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COMMENT

Comment on 'Fault tolerance analysis for stochastic systems using switching diffusion processes' by Yang, Jiang and Cocquempot

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Results are given in Yang et al. (2009) regarding the overall stability of switched diffusion processes based on stability properties of separate processes combined through stochastic switching. This paper argues two main results to be empty, in that the presented hypotheses are logically inconsistent.

Keywords: stochastic system; switching diffusion; stability

In Yang et al. (2009) stability results are presented for so called *switching diffusion processes* as indicated by the title. Such systems evolve in a hybrid state space, i.e. including both continuous and discrete state variable components. In each discrete state δ evolution of the continuous state $x \in R^n$ is governed by a diffusion process, i.e.

$$dx = f(x, u) + g(x, u)dW \tag{1}$$

where W is a Brownian motion, $u \in R^m$ is a control and f and g are appropriate mappings satisfying suitable smoothness conditions to ensure unique solutions to (1).
Evolution of the discrete state $\delta \in \mathcal{M}$ is governed by a continuous time Markov chain with an infinitesimal generator matrix $\Gamma = \{\rho_{ij}, i, j \in \mathcal{M}\}$ modelling stochastic transition between nominal and various faulty states.
The concept of input-to-state stability (ISS) is used in the presented analysis, where a stochastic system is said to be ISS *iff*

$$E[\alpha(|x(t)|)] \leq \beta(|x(0)|, t) + \gamma(||u||_{[0,t)}) \quad \forall t \geq 0 \tag{2}$$

where $\alpha, \gamma : [0, \infty) \to [0, \infty)$ are strictly increasing and continuous. In Yang et al. (2009) stochastic Lyapunov analysis is applied along with the infinitesimal generator $\mathcal{L}$, i.e. for the process (1)

$$\mathcal{L}V(y) = \lim_{h \to 0} E[V(x(t + h))|x(t) = y] \tag{3}$$

yielding the following sufficient stochastic ISS criterion

$$\mathcal{L}V(x(t)) \leq -\alpha_3 V(x(t)) + \gamma_1(|u|) \tag{4}$$

for $\alpha_3 > 0$. Combining a finite number of diffusion processes through Markovian switching naturally poses the question about overall stability. In Yang et al. (2009) three different situations are analyzed: all separate processes fulfil (4), only some do and lastly all processes fulfil (4). Our

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comment pertains primarily to results given for the first and last cases.
We repeat the main result for the first case given in Theorem 2 of Yang et al. (2009)

Theorem 0.7: *Let* $\bar{\lambda} = \max_i \rho_{ii}$, $\tilde{\lambda} = \max_{ij} \rho_{ij}$ *and* $\mu > 1$ *If*

$$\mathcal{L}V_q(x(t)) \leq -\lambda_0 V_q(x(t)) + \gamma_1(|u|) \quad \forall q \in \mathcal{M} \tag{5}$$

$$V_q(x) \leq \mu V_p(x) \quad \forall p, q \in \mathcal{M} \tag{6}$$

and

$$\mu < \frac{\tilde{\lambda}}{\bar{\lambda}} \tag{7}$$

Then the overall system is ISS

We do not argue the proof of the theorem, however we claim the theorem to be empty, since the hypothesis can never be fulfilled. From the definitions of $\tilde{\lambda}$ and $\bar{\lambda}$ as well as the structure of the generator matrix Γ we get (since all $\rho_{ii} \leq 0$))

$$\begin{aligned}\bar{\lambda} &= max_i|\rho_{ii}| = max_i(sum_{j\neq i}(\rho_{ij}))\\ &\geq max_i(max_{j\neq i}(\rho_{ij})) = max_{i,j\neq i}(\rho_{ij})\\ &= max_{i,j}(\rho_{ij}) = \tilde{\lambda}\end{aligned}$$

which together with (7) yields $\mu < 1$.
We cannot disprove the theorem nor can we give counter examples, since no examples would fulfil the hypothesis. However in the accompagning interpretation in Yang et al. (2009) it is stated: *Roughly speaking, if each mode is ISS, and the fault occurrence transition rate* $\max_{ij} \rho_{ij}$ *is large enough, then the ISS of the stochastic system is guaranteed.*
We find this statement highly counter intuitive, since stability arguments for switched systems under (5) and (11) would rely on long dwell times of separate systems to ensure sufficient decay of individual Lyapunov functions in between shifts.
Turning to the last situation, where no separate systems are assumed ISS, condition (5) is in Theorem 5 in Yang et al. (2009) replaced by

$$\mathcal{L}V_q(x(t)) \leq \lambda_1 V_q(x(t)) + \gamma_1(|u|) \tag{8}$$

where $\lambda_1 > 0$ and (7) by

$$\mu < \frac{\tilde{\lambda} - \lambda_1}{\bar{\lambda}} \tag{9}$$

which is equally inconsistent since $\lambda_1 > 0$.
Like in the former case the theorem cannot be disproved. However in the accompagning interpretation it is stated: *Theorem 5 shows that if the fault occurrence transition rate* $\max_{ij} \rho_{ij}$ *is larger than that of any previous cases (all ISS modes, partial ISS modes) and the ISS of SDP is achieved without any ISS mode. This result implies that, under the condition (9), we do not need to design the stabilising controller even if the stochastic system is not stable separately in the healthy and faulty situations.*
This statement can be met by a simple counterexample, where two identical unstable systems are combined by stochastic swiching. (In this case we may choose $V_q = V_p$, so $u = 1$ can be used and there are only 2 states $\frac{\tilde{\lambda}}{\bar{\lambda}} = 1$, so we are as close as possible to fulfilling (7)). However switching between identical unstable systems does not make the overall system stable. The above

statement is not only counter intuitive and wrong, it is also potentially harmful, to the extent that it may lead readers to refrain from safety related counter measures.

1 New result

Theorem 1.1: *If* $\exists L > 0$ *so that* $\forall x$*, where* $|x| \geq L$

$$\mathcal{L}V_q(x(t)) \leq \lambda_q V_q(x(t)) \quad \forall q \in \mathcal{M} \tag{10}$$

$$V_q(x) \leq \mu V_p(x) \quad \forall p, q \in \mathcal{M} \tag{11}$$

(11)

Then

Proof: Let the sequence $\{t_j\}$ be the transition instants of the switching process σ, such that $\sigma(t) = q_i$ for $t \in [t_i, t_{i+1})$ Define the process U by

$$\frac{d}{dt}U(t) = \lambda_{q_i} U(t) \tag{12}$$

for all $t \in [t_i, t_{i+1})$ and

$$U(t_{i+1}) = \mu U(t_{i+1}^-) \tag{13}$$

and finally

$$U(0) = E[V_{\sigma(0)}(x(0))] \tag{14}$$

then

$$E[V_{\sigma(t)}(x(t))] \leq U(t) \tag{15}$$

Proof:
First consider the case conditioned a fixed realization $\bar{\sigma}$ of the switching process. Assume $E[V_{q_i}(x(t_i))] \leq U(t_i)$. Then from (10) and (12) $E[V_{q_i}(x(t))] \leq U(t)$ for all $t \in [t_i, t_{i+1})$. Since x and V are continuous $V_{q_{i+1}}(x(t_{i+1})) \leq \mu V_{q_i}(x(t_{i+1})) = \mu V_{q_i}(x(t_{i+1}^-))$. Thus $E[V_{q_{i+1}}(x(t_{i+1}))] \leq \mu E[V_{q_i}(x(t_{i+1}^-))] \leq \mu U(t_{i+1}^-) = U(t_{i+1})$. Thus (15) is proved for all conditions $\bar{\sigma}$ and in turn unconditionally.

From (15) we have immediately

$$E[V_{\sigma(t)}(x(t))] \leq E[U(t)] \tag{16}$$

Define the processes γ_q by $\gamma_q(t) = I_{\sigma_t = q}U(t)$ and rewrite (12)

$$\frac{d}{dt}U(t) = \sum_q \lambda_q I_{\sigma_t = q} U(t) = \sum_q \lambda_q \gamma_q(t) \tag{17}$$

thus

$$\gamma_l(t+h) = I_{\sigma_{t+h}=l}[h\sum_j \lambda_j I_{\sigma_t=j} + \mu\sum_{j\neq l} I_{\sigma_t=j} + I_{\sigma_t=l}]U(t) \tag{18}$$

taking expected values gives

$$\begin{aligned} E[\gamma_l(t+h)] &= h\sum_j \lambda_j E[I_{\sigma_{t+h}=l}I_{\sigma_t=j}U(t)] + \mu\sum_{j\neq l} E[I_{\sigma_{t+h}=l}I_{\sigma_t=j}U(t)] + E[I_{\sigma_{t+h}=l}I_{\sigma_t=l}U(t)] \\ &= h(h\sum_{j\neq l}\lambda_j\rho_{jl}E[I_{\sigma_t=j}U(t)] + \lambda_l(1-h\sum_{j\neq l}\rho_{lj})E[I_{\sigma_t=l}U(t)]) \\ &+ h\mu\sum_{j\neq l}\rho_{jl}E[I_{\sigma_t=j}U(t)] + (1-h\sum_{j\neq l}\rho_{lj})E[I_{\sigma_t=l}U(t)] \\ &\approx h\lambda_l E[I_{\sigma_t=l}U(t)] + h\mu\sum_{j\neq l}\rho_{jl}E[I_{\sigma_t=j}U(t)] + (1-h\sum_{j\neq l}\rho_{lj})E[I_{\sigma_t=l}U(t)] \end{aligned}$$

so subtracting $E[I_{\sigma_t=l}U(t)]$ and taking limits for $h\to 0$ gives

$$\frac{d}{dt}E[\gamma_l(t)] = \lambda_l E[\gamma_l(t)] + \mu\sum_{j\neq l}\rho_{jl}E[\gamma_j(t)] - E[\gamma_l(t)]\sum_{j\neq l}\rho_{lj}$$

in general for $\epsilon>0$ we may write

$$\begin{aligned} \gamma_l^\epsilon(t+h) &= (I_{\sigma_{t+h}=l}[h\sum_j\lambda_j I_{\sigma_t=j} + \mu\sum_{j\neq l}I_{\sigma_t=j} + I_{\sigma_t=l}]U(t))^\epsilon \\ &= (I_{\sigma_{t+h}=l}[h(\sum_{j\neq l}\lambda_j I_{\sigma_t=j} + \lambda_l I_{\sigma_t=l}) + \mu\sum_{j\neq l}I_{\sigma_t=j} + I_{\sigma_t=l}]U(t))^\epsilon \\ &= (I_{\sigma_{t+h}=l}[\sum_{j\neq l}(h\lambda_j+\mu)I_{\sigma_t=j} + (h\lambda_l+1)I_{\sigma_t=l}]U(t))^\epsilon \\ &= I_{\sigma_{t+h}=l}[\sum_{j\neq l}(h\lambda_j+\mu)^\epsilon I_{\sigma_t=j} + (h\lambda_l+1)^\epsilon I_{\sigma_t=l}]U^\epsilon(t) \\ &\approx I_{\sigma_{t+h}=l}[\sum_{j\neq l}(\mu^\epsilon + h\epsilon\mu^{\epsilon-1}\lambda_j)I_{\sigma_t=j} + (1+h\epsilon\lambda_l)I_{\sigma_t=l}]U^\epsilon(t) \end{aligned} \tag{19}$$

taking expectations, subtracting $E[\gamma_l(t)^\epsilon]$ and taking limits for $h\to 0$ gives

$$\frac{d}{dt}E[\gamma_l(t)^\epsilon] = \mu^\epsilon\sum_{j\neq l}\rho_{jl}E[\gamma_j(t)^\epsilon] + (\epsilon\lambda_l - \sum_{j\neq l}\rho_{lj})E[\gamma_l(t)^\epsilon]$$

or more compactly

$$\frac{d}{dt}E[\gamma_l(t)^\epsilon] = \mu^\epsilon\sum_{j\neq l}\rho_{jl}E[\gamma_j(t)^\epsilon] + (\epsilon\lambda_l + \rho_{ll})E[\gamma_l(t)^\epsilon]$$

Leaving stability to the eigenvalues of the matrix $\Lambda_S(\mu,\lambda_1,..,\lambda_M,\epsilon)$, where $\Lambda_{Sjl} = \mu^\epsilon\rho_{jl}$ for $j\neq l$ and $\Lambda_{Sll} = \epsilon\lambda_l + \rho_{ll}$. Since $\Lambda_S(\mu,\lambda_1,..,\lambda_M,0) = \Lambda$, $\Lambda_S(\mu,\lambda_1,..,\lambda_M,0)$ has (for an irreducible Λ)

an eigenvalue in 0 of multiplicity 1, and all other eigenvalues in the left complex plane.
A sufficient criterion for the existence of an $\epsilon > 0$ such that $\Lambda_S(\mu, \lambda_1, .., \lambda_M, \epsilon)$ is stable is that the root locus of Λ_S for positive epsilon takes the unique root in 0 to the left half plane. Let $D(s, \epsilon)$ be the determinant of $sI - \Lambda_S$ then a first order approximation would give

$$D(s, \epsilon) = D(0, 0) + \frac{\partial D}{\partial s}|_{0,0} s + \frac{\partial D}{\partial \epsilon}|_{0,0} \epsilon = 0$$

or

$$s = -[\frac{\partial D}{\partial s}|_{0,0}]^{-1} \frac{\partial D}{\partial \epsilon}|_{0,0} \epsilon$$

Thus a sufficient stability criterion is

$$\frac{\partial D}{\partial s}|_{0,0} \frac{\partial D}{\partial \epsilon}|_{0,0} > 0 \tag{20}$$

Now since

$$D(s, \epsilon) = \sum_{i=0}^{M} d_i(\epsilon) s^i$$

we get in (20)

$$d_1(0) d_0'(0) > 0 \tag{21}$$

Now for the process U in (12)

$$U(t) = U(0) * exp(\int_0^t \lambda_{\sigma(t)} dt) * \mu^{N(t)}$$

Taking logarithms

$$log(U(t)) = log(U(0)) + \int_0^t \lambda_{\sigma(t)} dt + log(\mu) N(t)$$

Now from ergodicity of Λ, w.P.1 $\int_0^t \lambda_{\sigma(t)} dt = t \sum_{i=1}^M \pi_i \lambda_i + O(t)$ and $N(t) = -t \sum_{i=1}^M \pi_i \rho_{ii} + O(t)$. Thus an exact criterion for allmost sure convergence of U to 0 is

$$\sum_{i=1}^{M} \pi_i (\lambda_i - log(u) * \rho_{ii}) < 0 \tag{22}$$

It is readily shown that (20) conincides with (22) for $M = 2$. However for larger values (of M) this question is still unanswered (to me).

References

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