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Sriraman, Bharath; Dahl, Bettina

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Chapter 64

On Bringing Interdisciplinary Ideas to Gifted Education

Bharath Sriraman and Bettina Dahl

Abstract This chapter is based on the premise that the utopian goal of education is to unify various strands of knowledge as opposed to dividing it. Ideally education should nurture talent in the classroom and create well-rounded individuals akin to the great thinkers of the Renaissance. That is, individuals who are able to pursue multiple fields of research and appreciate both the aesthetic and the structural/scientific connections between mathematics, arts, and the sciences. We will explore an under addressed aspect of giftedness, namely the role of interdisciplinary activities and problems to foster talent in and across the disciplines of mathematics, science and humanities, increasingly important for emerging professions in the twenty-first century. Examples from the history of mathematics, science and arts will be used to argue for the value of such activities to foster polymathic traits in gifted individuals, particularly the questioning of paradigms. Recent findings from classroom studies will be used to illustrate the value of such an approach to gifted education.

Keywords Interdisciplinary ideas in gifted education · Domain general creativity · Domain general giftedness · History of science · Interdisciplinarity

Introduction: Why Interdisciplinarity?

Interdisciplinarity is a topic on which one finds scant literature in the field of education, particularly in gifted education. Although we live in an age where knowledge is increasingly being integrated in emerging domains such as mathematical genetics; bio-informatics; nanotechnology; modeling; ethics in genetics and medicine; ecology and economics in the age of globalization, the curriculum in most parts of the world is typically administered in discrete packages. The analogy of mice in a maze appropriately characterizes a day in the life of students, with mutually exclusive class periods for math, science, literature, languages, social studies etc. Yet reality does not function in this discrete manner. Although critical thinking, problem solving and communication are real world skills that cut across the aforementioned disciplines (Sriraman, 2003a, 2004a, 2004b; Sriraman & Adrian 2004a, 2004b) students are led to believe that these skills are context dependent. For instance teachers encourage critical thinking in debate, history, and literature, whereas problem solving is encouraged in the sciences and communication is traditionally valued in language classes. Even Mathematics is increasingly viewed as a highly specialized field in spite of its intricate connections to the arts and sciences. This being said, within the field of mathematics itself, a Danish project (Niss & Jensen, 2002) describes mathematics, not solely as various areas of content knowledge but also discusses eight mathematical competencies that comprise intuition and creativity across educational levels and topic areas throughout the education system. The thinking behind the Danish project[1] has influenced the mathematical domain of OECD's PISA project (OECD, 1999).

The thinkers of the Renaissance did not view themselves simply as mathematicians, or inventors

B. Sriraman (✉)
The University of Montana, Missoula, MT, USA
e-mail: sriramanb@mso.umt.edu

[1] Niss & Jensen (2002) see two groups of competencies, each comprising four competencies: (a) the ability to ask and answer

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or painters, or philosophers or political theorists, but thought of themselves as seekers of Knowledge, Truth and Beauty. In other words there was a Gestalt worldview that unified the arts and sciences as opposed to dividing it. An example of such thinkers is Sir Isaac Newton (1643–1727) who not only excelled in mathematics but did also in-depth and ground-breaking work in optics, physics, astronomy, and theology. His book entitled *Quaestiones Quaedam Philosophicae* was headed with a statement meaning "Plato is my friend, Aristotle is my friend, but my best friend is truth." Newton also did alchemy experiments. Alchemy (Holmyard, 1990) is both an early form of chemical technology exploring the nature of substances and a philosophy and spiritual discipline, combining elements of chemistry, metallurgy, physics, medicine, astrology, semiotics, mysticism, spiritualism and art., i.e., it was interdisciplinary in nature and builds on the assumption that everything is connected. Western alchemy became the forerunner of modern science and significant for the development of experimental sciences, and until the eighteenth century alchemy was considered a serious science.

Another example is René Descartes (1596–1650), a philosopher, mathematician, scientist, and lawyer. After crawling into a wall stove, Descartes had a vision and three dreams on the unification of science on November 10, 1619:

> He was possessed by a Genius, and the answers were revealed in a dazzling, unendurable light. Later, in a state of exhaustion, he went to bed and dreamed three dreams that had been predicted by this Genius. . . . He tells us that his third dream pointed to no less than the unification and the illumination of the whole of science, even the whole of knowledge, by one and the same method: the method of reason (Davis & Hersh, 1988, pp. 3–4).

Akin to this thinking is Gottfried Wilhelm von Leibniz (1646–1716), a philosopher, mathematician, and lawyer:

> The vision of Descartes became the new spirit. Two generations later, the mathematician and philosopher Leibniz talked about the 'characteristica universalis'. This was the dream of a universal method whereby all human problems, whether of science, law, or politics, could be worked out rationally, systematically, by logical computation. In our generation, the visions of Descartes and Leibniz are implemented on every hand (Davis & Hersh, 1988, pp. 7–8).

An earlier example is Girolamo Cardano (1501–1576), a mathematician and a medical doctor who wrote on a diversity of topics such as medicine, philosophy, astronomy, theology, and mathematics. He also published two encyclopedias of natural science, where we, as above, see an attempt to unity these various fields under the heading of science. Gliozzi (1970) comments about these encyclopedias that they essentially contain everything from cosmology and machine construction principles to the influence of demons. In other words, these encyclopedias were an amalgamation of fact and fiction, of science and imagination, of the real, and the occult.

The question for education, particularly gifted education, then is how do we re-create this worldview in the mathematics classroom? How can we create the experiences for students that lead to the realization and appreciation of the underlying unity of the arts and sciences? Is this even possible in the classroom? As mathematicians and mathematics educators our view of interdisciplinarity for gifted education is skewed toward the use of mathematics to develop interdisciplinary activities for students. This view is justified by the development of mathematics itself as a result of work in art, astronomy, natural and physical sciences, and theology. Our chapter will reveal that much of this historical work was conducted by eminently gifted individuals who were polymaths of the highest order.

In order to convey a unified view of knowledge to students who are used to viewing the arts and sciences through the "discrete" lens of disjoint school subjects unrelated to each other, one would almost have to create a paradigmatic shift in their mind sets. The unique nature of mathematics and philosophy and their intricate connections to the arts and sciences makes them the ideal bridge to unify the fragmented nature of students' curricular experiences. In this chapter, we demonstrate the possibility for students to create and challenge paradigms arising from fundamental philosophical questions common to both the arts and the sciences that lead to the consideration of a unified view of Knowledge. In today's world the urgency of preparing today's students adequately for future-oriented fields is increasingly being emphasized at the university level. Steen (2005) wrote that "as a science

question in and with mathematics: (1) mathematical thinking competency, (2) problem-handling competency, (3) modeling competency, (4) reasoning competency. (b) The ability to deal with mathematical language and tools: (5) representation competency, (6) symbol and formalism competency, (7) communication competency, (8) aids and tools competence.

biology depends increasingly on data, algorithms and models; in virtually every respect it is becoming... more mathematical" (xi). Both the National Research Council (NRC) and the National Science Foundation (NSF) in the United States is increasingly funding universities to initiate interdisciplinary doctoral programs between mathematics and the other sciences with the goal of producing scientists who are adept at "modeling reality." Yet many institutions find it difficult to recruit students capable of graduate level work in interdisciplinary fields such as mathematical biology and bio-informatics. This suggests that undergraduates feel under prepared to pursue careers in these emerging fields. Any educator with a sense of history foresees the snowball effect or the cycle of blaming inadequate preparation to high school onto middle school onto the very elementary grades, which suggests we work bottom up (Lesh & Sriraman, 2005a). That is, initiate and study school subjects from an interdisciplinary point of view; engage in the mathematics modeling of complex systems that occur in real-life situations from the early grades; and encourage the presentation of data and models through multimedias (Lesh, Hamilton, & Kaput, 2007; Lesh & Sriraman, 2005a, 2005b). Another objective of this chapter is to illustrate how students are capable of constructing a paradigm based on ontological, epistemological, and methodological assumptions (Sriraman, 2005a). In particular, we will demonstrate the value of and argue that the use of classical and contextual mathematics problems, mathematics fiction (literature), and paradoxes in mathematics classrooms can lead to desirable holistic outcomes such as broader student perspective, which is characterized by creativity (originality in thinking), critical thinking (philosophy), and problem solving. These outcomes create a potential for students to realize the unity of the arts and sciences.

Our preamble begs the question as to whether creativity is domain general or domain specific? We operate on the premise that creativity is domain general and that individuals are capable to making creative contributions across domains. Root-Bernstein (this volume) presents both sides of the issue and argues that creativity is domain general and that polymathy fosters creativity. He argues that "the observation that creativity is associated with polymathic ability has been validated by historians as well". We construct a working definition for interdisciplinary creativity based on the domain-specific and domain general arguments found in the literature.

In the general literature on creativity numerous definitions can be found. Craft (2002) used the term "life wide creativity" to describe the numerous contexts of day to day life in which the phenomenon of creativity manifests. Other researchers have described creativity as a natural "survival" or "adaptive" response of humans in an ever-changing environment (Gruber, 1989; Ripple, 1989). Craft (2003) has pointed out that it is essential we distinguish "everyday creativity" such as improvising on a recipe from "extraordinary creativity" which causes paradigm shifts in a specific body of knowledge. It is generally accepted that works of "extraordinary creativity" can be judged only by experts within a specific domain of knowledge (Csikszentmihalyi, 1988, 2000; Craft, 2003). For instance, Andrew Wiles' proof of Fermat's Last Theorem could only be judged by a handful of mathematicians within a very specific sub-domain of number theory. In the realm of educational psychology, one can also find a variety of definitions of creativity. For example, Weisberg (1993) suggested that creativity entails the use of ordinary cognitive processes and results in original and extraordinary products. Further, Sternberg & Lubart (2000) defined creativity as the ability to produce unexpected original work, which is useful and adaptive. Other definitions usually impose the requirement of novelty, innovation or unusualness of a response to a given problem (Torrance, 1974). Rather similar to this, Krutetskii (1976) stated that creative thinking is indicated by the product of mental activity having novelty and value both subjectively and objectively, that the thinking process requires a transformation of previously accepted ideas or a denial of them, and that the thinking process is characterized by the presence of strong motivation and stability. "A thorough, independent, and creative study of mathematics is a prerequisite for developing an ability for creative mathematical ability—for the independent formulation and solution of problems that have new and socially significant content" (Krutetskii, 1976, p. 69). Numerous confluence theories of creativity define creativity as a convergence of knowledge, ability, thinking style, motivational, and environmental variables (Sternberg & Lubart, 1996, 2000), an evolution of domain-specific ideas resulting in a creative outcome (Gruber & Wallace, 2000). For example, Csikszentmihalyi (2000) suggests creativity is one of several mutations resulting from a favorable interaction between an individual, domain, and field. Most recently, Plucker and Beghetto (2004) offered an empirical definition of creativity based on a survey and synthesis of numerous empirical studies in the field. They defined creativity as "the interplay between ability and process by which an individual or group produces an outcome or product that is both novel and useful as defined within some social context" (p. 156). The definition of Plucker and Beghetto is easily applicable to interdisciplinary creativity with the addition that processes and outcomes need not be domain specific.

Philosophers define a paradigm as the assumptions, concepts, and practices that create a view of reality in an intellectual discipline (Sriraman, 2004b, 2004c). Students naturally have an underlying (intuitive) but disjointed notion of what constitutes a paradigm in the liberal arts or mathematics or science based on observed practices in these disciplines during their schooling experiences. The question then is how we can use mathematics to change the disjointed assumptions students may have of the various disciplines. In other words how can we create experiences that eventually lead students to pose the ontological, epistemological, and methodological questions that constitute a paradigm? In doing so the approach to any content naturally takes on an interdisciplinary stance. We first present a historical perspective which suggests that an interdisciplinary outlook toward scholarship was more the norm than the exception. Our historical analysis will examine significant paradigmatic shifts which are didactically replicable particularly in the classroom with gifted students.

History of Connections in the Renaissance

The Intrinsic Connection Between Theology–Art–Science and Mathematics

The history and development of mathematics, science and arts are intricately connected to the rise and fall of ancient and modern civilizations. The progression of humanity from hunter-gather societies onto societies with sophisticated astronomical calendars, visually pleasing architectural forms (temples, mosques, cathedrals, etc.) reveals our quest to understand the cosmos, our attempts to represent and symbolize it via patterns, symmetries and structure. A common characteristic of many civilizations (both ancient and modern) is the quest to answer three basic philosophical questions:

(1) What is reality? Or what is the nature of the world around us? This is linked to the general ontological question of distinguishing objects (real versus imagined, concrete versus abstract, existent versus non-existent, independent versus dependent and so forth).
(2) How do we go about knowing the world around us? [the methodological question, which presents possibilities to various disciplines to develop methodological paradigms] and,
(3) How can we be certain in the "truth" of what we know? [the epistemological question].

The interplay of mathematics, arts and sciences is found in the attempts to answer these fundamental philosophical questions. In this sense philosophy can be viewed as the foundational bridge unifying mathematics, arts and the sciences. We will first focus on the attempts of the thinkers of the Renaissance, who did not view themselves simply as theologians or mathematicians or inventors or painters or philosophers or political theorists, but who thought of themselves as philosophers in the pursuit of Knowledge, Truth and Beauty. Then we will try to pinpoint the thinking characteristics of these natural philosophers, particularly the trait of polymathy (Root-Bernstein, 2003) which explains innovative behavior across numerous subject specific domains. Finally we will examine the implications of these findings for present day gifted education at the school and university levels. Given the dearth of literature on interdisciplinary approaches in gifted education, our goal is to use history and the limited number of interdisciplinary studies involving the innovative use of mathematics (content and pedagogy) within the realm of gifted education to bring interdisciplinary ideas to the field of gifted education.

Revisiting the Renaissance

The great intellectual gifts of the Renaissance can be viewed as the influence of Theology on Art and Science resulting in powerful mathematics as a consequence. The great works of Art during the Renaissance, particularly those of Italian painters (Massacio, Brunelleschi, Leonardo da Vinci, Michelangelo, Titian, Giotto, Raphael, etc.) immediately reveal the interplay of church doctrine with art in that these painters inspired devotion in people by painting divine Christian icons. In this process of trying to convey these divine images as realistically and beautifully as possible, these painters moved away from the medieval style of painting (very two dimensional) and essentially created the rules of perspectivity that allowed three-dimensional images to be projected onto a flat surface. This suggests that art was instrumental in initiating the mathematical foundations of the true rules of perspective. Geometrical optics also played a major role in how artists experimentally arrived at the mathematical rules of perspectivity.

Calter (1998) traces a rich lineage of the interaction of optics with art: Euclid's Optica (300 BC), Vitruvius' Ten Books on Architecture, Ptolemy's Optica (c. AD 140), (the Islamic) Alhazen's Perspectiva (c. AD 1000), Roger Bacon's Opus Majus (c. AD 1260), with sections on optics, "whose geometric laws, he maintained, reflected God's manner of spreading his grace throughout the universe—onto John Pecham's Perspectiva communis" (c. AD 1270). The amalgamation of mathematical ideas proposed in this lineage was formalized by Desargues (1593–1662) which is today studied in courses on projective geometry. Today the visual-artistic side of mathematics is completely lost under the rubble of formalization in most places although for instance the Danish compulsory education grades 3–9 has perspectivity included in the mathematics curriculum (Danish Ministry of Education, 2001). The visual side of mathematics has seen a revival in the twentieth century in the area of fractal geometry due to the work of Benoit Mandelbrot. This is explored further at a later junction in this chapter.

The Scientists-Mathematicians-Theologians of the Renaissance–Post-Renaissance

The relationship between science and theology can be traced back to the pre-Socratic Greeks. Pre-Socratic Greek society evolved from the typical "Sky-God" explanation of creation/reality onto a society that developed a rigorous and systematic philosophy to answer the three aforementioned questions (Sriraman & Benesch, 2005). The Pythagorean School (c. 500 BC) developed a mystical numeric system to designate and describe everything in the universe. They even went so far as to claim that numerical attributes could be used to describe everything in the universe. Numbers were designated abstract attributes: one, the number of reason; two, the first even or female number (the number of opinion); three, the first true male number (the number of harmony); four, the number of justice or retribution; five, marriage; six, creation; and so on. According to Brumbaugh (1981) each number had its own personality—masculine or feminine, perfect or incomplete, beautiful or ugly, which modern mathematics has deliberately eliminated, but we still find overtones of it in fiction and poetry. Further the motion of planets and musical notes was expressed as ratios of numbers. The Pythagorean School developed an elaborate numerical system consisting of even and odd numbers to describe the world around them. However Plato (429–327 BC) and Aristotle (384–322 BC) deviated from the mysticism of the Pythagoreans and instead attempted to understand the universe via reason. Plato suggested that the universe consisted of two realms, the visible realm which was deceptive because of its changing nature and an abstract realm which he believed was eternal and unchanging (Sriraman, 2004a). Within this "dualistic" ontology of reality, Plato answered the epistemological question by suggesting that knowledge derived empirically from the changing world was fallible, whereas knowledge derived from the abstract realm was infallible or absolute. Plato accorded a special place for mathematics in this pursuit of absolute knowledge by claiming that mathematics was derivable independent of the physical senses. Thus the purest form of "thought" was mathematical thought as it was deemed capable of deriving "eternal truths" or absolute knowledge. In spite of the alleged motto of the academy, Plato distinguished "numbers as ideas" from "numbers as mathematical objects." Unfortunately over time this important distinction faded and Platonism approached Pythagoreanism, which in turn influenced Renaissance philosophers, then modern natural science and thereby again modern philosophy.

Aristotle on the other hand was an empiricist whose prodigious work left a lasting impression until the thirteenth century (namely the dawn of the Renaissance). The Aristotelian approach to science was empirical and placed a heavy emphasis on perception through the senses. Aristotle rejected the Platonic notion of the mind's capacity to intuit/discern a priori reality and instead proposed an a posteriori or empirical methodology whereby knowledge is acquired by the mind. Aristotelian science was axiomatic and deductive in nature with the aim of explaining natural phenomenon. The underlying assumption of Aristotelian science was that all natural objects were fulfilling a potential determined by an actual prior natural object. For instance, a seed becomes a plant because it is merely fulfilling its potential of becoming a plant. Science historians today agree that Aristotle was an empiricist, who believed that knowledge is gained via observation, experimentation, and experience (Sriraman & Adrian, 2004a; Sriraman & Benesch, 2005). The question of whether or not Aristotelian science was the origin of dualism is still a matter of present day debate.[2] Recent scholarship on post-Renaissance science and natural philosophy traces a rich intellectual lineage centered on "scholasticized Aristotelianism" from seventeenth century natural philosophy onto medieval thinkers like Aquinas onto Aristotle (Sriraman & Benesch, 2005, p. 42).

Thomas Aquinas (1225–1274) synthesized "all that had been argued in Western thought up to his time and he showed it to be compatible with Christian beliefs" (Sharp, 2003, p. 346). His argument is superbly summarized by Sharp (2003) as follows:

> . . .Aquinas argued that all our rational knowledge of this world is acquired through sensory experience, on which our minds then reflect. When children are born, their minds are like a clean slate (tabula rasa). Aquinas developed a theory of knowledge which is uncompromisingly empirical. The world through which we gain our knowledge is God's creation, and therefore it is impossible for this gained knowledge to conflict with religious revelation (p. 346).

The Greek philosophers also stumbled upon the idea of the "infinite", a sophisticated mathematical abstraction, as evidenced in Zeno's paradoxes.[3] Bertrand Russell (1872–1970) observed that scholastic theology was one of the outcomes of mathematical abstraction (Russell, 1945, p. 37). His claim is supported by numerous historical examples, a handful of which are presented here. Sa'id ibn Yusuf (Saadia Gaon), a tenth century theologian and leader of the Babylonian Jews wrote a theological treatise called Kitab al-Amanat wa-al Itiqadat (the Book of Beliefs and Opinions) in which extensive use is made of mathematical arguments to answer cosmological questions, such as the existence of a creator. Gaon (1948) also cleverly reversed Zeno's paradox of Achilles and the tortoise[4] to prove that Creation occurred by arguing that if the world were uncreated, then time would be infinite. But infinite time could not be traversed. This implied that present moment could not have come about. But since the present moment exists, this implies that the world had a beginning. The ideas of the Greeks also had a profound influence on post-Renaissance mathematicians like Descartes (1596–1650), Pascal (1623–1662) and Leibniz (1646–1716) among others. One routinely comes across the use of mathematical analogies to prove the existence of God in the theological works of Descartes, Leibniz, and Pascal. For example, Descartes in the Fifth Meditation states,

> Certainly the idea of God, or a supremely perfect being is one which I find within me, just as surely the idea of any shape or number. And my understanding that it belongs to his nature that he always exists is no less clear and distinct than is the case when I prove of any shape or number that some property belongs to its nature. Hence, even if it turned out that not everything on which I have meditated in these past days is true, I ought still to regard the existence of God as having at least the same level of certainty as I have hitherto attributed to the truths in mathematics (Descartes, 1996, p. 45).

Leibniz in Theodicy argued that faith and reason were compatible:

> Theologians of all parties, I believe (fanatics alone excepted), agree that no article of faith must imply contradiction or contravene proofs as exact as those of mathematics, where the opposite of the conclusion can be reached ad absurdum, that is, to contradiction. It follows thence that certain writers have been too ready to grant that the Holy Trinity is contrary to that great principle, which states that two things, which are the same as the

[2] One could argue that Aristotle drew a distinction in the natural world between the animate and the inanimate, whereas Descartes was more focused on the human. Descartes' dualism dominated physics for a substantial period of time.

[3] Zeno (born around 495 BC) was a Greek philosopher and logician, and a student of the philosopher Parmenides. Zeno is remembered for paradoxes that stumped mathematicians for centuries. Zeno's paradoxes evolved from Parmenides' ideas about the illusory nature of motion, change, and time.

[4] Achilles and the tortoise: The running Achilles can never catch a crawling tortoise ahead of him because he must first reach where the tortoise started. However, when he reaches there, the tortoise has moved ahead, and Achilles must now run to the new position, which by the time he reaches the tortoise has moved ahead, etc. Hence the tortoise will always be ahead.

> third, are also the same as each other. For this principle is a direct consequence of that of contradiction, and forms the basis of all logic; and if it ceases, we can no longer reason with certainty (Leibniz, 1985, p. 87).

Leibniz's dissertation on the conformity of faith and reason can be interpreted to mean that it is logically contingent and intelligible for a human being to ask why an eternal being exists (Craig & Smith, 1995). Blaise Pascal, perhaps the most intriguing mathematical mystic argued for the use of "infinitesimal reasoning"[5] (or reasoning in infinitely small quantities) by proclaiming that the infinitely large and the infinitely small were mysteries of nature that man stumbled on by divine inspiration. Pascal is also remembered for his famous wager, where he argued that if God's existence has a probability of 0.5 (50/50 chance), then it is only rational for us to believe he does exist. Popkin (1989) paraphrasing Pascal writes that if you gain, you gain all; if you lose, you lose nothing!

The aforementioned historical examples, viz., the number mysticism of the Pythagoreans, the paradoxes of Zeno that brought forth the abstraction of the infinite, the attempts of medieval theologians like Saadia Gaon to systematize theology by constructing uniqueness proofs to theological theorems, and the use of mathematical arguments to prove the existence of a Creator by post-Renaissance mathematicians, illustrate that there has been a rich interplay between mathematics and theology (Sriraman, 2004a). While these aforementioned thinkers of the Renaissance and post-Renaissance who strongly believed in the existence of a creator invoked mathematical arguments to prove their beliefs in their philosophical writings, others such as Copernicus and Galileo, who were professed believers of the Catholic Church, found it increasingly difficult to believe in the prescribed view of the world (earth) as the center of the universe. Their model building during this time period reveals the interplay as well as the conflict between theology and science with mathematics replacing Aristotilean logic as the language of description.

[5] An infinitesimal is a number that is infinitely small but greater than zero. Infinitesimal arguments have historically been viewed as self-contradictory by mathematicians in the area of analysis. The infinitesimal Calculus of Newton and Leibniz was reformulated by Karl Weierstrass in the nineteenth century for the sole purpose of eliminating the use of infinitesimals. In the twentieth century Abraham Robinson revived the notion of infinitesimals and founded the subject of non-standard analysis to resolve the contradictions posed by infinitesimals within Calculus. Robinson attempted to use logical concepts and methods to provide a suitable framework for differential and integral Calculus.

Modeling the Universe: Copernicus–Galileo–Kepler

The Ptolemaic model (c. AD 87–150) of astronomy was based on the assumption that the earth was the center of the universe which was accepted by the Catholic Church as being compatible with its teachings. However this geocentric view of the world could not explain the curious planetary phenomenon observed by Nicholaus Copernicus (1473–1543). That is the retrograde motion (moving backwards and then forward) of Mars, Jupiter and Saturn, in addition to nearly invariant times that Venus and Mercury appeared in the sky which is shortly before sunrise and after sunset. However these queer motions were perfectly reasonable if one viewed the sun as the center of the "universe" as opposed to the earth. In such a model the peculiarities of the inner planets (Mercury and Venus) as well as the outer planets (Mars, Jupiter, and Saturn) in relation to the earth make perfect sense. The retrograde motion of outer planets is due to the fact that they get overtaken by the earth in its orbitary motion. Similarly Venus and Mercury appear static and only before sunrise and after sunset because their orbitary motions do not allow them to get behind the earth and manifest in the night sky. It is amazing what a little change in perspective does to our perceptions! However the conflicts of Copernicus' findings with Church dictum prevented a wider dissemination of his simpler planetary model until his death. Centuries later, the great German philosopher, writer, scientist Goethe (1749–1832) reflected on Copernicus' new perspective of our reality:

> Of all discoveries and opinions, none may have exerted a greater effect on the human spirit than the doctrine of Copernicus. The world had scarcely become known as round and complete in itself when it was asked to waive the tremendous privilege of being the center of the universe. Never, perhaps, was a greater demand made on mankind—for by this admission so many things vanished in mist and smoke! What became of our Eden, our world of innocence, piety and poetry; the testimony of the senses; the conviction of a poetic—religious faith? No wonder his contemporaries did not wish to let all this go and offered every possible resistance to a doctrine which

> in its converts authorized and demanded a freedom of view and greatness of thought so far unknown, indeed not even dreamed of.[6]

Galileo Galilei (1564–1642) pushed things further by using mathematics to explain interplanetary motion. In fact many science historians claim that Galileo was the first person to systematically use mathematics as the language of science instead of Aristotilean logic. Aristotle's conceptions of motion had several flaws which were rectified by Galileo by determining that velocity and acceleration were distinct. More importantly the question that vexed Copernicus of why the motion of the Earth was unfelt (if in fact it was moving) was answered by Galileo by suggesting that only acceleration is felt, whereas velocity is unfelt and invariant except when acted on by an external force (the notion of inertia). Thus, Galileo suggested that the Earth in addition to orbiting around the sun was also rotating on its own axis. Needless to say, his attempt to make his model public met with fierce resistance from the Church and led to his condemnation by the Inquisition.

During this same time period the German astronomer Johannes Kepler (1571–1630) confirmed and supported many of Galileo's well-formulated theories. Johannes Kepler was born in Weil der Stadt, Württemburg. While studying for Lutheran ministry at the University of Tübingen, he became familiar with the Copernican model, which he defended explicitly in the Mysterium Cosmographicum. The political forces of that time period with his unique personal circumstances, namely his strong adherence to the Augsburg Confession but rejection of several key Lutheran tenets, the use of the calendar introduced by Pope Gregory XIII, his rejection of the Formula of Concord, and finally his snub to Catholicism led to him to exile in Prague where he worked for the Danish astronomer Tycho Brahe. With the help of Brahe's data, Kepler made several seminal discoveries published in Astronomia Nova. The beauty of this work lies in the fact that Kepler arrived at the first two laws of planetary motion by working with incomplete/imperfect data (we must remember that this data was obtained before the invention of the telescope!). The first two laws were (1) planets move in ellipses with the Sun at one focus and (2) the radius vector describes equal areas in equal times. Finally the third law was published by Harmonices Mundi in 1619. The third law states that the squares of the periodic times are to each other as the cubes of the mean distances. Incidentally Newton's theory of gravitation grew out of Kepler's third law (and not a fallen apple as suggested by myth). Szpiro (2003) recently suggested that among the forces driving Kepler's work during his turbulent Tübingen years was to seek a theological explanation to his questions:

> Since God had created a perfect world, he thought it should be possible to discover and understand the geometric principles that govern the universe. After much deliberation Kepler believed he had found God's principles in the regular solids. . ..His explanations of the universe were based on an imaginary system of cubes, spheres and other solids that he thought were fitted between the sun and other planets. . .published in Mysterium Cosmographicum. This tome did not unveil any mysteries of the planetary system. . .since no solids exist that are suspended in the universe. But the book came to the attention of Tycho Brahe (Szpiro, 2003, p. 13).

And the rest is history. . .

Isaac Newton's (1642–1727) prodigious work included a mathematical model of the planetary system, in a sense suggesting that the universe was governed by certain laws, expressible via mathematics and discernible by humans (Sharp, 2003). This led to the development of natural philosophy as an answer to the ontological, methodological, and epistemological questions with mathematics becoming the medium of establishing truth. One consequence of Aristotle's empiricist tradition was the acceptance of the notion that knowledge of the external world was derived by an active soul,[7] which was in essence separate from that world (Polkinghorne, 1998). Salmon (1990) comments,

> It is illuminating to recognize that Cartesian dualism offered a way of resolving the conflict between science and religion—which had brought such great troubles to Galileo—by providing each with its own separate domain. Physical science could deal with matter, while religion could handle whatever pertains to the soul (p. 236).

This led to a growing acceptance among seventeenth century natural philosophers in the notion of duality (or dualism). René Descartes (1596–1650) is consid-

[6] Quote retrieved April 24, 2005, from http://www.blupete.com/Literature/Biographies/Science/Copernicus.htm

[7] Polkinghorne comments that Aristotle's view of the soul as the underlying "form" or pattern of the body was then taken up by Thomas Aquinas who rejected Platonic dualism that had dominated Western Christian thinking since Augustine. This view is corroborated by extant histories on the philosophy of dualism.

ered the founder of this belief system since he initiated the mind–body problem. Cartesian dualism essentially proclaims that we are composed of two distinct and basic substances, namely the mind (soul) and the matter. Matter was the material substance that extended into the world and took up space, whereas the mind (soul) was a thinking substance, which was not "localizable" in space. "If these two aspects (mind-matter) are to be held in equal balance, it seems that it will have to be in some way more subtle than mere juxtaposition" (Polkinghorne, 1998, p. 54). The problem of dualism can be reformulated as follows: One can think of subject and object as two unique and separate natures, neither of which is reducible to each other. The question of course in such a dualistic assumption is how do these two natures relate to each other? (Sriraman & Benesch, 2005).

The Modern Day Renaissance: Shifts in Perspective

Scientists in the twentieth and twenty-first centuries have developed the technical tools and the analytic and theoretical maturity necessary for analyzing nature at unprecedented micro and macrocosmic levels. By doing so, they have reaffirmed the dynamic nature of the whole that was reflected in the paradoxes of the ancients. The result is a view of nature in which processes have supplanted "things" in descriptions and explanations. By the end of the nineteenth century, limitations of the classical Newtonian/Euclidean worldview had become increasingly problematic as physicists began exploring nature at the subatomic level. The paradoxes posed by uncertainty, incompleteness, nonlocality, and wavicles, etc. let it seem apparent that in the subatomic world observations and observers are aspects of a whole. The physicist John Wheeler commented,

> . . .In the quantum principle we're instructed that the actual act of making an observation changes what it is that one looks at. To me, this is a perfectly marvelous feature of nature. . .. So the old word observer simply has to be crossed off the books, and we must put in the new word participator. In this way we've come to realize that the universe is a participatory universe (Buckley & David, 1979, pp. 53–54).

Biologists have found that methodological reductionism, i.e., going to the parts to understand the whole, which was central to the classical physical sciences, is less applicable when dealing with living systems. Such an approach may lead to a study not of the "living" but of the "dead" because in the examination of highly complex living systems "only by ripping apart the network at some point can we analyze life. We are therefore limited to the study of 'dead' things" (Cramer, 1993, p. 214). One of the most important shifts in the natural sciences in the modern period has been away from the view of a simple and complete separation between observer and observed to an awareness that an observer also represents a living aspect of that which is being observed—both as a product of nature and as the mental possibility in nature of observing, as in the notion of the "participatory universe" coined by John Wheeler. A synthesis of "product" and "process" are at the heart of the puzzles and paradoxes that we associate with ideas of "indeterminacy" in physics and with genes in biology. The very concept of "objectivity" maintains that the observed and observer are separate does not hold in the study of "highly complex biological processes such as evolution or the functioning of the central nervous system . . . we cannot distance ourselves from the object being considered; indeed, this is so at the very moment we start to think" (Cramer, 1993, p. 212). It is amazing how close in understanding, and that across six centuries, modern physics and biology are to the Neo-Confucian Philosopher, Wang Yang-Ming's continuum view of "innate knowledge":

> The innate knowledge of man is the same as that of plants and trees, tiles and stones. . .Heaven, Earth, the myriad things, and man form one body. The point at which this unity is manifested in its most refined and excellent form is the clear intelligence of the human mind (Chan, 1973, p. 221).

The very process of generalizing implies a belief in the unity of the world: "if the different parts of the universe were not like the members of one body, they would not act on one another. . .know nothing of one another, and we. . .would know only one of these parts. We do not ask if nature is one, but how it is one" (Poincaré, 1946, p. 130). The position on mind and nature of theoretical physicists seems consistent with that of the neo-Confucian philosopher. Another physicist suggests that the heliocentric universe is again becoming geo or human centered in that it is "formless potentia. . .and becomes manifest only when observed by conscious beings. . .Of course, we are not the geographical center, but that is not the issue. We are the center of the universe because we are its meaning" (Goswami, 1993, p. 141). This is yet another amazing shift in perspective.

Benoit Mandelbrot (1924–), a very recognizable name in twentieth century mathematics because of his seminal contributions to the development of fractal geometry, has repeatedly emphasized the need to reorient our perspectives to better understand the world around us. He has often very humbly characterized himself as an "accidental" mathematician. In spite of his early interest and precocity in the study of geometry he was "encouraged" by the French university establishment to embrace formalization which led him to leave the École Normale Supériere. He writes,

> I spent several years doing all kinds of things and became, in a certain sense, a specialist of odd and isolated phenomena. . .I did not know or care in which field I was playing. I wanted to find a place, a new field, where I could be the first person to introduce mathematics. Formalization had gone too far for my taste, in the mathematics favored by the establishment. . . (Mandelbrot, 2001, p. 192).

Mandelbrot made his astonishing mathematical discovery when working on an economics problem accidentally handed to him by a friend. Economists had long attempted to make sense of (and predict) stock market fluctuations and had proposed theories based on existing data which did not hold up when tested with primitive computers. Mandelbrot viewed fluctuations from the perspective of changing scales. That is the time scale can be in days or months or years. He suggested that the interchangeable nature of the time scales was the key to understanding the fluctuations:

> I cooked up the simplest mathematical formula I thought could explain this phenomenon. . .[making] no assumptions about people, markets or anything in the real world. It was based on a 'principle of invariance',—the hypothesis that, somehow economics is a world in which things are the same in the small as they are in the large except, of course for a suitable change of scale (Mandelbrot, 2001).

In 1960, Edward Lorenz, who was modeling the earth's atmosphere with non-linear equations at MIT, switched from rounding his equations to the sixth decimal point to doing so to the third. What emerged was a totally different system! He attributed the difference to a combination of the iteration of his equations plus the sensitivity of the system to initial conditions—in this case, the changes in the terminal decimal points. Lorenz named this randomness within his non-random weather models the "butterfly effect" in a paper he wrote entitled "Can the flap of a butterfly's wing stir up a tornado in Texas?" The discovery of "sensitive dependency on initial conditions" coupled with the "iteration of patterns or data" which produce random irregularities in deterministic systems is the beginning of the contemporary science of "deterministic chaos" (Peitgen, Juergens, & Saupe, 1992, p. 48).

The term "fractal" was coined from Latin fractua "irregular," to refer to the results of this combination of iteration and sensitivity. And it was Mandelbrot who provided the pictures of this deterministic chaos in his computer generated fractal images—what is described as ". . .a way of seeing infinity" (Gleick, 1987, p. 98). We discover these irregular non-linear fractal structures and patterns throughout nature, in the iterations of buds in Romanesco broccoli, the arterial and venous systems of kidneys, lungs, brains, coast lines, mountain ranges, root systems, and turbulences in fluids. For example, one might ask the length of a head of cauliflower or a coastline. At one level, the answer might be 8 in. or 580 miles. However, at the fractal level of iteration of growth patterns and/or ocean forces, both can be seen as infinite. In most, perhaps all of nature, we encounter a kind of deterministic chaos in a world described by "fractal geometries" which have ". . .become a way of measuring qualities that otherwise have no clear definition: the degree of roughness or brokenness or irregularity in an object" (Gleick, 1987, p. 98). This is the heterogeneous and nonlinear world of the branching of buds in the cauliflower head, the spongy tissue of the lungs, and the indentation on the beach. "Chaos is more like the rule in nature, while order (=predictability) is more like the exception" (Peitgen, Juergens, & Saupe, 1992, p. 48).

An unpredictable consequence of fractal geometry coupled with advances in computer graphics was that it was now possible for machines to produce geometric "art" based on very simple formulae which "shows surprising kinship to Old Masters paintings, Cubist paintings,[8] or Beaux Arts architecture. An ob-

[8] Cubism is a more modern art movement in which forms are abstracted by using an analytical approach to the object and painting the basic geometric solid of the subject. Cubism is a backlash to the impressionist period in which there is more of an emphasis of light and color. Cubism itself follows Paul Cezanne's statement that "Everything in nature takes its form from the sphere, the cone, and the cylinder" in which these three shapes are used to depict the object of the painting. Another way that the cu-

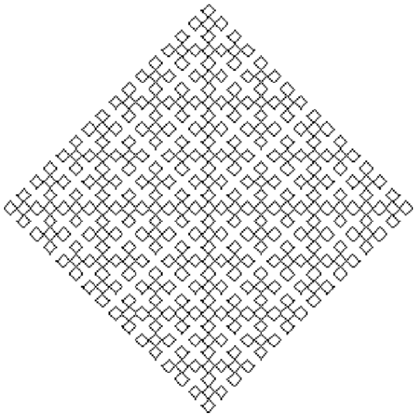

Fig. 64.1 Anklets of Krishna

vious reason is that classical visual arts, like fractals, involve very many scales of length and favor self-similarity (Mandelbrot, 1981, 1989). The discovery that self-similarity was an inherent property of nature as mathematically conceptualized by Mandelbrot was long written about and expressed by poets, satirists, writers, philosophers, and numerous religious traditions. For instance, in Southern India, Kolam is an art form used by women to decorate the entrance to homes and courtyards (see Fig. 64.1). These art forms go back over 6,000 years and consist of self-similar patterns repeated in different scales in very sophisticated fashion. Architecture in Hindu temples also reveals that the notion of self-similarity was used to create visually stunning forms.

Polymathy

The numerous examples of thinkers given thus far represent a unique sample of gifted individuals who made remarkable contributions to the arts, sciences, and mathematics, and who also happened to be philosophers. These individuals are best characterized as polymaths. The term polymath is in fact quite old and synonymous with the German term "Renaissance-mensch." Although this term occurs abundantly in the literature in the humanities, very few (if any) attempts have been made to isolate the qualitative aspects of thinking that adequately describe this term. Most cognitive theorists believe that skills are domain specific and typically non-transferable across domains. This implicitly assumes that "skills" are that which one learns as a student within a particular discipline. However such an assumption begs the question as to why polymathy occurs in the first place. Although the numerous historic and contemporary examples presented are of eminent individuals, it has been found that polymathy as a thinking trait occurs frequently in non-eminent samples (such as high school students) when presented with the opportunities to engage in trans-disciplinary behavior. In particular the use of unsolved classical problems and mathematics literature has been found to be particularly effective in fostering interdisciplinary thinking (see Sriraman, 2003a, 2003b, 2004b, 2004c). These are explored further in a subsequent section of the chapter.

Thinking Traits of Polymaths

Root-Bernstein (2003) has been instrumental in rekindling an interest in mainstream psychology in a systematic investigation of polymathy. That is the study of individuals, both historical and contemporary, and their trans-disciplinary thinking traits which enabled them to contribute to a variety of disciplines. His analysis of the works and biographies of numerous innovators both historical and contemporary reveals that arts advance the sciences and scientists are inspired by the arts (Fig. 64.2).

One recent example provided by Root-Bernstein (2003) is the effect of Escher's drawings on a young Roger Penrose, the mathematical physicist, who visited one of Escher's exhibitions in 1954. Stimulated by the seemingly impossible perspectives conveyed by Escher in two dimensions, Penrose began creating his own impossible objects such as the famous Penrose "impossible" tribar which shows a three-dimensional triangle that twists both forward and backward in two dimensions. Root-Bernstein writes,

> Roger Penrose showed his tribar to his father L.S. Penrose, a biologist who dabbled in art…[who] invented the impossible staircase in which stairs appear to spiral both up and down simultaneously…[and] sent Escher a

bist expressed their painting was by showing different views of an object put together in a way that you cannot actually see in real life. The Cubism period stated in Paris in 1908, reached its peak in 1914, and continued into the 1920s. Major cubists were Pablo Picasso and Georges Braque. For more information visit http://abstractart.20m.com/cubism.htm

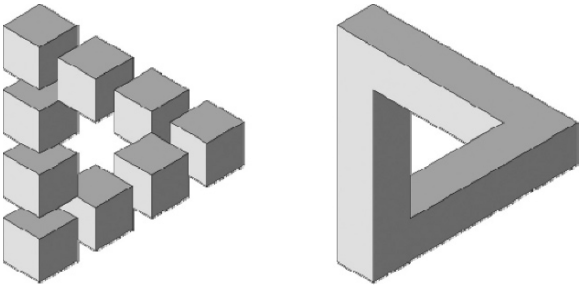

Fig. 64.2 The impossible Penrose Tribar[9]

> copy. . .[who] then developed artistic possibilities of the impossible staircase in ways that have since become famous (p. 274).

Another well-known consequence of Escher's artistic influence on mathematicians is the investigation of tiling problems (both periodic and aperiodic) popularized by both Roger Penrose and Martin Gardner, which helped crystallographers understand the structure of many metal alloys which are aperiodic (Peterson, 1985 as quoted by Root-Bernstein, 2003, p. 274).

Common thinking traits of the polymaths described in this chapter in conjunction with the thousands of polymaths (historical and contemporary) as analyzed by Root-Bernstein (1989, 1996, 2000, 2001, 2003) and reported in various chapters in Shavinina and Ferrari (2004) among others are (1) visual geometric thinking and/or thinking in terms of geometric principles, (2) frequent shifts in perspective, (3) thinking in analogies, (4) nepistemological awareness (that is, an awareness of domain limitations), (5) interest in investigating paradoxes (which often reveal interplay between language, mathematics, and science), (6) belief in Occam's Razor [simple ideas are preferable to complicated ones], (7) acknowledgment of Serendipity and the role of chance, and (8) the drive to influence the Agenda of the times.

A Model of Interdisciplinarity

In this section we present five overarching principles to maximize interdisciplinary creativity and general giftedness (see Fig. 64.3). In order to maximize the potential for creativity to manifest in the mathematics classroom, teachers can encourage mathematically creative students to share their synthetic insights on connections between seemingly diverse problems with the other students in the class (Sriraman, 2004e). Historic examples of synthetic thinking in mathematics, which connect seemingly diverse ideas/concepts can be used in the classroom to further illustrate the power and value of such insights. The scholarly, free market and aesthetic principles contain aspects of Sternberg's (1997) triachic view of giftedness. The five principles also encompass notions of polymathy which can foster creativity in general by connecting notions from the arts and sciences to mathematics and vice versa. These five overarching principles emerge from a synthesis and analysis of the historical literature which reveals the interdisciplinary creativity of individuals, particularly mathematicians (Sriraman, 2005b). The five principles are labeled as follows: (a) the Gestalt principle, (b) the Aesthetic principle, (c) the free market principle, (d) the scholarly principle, and (e) the uncertainty principle.

The Gestalt Principle

The eminent French mathematicians Hadamard (1945) and Poincaré (1948) viewed creativity as a process by which the mathematician makes choices between questions that lead to fruition as opposed to those that lead to nothing new. These mathematicians also viewed the act of creation as a combination of disparate latent ideas within the unconscious. Their conceptions of creativity were influenced by the Gestalt psychology of their time and they characterized mathematical creativity as a four-stage process consisting of preparation, incubation, illumination, and verification (Wallas, 1926; Hadamard, 1945). This combinatorial view of idea generation was also proposed by Hilbert, which unfortunately has been construed as a formalist view of mathematics. Both Poincaré and Hilbert were polymaths who had worked on numerous problems in physics, particularly the class equation. Hadamard made numerous forays into Gestalt psychology and his book was instrumental in popularizing Gestalt psychology from beyond the domain of psychology to the arenas of mathematics and cognition. Although psychologists have criticized the Gestalt model of cre-

[9] More information on the Penrose Tribar is found http://mathworld.wolfram.com/PenroseTriangle.html

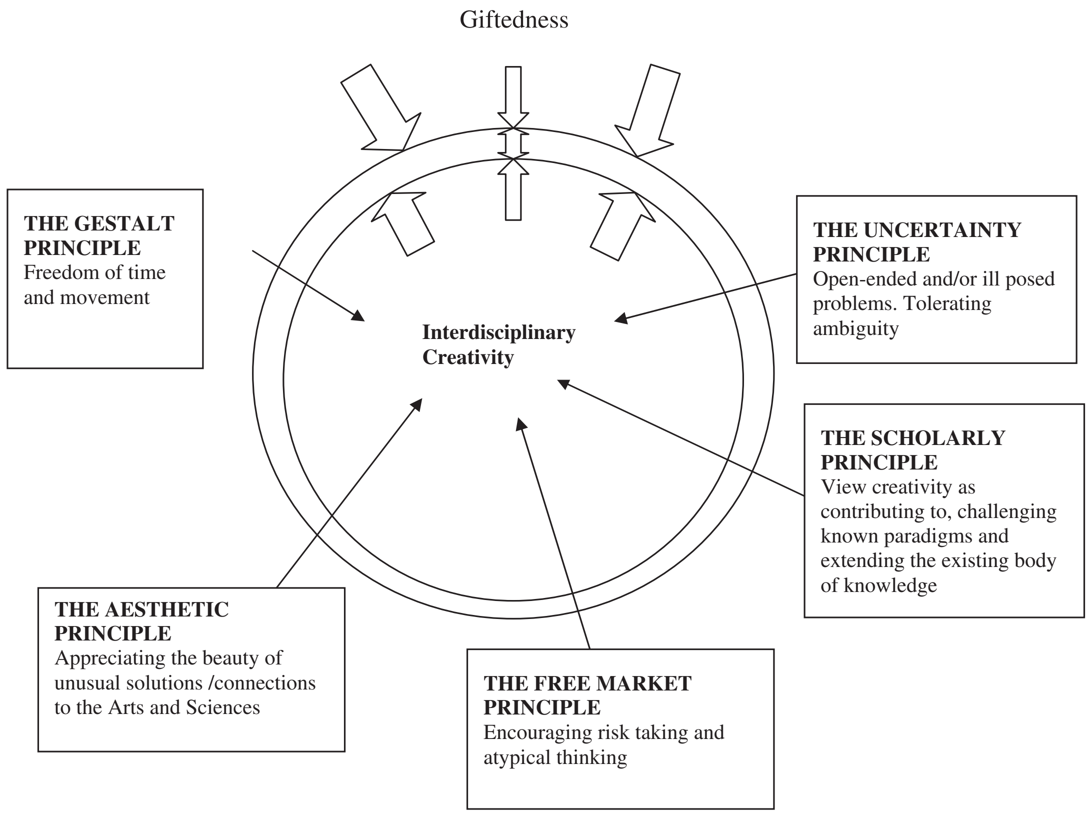

Fig. 64.3 Harmonizing interdisciplinary creativity and general giftedness

ativity because it attributes a large "unknown" part of creativity to unconscious drives during incubation, numerous studies with scientists and mathematicians (i.e., Burton, 1999a, 1999b; Davis & Hersh, 1981; Shaw, 1994; Sriraman, 2004d) have consistently validated this model. In all these studies after one has worked on a problem for a considerable time (preparation) without making a breakthrough, one puts the problem aside and other interests occupy the mind. Also Polya argued for the importance of the unconscious when stating that "conscious effort and tension seem to be necessary to set the subconscious work going" (Polya, 1971, p. 198). Hadamard put forth two hypotheses regarding the incubation phase: (1) the "rest-hypothesis" holds that a fresh brain in a new state of mind makes illumination possible. (2) The "forgetting-hypothesis" states that the incubation phase gets rid of false leads and makes it possible to approach the problem with an open mind (Hadamard, 1945, p. 33). Tall (1991) also argues that "working sufficiently hard on the problems to stimulate mental activity, and then relaxing . . . allow the processing to carry on subconsciously" (p. 15). Krutetskii explains that what is experienced as sudden inspiration "despite the apparent absence of a connection with his former experience, is the result of previous protracted thinking, of previously acquired experience, skills, and knowledge; it entails the processes and use of information the person amassed earlier" (Krutetskii, 1976, p. 305). Also Dahl (2004) has the unconscious as one of six essential themes in her CULTIS model [U for "unconscious"] of the psychology of learning mathematics. This period of incubation eventually leads to an insight on the problem, to the "Eureka" or the "Aha!" moment of illumination. Most of us have experienced this magical moment. Yet the value of this archaic Gestalt construct is ignored in the classroom. This implies that it is important that teachers encourage the gifted to engage in suitably challenging problems over a protracted time period thereby creating the opportunities for the discovery of an insight and to experience the euphoria of the "Aha!" moment.

The Aesthetic Principle

Many eminent creators (particularly mathematicians) have often reported the aesthetic appeal of creating a "beautiful" theorem that ties together seemingly disparate ideas, combines ideas from different areas of mathematics, or utilizes an atypical proof technique (Birkhoff, 1956, 1969; Dreyfus & Eisenberg, 1986; Hardy, 1940). Wedderburn's theorem that a finite division ring as a field is one instance of a unification of apparently random ideas because the proof involves algebra, complex analysis, and number theory. Cantor's argument about the uncountability of the set of real numbers is an often quoted example of a brilliant and atypical mathematical proof technique (Nickerson, 2000). The eminent English mathematician G.H. Hardy (1940) compared the professional mathematician to an artist, because like an artist, a mathematician was a maker of patterns in the realm of abstract ideas. Hardy (1940) said,

> A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas. . . . The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics (p. 13).

Also Krutetskii states that capable students often try to solve the problem in a more simple way or improve the solution. They do not show satisfaction before the solution is economical, rational, and elegant (Krutetskii, 1976, p. 285). Recent studies in Australia (Barnes, 2000) and Germany (Brinkmann, 2004) with middle and high school students revealed that students were capable of appreciating the "aesthetic beauty" of a simple solution to a complex mathematical problem.

The Free Market Principle

Scientists in an academic setting take a huge risk when they announce a new theory or medical breakthrough or proof to a long-standing unsolved problem. Often the reputation of the person is put to risk if a major flaw or refutation is discovered in their findings. For instance in mathematical folklore, Louis De Branges' announcement of a proof to the Riemann hypothesis[10] fell through upon scrutiny by the experts. This led to subsequent ignorance of his claim to a brilliant proof for the Bieberbach Conjecture.[11] The western mathematical community took notice of Louis De Brange's proof of the Bieberbach conjecture only after a prominent Soviet group of mathematicians supported his proof. On the other hand, Ramanujan's numerous intuitive claims, which lacked proof, were widely accepted by the community because of the backing of giants like G.H. Hardy and J.E. Littlewood. The implication of these anecdotes from professional mathematics for the classroom is that teachers should encourage students to take risks. In particular they should encourage the gifted/creative students to pursue and present their solutions to contest or open problems at appropriate regional and state math student meetings, allowing them to gain experience at defending their ideas upon scrutiny from their peers (Sriraman, 2005b).

The Scholarly Principle

K-12 teachers should embrace the idea of "creative deviance" as contributing to the body of mathematical knowledge, and they should be flexible and open to alternative student approaches to problems. In addition, they should nurture a classroom environment in which students are encouraged to debate and question the validity of both the teachers' and other students' approaches to problems. Gifted students should also be encouraged to generalize the problem and/or the solution as well as pose a class of analogous problems in other contexts. Enhancing students' ability to work with analogical problems has historically been

[10] The Riemann hypothesis states that the zeros of Riemann's zeta function all have a real part of one half. Conjectured by Riemann in 1859 and since then has neither been proved nor disproved. This is currently the most outstanding unsolved problem in mathematics.

[11] The Bieberbach conjecture is easily understood by undergraduate students with some exposure to complex analysis because of the elementary nature of its statement. A univalent function f transforms a point in the unit disk into the point represented by the complex number f(z) given by an infinite series $f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \ldots$ where the coefficients a_2, a_3, a_4, ... are fixed complex numbers, which specify f. In 1916 Bieberbach conjectured that no matter which such f we consider $|a_n| \leq n$. Loius de Branges proved this in 1985.

proposed by Polya (1954). Allowing students problem posing opportunities and understanding of problem design helps them to differentiate mathematical problems from non-mathematical problems, good problems from poor, and solvable from non-solvable problems. In addition, independent thinking can be cultivated by offering students the opportunity to explore problem situations without any explicit instruction. This is also supported by Carlson (1999) who found that one of the non-cognitive factors that play a major role in the high-achieving mathematics graduate students' success and further mathematical study is that they "enjoy the challenge of attempting complex mathematical tasks and believe that they possess abilities and strategies that facilitate their problem solving success" (Carlson, 1999, p. 242). Their teachers created a non-intimidating environment where students were encouraged to pose questions until they acquired understanding (Carlson, 1999, p. 244).

As Root-Bernstein (this volume) puts it,

> From the polymathy perspective, giftedness is the ability to combine disparate (or even apparently contradictory) ideas, sets of problems, skills, talents, and knowledge in novel and useful ways. Polymathy is therefore the main source of any individual's creative potential. The question of "who is creative" must then be reexamined in light of what is necessary for creative thinking. In light of this distinction, Santiago Ramon y Cajal, one of the first Nobel laureates in Medicine of Physiology, argued that it is not the precocious or monomaniacal student who is first in his class who we should expect to be creative, but the second tier of students who excel in breadth: "A good deal more worthy of preference by the clear-sighted teacher will be those students who are somewhat headstrong, contemptuous of first place, insensible to the inducements of vanity, and who being endowed with an abundance of restless imagination, spend their energy in the pursuit of literature, art, philosophy, and all the recreations of mind and body.

Teachers are also encouraged to engage in curriculum acceleration and compaction to lead gifted students into advanced concepts quickly to promote independent scholarly activity not simply from the point of depth but also from that of breadth. The longitudinal Study of Mathematically Precocious Youth (SMPY) started by Julian Stanley at the Johns Hopkins University in 1971 generated a vast amount of empirical data gathered over the last 30 years, and has resulted in many findings about the types of curricular and affective interventions that foster the pursuit of advanced coursework in mathematics. More than 250 papers have been produced in its wake, and they provide excellent empirical support for the effectiveness of curriculum acceleration and compaction in mathematics (Benbow, Lubinski, & Sushy, 1996). But the point we are making here is that many mathematical problems can be historically researched by students to find out the analogous problems tacked by individuals in history in order to make a breakthrough on the given problem. Kepler's conjecture is one outstanding example of such a problem and there is no shortage of such problems.

The Uncertainty Principle

Real-world problems are full of uncertainty and ambiguity as indicated in our analysis so far. Creating, as opposed to learning, requires that students be exposed to the uncertainty as well as the difficulty of creating original ideas in mathematics, science, and other disciplines. This ability requires the teacher to provide affective support to students who experience frustration over being unable to solve a difficult problem. Students should periodically be exposed to ideas from the history of mathematics and science that evolved over centuries and took the efforts of generations of mathematicians to finally solve (Einstein & Inheld, 1938; Kuhn, 1962). Cultivating this trait will ultimately serve the mathematically gifted student in the professional realm. Kiesswetter (1992) developed the so-called Hamburg Model in Germany, which is more focused on allowing gifted students to engage in problem-posing activities, followed by time for exploring viable and non-viable strategies to solve the posed problems. This approach captures an essence of the nature of professional mathematics, where the most difficult task is often to correctly formulate the problem (theorem). Conversely, some extant models within the United States, such as those used in the Center for Talented Youth (CTY), tend to focus on accelerating the learning of concepts and processes from the regular curriculum, thus preparing students for advanced coursework within mathematics (Barnett & Corazza, 1993). Having presented five principles that can maximize the interdisciplinary creativity we present examples from classroom studies[12] at the

[12] The classroom studies reported here were conducted by Sriraman with secondary students in the 1999–2002 time period. The study with pre-service teachers was conducted by Sriraman in the 2002–2005 time period.

secondary school and university level that illustrate these principles at work in a practical settings.

Can Paradigm Shifts Be Didactically Engineered?

The Value of Unsolved Classical Problems

The use of unsolved classical problems both conveys a sense of perspective on the origins of the problems and presents a natural opportunity to grapple with the question of allowable tools to tackle a given problem. Let us explain. Most students with even just a basic background in middle school mathematics would view most historical problems through the lens and the "resources" of the twenty-first century. One resource that many twenty-first century students rely on are calculators with powerful graphing and computing capabilities including computer algebra systems, not to mention freely available computing software and web-based applets. The didactic goal was to make students realize the limited nature of modern computing tools! Elementary number theoretic concepts such as prime numbers and tests for divisibility are introduced in most middle school curricula. However, in many parts of the world, particularly in the United States, at the high school level, the curriculum offers students very little opportunity to tackle number theoretic problems (Sriraman, 2003b). A famous example of one such unsolved problem would be Goldbach's conjecture. Goldbach (1742) casually wrote in a letter to Euler that numbers greater than 2 could be expressed as the sum of three primes, which Euler restated as follows: all even integers ≥ 4 could be expressed as the sum of two primes. Other good candidates are Catalan's conjecture (which was only resolved two years ago!). Catalan's conjecture asks whether 8 and 9 are the only consecutive powers? More generally it asks for all integer solutions to $x^a - y^b = 1$ in integers. Another beautiful problem is the Four Squares problem (resolved by Lagrange a long time ago), which states that every natural number n can be expressed as the sum of four squares. In other words $x^2 + y^2 + z^2 + w^2 = n$ is solvable for all integers n.

One problem used by Sriraman (2003b) with 13- to 15-year-old students is called the 5-tuple diophantine problem purportedly posed by Diophantus himself. In general a diophantine n-tuple is defined as a set of n positive integers such that the product of any two is one less than a perfect square. For instance (1,3,8) is a diophantine 3-tuple since the product of any integers is always one less than a perfect square. Over the course of the school year ninth-grade students tried to answer if a diophantine 5-tuple exists? This problem was tackled in parallel with the regular curriculum. The Lakatosian (1976) methodology of conjecture–proof–refutation was emphasized as students tackled this problem. The limitations of modern computing tools became obvious very quickly as we were working over the field of integers. The unimagined difficulties that arose from generating the general forms of the 4-tuples and the "futile" exhausting search for a 5-tuple led students to question the ontological status of the set of integers and more importantly their properties. It should be noted that this was a natural function of realizing the fallibility of all their known methodological tools, which were to a large extent that of trial and error before an insight occurred. Since the students did not have the mathematical sophistication to create number theoretic tools, a natural consequence was to question the question itself. In other words, it led to questions such as (1) why do positive integers have this strange property? (2) Does it matter if we are unable to find a 5-tuple? (3) Does a 5-tuple exist even if we are not able to find it? (4) Is this search real? The four aforementioned questions are philosophical in nature and pose the challenge of examining the ontological status of mathematics. In other words, Is mathematics real or imagined?

Mathematics and Literature

One could easily bias the students' natural line of philosophical questioning by pointing to the numerous connections and applications of mathematics to the arts and sciences. A better didactic approach is of course to let students make up their own minds. The question then was to find a didactic tool that would allow the teacher to scaffold this process. Mathematics fiction serves as the ideal tool to both awaken the imagination and explore more deeply philosophical questions.

Flatland (Abbot, 1984 reprint of 1884 edition) a book of mathematical fiction is the unusual marriage of literature and mathematics. Sriraman (2003a) used this book as a didactic tool to provide the ideal scaffolding for 13- to 15-year-old students to critically examine societal norms and biases in addition to exposing students to some very advanced mathematical ideas such as dimension. It also created the perfect setting to expose students to non-Euclidean geometries such as the Minkowskian space–time geometry and Fractal geometry. One purpose of introducing non-Euclidean geometries was to expose students to relatively modern ideas that were instrumental in the subsequent foundational problems that plagued mathematics in the early part of the twentieth century. The hope was that this exposure to non-intuitive mathematics would lead students to form a basis for an ontological and epistemological standpoint. A follow-up teaching experiment (Sriraman, 2004b) consisted of reading the first five chapters of *Flatterland*, a sequel to *Flatland*, in which Stewart (2001) brilliantly makes numerous non-Euclidean geometries accessible to the lay person in addition to introducing ideas such as encryption on the Internet, the taxi-cab metric, and fractal geometry. These readings further exposed students to the non-intuitive aspect of mathematics. For instance Stewart (2001) describes the non-intuitive possibility of fitting a cube of side length 1.06 into a unit cube, which some students did not accept as being possible in spite of the sound mathematical argument in the book:

> Mathematics is imagined by the human mind because you cannot physically create mathematics. There are many things that can be proven mathematically such as the larger cube in a smaller one but you cannot physically push a larger cube in a smaller one – Student comment

The origins of this mathematical problem of fitting a larger cube into a smaller one was a wager made and won by Prince Rupert in the late seventeenth century (Jerrard & Wetzel, 2004) about this bizarre possibility. The non-intuitive aspect of mathematics brought alive by *Flatterland* led students in Sriraman's studies to take an ontological position on the nature of mathematics. Eventually students also asked the epistemological question "How does one know truth in mathematics?" In other words students were questioning the nature of the truth of the non-intuitive possibilities of mathematics. The specific ontological question was whether the protagonist of *Flatterland* was discovering geometries that were present a priori or were the different geometries a figment of imagination made real via the use of a virtual reality device in the book (Sriraman, 2004b). The specific epistemological question was whether one could believe in the truth of these new geometries? For some students that were interested in science fiction, these questions also coincided with questions about the nature of reality and truth raised by the film *The Matrix* in which reality as is was quite different from reality experienced through a virtual interface (Sriraman, 2004b, 2004c). In other words, students were not only posing the analogous question for mathematics but were also willing to consider this question independent of the context of a specific intellectual discipline.

Conveying the Applied Nature of the Field of Mathematics

Another important aspect of this discussion is the question of the balance between pure and applied mathematics. The literature suggests that the nature of mathematics relevant for today's world has also changed. In spite of the rich and antiquated roots of mathematics, Steen (2001) suggested that mathematician's today should acknowledge the contributions of researchers in external disciplines like biology, physics, finance, information sciences, economics, education, medicine, and so on who successfully use mathematics to create models with far reaching and profound applications in today's world. These interdisciplinary and emergent applications have resulted in the field of mathematics thriving at the dawn of the twenty-first century. This wide range of application can also be seen in the TV series "Numb3rs" where a university professor in mathematics helps his brother who is an FBI agent solving crimes.

However, problem solving as it is implemented in the classroom does not contain this interdisciplinary approach and modeling of what is happening in the real world. Secondary mathematics is usually the gateway for an exposure to both breadth and depth of mathematical topics. However, most traditional mathematics curricula are still anchored in the traditional treatment of mathematics, as opposed to an interdisciplinary and modeling-based approach of mathematics used in the real world. The traditional treatment of mathematics has little or no emphasis on modeling-based activities, which require team work and communication. Additionally, the traditional mathematics has historically kept gifted girls from pursuing 4 years of high school mathematics. This deficit is difficult to remediate at the undergraduate level and results in the effect of low numbers of students capable of graduate level work in interdisciplinary fields such as mathematical biology and bio-informatics (see Steen, 2005). To remediate this deficit is also in line with UNESCO's Salamanca Statement that declares that "schools should accommodate all children regardless of their physical, intellectual, social, emotional, linguistic or other condition. This should include disabled and gifted children, street and working children, children from remote or nomadic populations, children from linguistic, ethnic or cultural minorities and children from other disadvantaged or marginalized areas or groups" (UNESCO, 1994, p. 6):

Any educator with a sense of history should foresee the snowball effect or the cycle of blaming inadequate preparation to high school onto middle school onto the very elementary grades, which suggests we work bottom up. That is, we should initiate and study the modeling of complex systems that occur in real-life situations from the very early grades. Lesh, Hamilton, & Kaput, (2007) reported that in projects such as Purdue University's Gender Equity in Engineering Project, when students' abilities and achievements were assessed using tasks that were designed to be simulations of real-life problem solving situations, the understandings and abilities that emerged as being critical for success included many that are not emphasized in traditional textbooks or tests. Thus, the importance of a broader range of deeper understandings and abilities and a broader range of students naturally emerged as having extraordinary potential. This finding coheres with the recommendations outlined by Renzulli and Reis (this volume). Surprisingly enough, these students also came from populations, specifically female and minority, that are highly underrepresented in fields that emphasize mathematics, science, and technology, and they were underrepresented because their abilities had been previously unrecognized (Lesh & Sriraman, 2005a, 2005b; Sriraman, 2005a, 2005b).

Thus it may be more fruitful to engage students in model-eliciting activities, which expose them to complex real-life systems, as opposed to contrived problem solving. The mathematical conceptual systems arising from such investigations have great potential for being pursued by mathematically gifted students purely in terms of their implications and because they create axiomatic structures through which theorems can be discovered that are analogous to what a pure mathematician does.

The Use of Paradoxes in Teacher Training

Given the lack of teacher training in interdisciplinarity, a vital component to promulgate in interdisciplinary approach in the curriculum is to initiate interdisciplinary thinking in the training of prospective teachers. In order to investigate whether prospective elementary school mathematics teachers display some of the thinking traits of polymaths, the first author conducted a 3-year study with approximately (Sriraman, 2009a, 2009b)120 prospective elementary school mathematics teachers enrolled in a year long mathematics content sequence required of elementary teachers for elementary certification. Students were presented with several set theoretic paradoxes in a linguistic form. For instance one of the tasks was

> The town barber shaves all those males, and only those males, who do not shave themselves. Assuming the barber is a male who shaves, who shaves the barber?
>
> First explain in your own words what this question is asking you? When you construct your response to the question, you must justify using clear language why you think your answer is valid?
>
> If you are unable to answer the question who shaves the barber?. . .you must again justify using clear language why you think the question cannot be answered.

This task is the well-known linguistic version of Russell's paradox, appropriately called the Barber Paradox. Students were given about 10 days to construct a written response to this task. The purpose of this task was to investigate whether students with no prior exposure to the paradox would be able to decipher the contradictions in the linguistic version of Russell's paradox, and whether they would be able to then construct their own set theoretic version of the paradox. All the students were also asked to complete the following affective tasks parallel to the mathematical task:

> Write one paragraph (200–300 words) about your impressions of a given question after you have read it, if possible while tackling it, and after you've finished it. In particular record things such as:
>
> a. The immediate feeling/mood about the question. (confidence, in confidence, ambivalence, happiness, tenseness etc.)
> b. After you've finished the question record the feeling/mood about the question (if you are confident about

your solution; why you are confident? Are you satisfied/unsatisfied? Are you elated/not elated? Are you frustrated? If so why?
c. Did you refer to the book, notes? Did you spend a lot of time thinking about what you were doing? Or was the solution procedural (and you simply went through the motions so to speak)
d. Was the question difficult, if so why? If not, why not?
e. Did you experience any sense of beauty in the question and/or your solution?

In this study besides the written responses of the students to the aforementioned tasks, the first author interviewed 20 students from the 120 students. These students were purposefully selected on the basis of whether they were able to unravel the paradox and formulate its set theoretic equivalent and those that were unsuccessful in their attempts to do so. During the interview clarifications were sought on the written solution and their affective responses. Students were allowed to speak at length on the nature of the problem and their struggles with it. The written artifacts (student solution and affective responses) and interview data were analyzed using a phenomenological-hermeneutic approach (Merleau-Ponty, 1962; Romme & Escher, 1993) with the purpose of re-creating the voice of the students. In addition, the constant comparative method from grounded theory was applied for the purposes of triangulating the categories which emerged from the phenomenological approach (e.g., Annells, 2006). The qualitative analysis of student solutions, affective responses, and interview transcripts indicated that nearly that 40% of the students displayed *polymathic* traits when engaged with the paradox. In particular students reported (1) frequent shifts in perspective (2) thinking with analogies, and (3) tendency toward nepistemology (i.e., questioning the validity of the question and its place in the domain of mathematics). The pre-service teachers also reported an increased interest in the place of paradoxes in mathematics, which they had believed as an infallible or absolutist science (Sriraman, 2009a, 2009b).

There have been recent attempts to classify works of mathematics fiction suitable for use by K-12 teachers in conjunction with science and humanities teachers to broaden student learning. Padula (2005) argues that although good elementary teachers have historically known the value of mathematical fiction, mainly picture books, through which children could be engaged in mathematical learning, such an approach also has considerable value at the secondary level. Padula (2005) provides a small classification of books appropriate for use at the middle and high school levels, which integrate paradoxes, art, history, literature, and science to "stimulate the interest of reluctant mathematics learners, reinforce the motivation of the student who is already intrigued by mathematics, introduce topics, supply interesting applications, and provide mathematical ideas in a literary and at times, highly visual context" (p. 13).

Concluding Thoughts

The tension between the disciplines that came out of the Renaissance, namely natural philosophy–art–alchemy–theology during the post-Renaissance continues today in the modern day antipathy between the ever increasing subdisciplines within arts, science, mathematics, and philosophy. Many of the thinking processes of polymaths who unified disciplines are commonly invoked by artists, scientists, mathematicians, and philosophers in their craft albeit the end products are invariably different. These disciplines explore our world for new knowledge. Literature is an excellent medium to create frequent shifts in perspective. Paradoxes can be easily investigated by exploring geometry motivated by Art. After all Art suggests new possibilities and pushes the limits of our imagination, whereas science verifies the actual limitations of these possibilities using mathematics. Both are driven by the need to understand reality with philosophy (and theology) often serving as the underlying framework linking the three. Models and Theory building lie at the intersection of art–science–mathematics. The history of model building in science conveys epistemological awareness of domain limitations. Arts imagine possibilities, science attempts to generate models to test possibilities, and mathematics serves as the tool. The implications for education today are to move away from the post-Renaissance snobbery rampant within individual disciplines at the school and university levels. By building bridges today between disciplines, the greatest benefactors are today's gifted children, the potential innovators of tomorrow.

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