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University students' retention of derivative concepts 14 months after the course: influence of 'met-befores' and 'met-afters'

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This article reports the concluding part of a larger study on retention of key procedural and conceptual concepts in differential and integral calculus among Croatian and Danish university students in non-mathematics study programmes. The first parts of the study examined the retention of the students' knowledge through a questionnaire testing core calculus concepts in derivative and integration given two and six months after the students had passed an exam testing those concepts. In the present article we continue to explore the retention of core concepts in derivative through a mixed method approach examining the knowledge of 10 second-year non-mathematics students 14 months after they took the course. The result showed that there were several negative met-befores and met-afters affecting the students' retention.

Keywords: calculus; derivative; retention; memory; met-befores; met-afters; conceptual; procedural

1. Introduction

Today industries and businesses worldwide make demands for a mathematically capable and trained work force, emphasizing mathematics as a powerful tool to understand, investigate, and make predictions in solving a wide range of problems [1]. There are numerous applications of mathematics in many scientific fields such as physics, chemistry, and engineering, especially the applications of calculus [2]. On the other hand, it has also been shown that university students in mathematics study programmes develop different mathematics, including calculus, concepts than other science students taking mathematics classes; henceforth called non-mathematics students [3–6]. These studies are the offspring of our research into the calculus knowledge of non-mathematics students. The study presented in this article is part of a larger study that includes students of natural sciences and engineering study programmes at one university in Croatia and one in Denmark. The first parts of the study examined the retention of these non-mathematics students' knowledge through a questionnaire testing core calculus concepts in derivative and integration two months after the students had passed an exam testing those concepts. The results

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were reported in several papers by the authors [7–10] and gave information, perspectives, and guidelines for further study. In the present article we continue to explore the retention of core concepts in calculus through a mixed method approach examining the knowledge of 10 second-year non-mathematics Croatian students 14 months after the course.

2. Theoretical background

In this section we will describe and discuss the theoretical frameworks for our study and analysis, how the present study relates to work previously done (by us and others), and finally our research questions.

2.1. *Procedural and conceptual knowledge*

When discussing the type of mathematical knowledge that a person possesses, one can use various categories. One categorization is into conceptual and procedural knowledge. Conceptual knowledge provides an understanding of the principles and relations between pieces of knowledge in a certain domain and procedural knowledge enables us to solve problems quickly and efficiently [11]. Star [12] argues that the terms conceptual and procedural knowledge confuse knowledge types with knowledge quality and therefore he states that conceptual knowledge could be better understood in terms of "knowledge of concepts and principles" and procedural knowledge as "knowledge of procedures". We use Stars' re-definitions to make it possible also to consider deep well-connected procedural knowledge and superficial and weakly connected conceptual knowledge.

Haapasalo and Kadijevich [13] argued that definitions such as the above of procedural and conceptual knowledge may lead to the conception that procedural knowledge is dynamic and conceptual knowledge static. Their definition instead highlights the dynamic nature of conceptual knowledge as well; i.e. procedural knowledge is a dynamic utilization of algorithms or procedures within a representation form, and conceptual knowledge denotes an ability to browse through a network consisting of concepts, rules, algorithms, procedures, and even solved problems in various representation forms. They emphasize that students' conceptual knowledge cannot be investigated by simply asking students what they know, but students must also be observed to see how they use that knowledge in problem solving. We will also adopt this understanding of conceptual knowledge as well as take the advice of also observing how they use the knowledge in problem solving. These various descriptions of procedural and conceptual knowledge are not mutually exclusive; instead we see them as emphasising various specifics within these types of knowledge. We therefore apply both definitions in order to have a range of tools to assist in analysing the results.

2.2. *Met-before and met-afters*

Professional mathematicians, who design a teaching of derivatives based on the limit concept, see limit as a *met-before*. A met-before is a mental construct that an individual uses at a given time based on prior experiences [14]. Using met-befores can sometimes be an advantage when learning a new mathematical concept and sometimes it can be an obstacle which causes severe difficulties. Hence, met-befores affect the learning of new concepts, but new mathematical concepts may also affect older knowledge. Such mental constructs are called *met-afters* [15]. Met-afters are those experiences met at a later time that affect the retention of old knowledge. Met-afters can also be both positive and negative, and the negative affect of some met-after shows fragility or inconsistency of the earlier learned knowledge.

Negative met-afters and met-befores can be identified either as being part of the procedural or the conceptual knowledge. New knowledge that builds on previous knowledge is much better remembered, but concepts that do not fit into earlier experience are learned temporarily and easily forgotten or not learned at all. According to McGowen and Tall [16], this can be observed when a student for instance is interested in acquiring only procedural knowledge. If there is no conceptual meaning, this kind of knowledge is stored improperly and is very fragile in the long-term memory. This previous knowledge makes it difficult to understand new subject matter, since the student is trying to distinguish among accessible rules and is trying to imbed new knowledge into his fragmented knowledge structure.

The structures met-befores and met-afters and aforementioned types of knowledge build the framework of our study. Together they provide the cornerstone of this article where we instigate retention of students' knowledge. Long-term retention as used in this study is in accordance with Sousa [17], who defines retention as a measure of how well a student remembers the learnt material over time. It can be considered as the extent to which one can successfully access and use the information from long-term memory.

2.3. *Previous studies on retention*

Some studies have examined the retention of school or university knowledge weeks, months, or years after students have passed the required exams. For instance Garner and Garner [18] investigated university students' knowledge seven months after the instructions in applied calculus. They compared two groups of students, one in a reform-oriented course and the other in a traditional course and found out that the students forgot most of the course content, but the performance of two groups was different. Students from the reform-oriented course had better results in the conceptual questions, while students from the traditional course were better at procedural questions. The authors of this article had a similar study comparing two groups of calculus students 2 months after the exam, one group coming from a traditional Croatian university and one other from a Danish university using a mix of traditional and reform-oriented teaching methods. The statistical data analysis showed that Danish students performed significantly better than the Croatian students on the conceptual questions, and vice versa for the procedural ones. Although, it seemed that students' success on the procedural and conceptual questions could be related with the teaching styles of each group [8], this factor could not be pointed out as the only one that explains the difference. Allen et al. [19] analysed students' results in procedural and conceptual knowledge one year after the instructions in two differential equation classes. Comparing the results of these two groups, they discovered that the group exposed to teaching for conceptual understanding achieved better results. Engelbrecht et al. [20] investigated the retention of basic techniques from the first-year calculus course, examining the knowledge of engineering students. Comparing their results from the pre-test with the post-test given two years two years after the instructions, they found a significant decline in their performance. Another reason for being interested in the long term retention of the students' knowledge is that it is not always linked with the actual course grade. In [7] we again compared the calculus teaching at a Croatian and Danish university in terms of the students' retention of key procedural and conceptual concepts of derivative two months after having passed similar Calculus 1 courses, and while being taught a Calculus 2 course. The results showed that for both countries, a large portion was forgotten and the passing grades of the Calculus 1 course did not predict the results in the test two months later. In fact, often students with the lowest passing grades had better results two months later, or there was no difference. In [9], we investigated the retention of key procedural and conceptual concepts in integral calculus among Croatian students in physics, food technology, and civil engineering study programmes. The students were surveyed two and six months after being taught integral calculus. The result showed that the students improved their answers to the conceptual questions while deteriorated their answers to most of the procedural questions. Physics students did comparably best.

Having interest in university non-mathematics students, in this study we investigated students' retained conceptual and procedural knowledge. Our aim was to see how students who use calculus conceptions further in their study programme, cope with questions connected directly with differential calculus.

2.4. *Research questions*

During 2009 we conducted studies at one Croatian university and one Danish university investigating students' procedural and conceptual knowledge in differential and integral calculus. These studies are reported in [7–9], which we also shortly refer to above. These studies were conducted through questionnaires two or six months after the instructions and examination of the course. It included more than 200 students from each university coming from various science study programmes. These studies provided interesting quantitative data but to get deeper into understanding the students' retention and knowledge of these concepts, we initiated the following study.

In this study, we investigated students' retention of conceptual and procedural knowledge of derivative concepts 14 months after the instruction of these concepts. We compared these results with results obtained two months after the examination [7–9]. Additionally, we discuss the mental structures met-befores and met-afters that appeared in the students' responses. Therefore our research focus is placed on the comparison of the students' performance two and 14 months after the course.

3. Methodology

In this section we explain the overall research strategy, how the study was designed, the questions we asked the students as well as some general methodological considerations.

3.1. *Mixed method approach*

We used a mix of research strategies in this study; which is what Robson [21] calls combining strategies. In our study, we combine a quantitative strategy of a questionnaire given in survey form with qualitative interviews. Robson [21] argues that the advantage of using multiple methods is that it provides triangulation, hence help validate the conclusions. Also, while the survey gave key information about the overall condition of the many students' retention, a qualitative interview gives information about how exactly some students' perceive the calculus concepts.

3.2. *Design of the new study*

A year after students were given the first questionnaire, we traced 10 students who were willing to participate in another examination of their knowledge of differentiation and derivatives. The students of this survey were all second-year Croatian students from physics (three students), food technology (one student), and civil engineering (six students) study programmes. Hence, the students are from the same study programmes as in all the previous studies, particularly [9] that also focused on these exact study programmes. Hence, the sampling method was *quota sampling* where we build up the sample with representative of the study programmes that we had particularly studied previously in order to make comparison. Within each category, *convenience sampling* was used since we had to rely on the willingness and availability of the participants.

Since we only have 10 students in this part of the study, we cannot make the same types of generalisations as in the first study which had answers to a questionnaire from over 200 students. Therefore, we replace the notion of generalizability with 'fittingness', "the degree to which the situation matches other situations in which we are interested" [22, p. 207]. And therefore, 'Thick descriptions' are therefore vital for others to be able to judge if the attributes compared are relevant [23, p. 33]. Below we therefore aim at providing very detailed descriptions and analysis.

All participants had used derivatives and differentiation in the second year of their studies in another calculus course and they had applied it through various non-mathematics courses. The civil engineering students had a mathematics course in which they examined various differential equations. The physics students had a mathematics course in which they dealt with functions in several variables, and during the second survey they were taking a course in differential equations. When the questionnaire took place, the food technology student was taking the first year calculus course in integrals and differential equations once again since he failed the exam.

The students in our new study were given a qualitative questionnaire with five questions. Given that our participants belonged to different study programmes, where each study programme has its own calculus course which had some differences but also a lot of alikeness, we designed the questions based on the common part of differential calculus. In this university, the non-mathematics study programmes do not have a joint calculus courses, and the courses are adapted according to the needs of the specific study programme and divided into several one-semester calculus courses. The study programmes investigated in this article have common differential and integral part in one variable calculus. The calculus course consists of lecture lessons and exercise lessons where the teaching approach is teacher-oriented.

In general, teaching strategies can roughly be divided into student-centred and teacher-centred teaching [24]. In the teacher-centred approach, the teacher has direct control over what is taught and how the learners are presented the information they should learn whereas in the student-centred approach, the learner is put at the focus and the teacher has less direct control over how and what students learn. Studies showed that teaching strategies employed in the class can influence the development of one type of knowledge more than another: teacher centred on procedural knowledge and student-oriented on conceptual knowledge (e.g. [18,19]).

Lectures are therefore given in a traditional form to a large group of students, and exercises are based on direct instructions, used in groups of 30 students where a problem solving or performance procedure is shown to the students. Conceptual ideas are taught in the context of procedural methods. The examination style is also the same for the investigated study programmes; i.e. in order to obtain a passing grade, students have to pass both a written and an oral exam. Besides solving tasks, the students' knowledge in formal mathematical theory is also examined.

3.3. *The five questions*

The question *Tangent* was concerned with the geometric interpretation of the derivative of the function at a given point. In *Quotient* our intention was to test the students' knowledge of differentiation of a simple rational function. *Composition* examined how students respond to a function which is composed of several basic functions. *Slope* incorporated several key concepts from differential calculus: the slope of a tangent line as the derivative of the function f at a given point and the process of differentiation. These four questions also appeared in the questionnaire given a year before, so we were able to compare these students' performance in this questionnaire with their results from the last year questionnaire, which gave us an opportunity to examine the retention of the students' procedural and conceptual knowledge.

In addition we inserted the question *Application* which is constructed differently than the other questions since we combined various topics on infinitesimals: continuity, local extrema, and shapes of the function into one question. The students have met questions such as *Application* before. This question focused on the conceptual side instead of the frequently used procedural one. We did not include this question in the first study, since the first questionnaire consisted of multiple choice questions. One of the reasons for this was that the first questionnaire had to be filled out by the students while they attended their class/lecture and to get permission to give the questionnaire here, it should not take too much time to fill out, thus multiple choice questions seemed as the most convenient way. Furthermore, there is no unique answer to this question and this was another reason why we did not include it in the first questionnaire.

We used Star's [12] categorization of procedural and conceptual knowledge and his classification into deep and superficial knowledge in our design and analysis of the questions. If one regards conceptual knowledge as knowledge of concepts and principles, it can be said that *Tangent* and *Slope* investigated superficial conceptual knowledge, while *Application* aimed at deeper conceptual knowledge. Similarly, if procedural knowledge is knowledge of procedures, *Quotient* investigated superficial procedural knowledge, while *Composition* investigated deeper procedural knowledge since several differentiation rules had to be connected.

However, the concepts procedural and conceptual are not absolute [25]. For example, in some cases *Composition* can be considered to be a conceptual question as it connects several differentiation rules. On the other hand, it is possible that some students had experienced tasks like *Slope*; hence their solution could be based only on recalling the method without any conceptual knowledge. In our case, the students were exposed to the chain rule of differentiation frequently, but not to questions like *Slope*. Hence, we detain the aforementioned classification on procedural and conceptual questions. The questions can be seen in Figure 1.

After completing the questionnaire, students participated in a semi-structured interview with the first author. The interview was held a week after the students had answered the questionnaire. During the interview, the students were asked to define the derivative of the function and to describe the applications of derivatives in mathematics and non-mathematics courses. Additionally, students were asked to comment on their solutions in the questionnaire.

This study involves participants wherefore we aimed at conforming to the BERA (British Educational Research Association, [26]) guidelines for ethical research. BERA was chosen since there is not something similar neither in Croatia nor Denmark. Therefore, we made it clear to the students that we were not evaluating their knowledge as examiners. Results and tapes are treated confidentially, and we asked for permission to tape-record before each interview. The head of each study programme gave the approval for conducting the study. Furthermore, the students' names are kept anonymous. The first author of the article did the interviews and she had also taught some of the students from the civil engineering study programme. However, she was not their teacher any longer and was not going to be their teacher again. Hence we judge that this did not affect the validity of the data. In any case, in terms of the problem of reactivity, the issue is not if the researcher affected the behaviour, but if she had affected it significantly in a way that was relevant to the claims made [27].

1. (Tangent) What is the geometric interpretation of the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ at the point x_0?
2. (Quotient) Find y', if given $y = \dfrac{x^2+2}{x^2}$.
3. (Composite) If given $y = \sin^2 6x$, find y'.
4. (Slope) What is the slope of the tangent line to the graph of the function $f(x) = (3x)^2$ at $x = 1$?
5. (Application) Sketch the graph of the function f which satisfies following conditions:
    a. f is discontinuous at $x = 0$, and $f(0) = 1$
    b. $f''(x) < 0$ for $x \in \langle -\infty, 0\rangle$, and $f''(x) > 0$ for $x \in \langle 0, +\infty\rangle$
    c. $f'(-1) = 0$ and $f'(x) \neq 0$ for $x \neq -1$

Figure 1. Questions from the questionnaire.

4. Analysis of data

In this section we describe how the students answered the questions, compare their results with the first survey, provide excerpts from the interview, and finally discuss this in relation to met-before and met-afters.

4.1. *Answers to the questionnaire*

In this section we analyse the data collected in the questionnaire for the four questions: *Tangent*, *Quotient*, *Composition*, and *Slope*.

Tangent was answered by six students. Three students had written that the answer is "the slope of the tangent to the curve at the given point", but they did not state which curve is involved. Three students were not sufficiently precise; their answer was "the tangent to the function in the point x_0" (Figure 2). Four students replied that they did not know or did not remember the answer.

All surveyed students answered question *Quotient*. Their answers could be placed in two groups. One group consisted of three students who simplified the given function and then calculated its derivative. Two of these students differentiated it completely accurately, while a third student wrote $4x^{-1}$ instead of $4x^{-3}$. Another group consisted of seven students who used the quotient differentiation rule. In this group, two students had made a mistake in the differentiation rule writing the plus sign in the numerator of the fraction instead of the minus sign. One of these students made a mistake on the operational level when he multiplied the expression in the bracket with the negative expression outside the bracket. Some of the students did not perform the cancelation of the opposite terms in the numerator or the cancelation of the terms in the numerator and the denominator in the obtained result.

For the question *Composition*, two students wrote that they did not know how to solve this task, without any attempt of solving, and seven students tried to calculate the derivative. Only two of these seven students differentiated the given function correctly without any mistakes in the application of the chain rule. Furthermore, two of these seven students made mistakes such as which can be seen in Figure 3. The tenth student did not recognize that the function is composed of several functions,

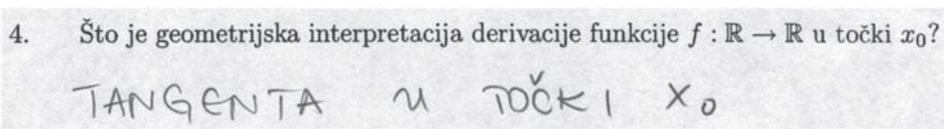

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Figure 2. Insufficiently precise answer to *Tangent*.

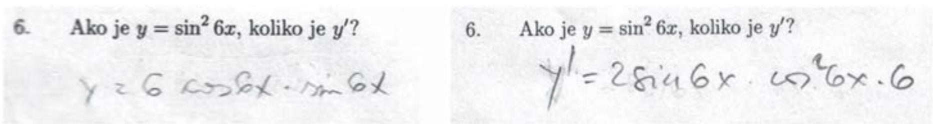

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Figure 3. Students' errors in the chain rule of differentiation.

hence had performed the differentiation as if it was not a composite function. He obtained “$\cos^2 6x$”.

Only five students attempted to solve the question *Slope*, and four of them gave the correct answer. Although their final result was correct, their mathematical notation may still not be a sign of a well-connected conceptual (or procedural) knowledge since no one wrote the answer in words explaining what the result represents. One student simply inserted $x = 1$ into the given function.

4.2. *Comparison of results*

In Table 1, we compared the students’ results obtained from the two questionnaires; one taken two months after the examination (see also [7]) and the other a year after that (the questionnaire reported here). We compared the results for the questions *Tangent*, *Quotient*, *Composition*, and *Slope* which were part of both studies. We have included only those answers that were precise and completely accurate. Students were most successful answering *Quotient*, which is in accordance with the overall results from the first questionnaire. Surprisingly, the lowest results are found in *Composition*, where only a few students were able to conduct the chain rule accurately or to remember it at all.

No one solved the question *Application* completely although some students attempted to. Some students sketched the graph of some function, e.g. the trigonometric function tangent (tan or t_g). One student sketched three figures for all three conditions separately. Other students stated that they did not know how to solve the given task.

4.3. *Interview results*

Students who used the quotient rule for differentiation in *Quotient* claimed that they felt more secure with this rule, in contrast to simplifying the given function and then performing the differentiation, see the interview excerpts quoted below (translated from Croatian by the first author):

> *Student* 9: I do not know why I discarded [simplifying the function]... I wasn’t sure if it’s right, and that rule [quotient rule] I know. And I think it’s safer to use it, than doing something else. Is this right? [pointing to his solution]
> *Student* 6: It was easier to me than separating [in two fractions].
> *Interviewer*: But, it is easier to separate [this fraction] in two parts, isn’t it?
> *Student* 6: Yes, but I got used to doing this when I see a fraction. I always use the quotient rule [for differentiation].

Table 1. Number of correct answers.

Questions	First questionnaire	Second questionnaire
Tangent	7	3
Quotient	10	6
Composition	10	2
Slope	4	4

Also, those students who used the quotient rule inaccurately were able to state the rule in its actual form during the interview.

In *Composition*, the students recognized that the function was composed of several functions, and they were also able to tell which functions they recognize. The students described theoretically what happens in the chain rule (differentiating inside out or vice versa), but most of them were not able to write that down in a formal mathematical language.

In the interview, we asked students to define the derivative of the function at some point. Their answers coincided with their answers to *Tangent*. Further, they said they associated the word derivative with differentiation of a function or, as they said, "with lowering the power of the function" e.g. transforming $f(x)=x^2$ into $2x$. Also, two students tried to write the formal definition using limit of the difference quotient, but had problems with either the form of the numerator or the denominator.

Most students had difficulty with the question *Slope*, but when asked if there was a connection between the questions *Tangent* and *Slope*, they were able to solve the question *Slope*, which indicated that the geometric interpretation of a derivative was learned by rote and mainly imprecisely. It seems that students connected the derivative with the tangent and this phenomenon of mixing the derivative and the tangent is also reported in [28,29].

Application was the most difficult question for all students. Students had, for instance, trouble with interpreting the conditions. Condition *a* was interpreted according to their image of discontinuous functions, see the excerpts quoted below:

> *Student* 9: The function does not go through the origin.
> *Student* 10: The function goes to one point, then stops a bit and then continues.
> The students concluded that conditions *b* and *c* talk about extreme values. They did not connect condition *b* with the concepts concave upward or concave downward, see, for instance, the excerpts below:
> *Student* 4: When the second derivative is greater than zero we have a minimum, and when the second derivative is less than zero we have maximum.

The students were asked to draw the graph of the function according to their interpretation but the majority refused, saying that their interpretation of the given conditions happens to be insufficient to sketch the graph. One student attempted to draw it, but he was unable to incorporate all conditions. At the end of the interview, students were asked if they could describe a procedure for finding local extrema. Many of them described the procedure in terms of one finding a derivative of the function and equating it with a zero.

All surveyed students answered correctly procedural questions *Quotient* and *Composition* two months after the course instructions, but only two out of ten were able to answer the aforementioned questions without mistakes 14 months after the instructions. Thus, one could say that only those two students retained the same level of procedural knowledge. Furthermore, students showed inflexibility of their knowledge when they held on to know rules. According to Star [12], the flexibility is the crucial component of deep procedural knowledge. This denotes knowledge of multiple methods for solving a class of problems and the ability to choose the most appropriate method for the given problem. The quotient rule provided security for seven students, even though they noticed that the function can be simplified and easier differentiated in its simpler form. This way, students did not show deep utilization of procedural knowledge.

The questions *Tangent* and *Slope*, and parts of *Application* are related, but it seemed that the students had difficulties to connect the appropriate concepts. Some students established a partial connection, but they did not use that knowledge to draw the graph of the function. Students therefore showed a very fragmented conceptual knowledge. When considering the question *Application*, the students have never discussed that one must first find a critical point in order to apply the "condition" for minimum or maximum. They connected $f'(-1)=0$ with the extrema of the function, but when they were asked which extrema $f'(-1)=0$ represents, they were not able to connect conditions *b* and *c*. The concept of the derivative was linked with the process of differentiation and the application of the derivatives was related to the procedure for finding local extrema of the function. Students remembered the procedure but did not understand the concepts involved in the procedure and had poor connections among those concepts. According to Haapasalo and Kadijevich [13], this revealed that their ability to browse knowledge network is very weak various since the concepts exist as separate entities in the students' minds. This also supports the findings of Viholainen [30] and Weber and Alcock [31], i.e. that students' reasoning is often too confined to one representational system, formal, visual, or algorithmic. Having good connections between the different representational systems would enable them to reconstruct the knowledge structure and flexibly change between representational systems.

4.4. *Met-befores and met-afters*

In this section we will address the negative met-befores and met-afters that were found in the students' attempts to solve the given questions. Negative met-befores can cause conflict in a new context and have a negative effect on learning, and negative met-afters demonstrate the fragile nature of what has earlier been learned [14,16].

First we will describe some negative met-befores noted in the questionnaires and observed during the interviews. These structures could be detected in the case of *Application*, the most difficult question for all participants. The students' interpretation of conditions revealed how they sought a procedural security of the calculation of any kind.

One student, for instance, came up with a function that she considered to satisfy the given conditions and wanted to use this "imaginary" formula. She tried to use the method she had learned in high school of inserting values and getting the function values to draw the graph of the required function. Another student sketched the trigonometric function tangent (tan or t_g) and claimed that the given conditions reminded her of the properties of a trigonometric function. It seems that the symbol $f(x)$ triggered a met-before where a function is defined by a concrete formula and is connected with various manipulation of it. Even if no defining formula was provided, the students attempted to find an appropriate one.

In the interview, several students connected conditions *b* and *c*, interpreting them in the same way. They claimed that "$f''(x) < 0$ for all $x \in \langle -\infty, 0\rangle$" stands for maximum, and "$f''(x) > 0$ for all $x \in \langle 0, \infty\rangle$" stands for minimum; also, they claimed that "$f'(-1)=0$" is either minimum or maximum. In the curriculum of this university's calculus course, the concept of extreme value precedes the concept of inflection point and a function being concave upward and concave downwards.

In the procedure for determining extrema, students usually calculated and used the first and second derivative of the function, i.e. the necessary and sufficient conditions [32]. Using the first derivative, they determined the critical points, and looking at the value of the second derivatives examined whether a certain critical point is a point of extrema. Once again, this met-before indicated the students' desire for procedural knowledge and the need of procedural security of calculation.

Another negative met-before was encountered during the discussion about the question *Application*. A form in which the function symbol f is equated with zero influenced two students in their interpretation of the first part of condition c. They said that the first part of condition c, "$f'(-1)=0$" indicated that -1 is a zero point of the function f. Zero points of various functions were frequently calculated through high school mathematics and this mental construct has anchored deeply in their knowledge.

Now we will discuss some negative met-afters that affected the students' long-term knowledge structure. Met-afters that appeared were connected with integrals and integration. In the question *Composition* some students tried to use some kind substitution in the chain rule of differentiation and during the interview they were still puzzled as to whether or not they should use it, see excerpts below:

> *Student* 6: I think we should use some substitution.
> *Interviewer*: Why?
> *Student* 6: Erm... we did this in Calculus 2 [referring to integration].
> *Interviewer*: Why did you want to use substitution here?
> *Student* 8: I don't know. It stayed [in my mind] from integrals.
> *Student* 3: We usually expressed $\sin^2 x$ using $\cos 2x$ in integrals [in Calculus 2].

The substitution method is a basic method used to calculate definite and indefinite integrals. Exposure to a new mathematical procedure seems to have left traces on students who probably were not comfortable with differentiation of composite functions when it is composed of three or more functions.

A student who separated the conditions in *Application* and sketched three separated figures for each condition, explained that he thought those were in fact three tasks. He explained that he used integration to "understand" what has to be drawn. He used integration twice in condition b, obtaining two linear functions and he marked the areas under their graphs as the solution (Figure 4). He connected differentiation and integration and used the main application of (definite) integral – calculating the area under the curve. This indicates that his knowledge of integrals had taken over his conceptions of derivatives. He gave similar arguments for using integration in condition c (Figure 4):

5. Discussion and conclusions

Students who participated in this survey claimed that they "know derivatives", and were willing to take part in our research. It seems that the students' perspectives would necessarily be related to the process of differentiation, since that process plays a large role in many applications and discussions of derivatives. Students' perspectives were determined almost exclusively by the process of differentiation as opposed to the definition of the derivative or the geometric interpretation of a derivative at a point or some other aspect of differentiation. However, many students had also rejected our invitation for participation in the research, claiming they

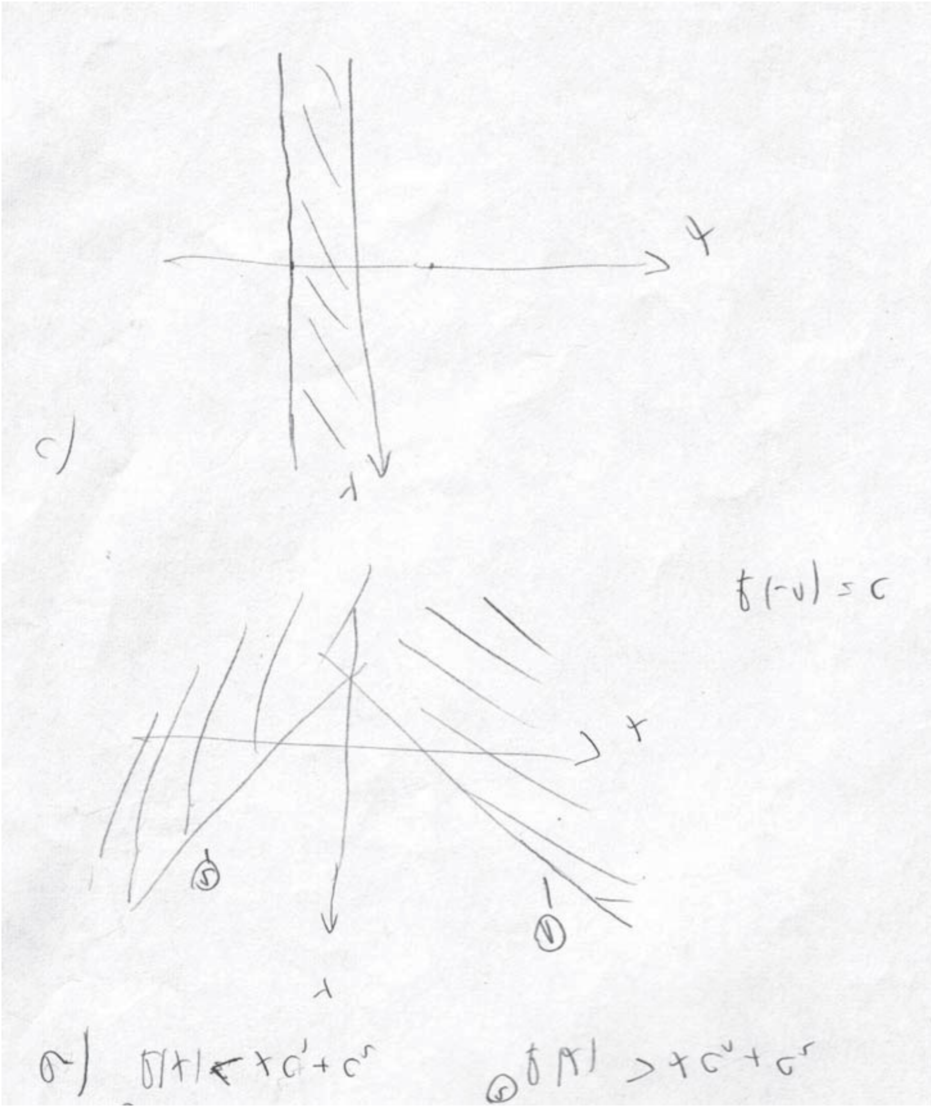

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Figure 4. Student's solution to *Application*.

"do not know derivatives" or "they are not good at derivatives". This makes us wonder what kind of knowledge retained those students who had refused our invitation. But since they did not want to participate in the study, we can only speculate about this.

In light of our results, it seems that the students' perspective "to know" or "not to know" derivatives is related with the process of differentiation. Students identified "knowing" with the ability to perform certain procedure, or to use an appropriate algorithm in the given context. Strong procedural approaches were visible when students did not know how to cope with the required conceptual reasoning and therefore interpreted the given problems in their own way, assigning them "appropriate" meanings. This is in line with the findings of Engelbrecht et al. [25]

who discovered students' preference for procedural solutions in conceptual questions. Students re-designed the tasks in a sense so they required a concrete function or formula that conformed the given conditions, and then solved the tasks in this new context.

Participants in our study remained in contact with derivatives and differentiation, taking various mathematics courses and using them in other non-mathematics courses. Engelbrecht et al. [20] found that there is a natural decline in knowledge that is not used regularly, but when the knowledge is encountered along the way, retention is strengthened. However, most of the time our students leaned on their procedural knowledge which turned out to be quite fragile even though they had in fact encountered it in other courses along the way.

As stated earlier, met-befores and met-afters can have positive and negative consequences in the long-term cognitive development of a person. We, however, described negative met-befores and met-afters in differential calculus, observed in students' written responses and during the interview. Those mental structures affected the stability on the retained calculus knowledge and showed the fragile nature of the previous learning. Most curricula address positive met-befores to broaden students' knowledge through earlier experience, disregarding negative ones that can cause great difficulties to many learners [14]. As it is important to reflect on negative met-befores, it is equally important to discover negative met-afters. Following Nogueira de Lima and Tall [33], identifying negative met-befores and met-afters in differential calculus could help lecturers to know what to re-address along all stages of learning calculus – limits, continuity, integrals, differential equations. Here we investigated met-befores and met-afters in small part of calculus, and further study is needed to broaden our knowledge of such structures: "Without addressing the problematic met-befores that remain under the surface, any chance of conceptual understanding is suppressed and the only way forward is to focus the student on the techniques required to succeed in performing correct methods of getting the answer" [16, p. 174].

6. Recommendations for practice

We would be hesitant to make too strong recommendations for practice due to the fact that what we have been studying is the problems of learning, not various attempts to remedy such problems. However, some thoughts may still be warranted.

In relation to long-term learning, Tall [14] states that procedural knowledge can be improved if one moves focus from the performed action to the effect of that action. One possible way, we suggest, is reformulating questions, subtly moving them from pure calculations to the questions that ask for explanation. As an example, we propose, that in differential calculus this could be achieved by substituting the question "Find the extreme value of $f(x) = x^2e^x$" with "Explain whether or not the point $(-2,1)$ can be an extreme value of $f(x) = x^2e^x$ if $f'\ (-2) = 0$?" or using more abstracts questions such as "Explain whether or not the point (x_0,y_0) can be extreme value of $f(x)$ for which $f'(x_0) = 0$ and $f(x_0) \neq y_0$?" Asking students for explanation triggers different kind of reasoning where relations are needed among several concepts. The students' weak conceptual knowledge and preference for procedural rules can be cause of illusion that in mathematics there is only one right way to solve mathematical problems. Giving more questions which do not have one unique answer, as the question *Application*, could enable better conceptualization of mathematics as a discipline rich in relationship, and not just a discipline of formulas and right and wrong answers. Further, the questions of this type demand intertwining learnt concepts into one meaningful whole. This is very beneficial for non-mathematics students and their future profession where they would encounter problems that can be solved in more than one way and using different methods.

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