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Published in:
Proceedings of the 47th SEFI Annual Conference 2019

Publication date:
2019

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Abou-Hayt, I., Dahl, B., & Rump, C. Ø. (2019). Teaching the limits of functions using The Theory of Didactical Situations and Problem-Based Learning. In B. V. Nagy, M. Murphy, H-M. Järvinen, & A. Kálmán (Eds.), *Proceedings of the 47th SEFI Annual Conference 2019* (pp. 58-69). SEFI: European Association for Engineering Education. https://www.sefi.be/wp-content/uploads/2019/10/SEFI2019_Proceedings.pdf

Teaching the limits of functions

using the theory of didactical situations and problem-based learning

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Conference Key Areas: Fundaments of Engineering Education: Mathematics and Physics, New Notions of Interdisciplinarity in Engineering Education

Keywords: Problem Based Learning, Theory of Didactical Situations, engineering mathematics, limits of functions

ABSTRACT

The concept of limit plays a central role in the foundation of modern mathematical analysis. However, the concept itself plays a minor role in both upper secondary and undergraduate engineering education, leaving the students with many misconceptions about the concept, resulting in poor performance in calculus and calculus-based engineering courses. Most emphasis in teaching has been on how to calculate the limit instead of on understanding its definition. In this paper, we will use the frameworks of Brousseau's theory of didactic situations (TDS) and Problem-Based Learning (PBL) to suggest a method to teach engineering students the concept of limit and explain its formal definition. The purpose is to enable the students to generate a precise definition of limit of a function that captures the intended meaning of the conventional ε-δ definition. Moreover, we will argue that TDS bears many similarities with PBL, as both frameworks require that the students act and engage in non-routine and realistic problems.

1 INTRODUCTION

Although mathematicians have long accepted the concept of limit as the foundation of modern calculus, the concept of limit itself has been marginalized in upper secondary schools and undergraduate engineering programs. Engineering students' understandings

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of calculus will greatly influence their ability to study more advanced analysis courses and engineering courses, such as dynamics, since these courses all require calculus as a prerequisite. One obstacle that can contribute to the difficulties of teaching and learning limits is the symbolic representation of the limit itself, $\lim_{x \to x_0} f(x)$. It can give rise to apparently contradictory processes such as: One that potentially never ends, and another of "getting close to". Nevertheless, as we will try to show in this paper, teaching the concept of limit successfully may not be an unattainable task if we use proper strategies and tools.

The focus of this paper will be on how the PBL and TDS frameworks helped us in structuring our approach of teaching the concept of limit at both upper secondary and college level. Moreover, we demonstrate how PBL, *in a mathematics teaching context*, is *compatible* with TDS.

2 WHY LIMITS ARE IMPORTANT?

Teaching and learning the concept of limit has long been a very important subject to mathematics educators. In fact, the concept itself has a long and interesting history [1]. Many mathematical and engineering concepts depend upon the concept of limit and without a proper definition of it, mathematical analysis as we know it today would simply not exist, since basic notions in mathematics and engineering are limits in some sense, e.g.,

- Instantaneous velocity and acceleration are the limits of average velocities and average accelerations, respectively [2].
- The area of a circle is the limit of areas of inscribed polygon as the number of sides increases infinitely.
- The slope of a tangent line to a curve is the limit of the slope of secant lines.

3 ON PROBLEM-BASED LEARNING (PBL)

In PBL, problems drive the learning. A teaching session begins with a problem to be solved, in such a way that students need to gain new knowledge before they can solve the problem. In contrast to a traditional teacher-centered pedagogy, PBL is a learner-centered educational method based on realistic problems encountered in the real world. These problems act as a stimulus for learning, integrating and organizing learned information in ways that will ensure its application to new, future problems [3]. Thus, PBL is not merely preparing problems for the students to solve in the class, but also about *creating opportunities* for the students to *construct* knowledge through effective interactions and collaborative inquiry. In PBL, an important task of the instructor is to initiate class discussions to enhance the students' reasoning skills and encourage them to apply their previous experiences to a novel case, thus enabling them to identify areas of gaps in their knowledge and prepare them to new knowledge acquisition. Through PBL, students are gradually given more and more responsibility for their own learning and become increasingly independent of the teacher in their understanding. The methodology of PBL will be illustrated when we design teaching situations that gradually guide the students to the formal definition of limits.

4 ON THE THEORY OF DIDACTIC SITUATIONS (TDS)

TDS is based on the idea that students construct *new* knowledge when they solve *non-routine* problems while adapting to what is called a didactical milieu [4]. Non-routine problems typically do not have an immediately apparent strategy for solving them. In TDS, the teacher's aim is to *engage* the students by *designing* didactical situations in such a way that the targeted mathematical knowledge would be the best means available for understanding the rules of the game and elaborating the winning strategy [5]. The withdrawal of the teacher and the subsequent transfer of the responsibility of the learning situation to the students is the essence of Brousseau's notion of *devolution*, where the students become the "owners" of a given problem, and thus enter the *adidactic* level, to produce the knowledge needed to solve it. [6] mentions four phases of didactic situations: *Action*, *formulation*, *validation* and *institutionalization*. These phases are exemplified below when we create didactical situations that eventually lead the students to capture the idea of limit.

5 RESARCH QUESTIONS

The main research questions of this article are

- How can we *design* didactical situations that lead to the rigorous ε-δ definition of limits in an introductory calculus course?
- Can the PBL and TDS frameworks be applied to teaching abstract notions in engineering mathematics, such as limits?

We will try to show that even seemingly theoretical notions in engineering mathematics are amenable to the PBL and TDS frameworks. The raison d'etre of this paper came from two similar teaching situations that the first author taught to engineering students at a higher education level in 2018. These students had no experience with any mathematically rigorous processes using the definition or proofs related to limits. The didactical situations described below require that the students participate in well-designed activities that use real-life problems, which presumably would guide the students to the correct conception of limits.

6 A PROBLEM-BASED APPROACH

6.1 Sources of Difficulties in the Teaching of the Concept of Limit

The differences between everyday language and the language of mathematics may contribute to the students' misconceptions, and hence also bring learning obstacles. For example, one may say that "my limit of running continuously is four kilometers". This everyday understanding of limit may suggest that a limit is some value one cannot exceed. The difficulties the students may encounter in understanding the concept of limit are discussed in [7], where three forms of obstacles to students' understanding of limits are mentioned:

- Epistemological obstacles related to the historical development and formalization of the limit concept.
- Cognitive obstacles related to the abstraction process involved in the formalization of the concept of limit.
- Didactical obstacles related to the ways the concept of limit is presented to students.

One consequence of these obstacles is that a formal definition of limits is not included in the Mathematics A curriculum in Denmark (the highest level possible), apparently due to its conceptual difficulty. Thus, upper secondary mathematics textbooks, such as the one by [8], give the following *informal* definition of limit:

If the values of the function $f(x)$ approaches the value L as x approaches x_0, we say that f has the limit L as x approaches x_0 and we write

$$\operatorname*{Lim}_{x \to x_0} f(x) = L$$

The real motive behind introducing the limits of functions in upper secondary school mathematics is its use in defining the derivative of a function at a point:

The derivative of a function f at a point x_0, denoted $f'(x_0)$, is given by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

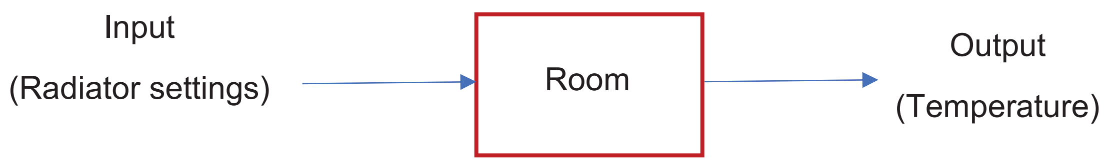
provided this limit exists.

The definition of the derivative is a so-called indeterminate form of type $\left[\frac{0}{0}\right]$ [9]. These forms can usually be evaluated by cancelling common factors, which is the usual method used in upper secondary school mathematics. Thus, it seems that the limit concept is reduced to an *algebra* of limits, suppressing the *topology* of limits, which is crucial in the formal definition: This didactical obstacle may lead to the misconception that the algebra of limits and topology of limits may be completely disconnected. The informal definition of limits therefore has its shortcomings. First, the definition does not precisely convey the mathematical meaning of the concept of limit. Second, the expression "approaches to" may result in the confusion whether limits are dynamic processes, where motion is involved, or static objects.

6.2 Teaching Situations Leading to the Concept of Limit

In response to the above-mentioned difficulties, we will show how we tackled teaching the concept of limit, using a terminology that is close to the one used in the formal definition, without sacrificing the topological aspect in the definition. Moreover, the concepts we use should be familiar to the students from their previous experiences. Specifically, we address the question: *Given a process or system, how can we control the error tolerance in the input, given that the output (or product) should have a given error tolerance*? So, in introducing the topic "*Introduction to Limits of Functions*" to the students, we started the lesson by giving the students five tasks. These tasks represent several teaching situations that may be needed to reach the institutionalized knowledge of limits of functions, i.e. the tasks can be regarded as a *gradual transition* from the students' personal knowledge to *institutionalized* knowledge.

Task 1: *Discussion*. How do you control the temperature of this classroom? Usually, we require that the room temperature to be the *ideal* $20^{\circ}\mathrm{C}$, but can we be sure that it is *precisely* $20^{\circ}\mathrm{C}$? If a temperature of exactly $20^{\circ}\mathrm{C}$ is practically unattainable, how can we keep the temperature of the room *close* to it? The discussion is open for all students. Many students gave the answer "We have to *continuously adjust* the settings of the radiator to guarantee that the temperature is always *near* $20^{\circ}\mathrm{C}$". Other students argued that "opening and closing the windows and the door also affect the temperature". All agreed that the temperature in the classroom is *dependent* on many factors. To make things simple, we *intervened* in the discussion and drew the following figure on the white board and asked the students to elaborate on it:



The purpose of this task is to guide students to reach the (simple) conclusion: To *control* the room temperature, one should *adjust* the settings of the radiator. Using TDS terminology, this task corresponds to the *formulation phase*, where the *milieu* is an open discussion. The students here construct *personal knowledge* about radiators and heat while *interacting* with the problem of maintaining a constant room temperature. Using the figure, the students' personal knowledge is being *validated* and becomes more formalized. Besides, this task encourages students to use relevant *experience-based knowledge* in order to arrive at a plausible conclusion, to use PBL terminology [10].

Task 2: The area of a circular plate is given by $A = \frac{\pi d^2}{4}$, where d is its diameter. A machinist is required to manufacture a circular metallic plate to be used in radio-controlled wall clocks. The area of the circular plate should be $169\pi\ \mathrm{cm}^2$. But since nothing is perfect, the machinist would be satisfied with an area *machined* within an error tolerance.

a) Within an error tolerance of $\pm 1\ \mathrm{cm}^2$ for the area, how *close* to $26\ \mathrm{cm}$ must the machinist control the diameter of the plate to achieve this?

b) Given a positive number ε. Within an error tolerance of $\pm\varepsilon\ \mathrm{cm}^2$ for the area, find a formula for the resulting tolerance δ of the diameter of the plate.

An excerpt of a student solution of Task 2 is shown in Fig. 1. The aim of Task 2 is twofold:

- To support the students' development of personal knowledge regarding the concepts of closeness and distance, which culminate in the result $|d - 26| < 0.024$ (Fig. 1).
- To help the students acquire new knowledge about tolerances, namely the fact that δ depends on ε.

To use TDS terminology, the teacher hands over the milieu to the students by presenting the problem and explaining the rules for solving it in such a way that the students can engage in the intended activities [4]. This corresponds to the *devolution* phase in TDS. This is also a PBL situation where teaching should offer the students the opportunity to engage in activities like those of a researcher. "PBL assumes that students learn best when applying theory and research-based knowledge in their work with an authentic problem" [3].

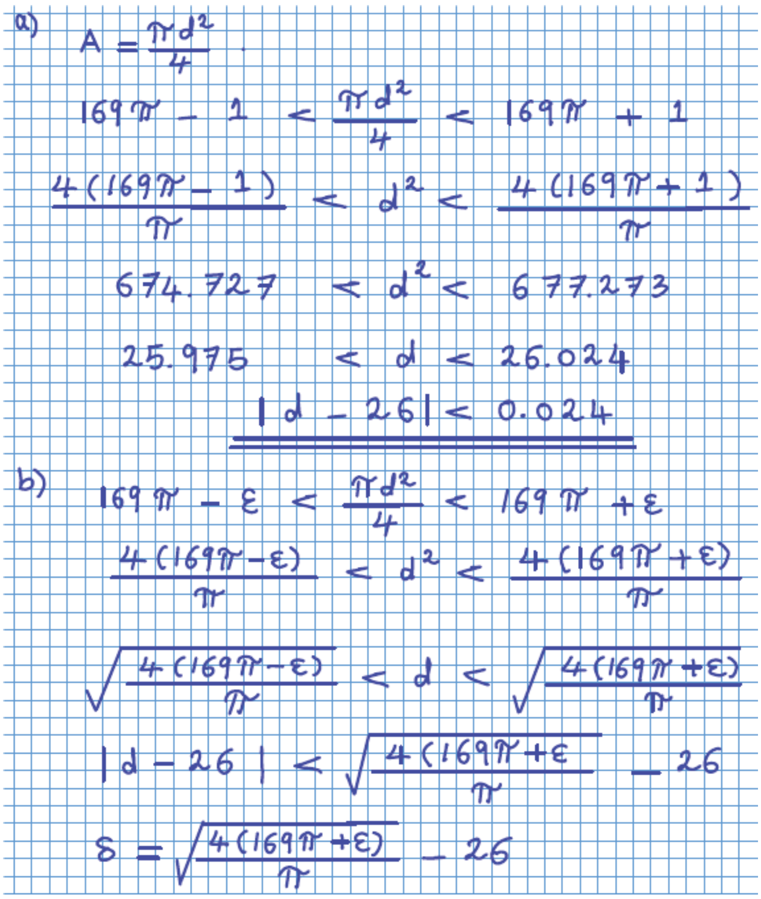


Fig. 1. An example of a student solution to Task 2.

Engaging in the task, the students employ their previously developed experience with inequalities and absolute values in order to solve the problem. In TDS, this corresponds to the *action* phase, where the situation is *adidactical*.

Task 3: This is a *didactical situation* where we explicitly interact with the students, in order to improve their understanding of error tolerances and provide them with some background for the independent acquisition of knowledge about dependent and independent variables. The function of the machine is to take metal sheets as *input* to produce circular plates as the final products, i.e., the *output*.

Input (metal sheets) → Machine → Output or Product (Circular plates)

The machinist must adjust the machine settings to satisfy the specifications of the products. The question now is: What error tolerance for the diameter *d* should be used so that the product (circular plates) requirements are met? Mathematically, Given $\varepsilon > 0$. Find δ such that if $|A - 169\pi| < \varepsilon$ then $|d - 26| < \delta$, where *A* is the area of the circular plate.

Task 4: This task is a partial generalization of the third one. This task is really a “didactical game” consisting of a challenge and a response. The “machine” now is a function f that transforms a number x (input) to another number $f(x)$ (output). Like the machinist’s work, we want the output $f(x)$ to be equal to a number L. In practice, we may be satisfied with an output $f(x)$ *somewhere* between $L - \varepsilon$ and $L + \varepsilon$, where ε is the error tolerance of $f(x)$. The question now is how accurate our control setting for x (the input) must be to guarantee this degree of accuracy in the function value $f(x)$. This error tolerance for x is usually denoted by δ. The function given to the students is $f(x) = 5x - 3$, together with the two numbers $L = 2$ and $x_0 = 1$. The challenge is to make $|f(x) - L|$ less than a given number $\varepsilon > 0$ by finding a number $\delta > 0$ such that $|x - x_0| < \delta$. The number ε itself is given in the following table:

Table 1. The challenge and the response

The challenge, ε	The response, δ
$\frac{1}{10}$	
$\frac{1}{100}$	
$\frac{1}{1000}$	
$\frac{1}{10000}$	

In the language of TDS, this task is the starting point directing the student's acquisition of the institutional knowledge of limits. Within the framework of PBL, it helps students acquire the skills required to tackle new problems involving limits. The students were required to work in groups of two to find the "response" δ of the challenge ε, by completing the table. A two-student group consisted of a *skeptic* and a *scholar*: The skeptic presented ε-challenges to show that there is room for doubt. The scholar should answer every challenge with a δ-interval around x_0 that keeps the function values within ε of L. The *culmination* of this task consisted of giving the students a new challenge: Find a formula for δ in terms of ε.

The series of the tasks mentioned above constitutes a TDS teaching process to arrive at the sought definition of a limit. This process also conforms to the essence of a PBL framework [11]. The *institutionalization* of all these tasks, where the students' personal knowledge finally reaches the state of institutional knowledge, is attained by confronting the students with the formal, rigorous definition of a limit of functions, as given in most engineering mathematics books, e.g. [9]:

We say that $f(x)$ approaches the **limit** L as x **approaches** x_0 if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$, such that for all x, if $|x - x_0| < \delta$ then $|f(x) - L| < \varepsilon$. And we write

$$\lim_{x \to x_0} f(x) = L$$

As a part of the *institutional* knowledge, we mentioned two remarks to this definition to the students:

- The definition does not ask for a "best" positive δ, just one that will work.
- Note that there is no need to evaluate $f(x_0)$. In fact, $f(x_0)$ may or may not equal L or may not exist at all! The limit L of the function $f(x)$ as $x \to x_0$ depends only on nearby values!

Task 5: This final task consists of some exercises, the purpose of which is to test if the students grasp the concept of limit: Use the formal definition of limit to prove the indicated limits. Due to page limits, we only discuss one of these:

1) Use a CAS tool to plot the graph of the function $f(x) = \frac{x^2-9}{x-3}$. Show, graphically, that $\lim_{x \to 3} \frac{x^2-9}{x-3} = 6$.

This task is a *validation* situation, i.e. students convey their ideas and the teacher plays a role of bridging their knowledge to achieve the intended knowledge [4]. Regarding this exercise, the students used *GeoGebra* and *Maple*. Both these CAS tools produced *wrong* plots of the function. The students, who are used to use CAS to solve mathematical problems, including trivial operations on numbers, were surprised that the CAS tools failed to draw the right graph. They were not aware of the limitations of these CAS tools. Here is the misleading graph which all students got (Fig. 2, left):

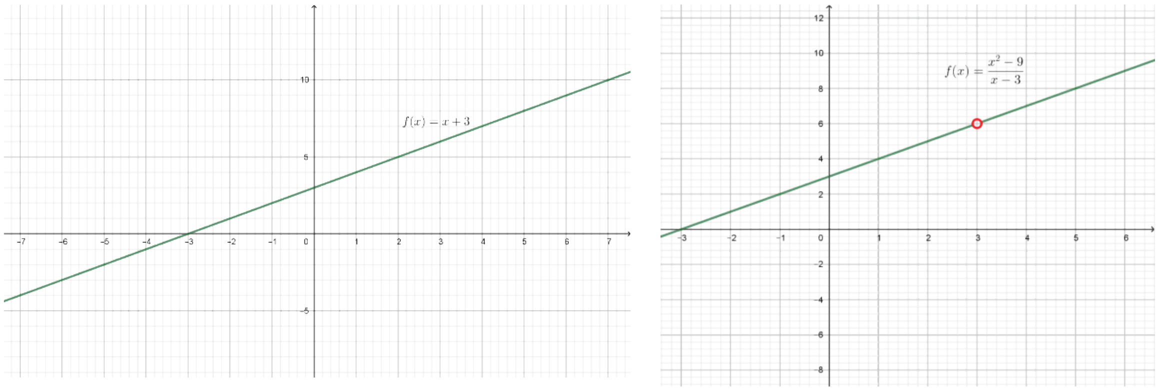


Fig. 2. Left: The "wrong" graph of the function $f(x) = \frac{x^2-9}{x-3}$. Right: The right graph.

The catch is that CAS tools automatically try to reduce an expression without showing the condition under which the reduction is valid. In our exercise, the function $f(x) = \frac{x^2-9}{x-3}$ is reduced to $f(x) = x + 3$, without further notice: The result is a straight line, where *x* can be any number! It is too easy to declare that one should have a critical attitude regarding the outputs of CAS tools, as this requires deeper insight and knowledge in the internal workings of these tools, something most students do not possess. The impact of CAS tools on mathematics teaching and learning is still subject to intensive research [12] and it is beyond the scope of this paper to account for the possible contribution of CAS tools in improving the students understanding of mathematical topics. It is crucial to present the true graph of the function $f(x) = \frac{x^2-9}{x-3}$ to the students (Fig. 2 right) and elaborate on the analogy with the previous tasks: A straight line with a "hole" at $x = 3$ means of course that $x \neq 3$. However, this does not prevent us from investigating the values of the function for values of x that are close to 3, like what we did in the previous tasks:

- In Task 1, it was beyond our reach to require a room temperature of *exactly* $20^{\circ}\mathrm{C}$, but we can get closer and closer to it.
- In Tasks 2 and 3, it was impossible to produce circular plates having a diameter of *exactly* $26\ \mathrm{cm}$ but we can get closer and closer to that.

Similarly, in the exercise in Task 5, we cannot give x the value 3, and hence the function cannot have the value 6. However, as the graph shows, the function *can* get closer and closer to 6 whenever x is sufficiently close to 3.

7 CONCLUDING REMARKS AND DISCUSSION

According to the PBL framework, "the problem is the starting point directing the student's learning process. A problem can be both theoretical and practical. It must also be authentic and scientifically based" [3]. The main requirement of the PBL framework is that the students seek new knowledge, through realistic problems. Thus, both PBL and TDS share the idea that a teacher provides students with the initial problem, so that the students act and formulate concepts related to the problem-solving activity.

At Aalborg University, both students and researchers are supposed to engage in problem-based, project-oriented approach in their academic work [13]. This paper itself can be regarded as a problem-based approach to applying TDS in introducing the theory of limits to engineering and upper secondary school students. In our own classes, many students were in fact able to prove that a given number is the limit of a given elementary function, using the formal definition. In fact, the problem that some students encounter was not in applying the definition, but rather in the algebra of inequalities involving absolute values. The ultimate purpose of the tasks mentioned is to make students capture the similarities between the following situations:

- We cannot guarantee a room temperature of exactly $20^{\circ}\mathrm{C}$, but we can get close to it.
- We cannot produce circular plates having an area of *exactly* $169\pi\ \mathrm{cm}^2$, but we can make their areas closer and closer to that.
- We cannot divide by zero, but it is possible to investigate the properties of a rational function[2] for values close to the zeros of its denominator.

We therefore do not believe that the ε-δ definition of limits is too advanced for the mathematics curriculum at the upper secondary school and undergraduate engineering programs. Since, by using carefully designed teaching situations and pedagogical approaches, it can be possible to equip the students with a proper understanding of the concept of limit, and we hypothesize that it will pay off in other mathematics and engineering science courses the students may encounter in their study.

8 FUTURE RESEARCH PERSPECTIVES

The methodology of this concept paper has been used in an introductory calculus course for engineering students at Aalborg University in Copenhagen, Denmark. However, no pre-tests or post-tests were conducted in the course. The first author has only tested the students understanding of the concept of limit through Task 5. The informal assessment of the course seemed to be promising. However, more research in teaching the concept of limits at upper secondary schools is still needed to get a nuanced understanding of the students' conceptions and misconceptions of the idea of limit and what it might mean to come to understand the limit concept. Therefore, in a future offering of the course, the first author plans to design empirical tests that would reveal the impact the TDS and PBL approaches might have on the students understanding of limits. This would be an interesting subject of a new research paper.

[2] Such as the function $f(x) = \frac{x^2-9}{x-3}$.

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