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UNIVERSITY STUDENTS' CONCEPT IMAGE AND RETENTION OF THE DEFINITE INTEGRAL

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This paper reports a part of a larger study researching the retention of key derivative and integration concepts months after the calculus course exam at a Croatian and Danish university. In this paper we focus on the students' long-term retention and concept image of the definite integral. 18 students in non-mathematics science study programmes were interviewed in pairs and presented with four tasks on the definite integral in order to expose their concept definition and concept image. Although it varied, no student had a coherent concept definition which caused problems solving the tasks. However, some students appeared to have a more coherent concept image, and solved some of the tasks, while others did not. We argue that the relation between the concept definition and the concept image varies from student to student.

INTRODUCTION

We have previously (Jukić & Dahl, 2010) investigated the relation between knowledge retention and exam results in differential calculus of university students in non-mathematics science study programmes. This survey showed that the students had forgotten a large part of the concepts and often those with the lowest grades had the better results two months later. Furthermore (Jukić & Dahl, 2011) the Danish students taught in a student-centred course statistically significantly outperformed the Croatian students on a teacher-centred course on the conceptual questions; vice versa for the procedural ones. In this paper we will further investigate the non-mathematics university students' retention and understanding of integrals by examining their concept image two months after the calculus exam.

THEORETICAL BACKGROUND

Concept image and concept definition

The term *concept image* includes all the non-verbal conceptions and associations that an individual has of a concept. It includes the mental pictures, properties, and processes related to the concept. The *concept definition* is the words used to specify the concept. To understand the formal concept definition, an individual creates his own personal interpretation of the definition. The personal concept definition may, or may not, be based on the formal definition (Tall & Vinner, 1981). Tall (2006) states that he and Vinner define a concept image differently. Tall regards a concept definition as part of the concept image whereas Vinner makes a distinction between them. Tall states that this difference had not had significant effect in the use of the

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terms in mathematics education research. Through this study, we will also add to this discussion about the relation between the concepts.

Vinner (1991) argues that in the long-term process of concept formation, the relation between the concept image and the concept definition should be established in both directions. However, teachers usually see this as a one-way process assuming that the concept definition will form the concept image. However, students frequently do not use the concept definition in building the concept image (Vinner & Dreyfuss, 1989) and they do not rely on the concept definition but on the concept image when solving problems (Vinner, 1991; Rasslan & Tall, 2002; Rösken & Rolka, 2007). Furthermore, Rasslan & Tall (2002) showed that some English high school students had a concept image of the definite integral but only a small number of students knew and used the concept definition. Similarly, Rösken & Rolka (2007) found that the concept definitions played a marginal role in some German secondary school students' conceptual knowledge of integral calculus and the students mainly leaned on their concept images, which caused difficulties in their reasoning and problem solving.

The concept image is not necessarily coherent but can include contradictions to the concept definition (Vinner & Tall, 1981; Viholainen, 2008). In fact, Juter (2005) showed that university students can have contradictory conceptions about the limit value of a function. Although the students claimed that a function cannot attain its limit values, they considered it possible in problem solving. Viholainen (2008) stated the following conditions for a concept image to be coherent: the conception must be clear; all conceptions are connected to each other; there are no internal contradictions, and the concept image is not contradictory to the formal definition. A coherent concept image is part of the conceptual knowledge where one understands the connections between different concepts, while a coherent concept image refers to a single concept (Viholainen, 2008).

Research questions

What are the Croatian students' concept image and retention of the definite integral, particularly the geometric interpretation of it? How is the relation between their concept definition and concept image?

METHODOLOGY

Two months after the calculus exam, we examined the students' knowledge of integral calculus. Students were interviewed in pairs in order to better establish an atmosphere of confidence and the interviewees may 'fill in gaps' for each other (Arksey & Knight, 1999). Schoenfeld (1985) further states that this kind of interviewing helps alleviate the pressure the students might otherwise feel solving tasks individually. The tasks (Figure 1) were on the geometric interpretation and analytical definition of the definite integral and the calculation of areas. Tasks 3-4 were indirect (Vinner, 1991) in order to expose their concept images while Task 1-2 were directly about their concept definitions in order to see the difference. Task 4 is from Mahir (2009). She studied first-year university students who passed a calculus course. Only 16 of 62 had solved this correctly. We find the task good at detecting if the students recognize the areas that constitute the whole area as the graph is both above and below the *x*-axis. We will compare our results with Mahir's.

Task 1. What is a geometric interpretation of the integral $\int_b^a f(x)dx$? Give an illustration.

Task 2. What is the analytical definition of the definite integral?

Task 3. Calculate the area bounded by the curves $y = \frac{x^2}{2}$ and $y = \sqrt{2x}$ under the line $y = 2$.

Task 4. The graph of f is sketched to the right. Given that $\int_{-2}^{5} f(x)dx = \frac{39}{8}$, find the value v.

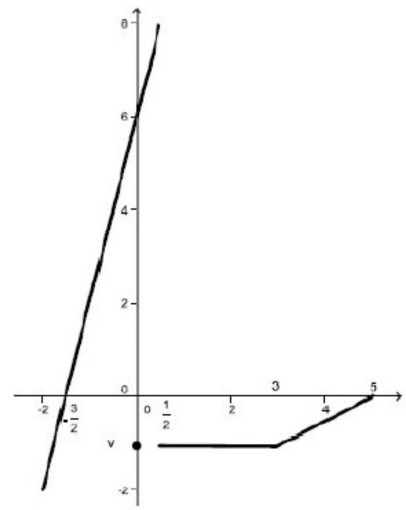

Figure 1: The four tasks (translated from Croatian).

The interviewed students were first-year non-mathematics students belonging to the civil engineering, electrical engineering, or physics study programmes. They were chosen randomly and paired according to study programme affiliation and therefore not strangers but acquaintances, hence we assume more comfortable in the interview where they had to re-create or remember their knowledge (Morgan, 1988). We interviewed nine pairs, eight females and ten males. Before the interview, they were informed that their identity will be kept safe and were acquainted with the purpose of the research. The students received a sheet with the tasks and plenty of empty space to write the answers. They were asked to think their solutions out aloud and if there was a long silence, they were asked what they were thinking at the moment. During the interview, students were also asked to elaborate their claims when they solved the tasks. The interviews were recorded and transcribed. The transcript pieces below are translated from Croatian.

RESULTS OF THE INTERVIEWS

Task 1: Geometric interpretation of the definite integral

All students defined the definite integral as a tool to calculate the area under the curve. However, their geometric interpretation stated only "area under the curve" and not the area between the curve and the *x*-axis. Their illustrations represented mainly the area above the *x*-axis on the interval [*a*,*b*] except students J & P (Figure 2). When asked, J & P said they would put a negative sign in front of the integral to calculate this area. The figures drawn were very simple and noone addressed the possibility that part of the area could be above the *x*-axis and another part below the *x*-axis. The answers showed that the students' concept definition is not coherent but strongly linked with the image of a certain function and their concept definition seems not to be based on the formal definition. This is similar to Rösken & Rolka (2007) who found, among 24 secondary school students, that the geometric interpretation of the definite integral is connected mainly with the image of the area above the *x*-axis.

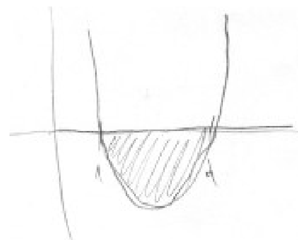

Figure 2: Area image drawn by students J & P.

Task 2: Analytical definition of the definite integral

All students knew that the area under the curve is approximated by rectangles and that the approximation becomes more accurate as the number of rectangles increases and they all used visual representations to elaborate their statement (Figure 3).

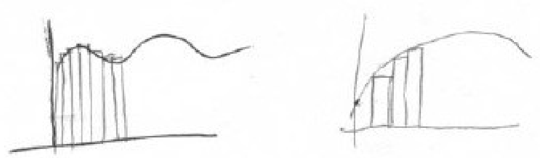

Figure 3: Approximating the area with rectangles, students M & D.

11 of 18 students had a coherent concept definition about the position of the rectangles – i.e. if they can be place above or below the curve or mixed. This number emerged due to a conflict between students V & W who disputed if the rectangles can be positioned just above the curve or under the curve as well.

W: These are rectangles [draws a figure with rectangles about the curve]. So so small rectangles. One part is always sticking out. We strive to get as many rectangles as possible to divide the curve, i.e. the area under the curve, to minimize the loss.

I: Is it possible that the rectangles can be inscribed, under the curve?

W: No. I think that cannot be the case. Then you get a smaller area than what we are looking for.

V: And here you have larger one.

W: But here are minor losses.

V: How do you know? They can be equal.

W: So they taught us [laughs]. If this is the curve, and these are the rectangles [draws inscribed rectangles], there will be always some losses …

V: But, if you draw very small rectangles, and they [losses] will be very small, so it will not affect much [draws two figures similar to Figure 3]. The area is bigger here [pointing to the first figure, to the circumscribed rectangles]… and here it is smaller [pointing to the second figure, to the inscribed rectangles]. In both cases, it is not the same as the area under the curve… but they are very close… So the rectangles can be above and under the curve.

W: Hm… I do not know… I do not remember it that way.

The students knew the constitutive connections in the analytic definition but most were unable to define it adequately. For instance the students knew that they had to divide it into rectangles but they did not know what to do next. Just one student gave an appropriate definition using formal mathematics expressions. This suggests that the students' concept definitions are weakly related to the formal analytic definition.

Task 3: Calculation of the area

Six pairs had serious difficulties with the shape of the curves. To overcome this, they calculated a table of *x,y* values or squared the expression and used "the new" curve. They knew that the area was to be calculated using an integral but they had problems defining the limits for the integration. The shape of the area caused difficulties since both the curve $y = 2\sqrt{x}$ and the line y = 2 represented the top of the area. In the calculus course, students learnt to "subtract" the upper and lower curves. The combination of two curves both being the upper bound for the area made a conflict in their concept image of the geometric interpretation of the definite integral as the area for seven pairs. One pair, V & W, wanted to calculate the area above y = 2. However, two pairs avoided the problem with the two upper bounds by switching variable for integration. They rotated the figure and posed the problem in terms of y. This way they had only one upper curve and one lower curve (Figure 4). Hence, their concept image may be more coherent since they had a more flexible use of the concept.

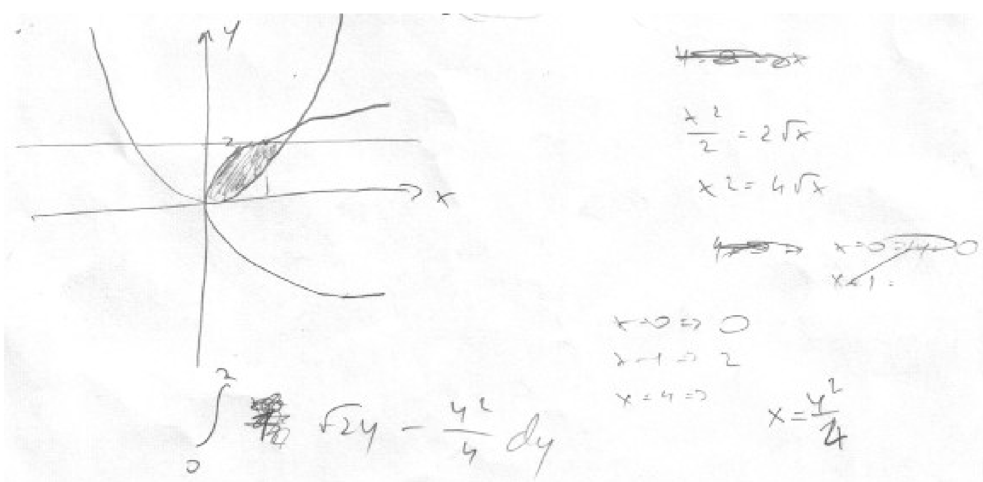

Figure 4: Solution to Tasks 3 by students M & D integrating by *y*.

Task 4: Using areas to determine the desired value

The students had problems determining the limits of the areas and six pairs marked the area from $x = -3/2$ to $x = 0$ not bounded from the bottom (an example of the six pairs is seen in Figure 5). The second difficulty arose between $x = 0$ and $x = ½$ where some marked the area under the x-axis, not bounded for the bottom. A seventh pair, D & M, claimed that the task was not well-defined due to the part under the x-axis.

M: I do not see which areas are included in the whole area.
D: And this under the x-axis between 0 and ½?
I: No
M: How come this under the x-axis from -3/2 to -2 is included and from 0 to ½ is not?

Also, M & D wanted to include the part between the y-axis and the line $y = 4x + 6$ on the interval [0, ½] (Figure 6). Their confusion made them give up the task. This is interesting as M & D made a creative solution to Task 3 (Figure 4), but we now see that this might have been their way of coping with a "hole" in their concept image. Task 4 exposed the hole and left no way around it. Only two pairs marked the area correctly.

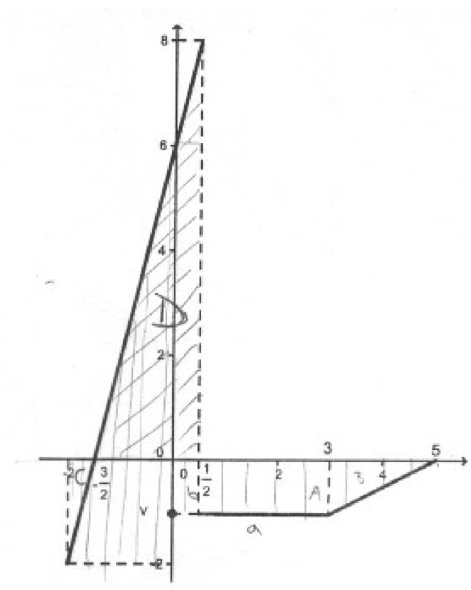

Figure 5: Areas marked by J & P.

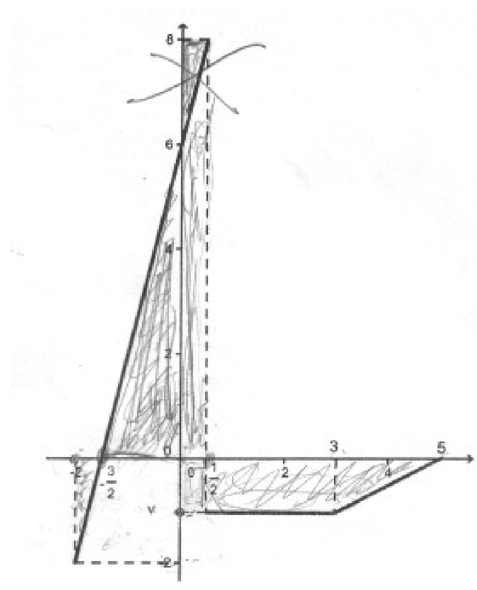

Figure 6: Areas marked by M & D.

It seems that the students' concept definition of the integral influenced their ability to see, or not to see, which areas constitute the whole area. This was seen when the students were told by the interviewer that they had marked the wrong area parts. Six students repeated the geometric definition of the integral as the area under the curve and gave this as an argument for their wrong answer. The students did not discuss the fact that the area must be bounded both from above and below, although they used this fact in Task 3, and therefore had difficulty understanding why they should mark the area only above the x-axis from $x = - 3/2$ to $x = ½$. Some pointed with fingers to the figure and argued that the line stops at $y = -2$, and "therefore this part under is also included".

I: What was confusing for you?

V: This part… the line [$y = 4x + 6$]… when it goes under the x-axis.

Furthermore, all students claimed that the task was too difficult to solve even though it included only lines and not other types of curves. All had trouble with calculating the value v and 14 students did not by themselves mark the appropriate area, but they had less trouble in Task 3 than with Task 4. Hence, facing the area problem from a different perspective exposed a conflict within their concept image. They were accustomed to tasks like Task 3, which suggests that their reasoning sequence follows the following pattern "given function - figure to draw - area to be calculated - integral as tool for calculating the area". Changing the order exposed deficiencies in the students' concept image.

CONCLUSIONS

From the first two tasks it appeared that no student had a coherent concept definition of the geometric interpretation of the definite integral, although some students' definitions were more coherent than others. Tasks 3-4 aimed at exposing their concept image. In Task 3, all students had problems coping with two upper boundaries although two pairs found it by integrating by y. In Task 4, all but two pars made erroneous conclusions concerning what constitutes the given area. The students' incoherence of the area concept image might, on the one hand, have been foreseen from their answers to Task 1, which asked for the concept definitions and where all students only drew the area on one side of the x-axis. On the other hand, as discussed above, the concept definitions may not always be part of the concept images. The study therefore also adds to the Tall-Vinner discussion about if the concept definition is part of the concept image or different from it. We saw that six students in Task 4 referred to the concept definition when the interviewer told them that the area marked was wrong. At least for these students, their concept definition was part of their concept image. On the other hand, four students marked the appropriate are in Task 4, hence exposed a coherent concept image without having shown a coherent concept definition in Tasks 1-2. Hence, we argue that the relation between concept definition and concept image varies from student to student. It also reveals that the long-term retention of the concept image of the definite integral is weak even though the students two months earlier had passed the calculus course.

Task 4 was from Mahir (2009) where 16/62 students solved it correctly. Mahir did not focus on the concept image but we assume that the 16 students had a coherent concept image since they could not otherwise have solved it. These were mathematics students, hence they usually develop different mathematical concepts than non-mathematics students (Maull & Berry, 2001; Bingolbali et al., 2007). The two samples are not representative, hence not comparable, however the correct understanding rate was quite alike since 4/18 of the non-mathematics students in our study marked the correct area. Hence, the task was difficult to both student groups.

Acknowledgement

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