |\hat{g}_m(\xi)|\sum_{I\in H_{l,q}}\int_I |G(x)-G(x_I)|\,\mathrm{d}x \\ \le& \frac{C2^{2q-l}}{1+|\xi/m|}\max_{\substack{x\in\mathbb{R}^2\\ |\kappa'|\le 1}} |G^{(\kappa')}(x)| \le Cm2^{2q-l}. \end{aligned} \tag{5.21}$$

By combining (5.19)-(5.21) for sufficiently large l, q, m, we get

$$|\hat{a}_\mu(\xi) - \hat{b}_{l,q,m}(\xi)| \le \varepsilon\min(1, 2^{-j} + |\xi_1| + 2^{-\frac{j}{2}}|\xi_2|)^M.$$

It follows that by choosing l, q, m large enough $b_{l,q,m}$ fulfills both (5.3) and (5.4). Furthermore, we have $b_{l,q,m} \in \Theta_{K,m}$, $K = 2^{q+l+1}$.

■

6. Discussion

In this paper we studied a flexible method for generation curvelet type systems with the same sparse representation properties as curvelets when measured in $g^s_{p,q}$. With Proposition 4.2, Lemma 4.3 and Theorem 4.4 we proved that a system of curvelet molecules which is close enough to curvelets has these sparse representation properties. Furthermore, with Proposition 5.1 we constructed such a system of curvelet molecules from finite linear combinations of shifts and dilates for a single function with sufficient smoothness and decay.

Examples of functions with sufficient smoothness and decay are the exponential function $e^{-|\cdot|^2}$ and the rational functions $(1+|\cdot|^2)^{-N}$ with N sufficiently large. An example with compact support can be constructed by using a spline with compact support. Furthermore as the system is constructed using finite linear combinations of splines, we get a system consisting of compactly supported splines.

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