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Plug & Play Control of Hydraulic Networks

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Publication date:
2012

Document Version
Accepted author manuscript, peer reviewed version

Citation for published version (APA):
Jensen, T. N. (2012). *Plug & Play Control of Hydraulic Networks*.

Plug & Play Control of Hydraulic Networks

TOM NØRGAARD JENSEN

'This work is a part of the research project *Plug & Play Process Control*. An industrial case study involving a large-scale hydraulic network with non-linear dynamics is studied. The hydraulic network underlies a district heating system, which provides heating water to a number of end-users in a city district. The case study considers a novel approach to the design of district heating systems in which the diameter of the pipes used in the system is reduced in order to reduce the heat losses in the system, thereby making it profitable to provide district heating to areas with low energy demands. The new structure has the additional benefit that structural changes such as the addition or removal of end-users are easily implementable. In this work, the problem of controlling the pressure drop at the end-users to a constant reference value is considered. This is done by the use of pumps located both at the end-users and at designated places across the network. The control architecture which is used consists of a set of decentralized linear control actions. The control actions use only the measurements obtained locally at each end-user. Both proportional and proportional-integral control actions are considered. The results consist of a series of global stability results of the closed-loop system. The stability analysis is complicated by the non-linearities present in the system process. Specifically, global practical output regulation is shown when using proportional control actions, while global asymptotical output regulation is shown when using proportional-integral control actions. Since the results are global in the state space, it is concluded that the closed-loop system maintains its stability properties when structural changes are implemented.

Plug & Play Control of Hydraulic Networks

AAU 2012

Tom Nørgaard Jensen

Plug & Play Control of Hydraulic Networks

Plug & Play Control of Hydraulic Networks
Ph.D. thesis

To Lilly and Marie.

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Preface and Acknowledgements

This thesis is submitted as a collection of papers in partial fulfillment of the requirements for a Doctor of Philosophy at the Section of Automation and Control, Department of Electronic Systems, Aalborg University, Denmark. The work has been carried out in the period August 2008 to November 2011 under the supervision of Professor Rafał Wisniewski and Professor Claudio DePersis.

Aalborg University
November, 2011
Tom Nørgaard Jensen

Abstract

Typically, control systems are designed with little or no consideration for possible changes in the structure of the system process to be controlled. In classic control design, a monolithic approach is taken where structural changes in the system process require the development of a new mathematical model of the system and a subsequent redesign of the control system. This process can be expensive and time consuming. Therefore, an attractive alternative is to design the control system such that it automatically reconfigures whenever structural changes occur. This is the aim of the *Plug & Play Process Control* research program, which the work presented here is a part of.

An industrial case study involving a large-scale hydraulic network with non-linear dynamics is studied. The hydraulic network underlies a district heating system, which provides heating water to a number of end-users in a city district. The case study considers a novel approach to the design of district heating systems in which the diameter of the pipes used in the system is reduced in order to reduce the heat losses in the system, thereby making it profitable to provide district heating to areas with low energy demands. The new structure has the additional benefit that structural changes such as the addition or removal of end-users are easily implementable. In this work, the problem of controlling the pressure drop at the end-users to a constant reference value is considered. This is done by the use of pumps located both at the end-users and at designated places across the network.

The control architecture which is used consists of a set of decentralized linear control actions. The control actions use only the measurements obtained locally at each end-user. Both proportional and proportional-integral control actions are considered. Some of the work considers control actions which are constrained to non-negative values only. This is due to the fact that the actuators in this type of system typically consist of centrifugal pumps which are only able to deliver non-negative actuation. Other parts of the work consider control actions which have been quantized. That is, they are restricted to piecewise constant signals taking value in a bounded set. This is done in order to facilitate sending the control signals across a finite bandwidth communication network. This is necessary since the actuators in the system are geographically separated from the logic circuitry implementing the control actions.

The results presented here consist of a series of global stability results of the closed-loop system using the control actions described above. The stability analysis is complicated by the non-linearities present in the system process. Specifically, global practical output regulation can be shown when using proportional control actions, while global asymptotical output regulation can be shown when using proportional-integral control actions. Since the results are global in the state space, it is concluded that the closed-loop system maintains its stability properties when structural changes are implemented.

Synopsis

Kontrol systemer bliver typisk designet med få eller ingen hensyn til mulige strukturelle ændringer i system processen der skal reguleres. I klassisk kontrol design anvendes en monolitisk tilgang, hvor strukturelle ændringer i system processen kræver udvikling af en ny matematisk model af systemet med efterfølgende re-design af kontrol systemet. Denne proces kan være bekostelig og tidskrævende. Et attraktivt alternativ er derfor at designe kontrol systemet således at det automatisk re-konfigurerer når strukturelle ændringer forekommer. Dette er målsætningen for *Plug & Play Process Control* forskningsprojektet, som dette værk er en del af.

Arbejdet omhandler et case study fra industrien, som involverer et stor-skala hydraulisk netværk med ulineær dynamik. Det hydrauliske netværk udgør et fjernvarmesystem, som forsyner et antal slutbrugere i et bydistrikt med varmt vand. Der tages udgangspunkt i en ny tilgang til design af fjernvarmesystemer, hvor diameteren af de rør der anvendes i systemet reduceres for at reducere varmetabene i systemet, og derved gøre det rentabelt at tilbyde fjernvarme i områder med lavt energibehov. Det nye design har ydermere den fordel at strukturelle ændringer i systemet, såsom tilføjelse eller fjernelse af slutbrugere, er nemme at implementere. Dette værk omhandler kontrol opgaven i systemet, som er at regulere trykfaldet hos slutbrugerne til en konstant reference. Til dette anvendes pumper placeret både hos slutbrugerne og udvalgte steder i netværket.

Den anvendte kontrol arkitektur består af et sæt af decentraliserede lineære kontrol virkninger. Kontrol virkningerne gør kun brug af signaler, som er målt lokalt hos den enkelte slutbruger. Både proportionale og proportional-integrale kontrol virkninger bliver undersøgt. Dele af værket omhandler kontrol virkninger som er begrænset til ikke-negative værdier. Dette skyldes at aktuatorerne i denne type systemer typisk består af centrifugal pumper, som kun er i stand til at levere ikke-negativ aktivering. Andre dele af værket omhandler kvantiserede kontrol virkninger, hvilket vil sige kontrol virkninger som er begrænset til at antage stykvis konstante værdier fra en begrænset mængde. Dette er for at muliggøre afsendelsen af kontrol signalerne over et kommunikationsnetværk med begrænset båndbredde. Dette er nødvendigt da aktuatorerne i systemet er geografisk adskilt fra elektronikken, som beregner kontrol signalerne.

Værkets resultater består af en række globale stabilitets resultater fra lukket-sløjfe systemet med de føromtalte kontrol virkninger. Stabilitets analysen kompliceres af de ulineariteter som er til stede i system processen. Mere specifikt kan global praktisk output regulering bevises når der anvendes proportional kontrol, mens global asymptotisk output regulering kan bevises når der anvendes proportional-integral kontrol. Da resultaterne er globale i tilstandsrummet, kan det konkluderes at lukket-sløjfe systemet beholder sine stabilitetsegenskaber når strukturelle ændringer bliver implementeret.

| Nomenclature

$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of positive real numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Z}_+$	The set of positive integers
$\mathbb{R}^n$	The n-dimensional Euclidean space
x_i	The ith component of the vector x
$\langle x, y \rangle$	The scalar product between vectors x and y
$|a|$	The absolute value of the scalar a
$||x||$	The Euclidean norm of the vector x
$B_r(x)$	The open ball of radius r centred in x, that is $B_r(x) = \{y \in \mathbb{R}^n \mid ||y - x|| < r\}$
$M(n, m; \mathbb{R})$	The set of n-by-m matrices with real entries, also $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$
A^T	The transpose of the matrix A
A_{ij}	The entry in the ith row and jth column of the matrix A
$A > 0$	The matrix A is positive definite, that is $x^T A x > 0$ for every $x \neq 0$
X^c	The complement of the set X
$X \subset Y$	The set X is a proper subset of the set Y
$X \times Y$	The Cartesian product between the sets X and Y
$d(X, Y)$	The Hausdorff metric between the sets X and Y
$\dfrac{d}{dt}x = \dot{x}$	The time derivative of variable x
$\rightarrow$	Mapping from a domain into a range, but also "tends to"
C^1	The set of continuously differentiable functions, also a map $f(\cdot)$ will be said to be C^1 if $f(\cdot) \in C^1$
$\nabla f(\cdot)$	The gradient of the function $f(\cdot)$
$Df(\cdot)$	The Jacobian matrix of the map $f(\cdot)$

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be monotonically increasing if it is natural order preserving, *i.e.*, for all a and b such that $a < b$ then $f(a) < f(b)$.
A continuous map $f : X \to Y$ is said to be:
an *injection* if it is into, *i.e.*, for every $x, y \in X$, if $f(x) = f(y)$ then $x = y$
a *surjection* if it is onto, *i.e.*, if for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = y$
a *bijection* if it is both an *injection* and a *surjection*
a *homeomorphism* if it is a *bijection* with a continuous inverse f^{-1}
proper if the inverse image of a compact set is compact
monotonically increasing if $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^n$ and $\langle x - y, f(x) - f(y) \rangle > 0$.

1 | Introduction

The work presented here regards stability analysis of a feedback control system, in which a set of linear decentralized event-based controllers are used for output regulation of a large-scale non-linear system process.

The system process represents an industrial case study which involves a novel paradigm for the design of district heating systems (see [Kallesøe, 2007, Bruus et al., 2004]). The new paradigm is motivated by the assessment that a reduction in the diameter of the pipes used in the system can lead to a reduction in the heat loss in the system of up to 50 %, thereby making it profitable to offer district heating to areas with low demand.

Furthermore, by introducing a multi-pump architecture, the structure of the district heating system becomes more flexible as end-users can be added to or removed from the system online. The system will be described in detail later.

The case study has been proposed by one of the industrial partners in the research program *Plug & Play Process Control* [Stoustrup, 2009, Stoustrup, 2006]. The research program focus on a novel concept for process control where the control system automatically reconfigures when an intelligent sensor or actuator is added to or removed from the system.

1.1 Motivation

Powerful tools exist to design feedback control for a system with known structure, especially for linear systems where [Franklin et al., 2002] and [Franklin et al., 1998] comes to mind. However, they come in short if the structure of the system to be controlled for some reason is required to change over time. Depending on the complexity and nature of the system this might require the mathematical model describing the system behaviour to be changed and a new feedback controller to be designed.

Typically, control systems are designed without much care for possible future changes in the structure of the system being controlled. Changes in system structure alter the way that the closed-loop system performs regarding its control task and can result in suboptimal or unwanted behaviour. An off-the-shelf solution to the control problem would be to construct a new model of the system, which can be time consuming. Another way is to let the structure of the controller change automatically whenever changes in the system are detected. The latter is the aim of the *Plug and Play Process Control* research program. The *Plug and Play Process Control* research program aims at providing general theories for designing and analysing the stability of feedback controllers for systems with varying structure. For instance, take the following example of such a system, taken from the web

page of the research program [Stoustrup, 2006]:

> *"Imagine a farmer observing some region in his stable, where the pigs are not comfortable. He plugs a new intelli-sensor in a vacant socket in that part of the stable. The stable ventilation system automatically registers the new component and in response reconfigures itself in order to stabilize the indoor climate in the proximity of this sensor, leading to animal comfort and increased productivity."*

Five companies participate in the *Plug and Play Process Control* research program; Danfoss, Grundfos, Skov, DONG Energy and FLSmidth Automation, each providing one or more case studies. The work is divided into a number of work packages, the contents of which will be described briefly in the following.

WP1: **Integration of hardware, networks, and protocols for flexible control systems.** This work package deals with the communication network needed for a plug and play control system. This includes reconfiguration of the communication topology whenever new devices such as sensors or actuators are introduced to the system. Literature on the work from the work package includes [Meybodi et al., 2011b], [Meybodi et al., 2011a] and [Meybodi et al., 2012].

WP2: **Correlation based sensor/actuator awareness.** This work package deals with identifying the nature of a newly attached component, whether it be a sensor or an actuator. That is, given a new component identify the system state/variables it affects/measures and update the system model accordingly. Both black and white box models are considered depending on the situation. Literature from the work package includes [Knudsen, 2009a], [Knudsen, 2009b], [Knudsen and Trangbæk, 2008], [Bendtsen et al., 2008] and [Knudsen et al., 2012].

WP3: **Structurally based reconfiguration.** The work package deals with automatically reconfiguring an existing controller whenever structural changes, such as the addition/removal of sensors or actuators, are introduced in the system being controlled. Literature on the work from this work package includes [Stoustrup et al., 2009], [Trangbæk, 2009], [Trangbæk et al., 2009], [Trangbæk et al., 2008], [Trangbæk and Bendtsen, 2009], [Trangbæk and Bendtsen, 2010], [Bendtsen et al., 2011], [Trangbæk, 2010b] and [Trangbæk, 2010a].

WP4: **Model-based control performance optimization through flexible sensor/actuator configuration.** This work package deals with model based control and performance of the control when introducing structural changes. When new components are introduced, the control algorithms are changed to achieve optimal performance. Literature from the work package includes [Michelsen et al., 2008], [Michelsen et al., 2009], [Michelsen and Trangbæk, 2009] and [Michelsen and Stoustrup, 2010].

WP5: **Survivability and performance measures.** This work package deals with the evaluation of the available sensors/actuators with the aim of achieving the optimal performance of the system. Literature from the work package includes [Kragelund, 2010], [Kragelund et al., 2008], [Kragelund et al., 2010b], [Kragelund

et al., 2009a], [Kragelund et al., 2011], [Kragelund et al., 2009b], [Kragelund et al., 2010d], [Kragelund et al., 2010a] and [Kragelund et al., 2010c].

WP6: Decentralized event-based networked non-linear control for Plug-and-Play Process Control. This work package deals with decentralized and event-based control of large-scale systems subject to structural changes such as the addition/removal of sensors or actuators. Literature from the work package includes [DePersis and Kallesøe, 2008], [DePersis and Kallesøe, 2009a], [DePersis and Kallesøe, 2009b] and [DePersis and Kallesøe, 2011]. Furthermore, the work presented here is a part of the work package.

Since this work is focused on one of the case studies in the *Plug and Play Process Control* research program, the motivation for the case study will be introduced in the following.

As previously mentioned the case study involves a new paradigm for the design of district heating systems. Traditionally, district heating systems are designed to have few pump stations, with the hydraulic dynamics between pump stations decoupled using heat exchangers. This has the advantage that it is easy to maintain and supervise pumps in the system, and control is easy since the dynamics are decoupled. However, since there are few pumps in the system, pipes with large diameter, and thus small pressure gradients, are needed. Furthermore, the structure of the overall system is inflexible and designing a system which can handle expansions can be expensive [Kallesøe, 2007]. An example of the network structure for a traditional district heating system is illustrated in Fig. 1.1. As it is evident from the figure, pumps are separated by heat exchangers.
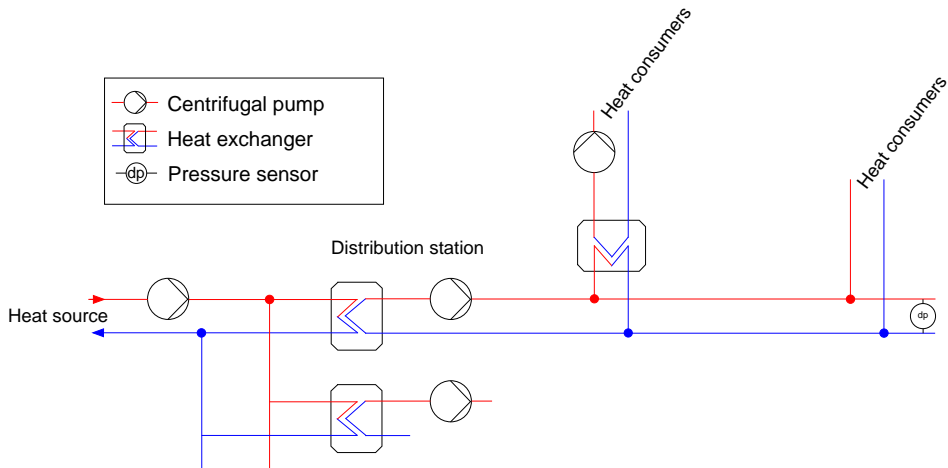
Figure 1.1: Example of the structure of a traditional district heating system [Kallesøe, 2007].

On the other hand, by reducing the diameter of the pipes used in the system, the heat losses, due to heat dispersion from the pipes, can be reduced. However, the pressure gradients of the pipes are increased with the risk of violating pressure constraints of the pipes. This issue can be overcome by placing so-called pressure boosting pumps along the pipeline. This has the additional benefit that the structure of the system becomes more

flexible in the sense that end-users can easily be added to or removed from the system [Kallesøe, 2007]. The added flexibility calls for a control architecture which is able to handle structural changes in the system while the system is kept online. An example of the network structure for the novel district heating system paradigm is illustrated in Fig. 1.2. As can be seen in the figure multiple pumps can be found on the same pipeline.
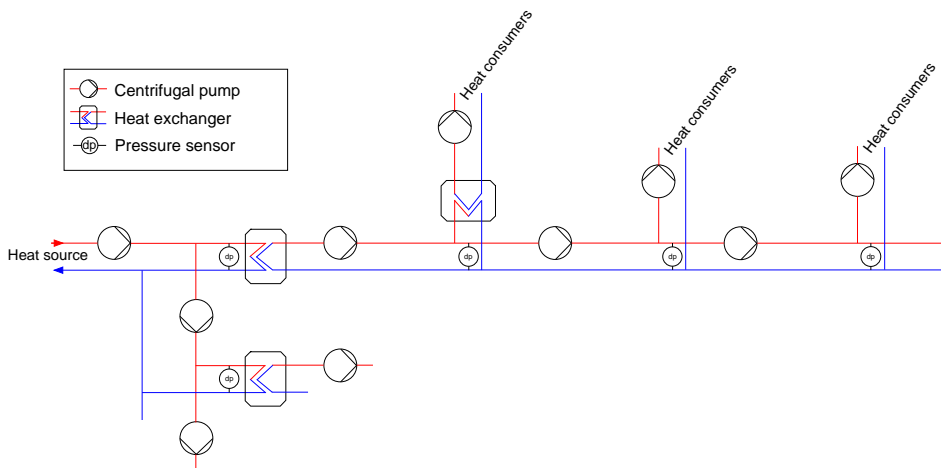
Figure 1.2: Example of the structure of a district heating system in the novel design paradigm [Kallesøe, 2007].

1.2 State of the Art and Background

Work on control problems involving fluid flow networks can roughly be separated into two categories; works involving open networks with no cycles and works involving closed networks with cycles.

Examples of open networks include irrigation networks as considered in [Cantoni et al., 2007]. Here the problem of minimizing distribution losses due to oversupply is considered. Another example is considered in [Polycarpou et al., 2002] and [Wang et al., 2006], where the problem of controlling the water quality in drinking water distribution networks using disinfectants is considered. A final example is [Marinaki, 1999] and [Wan and Lemmon, 2007] where flow control in sewer networks is considered. Common for these networks is the presence of capacitive elements which is not present in the district heating system. Furthermore, the district heating system constitutes a closed network.

Examples of closed networks include mine ventilation networks which are considered in [Hu et al., 2003]. Here non-linear model based feedback control of the air quality in mines is considered. This work is extended in [Koroleva et al., 2006], where decentralized feedback control of more general fluid flow networks is considered. The dynamics of these networks are closely related to the dynamics of the district heating system.

However, in the case of the district heating system, it is desired to use a set of simple decentralized linear control actions for the purpose of output regulation. Results on the problem of feedback control of the district heating system considered here have appeared

in [DePersis and Kallesøe, 2008], [DePersis and Kallesøe, 2009a] and [DePersis and Kallesøe, 2009b]. These results have been collected in the recent paper [DePersis and Kallesøe, 2011], in which also the mathematical model, which describes the behaviour of the system, can be found. Before outlining the results obtained in these works, the model along with the control problem will be stated.

The system under consideration is a hydraulic network comprising a district heating system. Figure 1.3 illustrates a small district heating system with two apartment buildings which constitutes the end-users. Figure 1.4 shows the underlying hydraulic network diagram.
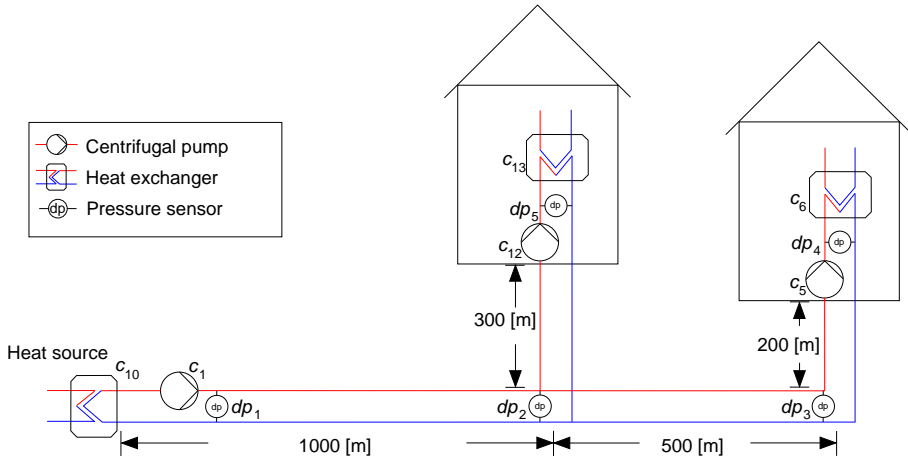
Figure 1.3: A sketch of a small district heating system.
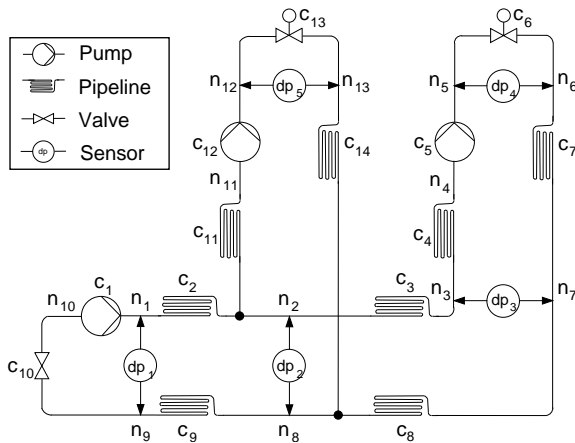
Figure 1.4: The hydraulic network diagram.

In the following the mathematical model of the hydraulic network will be described along with the presentation of the output regulation problem and the proposed strategy

for control and dealing with the structural changes which may occur in the system.

System Model

The hydraulic network consists of a number of connections between two-terminal components, which are: valves, pipes and pumps. The kth system component is characterized by dual variables, the first of which is the pressure drop Δh_k across it

$$\Delta h_k = h_i - h_j, \tag{1.1}$$

where i, j are nodes in the network; h_i, h_j are the relative pressures at the nodes.

The second variable characterizing the component is the fluid flow q_k through it. The components have algebraic or dynamic expressions governing the relationships between the two variables. The stability analysis presented in this thesis relies on the system model derived in [DePersis and Kallesøe, 2011], which reposes on the assumption that the fluid in the system is incompressible and that pipe diameter is constant along a pipe. For additional details on the modelling of the system, the interested reader is referred to [DePersis and Kallesøe, 2011].

Valves

The behaviour of valves in the network is governed by the following algebraic expression

$$h_i - h_j = \mu_k(q_k) \equiv \mu_k(v_k, q_k), \tag{1.2}$$

where v_k is the hydraulic resistance of the valve; $\mu_k(\cdot)$ is a C^1 and proper function, which for any fixed value of v_k is zero at $q_k = 0$, monotonically increasing and $\mu_k(v_k, \cdot) = 0$ for $v_k = 0$.

Pipes

The behaviour of pipes in the network is governed by the dynamic equation

$$\mathcal{J}_k \dot{q}_k = (h_i - h_j) - \lambda_k(q_k) \tag{1.3}$$

where $\lambda_k(q_k) \equiv \lambda(p_k, q_k)$; $\mathcal{J}_k$ and p_k are parameters representing mass inertia of the fluid in the pipe and friction in the pipe respectively; $\lambda_k(\cdot)$ is a function with the same properties as $\mu_k(\cdot)$.

Pumps

A (typically centrifugal) pump is a component which delivers a desired pressure difference Δh_k regardless of the value of the fluid flow through it. Thus, the behaviour of pumps in the network is governed by the following expression

$$h_i - h_j = -\Delta h_{p,k}, \tag{1.4}$$

where $\Delta h_{p,k}$ is a non-negative control input.

Component Model

A generalized component model can be derived using the following expression

$$\Delta h_k = \mathcal{J}_k \dot{q}_k + \lambda_k(q_k) + \mu_k(q_k) - \Delta h_{p,k}, \tag{1.5}$$

where $\mathcal{J}_k, p_k$ are non-zero for pipe components and zero for other components; v_k is non-zero for valve components and zero for other components; $\Delta h_{p,k}$ is non-zero for pump components and zero for other components.

The values of the parameters p_k and v_k are typically unknown, but they will be assumed to take values in a compact set of non-negative values. Likewise, the functions $\mu_k(q_k)$ and $\lambda_k(q_k)$ are not precisely known, only their properties of being C^1, monotone, zero in $q_k = 0$ and proper are guaranteed. The varying heating demand of the end-users, which is the main source of disturbances in the system, is modelled by a (end-user) valve with variable hydraulic resistance. In the network model, a distinction is to be made between end-user valves and the rest of the valves in the network. Two types of pumps are present in the network; the end-user pumps, which are mainly used to meet the demand at the end-users, and booster pumps which are used to meet constraints on the relative pressures in the network [DePersis and Kallesøe, 2009b].

Network Model

The network model has been derived using standard circuit theory, see e.g. [Desoer and Khu, 1969] or [Brayton and Moser, 1964a, Brayton and Moser, 1964b]. The hydraulic network consists of m components and n end-users ($m > n$). The network is associated with a graph $\mathcal{G}$ which has nodes coinciding with the terminals of the network components. The edges of the network are the components themselves. The graph satisfies the following:

Assumption 1. *[DePersis and Kallesøe, 2011] The graph $\mathcal{G}$ is connected.*

By the use of graph theory, a set of n independent flow variables q_i have been identified. These flow variables have the property that their values can be set independently from other flows in the network. The independent flow variables coincide with the flows through the chords[1] of the graph [DePersis and Kallesøe, 2009a]. To each chord in the graph, a fundamental (flow) loop is associated, and along this loop Kirchhoffs voltage law holds. This means that the following equality applies

$$B\Delta h = 0, \tag{1.6}$$

where $B \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; Δh is a vector consisting of the pressure drops across the components in the network.

The entries of the fundamental loop matrix B are $-1, 1$ or 0, depending on the network topology. Here $B_{ij} = 1$ if the jth component belongs to the ith fundamental flow loop and flow directions agree, $B_{ij} = -1$ if the jth component belongs to the ith fundamental flow loop and flow directions disagree and $B_{ij} = 0$ if the jth component does not

---

[1]Let $\mathcal{T}$ denote the spanning tree of $\mathcal{G}$, i.e. a connected subgraph which contains all nodes of $\mathcal{G}$ but no cycles. Then the edges of $\mathcal{G}$ which are not included in $\mathcal{T}$ are the chords of $\mathcal{G}$ (see [Desoer and Khu, 1969]).

belong to the ith fundamental flow loop. For the case study in question, the hydraulic network underlies a district heating system. Because of the latter, the following statements can be made regarding the network.

Assumption 2. *[DePersis and Kallesøe, 2011] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 1.5. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 3. *[DePersis and Kallesøe, 2011] There exists one and only one component called the heat source. It corresponds to a valve[2] of the network, and it lies in all the fundamental loops.*
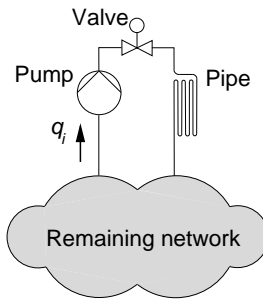
Figure 1.5: The series connection associated with each end-user [DePersis and Kallesøe, 2009a].

Proposition 1. *[DePersis and Kallesøe, 2011] Any hydraulic network satisfying Assumptions 1 and 2 admits the representation*

$$J\dot{q} = f(B^T q) + u \tag{1.7}$$

$$y_i = \mu_i(q_i) \, , \; i = 1, \ldots, n, \tag{1.8}$$

where $q \in \mathbb{R}^n$ is the vector of independent flows; $u \in \mathbb{R}^n$ is a vector of independent inputs consisting of a linear combination of the delivered pump pressures; y_i is the measured pressure drop across the ith end-user valve; $J \in M(n; \mathbb{R})$, $J > 0$; $f(\cdot)$ is a C^1 map; $\mu_i(\cdot)$ is the fundamental law of the ith end-user valve. In (1.8), it is assumed that the first n components coincide with the end-user valves.

Under Assumptions 1-3, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix B are equal to 1 or 0, where B_{ij} is 1 if component j belongs to fundamental flow loop i and 0 otherwise.

Defining the vector of flows through the components in the system as $x = B^T q \in \mathbb{R}^m$, the map $f(\cdot)$ can be written as [DePersis and Kallesøe, 2009a]

$$f(x) = -B(\lambda(x) + \mu(x)), \, \forall x \in \mathbb{R}^m, \tag{1.9}$$

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[2]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

where $\lambda(x) = [\lambda_1(x_1), \ldots, \lambda_m(x_m)]^T$; $\mu(x) = [\mu_1(x_1), \ldots, \mu_m(x_m)]^T$ and $\lambda_i(\cdot)$ is non-zero for pipe components and $\mu_i(\cdot)$ is non-zero for valve components.

The matrix J in (1.7) is given by

$$J = B\mathcal{J}B^T \tag{1.10}$$

where $\mathcal{J} = \text{diag}(\mathcal{J}_1, \ldots, \mathcal{J}_m)$.

Let $\Delta h_e \in \mathbb{R}^n$ and $\Delta h_b \in \mathbb{R}^o$ denote the vectors of pressures delivered by the end-user pumps and boosting pumps respectively. Then the input u in (1.7) can be written as

$$u = \Delta h_e + F\Delta h_b \tag{1.11}$$

$$= \begin{bmatrix} I_n & F \end{bmatrix} \begin{pmatrix} \Delta h_e \\ \Delta h_b \end{pmatrix} \tag{1.12}$$

$$= \bar{B} \begin{pmatrix} \Delta h_e \\ \Delta h_b \end{pmatrix} \tag{1.13}$$

where $F \in M(n, o; \mathbb{R})$ consisting of 1,0 is the sub-matrix of B mapping boosting pumps to the fundamental flow loops. That is, $F_{ij} \neq 0$ if and only if Δh_{bj} is present in the ith fundamental flow loop. Since $o \neq 0$, it is evident from (1.11) and (1.7) that the system is over actuated.

Now, the purpose of the control can be defined as follows.

Definition 1. *Output regulation problem:* Given a vector r of reference values, where $r \in \mathcal{R} = \{x \in \mathbb{R}^n \mid 0 < r_m \leq x_i \leq r_M\}$, and $\varepsilon > 0$ design control signals $u_i(t)$ such that $\lim_{t \to \infty} |y_i - r_i| < \varepsilon$.

The district heating system is subject to changes in the structure of the network. Examples of actions which will result in changes in the network structure is the addition or removal of an end-user in the system or pumps being decommissioned due to failures. The type of changes which are considered in the work presented here are the former. To ease the handling of structural changes in the hydraulic network a control architecture which consists of a set of decentralized proportional control actions have been proposed in [DePersis and Kallesøe, 2011]. These control actions are given as

$$u_i = -N_i(y_i - r_i), \tag{1.14}$$

where $N_i > 0$ and $i = 1, 2, \ldots, n$.

The control actions have also been extended to provide integral action as follows [DePersis et al., 2011]

$$\begin{aligned} \dot{\xi}_i &= -K_i(y_i - r_i) \\ u_i &= \xi_i - N_i(y_i - r_i) \end{aligned} \tag{1.15}$$

where $K_i > 0$.

This architecture has the benefit that the control signal for the individual fundamental flow loop uses information from only said flow loop. Since individual end-users can be associated with individual fundamental flow loops, this means that whenever an end-user

is taken into or out of commission the corresponding control signal can immediately be taken into or out of commission.

In [DePersis and Kallesøe, 2008] a simple system limited to two end-users are considered. The main result in [DePersis and Kallesøe, 2008] shows that when using the proportional control actions semi-global practical output regulation is achievable under time varying demand from the end-users. In [DePersis and Kallesøe, 2009a] the general network model which has been repeated in this section is derived. The paper also provides a proof of semi-global practical output regulation when using the proportional control actions and constraining the control actions to non-negative values only. The paper [DePersis and Kallesøe, 2009b] extends the result from [DePersis and Kallesøe, 2009a] to showing semi-global output regulation when using non-negative constrained binary control actions. Lastly, the result in [DePersis and Kallesøe, 2009b] is extended in [DePersis et al., 2010] to show semi-global practical output regulation when using non-negative constrained and quantized proportional control actions. An elaboration of the result in [DePersis et al., 2010] is given in Chapter 2 to provide a comparison between the approach used to show the semi-global results and the approach used here to provide global results. All of the results mentioned here have been collected in the resent paper [DePersis and Kallesøe, 2011].

Since the results described above are semi-global no guarantees about the stability of the closed-loop system can be given when end-users are added to or removed from the system. This is because the initial conditions of the newly obtained system are not guaranteed to belong to the compact attractor set.

Dealing with Structural Changes

Since the new paradigm for the design of district heating networks provide the possibility of having systems with varying network structure, it is necessary to examine the stability properties of the closed-loop system, when it undergoes changes in the network structure.

To assure that problems with instability of the closed-loop system, whenever structural changes are implemented does not arise, the strategy in the work presented here is to show that the closed-loop system is inherently robust towards this type of changes. Specifically, if global stability of the closed-loop system can be shown to hold for an arbitrary number of end-users in the system, then the system will be robustly stable with respect to the structural changes mentioned above. The contribution of the papers written in the duration of the PhD project is a number of results which show global stability properties of the closed-loop system using the proposed feedback control actions. In some cases the control actions have been extended to being quantized or non-negatively constrained or both. This will be emphasized in Chapter 2.

By (1.11) it is evident that multiple pump pressure inputs contribute to the 'virtual' input u_i. This means that 1) a strategy for distributing the control signal u_i to the pumps should be developed and 2) information regarding the control signal u_i needs to be communicated across the network. Regarding 1), the papers A and G described in Chapter 2 provide suggestions to such a strategy. Regarding 2), the information on the structure of the network needed for knowing which pumps to communicate the signal u_i to can be assumed to be known before taking end-user i into commission. The papers A, C and E described in Chapter 2 consider event-based control signals which are considered eligible for being communicated across a finite bandwidth communication network.

The introduced control actions constitutes a passive system. Furthermore, the behaviour of the pipe and valve components in the hydraulic network is governed by passive functions, thus the hydraulic network is a passive system. Stability theorems for the negative feedback interconnection of passive systems can be found in [Khalil, 2002, van der Schaft, 1999, Isidori, 1999] among others. However, as it is derived in the following, the passive output of the hydraulic network is given by the system state q and not the actual output y. That is, to apply a traditional global stability result relying on said theorems, one has to assume that the state (fundamental flows) are measured and the reference is given as a vector of desired fundamental flows. In the following, the passivity properties of the hydraulic network will be derived along with a study of the closed loop stability properties based on these properties. This study is not documented in the papers, but has been used as a starting point for some of the analysis subsequently carried out and documented in the papers.

Passivity of Hydraulic Networks

A block diagram of the closed-loop system is shown in Fig. 1.6. In the block diagram, the block representing the hydraulic network has been split into two subsystems. The first subsystem H_1 represents the model from the input vector u to the flow vector q. The second subsystem $h(\cdot)$ represents the output map, which maps the flow vector q to the vector of measured outputs $y = h(q)$. Furthermore, a block N representing the proportional control actions is present, where N is a diagonal matrix with positive entries.
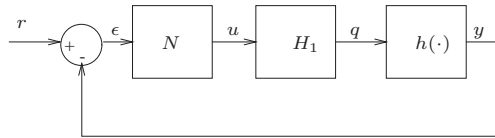
Figure 1.6: Feedback connection of system with proportional control actions with gain matrix N.

If $r = 0$, the system in Fig. 1.6 is equivalent to the system illustrated in Fig. 1.7. In the following it will be shown that it is equivalent to the feedback interconnection of a strictly passive system with a passive memoryless system. Thereby rendering the origin globally asymptotically stable. Later, analysis for non-zero r will be done.
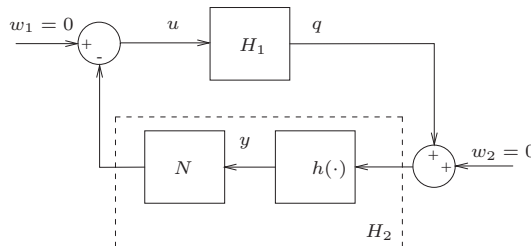
Figure 1.7: This system is equivalent to the one illustrated in Fig. 1.6 with $r = 0$.

First, the passivity properties of the subsystem H_1 is considered, where H_1 is given by

$$H_1 : \begin{cases} J\dot{q} = f(B^T q) + u \\ y_{H_1} = q \end{cases} \tag{1.16}$$

The power P_{in} which is supplied externally to the system can be calculated as in [Khalil, 2002]

$$P_{in} = u^T y \tag{1.17}$$

Integrating the power supplied to the system over time, an expression of the energy supplied to the system E_{in} can be obtained

$$E_{in}(t) = \int_0^t u^T(s)y(s)ds \tag{1.18}$$

In order for the system to be passive, the energy absorbed in the network over any period of time is required to be greater than or equal to the energy stored in the network over the same period of time, which corresponds to

$$\int_0^t u^T(s)y(s)ds \geq V(q(t)) - V(q(0)) \tag{1.19}$$

where $V(q)$ is an energy storage function for the system.

The inequality in (1.19) must hold for every $t \geq 0$, which corresponds to the instantaneous power inequality must hold for all t

$$u^T(t)y(t) \geq \dot{V}(q) \tag{1.20}$$

The energy storage function for the system is chosen as the following

$$V(q) = \frac{1}{2}q^T J q \Rightarrow \tag{1.21}$$

$$\dot{V}(q) = q^T J\dot{q} \tag{1.22}$$

Multiplying the system state equation in (1.16) from the left by q^T gives the following

$$q^T J\dot{q} = q^T f(B^T q) + q^T u \Leftrightarrow \tag{1.23}$$

$$q^T u = q^T J\dot{q} - q^T f(B^T q) \tag{1.24}$$

Using (1.9) gives the following

$$u^T q = \dot{V}(q) + q^T B \left(\lambda(B^T q) + \mu(B^T q)\right) \tag{1.25}$$

Both the maps $\lambda(\cdot)$ and $\mu(\cdot)$ consist of smooth, monotonic increasing functions $\lambda_i(\cdot)$ and $\mu_i(\cdot)$ which are zero in $x_i = 0$ [DePersis and Kallesøe, 2009a], because of this the following applies

$$u^T q \geq \dot{V}(q) \tag{1.26}$$

which shows that the system H_1 is passive.

Specifically, since the following inequality holds

$$u^T q \geq \dot{V}(q) + \psi(q) \tag{1.27}$$

and $\psi(q) > 0$ for every $q \neq 0$, the system H_1 is strictly passive [Khalil, 2002].

Next, take the system H_2 in Fig. 1.7. Since the matrix N is diagonal with positive entries and the output functions $h_i(q_i) = \mu_i(q_i)$ are monotonically increasing and zero in $q_i = 0$ it follows

$$N_i q_i h_i(q_i) > 0 \;,\; \forall q_i \neq 0 \Rightarrow \tag{1.28}$$

$$\sum_{i=1}^{n} N_i q_i h_i(q_i) > 0 \;,\; \forall q \neq 0 \Rightarrow \tag{1.29}$$

$$q^T N h(q) > 0 \;,\; \forall q \neq 0. \tag{1.30}$$

From this it is concluded that the system H_2 is passive memoryless. This shows that $q = 0$ is the globally asymptotically stable equilibrium point of the closed-loop system, since the energy storage function $V(q)$ is radially unbounded, see *Theorem 6.4* in [Khalil, 2002].

Passivity of Incremental Model

The result derived in the previous section states that the system can be asymptotically stabilized towards the origin. Since this case is not of interest, it is examined if the system can be stabilized towards an arbitrary point in the state space.

Take a general non-linear passive system of the form

$$\dot{x} = F(x) + Gu \tag{1.31}$$

$$y = H(x) \tag{1.32}$$

Comparing to the system equations in (1.16) it is seen that

$$F(q) = J^{-1} f(B^T q) \tag{1.33}$$

$$G = J^{-1} \tag{1.34}$$

$$H(q) = Iq \tag{1.35}$$

The incremental model which describes the system around a desired equilibrium point x^* is given by the following set of equations

$$\dot{x} = F(x) + Gu^* + G\tilde{u} \tag{1.36}$$

$$\tilde{y} = H(x) - H(x^*) \tag{1.37}$$

where $(\tilde{\cdot}) = (\cdot) - (\cdot)^*$ are the incremental variables [Jayawardhana et al., 2007].

The constant input and output vectors (u^* resp. y^*) associated with the desired equilibrium state x^* are in general defined as

$$u^* := -G^{\dagger} F(x^*) \tag{1.38}$$

$$y^* := H(x^*) \tag{1.39}$$

where $G^\dagger = (G^T G)^{-1} G^T$ is the pseudo inverse of the matrix G given that G has full column rank [Jayawardhana et al., 2007].

A block diagram of the incremental feedback interconnected system is given in Fig. 1.8. Here $\tilde{H}_i$ denotes the incremental version of the system H_i for $i = 1, 2$.
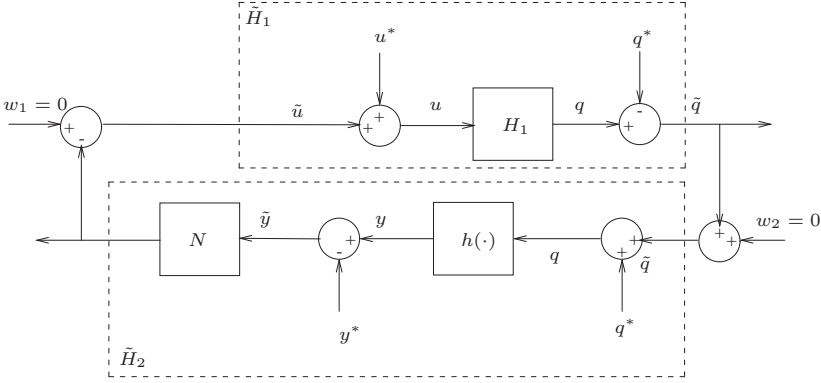
Figure 1.8: Block diagram of the incremental feedback system.

As in the previous subsection, the stability analysis of the closed loop system will be done by deriving the passivity properties of the incremental systems $\tilde{H}_1$ and $\tilde{H}_2$.

First, the passivity properties of the incremental system $\tilde{H}_1$ are examined. In the system H_1 the matrix $G = J^{-1}$ has the inverse, which is J, therefore u^* is given as

$$u^* = -JJ^{-1}f(B^T q^*) \Leftrightarrow \tag{1.40}$$

$$u^* = -f(B^T q^*) \tag{1.41}$$

If the system satisfies the property

$$[F(x) - F(x^*)]^T [\nabla V(x) - \nabla V(x^*)] \le 0 \tag{1.42}$$

where $\nabla V(x)$ is the gradient of the storage function as a column vector, then the incremental model of the system is passive with the energy storage function $V_0(x)$ [Jayawardhana et al., 2007]

$$V_0(x) = V(x) - x^T \nabla V(x^*) - \left[V(x^*) - (x^*)^T \nabla V(x^*) \right] \tag{1.43}$$

Now, set $x = B^T q \in \mathbb{R}^m$ and consider the functions $\lambda_i(x_i)$, which has the properties that they are monotonic increasing and zero in $x_i = 0$.

Because of these properties, the following applies

$$- [\lambda_i(x_i) - \lambda_i(x_i^*)] (x_i - x_i^*) < 0 , \ \forall x_i \ne x_i^* \Rightarrow \tag{1.44}$$

$$- \sum_{i=1}^m [\lambda_i(x_i) - \lambda(x_i^*)] (x_i - x_i^*) < 0 , \ \forall x \ne x^* \Rightarrow \tag{1.45}$$

$$- [\lambda(x) - \lambda(x^*)]^T (x - x^*) < 0 , \ \forall x \ne x^* \tag{1.46}$$

The map $\mu(\cdot)$ has the same properties as $\lambda(\cdot)$, i.e. it consists component-wise of monotonic increasing functions which are zero for $x_i = 0$. Furthermore, using the identity in (1.9) one can see that the following applies

$$\left[f(B^T q) - f(B^T q^*) \right]^T (q - q^*) < 0 \ , \ \forall q \neq q^* \tag{1.47}$$

Multiplying this expression with I_n in between the terms gives

$$\left[f(B^T q) - f(B^T q^*) \right]^T I_n (q - q^*) < 0 \ , \ \forall q \neq q^* \tag{1.48}$$

which in turn can be rewritten to

$$\left[J^{-1} f(B^T q) - J^{-1} f(B^T q^*) \right]^T (J^T q - J^T q^*) < 0 \ , \ \forall q \neq q^* \tag{1.49}$$

The expression $J^{-1} f(B^T q)$ corresponds to $F(x)$ in the general system model given in (1.31), and the expression $J^T q$ corresponds to $\nabla V(x)$ in (1.42). Thus it is shown that the incremental system model of the subsystem H_1 is passive. Let $\tilde{q} = q - q^*$, then it can be verified that the expression in (1.43) corresponds to

$$V_0(q) = \frac{1}{2} \tilde{q}^T J \tilde{q} \tag{1.50}$$

for the system H_1.

It can furthermore be verified that the incremental model of H_1 is strictly passive with respect to the storage function $V_0(q)$. To this end, take the time derivative of $V_0(q)$, which is given as

$$\dot{V}_0(q) = \tilde{q}^T J \dot{q} \tag{1.51}$$

$$= \tilde{q}^T f(B^T q) + \tilde{q}^T u^* + \tilde{q}^T \tilde{u} \tag{1.52}$$

$$= \tilde{q}^T \left(f(B^T q) - f(B^T q^*) \right) + \tilde{q}^T \tilde{u} \tag{1.53}$$

$$= -\tilde{q}^T B \left(\lambda(B^T q) + \mu(B^T q) - \lambda(B^T q^*) - \mu(B^T q^*) \right) + \tilde{q}^T \tilde{u} \tag{1.54}$$

Again, the maps $\lambda(\cdot)$ and $\mu(\cdot)$ consist of monotonic increasing functions which are zero for $x = 0$, thus the following inequality is fulfilled

$$(q - q^*)^T B \left(\lambda(B^T q) + \mu(B^T q) - \lambda(B^T q^*) - \mu(B^T q^*) \right) > 0 \ , \ \forall q \neq q^* \tag{1.55}$$

and strict passivity of the incremental system $\tilde{H}_1$ follows.

Now, the passivity properties of the incremental system $\tilde{H}_2$ are examined. Again, it is recalled that the functions $h_i(q_i) = \mu_i(q_i)$ are monotonically increasing and zero in $q_i = 0$. Since, N is diagonal with positive entries it follows that

$$N_i(q_i - q_i^*)(h_i(q_i) - h_i(q_i^*)) > 0 \ , \ \forall q_i \neq q_i^* \Rightarrow \tag{1.56}$$

$$\sum_{i=1}^n N_i(q_i - q_i^*)(h_i(q_i) - h_i(q_i^*)) > 0 \ , \ \forall q \neq q^* \Rightarrow \tag{1.57}$$

$$(q - q^*)^T N(h(q) - h(q^*)) > 0 \ , \ \forall q \neq q^* \tag{1.58}$$

which shows that $\tilde{H}_2$ is passive memoryless. Again, since $V_0(q)$ is radially unbounded, $\tilde{H}_1$ is strictly passive and $\tilde{H}_2$ is passive memoryless, the closed-loop system is globally asymptotically stable with $q = q^*$ as the equilibrium point. Although the result derived in the above shows that the closed-loop system in Fig. 1.8 is globally asymptotically stable some issues still remain, which will be illustrated in the following.

Practical output regulation

Since the functions $\mu_i(\cdot)$ are monotonically increasing and proper they admit global inverses $\mu_i^{-1}(\cdot)$. Now, let r be the vector of reference values and let $\hat{q}$ be the vector defined by

$$\hat{q}_i = \mu_i^{-1}(r_i) \ , \ i = 1, 2, \ldots, n. \tag{1.59}$$

Referring to Fig. 1.8, set $q^* = \hat{q}$ and $y^* = \mu(\hat{q}) = r$, now what should u^* be in order to render $\hat{q}$ the global asymptotically stable equilibrium point? The answer comes from the steady state expression of (1.7)

$$0 = f(B^T\hat{q}) + u^* \Leftrightarrow \tag{1.60}$$

$$u^* = -f(B^T\hat{q}). \tag{1.61}$$

However, since an exact expression for the steady state input $-f(B^T\hat{q})$ is generally unknown, it is in general impossible to achieve asymptotic output regulation using only proportional control actions, as would be expected.

Instead, consider the map $F : \mathbb{R}^n \to \mathbb{R}^n$

$$F(z) = \mu(z) - N^{-1}f(B^Tz). \tag{1.62}$$

If $F(\cdot)$ is surjective onto the set $\mathcal{R}$ of possible reference values, then for every vector $r \in \mathcal{R}$, there exists a vector q' such that

$$r = \mu(q') - N^{-1}f(B^Tq'). \tag{1.63}$$

This in turn means that the block diagram in Fig. 1.8 with $q^* = q'$, $y^* = r$ and $u^* = 0$ is equivalent to the same block diagram just with $q^* = q'$, $y^* = \mu(q')$ and $u^* = -f(B^Tq')$, which shows that $q = q'$ is the global asymptotically stable equilibrium point of the closed-loop system, and furthermore

$$r_i - y_i' = -\frac{1}{N_i}f_i(B^Tq') \tag{1.64}$$

where $y' = \mu(q')$.

What now remains to be shown is that $F(\cdot)$ in fact is surjective onto $\mathcal{R}$, which is the starting point for the analysis carried out in Paper B (see Chapter 2). The result of Paper B is that for functions $\mu_k(\cdot)$ and $\lambda_k(\cdot)$ with certain properties, the map $F(\cdot)$ is a global homeomorphism, and thus surjective onto $\mathcal{R}$.

Asymptotic Output Regulation

Additional analysis based on passive systems theory can also be used to show stability of the desired equilibrium point of the closed-loop system when using the proportional-integral controllers in (1.15). To that end, consider the feedback interconnection system in Fig. 1.7, but now let the block H_1 denote the system $\dot{u} \to \dot{q}$, that is

$$H_1 : \begin{cases} J\ddot{q} = Df(B^Tq)\dot{q} + \dot{u} \\ y_{H_1} = \dot{q} \end{cases} \tag{1.65}$$

where $Df(B^T q)$ denotes the Jacobian of $f(B^T q)$ with respect to q.

Define the storage function $V_1(\dot{q})$ as

$$V_1(\dot{q}) = \frac{1}{2}\dot{q}^T J \dot{q} \tag{1.66}$$

then the time derivative of $V_1(\dot{q})$ is

$$\dot{V}_1(\dot{q}) = \dot{q}^T J \ddot{q}. \tag{1.67}$$

From (1.65) the following is true

$$\dot{V}_1(\dot{q}) = \dot{q}^T Df(B^T q)\dot{q} + \dot{q}^T \dot{u}. \tag{1.68}$$

If it is assumed that the derivatives of the functions $\lambda_k(\cdot)$ are bounded away from zero[3], it can be shown that the matrix $-Df(B^T q)$ is positive definite for any q (see [DePersis et al., 2011]), and thus it follows that

$$\dot{q}^T \dot{u} \geq \dot{V}_1(\dot{q}) + \psi(\dot{q}) \tag{1.69}$$

where $\psi(\cdot)$ is some positive definite function. Since $V_1(\cdot)$ is radially unbounded it follows that the system H_1 is strictly passive.

Likewise, let the system H_2 in the system denote the system $\dot{q} \to \dot{u}$ where (by (1.15))

$$-\dot{u} = -\dot{\xi} + ND\mu(q)\dot{q} \tag{1.70}$$
$$= K(\mu(q) - r) + ND\mu(q)\dot{q} \tag{1.71}$$

where $D\mu(q)$ denote the Jacobian of $\mu(q)$ with respect to q.

Again, let $\hat{q} = \mu^{-1}(r)$, define the change of coordinates $\tilde{q} = q - \hat{q}$, and let the map $\tilde{\mu} : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$\tilde{\mu}(\tilde{q}) = \mu(\tilde{q} + \hat{q}) - \mu(\hat{q}) \tag{1.72}$$
$$= \mu(q) - \mu(\hat{q}) \tag{1.73}$$
$$= y - r. \tag{1.74}$$

By the properties of $\mu(\cdot)$ it follows that $\tilde{\mu}(\tilde{q})$ is monotonically increasing and zero in $\tilde{q} = 0$.

Define the storage function $V_2(\tilde{q})$ as

$$V_2(\tilde{q}) = \sum_{i=1}^{n} K_i \int_0^{\tilde{q}_i} \tilde{\mu}_i(s)ds \tag{1.75}$$

which is positive definite and radially unbounded by the properties of $\tilde{\mu}(\cdot)$.

Then the time derivative of $V_2(\tilde{q})$ is

$$\dot{V}_2(\tilde{q}) = \dot{q}^T K \tilde{\mu}(\tilde{q}). \tag{1.76}$$

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[3]This assumption is motivated by the fact that for small values, the flow through the pipes can be considered laminar [Roberson and Crowe, 1993]

Furthermore,

$$-\dot{q}^T \dot{u} = \dot{q}^T K \tilde{\mu}(\tilde{q}) + \dot{q}^T N D \mu(q) \dot{q} \tag{1.77}$$

$$= \dot{V}_2(\tilde{q}) + \dot{q}^T N D \mu(q) \dot{q}. \tag{1.78}$$

Since the functions $\mu_i(q_i)$ are monotonically increasing and N is a diagonal matrix with positive entries it follows that the matrix $N D \mu(q)$ is positive semi-definite from which it follows that the system H_2 is input feed-forward passive, see [Khalil, 2002].

Since, $\dot{q} = 0$ is a strict minimum of $V_1(\dot{q})$ and $\tilde{q} = 0$ is a strict minimum of $V_2(\tilde{q})$ it follows by Proposition A.10 in [Ortega et al., 1998] that $(\dot{q}, \tilde{q}) = (0, 0)$ is a stable equilibrium of the feedback interconnection system.

The analysis above is the starting point of Paper F, where global asymptotic output regulation is shown using similar arguments. However, the proof is done by showing that the second order dynamics of the closed-loop system is similar to an Euler-Lagrange mechanical system with Rayleigh dissipation.

Euler-Lagrange Systems

The motion of a mechanical system can be described by the Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}} \mathcal{L}(q, \dot{q}) \right) - \frac{\partial}{\partial q} \mathcal{L}(q, \dot{q}) = Q \tag{1.79}$$

where $q \in \mathbb{R}^n$ is a vector of generalized coordinates; $\dot{q} \in \mathbb{R}^n$ is the corresponding vector of generalized velocities; $Q \in \mathbb{R}^n$ is a vector of external forces acting on the system; $\mathcal{L} : \mathbb{R}^{2n} \to \mathbb{R}$ is the Lagrangian function given by

$$\mathcal{L}(q, \dot{q}) = \mathcal{T}(q, \dot{q}) - \mathcal{V}(q) \tag{1.80}$$

where $\mathcal{T} : \mathbb{R}^{2n} \to \mathbb{R}$ is the kinetic energy function and $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ is the potential energy function. In the specific case considered here, the only forces acting on the system are the dissipative forces, and as a consequence

$$Q = -\frac{\partial}{\partial \dot{q}} \mathcal{F}(\dot{q}) \tag{1.81}$$

where $\mathcal{F}(\dot{q})$ is the Rayleigh dissipation, which satisfies

$$\dot{q}^T \frac{\partial}{\partial \dot{q}} \mathcal{F}(\dot{q}) \geq 0. \tag{1.82}$$

Furthermore, the system is said to be fully-damped if the Rayleigh dissipation function further satisfies

$$\dot{q}^T \frac{\partial}{\partial \dot{q}} \mathcal{F}(\dot{q}) \geq \sum_{i=1}^{n} \alpha_i \dot{q}_i^2. \tag{1.83}$$

with $\alpha_i > 0$ for $i = 1, 2, \ldots, n$.

The Hamiltonian function $\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R}$ is defined as

$$\mathcal{H}(q, \dot{q}) = \left(\frac{\partial}{\partial \dot{q}} \mathcal{L}(q, \dot{q}) \right)^T \dot{q} - \mathcal{L}(q, \dot{q}). \tag{1.84}$$

In a standard mechanical system, the kinetic energy $\mathcal{T}(\cdot)$ is of the form

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}, \tag{1.85}$$

and as a consequence, the Hamiltonian function is the sum of the kinetic and potential energy functions

$$\mathcal{H}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) + \mathcal{V}(q). \tag{1.86}$$

The time derivative of the Hamiltonian function is given as

$$\frac{d}{dt}\mathcal{H}(q,\dot{q}) = \frac{d}{dt}\left(\left(\frac{\partial}{\partial\dot{q}}\mathcal{L}(q,\dot{q})\right)^T\dot{q} - \mathcal{L}(q,\dot{q})\right) \tag{1.87}$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}\mathcal{L}(q,\dot{q})\right)^T\dot{q} + \left(\frac{\partial}{\partial\dot{q}}\mathcal{L}(q,\dot{q})\right)^T\ddot{q} +$$
$$- \left(\left(\frac{\partial}{\partial\dot{q}}\mathcal{L}(q,\dot{q})\right)^T\ddot{q} + \left(\frac{\partial}{\partial q}\mathcal{L}(q,\dot{q})\right)^T\dot{q}\right) \tag{1.88}$$

$$= \left(\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}\mathcal{L}(q,\dot{q})\right) - \frac{\partial}{\partial q}\mathcal{L}(q,\dot{q})\right)^T\dot{q} \tag{1.89}$$

$$= Q^T\dot{q} \tag{1.90}$$

$$= -\left(\frac{\partial}{\partial\dot{q}}\mathcal{F}(\dot{q})\right)^T\dot{q} \leq -\sum_{i=1}^{n}\alpha_i\dot{q}_i^2. \tag{1.91}$$

If additionally $\mathcal{V}(\cdot)$ has a strict minimum at some point, say $q_0 \in \mathbb{R}^n$, then $\mathcal{H}(\cdot)$ will attain a strict minimum at $(q_0, 0)$. Then, by (1.91) and the LaSalle invariance principle, $(q_0, 0)$ is the global asymptotically stable equilibrium point of the system.

How this applies to the hydraulic network will be elaborated in the following chapter.

1.3 Outline of the Thesis

This thesis is written as a collection of the papers, which have been produced during the course of the PhD project. With the state-of-the-art and background now covered, the remainder of the thesis will proceed as follows. The next chapter contains an overview of the content of the papers. Following this, Chapter 3 will provide a conclusion on the project and give some suggestions to issues which are interesting to address in the future. Lastly, the remainder of the thesis consists of the papers themselves. As such, some repetition of introductory sections should be expected.

2 | Summary of contributions

This chapter presents a summary of the contributions made during the course of the project. The contributions can be divided into two categories. The first category are results describing the stability properties of the closed-loop system when using only proportional control actions in the system, this category is presented in Section 2.1. The second category are results describing the stability properties of the closed-loop system when both proportional and integral control actions are used, this category is presented in Section 2.2.

Generally, the control structure is the one illustrated in Fig. 2.1. Here the block C represents the controller which provides either proportional or proportional-integral control actions. For both the proportional and the proportional-integral control actions the control architecture is completely decentralized in the sense that the control action for each fundamental flow loop is using information from said flow loop only.
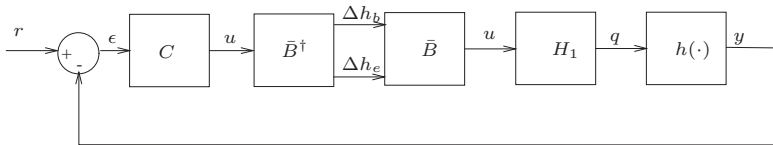
Figure 2.1: General structure of the closed-loop system considered in the papers.

Most of the results only consider the part of the control which involves the generation of the 'virtual' control signal (u in Fig. 2.1). A simple way of mapping the virtual input u to the actual input pressure vectors Δh_b and Δh_e, which are the vectors of pressures delivered by the booster pumps and end-user pumps respectively, would be to simply use the Moore-Penrose pseudo-inverse of $\bar{B}$. However, papers A and G consider mappings with different prudent properties.

Furthermore, papers A, C, and E also considers the closed-loop system with a quantization of the control signals. This is illustrated in Figure 2.2. Here the block Q constitutes the quantizer. The quantized version of the control signals are piecewise constant signals taking value in a finite set. This allows them to be transmitted across a communication network of finite bandwidth. This is necessary since the actuators in the system are geographically separated from the logic circuitry implementing the control actions.

For the closed-loop system with the quantized control signals, the dynamics will be described by discontinuous equations. For these systems, the solutions will be considered in the form of Krasovskii solutions to discontinuous differential equations.
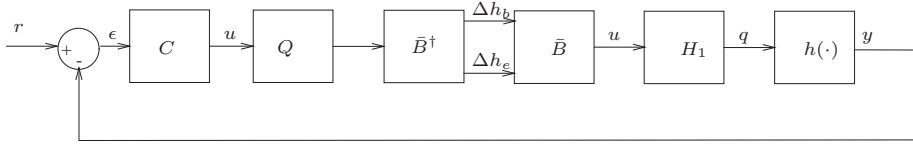
Figure 2.2: Structure of the system with quantized control signals.

Definition 2. [Hájek, 1979] A map $\varphi : I \to \mathbb{R}^n$ is a Krasovskii solution of an autonomous system of ordinary differential equations $\dot{x} = G(x)$, where $G : \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \in I$ it satisfies the differential inclusion $\dot{\varphi}(t) \in KG(\varphi(t))$, where $KG(x) = \bigcap_{\eta > 0} \overline{\text{co}}\, G(B_\eta(x))$ and $\overline{\text{co}}\, G$ is the convex closure of the set G.

Here I is an interval of real numbers, possibly unbounded. By definition, the operators K associates to $G(x)$ a set valued map which is compact for every $x \in \mathbb{R}^n$. Furthermore, if $G(x)$ is locally bounded this set valued map is upper semi-continuous with convex values. Then, for each initial state x_0, there exists at least one Krasovskii solution of $\dot{x} = G(x)$ [Aubin and Cellina, 1984].

In the following, proofs or proof strategies for some of the results documented in the papers will be given. These are provided to make the chapter self contained. For the interested reader, the full versions of the proofs are found in the contributions part of the thesis, which contains the full papers.

2.1 Practical output regulation in hydraulic networks

This section presents the main results of the papers on the stability properties of the closed-loop system when proportional output feedback control is used in the system.

Paper A: [DePersis et al., 2010]

The result presented in Paper A shows that the closed-loop system with proportional control actions constrained to non-negative values and with logarithmic quantization, can provide semi-global practical output regulation. That is, for any compact set of initial conditions of the system, there exist gains of the proportional controller and parameters of the quantizer such that the basin of attraction contain the initial conditions and the attractor set can be designed as an arbitrarily small neighborhood of the desired steady state. It should be mentioned that the author of the thesis have not contributed to the stability result, and that it is merely included here to illustrate the difference in the approaches used in this result and the subsequent results on global stability. The contribution from the author of the thesis to the paper will be stated immediately after the result on stability.

The control signals considered in the paper are the following

$$u_i = \psi(\tilde{u}_i), \tag{2.1}$$

where the map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\psi(x) = \begin{cases} \psi_i , & \frac{\psi_i}{1+\delta} < x \leq \frac{\psi_i}{1-\delta} , \ 0 \leq i \leq j \\ 0 , & 0 \leq x \leq \frac{\psi_j}{1+\delta} \end{cases} \qquad (2.2)$$

and

$$\tilde{u}_i = \begin{cases} -N_i(y_i - r_i) , & y_i - r_i \leq 0 \\ 0 , & y_i - r_i \geq 0 \end{cases} \qquad (2.3)$$

where $j \in \mathbb{Z}_+$, $\delta \in (0,1)$, $\psi_i = \psi_{i-1} \frac{1-\delta}{1+\delta}$ for every $i = 1, 2, \ldots, j$ and $\psi_0 > 0$ are parameters of the (logarithmic) quantizer, with j, δ and ψ_0 to be designed. The parameter $N_i > 0$ is the proportional controller gain.

This gives the following expression for the closed-loop system

$$J\dot{q} = f(B^T q) + \Psi(\tilde{u}) \qquad (2.4)$$

where $\Psi(\tilde{u}) = (\psi(\tilde{u}_1), \ldots, \psi(\tilde{u}_n))$. The right hand side of (2.4) is discontinuous.

The Krasovskii solutions to (2.4) are absolutely continuous functions satisfying the differential inclusion

$$J\dot{q} \in f(B^T q) + K\Psi(\tilde{u}) \qquad (2.5)$$

where $K\Psi(\tilde{u}) \subseteq \times_{i=1}^{n} K\psi(\tilde{u}_i)$ with

$$K\psi(\tilde{u}_i) \subseteq \begin{cases} \{(1 + \lambda\delta)\tilde{u}_i , \ \lambda \in [-1,1]\} , & \frac{\psi_j}{1+\delta} < \tilde{u}_i \leq \frac{\psi_0}{1-\delta} \\ \{\lambda(1 + \delta)\tilde{u}_i , \ \lambda \in [0,1]\} , & 0 \leq \tilde{u}_i \leq \frac{\psi_j}{1+\delta} \end{cases} . \qquad (2.6)$$

Then the main result of Paper A is

Proposition 2. *[DePersis et al., 2010] For any choice of the parameter $q_M > 0$, any compact set $\mathcal{R} \subset \mathbb{R}_+$, any compact set $\mathcal{Q}$ of initial conditions described by*

$$\mathcal{Q} = \{q \in \mathbb{R}^n \mid |q_i| \leq q_M , \ i = 1, \cdots, n\} , \qquad (2.7)$$

for any arbitrarily small positive number γ, and for any value of the quantization parameter $\delta \in (0,1)$ there exist gains $N_i^* > 0$ and parameters ψ_0, j of the quantizer such that for all $N_i > N_i^*$, for any $r \in \mathcal{R}$, any Krasovskii solution $q(t)$ of the closed-loop system (2.4), with initial condition in $\mathcal{Q}$ is attracted by the set $\{\epsilon \in \mathbb{R}^n \mid |\epsilon_i| \leq \gamma , \ i = 1, 2, \ldots, n\}$, where $\epsilon_i = y_i - r_i$.

Proof of Proposition 2. The proof is somewhat technical, so only the strategy of the proof will be given here. For the full version of the proof see [DePersis and Kallesøe, 2011].

Since the functions $\mu_i(\cdot)$ are monotonically increasing and proper they admit global inverses $\mu_i^{-1}(\cdot)$. Then, the desired equilibrium point is

$$\hat{q} = \mu^{-1}(r) \qquad (2.8)$$

where $\mu^{-1}(r) = (\mu_1^{-1}(r_1), \ldots, \mu_n^{-1}(r_n))$.

Define the error coordinates e as $e = q - \hat{q}$ and the positive definite Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(e) = \frac{1}{2}e^T J e \qquad (2.9)$$

with the time derivative

$$\dot{V}(e) = e^T J \dot{e} \tag{2.10}$$

$$= e^T J \dot{q} \tag{2.11}$$

$$= e^T f(B^T q) + e^T \nu \ , \ \forall \nu \in K\Psi(\tilde{u}). \tag{2.12}$$

Then a compact set S is constructed where $S = \{e \in \mathbb{R}^n \mid \varrho \leq V(e) \leq \sigma\}, 0 < \varrho < \sigma$ and $\mathcal{Q} \subseteq \{q \in \mathbb{R}^n \mid V(e)|_{e=q-\hat{q}} \leq \sigma\}$. The remainder of the proof consists of showing that there exists parameters ψ_0 and j of the quantizer and N_i^* of the proportional controller such that for all $N_i \geq N_i^*$, $\dot{V}(e) < 0$ for every $e \in S$ and $\nu \in K\Psi(\tilde{u})$. □

Additional to the main result in Proposition 2, the paper also provides a suggestion to a graph based approach to solve the problem of distributing the control signal u_i to the multiple pumps contributing to it, which is the problem of designing the map $\bar{B}^\dagger$ in Fig. 2.2. The designed map has the property that if the components of u are non-negative, then the components of the vectors Δh_b and Δh_e are non-negative as well. Furthermore, the mapping defines a graph which in turn can be used to define the communication topology which should be used to communicate control signals across the network. The derivation of this mapping constitutes the contribution to Paper A from the author of the thesis.

Paper B: [Jensen and Wisniewski, 2011b]

It is found that the closed-loop system with proportional control and no actuator constraints is globally practically stable. While the focus of the paper is on the application of the hydraulic network, the result presented in Paper B, can be extended to a general class of systems.

Consider the following system

$$A\dot{x} = f(x) + u \tag{2.13}$$

$$y_i = h_i(x_i) \tag{2.14}$$

where $x \in \mathbb{R}^n$, $A \in M(n; \mathbb{R})$ with $A > 0$, the map $-f(\cdot)$ is continuous, monotonically increasing and proper, the function $h_i(\cdot)$ is continuous, monotonically increasing and proper and given a positive definite diagonal matrix N, the map $F : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$F(z) = h(z) - N^{-1}f(z) \tag{2.15}$$

is proper.

Let $r \in \mathbb{R}^n$ be a vector of reference values, and let

$$u = -N(y - r), \tag{2.16}$$

then the following result is true.

Proposition 3. *There exists a unique point $x^* \in \mathbb{R}^n$ which is the globally asymptotically stable equilibrium point of the closed-loop system (2.13), (2.14) and (2.16). Furthermore,*

$$h(x^*) - r = N^{-1}f(x^*). \tag{2.17}$$

Proof of Proposition 3. The proof is done in two steps, which follows along the lines of the proofs of Proposition 11 and Proposition 12 in Paper B on page 61. The first step is to prove that the map $F(\cdot)$ is a global homeomorphism. This can be done following the technique used to prove Proposition 11 in Paper B. Since $F(\cdot)$ is a global homeomorphism it follows that for every vector $r \in \mathbb{R}^n$ of reference values, there exists a unique vector $x^* \in \mathbb{R}^n$ such that

$$r = h(x^*) - N^{-1}f(x^*) \tag{2.18}$$

which in turn means that the closed-loop system can be written as

$$A\dot{x} = f(x) - f(x^*) - N(y - y^*) \tag{2.19}$$

where $y^* = h(x^*)$.

Secondly, it can be proved that x^* is a globally asymptotically stable equilibrium point of the closed-loop system using the technique in the proof of Proposition 12. Specifically, using the Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$

$$V(x) = \frac{1}{2}(x - x^*)^T A(x - x^*), \tag{2.20}$$

along with the monotonicity properties of $f(\cdot)$ and $h(\cdot)$ the thesis follows. $\qquad\square$

Since the result is global and independent on the number n of end-users it is concluded that it is possible to add or remove end-users from the system and still have a global asymptotically stable equilibrium point x^* of the newly obtained system. However, to keep the same level of performance it may be necessary to tune the gains N_i.

Paper C: [Jensen and Wisniewski, 2011c]

The result in Paper C is an extension of the result in Paper B and partly of the result in Paper A. The proportional control actions are used and a quantization of the control signals is introduced. The result shows that the trajectories of closed-loop system are bounded and globally asymptotically stable towards a compact set of the state space.

As in Paper A, the control signals considered in the paper are the following

$$u_i = \psi(\tilde{u}_i), \tag{2.21}$$

with the slight modifications

$$\tilde{u}_i = -N_i(y_i - r_i) \tag{2.22}$$

and

$$\psi(-x) = -\psi(x), \tag{2.23}$$

which means that the control signals are not constrained to non-negative values.

Again, the closed-loop system is given as

$$J\dot{q} = f(B^T q) + \Psi(\tilde{u}). \tag{2.24}$$

The solutions are again considered in the sense of Krasovskii solutions to the differential inclusion

$$J\dot{q} \in f(B^T q) + K\Psi(\tilde{u}), \tag{2.25}$$

where $K\Psi(\tilde{u}) \subseteq \times_{i=1}^{n} K\psi(\tilde{u}_i)$ with the set valued map K modified to

$$K\psi(\tilde{u}_i) \subseteq \begin{cases} \psi_0 \,, & \tilde{u}_i > \frac{\psi_0}{1-\delta} \\ \psi_k \,, & \frac{\psi_k}{1+\delta} < \tilde{u}_i < \frac{\psi_k}{1-\delta} \,, \ k = 0, \ldots, j \\ \{\frac{1-\lambda\delta}{1+\lambda\delta}\psi_k \,, \ \lambda \in [0,1]\} \,, & \tilde{u}_i = \frac{\psi_k}{1-\delta} \,, \ k = 0, \ldots, j \\ 0 \,, & 0 \leq \tilde{u}_i < \frac{\psi_j}{1+\delta} \\ -K\psi(-\tilde{u}_i) \,, & \tilde{u}_i \leq 0 \end{cases} \,, \quad (2.26)$$

that is, $K\psi(\cdot)$ is only set-valued at points of discontinuity of $\psi(\cdot)$.

The main result of Paper C can then be summarized in the following proposition

Proposition 4. *For any gain $N_i > 0$ and for any value $j \in \mathbb{Z}_+$ of the quantizer, there exist parameter ψ_0 of the quantizer and a compact set $\mathcal{Q}$, with the property that the Krasovskii solutions $q(t)$ of the closed-loop system (2.24) are attracted to $\mathcal{Q}$.*

Proof of Proposition 4. The proof is somewhat technical, so only the strategy of the proof will be given here. For the full proof, the interested reader is referred to [Jensen and Wisniewski, 2011c].

Using the facts that N_i and r_i in (2.22) are constants, (2.24) can be rewritten to

$$J\dot{q} = f(B^T q) - N(Y(y) - r) \tag{2.27}$$

where the following identities has been used

$$Y(y) = (\Upsilon_1(y_1), \ldots, \Upsilon_n(y_n)) \tag{2.28}$$

and

$$\Upsilon_i(y_i) = -\frac{\psi(\tilde{u}_i)}{N_i} + r_i. \tag{2.29}$$

Then (2.25) can be rewritten as

$$J\dot{q} \in f(B^T q) - N(KY(y) - r) \tag{2.30}$$

where $KY(\cdot)$ can be defined in a manner similar to the definition of $K\psi(\cdot)$.

Recalling the map $F : \mathbb{R}^n \to \mathbb{R}^n$

$$F(z) = \mu(z) - N^{-1}f(B^T z), \tag{2.31}$$

which is a global homeomorphism, the differential inclusion (2.30) can be rewritten as

$$J\dot{q} \in f(B^T q) - f(B^T q^*) - N(KY(y) - y^*) \tag{2.32}$$

where $q^* = F^{-1}(r)$ and $y^* = \mu(q^*)$.

Now, the quantizer is designed such that $-\psi_0 < u_i^* < \psi_0$ where

$$u_i^* = -N_i(y_i^* - r_i) \tag{2.33}$$

which means that when the output $y_i = y_i^*$ the input to the quantizer does not go beyond the maximum or below the minimum output of the quantizer.

Consider the Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$

$$V(q) = \frac{1}{2}(q - q^*)^T J (q - q^*) \tag{2.34}$$

with the time derivative

$$\dot{V}(q) = (q - q^*)^T J \dot{q} \tag{2.35}$$

$$= (q - q^*)(f(B^T q) - f(B^T q^*) - N(\nu - y^*)) \ , \ \forall \nu \in KY(y). \tag{2.36}$$

Then by using the monotonicity properties of $f(\cdot)$ and $\mu(\cdot)$ it can be shown that there exists a compact set $\mathcal{Q}$ with the property that $\dot{V}(q) < 0$, for every $q \in \mathcal{Q}^c$. $\qquad \square$

By applying a result similar to Proposition 2 from Paper A it is concluded that global practical output regulation is possible. Furthermore, since the result is global and independent on the number n of end-users, these can be added to or removed from the system and the trajectories of the newly obtained system will be bounded. However, to keep the same level of performance it might be necessary to tune the gains N_i.

Paper D: [Jensen and Wisniewski, 2011a]

The result in Paper D is an extension of the result in Paper B. Here the control signals are constrained to non-negative values to take into account the fact that the actuators in the system are typically able to deliver non-negative actuation only.

Just as the result in Paper B the result in Paper D can be generalized to a larger class of systems than the hydraulic network considered in the case study. Again, these systems are described by (2.13) and (2.14). The following control will be used

$$u_i = s(\tilde{u}_i) \tag{2.37}$$

where

$$\tilde{u}_i = -N_i(y_i - r_i) \tag{2.38}$$

$$s(z) \begin{cases} z \ , & z \geq 0 \\ 0 \ , & z \leq 0 \end{cases} . \tag{2.39}$$

Again it is assumed that $F(\cdot)$ in (2.15) is proper.

The main result of Paper D can then be stated as

Proposition 5. *If the equilibrium point $x^* \in \mathbb{R}^n$ which is the globally asymptotically stable equilibrium point of the closed-loop system (2.13), (2.14) and (2.16) fulfils $h_i(x_i^*) < r_i$, then it is the globally asymptotically stable equilibrium point of the closed-loop system (2.13), (2.14) and (2.37). Furthermore,*

$$h(x^*) - r = N^{-1} f(x^*). \tag{2.40}$$

Proof. Since N_i and r_i are constants, the closed-loop system can be written as

$$A\dot{x} = f(x) - N(\bar{S}(y) - r) \tag{2.41}$$

where $\bar{S}(y) = (\bar{s}_1(y_1), \ldots, \bar{s}_n(y_n))$ and

$$\bar{s}_i(z) \begin{cases} z, & z \leq r_i \\ r_i, & z \geq r_i \end{cases}. \tag{2.42}$$

Since $F(\cdot)$ is proper it is a global homeomorphism as shown in [Jensen and Wisniewski, 2011b], and there exists $x^* = F^{-1}(r)$ such that

$$A\dot{x} = f(x) - f(x^*) - N(\bar{S}(y) - y^*) \tag{2.43}$$

where $y^* = h(x^*)$.

The rest of the proof follows along the lines of the proof of Proposition 3 and exploiting that $\bar{s}_i(y_i^*) = y_i^*$ because of the additional assumption $h_i(x_i^*) < r_i$. $\qquad\square$

Again, since the result is global and independent on the number n of end-users in the system, end-users can be added to or removed from the system and the newly obtained system will have a global asymptotically stable equilibrium point x^*. Again, to keep the same level of performance it may be necessary to adjust the gains N_i.

In the specific case of the hydraulic network underlying the district heating system it has yet to be proved that indeed $y_i^* < r_i$ in the general case (arbitrary n). However, a proof for the case $n = 2$ can be found in [Jensen and Wisniewski, 2011a]. Furthermore, that $y_i^* < r_i$ has been supported by simulations and proved for systems with up to four end-users ($n = 4$).

Paper E: [Jensen and Wisniewski, 2011d]

This paper collects the results from papers B-D and can be seen as an extension of the results in [DePersis and Kallesøe, 2011]. The main result of Paper E state that the trajectories of the closed-loop system with quantized proportional control constrained to non-negative values are bounded and globally asymptotically attracted to a compact set of the state space. The quantization map used in the paper describes a general set of monotonically increasing quantization maps with hysteresis. Thus, the logarithmic quantizer (if hysteresis is included) used in paper A and C is included in this set, but also other types such as the uniform quantizer for instance.

The control used in the paper is the following

$$u_i = \psi_m(\tilde{u}_i) \tag{2.44}$$

where

$$\tilde{u}_i = \begin{cases} -N_i(y_i - r_i), & y_i - r_i \leq 0 \\ 0, & y_i - r_i \geq 0 \end{cases} \tag{2.45}$$

and the quantization map $\psi_m(\cdot)$ will be described in the following.

First, for $l \in \mathbb{Z}_+$ let $A = \{A_0, A_1, \ldots, A_l\}$ and $B = \{B_0, B_1, \ldots, B_{l+1}\}$ be the following family of intervals

$$A = \{(-\infty, \alpha_0], (\alpha_0, \alpha_1], \ldots, (\alpha_{l-2}, \alpha_{l-1}], (\alpha_{l-1}, \infty)\} \tag{2.46}$$

$$B = \{(-\infty, \beta_0], (\beta_0, \beta_1], \ldots, (\beta_{l-2}, \beta_{l-1}], (\beta_{l-1}, \beta_l], (\beta_l, \infty)\} \tag{2.47}$$

where l, α_i and β_j for $i = 0, 1, \ldots, l-1$ and $j = 0, 1, \ldots, l$ are design parameters of the quantizer and such that $\beta_i < \alpha_i < \beta_{i+1}$ for $i = 0, 1, \ldots, l-1$. Note that

$$\mathbb{R} = \bigcup_{i=0}^{l} A_i = \bigcup_{j=0}^{l+1} B_j.$$

Let $\psi_m : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi_m(x(t)) = \begin{cases} \psi_k^A \,, & \text{if } t = t_0 \wedge x(t_0) \in A_k \\ \psi_k^A \,, & \begin{aligned} &\text{if } x(t) = \beta_k \wedge \psi_m(x(t^-)) = \psi_{k+1}^B \text{ or} \\ &x(t) = \beta_k \wedge \psi_m(x(t^-)) = \psi_k^B, \, 1 \le k \le l \end{aligned} \\ \psi_k^B \,, & \begin{aligned} &\text{if } x(t) = \alpha_{k-1} \wedge \psi_m(x(t^-)) = \psi_k^A \text{ or} \\ &x(t) = \alpha_{k-1} \wedge \psi_m(x(t^-)) = \psi_{k-1}^A, \, 1 \le k \le l \end{aligned} \\ \psi_0^A \,, & \text{if } x(t) = \beta_0 \wedge \psi_m(x(t^-)) = \psi_1^B \\ \psi_m(x(t^-)) \,, & \text{otherwise} \end{cases}$$

(2.48)

where ψ_k^A and ψ_k^B are design parameters of the quantizer, with $\psi_0^A = 0$ and $\psi_{k-1}^A < \psi_k^B < \psi_k^A$ for all $k = 1, 2, \ldots, l$.

Remark 1: The map $\psi_m(\cdot)$ is defined for piecewise monotone signals $x : [t_0, t] \to \mathbb{R}$. There is a family of k partitions of $[t_0, t]$ denoted $I_1, I_2, \ldots, I_k$ where $I_1 = [t_0, t_1), I_2 = [t_1, t_2), \ldots, I_k = [t_{k-1}, t]$ and $t_i < t_{i+1} < t$ for $i = 0, 1, \ldots, k-2$, such that $x(\tau)$ is monotone for $\tau \in I_j$ for $j = 1, 2, \ldots, k$. Then t^- is defined as $t^- = \tau$ if $\tau \in \text{int}(I_{k-1})$.

This gives the closed-loop system

$$J\dot{q} = f(B^T q) + \Psi_m(\tilde{u}) \tag{2.49}$$

where $\Psi_m(\tilde{u}) = (\psi_m(\tilde{u}_1), \ldots, \psi_m(\tilde{u}_n))$.

The Krasovskii solutions to (2.49) are absolutely continuous functions $q(t)$ which solves the Cauchy problem

$$J\dot{q} \in f(B^T q) + K\Psi_m(\tilde{u}) \,, \quad q(0) = q_0 \tag{2.50}$$

where $K(\Psi(\tilde{u})) \subseteq \times_{i=1}^{n} K(\psi_m(\tilde{u}_i))$ and $K(\psi_m(x))$ is given by

$$K(\psi_m(x)) = \begin{cases} \psi_l^A \,, & x > \beta_l \\ \{\lambda \psi_l^A, \, \lambda \in [0,1]\} \,, & x \in [\beta_0, \beta_l] \\ 0 \,, & x < \beta_0 \end{cases} \tag{2.51}$$

Again, let

$$F(z) = \mu(z) - N^{-1} f(B^T z) \tag{2.52}$$

and

$$q^* = F^{-1}(r), \tag{2.53}$$

with the following conjecture, which has been supported by numerical simulations and proved to hold for systems with up to four end-users ($n = 4$)

Conjecture 1. *The point q^* defined by (2.53) fulfils $\mu_i(q_i^*) < r_i$.*

Given these preliminaries, the following result can be proved

Proposition 6. *For any gain $N_i > 0$ and for any value $l \in \mathbb{Z}_+$ and α_j, β_j, where $j = 0, 1, \ldots, l$, of the quantizer, such that $\beta_j < \alpha_j < \beta_{j+1}$, if the parameter ψ_l^A of the quantizer fulfils $\psi_l^A > -f_i(B^T q^*)$, where q^* is defined by (2.53), then a compact set $\mathcal{Q}$ exists, with the property that the Krasovskii solutions $q(t)$ to the Cauchy problem (2.50) are attracted to $\mathcal{Q}$.*

Remark 2: Conjecture 1 and the fact that $\psi_l^A > -f_i(B^T q^*)$ assures that the input to the quantizer when $y_i = y_i^*$ does not go beyond the maximum or below the minimum output of the quantizer. That is, $0 < u_i^* < \psi_l^A$ where $u_i^* = -N_i(y_i^* - r_i)$ (by (2.52)).

Proof of Proposition 6. Again, the proof is quite technical, so only the strategy of the proof will be given. For the full version of the proof, the interested reader is referred to [Jensen and Wisniewski, 2011d].

Since N_i and r_i are constants a map $\Upsilon_i : \mathbb{R} \to \mathbb{R}$ with the following property exists

$$\Upsilon_i(y_i) = -\frac{\psi_m(\tilde{u}_i)}{N_i} + r_i. \tag{2.54}$$

Using this and the identity in (2.53), the closed-loop system (2.49) can be written as

$$J\dot{q} = f(B^T q) - f(B^T q^*) - N(Y(y) - y^*) \tag{2.55}$$

where $Y(y) = (\Upsilon_1(y_1), \ldots, \Upsilon_n(y_n))$.

The Krasovskii solutions $q(t)$ to the Cauchy problem (2.50) are then the solutions to the problem

$$J\dot{q} \in f(B^T q) - f(B^T q^*) - N(KY(y) - y^*) \tag{2.56}$$

where $KY(y)$ can be defined in a manner similar to $K\Psi_m(\tilde{u})$.

Then the Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$

$$V(q) = (q - q^*)^T J(q - q^*) \tag{2.57}$$

will be used. The function $V(\cdot)$ has the time derivative

$$\dot{V}(q) = (q - q^*)^T J\dot{q} \tag{2.58}$$

$$= (q - q^*)^T (f(B^T q) - f(B^T q^*) - N(\nu - y^*)) , \ \forall \nu \in KY(y). \tag{2.59}$$

Then by using the monotonicity properties of the maps $f(\cdot)$ and $\mu(\cdot)$ it can be shown that there exists a compact set $\mathcal{Q}$ such that $\dot{V}(q) < 0$ for every $q \in \mathcal{Q}^c$. □

By applying Proposition 2 from Paper A it can furthermore be shown that practical output regulation is possible if logarithmic quantizers are used.

2.2 Asymptotic output regulation in hydraulic networks

This section presents the main results of the papers on the stability properties of the closed-loop system when proportional-integral feedback control is used in the system.

Paper F: [DePersis et al., 2011]

The result in Paper F shows that the closed-loop system with the proportional-integral control actions is global asymptotically stable towards the desired reference point if no actuator constraints are assumed. The result is proved by showing that the closed-loop system can be described as an Euler-Lagrange mechanical system. Specifically, it is shown that the second order dynamics of the closed-loop system describes a fully-damped Euler-Lagrange mechanical system with Rayleigh dissipation and no inputs. For literature in these types of systems see for instance [Ortega et al., 1998, van der Schaft, 1999].

The control used in the paper is

$$\dot{\xi}_i = -K_i(y_i - r_i) \tag{2.60}$$

$$u_i = \xi_i - N_i(y_i - r_i) \tag{2.61}$$

which gives the closed-loop system

$$J\dot{q} = f(B^T q) + \xi - N(y - r) \tag{2.62}$$

$$\dot{\xi} = -K(y - r). \tag{2.63}$$

Let $q_i^* = \mu_i^{-1}(r_i)$ and define the transformation of coordinates $\tilde{q}_i = q_i - q_i^*$. Assuming that the derivatives of the functions $\lambda_k(\cdot)$ describing the behaviour of the pipes are bounded away from zero[1], the following result can be proved

Proposition 7. *The point $(\tilde{q}, \dot{q}) = 0$ is a globally asymptotically stable equilibrium point of the closed-loop system given by (2.62) and (2.63).*

Proof of Proposition 7. The strategy of the proof is to show that the second order dynamics of (2.62) describes an Euler-Lagrange mechanical system with Rayleigh dissipation and then use the analysis carried out in the end of Section 1.2.

The second order dynamics of (2.62) is

$$J\ddot{q} = \left(Df(B^T q) - ND\mu(q)\right)\dot{q} + \dot{\xi} \tag{2.64}$$

$$= -G(q)\dot{q} - K(\mu(q) - r) \tag{2.65}$$

where $Dg(x)$ denotes the Jacobian of the map $g(x)$ and $G(q) = -Df(B^T q) + ND\mu(q)$.

Using that the derivatives of the functions $\lambda_k(\cdot)$ are bounded away from zero it can be shown that $G(q)$ is a positive definite matrix, see [DePersis et al., 2011].

Define the function $\tilde{\mu}_i : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{\mu}_i(\tilde{q}_i) = \mu_i(\tilde{q}_i + q_i^*) - \mu_i(q_i^*) \tag{2.66}$$

$$= \mu_i(q_i) - r \tag{2.67}$$

then by using the properties of $\mu_i(\cdot)$ it can be shown that $\tilde{\mu}_i(x)$ is monotonically increasing and zero in $x = 0$.

Now, let the kinetic energy function $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}$ be given as

$$\mathcal{T}(\dot{q}) = \frac{1}{2}\dot{q}^T J\dot{q} \tag{2.68}$$

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[1]This assumption is motivated by the fact that for low values the flow can be considered laminar [Roberson and Crowe, 1993].

and the potential energy function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ be

$$\mathcal{V}(\tilde{q}) = \sum_{i=1}^{n} K_i \int_0^{\tilde{q}_i} \tilde{\mu}_i(s) ds \qquad (2.69)$$

then the thesis follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As an additional result in Paper F it is also shown that the desired equilibrium point of the closed-loop system is semi-global exponentially stable.

Paper G: [Jensen et al., 2011]

This paper represents an extension of the result presented in Paper F, in which the extra degree of freedom coming from the fact that the district heating system is over actuated (see (1.11)), is exploited to expand the controllers introduced in Paper F such that the steady state electrical power consumption of the pumps in the system is minimal.

First, let (q^*, ξ^*) denote the steady state of the closed-loop system (2.62)-(2.63) and consider the change of coordinates

$$\begin{aligned} \tilde{q} &= q - q^* \\ \tilde{\xi} &= \xi - \xi^*. \end{aligned} \qquad (2.70)$$

The control used in the paper is

$$\Delta \dot{h}_{bj} = -L_j \left(\frac{\partial}{\partial \Delta h_{bj}} P(\Delta h_b, \hat{q}, \xi) \right) \qquad (2.71)$$

$$\dot{\xi}_i = -K(y_i - r_i) \qquad (2.72)$$

$$\Delta h_{ei} = \xi_i - N_i(y_i - r_i) - F_i^T \Delta h_b \qquad (2.73)$$

where $L_j > 0$ with $j = 1, 2, \ldots, o$; $P(\Delta h_b, \hat{q}, \xi)$ is a simplified version of the electrical power function of the pumps in the system; $\hat{q}$ is an estimate of q fulfilling $\hat{q} = \alpha q$ with $\alpha > 0$; F_i^T is the ith row of F.

The function $P(\cdot, \hat{q}, \xi)$ is a sum of an bi-linear function and quadratic penalty terms (see [Fletcher, 1975]) designed to make the minimum of $P(\cdot, \hat{q}^*, \xi^*)$ belong to some desired set. The function $P(\cdot, \hat{q}, \xi)$ is convex and radially unbounded. Furthermore, $P(\cdot, \hat{q}^*, \xi^*)$ has a closed and convex set of minimizers. That is, the set

$$\mathcal{M} = \{x \in \mathbb{R}^o \mid P(x, \hat{q}^*, \xi^*) \le P(y, \hat{q}^*, \xi^*)\}, \ \forall y \in \mathbb{R}^o \qquad (2.74)$$

is compact and convex. The penalty functions has a design parameter $\kappa > 0$ and it can be shown that there exists finite $\kappa^* > 0$ such that for all $\kappa > \kappa^*$, $P(\cdot, \hat{q}^*, \xi^*)$ is positive definite.

The closed-loop system is

$$J\dot{q} = f(B^T q) + \xi - N(y - r) \qquad (2.75)$$

$$\dot{\xi} = -K(y - r) \qquad (2.76)$$

$$\Delta \dot{h}_{bj} = -L_j \left(\frac{\partial}{\partial \Delta h_{bj}} P(\Delta h_b, \hat{q}, \xi) \right) \qquad (2.77)$$

The closed-loop system can be seen as an interconnection of two separate systems, where the state of the first system is an external input to the second system. This is illustrated in Fig. 2.3. Comparing with Fig. 2.3, $z = (\tilde{q}, \tilde{\xi})$ and $x = \Delta h_b$.
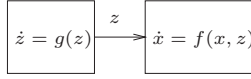
Figure 2.3: Block diagram of the cascaded system.

As shown in Paper F ([DePersis et al., 2011]) the point $\tilde{q} = 0$, $\tilde{\xi} = 0$ is a global asymptotically stable equilibrium of the closed-loop system (2.75)-(2.76), which in turn means that the external input to the second system in Fig. 2.3 decays to zero.

The set $\mathcal{M}$ can be shown to be global asymptotically stable for the system

$$\dot{x} = -L\left(\nabla P(x, \hat{q}^*, \xi^*)\right). \tag{2.78}$$

The main result of the paper relies on the following theorem

Theorem 1. *Consider the system in Fig. 2.3*

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z), \end{aligned} \tag{2.79}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(y, 0) = 0$, $\forall y \in Y$, $g(0) = 0$ and $Y \subset \mathbb{R}^n$ is non-empty, compact and connected and $f(x, z)$, $g(z)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$.

Suppose $Y \subset \mathbb{R}^n$ is a globally asymptotically stable set of $\dot{x} = f(x, 0)$ and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is globally asymptotically stable. Suppose the integral curves of the composite system are defined for all $t \geq 0$ and bounded. Then, the state set $(x, z) \in (Y, 0)$ of (2.79) is globally asymptotically stable.

Proof of Theorem 1. The proof follow along the lines of the proof of Theorem 10.3.1, Corollary 10.3.3 in [Isidori, 1999]. Specifically, $\|x(t)\|$ should be replaced by $d(x(t), Y)$. $\qquad\square$

If $o \leq 2$ it can be shown that the trajectories of the closed-loop system are bounded, and by applying Theorem 1 the main result of the paper can be proved

Theorem 2. *Let $o \leq 2$, $\kappa > \kappa^*$, $\tilde{q} = q - q^*$ and $\tilde{\xi} = \xi - \xi^*$. The state set*

$$M = \{(\Delta h_b, q, \xi) \in \mathbb{R}^o \times \mathbb{R}^n \times \mathbb{R}^n \mid \Delta h_b \in \mathcal{M} \wedge \tilde{q} = \tilde{\xi} = 0\}. \tag{2.80}$$

is globally asymptotically stable for the closed loop system (2.75)-(2.77). In particular

$$\lim_{t \to \infty} d(\zeta(t), M) = 0, \tag{2.81}$$

and

$$\dot{\zeta} = 0, \ \forall \zeta \in M, \tag{2.82}$$

where $\zeta = (\Delta h_b, q, \xi)$.

The theorem above concludes the summary of contributions. In the following chapter, conclusions on the project will be drawn and suggestions to future work will be given.

3 | Conclusion

In this chapter, the main conclusions from the work presented in the previous chapters will be drawn. Following these, some suggestions to possible future research directions will be given.

3.1 Conclusion

The work presented in this thesis considered an industrial case study consisting of a large-scale hydraulic network underlying a district heating system subject to structural changes. The problem of regulating the pressure drop at the end-users using a set of pumps in the system was described, along with a set of decentralized control actions used in solving the problem. The results from this work regards stability properties of the closed-loop system, and can be divided into two main categories; practical output regulation using proportional control actions and asymptotic output regulation using proportional-integral control actions. Since the actuators in the system are constrained to non-negative actuation, parts of the work considered control constrained to non-negative values. Other parts considered quantized control actions because of the need to send these across a finite bandwidth network. Lastly, some of the results considered suggestions to mappings from the control actions to the actuator inputs since the system is over actuated.

The results regarding the proportional control actions were collected in [Jensen and Wisniewski, 2011d], where it was shown that the trajectories of the closed-loop system remains bounded when using constrained and quantized proportional control actions. Furthermore, with high gain control, the output regulation error can be made arbitrarily small. These results are global in the state space and valid for an arbitrary number of end-users. Therefore, it is concluded that end-users can be added to or removed from the system while maintaining the stability properties. Lastly, a suggestion to a mapping from the control actions to the actuator inputs was given in [DePersis et al., 2010]. This mapping has the property that it guarantees that non-negative control actions are mapped to non-negative actuator inputs. Thus, it guarantees that the constraints on the actuators are not violated.

The result in [DePersis et al., 2011] regarding proportional-integral control actions showed that when no positivity constraints on the actuators are assumed, then the desired steady state is global asymptotically stable for the closed-loop system with arbitrary positive control gains. This result was extended in [Jensen et al., 2011], where a dynamic mapping from the control actions in [DePersis et al., 2011] to the actuator inputs was introduced. The purpose of this mapping was to minimize the steady state electrical power

consumption of the actuators in the system. The results showed that for systems with two or less of the so-called boosting pumps, global asymptotic output regulation with minimal power consumption can be proved. Again, since these results are global in the state space and valid for an arbitrary number of end-users, it is concluded that end-users can be added to or removed from the system, while maintaining the closed-loop stability properties.

3.2   Future Work

This section provides some suggestions to future research directions on the system considered in the paper. These suggestions are based both on limitations on some of the presented results and on more general issues which have yet to be addressed.

- Since the control signals needs to be sent over a communication network, stability analysis of the closed-loop system with delays in the communication network will be relevant.

- The results in [Jensen and Wisniewski, 2011d] and [Jensen and Wisniewski, 2011a] rely on the conjecture that $y_i^* < r_i$ for $i = 1, 2, \ldots, n$, which has yet to be proved for an arbitrary number n of end-users.

- The results regarding the proportional-integral control actions will need to be extended to the case of control constrained to non-negative values and quantized control, similar to the way it has been done for the case of proportional control actions, since the actuators in this type of system will typically be restricted to provide non-negative actuation only.

- The result regarding the proportional-integral control actions with the steady state energy minimization scheme holds for networks with two or less boosting pumps. A generalization to an arbitrary number of boosting pumps will be preferable. Furthermore, the result relies on a simplified version of the system power function. Future work could consider a similar result but with a more realistic power function of the system.

- The results presented have been focused on the district heating system, which is a closed network without capacitive elements. An extension to open networks and networks with capacitive elements could be included in future work. Examples of these types of networks include irrigation networks and water supply systems, where reduction of the water losses in these systems could be of interest.

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Contributions

Paper A

Quantized pressure control in large-scale nonlinear hydraulic networks

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Abstract

It was shown previously that semi-global practical pressure regulation at designated points of a large-scale nonlinear hydraulic network is guaranteed by distributed proportional controllers. For a correct implementation of the control laws, each controller, which is located at these designated points and which computes the control law based on local information only (measured pressure drop), is required to transmit the control values to neighbor pumps, i.e. auxiliary pumps which are found along the same fundamental circuit. In this paper we show that quantized controllers can serve well to this purpose. Besides a theoretical analysis of the closed-loop system, we provide experimental results obtained in a laboratory district heating system. This approach is fully compatible with plug-and-play control strategies.

1   Introduction

This work is part of an on-going research on the design of control laws for large-scale non-linear hydraulic networks required to be implementable in a *plug-and-play* fashion, namely to be easily reconfigurable when new sensors, actuators or components are added to the existing control system.

The large-scale hydraulic network underlies a district heating system with an arbitrary number of end-users. The problem consists of regulating the pressure at the end-users to a constant value despite the unknown demands of the users themselves. The regulation problem is addressed for a new generation of district heating systems, where multiple pumps are distributed across the network at the end-users. In these new large-scale heating systems, the diameter of the pipes is decreased in order to reduce heat dispersion. The reduced diameter of the pipes increases the pressure losses which must be compensated by a larger pump effort. The latter can be achieved only with the multi-pump architecture ([1]). Besides the reduced heat losses, having multiple pumps distributed across the network makes it robust to the failure of one or more pumps. However, this issue is not considered in the paper. Moreover, we do not take into account the problem of damping fast pressure transients due to water hammering, as this problem is not to be handled by our controller, but by well-placed passive dampers in the network.

There is a large number of works devoted to large-scale hydraulic networks, and more in particular to water supply systems. A recent paper with an extended bibliography on the modeling and control of hydraulic networks is [2], in which the emphasis is on "open" hydraulic networks, as found in irrigation channels, sewer networks and water distribution systems. Papers which deal with various control problems for open hydraulic networks include [3], [4] and references therein.

In our application, however, the network is "closed". Similar networks and models arise for instance in mine ventilation networks and cardiovascular systems. These classes of systems are the motivation for the works [5], [6], [7], where nonlinear adaptive controllers are proposed to deal with the presence of uncertain parameters. Other systems close to the one considered here are nonlinear RLC circuits (see e.g. [8] and references therein).

Preliminary results on the case study of interest in this paper have appeared in [9], [10] and [11] (the contribution of the latter compared with the present paper is discusses later below). In [9], the control law was designed for a reduced-scale laboratory set-up

of a district heating system. A general model for a large class of hydraulic networks was derived in [10], and distributed proportional controllers were designed. Since centrifugal pumps are used in these hydraulic networks and those are pumps which can only provide a positive control action, the positivity constraint on the control law was explicitly taken into account.

In this paper, we face the following control problem. For a correct implementation of the control laws of [10], each controller, which is located at the end-user and which computes the control law based only on local information (measured pressure drop), is required to transmit the control values to "neighbor" pumps, i.e. auxiliary pumps which are found along the same circuit where the end-user lies. Due to physical constraints and the large-scale nature of the system, it is convenient to transmit information "sporadically". This motivates us to investigate the possibility to achieve the previous control objective (pressure regulation) by quantized controllers ([13], [14], [15], [16]). These controllers take value in a finite set (and therefore control values can be transmitted over a finite-bandwidth communication channel) and change their values only when certain boundaries in the state space are crossed. Since the feedback control action delivered by each pump makes use of local information only (pressure drop measured at the pump itself), it lends itself to be fully compatible with the plug-and-play-control strategy.

Controllers motivated by a similar need of being implemented in an industrial networked environment have been investigated in [4], as a result of an optimal control problem, and in [11], where binary controllers were employed. Quantized controllers were introduced as well, but no explicit proof was given.

Quantized controllers change their values less abruptly than binary controllers, thus reducing the fatigue of the actuators. Moreover, the prescribed control goal by quantized controllers is achieved with less control effort at steady state. Finally, while in [11] only simulations were presented, here we discuss experimental results obtained in a laboratory district heating system.

In Section 2, the class of hydraulic networks of interest in this paper is recalled. In Section 3, the quantized control strategy is analyzed. Experimental results are discussed in Section 4. Conclusions are drawn in Section 5.

2 Large-scale hydraulic networks

We introduce the model of a large-scale hydraulic network underlying a district heating system. The model is taken from [10], [11] to which we refer the reader for further details.

Hydraulic networks

An hydraulic network is a connection of two-terminal components such as valves, pipes and pumps (see Fig. 4.3, for a diagram of an hydraulic network), whose constitutive laws put in relation the pressure drop $\Delta h = h_i - h_j$ across the element and the flow q through the element. We briefly recall the constitutive laws of these components.

A *valve* is characterized by the algebraic relation

$$h_i - h_j = \mu(K_v, q)$$

where K_v is the hydraulic resistance of the valve, and μ is a smooth function of its arguments which, for each fixed value of K_v is zero at zero and strictly increasing. The

constitutive law of a *pipe* is a dynamic relation of the type

$$J\frac{dq}{dt} = (h_i - h_j) - \lambda(K_p, q)$$

with J, K_p parameters and λ a function which enjoys the same properties of the function μ. Finally, a (centrifugal) *pump* is a device which delivers the desired pressure difference $h_i - h_j$ no matter what is the flow through it. The constitutive law of the pump is

$$h_i - h_j = -\Delta h_p$$

where Δh_p is a nonnegative function of time which is viewed as a control input.

The value of the parameters K_v, K_p are typically unknown and we shall assume they range over a compact sets of strictly positive values, denoted by $\mathcal{P}$. Similarly, the functions μ, λ are not precisely known, and in fact knowing them is not necessary for the analysis, at least as far as the two properties of smoothness and monotonicity are guaranteed. We will distinguish between end-user valves and the other valves, allowing the former to change the value of the hydraulic resistance in a piece-wise constant fashion, and between the end-user pumps – located in the vicinity of the end-user valves – and the boosting pumps, that is pumps used to fulfill constraints on the relative pressures across the network which the end-user pumps alone – mainly used to meet the demands of the end-users – could not fulfill.

Model

To derive a model for these systems, it is convenient and natural to resort to tools in circuit theory ([17]). We will not review all the details here, referring the interested reader to [17], [10]. Rather, we will only recall the few notions which are needed to follow the developments below. In particular, we associate to the hydraulic network a graph $\mathcal{G}$ whose nodes are the terminals of the network's components and whose edges are the components themselves. Then a set of $n \geq 1$ *independent* flow variables (i.e. a set of flow variables whose value can be set independently from all the other flows in the network) are singled out. These independent flows coincide with the flows through the so-called *chords* of the graph ([17], [10]). A fundamental loop is associated to each chord, and along each fundamental loop Kirchhoff's voltage law holds, that is $B\Delta h = \mathbf{0}_{n \times 1}$, where B is called the *fundamental loop matrix*, i.e. a matrix of $-1, 1, 0$ whose value depends on the topology of the circuit, and Δh is the vector of all the pressure drops across the components of the network.

The class of hydraulic networks which are important for our case study satisfies the following two assumptions:

Assumption 4. *Each user valve is in series with a pipe and a pump, see Fig. 4.1. Moreover, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 5. *There exists one and only one component called the* heat source. *It corresponds to a valve[1] of the network, and it lies in all the fundamental loops.*

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[1]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.
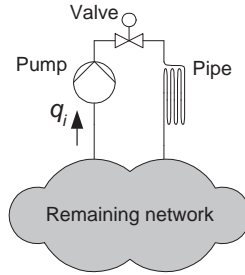
Figure 4.1: The series connection associated with each end-user.

The following result holds ([10]):

Proposition 8. *Any hydraulic network satisfying Assumption 4 obeys the equations*

$$
\begin{aligned}
J\dot{q} &= f(K_p, K_v, B^T q) + \tilde{u} \\
y_i &= \mu_i(k_{vi}, q_i) , \; i = 1, 2, \ldots, n
\end{aligned}
\tag{4.1}
$$

with $q \in \mathbb{R}^n$ the vector of independent flows, $\tilde{u} \in \mathbb{R}^n$ a vector of n independent inputs, y_i the measured pressure drop across the ith end-user valve, $J = J^T > 0$ an $n \times n$ matrix, K_p, K_v vectors of parameters, $f(K_p, K_v, B^T q)$ a smooth vector field, $\mu_i(k_{vi}, q_i)$ the constitutive law of the ith end-user valve.

The model has some nice features among which we recall the following, which states that if all the flows in the network have positive sign and there is no input action, then all the entries of the flow velocity vector $J\dot{q}$ are strictly negative. Namely we have ([10]):

Lemma 1. *Under Assumptions 4 and 5, $q \in \mathbb{R}_+^n$ [2] implies $-f(K_p, K_v, q) \in \mathbb{R}_+^n$.*

The input vector $\tilde{u}$ deserves a few comments too. As a matter of fact, it can be shown ([10]) that $\tilde{u} = B\Delta h_p$, with B, the fundamental matrix recalled above, and Δh_p, the vector of pump pressures, taking respectively the form

$$
B = \begin{pmatrix} I & I & F' \end{pmatrix} , \; \Delta h_p = \begin{pmatrix} 0 \\ \Delta h_p^e \\ \Delta h_p^b \end{pmatrix} ,
$$

with Δh_p^e, Δh_p^b the vectors of pressures delivered by the end-user pumps and, respectively, the boosting pumps. The sub-matrix F' turns out to have all *non-negative* entries as a consequence of Assumption 2.

Communication Topology

We have just established that the control law $\tilde{u}$ is a linear combination of vectors Δh_p^b and Δh_p^e. Since the pumps in the network are centrifugal pumps which cannot deliver

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[2]$\mathbb{R}_+^n$ denotes the positive orthant of $\mathbb{R}^n$, i.e. the set $\{q \in \mathbb{R}^n \; : \; q_i > 0 , \; i = 1, \ldots, n\}$.

negative pressures, having a positive control law $\tilde{u}$ is essential. Suppose a control law exists which produce non-negative control actions (see [10] and Section 3 below). How should these control actions be mapped to pump pressures in order to keep the positivity constraints? Below we build a graph which describes this mapping. It also results in a way to distribute the control effort among the end-user pumps and the boosting pumps.

Define the following k sets:

$$\mathcal{H}_j^b = \{\tilde{u}_i \in \{\tilde{u}_1, \ldots, \tilde{u}_n\} : F_{ij}' \neq 0\}, \quad j = 1, \ldots, k$$

where k is the number of boosting pumps in the system. That is: $\mathcal{H}_j^b$ is the subset of the control actions to which Δh_{pj}^b contributes. The following assumptions are made regarding the sets:

Assumption 6. *There exists one boosting pump Δh_{pi}^b for which $\mathcal{H}_i^b = \{\tilde{u}_1, \ldots, \tilde{u}_n\}$.*

This assumption corresponds to the statement that there exists one boosting pump which is providing actuation to all the fundamental flow loops. Since a boosting pump will be located in connection with the heat source this assumption will generally be fulfilled.

A hierarchy (tree) among the boosting pumps is now constructed. The starting point is the forward tree $\mathcal{T}_f$ (see [10]). The tree is constructed by removing all edges from $\mathcal{T}_f$ which does not correspond to a boosting pump. The boosting pump Δh_{pi}^b for which $\mathcal{H}_i^b = \{\tilde{u}_1, \ldots, \tilde{u}_n\}$ will be the root of the tree.

Using this tree it is then possible to define the pressures which each pump must deliver. Each boosting pump needs to calculate:

$$\Delta h_{pi}^b = \kappa_i \left(\min_{\tilde{u}_j \in \mathcal{H}_i^b} \tilde{u}_j - \Delta h_{p+}^b \right) \tag{4.2}$$

where Δh_{p+}^b is the actuation provided by the boosting pumps located above Δh_{pi}^b in the tree ($\Delta h_{p+}^b = 0$ if the boosting pump Δh_{pi}^b is the root of the tree), and $0 < \kappa_i < 1$ is the scaling factor which leaves some fraction of the actuation to the end-user pumps.

Remark 3: The signal Δh_{p+}^b can be calculated at the boosting pump located immediately above Δh_{pi}^b in the tree and thus communicated from here.

Each end-user pump then only need to subtract the boosting pump actions from their respective control actions:

$$\Delta h_{pj}^e = \tilde{u}_j - \sum_{i=1}^{k} F_{ji}' \Delta h_{pi}^b, \quad j = 1, \ldots, n . \tag{4.3}$$

To implement (4.2), the boosting pumps must communicate each other their control effort according to the topology described by the tree. Moreover, let $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, n\}$ be such that $\tilde{u}_j \in \mathcal{H}_i^b$. Then the controller j, which is located at the end-user pump j and which computes the control law $\tilde{u}_j$, must communicate $\tilde{u}_j$ to the boosting pump i. Finally, each boosting pump i must also communicate Δh_{pi}^b to the end-user pumps j for which $F_{ji}' \neq 0$ (see (4.3)). In the next section, we propose control laws which the pumps can communicate each other.

3 Pressure regulation by quantized control

Motivation

We are interested in the problem of designing a set of *distributed* controllers which regulates each output (the pressure drop at the end-user valve) y_i to the positive set-point reference value r_i, with $r = (r_1, \ldots, r_n) \in \mathcal{R}$ ranging in a known compact set, namely $\mathcal{R} = \{r \in \mathbb{R}^n : 0 < r_m \leq r_i \leq r_M, i = 1, \ldots, n\}$ (although typically $r_1 = \ldots = r_n = 0.5$ [*bar*]). We start from a set of proportional controllers of the following form

$$\tilde{u}_i = \begin{cases} -N_i(y_i - r_i), & y_i - r_i \leq 0 \\ 0, & y_i - r_i \geq 0, \end{cases} \tag{4.4}$$

where $N_i > 0$ is the controller gain. These controllers were studied in [10].

Since controllers and pumps are distributed across the network and hence geographically separated, it is important to investigate a way in which the control laws (4.4) can actually be communicated to the pumps (see Subsection 2). In this section, we propose to use *quantized* control laws and prove that a quantized version of (4.4) achieves the control objectives.

By quantized control is meant a piece-wise constant control law which takes values in a finite set. The state place is partitioned into a finite number of regions, and a control value is assigned to each one of the regions. The transitions from one control value to another take place when the state crosses the boundaries of the regions. Since quantized control laws take value in a finite set, in principle these values can be transmitted over a finite bandwidth communication channel. Quantized control for nonlinear systems has been investigated in a number of papers, among which we recall [18], [13], [14], [15], [16]. Here, we extend the results of [16], where a quantized version of the so-called semi-global backstepping lemma was proven, to the case in which *multiple positive* inputs are present. To the best of our knowledge, this is the first time a class of quantized controllers for a nonlinear multi-input industrial process is investigated.

Quantized controllers

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be the map (remember that $\tilde{u}_i \in \mathbb{R}_+$, see (4.4))

$$\psi(u) = \begin{cases} \psi_i, & \dfrac{\psi_i}{1+\delta} < u \leq \dfrac{\psi_i}{1-\delta}, \ 0 \leq i \leq j \\ 0, & 0 \leq u \leq \dfrac{\psi_j}{1+\delta} \end{cases} \tag{4.5}$$

In the definition above, j is a positive integer, u_0 is a positive real number, $\delta \in (0,1)$, and $\psi_i = \rho^i \psi_0$ for $i = 1, 2, \ldots, j$ with $\rho = \frac{1-\delta}{1+\delta}$. The parameters j, ψ_0, δ are to be designed. The map ψ is known as *logarithmic* quantizer ([18]).

Consider now the quantized version of the control law (4.4) and the resulting closed-loop system, namely

$$J\dot{q}_f = f(K_p, K_v, B^T q_f) + \Psi(\tilde{u}), \tag{4.6}$$

with $\tilde{u}$ as in (4.4) and $\Psi(\tilde{u}) = (\psi(\tilde{u}_1) \ldots \psi(\tilde{u}_n))^T$. Since $\Psi(\tilde{u})$ is a discontinuous function of the state variables, the closed-loop system (4.6) is a system with discontinuous

right-hand side. For this system the solutions are intended in the Krasowskii sense, a notion which is here briefly recalled:

Definition 3. A curve $\varphi : [0, +\infty) \to \mathbb{R}^n$ is a *Krasowskii solution* of a system of ordinary differential equations $\dot{x} = G(t, x)$, where $G : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \geq 0$ it satisfies the differential inclusion $\dot{x} \in K(G(t, x))$, where $K(G(t, x)) = \cap_{\delta > 0} \overline{co} \, G(t, B_\delta(x))$ and $\overline{co}G$ is the convex closure of the set G.

Recalling [19], Theorem 1, Properties 2), 3) and 7), we can state that the Krasowskii solutions of (4.6) are absolutely continuous functions which satisfy the differential inclusion

$$J\dot{q}_f \in f(K_p, K_v, B^T q_f) + v , \tag{4.7}$$

where $v \in K(\Psi(\tilde{u}))$, $K(\Psi(\tilde{u})) \subseteq \times_{i=1}^n K(\psi(\tilde{u}_i))$ and ([16])

$$K(\psi(\tilde{u}_i)) \subseteq \begin{cases} \{(1 + \lambda\delta)\tilde{u}_i , \ \lambda \in [-1, 1]\} \\ \qquad\qquad \frac{u_j}{1+\delta} < \tilde{u}_i \leq \frac{u_0}{1-\delta} \\ \{\lambda(1 + \delta)\tilde{u}_i , \ \lambda \in [0, 1]\} \\ \qquad\qquad 0 \leq \tilde{u}_i \leq \frac{u_j}{1+\delta} \end{cases} \tag{4.8}$$

The result below proves that quantized controllers can steer any initial state included in an arbitrarily large set to an arbitrarily small neighborhood of the desired reference value. In what follows, the following terminology will be in use: a trajectory is *attracted* by a set S if it is defined for all $t \geq 0$, and it belongs to S for all $t \geq T$, with $T > 0$ a finite time. Our control goal is the following:

Proposition 9. *For any choice of the parameter $q_M > 0$, any compact set $\mathcal{R} \subset \mathbb{R}_+$, any compact set $\mathcal{Q}$ of initial conditions described by*

$$\mathcal{Q} = \{q \in \mathbb{R}^n \ : \ |q_i| \leq q_M , \ i = 1, \cdots, n\} , \tag{4.9}$$

for any arbitrarily small positive number γ, and for any value of the quantization parameter $\delta \in (0, 1)$ there exist gains $N_i^* > 0$ and parameters ψ_0, j of the quantizer such that for all $N_i > N_i^*$, for any $r \in \mathcal{R}$, any Krasowskii solution $q_f(t)$ of the closed-loop system (4.6), (4.4) with initial condition in $\mathcal{Q}$ is attracted by the set $\{\epsilon \in \mathbb{R}^n \ : \ |\epsilon_i| \leq \gamma , \ i = 1, 2, \ldots, n\}$, where $\epsilon_i = y_i - r_i$.

The proof is omitted due to space limitation and it can be found in [12].

We cannot exclude that sliding modes may arise along those (switching) surfaces where $-N_i(\mu_i(K_{vi}, q_{fi}) - r_i) = \psi_j(1 + \delta)^{-1}$ for some i, j. This would give raise to chattering and it would jeopardize the possibility of transmitting the control values over a communication network, since a large bandwidth would be required. To this regard, we observe that it is always possible to replace the quantizers (4.5) with quantizers for which sliding modes are guaranteed to never occur. We follow the arguments of [16] and [14].

Let us introduce a new quantizer described by the following multi-valued map:

$$
\psi_m(u) = \begin{cases}
\psi_i , & \dfrac{\psi_i}{1+\delta} < u \le \dfrac{\psi_i}{1-\delta} , \\
& \qquad 0 \le i \le j \\
\dfrac{\psi_i}{1+\delta} , & \dfrac{\psi_i}{(1+\delta)^2} < u \le \dfrac{\psi_i}{1-\delta^2} , \\
& \qquad 0 \le i \le j \\
0 , & 0 \le u \le \dfrac{\psi_j}{1+\delta}
\end{cases}
\tag{4.10}
$$

Fig. 4.2 gives a pictorial representation of the map in the case $j = 1$. Compared with the previous quantizer, in the quantizer (4.10) there are additional quantization levels equal to $\pm\frac{\psi_i}{1+\delta}$, $i = 0, 1, \ldots, j$. The figure helps to understand how the switching occurs with these quantizers. Suppose for instance that $\psi_m(u) = \psi_1$, u is decreasing and hits the point $\psi_1(1+\delta)^{-1}$ (in the Figure this situation corresponds to point o). Then a switching occurs and $\psi_m(u) = \psi_1(1+\delta)^{-1}$ (i.e. there is a jump from o to a in the Figure). If u decreases and becomes equal to $\psi_1(1+\delta)^{-2}$ (point b), then a new transition occurs (b→c). If, on the other hand, u increases until it reaches the value $\psi_0(1+\delta)^{-2}$ (point e) then a transition takes place from e to p.

From the above description it should be clear that the new quantization levels and the new switching mechanism prevent the system to experience sliding modes and chattering. For the sake of simplicity we shall refer to these quantizers as quantizers with hysteresis. One may then wonder whether Proposition 9 still holds. The answer is positive since the new quantization levels belong to the sets on the right-hand side of (4.8), and Proposition 9 was proven letting each component v_i of v range over these sets. Hence Proposition 9 is still valid if we replace the quantizers (4.5) with the quantizers (4.10). The experimental results we present below are obtained using the quantizers with hystersis just introduced.

4   Experiments

This section presents experimental results obtained using the proposed controllers on a specially designed setup. The setup corresponds to a "small" district heating system with four end-users with a network layout as the system shown in Fig. 4.3. Although this number is by far less than the number of end-users expected in real district heating systems, it makes it possible to build an operational setup in a laboratory, and it covers the main features of a real system. A picture of the test setup is shown in Fig. 4.4.

The design of the piping of the test setup is aimed at emulating the dynamics of a real district heating system. However, due to physical constraints, the dynamics of the setup are approximately 5 to 10 times faster than the dynamics expected in a real system.

The natural disturbances in district heating systems are valve changes. However, the valves on the test setup are slow motor valves that are unable to excite the dynamics of the system. Therefore, to exemplify the performance of the controllers, the system response to a step in the references is tested. The references are changed from 0.2 [bar] to 0.45 [bar] and then back to 0.2 [bar].
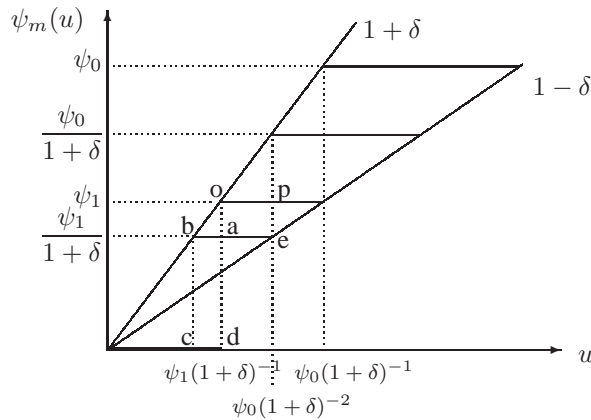
Figure 4.2: The multi-valued map $\psi_m(u)$ for $u > 0$, and with $j = 1$.
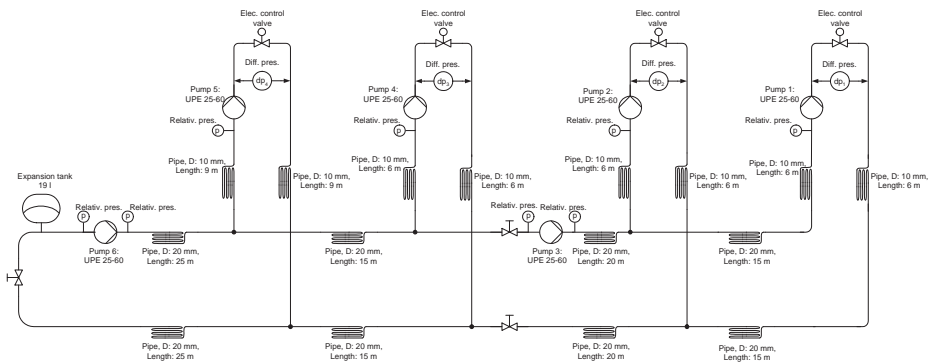
Figure 4.3: A diagram of the hydraulic network of the test setup in Fig. 4.4. The system contains four end-user pumps and two booster pumps.

Results obtained with the quantized controllers given by Proposition 9 are shown in Fig. 4.5. The design parameters of the quantizers (4.5) are chosen as $\psi_0 = 1$, $\delta = 0.25$, and $j = 3$. The gains of the controllers are set to $N_i = 1.5$, $i = 1, \ldots, 4$.

From the test results it is immediately seen that there is a steady state error between the measured pressures and the reference pressures. This is due to the use of quantized *proportional* controllers. Such steady state errors can be reduced by adjusting the gains of the controllers. From the behavior of both the controlled pressures and the controller inputs it is seen that the control system well-behaves and that the steady state is achieved within a reasonably short period of time.

The experimental results confirm the theoretical analysis, namely that semi-global practical regulation of the plant is guaranteed by distributed quantized proportional controllers. The experiments emphasize that relatively large delays (as those introduced in
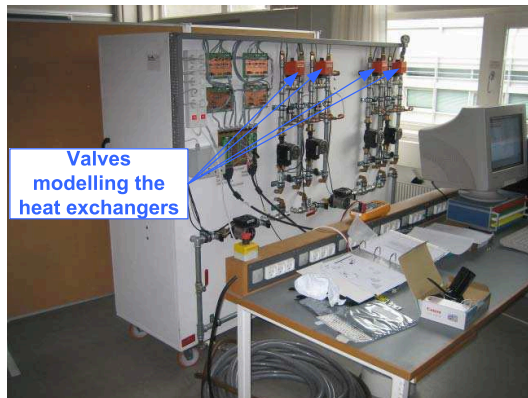
Figure 4.4: A picture of the test setup. The marked valves model the primary side of the heat exchanger of the end-users.
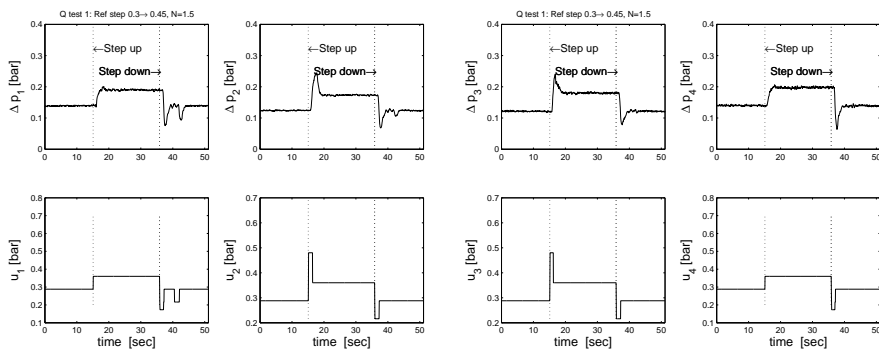
Figure 4.5: The results obtained using the proposed quantized controllers. The top plot shows the controlled pressures and the bottom plot shows the quantized control inputs.

these experiments by the hardware setup) can impose restrictions on the performance (oscillations) and on the accuracy of the controllers (large delays prevent from increasing the gains of the controllers and in turn from reducing the regulation error).

5   Conclusions

The paper deals with the study of an industrial system distributed over a network. Positive quantized controllers have been proposed to practically regulate the pressure at the end-users and experimental validation of the results has been provided. The actual implementation of the quantized controller over an actual communication network in a urban environment is currently under investigation.

We plan to extend our findings to the case of proportional-integral controllers ([20], [21], [22]), and to include constraints on the sign of the flows as well ([9]). Other research di-

rections will focus on controller redesign when new end-users are added to the network, extension of the results to the case of open hydraulic networks ([2]), and robustification of the controllers to delays, the latter being a very important and challenging problem.

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Paper B

Global practical stabilization of large-scale hydraulic networks

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Abstract

Proportional feedback control of a large scale hydraulic network which is subject to structural changes is considered. Results regarding global practical stabilization of the non-linear hydraulic network using a set of decentralized proportional control actions are presented. The results show that closed loop stability of the system is maintained when structural changes are introduced to the system.

1 Introduction

An industrial case study involving a system distributed over a network is investigated. The system is a large-scale hydraulic network which underlies a district heating system with an arbitrary number of end-users. The case study considers a new paradigm for constructing district heating systems [1]. The new paradigm is motivated by the possibility of reducing the overall energy consumption of the system while making the network structure more flexible. However, the new system paradigm also calls for a new control architecture, which is able to handle the flexible network structure [1].

The case study is a part of the research program *Plug & Play Process Control* [2] and has been proposed by one of the industrial partners involved in the research program. The goal of the research program is automatic reconfiguration of the control system whenever components, such as sensors or actuators, are added to or removed from the system. In the case of the district heating system, the addition (removal) of components could, for instance, be due to the addition (removal) of one or more end-users to (from) the system. Whenever such an addition or removal is made, the structure of the system is changed and the control should accommodate the changes.

The control objective of the system in question is to regulate the pressure drops across the so-called end-user valves in the hydraulic network to a given piecewise constant reference point. This goal shall be obtained in spite of the unknown demand of the end-users. The controllers, which will be considered here, are a set of decentralized proportional controllers, which use only locally available information. This control architecture has been chosen, since it is expected that changes in the system structure can be easily handled [3].

Previous work on a simple system with two end-users has shown that high-gain proportional controllers semi-globally stabilizes the closed loop system towards a set of attractors [4]. The results show that whenever the controller gains are large enough, the basin of attraction contains the set of all possible initial conditions of the system. However, if changes to the structure of the system is introduced, such as the addition or removal of end-users, the results cannot guarantee closed loop stability of the newly obtained system without proper redesign of the controller gains.

The results presented here are threefold. First, the results are applicable for a large-scale hydraulic network, since no assumptions are made regarding the number of end-users in the system. Secondly, the proposed control architecture is decentralized in the sense that the individual controllers use only locally available information. Thirdly, the results show that the closed loop system is globally practically stable with a unique equilibrium point using a set of arbitrary positive controller gains.

Compared to previous results in [4], which are semi-global, the global result here shows that the closed loop system will be stable regardless of the initial conditions. Fur-

thermore, since the result is independent of the number of end-users, the system will also be stable whenever components are added to or removed from the system, since the initial conditions of the newly obtained system are guaranteed to belong to the basin of attraction. This, along with the fact that the control scheme is decentralized, makes structural changes in the system easy to implement.

In Section 2, the models of the individual system components as well as the model of the hydraulic network are presented along with the proposed controllers. The closed loop properties of the system is derived in Section 3. Section 4 provides a proof of an important intermediate proposition, which is used to derive the closed loop stability properties of the system.

Nomenclature

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, with the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, x_i denotes the i'th element of x. Let $M(n, m; \mathbb{R})$ denote the set of $n \times m$ matrices with real entries, and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$. For a matrix $\mathbf{A}$, the notation A_{ij} will be used to denote the entry in the i'th row and j'th column of $\mathbf{A}$. For a square matrix $\mathbf{A}$, $\mathbf{A} > 0$ means that $\mathbf{A}$ is positive definite, i.e., $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \, \forall \mathbf{x} \neq 0$. For a square matrix $\mathbf{A}$, $\mathbf{A} = \text{diag}(x_i)$ means that $\mathbf{A}$ has x_i as entries on the main diagonal and zero elsewhere. Throughout the following, C^1 denotes a continuously differentiable function (map), and all functions (maps) introduced will be assumed C^1. A continuous function (map) $f : X \to Y$ is said to be: an *injection* if it is into, i.e., for every $a, b \in X$, if $f(a) = f(b)$ then $a = b$; a *surjection* if it is onto, i.e., if for every $\mathbf{y} \in Y$ there exists at least one $\mathbf{x} \in X$ such that $f(\mathbf{x}) = \mathbf{y}$; a *bijection* if it is both an *injection* and a *surjection*; a *homeomorphism* if it is a *bijection* with a continuous inverse f^{-1}; a *diffeomorphism* if it is a *bijection* with a C^1 inverse f^{-1}. A continuous function (map) is said to be *proper* if the inverse image of a compact set is compact. A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is order preserving, i.e., for all x and y such that $x \leq y$ then $f(x) \leq f(y)$. The open ball with radius r and centred in $\mathbf{x}$ is denoted $B_r(\mathbf{x})$, i.e., $B_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n | |\mathbf{y} - \mathbf{x}| < r\}$. Likewise, the corresponding closed ball is denoted $\bar{B}_r(\mathbf{x})$, i.e., $\bar{B}_r(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n | |\mathbf{y} - \mathbf{x}| \leq r\}$.

2   System Model

The system under consideration is a hydraulic network underlying a district heating system. The model has been derived in detail in [3] and will be recalled here but in fewer details.

The hydraulic network consists of a number of connections between two-terminal components, which can be valves, pipes and pumps. The system components are characterized by dual variables, the first of which is the pressure drop Δh across them

$$\Delta h = h_i - h_j, \tag{5.1}$$

where i, j are nodes in the network; h_i, h_j are the relative pressures at the nodes.

The second variable characterizing the components is the fluid flow q through them. The components have algebraic or dynamic expressions governing the relationships between the two variables.

Valves

Valves in the network are governed by the following algebraic expression

$$h_i - h_j = \mu(k_v, q), \tag{5.2}$$

where k_v is the hydraulic resistance of the valve; $\mu(k_v, q)$ is a C^1 and proper function, which for any fixed value of k_v is zero at $q = 0$ and monotonically increasing. Furthermore, $\mu(0, \cdot) = 0$.

Pipes

Pipes in the network are governed by the dynamic equation

$$\mathcal{J}\dot{q} = (h_i - h_j) - \lambda(k_p, q), \tag{5.3}$$

where $\mathcal{J}$ and k_p are parameters of the pipe; $\lambda(k_p, q)$ is a function with the same properties as $\mu(k_v, q)$.

Pumps

A (typically centrifugal) pump is a component which delivers a desired pressure difference Δh regardless of the value of the fluid flow through it. Thus, the pumps in the network are governed by the following expression

$$h_i - h_j = -\Delta h_p, \tag{5.4}$$

where Δh_p is a non-negative control input.

Component Model

A generalised component model can be made using the following expression

$$\Delta h = \mathcal{J}\dot{q} + \lambda(k_p, q) + \mu(k_v, q) - \Delta h_p \tag{5.5}$$

where $\mathcal{J}, k_p$ are non-zero for pipe components and zero for other components; k_v is non-zero for valve components and zero for other components; Δh_p is non-zero for pump components and zero for other components.

The values of the parameters k_p and k_v are typically unknown, but they will be assumed to be piecewise constant functions of time ranging over a compact set of non-negative values. Likewise, the functions $\mu(k_v, q)$ and $\lambda(k_p, q)$ are not precisely known, only their properties of being C^1, monotone and proper are guaranteed. The varying heating demand of the end-users, which is the main source of disturbances in the system, is modelled by a (end-user) valve with variable hydraulic resistance. In the network model, a distinction is to be made between end-user valves and the rest of the valves in the network. Two types of pumps are present in the network; the end-user pumps, which are mainly used to meet the demand at the end-users, and booster pumps which are used to meet constraints on the relative pressures in the network [5].

Network Model

The network model has been derived using standard circuit theory [3]. The hydraulic network consists of m components and n end-users ($m > n$). The network is associated with a graph $\mathcal{G}$ which has nodes coinciding with the terminals of the network components. The edges of the network are the components themselves. By the use of graph theory, a set of n independent flow variables q_i have been identified. These flow variables have the property that their values can be set independently from other flows in the network. The independent flow variables coincide with the flows through the chords of the graph [3]. To each chord in the graph, a fundamental (flow) loop is associated, and along this loop Kirchhoffs voltage law holds. This means that the following equality holds

$$\mathbf{B}\Delta\mathbf{h} = \mathbf{0}, \tag{5.6}$$

where $\mathbf{B} \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; $\Delta\mathbf{h}$ is a vector consisting of the pressure drops across the components in the network.

The entries of the fundamental loop matrix $\mathbf{B}$ are $-1, 1$ or 0, dependent on the network topology. For the case study in question, the hydraulic network underlies a district heating system. Because of this, the following statements can be made regarding the network.

Assumption 2.1: [3] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 5.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.

Assumption 2.2: [3] There exists one and only one component called the heat source. It corresponds to a valve[1] of the network, and it lies in all the fundamental loops.
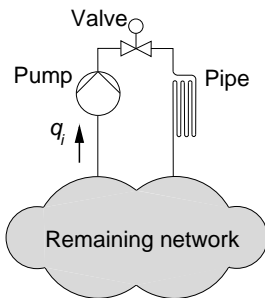
Figure 5.1: The series connection associated with each end-user [3].

Proposition 10. *[3] Any hydraulic network satisfying Assumption 2.1 admits the representation*

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{B}^T\mathbf{q}) + \mathbf{u} \tag{5.7}$$
$$y_i = \mu_i(k_{vi}, q_i) , \ i = 1, \ldots, n, \tag{5.8}$$

---

[1]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of independent flows; $\mathbf{u} \in \mathbb{R}^n$ is a vector of independent inputs consisting of a linear combination of the delivered pump pressures; y_i is the measured pressure drop across the ith end-user valve; $\mathbf{J} > 0 \in M(n; \mathbb{R})$; $\mathbf{K}_p, \mathbf{K}_v$ are vectors of system parameters; $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ is a C^1 vector field; $\mu_i(k_{vi}, \cdot)$ is the fundamental law of the ith end-user valve. In (5.8), it is assumed that the first n components coincide with the end-user valves.

Under Assumption 2.1 and Assumption 2.2, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix $\mathbf{B}$ are equal to 1 or 0, where B_{ij} is 1 if component j belongs to fundamental flow loop i and 0 otherwise.

Defining the vector of flows through the components in the system as $\mathbf{x} = \mathbf{B}^T \mathbf{q} \in \mathbb{R}^m$, the vector field $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ can be written as [3]

$$\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{x}) = -\mathbf{B}(\boldsymbol{\lambda}(\mathbf{K}_p, \mathbf{x}) + \boldsymbol{\mu}(\mathbf{K}_v, \mathbf{x})),$$

(5.9)

$\forall \mathbf{x} \in \mathbb{R}^m$,
where $\boldsymbol{\lambda}(\mathbf{K}_p, \cdot) = [\lambda_1(k_{p1}, x_1), \dots, \lambda_m(k_{pm}, x_m)]^T$;
$\boldsymbol{\mu}(\mathbf{K}_v, \cdot) = [\mu_1(k_{v1}, x_1), \dots, \mu_m(k_{vm}, x_m)]^T$, and k_{pi} is non-zero for pipe components and k_{vi} is non-zero for valve components.

The matrix $\mathbf{J}$ in (5.7) is given by

$$\mathbf{J} = \mathbf{B}\boldsymbol{\mathcal{J}}\mathbf{B}^T$$

(5.10)

where $\boldsymbol{\mathcal{J}} = \operatorname{diag}(\mathcal{J}_1, \dots, \mathcal{J}_m)$ and $\mathcal{J}_i$ is non-zero for pipe components.

The input $\mathbf{u}$ to the system deserves a few comments as well. Define the vectors $\boldsymbol{\Delta}\mathbf{h}_{pe}$ and $\boldsymbol{\Delta}\mathbf{h}_{pb}$ as the vectors of pump pressures delivered by respectively the end-user pumps and the booster pumps. Then $\mathbf{u}$ is given as

$$\mathbf{u} = \boldsymbol{\Delta}\mathbf{h}_{pe} + \mathbf{F}\boldsymbol{\Delta}\mathbf{h}_{pb}$$

(5.11)

where $\mathbf{F} \in M(n, k; \mathbb{R})$ is the sub-matrix of $\mathbf{B}$ which maps the booster pumps to the fundamental flow loops; k is the number of booster pumps in the network.

A sketch of a simple district heating system with a heat source and two apartment buildings is illustrated in Fig. 5.2. The corresponding hydraulic network is illustrated in Fig. 5.3. The two end-users are represented by the series connections $\{c_{12}, c_{13}, c_{14}\}$ and $\{c_5, c_6, c_7\}$. The heat source is represented by the valve $\{c_{10}\}$ which models the pressure losses in the secondary side of the heat exchanger of the heat source.

It is desired to regulate the pressure y_i across the ith end-user valve to a given reference value r_i with the use of a feedback controller using locally available information only. The desired reference value of the pressure across the end-user valve is assumed to be a piecewise constant function of time, and it ranges in a known set $[r_m, r_M]$. Thus, the vector $\mathbf{r} = (r_1, \dots, r_n)$ of reference values takes values in a known compact set $\mathcal{R}$

$$\mathcal{R} = \{\mathbf{r} \in \mathbb{R}^n | r_m \le r_i \le r_M\}.$$

(5.12)

For the purpose of practical output regulation, a set of decentralized proportional controllers will be the focus of the work presented here. The controllers considered will be of the form

$$u_i = -\gamma_i(y_i - r_i), \ i = 1, \dots, n,$$

(5.13)
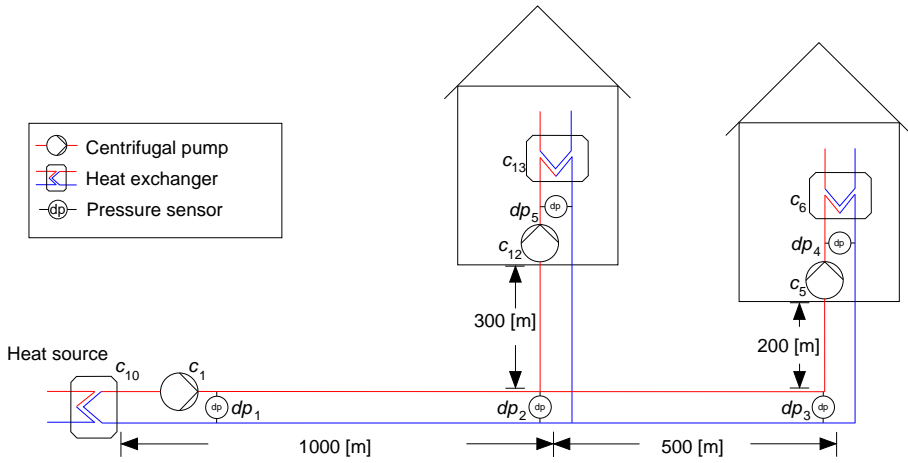
Figure 5.2: A sketch of a small district heating system.
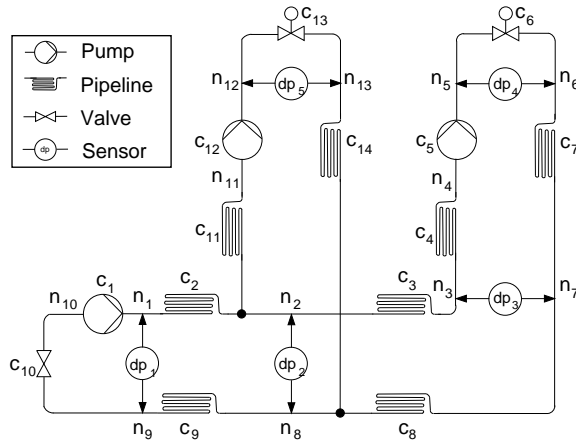
Figure 5.3: The hydraulic network diagram.

where $\gamma_i > 0$ is the controller gain at end-user i.

The pressure control for the ith end-user valve use only the pressure measurement obtained at said valve. Thus, the controllers are decentralized in the sense that the individual controller use only locally available information.

3   Stability Properties of Closed Loop System

In this section, the main result regarding the closed loop stability properties of the feedback control system introduced in the previous section will be presented.

To simplify the notation, $\mathbf{f}^K(\cdot)$ will be used to denote $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$. Likewise, $\boldsymbol{\lambda}^K(\cdot)$ and $\boldsymbol{\mu}^K(\cdot)$ will be used to denote $\boldsymbol{\lambda}(\mathbf{K}_p, \cdot)$ and $\boldsymbol{\mu}(\mathbf{K}_v, \cdot)$. The closed loop system defined

by (5.7), (5.8) and (5.13) is given by

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}^K(\mathbf{B}^T\mathbf{q}) - \mathbf{\Gamma}(\mathbf{y}(\mathbf{q}) - \mathbf{r}). \tag{5.14}$$

where $\mathbf{\Gamma} = \mathrm{diag}(\gamma_i)$.

Subsequently, a more specific class of functions will be used in the expressions of $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$. This more specific class is motivated by the presence of turbulent[2] flows in the system [3]. The class of functions, which will be considered, is the following

$$\mu_i(k_{vi}, x_i) = k_{vi}|x_i|x_i \tag{5.15}$$

$$\lambda_i(k_{pi}, x_i) = k_{pi}|x_i|x_i \tag{5.16}$$

An important intermediate result, which will be used for establishing the stability properties of the closed loop system, is presented below.

Define the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\mathbf{F}(\mathbf{z}) = \mathbf{y}(\mathbf{z}) - \mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{z}). \tag{5.17}$$

Proposition 11. *For the class of functions defined in (5.15) and (5.16), the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (5.17) is a homeomorphism.*

The proof of Proposition 11 has been left out of this section to maintain the flow of the exposition, but can be found in Section 4.

As a consequence of Proposition 11, for any vector $\mathbf{r} \in \mathcal{R}$ of output reference values, there exists a unique vector of flows $\mathbf{q}^* \in \mathbb{R}^n$ such that

$$\mathbf{q}^* = \mathbf{F}^{-1}(\mathbf{r}), \tag{5.18}$$

which means that $\mathbf{r}$ can be expressed in terms of $\mathbf{q}^*$ as

$$\mathbf{r} = \mathbf{y}(\mathbf{q}^*) - \mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*). \tag{5.19}$$

Using the identity in (5.19), the expression of the closed loop system in (5.14) can be replaced by

$$\mathbf{J}\dot{\mathbf{q}} = \tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) - \mathbf{\Gamma}(\mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{q}^*)), \tag{5.20}$$

where $\tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) = \mathbf{f}^K(\mathbf{B}^T\mathbf{q}) - \mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*)$. Recall that the vector $\mathbf{q}^* \in \mathbb{R}^n$ is some unknown but unique vector of flows, which is constant for every constant vector $\mathbf{r}$ of reference values.

Proposition 12. *The point $\mathbf{q}^*$ defined by (5.18) is a globally asymptotically stable equilibrium point of the closed loop system defined by (5.7), (5.8) and (5.13).*

Proof of Proposition 12. Define the variable $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*$, and the function $V(\tilde{\mathbf{q}})$ as

$$V(\tilde{\mathbf{q}}) = \frac{1}{2}\langle \tilde{\mathbf{q}}, \mathbf{J}\tilde{\mathbf{q}} \rangle, \tag{5.21}$$

which has the properties

---

[2]Since the motivation for considering the new paradigm is reducing the diameters of the pipes used in the network, the likelihood for turbulent flows increases.

- $V(\mathbf{0}) = 0$

- $V(\tilde{\mathbf{q}}) > 0$, $\forall \tilde{\mathbf{q}} \neq 0$

- $\lim_{||\tilde{\mathbf{q}}|| \to \infty} V(\tilde{\mathbf{q}}) = \infty$.

The time derivative of $V(\tilde{\mathbf{q}})$ is given by

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) = \langle \tilde{\mathbf{q}}, \mathbf{J}\dot{\mathbf{q}} \rangle \Leftrightarrow \tag{5.22}$$

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) = \langle \mathbf{q} - \mathbf{q}^*, \tilde{\mathbf{f}}^K(\mathbf{q}, \mathbf{q}^*) - \mathbf{\Gamma}[\mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{q}^*)] \rangle. \tag{5.23}$$

The functions $\lambda_i(k_{pi}, x_i)$ have the properties that they are monotonically increasing and zero for $x_i = 0$, consequently it applies that

$$-(x_i - x_i^*) \left[\lambda_i(k_{pi}, x_i) - \lambda_i(k_{pi}, x_i^*) \right] < 0, \forall x_i \neq x_i^* \Rightarrow \tag{5.24}$$

$$-\left\langle \mathbf{x} - \mathbf{x}^*, \boldsymbol{\lambda}^K(\mathbf{x}) - \boldsymbol{\lambda}^K(\mathbf{x}^*) \right\rangle < 0, \forall \mathbf{x} \neq \mathbf{x}^*. \tag{5.25}$$

The map $\boldsymbol{\mu}^K(\mathbf{x})$ has the same properties as $\boldsymbol{\lambda}^K(\mathbf{x})$, i.e., it consists of monotonically increasing functions which are zero for $x_i = 0$. Due to these properties, the fact that $\mathbf{x} = \mathbf{B}^T\mathbf{q}$ and the identity in (5.9), the following inequality holds

$$\left\langle \mathbf{q} - \mathbf{q}^*, \mathbf{f}^K(\mathbf{B}^T\mathbf{q}) - \mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*) \right\rangle < 0, \forall \mathbf{q} \neq \mathbf{q}^*. \tag{5.26}$$

Furthermore, since $y_i(q_i)$ is a monotonically increasing function which is zero at $q_i = 0$, the inequality below is true

$$\langle \mathbf{q} - \mathbf{q}^*, \mathbf{y}(\mathbf{q}) - \mathbf{y}(\mathbf{q}^*) \rangle > 0, \forall \mathbf{q} \neq \mathbf{q}^*. \tag{5.27}$$

Using (5.26) and (5.27) in (5.23) and observing that $\mathbf{\Gamma} > 0$, the following inequality is obtained

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) < 0, \forall \tilde{\mathbf{q}} \neq 0. \tag{5.28}$$

As a consequence of the properties of $V(\tilde{\mathbf{q}})$ and (5.28), the point $\tilde{\mathbf{q}} = 0$ is a globally asymptotically stable equilibrium point of the closed loop system (see for instance [6], Theorem 4.2). Considering the change of coordinates $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*$ it is concluded that $\mathbf{q} = \mathbf{q}^*$ is a globally asymptotically stable equilibrium point of the closed loop system. $\square$

Proposition 12 shows that for every constant vector $\mathbf{r}$ and gain $\gamma_i > 0$, there exists a unique constant vector $\mathbf{q}^*$ such that $\mathbf{q}^*$ is a globally asymptotically stable equilibrium point of the closed loop system. Note that only the properties of the functions $\mu(k_v, q)$ and $\lambda(k_p, q)$ being monotonically increasing and zero at $q = 0$ are used in the proof of Proposition 12. This means that the control system is robust towards uncertainties in the system parameters.

With the flows in the system converging to $\mathbf{q}^*$, the output of the system will converge to the value $\mathbf{y}^* = \mathbf{y}(\mathbf{q}^*)$. Using (5.19), the following relation is given between the vector $\mathbf{r}$ of reference values and $\mathbf{q}^*$

$$\mathbf{r} - \mathbf{y}(\mathbf{q}^*) = -\mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{q}^*). \tag{5.29}$$

Using the definition of $\mathbf{\Gamma}$, the i'th component is

$$r_i - y_i(q_i^*) = -\frac{1}{\gamma_i} f_i^K(\mathbf{B}^T \mathbf{q}^*). \tag{5.30}$$

Letting $\gamma_i \to \infty$, the right hand side of (5.30) will converge to zero. From this it can be seen that the use of large gains in the controller will let the output regulation error become small.

Since the system is globally asymptotically stable at $\mathbf{q}^*$, the system state will converge to $\mathbf{q}^*$ regardless of the initial conditions. Furthermore, the stability property is independent of the number n of end-users. This has the consequence that flow loops along with their respective controllers can be added to or removed from the system without the need for redesigning the controller gains in order for the system to be stable. However, controller gains may have to be redesigned for the purpose of fulfilling some specifications on the regulation error. From 5.30, it can be seen that each individual controller can adjust its own gain freely.

4 Properties of $\mathbf{F}(\mathbf{q}^*)$

This section provides a proof of Proposition 11, which has been used in deriving the closed loop properties of the system.

For the specific class of $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ defined in (5.15) and (5.16), the output map (5.8) can be rewritten as

$$\mathbf{y} = (k_{v1}|q_1|q_1, \ldots, k_{vn}|q_n|q_n)^T, \tag{5.31}$$

which in turn can be rewritten as

$$\mathbf{y} = \mathbf{H}(\mathbf{q})\mathbf{q}, \tag{5.32}$$

where $\mathbf{H}(\mathbf{q}) \in M(n; \mathbb{R})$ is given by

$$\mathbf{H}(\mathbf{q}) = \mathrm{diag}(k_{vi}|q_i|), \tag{5.33}$$

$i = 1, \ldots, n$.

Likewise, by substituting back $\mathbf{x}$ with $\mathbf{B}^T \mathbf{q}$, the expression for $\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \cdot)$ in (5.9) can be rewritten as

$$\mathbf{f}(\mathbf{K}_p, \mathbf{K}_v, \mathbf{B}^T \mathbf{q}) = -\mathbf{B}\mathbf{N}(\mathbf{B}^T \mathbf{q})\mathbf{B}^T \mathbf{q}, \tag{5.34}$$

where $\mathbf{N}(\mathbf{B}^T \mathbf{q}) \in M(m; \mathbb{R})$ is given by

$$\mathbf{N}(\mathbf{B}^T \mathbf{q}) = \mathrm{diag}((k_{vj} + k_{pj})|\mathbf{B}_j^T \mathbf{q}|), \tag{5.35}$$

$j = 1, \ldots, m$, where k_{vj} is non-zero for valve components and k_{pj} is non-zero for pipe components; $\mathbf{B}_j$ is the jth column of $\mathbf{B}$.

Define the function $\bar{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\bar{\mathbf{F}}(\mathbf{z}) = \mathbf{\Gamma}\mathbf{y}(\mathbf{z}) - \mathbf{f}^K(\mathbf{z}). \tag{5.36}$$

For the specified class of $\mu(k_v, q)$ and $\lambda(k_p, q)$, $\bar{\mathbf{F}}(\mathbf{z})$ can be written as

$$\bar{\mathbf{F}}(\mathbf{z}) = \mathbf{\Gamma}\mathbf{H}(\mathbf{z})\mathbf{z} + \mathbf{B}\mathbf{N}(\mathbf{B}^T\mathbf{z})\mathbf{B}^T\mathbf{z}, \qquad (5.37)$$

From the above, it can be established that $\bar{\mathbf{F}}(\mathbf{z})$ scales in the sense stated below

$$\bar{\mathbf{F}}(\lambda\mathbf{z}) = \lambda|\lambda|\bar{\mathbf{F}}(\mathbf{z}), \qquad (5.38)$$

where $\lambda \in \mathbb{R}$.

Furthermore, note that $g(\lambda) = \lambda|\lambda|$ is bijective, i.e. for every $\kappa \in \mathbb{R}$ there exists a unique $\lambda \in \mathbb{R}$ such that

$$\kappa = \lambda|\lambda|. \qquad (5.39)$$

The properties (5.38) and (5.39) are instrumental in the proof of Proposition 11.

Proof of Proposition 11. As a consequence of (5.27) and the fact that $\mathbf{\Gamma} > 0$, the following inequality is satisfied

$$\langle \mathbf{z} - \mathbf{z}^*, \mathbf{\Gamma}[\mathbf{y}(\mathbf{z}) - \mathbf{y}(\mathbf{z}^*)]\rangle > 0, \ \forall \mathbf{z} \neq \mathbf{z}^*. \qquad (5.40)$$

Likewise, from (5.26) the following inequality is obtained

$$-\langle \mathbf{z} - \mathbf{z}^*, \mathbf{f}^K(\mathbf{B}^T\mathbf{z}) - \mathbf{f}^K(\mathbf{B}^T\mathbf{z}^*)\rangle > 0, \ \forall \mathbf{z} \neq \mathbf{z}^*. \qquad (5.41)$$

A combination of these two inequalities gives

$$\langle \mathbf{z} - \mathbf{z}^*, \bar{\mathbf{F}}(\mathbf{z}) - \bar{\mathbf{F}}(\mathbf{z}^*)\rangle > 0, \ \forall \mathbf{z} \neq \mathbf{z}^*. \qquad (5.42)$$

Definition 4.1: [7]. Let $\mathbf{f} : X \to Y$, $X \subset \mathbb{R}^n$, $Y = \mathbb{R}^n$. Let the following inner product be denoted by

$$\langle \mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2\rangle \equiv \alpha(\mathbf{x}_1, \mathbf{x}_2).$$

Then $\mathbf{f}$ is said to be increasing on X, or simply an increasing function if and only if

$$\alpha(\mathbf{x}_1, \mathbf{x}_2) > 0, \ \forall \mathbf{x}_1, \mathbf{x}_2 \in X \text{ and } \mathbf{x}_1 \neq \mathbf{x}_2.$$

From (5.42) and Definition 4.1, it can be seen that $\bar{\mathbf{F}}(\mathbf{z})$ is an increasing function for every point $\mathbf{z} \in \mathbb{R}^n$.

Lemma 4.1: [7]. Let $\mathbf{f} : U \to \mathbb{R}^n$, where U is an open convex subset of $\mathbb{R}^n$.

  (a) *If $\mathbf{f}$ is increasing on U, then $\mathbf{f}$ is injective on U.*
  (b) *If $\mathbf{f}$ is continuous and increasing on U, then $\mathbf{f}$ is a homeomorphism on U[3] and its inverse function $\mathbf{f}^{-1} : \mathbf{f}(U) \to U$ is also increasing on $\mathbf{f}(U)$.*

Since $\bar{\mathbf{F}}(\mathbf{z})$ is continuous and increasing for every point $\mathbf{z} \in \mathbb{R}^n$, it follows from Lemma 4.1 that $\bar{\mathbf{F}}(\mathbf{z})$ is a local homeomorphism.

Proposition 13. *For the specified class of $\mu(k_v, q)$ and $\lambda(k_p, q)$ defined in (5.15) and (5.16), the map $\bar{\mathbf{F}} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (5.17) is proper.*

---

[3]$\mathbf{f}$ is a homeomorphism on U if and only if $\mathbf{f} : U \to V$ is a homeomorphism, where $V = \mathbf{f}(U)$

Proof of Proposition 13. In the proof, the following lemma will be used.

Lemma 4.2: [8]. Let $\mathbf{f}$ be a continuous map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then $\mathbf{f}$ is a proper map if and only if:

$$\lim_{|\mathbf{x}|\to\infty} |\mathbf{f}(\mathbf{x})| = \infty$$

Thus, if $\bar{\mathbf{F}}(\cdot)$ is proper it should fulfil

$$\lim_{|\mathbf{z}|\to\infty} |\bar{\mathbf{F}}(\mathbf{z})| = \infty. \tag{5.43}$$

Suppose by contradiction, that some sequence $\{\mathbf{z}_n\}_{n\in\mathbb{N}}$ exists, where

$$\lim_{n\to\infty} |\mathbf{z}_n| = \infty \tag{5.44}$$

and

$$\bar{\mathbf{F}}(\mathbf{z}_n) \in B_r(0), \ \forall n \in \mathbb{N}, \tag{5.45}$$

for some $r \in \mathbb{R}$.

Since $\bar{\mathbf{F}}(\cdot)$ is a local homeomorphism, there exits some open set $\mathcal{U} \subset \mathbb{R}^n$ containing 0 and an open set $\mathcal{V} \subset \mathbb{R}^n$, such that $\bar{\mathbf{F}} : \mathcal{U} \to \mathcal{V}$ is a homeomorphism. Furthermore, it is known that $0 \in \mathcal{V}$ since $\bar{\mathbf{F}}(0) = 0$.

Because of the scaling property (5.38) of $\bar{\mathbf{F}}(\cdot)$, there exists some $\hat{\mathbf{z}}_n \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $r_\mathcal{V} \in \mathbb{R}$, such that

$$\lambda\hat{\mathbf{z}}_n = \mathbf{z}_n, \tag{5.46}$$

$$\bar{\mathbf{F}}(\mathbf{z}_n) = \bar{\mathbf{F}}(\lambda\hat{\mathbf{z}}_n) = \lambda|\lambda|\bar{\mathbf{F}}(\hat{\mathbf{z}}_n) \tag{5.47}$$

and

$$\bar{\mathbf{F}}(\hat{\mathbf{z}}_n) \in \bar{B}_{r_\mathcal{V}}(0) \subset \mathcal{V}, \tag{5.48}$$

where $\hat{\mathbf{z}}_n$ is unique for a specific choice of λ.

However, this indicates that

$$\hat{\mathbf{z}}_n \in \bar{K} \subset \mathcal{U} \tag{5.49}$$

where $\bar{K} = \bar{\mathbf{F}}^{-1}(\bar{B}_{r_\mathcal{V}}(0))$ is some compact and thus bounded set.

This is a contradiction since

$$\lim_{n\to\infty} |\hat{\mathbf{z}}_n| = |\frac{1}{\lambda}| \lim_{n\to\infty} |\mathbf{z}_n| = \infty \tag{5.50}$$

$\square$

Theorem 3. *[8]. Let $\mathbf{f}$ be a map from $\mathbb{R}^n$ into $\mathbb{R}^n$, then $\mathbf{f}$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if $\mathbf{f}$ is:*

1) a local homeomorphism

2) a proper map

From Theorem 3 it follows that $\bar{\mathbf{F}}(\mathbf{z})$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Now, consider the linear transformation $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbf{T}(\mathbf{v}) = \mathbf{\Gamma}^{-1}\mathbf{v}$. Since $\mathbf{\Gamma}^{-1}$ is non-singular, the transformation $\mathbf{T}(\cdot)$ is a diffeomorphism. Thus, the composition

$$(\mathbf{T} \circ \bar{\mathbf{F}})(\mathbf{z}) = \mathbf{y}(\mathbf{z}) - \mathbf{\Gamma}^{-1}\mathbf{f}^K(\mathbf{B}^T\mathbf{z}), \tag{5.51}$$

is a homeomorphism.

$\square$

Since $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism it is bijective and has a continuous inverse $\mathbf{F}^{-1}$

$$\mathbf{F}(\mathbf{F}^{-1}(\mathbf{r})) = \mathbf{r}. \tag{5.52}$$

5 Numerical Results

The proposed proportional controllers have been tested using numerical simulations. The results of the simulations are shown in Fig. 5.4 and Fig. 5.5. The simulated system consists of two end-users corresponding to the hydraulic network illustrated in Fig. 5.3. The parameters used in the system are: $J_{11} = 0.3697, J_{12} = J_{21} = 0.0559, J_{22} = 0.2738$; $k_{p2} = k_{p9} = 0.0024$; $k_{p3} = k_{p8} = 0.0012$; $k_{p4} = k_{p7} = 0.0014$; $k_{p11} = k_{p14} = 0.0021$; $k_{v6} = k_{v13} = 0.01$; $k_{v10} = 0.0013$. Furthermore, the functions $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ used in the simulation are the ones introduced in Section 4.
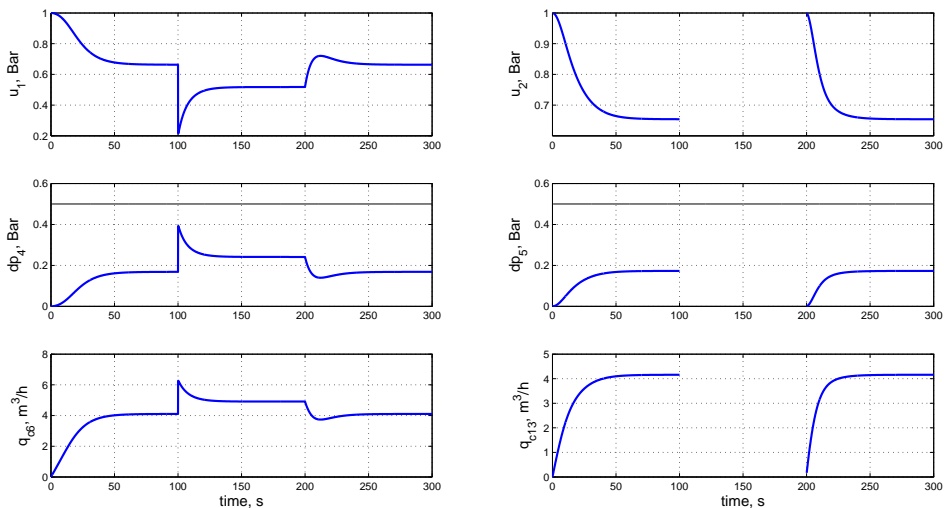
Figure 5.4: The figure shows the result of a numerical simulation of the system in Fig. 5.3. The figure shows control inputs u_1 and u_2, the controlled variable dp_4 and dp_5, and the flow through valve c_6 and c_{13} obtained with the proportional feedback control. At time 100 s, the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from the system. At time 200 s the end-user connection is re-inserted into the system.

First, a scenario where the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from and later re-inserted into the system has been simulated. This is simulated by increasing the hydraulic resistance k_{v13} of c_{13} to a large value and thereby reducing q_2 to close to zero. The results are shown in Fig. 5.4. Explicitly, the end-user connection is removed at time 100 s and re-inserted at time 200 s. All system parameters are maintained at the same values throughout the simulation, and the controller gain $\gamma_1 = \gamma_2 = 2$ has been used. The reference value for the pressure across the end user valves is indicated by the solid line at 0.5 Bar in the two plots in the middle.

In Fig. 5.4, it can be seen that a steady state, with an equilibrium point $\mathbf{q}^* = (q_{c6}^*, q_{c13}^*) \approx (4.1, 4.2)$ Bar, has been reached at time 100 s. Later, when the above mentioned end-user connection is re-inserted, the same equilibrium point $\mathbf{q}^*$ has been reached again at time 300 s. Since the same system parameters are used throughout the simulation, it is expected that the same equilibrium point will be reached since the relation between the reference value and the equilibrium point is the homeomorphism given by the expression in (5.19). Furthermore, when only one end-user is present, it can be seen that a steady state with an equilibrium point $q^* = q_{c6}^* \approx 4.9$ Bar is reached between time 100 s and 200 s.

Secondly, a scenario has been simulated where steps in the hydraulic resistance k_{v6}, k_{v13} of the end-user valves c_6, c_{13} are made. This corresponds to a varying demand for heating at the end-users. The steps are between the values 0.01 and 0.11. The results of the simulation are seen in Fig. 5.5. Again, $\gamma_1 = \gamma_2 = 2$ and the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from and later re-inserted into the system. Specifically, it is removed between time 300 s and time 600 s.

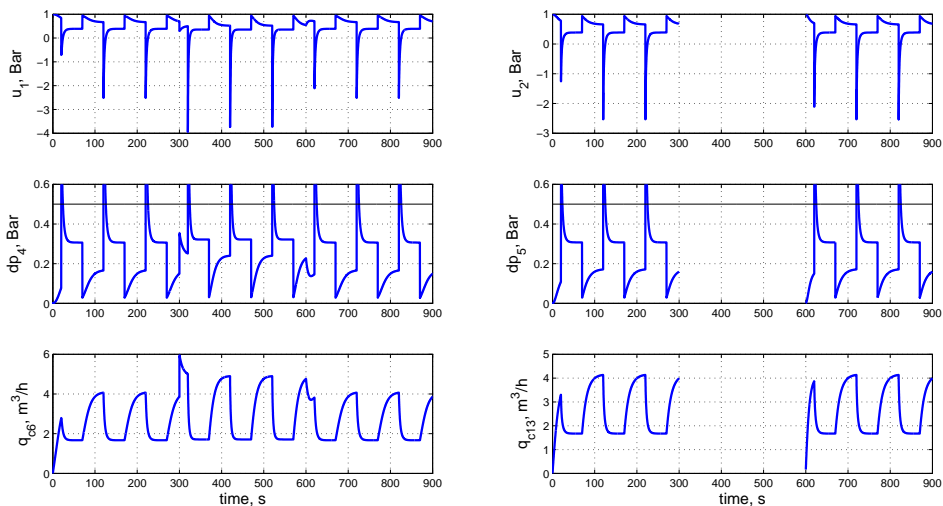Figure 5.5: The figure shows the result of a numerical simulation of the system in Fig. 5.3. Throughout the simulation, steps between values 0.01 and 0.11 are made in the hydraulic resistance (k_{v6}, k_{v13}) of the end-user valves c_6, c_{13}. At time 300 s, the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from the system. At time 600 s the end-user connection is re-inserted into the system.

In Fig. 5.5, it can be seen that the system remains stable when a step is made in the hydraulic resistance of the end-user valves.

6   Conclusion

An industrial case study involving a large-scale hydraulic network underlying a district heating system was investigated. The system under investigation is subject to structural changes. A set of decentralized proportional controllers for practical output regulation were proposed. The results show that the closed loop system is globally practically stable with a unique equilibrium point. The decentralized architecture of the control design and the fact that the closed loop system is globally stable make it easy to implement structural changes in the system, while maintaining closed loop stability. The results were supported by numerical simulations of a simple two end-user system.

Some natural future extensions of the work presented here will be restricting the control actions to only positive values and the incorporation of integral control actions. Since the (centrifugal) pumps used in the network are able to deliver only positive pressures, it should be examined if the stability properties of the system are kept when this restriction is taken into consideration. The incorporation of integral control actions would be interesting with respect to accommodating for the output regulation error which is present with the proportional control actions.

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Paper C

Global Stabilization of Large-Scale Hydraulic Networks Using Quantized Proportional Control

Tom Nørgaard Jensen, Rafał Wisniewski

Abstract

An industrial case study involving a hydraulic network underlying a district heating system is investigated. The flexible structure of the network calls for control structure which is able to handle changes in the network structure. For this purpose a set of decentralized proportional controllers have been proposed. These controllers make use only of locally available information, and in order to make implementation of the control laws possible, the control signals are required to be communicated across the network. To accommodate this a quantized version of the control laws are considered, and the results show that the designed closed loop system maintains its stability properties despite the structural changes introduced in the system.

1 Introduction

The work presented here considers the investigation of an industrial case study. The case study involves a large-scale hydraulic network which underlies a district heating system. Specifically, the case study regards a new paradigm for the design of district heating systems. By reducing the diameters of the pipes in the network the heat dispersion can be reduced, making it possible to reduce the heat losses in the system by 20 % to 50 % [Kallesøe(2007)]. Furthermore, the new paradigm allows for a more flexible network structure, which calls for a new control structure which is able to handle structural changes in the network, such as the addition or removal of end-users [Kallesøe(2007)]. The case study is part of the ongoing research program *Plug & Play Process Control* [Stoustrup(2009)], which considers automatic reconfiguration of the control system whenever components such as actuators or sensors are added to or removed from the system. The case study has been proposed by one of the industrial partners involved in the program.

A set of decentralized proportional control actions are proposed to meet the control objective in the system, which is to maintain the pressure across the so-called end-user valves at a piecewise constant reference point. The controllers use only locally available information, which is the pressure measurement at each end-user.

Reducing the pipe diameter in the district heating system, has the consequence that the pressure losses across the pipes are increased. This is compensated by distributing a number of (boosting) pumps across the network in order to meet pressure constraints [Kallesøe(2007)]. This means that the actuators are geographically separated from the controllers, making it necessary to communicate the control signals over a communication network. In order to accomplish this, the control signals are quantized in the sense that they are piecewise constant taking value in a finite set. This makes it possible to send them across a finite bandwidth network.

The result presented here shows that, given a properly designed quantizer, the closed loop system with the quantized control actions is globally attracted to a compact set, which can be made arbitrarily small by a proper design of the controller gains and quantization parameters. Furthermore, since the result is independent of the number of end-users in the system, the closed loop system will maintain these stability properties whenever end-users are added to or removed from the system.

The model of the system is introduced in Section 2. In Section 3, the control objective is introduced along with the proposed controllers and the quantization map. In Section 4, the stability properties of the closed loop system are analysed. Section 5 presents the

result of numerical simulations performed on the closed loop system. Finally, conclusions are drawn in Section 6.

Preliminaries

- Throughout the following, C^1 denotes the set of continuously differentiable functions.

- A continuous function (map) is said to be proper if the inverse image of a compact set is compact.

- A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is natural order preserving, *i.e.*, for all x and y such that $x \le y$ then $f(x) \le f(y)$.

- $M(n, m; \mathbb{R})$ denotes the set of $n \times m$ matrices with real entries and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$.

- $A > 0$ means that A is a positive definite matrix, *i.e.*, $A = A^\top$ and $x^\top A x > 0, \forall x \ne 0$.

- $A = \mathrm{diag}(x_i)$ means that A has entries x_i on the main diagonal and zero elsewhere.

- For two vectors $a, b \in \mathbb{R}^n$, $\langle a, b \rangle$ denotes the Euclidean scalar product.

- $B_r(x) = \{ y \in \mathbb{R}^n \mid |y - x| < r \}$.

2   System Model

In this section, the model of the large-scale hydraulic network will be recalled. The model is fully described in [DePersis and Kallesøe(2009)].

Component Models

The hydraulic network is comprised of three types of two-terminal components: valves, pipes and pumps as well as a number of interconnections between these components. These components are characterized by dual variables, the first of which is the pressure drop Δh across them

$$\Delta h = h_i - h_j, \tag{6.1}$$

where i, j are nodes of the network; h_i, h_j are the relative pressures at the nodes.

The other variable characterizing the components is the fluid flow q through them. The components in the network are governed by dynamic or algebraic equations describing the relation between the two dual variables.

Valves

A valve in the hydraulic network is described by the following algebraic relation

$$h_i - h_j = \mu(k_v, q), \tag{6.2}$$

where k_v is the hydraulic resistance of the valve; $\mu(k_v, \cdot) \in C^1$ is proper and for any constant value of k_v is zero at $q = 0$ and monotonically increasing.

Pipes

A pipe is described by the dynamic equation

$$\mathcal{J}\dot{q} = (h_i - h_j) - \lambda(k_p, q) \tag{6.3}$$

where $\mathcal{J}$ and k_p are parameters of the pipe; $\lambda(k_p, \cdot) \in C^1$ have the same properties as $\mu(k_v, \cdot)$.

Pumps

A (centrifugal) pump is a component which is able to maintain a desired pressure difference Δh across it regardless of the value of the fluid flow through it. This means that the constitutive law of the pump is

$$h_i - h_j = -\Delta h_p \tag{6.4}$$

where Δh_p is a signal, which for the purpose of the present exposition, is viewed as a control input.

Typically, exact values of the parameters k_v and k_p are not known but will be assumed to be positive and to take values in a known compact set. Furthermore, the functions $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ are not precisely known. Only their properties of being in C^1, proper, monotonic increasing and zero for $q = 0$ will be guaranteed.

The varying demand for heating at the end-users in the hydraulic network is modelled by a (end-user) valve for which the hydraulic resistance can be changed in a piecewise constant way. Thus, a distinction is to be made between the end-user valves and the remaining valves in the network. Likewise, a distinction is made between end-user pumps and booster pumps in the network. The later are pumps placed in the network to meet constraints on the relative pressures across the network. The former are pumps located in the vicinity of the end-user valves and are mainly used to meet the demands of the end-users.

Network Model

The model of the hydraulic network has been derived by using tools from circuit theory [DePersis and Kallesøe(2009)]. The network is comprised of m components and n end-users, where $m > n$. To the network is associated a graph $\mathcal{G}$, where the nodes of $\mathcal{G}$ coincides with the terminals of the components and the edges of $\mathcal{G}$ coincides with the components themselves. A vector of independent flow variables is identified as the flows through the chords of $\mathcal{G}$. These flow variables have the property that they can be set independently of all other flow variables in the network. To each chord in $\mathcal{G}$ (*i.e.* to each independent flow variable) a fundamental flow loop is associated. Along each of the fundamental flow loops Kirchhoff's voltage law holds, which can be expressed as

$$B\Delta h = 0, \tag{6.5}$$

where $B \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; Δh is a vector consisting of the pressure drops across the components in the network. The fundamental loop matrix B consists of $-1, 0, 1$, depending on the structure of the network.

The class of hydraulic networks which are considered here satisfy the following two assumptions:

Assumption 7. *[DePersis and Kallesøe(2009)] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 6.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 8. *[DePersis and Kallesøe(2009)] There exists one and only one component called the heat source. It corresponds to a valve[1] of the network, and it lies in all the fundamental loops.*
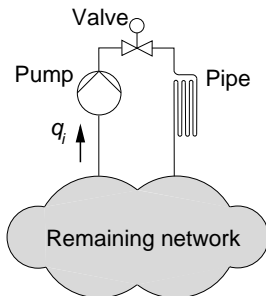
Figure 6.1: The series connection associated with each end-user.

Proposition 14. *[DePersis and Kallesøe(2009)] Any hydraulic network satisfying Assumption 7 admits the representation:*

$$J\dot{q} = f(K_p, K_v, B^\top q) + u \tag{6.6}$$

$$y_i(q_i) = \mu_i(k_{vi}, q_i) \, , \; i = 1, 2, \ldots, n \tag{6.7}$$

where $q \in \mathbb{R}^n$ is the vector of independent flows; $u \in \mathbb{R}^n$ is a vector of independent inputs, which is a linear combination of the delivered pump pressures; y_i is the measured pressure drop across the ith end-user valve (see (6.2)); $J \in M(n; \mathbb{R})$ and $J > 0$; K_p, K_v are vectors of system parameters; $f(K_p, K_v, B^\top q) \in C^1$; $\mu_i(k_{vi}, q_i)$ is the constitutive law of the ith end-user valve. In (6.7), it is assumed that the first n components coincide with the end-user valves.

Under Assumption 7 and Assumption 8, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix B are equal to 1 or 0.

A sketch of a simple district heating system with a heat source and two apartment buildings is illustrated in Fig. 6.2. The corresponding hydraulic network is illustrated in Fig. 6.3. The two end-users are represented by the series connections $\{c_{12}, c_{13}, c_{14}\}$ and $\{c_5, c_6, c_7\}$. The heat source is represented by the valve $\{c_{10}\}$ which models the pressure losses in the secondary side of the heat exchanger of the heat source.

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[1]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.
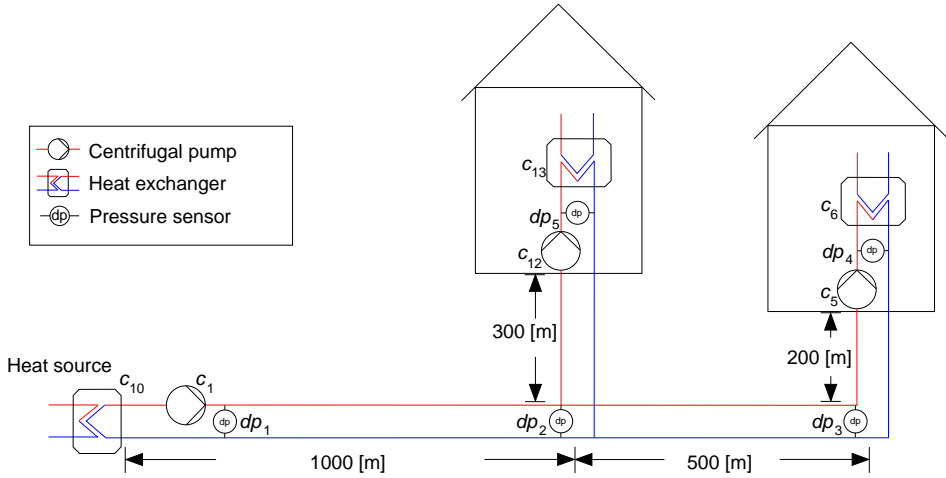
Figure 6.2: A sketch of a small district heating system.
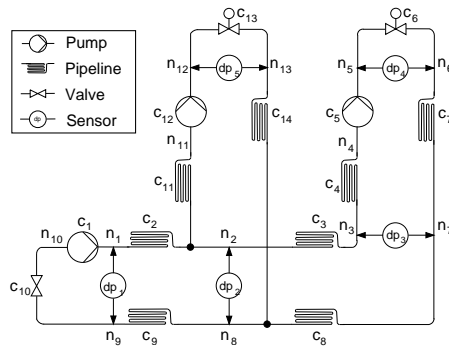
Figure 6.3: The hydraulic network diagram.

3 Pressure Regulation by Quantized Control Actions

This section introduces the control objective for the system in question along with a set of proposed control actions to accommodate this objective. Furthermore, a quantization map is introduced, which lets the control signals be piecewise constant taking values in a finite set.

Pressure Regulation Problem

It is desired to regulate the pressure (y_i) across the ith end-user valve to a given reference value (r_i) with the use of a feedback controller using locally available information only. The vector $r = (r_1, \ldots, r_n)$ of reference values take values in a known compact set $\mathcal{R}$:

$$\mathcal{R} = \{r \in \mathbb{R}^n \mid 0 < r_m \leq r_i \leq r_M\} \tag{6.8}$$

For the purpose of practical output regulation, a set of decentralized proportional controllers will be the focus of the work presented here. The controllers considered will be of the form:

$$u_i = -\gamma_i(y_i(q_i) - r_i) , \ i = 1, 2, \ldots, n \tag{6.9}$$

where $\gamma_i > 0$ is the controller gain.

The controllers are decentralized in the sense that the individual controller use locally available information only. Thus, the control for the ith end-user uses information obtained only at the ith end-user, which is the measurement of the pressure across the end-user valve.

Quantization Map

This section describes the quantizers which will be used. To that end, let l be a positive integer, ψ_0 a positive real number, $\delta \in (0,1)$, and $\psi_k = \rho^k \psi_0$ for $k = 1, 2, \ldots, l$ with $\rho = \frac{1-\delta}{1+\delta}$ (i.e. $\psi_k = \frac{1-\delta}{1+\delta}\psi_{k-1}$). The following quantizer is then proposed [DePersis et al.(2010)]:

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi(u_i) = \begin{cases} \psi_0 , & \frac{\psi_0}{1-\delta} < u_i \\ \psi_k , & \frac{\psi_k}{1+\delta} < u_i \leq \frac{\psi_k}{1-\delta}, \ 0 \leq k \leq l \\ 0 , & 0 \leq u_i \leq \frac{\psi_l}{1+\delta} \\ -\psi(-u_i) , & u_i < 0 \end{cases} \tag{6.10}$$

The parameters l, ψ_0 and δ of the map (quantizer) are to be designed.

Define $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ as $\Psi(u) = (\psi(u_1), \ldots, \psi(u_n))^\top$, then the closed loop system with the quantized version of the proportional control actions is given as

$$J\dot{q} = f(K_p, K_v, B^\top q) + \Psi(u) \tag{6.11}$$

The piecewise constant map $\psi(\cdot)$ changes value whenever the continuous control signal u_i crosses some boundary, as defined in (6.10). The control signal u_i is governed by the expression (6.9), where r_i and γ_i are constant parameters. Thus, the quantized version $(\psi(u_i))$ of the control signal can be replaced with an expression depending on a quantized version of the system output $(\Upsilon(y_i))$ such that

$$\psi(-\gamma_i(y_i(q_i) - r_i)) = -\gamma_i(\Upsilon(y_i) - r_i). \tag{6.12}$$

To this end, the following quantized version of the output $y_i(q_i)$ is considered.

Define $\epsilon_i = y_i - r_i$ and let $\Upsilon : \mathbb{R} \to \mathbb{R}$ be the map

$$\Upsilon(y_i) = r_i + \begin{cases} \frac{\psi_0}{\gamma_i} , & \epsilon_i > \frac{\psi_0}{(1-\delta)\gamma_i} \\ \frac{\psi_k}{\gamma_i} , & \frac{\psi_k}{(1-\delta)\gamma_i} \geq \epsilon_i > \frac{\psi_k}{(1+\delta)\gamma_i}, \\ & \quad\quad 0 \leq k \leq l \\ 0 , & \frac{\psi_l}{(1+\delta)\gamma_i} \geq \epsilon_i \geq 0 \\ r_i - \Upsilon(r_i - \epsilon_i) , & \epsilon_i \leq 0 \end{cases} \tag{6.13}$$

Define $Y : \mathbb{R}^n \to \mathbb{R}^n$ as $Y(y) = (\Upsilon(y_1), \ldots, \Upsilon(y_n))^\top$, and $\Gamma = \text{diag}(\gamma_i)$, then the closed loop system (6.11) can be rewritten to

$$J\dot{q} = f(K_p, K_v, B^\top q) - \Gamma(Y(y) - r) \tag{6.14}$$

since the identity in (6.12) is fulfilled.

The closed loop system in (6.14) has a discontinuous right hand side. Solutions to this system will here be considered in the sense of Krasovskii solutions.

Definition 3.1: [Bacciotti(2004), Bacciotti and Ceragioli(2006)] A map $\varphi : I \to \mathbb{R}^n$ is a Krasovskii solution of an autonomous system of ordinary differential equations $\dot{x} = G(x)$, where $G : \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \in I$ it satisfies the differential inclusion $\dot{\varphi}(t) \in KG(\varphi(t))$, where $KG(x) = \bigcap_{\delta > 0} \overline{\mathrm{co}}\, G(B_\delta(x))$ and $\overline{\mathrm{co}}\, G$ is the convex closure of the set G.

Here, I is an interval of real numbers, possibly unbounded. If $G(x)$ is Lebesgue measurable and locally bounded, the operators K associates to $G(x)$ a set valued map which is upper semi-continuous, compact and convex valued. In particular, for each initial state x_0 there exists at least one Krasovskii solution of $\dot{x} = G(x)$ [Bacciotti and Ceragioli(2006)].

The Krasovskii solutions of (6.14) are absolutely continuous functions which satisfy the differential inclusion [Paden and Sastry(1987)]

$$J\dot{q} \in f(K_p, K_v, B^\top q) - \Gamma(K(Y(y)) - r), \tag{6.15}$$

where $K(Y(y)) \subseteq \times_{i=1}^n K(\Upsilon(y_i))$ and $K(\Upsilon(y_i))$ is given by

$$K(\Upsilon(y_i)) = r_i + \begin{cases} \frac{\psi_0}{\gamma_i}, & \epsilon_i > \frac{\psi_0}{(1-\delta)\gamma_i} \\ \frac{\psi_k}{\gamma_i}, & \frac{\psi_k}{(1-\delta)\gamma_i} > \epsilon_i > \frac{\psi_k}{(1+\delta)\gamma_i}, \\ & \qquad\qquad 0 \le k \le l \\ \Delta \frac{\psi_k}{\gamma_i}, & \epsilon_i = \frac{\psi_k}{(1+\delta)\gamma_i}, \ 0 \le k \le l \\ 0, & \frac{\psi_l}{(1+\delta)\gamma_i} > \epsilon_i \ge 0 \\ r_i - K(\Upsilon(r_i - \epsilon_i)), & \epsilon \le 0 \end{cases} \tag{6.16}$$

for all $\Delta \in \{\frac{1-\lambda\delta}{1+\lambda\delta}, \ \lambda \in [0,1]\}$.

4 Stability Properties of Closed Loop System

In this section, the stability properties of the closed loop system introduced above will be examined. Subsequently, $f_K(\cdot)$ will be used to denote $f(K_p, K_v, \cdot)$. Furthermore, a more specific class of functions will be used in the expressions of $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$. This more specific class is motivated by the presence of turbulent[2] flows in the system [DePersis and Kallesøe(2009)]. The class of functions, which will be considered, is the following

$$\mu_i(k_{vi}, x_i) = k_{vi}|x_i|x_i \tag{6.17}$$

$$\lambda_i(k_{pi}, x_i) = k_{pi}|x_i|x_i \tag{6.18}$$

First, let the map $F : \mathbb{R}^n \to \mathbb{R}^n$ be given as

$$F(z) = y(z) - \Gamma^{-1} f_K(B^\top z). \tag{6.19}$$

---

[2]Since the motivation for considering the new paradigm is reducing the diameters of the pipes used in the network, the likelihood for turbulent flows increases.

Proposition 15. *[Jensen and Wisniewski(2010)] For the class of functions defined in (6.17) and (6.18), the map $F : \mathbb{R}^n \to \mathbb{R}^n$ defined in (6.19) is a homeomorphism.*

As a consequence of Proposition 15, there exists a unique vector $q^* \in \mathbb{R}^n$ for each vector of reference values $r \in \mathbb{R}^n$, and the relation between r and q^* is

$$r = y(q^*) - \Gamma^{-1} f_K(B^\top q^*). \tag{6.20}$$

This means that the expression for the closed loop system given in (6.14) can be replaced by

$$J\dot{q} \in \tilde{f}_K(\tilde{q}) - \Gamma(K(Y(y)) - y(q^*)) \tag{6.21}$$

where $\tilde{f}_K(\tilde{q}) = f_K(B^\top q) - f_K(B^\top q^*)$.

The following change of coordinates is made

$$\tilde{q} = q - q^*, \tag{6.22}$$

and the (Lyapunov) function $V : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$V(\tilde{q}) = \frac{1}{2} \langle \tilde{q}, J\tilde{q} \rangle. \tag{6.23}$$

The time derivative of $V(\tilde{q})$ is then given as

$$\frac{d}{dt} V(\tilde{q}) = \langle \tilde{q}, J\dot{q} \rangle \tag{6.24}$$

$$\frac{d}{dt} V(\tilde{q}) \in \left\langle \tilde{q}, \tilde{f}_K(\tilde{q}) - \Gamma(K(Y(y)) - y(q^*)) \right\rangle \tag{6.25}$$

$$\frac{d}{dt} V(\tilde{q}) \in \left\langle \tilde{q}, \tilde{f}_K(\tilde{q}) \right\rangle - \langle \tilde{q}, \Gamma(K(Y(y)) - y(q^*)) \rangle \tag{6.26}$$

It can be shown that the following inequality holds [Jensen and Wisniewski(2010)]

$$w(\tilde{q}) \equiv \left\langle \tilde{q}, \tilde{f}_K(\tilde{q}) \right\rangle < 0, \ \forall \tilde{q} \neq 0. \tag{6.27}$$

Now, the properties of the second term on the right hand side of (6.26) are examined. To that end, the parameter ψ_0 of the quantizer is first designed such that

$$r_i - \frac{\psi_0}{\gamma_i} \leq y_i(q_i^*) \leq r_i + \frac{\psi_0}{\gamma_i}, \ i = 1, 2, \ldots, n \tag{6.28}$$

Remark 4: Since the output functions are monotonic increasing and zero in $q_i = 0$, the following inequality holds:

$$(q_i - q_i^*)(y_i(q_i) - y_i(q_i^*)) > 0, \ i = 1, 2, \ldots, n. \tag{6.29}$$

Now, consider two different situations for $y_i(q_i^*)$ (the output of the system when $q = q^*$):

1) $y_i(q_i^*)$ is exactly equal to one of the quantization levels.

This is the case if the parameters γ_i, ψ_0, δ and l are designed such that $y_i(q_i^*) = r_i$ or such that there exist some $k \in \{0, 1, \ldots, l\}$ so either $y_i(q_i^*) = r_i + \frac{\psi_k}{\gamma_i}$ if $y_i(q_i^*) > r_i$ or $y_i(q_i^*) = r_i - \frac{\psi_k}{\gamma_i}$ if $y_i(q_i^*) < r_i$.

2) $y_i(q_i^*)$ lies between two quantization levels.

This is the case if for $y_i(q_i^*) > r_i$, either $r_i < y_i(q_i^*) < r_i + \frac{\psi_l}{\gamma_i}$ or there exist some $k \in \{1, \ldots, l\}$ such that $r_i + \frac{\psi_k}{\gamma_i} < y_i(q_i^*) < r_i + \frac{\psi_{k-1}}{\gamma_i}$. Or if for $y_i(q_i^*) < r_i$, either $r_i - \frac{\psi_l}{\gamma_i} < y_i(q_i^*) < r_i$ or there exist some $k \in \{1, \ldots, l\}$ such that $r_i - \frac{\psi_{k-1}}{\gamma_i} < y_i(q_i^*) < r_i - \frac{\psi_k}{\gamma_i}$.

First, consider situation 1). In the range where $\Upsilon(y_i) = y_i(q_i^*)$, the following is fulfilled

$$(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) = 0 \tag{6.30}$$

and outside the above mentioned range

$$(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > 0. \tag{6.31}$$

If situation 1) is fulfilled for every $i = 1, 2, \ldots, n$, then

$$-\langle q - q^*, \Gamma(v - y(q^*)) \rangle \leq 0, \ \forall v \in K(Y(y)), \tag{6.32}$$

since $\Gamma > 0$.

This shows $q = q^*$ is a globally asymptotically stable equilibrium point of the closed loop system, since

$$\frac{d}{dt}V(\tilde{q}) \leq w(\tilde{q}) < 0, \ \forall q \neq q^* \tag{6.33}$$

where $\frac{d}{dt}V(\tilde{q})$ is given in (6.26) and $w(\tilde{q})$ is as defined in (6.27).

A more realistic situation is that there exist some $p \in \{1, 2, \ldots, n\}$ (of course with a proper rearrangement of q) such that situation 2) is fulfilled for $q_1^*, q_2^*, \ldots, q_p^*$.

Now, consider situation 2) for q_i^*. Denote the bounds in 2) α_i, β_i such that $\alpha_i < y_i(q_i^*) < \beta_i$. Whenever $y_i(q_i)$ is outside the range (α_i, β_i)

$$(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > 0. \tag{6.34}$$

For a subset of the range (α_i, β_i) the sign of the product above changes.

Thus for the set $\mathcal{S} = \{q \in \mathbb{R}^n \mid y_i(q_i) \notin (\alpha_i, \beta_i), \ i = 1, \ldots, p\}$, it can be guaranteed that $\frac{d}{dt}V(\tilde{q}) < w(\tilde{q}) < 0$.

Define $\mathcal{S}_1 = \mathbb{R}^n \setminus \mathcal{S}$. For a given point in the set $\mathcal{S}_1$, there exists an index $s \leq p$ (with a proper rearrangement of q), such that

$$y_i(q_i) \in (\alpha_i, \beta_i), \ i = 1, 2, \ldots, s \tag{6.35}$$

Since $y_i(q_i)$ is proper, monotonically increasing and zero in $q_i = 0$, it admits a continuous inverse. Thus, the bound on $y_i(q_i)$ means that q_i is also bounded. Therefore, there exist some finite $m > 0$ such that

$$(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -m \tag{6.36}$$

and consequently, for each point $q \in \mathcal{S}_1$, there exist a finite $M > 0$ such that

$$\sum_{i=1}^{s}(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -M. \tag{6.37}$$

Let $M_{\mathcal{S}_1} > 0$ be the bound which fulfils

$$\sum_{i=1}^{s}(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -M_{\mathcal{S}_1}, \ \forall q \in \mathcal{S}_1, \tag{6.38}$$

which exists, since $\alpha_i < y_i(q_i) < \beta_i$ for $i = 1, \ldots, s$.

Let the set $\mathcal{S}_2 \subset \mathcal{S}_1$ denote the set for which the following holds

$$\sum_{i=s+1}^{p}(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > M_{\mathcal{S}_1}. \tag{6.39}$$

Note that q_i^* is constant and $\Upsilon(y_i)$ is bounded, thus there exists finite q_i such that (6.39) is fulfilled, since q_i is unbounded for $i = s + 1, \ldots, p$.

Thus in the set $\mathcal{S}_2$, the following inequality holds

$$-\langle q - q^*, \Gamma(v - y(q^*))\rangle \leq 0, \ \forall v \in K(Y(y)), \tag{6.40}$$

since $\Gamma > 0$.

Consequently $\frac{d}{dt}V(\tilde{q}) < w(\tilde{q}) < 0$ on the set $\mathcal{S}_2$.

From the analysis above it is concluded that there exists some compact set $\mathcal{Q} \subset \mathbb{R}^n$, where $\mathcal{S}_1 \setminus \mathcal{S}_2 \subset \mathcal{Q}$, with the property that all trajectories of the system is attracted to $\mathcal{Q}$.

Furthermore, whenever the initial conditions of the closed loop system belong to a compact set, say $\mathcal{Q}$, it can be shown by applying Lyapunov arguments that practical output regulation of the system is achievable. That is: for any arbitrarily small positive number ε, and for any value of the quantization parameter $\delta \in (0,1)$ there exist gains $\gamma_i^* > 0$ and parameters l and ψ_0 of the quantizer such that for all $\gamma_i > \gamma_i^*$, for any $r \in \mathcal{R}$, any Krasovskii solution $q(t)$ of the closed loop system with initial condition in $\mathcal{Q}$ is attracted by the set $\{\epsilon \in \mathbb{R}^n \mid |\epsilon_i| \leq \varepsilon, \ i = 1, 2, \ldots, n\}$, where $\epsilon_i = y_i - r_i$. The proof is similar to the one presented in [DePersis and Kallesøe(2010)] and is left out for brevity.

Since the result is global, the basin of attraction of the set $\mathcal{Q}$ is the entire state space $\mathbb{R}^n$. Furthermore, since the result is independent on the number n of end-users, it will be possible to add or remove end-users in the system while maintaining stability in the sense that for the newly obtained system a compact set $\mathcal{Q}$ which attracts the system trajectories will exist, given that (6.28) is fulfilled.

Quantization with Hysteresis

Using the quantizers defined in (6.13) may result in sliding modes arising along the switching surfaces, resulting in chattering and consequently the requirement for a large bandwidth. However, it is possible to replace the quantizer in (6.10) with an alternative for which it can be guaranteed that no sliding modes will arise [DePersis et al.(2010)].

Due to space limitations no explicit proof of stability of the closed loop system using this alternate quantizer will be provided here. However, the proof can be done by a proper redefinition of the bounds α_i and β_i in the previous section.

5 Numerical Results

A numerical simulation of the system in Fig. 6.3 in closed loop with the proposed control has been performed, and the results are shown in Fig. 6.4. The proportional control
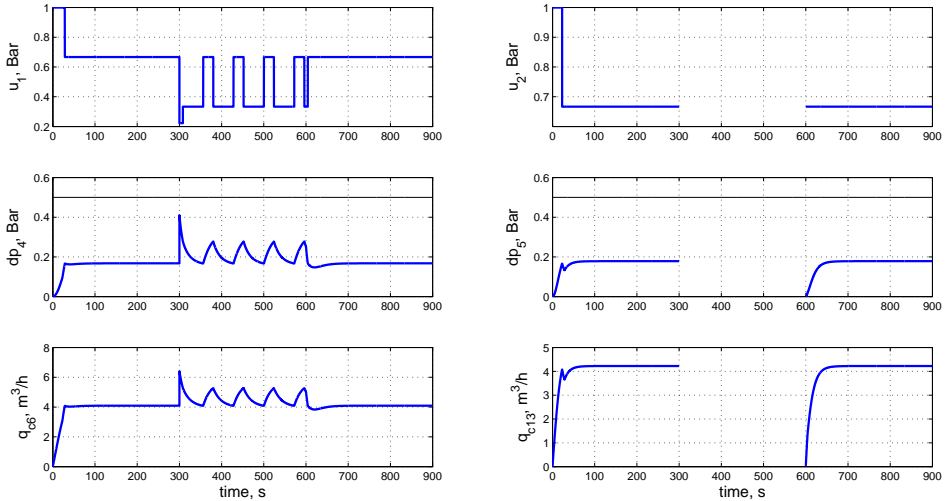
Figure 6.4: Result of a numerical simulation of the two end-user system in Fig. 6.3. The figure shows control inputs u_1 and u_2, the controlled variable dp_4 and dp_5, and the flow through valve c_6 and c_{13} obtained with the proportional feedback control. At time 300 s, the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$ is removed from the system. At time 600 s the end-user connection is re-inserted into the system.

actions defined in (6.9) and the quantizers including hysteresis has been used. A scenario, where the end-user connection consisting of $\{c_{12}, c_{13}, c_{14}\}$, has been removed from and later re-inserted into the system has been simulated. The Figure shows that the end-user connection is removed at time 300 s and re-inserted again at time 600 s. The parameters used in the simulation are; $\gamma_1 = \gamma_2 = 2$, $\delta = 0.5$, $\psi_0 = 1$ and $l = 2$. The reference values are $r_1 = r_2 = 0.5$ Bar, which is indicated by the solid line in the middle two plots in Fig. 6.4. Contrary to the result with the continuous proportional control actions [Jensen and Wisniewski(2010)], it is evident from Fig. 6.4 that a single unique equilibrium point can generally not be achieved when the quantized version of the proportional control actions are used. For instance a limit cycle-type behaviour is achieved for the single end-user system between time 300 s and 600 s.

6 Conclusion

An industrial case study involving a large-scale hydraulic network underlying a district heating system was investigated. The results show that the closed loop system using a set of quantized proportional feedback control actions is globally stable in the sense that there exists a compact set $\mathcal{Q}$ which attracts all system trajectories. Furthermore, it has been shown that this set can be made arbitrarily small by choosing a proper set of parameters for the feedback controller and quantizer. Specifically, for the result to hold, the bounds in (6.28) has to be fulfilled. Since the result is global and independent on the number of end-users in the system, a set $\mathcal{Q}$ with the above mentioned properties will also exist for the newly obtained system if it should be necessary to add or remove end-users

to/from the system. This, along with the decentralized nature of the control structure, will make it easy to implement structural changes in the system, while maintaining closed loop stability.

Future extensions of the results presented in this paper, will consist of an investigation of quantized proportional controllers, which are constrained to deliver only positive control signals. This is important since the (centrifugal) pumps used in the network are only able to deliver positive pressure inputs to the system. Furthermore, it will be interesting to investigate closed loop stability using proportional-integral control actions in order to accommodate for the output regulation error, which is present with the proportional control actions.

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7 Errata

1. Below (6.32) instead of: *since* $\Gamma > 0.$, the argument should be changed to: *since* $\Gamma = \mathrm{diag}(\gamma_i)$ *and* $\gamma_i > 0$.

2. Below (6.34) instead of: *Thus for the set* $\mathcal{S} = \{q \in \mathbb{R}^n \mid y_i(q_i) \notin (\alpha_i, \beta_i) \ , \ i = 1, \ldots, p\}$, *it can be guaranteed that* $\frac{d}{dt}V(\tilde{q}) < w(\tilde{q}) < 0.$, the argument is changed to: *Thus for the set* $\mathcal{S} = \{q \in \mathbb{R}^n \mid y_i(q_i) \notin (\alpha_i, \beta_i) \ , \ i = 1, \ldots, p\}$, *it can be guaranteed that* $\frac{d}{dt}V(\tilde{q}) < w(\tilde{q}) < 0$, *since* $\Gamma = \mathrm{diag}(\gamma_i)$ *and* $\gamma_i > 0$.

3. (6.36) should be changed to

$$\gamma_i(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -m$$

4. (6.37) should be changed to

$$\sum_{i=1}^{s} \gamma_i(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -M$$

5. (6.38) should be changed to

$$\sum_{i=1}^{s} \gamma_i(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > -M_{\mathcal{S}_1}, \ \forall q \in \mathcal{S}_1$$

6. (6.39) should be changed to

$$\sum_{i=s+1}^{p} \gamma_i(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > M_{\mathcal{S}_1}$$

7. Below (6.40) instead of: *since* $\Gamma > 0.$, the argument should be changed to: *since* $\Gamma = \mathrm{diag}(\gamma_i)$ *and* $\gamma_i > 0$.

Paper D

**Global Practical Pressure Regulation in Non-linear Hydraulic
Networks by Positive Controls**

Tom Nørgaard Jensen, Rafał Wisniewski

Abstract

An industrial case study involving a large-scale hydraulic network is considered. The hydraulic network underlies a district heating system. The network is subject to structural changes, such as the addition or removal of end-users. The actuators (pumps) in the system are limited to non-negative actuation values. The problem of controlling the pressures across the so-called end-user valves to a vector of desired reference values is described. The results show that global practical output regulation can be achieved using a set of proportional control actions which are constrained to non-negative values. Since the result is global, structural changes can be implemented while maintaining closed loop stability of the system.

1 Introduction

This work investigates an industrial case study of a large scale hydraulic network underlying a district heating system. Specifically, a new paradigm for the design of district heating systems is considered in the case study. It has been assessed that a reduction of the pipe diameter used in district heating systems, which will reduce heat dispersion, can reduce the heat losses with up to 50 % [1]. Furthermore, the new paradigm also leads to a more flexible network structure, which calls for a control architecture which is able to handle structural changes in the system, such as the addition or removal of end-users. The case study has been proposed by one of the industrial partners involved in the ongoing research program *Plug & Play Process Control* [5]. This research program considers automatic reconfiguration of the control whenever components such as sensors, actuators or maybe even entire subsystems are added to or removed from a control system.

A set of decentralized proportional control actions will be utilized to accommodate the control objective, which is to keep the pressure drop across the so-called end-user valves at a desired reference value. The controllers are decentralized as they use only locally available information, which is the pressure measurement at each of the end-users. The actuators (typically centrifugal pumps) used in the hydraulic network are limited in their actuation in the sense that they are only able to deliver non-negative pressures. Therefor, the control actions are subject to a non-negativity constraint.

The result presented here comprises an important extension of the result presented in [2], where it was shown that the closed loop system is semi-globally attracted to a neighbourhood of the desired equilibrium. That is; for any compact set of initial conditions, say $\mathcal{Q}$, the basin of attraction can be designed to contain $\mathcal{Q}$ by increasing the gains. Furthermore, the attractor set can be made arbitrarily small by increasing the gains of the controller. However, if structural changes, such as the addition or removal of end users, are introduced in the system, the results cannot guarantee closed loop stability of the newly obtained system without a proper redesign of the control gains.

Whereas, the analysis presented here shows that the decentralized proportional controllers subjected to the constraints, leads to a closed loop system which is globally asymptotically stable with a unique equilibrium point. Although the attained equilibrium point is different from the desired one. By adjusting the gains used in the controllers, the output regulation error can be made arbitrarily small. This result, which is the original contribution of this paper, is independent on the number of end-users in the system; and as a consequence, end-users can be added to or removed from the system while still

maintaining the stability properties of the closed loop system, since the initial conditions of the newly obtained system are guaranteed to belong to the basin of attraction.

The model of the system is introduced in Section 2. In Section 3, the control objective is introduced along with the proposed controllers and the non-negative constraints. In Section 4, the stability properties of the closed loop system are analysed. Section 5 presents the result of numerical simulations performed on the closed loop system. Finally, conclusions are drawn in Section 6.

Preliminaries

- Throughout the following, C^1 denotes a continuously differentiable function.

- A continuous map is said to be proper if the inverse image of a compact set is compact.

- For a vector $x \in \mathbb{R}^n$, x_i denotes the ith component of x.

- For two vectors $x, y \in \mathbb{R}^n$, $\langle x, y \rangle$ denotes the Euclidean scalar product.

- $M(n, m; \mathbb{R})$ denotes the set of $n \times m$ matrices with real entries and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$.

- $A > 0$ means that A is a positive definite matrix.

- For a square matrix A, $A = \text{diag}(x_i)$ means that A has x_i as entries on the main diagonal and zero elsewhere.

2   System Model

In this section, the model of the large-scale hydraulic network will be described. The model is derived in [2], which the interested reader can refer to for more details.

Component Models

The hydraulic network is comprised of three types of two-terminal components: valves, pipes and pumps as well as a number of interconnections between these components. These components are characterized by dual variables, the first of which is the pressure drop Δh across them

$$\Delta h = h_i - h_j, \tag{7.1}$$

where i, j are nodes of the network; h_i, h_j are the relative pressures at the nodes.

The other variable characterizing the components is the fluid flow q through them. The components in the network are governed by dynamic or algebraic equations describing the relation between the two dual variables.

Valves

A valve in the hydraulic network is described by the following algebraic relation

$$h_i - h_j = \mu(k_v, q), \tag{7.2}$$

where $k_v > 0$ is the hydraulic resistance of the valve; $\mu(k_v, \cdot) \in C^1$ is proper and for any constant value of k_v is zero at $q = 0$ and monotonically increasing.

Pipes

A pipe is described by the dynamic equation

$$\mathcal{J}\dot{q} = (h_i - h_j) - \lambda(k_p, q) \tag{7.3}$$

where $\mathcal{J} > 0$ and $k_p > 0$ are parameters of the pipe; $\lambda(k_p, \cdot) \in C^1$ have the same properties as $\mu(k_v, \cdot)$.

Pumps

A (centrifugal) pump is a component which is able to maintain a desired pressure difference Δh across it regardless of the value of the fluid flow through it. This means that the constitutive law of the pump is

$$h_i - h_j = -\Delta h_p \tag{7.4}$$

where Δh_p is a non-negative signal, which for the purpose of the present exposition, is viewed as a control input.

Typically, exact values of the parameters k_v and k_p are not known but will be assumed to be positive and to take values in a known compact set. Furthermore, the functions $\mu(k_v, \cdot)$ and $\lambda(k_p, \cdot)$ are not precisely known. Only their properties of being C^1, proper, monotonic increasing and zero for $q = 0$ will be guaranteed.

The varying demand for heating at the end-users in the hydraulic network is modelled by a (end-user) valve for which the hydraulic resistance can be changed in a piecewise constant way. Thus, a distinction is to be made between the end-user valves and the remaining valves in the network. Likewise, a distinction is made between end-user pumps and booster pumps in the network. The latter are pumps placed in the network to meet constraints on the relative pressures across the network. The former are pumps located in the vicinity of the end-user valves and are mainly used to meet the demands of the end-users.

Subsequently, $\mu(\cdot)$ $(\lambda(\cdot))$ will be used to denote $\mu(k_v, \cdot)$ $(\lambda(k_p, \cdot))$.

Network Model

The model of the hydraulic network has been derived by using tools from circuit theory [2]. The network is comprised of m components and n end-users, where $m > n$. To the network there is associated a graph $\mathcal{G}$, where the nodes of $\mathcal{G}$ coincides with the terminals of the components and the edges of $\mathcal{G}$ coincides with the components themselves. A vector of independent flow variables is identified with the flows through the chords[1] of $\mathcal{G}$. These flow variables have the property that they can be set independently of all other flow variables in the network. A fundamental flow loop is associated to each chord in $\mathcal{G}$ (i.e. to

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[1]Let $\mathcal{T}$ denote the spanning tree of $\mathcal{G}$, i.e. a connected subgraph which contains all nodes of $\mathcal{G}$ but no cycles. Then the edges of $\mathcal{G}$ which are not included in $\mathcal{T}$ are the chords of $\mathcal{G}$ (see [2]).

each independent flow variable). Along each of the fundamental flow loops Kirchhoff's voltage law holds, which can be expressed as

$$B\Delta h = 0, \tag{7.5}$$

where $B \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; Δh is a vector consisting of the pressure drops across the components in the network. The entries of the fundamental loop matrix B consist of $-1, 0, 1$, and the values depend on the structure of the network.

The class of hydraulic networks which are considered here satisfy the following two assumptions:

Assumption 2.1*: [2] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 7.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 2.2*: [2] There exists one and only one component called the heat source. It corresponds to a valve[2] of the network, and it lies in all the fundamental loops.*
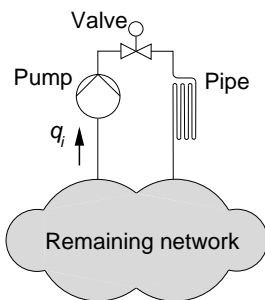
Figure 7.1: The series connection associated with each end-user.

Proposition 16. *[2] Any hydraulic network satisfying Assumption 2.1 admits the representation:*

$$J\dot{q} = f(B^T q) + u \tag{7.6}$$

$$y_i(q_i) = \mu_i(q_i) \, , \, i = 1, 2, \ldots, n \tag{7.7}$$

where $q \in \mathbb{R}^n$ is the vector of independent flows; $u \in \mathbb{R}^n$ is a vector of independent inputs, which is a linear combination of the delivered pump pressures; y_i is the pressure drop measured across the ith end-user valve (see (7.2)); $J \in M(n; \mathbb{R})$ and $J > 0$; $f(B^T q)$ is a C^1 vector field; $\mu_i(q_i)$ is the constitutive law of the ith end-user valve. In (7.7), it is assumed that the first n components coincide with the end-user valves.

Defining $x = B^T q$, the vector field $f(x)$ can be written as [2]:

$$f(x) = -B(\lambda(x) + \mu(x)) \tag{7.8}$$

where $\lambda(x) = [\lambda_1(x_1), \ldots, \lambda_m(x_m)]^T$; $\mu(x) = [\mu_1(x_1), \ldots, \mu_m(x_m)]^T$.

---

[2]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

Under Assumption 2.1 and Assumption 2.2, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix B are equal to 1 or 0.

A sketch of a simple district heating system with a heat source and two apartment buildings is illustrated in Fig. 7.2. The corresponding hydraulic network is illustrated in Fig. 7.3. The two end-users are represented by the series connections $\{c_{12}, c_{13}, c_{14}\}$ and $\{c_5, c_6, c_7\}$. The heat source is represented by the valve $\{c_{10}\}$ which models the pressure losses in the secondary side of the heat exchanger of the heat source.
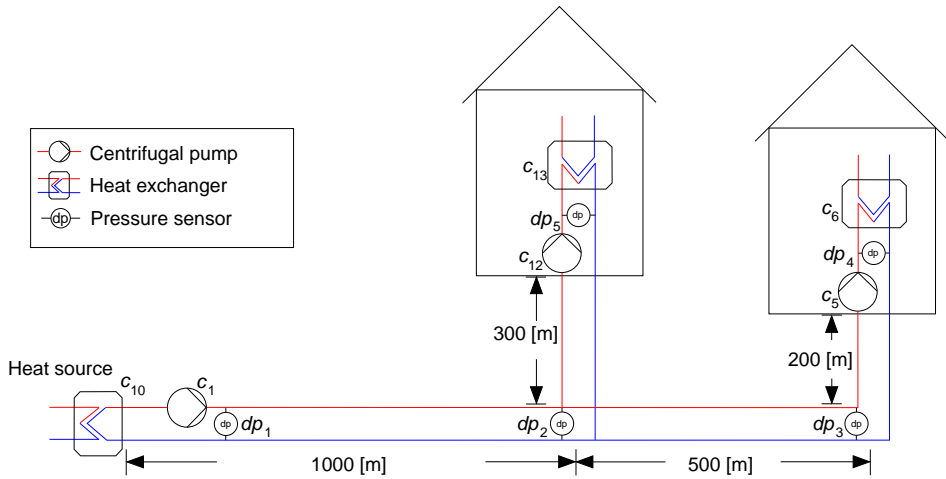
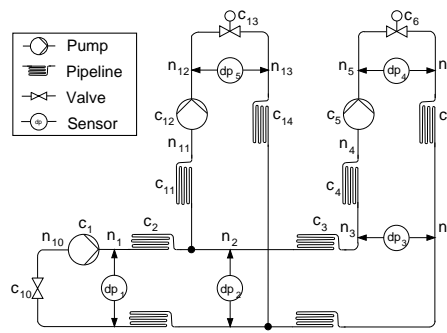Figure 7.2: A sketch of a small district heating system.

Figure 7.3: The hydraulic network diagram.

3 Pressure Regulation by Positive Constrained Proportional Control

In this section, the pressure regulation problem is introduced along with a number of proposed control actions to accommodate the control objective. Furthermore, a saturation map is introduced, in order to take into account the fact that the (centrifugal) pumps used in the system are only able to deliver non-negative actuation. This means that the control actions are subject to constraints.

Pressure Regulation Problem

It is desired to regulate the pressure $(y_i(q_i))$ across the ith end-user valve to a given reference value (r_i) with the use of a feedback controller using locally available information only. The vector $r = (r_1, \ldots, r_n)$ of desired reference values is assumed to be piecewise constant, taking values in a known compact set $\mathcal{R}$:

$$\mathcal{R} = \{r \in \mathbb{R}^n \mid 0 < r_m \leq r_i \leq r_M\} \qquad (7.9)$$

For the purpose of practical output regulation, a family of decentralized proportional controllers will be the focus of the work presented here. The controllers considered will be of the form:

$$u_i = -\gamma_i(y_i(q_i) - r_i) , \ i = 1, 2, \ldots, n \qquad (7.10)$$

where $\gamma_i > 0$ is the controller gain.

The controllers are decentralized in the sense that the individual controller uses locally available information only. Thus, the control for the ith end-user uses information obtained only at the ith end-user, which is the measurement of the pressure across the end-user valve.

Constraint Map

Since the pumps in the hydraulic network are only able to deliver positive pressures, it is desired to constrain the control signals u_i to positive values. To that end, let the constraint map $s : \mathbb{R} \to \mathbb{R}$ be given as

$$s(x) = \left\{ \begin{array}{ll} x & x \geq 0 \\ 0 & x \leq 0 \end{array} \right. . \qquad (7.11)$$

Now, define $S(u) = (s(u_1), \ldots, s(u_n))^T$, then the closed loop system with the constrained control is given as

$$J\dot{q} = f(B^T q) + S(u). \qquad (7.12)$$

The control signal u_i is governed by the expression in (7.10), where γ_i and r_i are constant. Thus the constrained version of the control signal $s(u_i)$ given by (7.11) can be replaced by an expression depending on a constrained version of the system output $\bar{s}_i(y_i(q_i))$, defined by

$$s(u_i) = -\gamma_i(\bar{s}_i(y_i(q_i)) - r_i) \qquad (7.13)$$

which, recalling (7.10), can be rewritten as

$$s(-\gamma_i(y_i(q_i) - r_i)) = -\gamma_i(\bar{s}_i(y_i(q_i)) - r_i). \qquad (7.14)$$

The constraint map $\bar{s}_i : \mathbb{R} \to \mathbb{R}$ fulfilling (7.14) is

$$\bar{s}_i(x) = \begin{cases} x & x \leq r_i \\ r_i & x \geq r_i \end{cases} \tag{7.15}$$

Let $\bar{S}(y(q)) = (\bar{s}_1(y_1(q_1)), \ldots, \bar{s}_n(y_n(q_n)))^T$, then by using the relation in (7.14), the closed loop system (7.12) can be written as

$$J\dot{q} = f(B^T q) - \Gamma(\bar{S}(y(q)) - r) \tag{7.16}$$

where $\Gamma = \text{diag}(\gamma_i)$.

4 Stability Properties of Closed Loop System

Subsequently, a more specific class of functions will be used in the expressions of $\mu(\cdot)$ and $\lambda(\cdot)$. This more specific class reflects the presence of turbulent[3] flows in the system [2]. The following class of functions will be considered

$$\mu_i(x_i) = k_{vi}|x_i|x_i \tag{7.17}$$

$$\lambda_i(x_i) = k_{pi}|x_i|x_i \tag{7.18}$$

Now, let the map $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows

$$F(z) = y(z) - \Gamma^{-1}f(B^T z). \tag{7.19}$$

Proposition 17. *[6] For the class of functions defined in (7.17) and (7.18), the map $F : \mathbb{R}^n \to \mathbb{R}^n$ defined in (7.19) is a homeomorphism.*

As a consequence of Proposition 17, there exists a unique vector $q^* \in \mathbb{R}^n$ for every vector $r \in \mathbb{R}^n$ of reference values, such that

$$q^* = F^{-1}(r) \tag{7.20}$$

and

$$r = y(q^*) - \Gamma^{-1}f(B^T q^*). \tag{7.21}$$

To simplify notation, define $\tilde{q} = q - q^*$ and $\tilde{f}(\tilde{q}) = f(B^T(\tilde{q} + q^*)) - f(B^T q^*)$.

Using (7.21), it is possible to rewrite the closed loop system (7.16) as

$$J\dot{q} = \tilde{f}(\tilde{q}) - \Gamma(\bar{S}(y(q)) - y(q^*)). \tag{7.22}$$

The following conjecture will be instrumental in the derivation of the stability properties of the closed loop system.

Conjecture 2. *Under Assumption 2.1 and Assumption 2.2 the vector q^* defined by (7.20), fulfils that $y_i(q_i^*) < r_i$, when $\Gamma = \text{diag}(\gamma_i)$, $\gamma_i > 0$ and $r \in \mathcal{R}$.*

---

[3]Since the motivation for considering the new paradigm is reducing the diameters of the pipes used in the network, the likelihood for turbulent flows increases.

Conjecture 2 is supported by simulation results similar to those presented in Section 5. Furthermore, the conjecture can be proved to hold for a two end-user system ($n = 2$).

Proof of Conjecture 2 for $n = 2$. Let C denote the set of components in the network. Furthermore, let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the chords of the graph associated to the network. Create a partition of C denoted C_1, C_2, C_{12} such that the flow through components in C_1 equals the flow through $\mathcal{L}_1$, the flow through components in C_2 equals the flow through $\mathcal{L}_2$ and $C_{12} = C \setminus (C_1 \bigcup C_2)$.

Consider the system in Fig. 7.3 then $C = \{c_1, \ldots, c_{14}\}$. By Assumption 2.1 let $\mathcal{L}_1$ be given by the series connection $\{c_6, c_7\}$ and $\mathcal{L}_2$ be given by the series connection $\{c_{13}, c_{14}\}$. Then the partition is given as $C_1 = \{c_3, \ldots, c_8\}$, $C_2 = \{c_{11}, \ldots, c_{14}\}$ and $C_{12} = \{c_1, c_2, c_9, c_{10}\}$.

For a two end-user system the right hand side of (7.21) can be rewritten as

$$
\begin{pmatrix} y_1(q_1^*) \\ y_2(q_2^*) \end{pmatrix} + \begin{bmatrix} \frac{1}{\gamma_1} & 0 \\ 0 & \frac{1}{\gamma_2} \end{bmatrix} \begin{pmatrix} \lambda\mu_1(q_1^*) + \lambda\mu_{12}(q_1^* + q_2^*) \\ \lambda\mu_2(q_2^*) + \lambda\mu_{12}(q_1^* + q_2^*) \end{pmatrix} \tag{7.23}
$$

where $\lambda\mu_1(q_1^*) = \sum_{i \in C_1} \lambda_i(q_1^*) + \mu_i(q_1^*)$; $\lambda\mu_2(q_2^*) = \sum_{i \in C_2} \lambda_i(q_2^*) + \mu_i(q_2^*)$; $\lambda\mu_{12}(q_1^* + q_2^*) = \sum_{i \in C_{12}} \lambda_i(q_1^* + q_2^*) + \mu_i(q_1^* + q_2^*)$. Note, that the functions $\lambda\mu_1(\cdot)$, $\lambda\mu_2(\cdot)$ and $\lambda\mu_{12}(\cdot)$ are monotonically increasing and zero when the argument is zero.

The proof of the proposition is by contradiction. Suppose that $y_1(q_1^*) \geq r_1 > 0$. Then, since $y_i(\cdot)$ is monotonically increasing and zero in zero, it follows that $q_1^* > 0$ and consequently that $\lambda\mu_1(q_1^*) > 0$. Since $\gamma_i > 0$ and (7.21) needs to be fulfilled, it is concluded that $\lambda\mu_{12}(q_1^* + q_2^*) < 0$ and consequently that $q_2^* < 0$. However, this means that

$$
r_2 \neq y_2(q_2^*) + \frac{1}{\gamma_2}(\lambda\mu_2(q_2^*) + \lambda\mu_{12}(q_1^* + q_2^*)) < 0, \tag{7.24}
$$

since $r_2 > 0$. This gives a contradiction since (7.21) is not fulfilled. $\square$

Proofs similar to the one above have been made for $n = 3$ and $n = 4$.

Proposition 18. *Suppose Conjecture 2 holds, that is;* $y_i(q_i^*) < r_i$, *with the point* q^* *defined by (7.20). Then* q^* *is the global asymptotically stable equilibrium point of the closed loop system (7.16).*

Proof of Proposition 18. First, let the Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$ be given as

$$
V(\tilde{q}) = \frac{1}{2}\langle \tilde{q}, J\tilde{q} \rangle, \tag{7.25}
$$

which has the properties

- $V(0) = 0$

- $V(\tilde{q}) > 0$, $\forall \tilde{q} \neq 0$

- $\lim_{||\tilde{q}|| \to \infty} V(\tilde{q}) = \infty$.

The time derivative of $V(\tilde{q})$ is given by

$$\frac{d}{dt}V(\tilde{q}) = \langle \tilde{q}, J\dot{q} \rangle. \tag{7.26}$$

Using the expression (7.16) for the closed loop system, the time derivative (7.26) of $V(\tilde{q})$ can be expressed as

$$\frac{d}{dt}V(\tilde{q}) = \langle \tilde{q}, \tilde{f}(\tilde{q}) \rangle - \langle \tilde{q}, \Gamma(\bar{S}(y(q)) - y(q^*)) \rangle \tag{7.27}$$

It can be shown that the following inequality holds [6]

$$W(\tilde{q}) \equiv -\langle \tilde{q}, \tilde{f}(\tilde{q}) \rangle > 0, \ \forall \tilde{q} \neq 0. \tag{7.28}$$

The second term on the right hand side of (7.27) can be written as

$$-\langle \tilde{q}, \Gamma(\bar{S}(y(q)) - y(q^*)) \rangle = -\sum_{i=1}^{n} \gamma_i \tilde{q}_i (\bar{s}_i(y_i(q_i)) - y_i(q_i^*)) \tag{7.29}$$

since Γ is diagonal with entries γ_i.

Recall that the functions $y_i(q_i) = \mu_i(q_i)$ are monotonically increasing and zero for $q_i = 0$. Because of this, the following holds

$$q_i < q_i' \Leftrightarrow y_i(q_i) < y_i(q_i'). \tag{7.30}$$

Now, the following two situations for q_i are examined

1) $q_i \leq q_i^*$

2) $q_i \geq q_i^*$

for the product
$$\gamma_i(q_i - q_i^*)(\bar{s}_i(y_i(q_i)) - y_i(q_i^*)). \tag{7.31}$$

First, consider situation 1). Since $q_i \leq q_i^*$, using the property (7.30), it follows that $y_i(q_i) \leq y_i(q_i^*)$. Furthermore, by Conjecture 2 $y_i(q_i) < r_i$, then it follows from (7.15) that $\bar{s}_i(y_i(q_i)) = y_i(q_i)$. In conclusion for situation 1)

$$\gamma_i(q_i - q_i^*)(\bar{s}_i(y_i(q_i)) - y_i(q_i^*)) \geq 0 \tag{7.32}$$

since $\gamma_i > 0$. Furthermore, the inequality is strict for $q_i \neq q_i^*$.

Now, consider situation 2). Again, since $q_i \geq q_i^*$, from (7.30), it follows that $y_i(q_i) \geq y_i(q_i^*)$. By Conjecture 2 and (7.15), it follows that $\bar{s}_i(y_i(q_i)) \geq y_i(q_i^*)$. In conclusion for situation 2)

$$\gamma_i(q_i - q_i^*)(\bar{s}_i(y_i(q_i)) - y_i(q_i^*)) \geq 0 \tag{7.33}$$

again with strict inequality for $q_i \neq q_i^*$.

From the consideration above it is concluded that

$$\frac{d}{dt}V(\tilde{q}) < -W(\tilde{q}) < 0, \ \forall \tilde{q} \neq 0. \tag{7.34}$$

with $W(\tilde{q})$ as defined in (7.28) and consequently that $\tilde{q} = 0$ is a global asymptotically stable equilibrium point of the closed loop system (7.16). □

Using (7.21) and that Γ is diagonal with entries $\gamma_i > 0$, it can be seen that the following holds

$$r_i = y_i(q_i^*) - \frac{1}{\gamma_i} f_i(B^T q^*), \qquad (7.35)$$

which shows that the use of large gains γ_i will let the output regulation error become small.

Furthermore, since Proposition 18 is independent of the number n of end-users in the system, the closed loop system will remain stable when adding or removing end-users to or from the system. However, note that the equilibrium point q^* will change when structural changes are made, so it may be necessary to adjust the controller gains in order to keep the same level of performance.

5   Numerical Results

The proportional controllers with the non-negativity constraints have been tested using numerical simulations. In the simulations, a four end-user system like the one illustrated in Fig. 7.4 has been used. The end-users in the system are comprised of the connections $\{c_4, c_5, c_6\}$, $\{c_9, c_{10}, c_{11}\}$, $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$. The gains $\gamma_1, \gamma_2, \gamma_3, \gamma_4 =$
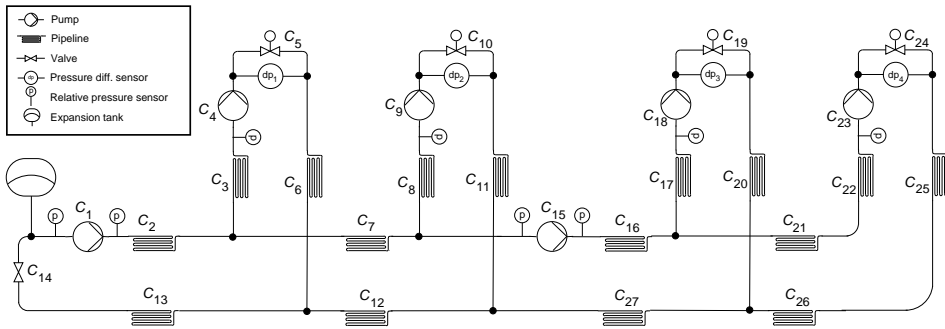
Figure 7.4: The hydraulic network diagram for the system with four end-users which has been used in the simulations.

2 and references $r_1, r_2, r_3, r_4 = 0.2$ Bar have been used.

First, a scenario, where the two end-users consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ have been removed from and later re-inserted into the system, has been simulated. The results of the simulation are shown in Fig. 7.5. As can be seen in this figure a steady state $q^*(100) = (0.1517, 0.1502, 0.1432, 0.1424)$ m^2/h has been attained before the end-user connections are removed at time 100 s. After the end-user connections have been removed a new steady state $q^*(200) = (0.1707, 0.1756)$ m^2/h is attained before the re-insertion of the end-user connections at time 200 s. At time 300 s, the steady state $q^*(300) = q^*(100)$ is attained. This is expected since the map $F(\cdot)$ defined in (7.19) is a homeomorphism. Furthermore, notice that the steady state values of dp_1, dp_2, dp_3, dp_4 fulfils Conjecture 2.

Secondly, a scenario has been simulated in which the hydraulic resistance (k_{v5}, k_{v10}, k_{v19} and k_{v24}) of the end-user valves is varied. This corresponds to a variation in the heating demands at the end-users, and is considered the main disturbance in the system.
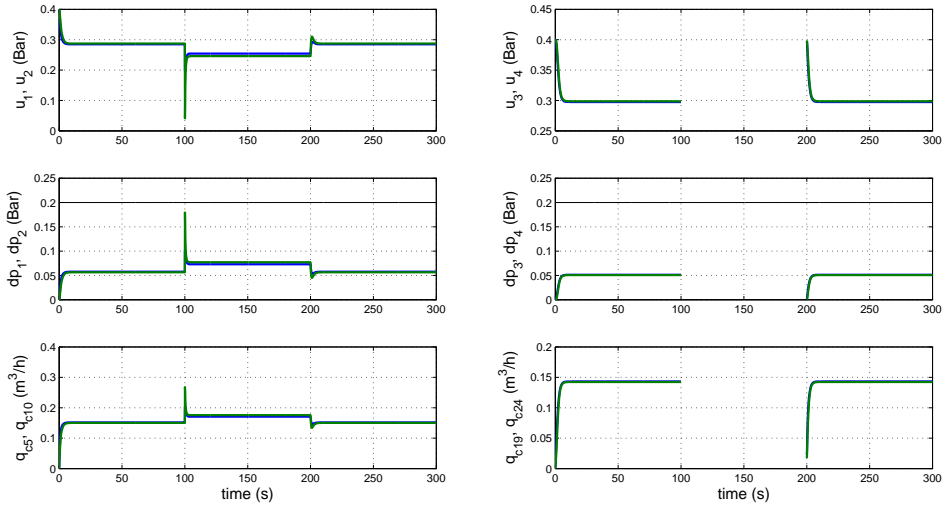
Figure 7.5: Result of a numerical simulation of the four end-user system in Fig. 7.4. The figure shows control inputs u_1, u_2, u_3, u_4, the controlled variable dp_1, dp_2, dp_3, dp_4, and the flow through valves $c_{24}, c_{19}, c_{10}, c_5$ obtained with the proportional feedback control. At time 100 s, the end-user connections consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are removed from the system. At time 200 s the end-user connections are re-inserted into the system. The solid line at 0.2 Bar in the two middle plots indicates the reference value.

The results of the simulation are given in Fig. 7.6. As seen in this figure, the closed loop system remains stable also with the disturbances present in the system.

6   Conclusion

An industrial case study involving a large scale hydraulic network underlying a district heating system has been examined. A set of decentralized proportional control actions to accommodate the output regulation problem were presented. The control actions were modified to take into consideration non-negative constraints on the actuators in the system. The results show that the proposed control actions are able to provide global practical asymptotic output regulation. Furthermore, since the result is independent on the number of end-users in the system, end-users can be added to or removed from the system while maintaining the closed loop stability properties of the system.

Some natural future extensions of the work presented here is to incorporate event based control actions as in [7] and [3, 4], which explicitly take the non-negativity constraints into consideration. Furthermore, incorporation of integral control action is seen natural in order to eliminate the output regulation error, which is present with the proportional control actions.
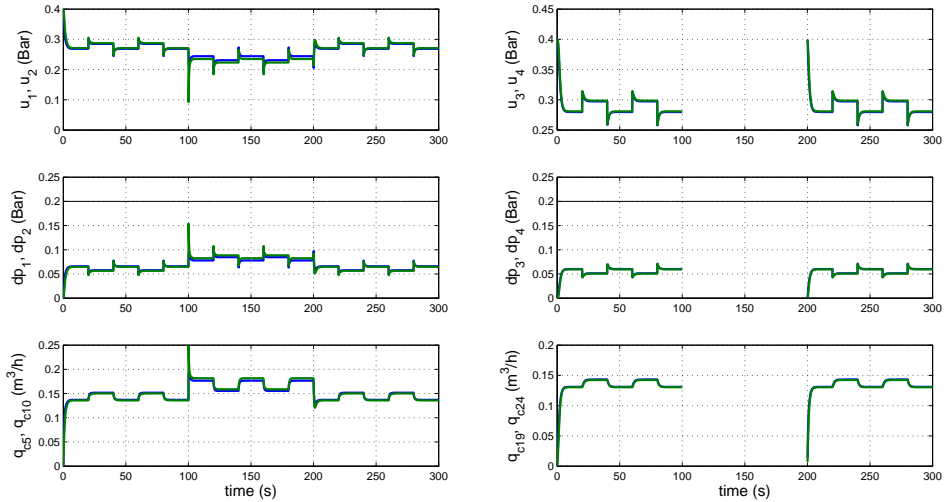
Figure 7.6: Result of a numerical simulation of the four end-user system in Fig. 7.4. Throughout the simulation steps are made in the hydraulic resistance of the end-user valves. At time 100 s, the end-user connections consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are removed from the system. At time 200 s the end-user connections are re-inserted into the system.

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Paper E

Global Stabilization of Large-Scale Hydraulic Networks with Quantized and Positive Proportional Controls

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Abstract

The work presented here considers an industrial case study. The case study involves a large-scale hydraulic network which underlies a district heating system. The structure of the network is subject to change, such as the removal or addition of end-users. The problem of controlling the output of the system to a desired reference point is addressed. The actuators in the system are geographically separated from the controllers, which means that control signals should be communicated via a communication network with finite bandwidth. Furthermore, the actuators are limited to positive actuation only. This is solved by using decentralized control architecture using quantized proportional control signals limited to positive values. The results show that practical output regulation is achievable. Furthermore, it is possible to add and remove end-users while the system is on-line without destabilizing the closed loop system.

1 Introduction

An industrial case study involving a large-scale hydraulic network is considered. The hydraulic network underlies a district heating system. The case study considers a new paradigm for designing district heating systems. By reducing the diameter of the pipes used in the network and using a multi-pump architecture, it has been assessed that a reduction in the heat losses in the system of up to 50 % is possible [1]. Furthermore, a more flexible network structure is achievable, in which for instance end-users can be arbitrarily added to or removed from the system.

The added flexibility in the network structure calls for a control architecture which is able to handle these types of changes in the network structure. In the work presented here, a set of decentralized proportional control actions will be in the focus. The individual control signal relies only on information obtained at the individual end-user. Furthermore, since the (centrifugal) pumps used in the system are only able to deliver non-negative actuation to the hydraulic network, the control signals will be limited to non-negative values. Lastly, the multi-pump architecture leads to the actuators being geographically separated from the controllers. This means that it is necessary to communicate the control signals over a communication network. To accommodate this need, the control signals are quantized in the sense that they are piecewise constant and take value in a finite set. This has the benefit that it is possible to communicate them across a finite bandwidth communication network.

The results presented here represents an important extension of the results presented in [2]. In [2], it was shown that semi-global practical output regulation is achievable using the proposed control architecture. That is, the trajectories of the system are locally attracted to a neighborhood of the desired equilibrium, and for every initial condition contained within some compact set, say $\mathcal{Q}$, the basin of attraction can be designed to cover $\mathcal{Q}$ by increasing the gains of the controllers. Furthermore, the attractor set can be made an arbitrarily small neighborhood of the desired equilibrium by increasing the gains of the controllers.

On the other hand, the results presented here show that, given a properly designed maximum quantization level, the trajectories of the closed loop system are globally attracted to a compact set. That is, for an arbitrary value of the controller gain, a compact

attractor set exists with a global basin of attraction, given a proper design of the maximum quantization level. Furthermore, the attractor set can be made arbitrarily small with a proper design of the controller gains and quantization parameters.

Since the result presented here is global and independent of the number of end-users in the system, it is possible to add and remove end-users to/from the system while maintaining the closed loop stability properties. That is, for the newly obtained system a compact set of attractors with a global basin of attraction will exist, however, to keep the same level of performance it may be necessary to adjust the controller gains and the parameters of the quantizer. Furthermore, since the controllers are decentralized, changes in the network structure are easy to implement.

The outline of the paper is as follows. The component and network models are described in Section 2. In Section 3, the output regulation problem is described along with the proposed set of controllers. The stability properties of the closed loop system are derived in Section 4. The results of tests performed on closed loop system in a laboratory setup are presented in Section 5. Finally, conclusions are drawn in Section 6.

Nomenclature

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space, with the standard scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$ between two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. For a vector $x \in \mathbb{R}^n$, x_i denotes the i'th element of x. The notation $\mathbb{R}^n_+$ denotes the positive orthant of $\mathbb{R}^n$, that is $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i > 0\}$, $i \in \{1, 2, \ldots, n\}$. The symbol $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}_+$ the set of integers greater than zero. Let $M(n, m; \mathbb{R})$ denote the set of $n \times m$ matrices with real entries, and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$. For a matrix $\mathbf{A}$, the notation A_{ij} will be used to denote the entry in the i'th row and j'th column of $\mathbf{A}$. For a square matrix $\mathbf{A}$, $\mathbf{A} > 0$ means that $\mathbf{A}$ is positive definite, i.e., $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \, \forall \mathbf{x} \neq 0$. For a square matrix $\mathbf{A}$, $\mathbf{A} = \mathrm{diag}(x_i)$ means that $\mathbf{A}$ has x_i as entries on the main diagonal and zero elsewhere. Throughout the following, C^1 denotes the set of continuously differentiable maps. A continuous function (map) is said to be *proper* if the inverse image of a compact set is compact. A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is order preserving, i.e., for all x and y such that $x \leq y$ then $f(x) \leq f(y)$. The open ball with radius r and centred in $\mathbf{x}$ is denoted $B_r(\mathbf{x})$.

2   System model

In this section, the model of the large-scale hydraulic network will be described. The model is derived in [2], which the interested reader can refer to for more details.

Component Models

The hydraulic network is comprised of three types of two-terminal components: valves, pipes and pumps as well as a number of interconnections between these components. These components are characterized by dual variables, the first of which is the pressure drop Δh across them

$$\Delta h = h_i - h_j, \tag{8.1}$$

where i, j are nodes of the network; h_i, h_j are the relative pressures at the nodes.

The other variable characterizing the components is the fluid flow q through them. The components in the network are governed by dynamic or algebraic equations describing the relation between the two dual variables.

Valves

A valve in the hydraulic network is described by the following algebraic relation

$$h_i - h_j = \mu(q) \equiv \mu(v, q), \tag{8.2}$$

where $v > 0$ is the hydraulic resistance of the valve; $\mu(v, \cdot) \in C^1$ is proper and for any constant value of v is zero at $q = 0$ and monotonically increasing.

Pipes

A pipe is described by the dynamic equation

$$\mathcal{J}\dot{q} = (h_i - h_j) - \lambda(q) \tag{8.3}$$

where $\lambda(q) \equiv \lambda(p, q)$; $\mathcal{J} > 0$ and $p > 0$ are parameters of the pipe; $\lambda(p, \cdot) \in C^1$ have the same properties as $\mu(v, \cdot)$.

Pumps

A (centrifugal) pump is a component which is able to maintain a desired pressure difference Δh across it regardless of the value of the fluid flow through it. This means that the constitutive law of the pump is

$$h_i - h_j = -\Delta h_p \tag{8.4}$$

where Δh_p is a non-negative control input.

Typically, exact values of the parameters v and p are not known but will be assumed to be positive and to take values in a known compact set. Furthermore, the functions $\mu(\cdot)$ and $\lambda(\cdot)$ are not precisely known. Only their properties of being in C^1, proper, monotonic increasing and zero for $q = 0$ will be guaranteed.

The varying demand for heating at the end-users in the hydraulic network is modelled by a (end-user) valve for which the hydraulic resistance can be changed in a piecewise constant way. Two types of pumps are present in the network; the end-user pumps, which are mainly used to meet the demand at the end-users, and booster pumps which are used to meet constraints on the relative pressures in the network.

Network Model

The model of the hydraulic network has been derived by using tools from circuit theory [2]. The network is comprised of m components and n end-users, where $m > n$. To the network there is associated a graph $\mathcal{G}$, where the nodes of $\mathcal{G}$ coincides with the terminals of the components and the edges of $\mathcal{G}$ coincides with the components themselves. A vector of independent flow variables is identified with the flows through the chords[1] of $\mathcal{G}$.

---

[1]Let $\mathcal{T}$ denote the spanning tree of $\mathcal{G}$, i.e. a connected subgraph which contains all nodes of $\mathcal{G}$ but no cycles. Then the edges of $\mathcal{G}$ which are not included in $\mathcal{T}$ are the chords of $\mathcal{G}$ (see [2]).

These flow variables have the property that they can be set independently of all other flow variables in the network. A fundamental flow loop is associated to each chord in $\mathcal{G}$ (i.e. to each independent flow variable). Along each of the fundamental flow loops Kirchhoff's voltage law holds, which can be expressed as

$$\mathbf{B}\Delta\mathbf{h} = 0, \qquad (8.5)$$

where $\mathbf{B} \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; $\Delta\mathbf{h}$ is a vector consisting of the pressure drops across the components in the network. The entries of the fundamental loop matrix $\mathbf{B}$ consist of $-1, 0, 1$, and its value depends on the structure of the network.

The class of hydraulic networks which are considered here satisfy the following three assumptions:

Assumption 2.1: [2] The graph $\mathcal{G}$ is connected.

Assumption 2.2: [2] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 8.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.

Assumption 2.3: [2] There exists one and only one component called the heat source. It corresponds to a valve[2] of the network, and it lies in all the fundamental loops.
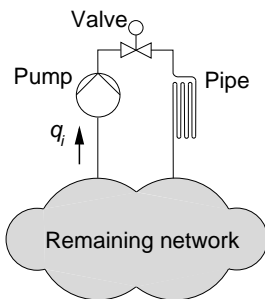
Figure 8.1: The series connection associated with each end-user.

Proposition 19. *[2] Any hydraulic network satisfying Assumptions 2.1 and 2.2 admits the representation:*

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}(\mathbf{B}^T\mathbf{q}) + \mathbf{u} \qquad (8.6)$$

$$y_i = \mu_i(q_i), \; i = 1, 2, \ldots, n \qquad (8.7)$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of independent flows; $\mathbf{u} \in \mathbb{R}^n$ is a vector of independent inputs, which is a linear combination of the delivered pump pressures; y_i is the pressure drop measured across the ith end-user valve (see (8.2)); $\mathbf{J} \in M(n; \mathbb{R})$ and $\mathbf{J} > 0$; $\mathbf{f}(\mathbf{B}^T\mathbf{q}) \in C^1$; $\mu_i(q_i)$ is the constitutive law of the ith end-user valve. In (8.7), it is assumed that the first n components coincide with the end-user valves.

---

[2]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

Defining $\mathbf{x} = \mathbf{B}^T\mathbf{q}$, *the map* $\mathbf{f}(\mathbf{x})$ *can be written as [2]:*

$$\mathbf{f}(\mathbf{x}) = -\mathbf{B}(\boldsymbol{\lambda}(\mathbf{x}) + \boldsymbol{\mu}(\mathbf{x})) \tag{8.8}$$

where $\boldsymbol{\lambda}(\mathbf{x}) = [\lambda_1(x_1), \ldots, \lambda_m(x_m)]^T$; $\boldsymbol{\mu}(\mathbf{x}) = [\mu_1(x_1), \ldots, \mu_m(x_m)]^T$.

Under Assumption 2.2 and Assumption 2.3, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix $\mathbf{B}$ are equal to 1 or 0.

A sketch of a simple district heating system with a heat source and two apartment buildings is illustrated in Fig. 8.2. The corresponding hydraulic network is illustrated in Fig. 8.3. The two end-users are represented by the series connections $\{c_{12}, c_{13}, c_{14}\}$ and $\{c_5, c_6, c_7\}$. The heat source is represented by the valve $\{c_{10}\}$ which models the pressure losses in the secondary side of the heat exchanger of the heat source.
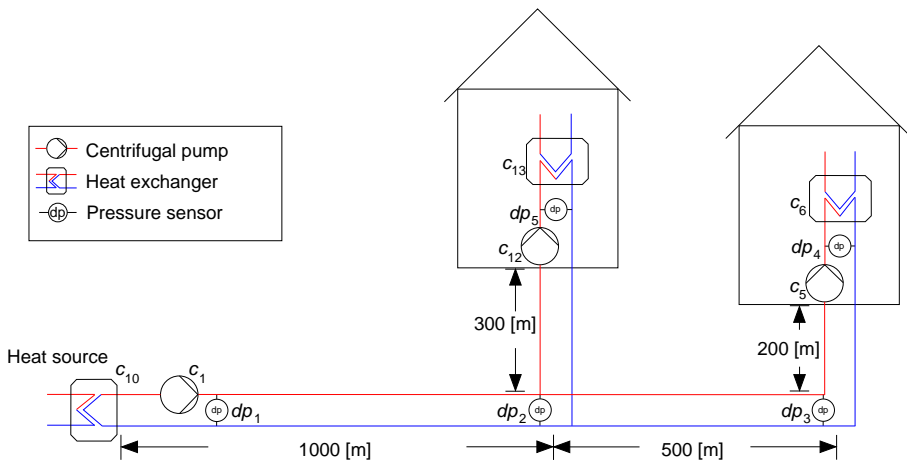
Figure 8.2: A sketch of a small district heating system.

3 Stabilization by Positive and Quantized Proportional Control

Pressure Regulation Problem

It is desired to regulate the pressure y_i across the ith end-user valve to a given reference value r_i with the use of a feedback controller using locally available information only. The vector $\mathbf{r} = (r_1, \ldots, r_n)$ of desired reference values is assumed to be piecewise constant, taking values in a known compact set $\mathcal{R}$:

$$\mathcal{R} = \{\mathbf{r} \in \mathbb{R}^n \mid 0 < r_m \leq r_i \leq r_M\} \tag{8.9}$$

For the purpose of practical output regulation, a set of decentralized proportional controllers will be the focus of the work presented here. The controllers considered will be of the form:

$$u_i = \begin{cases} -\gamma_i(y_i - r_i), & y_i \leq r_i \\ 0, & y_i \geq r_i \end{cases}, \ i = 1, 2, \ldots, n \tag{8.10}$$

where $\gamma_i > 0$ is the controller gain.
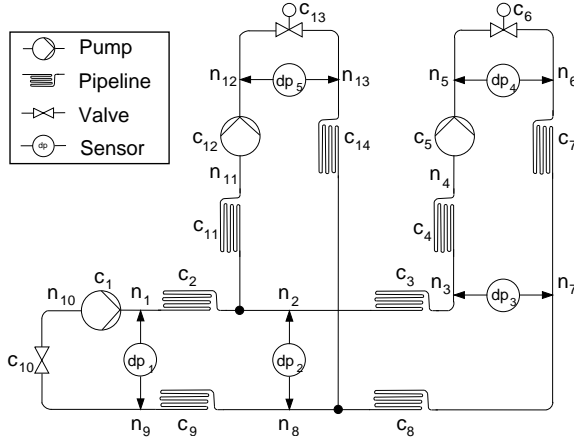
Figure 8.3: The hydraulic network diagram.

Quantization Map

This section describes the family of quantizers which will be considered in the exposition, which are a set of piecewise constant, non-decreasing functions taking non-negative values in a finite set. Furthermore, the quantizers will have hysteresis in order to prevent sliding modes and thereby chattering.

First, for $l \in \mathbb{Z}_+$ let $A = \{A_0, A_1, \ldots, A_l\}$ and $B = \{B_0, B_1, \ldots, B_{l+1}\}$ be the following family of intervals

$$A = \{(-\infty, \alpha_0], (\alpha_0, \alpha_1], \ldots, (\alpha_{l-2}, \alpha_{l-1}], (\alpha_{l-1}, \infty)\} \tag{8.11}$$

$$B = \{(-\infty, \beta_0], (\beta_0, \beta_1], \ldots, (\beta_{l-2}, \beta_{l-1}], (\beta_{l-1}, \beta_l], (\beta_l, \infty)\} \tag{8.12}$$

where l, α_i and β_j for $i = 0, 1, \ldots, l-1$ and $j = 0, 1, \ldots, l$ are design parameters of the quantizer and such that $\beta_i < \alpha_i < \beta_{i+1}$ for $i = 0, 1, \ldots, l-1$. Note that $\mathbb{R} = \bigcup_{i=1}^{l} A_i = \bigcup_{j=1}^{l+1} B_j$.

Let $\psi_m : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi_m(x(t)) = \begin{cases} \psi_k^A, & \text{if } t = t_0 \wedge x(t_0) \in A_k \\ \psi_k^A, & \text{if } x(t) = \beta_k \wedge \psi_m(x(t^-)) = \psi_{k+1}^B \text{ or} \\ & x(t) = \beta_k \wedge \psi_m(x(t^-)) = \psi_k^B, \ 1 \le k \le l \\ \psi_k^B, & \text{if } x(t) = \alpha_{k-1} \wedge \psi_m(x(t^-)) = \psi_k^A \text{ or} \\ & x(t) = \alpha_{k-1} \wedge \psi_m(x(t^-)) = \psi_{k-1}^A, \ 1 \le k \le l \\ \psi_0^A, & \text{if } x(t) = \beta_0 \wedge \psi_m(x(t^-)) = \psi_1^B \\ \psi_m(x(t^-)), & \text{otherwise} \end{cases}$$

$$\tag{8.13}$$

where ψ_k^A and ψ_k^B are design parameters of the quantizer, $\psi_0^A = 0$ and $\psi_{k-1}^A < \psi_k^B < \psi_k^A$ for all $k = 1, 2, \ldots, l$.

Remark 5: The map $\psi_m(\cdot)$ is defined for piecewise monotone signals $x : [t_0, t] \to \mathbb{R}$. There is a family of k partitions of $[t_o, t]$ denoted $I_1, I_2, \ldots, I_k$ where $I_1 = [t_0, t_1), I_2 = [t_1, t_2), \ldots, I_k = [t_{k-1}, t]$ and $t_i < t_{i+1} < t$ for $i = 0, 1, \ldots, k-2$, such that $x(\tau)$ is monotone for $\tau \in I_j$ for $j = 1, 2, \ldots, k$. Then t^- is defined as $t^- = \tau$ if $\tau \in \text{int}(I_{k-1})$.

Define $\mathbf{\Psi}_m : \mathbb{R}^n \to \mathbb{R}^n$ as $\mathbf{\Psi}_m(\mathbf{x}) = (\psi_m(x_1), \ldots, \psi_m(x_n))^T$, then the closed loop system with the quantized version of the proportional control actions is given as

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}(\mathbf{B}^T\mathbf{q}) + \mathbf{\Psi}_m(\mathbf{u}) \tag{8.14}$$

The quantized version $(\psi_m(u_i))$ of the control signal can be replaced with an expression depending on a quantized version of the system output $(\Upsilon_i(y_i))$ such that

$$\psi_m(-\gamma_i(y_i - r_i)) = -\gamma_i(\Upsilon_i(y_i) - r_i). \tag{8.15}$$

To this end, the following map is considered

$$\Upsilon_i(x(t)) = r_i + \begin{cases} -\dfrac{\psi_k^A}{\gamma_i}, & \text{if } t = t_0 \wedge -\gamma_i(x(t_0) - r_i) \in A_k \\[2mm] -\dfrac{\psi_k^A}{\gamma_i}, & \begin{aligned}&\text{if } -\gamma_i(x(t) - r_i) = \beta_k \wedge \Upsilon_i(x(t^-)) = r_i - \dfrac{\psi_{k+1}^B}{\gamma_i} \text{ or}\\ & -\gamma_i(x(t) - r_i) = \beta_k \wedge \Upsilon_i(x(t^-)) = r_i - \dfrac{\psi_k^B}{\gamma_i},\\ &\hspace{5cm} 1 \leq k \leq l\end{aligned} \\[6mm] -\dfrac{\psi_k^B}{\gamma_i}, & \begin{aligned}&\text{if } -\gamma_i(x(t) - r_i) = \alpha_{k-1} \wedge \Upsilon_i(x(t^-)) = r_i - \dfrac{\psi_k^A}{\gamma_i} \text{ or}\\ & -\gamma_i(x(t) - r_i) = \alpha_{k-1} \wedge \Upsilon_i(x(t^-)) = r_i - \dfrac{\psi_{k-1}^A}{\gamma_i},\\ &\hspace{5cm} 1 \leq k \leq l\end{aligned} \\[6mm] -\dfrac{\psi_0^A}{\gamma_i}, & \text{if } -\gamma_i(x(t) - r_i) = \beta_0 \wedge \Upsilon_i(x(t^-)) = r_i - \dfrac{\psi_1^B}{\gamma_i} \\[2mm] \Upsilon_i(x(t^-)), & \text{otherwise} \end{cases} \tag{8.16}$$

Define $\mathbf{Y} : \mathbb{R}^n \to \mathbb{R}^n$ as $\mathbf{Y}(\mathbf{x}) = (\Upsilon_1(x_1), \ldots, \Upsilon_n(x_n))^T$, and $\mathbf{\Gamma} = \text{diag}(\gamma_i)$, then the closed loop system (8.14) can be rewritten to

$$\mathbf{J}\dot{\mathbf{q}} = \mathbf{f}(\mathbf{B}^T\mathbf{q}) - \mathbf{\Gamma}(\mathbf{Y}(\mathbf{y}) - \mathbf{r}) \tag{8.17}$$

since the identity in (8.15) is fulfilled.

The closed loop system in (8.17) has a discontinuous right hand side. Solutions to this system will here be considered in the sense of Krasovskii solutions.

Definition 3.1: [3] A map $\varphi : I \to \mathbb{R}^n$ is a Krasovskii solution of an autonomous system of ordinary differential equations $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$, where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$, if it is absolutely continuous and for almost every $t \in I$ it satisfies the differential inclusion $\dot{\varphi}(t) \in K\mathbf{G}(\varphi(t))$, where $K\mathbf{G}(\mathbf{x}) = \bigcap_{\delta>0} \overline{co}\,\mathbf{G}(B_\delta(\mathbf{x}))$ and $\overline{co}\,\mathbf{G}$ is the convex closure of the set $\mathbf{G}$.

Here I is an interval of real numbers, possibly unbounded. By definition, the operators K associates to $\mathbf{G}(\mathbf{x})$ a set valued map which is compact for every $\mathbf{x} \in \mathbb{R}^n$. Furthermore, if $\mathbf{G}(\mathbf{x})$ is locally bounded this set valued map is upper semi-continuous with convex values [4]. Then, for each initial state $\mathbf{x}_0$, there exists at least one Krasovskii solution of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ [4].

Using the calculus given in [5] it can be calculated that the Krasovskii solutions of (8.17) are absolutely continuous functions which solves the Cauchy problem

$$\mathbf{J}\dot{\mathbf{q}} \in \mathbf{f}(\mathbf{B}^T\mathbf{q}) - \mathbf{\Gamma}(K(\mathbf{Y}(\mathbf{y})) - \mathbf{r}), \ \mathbf{q}(0) = \mathbf{q}_0 \tag{8.18}$$

where $K(\mathbf{Y}(\mathbf{y})) \subseteq \times_{i=1}^n K(\Upsilon_i(y_i))$ and $K(\Upsilon_i(y_i))$ is given by

$$K(\Upsilon_i(y_i)) = r_i + \begin{cases} -\frac{\psi_l^A}{\gamma_i}, & -\gamma_i(y_i - r_i) > \beta_l \\ \{-\lambda\frac{\psi_l^A}{\gamma_i}, \ \lambda \in [0,1]\}, & -\gamma_i(y_i - r_i) \in [\beta_0, \beta_l] \\ 0, & -\gamma_i(y_i - r_i) < \beta_0 \end{cases} \tag{8.19}$$

4 Stability Properties of Closed Loop System

In this section, the stability properties of the closed loop system introduced above will be examined. Subsequently, a more specific class of functions will be used in the expressions of $\boldsymbol{\mu}(\cdot)$ and $\boldsymbol{\lambda}(\cdot)$. This more specific class reflects the presence of turbulent[3] flows in the system [6]. The class of functions, which will be considered, is the following

$$\mu_i(x_i) = k_{vi}|x_i|x_i \tag{8.20}$$

$$\lambda_i(x_i) = k_{pi}|x_i|x_i \tag{8.21}$$

Let the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be given as

$$\mathbf{F}(\mathbf{z}) = \mathbf{y}(\mathbf{z}) - \mathbf{\Gamma}^{-1}\mathbf{f}(\mathbf{B}^T\mathbf{z}). \tag{8.22}$$

Proposition 20. *[7] For the class of functions defined in (8.20) and (8.21), the map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (8.22) is a homeomorphism.*

As a consequence of Proposition 20, there exists a unique vector $\mathbf{q}^* \in \mathbb{R}^n$ for each vector of reference values $\mathbf{r} \in \mathbb{R}^n$, and the relation between $\mathbf{r}$ and $\mathbf{q}^*$ is

$$\mathbf{q}^* = \mathbf{F}^{-1}(\mathbf{r}), \tag{8.23}$$

furthermore

$$\mathbf{r} = \mathbf{y}(\mathbf{q}^*) - \mathbf{\Gamma}^{-1}\mathbf{f}(\mathbf{B}^T\mathbf{q}^*). \tag{8.24}$$

Define $\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*$, then the expression for the closed loop system given in (8.17) can be replaced by

$$\mathbf{J}\dot{\mathbf{q}} \in \tilde{\mathbf{f}}(\tilde{\mathbf{q}}) - \mathbf{\Gamma}(K(\mathbf{Y}(\mathbf{y})) - \mathbf{y}(\mathbf{q}^*)) \tag{8.25}$$

where $\tilde{\mathbf{f}}(\tilde{\mathbf{q}}) = \mathbf{f}(\mathbf{B}^T(\tilde{\mathbf{q}} + \mathbf{q}^*)) - \mathbf{f}(\mathbf{B}^T\mathbf{q}^*)$.

The following conjecture will be instrumental in the derivation of the stability properties of the closed loop system.

Conjecture 3. *Under Assumption 2.2 and Assumption 2.3 the vector $\mathbf{q}^*$ defined by (8.23), with $\mathbf{\Gamma} = \text{diag}(\gamma_i)$, $\gamma_i > 0$ and $r_i > 0$ fulfils that $y_i(q_i^*) < r_i$.*

---

[3]Since the motivation for considering the new paradigm is reducing the diameters of the pipes used in the network, the likelihood for turbulent flows increases.

A proof of Conjecture 3 for $n = 2$ is given in [8]. Furthermore, the conjecture has been supported by numerical simulations of a small scale system with up to four end-users.

By (8.24), Conjecture 3 corresponds to $-f_i(\mathbf{B}^T\mathbf{q}^*) > 0$. Before stating the main result, the following Lemma will be given.

Lemma 4.1: *Let $\mathbf{q}^*$ be defined by (8.23) and $-f_i(\mathbf{B}^T\mathbf{q}^*) > 0$ by Conjecture 3, then for every $\mathbf{r} \in \mathcal{R}$, if $\psi_l^A > -f_i(\mathbf{B}^T\mathbf{q}^*)$ for every $i = 1, 2, \ldots, n$, there exists a bounded interval $I_i \subset \mathbb{R}$, such that for every $q_i \in I_i^c$*

$$(q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) > 0. \tag{8.26}$$

Furthermore

$$|q_i| \to \infty \Rightarrow (q_i - q_i^*)(\Upsilon(y_i) - y_i(q_i^*)) \to \infty. \tag{8.27}$$

Proof of Lemma 4.1. By (8.24), the property

$$\psi_l^A > -f_i(\mathbf{B}^T\mathbf{q}^*) \tag{8.28}$$

corresponds to

$$r_i - \frac{\psi_l^A}{\gamma_i} < y_i(q_i^*) \tag{8.29}$$

since $\mathbf{\Gamma} = \mathrm{diag}(\gamma_i)$ and $\gamma_i > 0$.

Furthermore, by Conjecture 3

$$r_i - \frac{\psi_l^A}{\gamma_i} < y_i(q_i^*) < r_i. \tag{8.30}$$

By the definition of $\Upsilon_i(y_i)$ in (8.16)

$$\Upsilon_i(y_i) = \begin{cases} r_i, & \forall y_i > r_i - \frac{\beta_0}{\gamma_i} \\ r_i - \frac{\psi_l^A}{\gamma_i}, & \forall y_i < r_i - \frac{\beta_l}{\gamma_i} \end{cases}. \tag{8.31}$$

Now, define the interval $I_i = \{q_i \in \mathbb{R} \mid y_i \in [r_i - \frac{\beta_l}{\gamma_i}, r_i - \frac{\beta_0}{\gamma_i}]\}$ which is bounded by continuity of $\mu_i(\cdot)$.

Since $\mu_i(\cdot)$ is monotonically increasing it follows that

$$(q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) > 0, \ \forall q_i \in I_i^c. \tag{8.32}$$

Furthermore, since q_i^* and $y_i(q_i^*)$ are constant and $\Upsilon_i(y_i)$ is bounded, from (8.32) it follows that

$$|q_i| \to \infty \Rightarrow (q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) \to \infty \tag{8.33}$$

which completes the proof. $\square$

The following proposition regarding the stability properties of the closed loop system can now be proved. The proposition states that for any gain Γ of the proportional control actions, there exists a value ψ_l^A of the quantizer and a compact set $\mathcal{Q}$, such that the trajectories of the closed loop system are attracted to $\mathcal{Q}$.

Proposition 21. *For any gain $\gamma_i > 0$ and for any value $l \in \mathbb{Z}_+$ and α_j, β_j, where $j = 0, 1, \ldots, l$, of the quantizer, such that $\beta_j < \alpha_j < \beta_{j+1}$, if the parameter ψ_l^A of the quantizer fulfils $\psi_l^A > -f_i(\mathbf{B}^T \mathbf{q}^*)$, where $\mathbf{q}^*$ is defined by (8.24), then a compact set $\mathcal{Q}$ exists, with the property that the Krasovskii solutions $\mathbf{q}(t)$ to the Cauchy problem (8.18) are attracted to $\mathcal{Q}$.*

Proof of Proposition 21. Recall, that $\tilde{\mathbf{q}}$ is defined by the following change of coordinates

$$\tilde{\mathbf{q}} = \mathbf{q} - \mathbf{q}^*. \tag{8.34}$$

The Lyapunov candidate function $V : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$V(\tilde{\mathbf{q}}) = \frac{1}{2} \langle \tilde{\mathbf{q}}, \mathbf{J}\tilde{\mathbf{q}} \rangle . \tag{8.35}$$

The time derivative of $V(\tilde{\mathbf{q}})$ is then given as

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) = \langle \tilde{\mathbf{q}}, \mathbf{J}\dot{\mathbf{q}} \rangle \tag{8.36}$$

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) \in \left\langle \tilde{\mathbf{q}}, \tilde{\mathbf{f}}(\tilde{\mathbf{q}}) - \mathbf{\Gamma}(K(\mathbf{Y}(\mathbf{y})) - \mathbf{y}(\mathbf{q}^*)) \right\rangle \tag{8.37}$$

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) \in \left\langle \tilde{\mathbf{q}}, \tilde{\mathbf{f}}(\tilde{\mathbf{q}}) \right\rangle - \langle \tilde{\mathbf{q}}, \mathbf{\Gamma}(K(\mathbf{Y}(\mathbf{y})) - \mathbf{y}(\mathbf{q}^*)) \rangle \tag{8.38}$$

It can be shown that the following inequality holds [7]

$$W(\tilde{\mathbf{q}}) \equiv - \left\langle \tilde{\mathbf{q}}, \tilde{\mathbf{f}}(\tilde{\mathbf{q}}) \right\rangle > 0, \tag{8.39}$$

from which it follows

$$\frac{d}{dt} V(\tilde{\mathbf{q}}) < - \langle \tilde{\mathbf{q}}, \mathbf{\Gamma}(\boldsymbol{v} - \mathbf{y}(\mathbf{q}^*)) \rangle , \ \forall \boldsymbol{v} \in K(\mathbf{Y}(\mathbf{y})). \tag{8.40}$$

Define the set $I = \{\mathbf{q} \in \mathbb{R}^n \mid q_i \in I_i\}$, with I_i defined by Lemma 4.1, then it follows that there exists a finite $M > 0$ such that

$$\sum_{i=1}^{n} \gamma_i (q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) > -M, \ \forall \mathbf{q} \in I, \tag{8.41}$$

since $\gamma_i > 0$, q_i^* and $y_i(q_i^*)$ are constants and q_i and $\Upsilon_i(y_i)$ belong to a bounded set.

Furthermore, since $(q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) > 0$ for every $q_i \in I_i^c$, consequently

$$\sum_{i=1}^{n} \gamma_i (q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) > -M, \ \forall \mathbf{q} \in \mathbb{R}^n. \tag{8.42}$$

From Lemma 4.1, and (8.42) it follows that

$$|\mathbf{q}| \to \infty \Rightarrow \sum_{i=1}^{n} \gamma_i (q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) \to \infty. \tag{8.43}$$

From (8.43) and Lemma 4.1 it is concluded that there exists a compact set $\mathcal{Q} \supset I$, with the property

$$\langle \tilde{\mathbf{q}}, \mathbf{\Gamma}(\mathbf{Y}(\mathbf{y}) - \mathbf{y}(\mathbf{q}^*)) \rangle = \sum_{i=1}^{n} \gamma_i (q_i - q_i^*)(\Upsilon_i(y_i) - y_i(q_i^*)) > 0, \ \forall \mathbf{q} \in \mathcal{Q}^c, \quad (8.44)$$

and consequently for every $\mathbf{q} \in \mathcal{Q}^c$

$$- \langle \tilde{\mathbf{q}}, \mathbf{\Gamma}(\boldsymbol{v} - \mathbf{y}(\mathbf{q}^*)) \rangle < 0, \ \forall \boldsymbol{v} \in K(\mathbf{Y}(\mathbf{y})), \quad (8.45)$$

and the thesis follows. $\qquad\square$

Remark 6: Since the result is global and independent on the number n of end-users in the system it follows that end-users can be added to or removed from the system while maintaining stability in the sense that a compact set $\mathcal{Q}$ which attracts system trajectories will exist for the newly obtained system, given that (8.28) is fulfilled. However, to keep the same level of performance it may be necessary to adjust the gains γ_i and quantization parameters ψ_i^A and l.

Furthermore, if logarithmic quantizers are considered, practical output regulation have been proved in [2].

To that end, let l be a positive integer, ψ_0 a positive real number, $\delta \in (0, 1)$, and $\psi_k = \rho^k \psi_0$ for $k = 1, 2, \ldots, l$ with $\rho = \frac{1-\delta}{1+\delta}$ (i.e. $\psi_k = \frac{1-\delta}{1+\delta} \psi_{k-1}$). The following (logarithmic) quantizer is then proposed [2]:

Let $\psi : \mathbb{R} \to \mathbb{R}$ be the map

$$\psi(x) = \begin{cases} \psi_0 , & \frac{\psi_0}{1-\delta} < x \\ \psi_k , & \frac{\psi_k}{1+\delta} < x \leq \frac{\psi_k}{1-\delta}, \ 0 \leq k \leq l \\ \frac{\psi_k}{1+\delta} , & \frac{\psi_k}{(1+\delta)^2} < x \leq \frac{\psi_k}{1-\delta^2}, \ 0 \leq k \leq l \\ 0 , & 0 \leq x \leq \frac{\psi_l}{1+\delta} \end{cases} \quad (8.46)$$

The parameters l, ψ_0 and δ of the map (quantizer) are to be designed.

Then, the following proposition is proved in [2]:

Proposition 22. *For any choice of the parameter $q_M > 0$, any compact set $\mathcal{R} \subset \mathbb{R}_+$, any compact set $\mathcal{Q}$ of initial conditions described by*

$$\mathcal{Q} = \{\mathbf{q} \in \mathbb{R}^n \mid |q_i| \leq q_M , \ i = 1, \ldots, n\} , \quad (8.47)$$

for any arbitrarily small positive number ε, and for any value of the quantization parameter $\delta \in (0, 1)$ there exist gains $\gamma_i^* > 0$ and parameters ψ_0, l of the quantizer such that for all $\gamma_i > \gamma_i^*$, for any $\mathbf{r} \in \mathcal{R}$, any Krasovskii solution $\mathbf{q}(t)$ of the closed loop system (8.14) with initial condition in $\mathcal{Q}$ is attracted by the set $\{\boldsymbol{\epsilon} \in \mathbb{R}^n \mid |\epsilon_i| \leq \varepsilon , \ i = 1, \ldots, n\}$, where $\epsilon_i = y_i - r_i$.

Furthermore, it is remarked in [2] that Proposition 22 holds for other quantizers as well, such as the uniform quantizer for instance.

5 Numerical Results

In this section, the results of numerical simulations performed on the closed loop system, are presented. The hydraulic network diagram of the system used in the simulations is shown in Fig. 8.4. The system is a laboratory scale system, in which four end-users are present. The end-users are represented by the series connections $\{c_4, c_5, c_6\}$, $\{c_9, c_{10}, c_{11}\}$, $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$. The quantization map used throughout
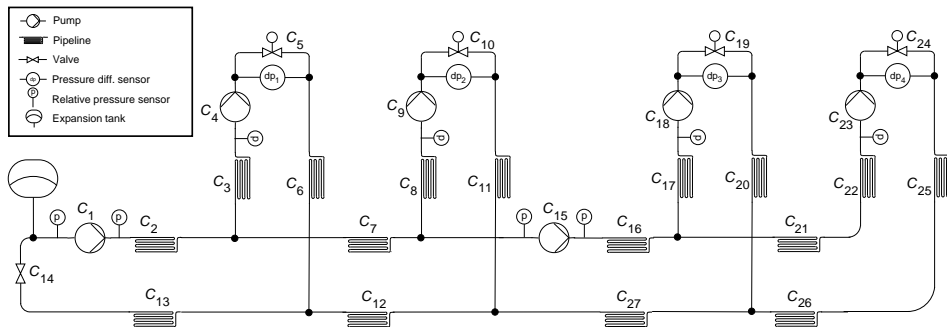

Figure 8.4: Diagram of the hydraulic network used in the simulation.

the simulations is the logarithmic quantizer introduced in (8.46). The parameters used in the simulation are: $J_{11} = 1.0787$, $J_{12} = J_{13} = J_{14} = J_{21} = J_{31} = J_{41} = 0.4421$, $J_{22} = 1.1318$, $J_{23} = J_{24} = J_{32} = J_{42} = 0.7074$, $J_{33} = 1.4854$, $J_{34} = J_{43} = 1.061$, $J_{44} = 1.7507$; $p_2 = p_{13} = 0.0586$, $p_3 = p_6 = 0.6755$, $p_7 = p_{12} = p_{21} = p_{26} = 0.0352$, $p_8 = p_{11} = p_{17} = p_{20} = p_{22} = p_{25} = 0.4503$, $p_{16} = p_{27} = 0.0469$; $v_5 = v_{10} = v_{19} = v_{24} = 0.005$, $v_{14} = 0.0013$; $\mathbf{r} = 0.2\mathbb{1}_4$; $\mathbf{\Gamma} = 2I_4$; $\psi_0 = 0.5$; $l = 2$; $\delta = 0.5$.

A scenario is simulated, where the end-user connections $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are first removed from and then later re-introduced into the system. The removal of the end-users are simulated by changing the parameters v_{19} and v_{24} to a large value $(0.005 + 1.25 \cdot 10^3)$, thereby reducing the flows q_{c19} and q_{c24} to close to zero.

In Fig. 8.5 the results of the simulations are shown. As it is evident from the figure, system trajectories are bounded and practical output regulation is achieved, both in the situation where all four end-users are present as well as when only two are. However, as shown in Section 4 asymptotic stability is generally not achievable, and limit-cycle-type behaviour is possible as shown in [9].

6 Conclusion

An industrial case study involving a large scale hydraulic network has been examined. The hydraulic network underlies a district heating system. Specifically, stability properties of the closed loop system using quantized proportional control actions constrained to non-negative values were investigated. Particularly, the quantized control actions was constrained to take values in a finite set, thereby making it possible to send them across a communication network using a finite bandwidth. The stability analysis shows that given a properly chosen upper level of the quantizer, a compact set $\mathcal{Q}$ exists with the property that all closed loop system trajectories are globally attracted to it. Furthermore, by
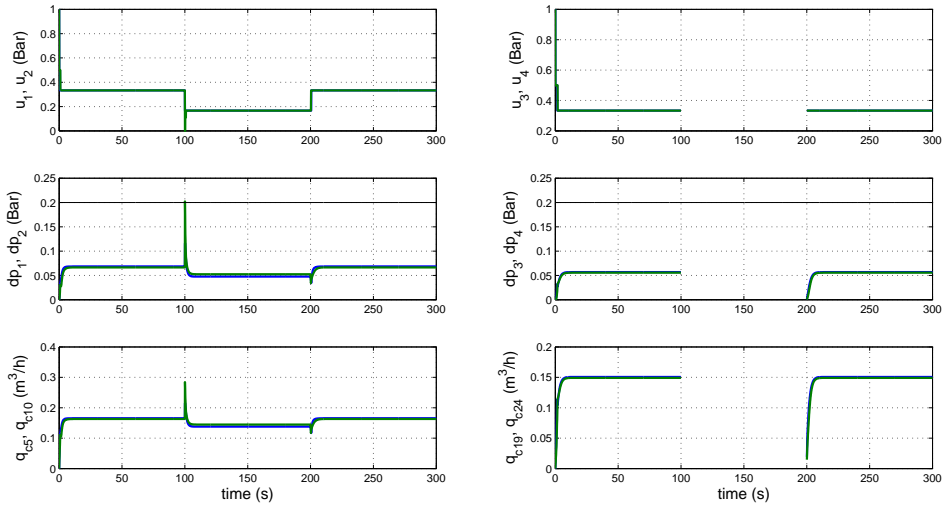
Figure 8.5: Results of the simulation performed on the system in Fig. 8.4 in closed loop with the proposed controllers. The figure shows control inputs u_1, u_2, u_3, u_4, the controlled variable dp_1, dp_2, dp_3, dp_4, and the flow through valves $c_{24}, c_{19}, c_{10}, c_5$ obtained with the quantized proportional feedback control. At time 100 s, the end-user connections consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are removed from the system. At time 200 s the end-user connections are re-inserted into the system. The solid line at 0.2 Bar in the two middle plots indicates the reference value.

a proper design of the parameters of the quantizer and the control gain, practical output regulation is achieved. Since these results are both global and independent of the number of end-users in the system, it is concluded that end-users can be added to and removed from the system, while still maintaining the property that a compact set $\mathcal{Q}$, which globally attracts system trajectories, exists for the newly obtained system. However, to keep the same level of performance may require an adjustment of the parameters of the quantizer and the controller gains.

Some natural future extensions of the work presented here are considered the introduction of delays in the communication network as well as the introduction of integral control actions. Since a communication network in practice is likely to introduce delays in the control loop, it is considered necessary to examine the stability properties of the closed loop system when such delays are introduced. The addition of integral control actions are considered natural to accommodate for the steady state output regulation error which is present with the proportional control actions only.

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Paper F

Output Regulation of Large-Scale Hydraulic Networks

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Abstract

An industrial case study involving a large-scale hydraulic network is examined. The hydraulic network underlies a district heating system. The network is subject to structural changes in the sense that end-users may be added to or removed from the network. The problem of output regulation in the network is addressed. The results show that semi-global exponential output regulation is achievable using a set of decentralized proportional-integral control actions. Furthermore, by adding an assumption about the behaviour of the components in the system, which is justified in practice, global asymptotic output regulation is shown. The fact that the result is global and independent on the number of end-users has the consequence that structural changes can be made in the network while maintaining the stability properties of the system. Furthermore, the decentralized nature of the control architecture eases the implementation of structural changes in the network.

1   Introduction

The work presented here concerns an industrial case study involving a large-scale hydraulic network. The hydraulic network underlies a district heating system. The case study considers a new paradigm for the design of district heating systems, in which it has been proposed to reduce the diameter of the pipes in the network. By reducing the pipe diameter, it is possible to reduce the heat dispersion from the pipes and thereby reduce the energy losses in the system [2]. On the other hand, the reduced diameters induce increased pressure losses throughout the network which must be compensated by multiple pumps. Studies held that the multi-pump architecture is the technology which can compensate for the increased pressure losses while still achieving a substantial energy saving ([1]). The multi-pump architecture raises the question of how the pumps should be operated to control the network in appropriate way. The new paradigm also gives rise to a flexible network structure in which end-users can be added to or removed from the network. The case study is part of the ongoing research program *Plug & Play Process Control* [3] which considers automatic reconfiguration of the control system if components such as sensors, actuators or subsystems are added to or removed from a system.

To fulfil the control objective, which is to keep the pressure across the so-called end-user valves at a constant reference, a set of proportional-integral control actions is proposed. The control actions are decentralized in the sense that the individual controllers use only locally available information, which is the pressure measurement at the end-user. The results show that it is possible to achieve semi-global exponential output regulation using this control architecture. By adding an additional assumption regarding the constitutive relation of the components in the system, it is, furthermore, possible to show global asymptotic output regulation. These two results represent an important extension of [5], where semi-global practical stability was achieved by proportional controllers, and we believe they are instrumental for further developments which are briefly discussed in conclusions of the paper.

In Section 2, the model of the system is introduced along with the output regulation problem. The main results of the paper are presented in Section 3. In Section 4, the results of numerical simulations of the closed loop system are presented. Finally, conclusions are given in Section 5 along with possible future research directions.

Nomenclature: For a vector $x \in \mathbb{R}^n$, x_i denotes the ith element of x. Let $M(n, m; \mathbb{R})$ denote the set of $n \times m$ matrices with real entries, and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$. For a square matrix A, $A > 0$ means that A is positive definite. For a square matrix A, $A = \text{diag}(x_i)$ means that A has x_i as entries on the main diagonal and zero elsewhere. For a matrix A, A_{ij} will be used to denote the entry in the ith row and jth column of A. A matrix A is said to be *Hurwitz* if all eigenvalues of A have strictly negative real part. Throughout the paper, C^1 denotes a continuously differentiable map. A continuous map is said to be *proper* if the inverse image of a compact set is compact. A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is order preserving, i.e., for all x and y such that $x \leq y$ then $f(x) \leq f(y)$. For a map $f : \mathbb{R}^n \to \mathbb{R}^m$, let $Df(\cdot)$ denote the Jacobian matrix of $f(\cdot)$.

2 System Model

The system under consideration is a hydraulic network comprising a district heating system. The model has been derived in detail in [4] and will be recalled here, but in fewer details.

The hydraulic network consists of a number of connections between two-terminal components, which in this work are: valves, pipes and pumps. The kth system component is characterized by dual variables, the first of which is the pressure drop Δh_k across it

$$\Delta h_k = h_i - h_j, \tag{9.1}$$

where i, j are nodes in the network; h_i, h_j are the relative pressures at the nodes.

The second variable characterizing the component is the fluid flow q_k through it. The components have algebraic or dynamic expressions governing the relationships between the two variables.

Valves

The behaviour of valves in the network is governed by the following algebraic expression

$$h_i - h_j = \mu_k(q_k) \equiv \mu_k(v_k, q_k), \tag{9.2}$$

where v_k is the hydraulic resistance of the valve; $\mu_k(\cdot)$ is a C^1 and proper function, which for any fixed value of v_k is zero at $q_k = 0$, strictly monotonically increasing and $\mu_k(v_k, \cdot) = 0$ for $v_k = 0$.

Pipes

The behaviour of pipes in the network is governed by the dynamic equation

$$\mathcal{J}_k \dot{q}_k = (h_i - h_j) - \lambda_k(q_k) \tag{9.3}$$

where $\lambda_k(q_k) \equiv \lambda(p_k, q_k)$; $\mathcal{J}_k$ and p_k are parameters of the pipe; $\lambda_k(\cdot)$ is a function with the same properties as $\mu_k(\cdot)$.

Pumps

A (typically centrifugal) pump is a component which delivers a desired pressure differ-
ence Δh_k regardless of the value of the fluid flow through it. Thus, the behaviour of
pumps in the network is governed by the following expression

$$h_i - h_j = -\Delta h_{p,k}, \tag{9.4}$$

where $\Delta h_{p,k}$ is a non-negative control input. In this paper we disregard the constraint
on control input and refer the interested reader to [5] where positive *proportional* control
laws have been studied.

Component Model

A generalised component model can be derived using the following expression

$$\Delta h_k = \mathcal{J}_k \dot{q}_k + \lambda_k(q_k) + \mu_k(q_k) - \Delta h_{p,k}, \tag{9.5}$$

where $\mathcal{J}_k, p_k$ are non-zero for pipe components and zero for other components; v_k is non-
zero for valve components and zero for other components; $\Delta h_{p,k}$ is non-zero for pump
components and zero for other components.

The values of the parameters p_k and v_k are typically unknown, but they will be as-
sumed to take values in a compact set of non-negative values. Likewise, the functions
$\mu_k(q_k)$ and $\lambda_k(q_k)$ are not precisely known, only their properties of being C^1, monotone,
zero in $q_k = 0$ and proper are guaranteed. The varying heating demand of the end-users,
which is the main source of disturbances in the system, is modelled by a (end-user) valve
with variable hydraulic resistance. In the network model, a distinction is to be made be-
tween end-user valves and the rest of the valves in the network. Two types of pumps are
present in the network; the end-user pumps, which are mainly used to meet the demand
at the end-users, and booster pumps which are used to meet constraints on the relative
pressures in the network [6].

Network Model

The network model has been derived using standard circuit theory [4]. The hydraulic net-
work consists of m components and n end-users $(m > n)$. The network is associated with
a graph $\mathcal{G}$ which has nodes coinciding with the terminals of the network components. The
edges of the network are the components themselves. The graph satisfies the following:

Assumption 2.1: *[5]* $\mathcal{G}$ *is a connected graph.*

By the use of graph theory, a set of n independent flow variables q_i have been iden-
tified. These flow variables have the property that their values can be set independently
from other flows in the network. The independent flow variables coincide with the flows
through the chords[1] of the graph [4]. To each chord in the graph, a fundamental (flow)

---

[1] Let $\mathcal{T}$ denote the spanning tree of $\mathcal{G}$, i.e. a connected subgraph which contains all nodes of $\mathcal{G}$ but no
cycles. Then the edges of $\mathcal{G}$ which are not included in $\mathcal{T}$ are the chords of $\mathcal{G}$ (see [4]).

loop is associated, and along this loop Kirchhoffs voltage law holds. This means that the following equality applies

$$B\Delta h = 0, \tag{9.6}$$

where $B \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; Δh is a vector consisting of the pressure drops across the components in the network.

The entries of the fundamental loop matrix B are $-1, 1$ or 0, depending on the network topology. For the case study in question, the hydraulic network underlies a district heating system. Because of the latter, the following statements can be made regarding the network.

Assumption 2.2: [5] Each end-user valve is in series with a pipe and a pump, as seen in Fig. 9.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.

Assumption 2.3: [5] There exists one and only one component called the heat source. It corresponds to a valve[2] of the network, and it lies in all the fundamental loops.
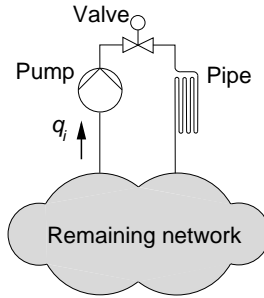
Figure 9.1: The series connection associated with each end-user [4].

Proposition 23. *[5] Any hydraulic network satisfying Assumptions 2.1 and 2.2 admits the representation*

$$J\dot{q} = f(B^T q) + u \tag{9.7}$$

$$y_i(q_i) = \mu_i(q_i) , \ i = 1, \ldots, n, \tag{9.8}$$

where $q \in \mathbb{R}^n$ is the vector of independent flows; $u \in \mathbb{R}^n$ is a vector of independent inputs consisting of a linear combination of the delivered pump pressures; y_i is the measured pressure drop across the ith end-user valve; $J \in M(n; \mathbb{R})$, $J > 0$; $f(\cdot)$ is a C^1 vector field; $\mu_i(\cdot)$ is the fundamental law of the ith end-user valve. In (9.8), it is assumed that the first n components coincide with the end-user valves.

Under Assumptions 2.1-2.3, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix B are equal to 1 or 0, where B_{ij} is 1 if component j belongs to fundamental flow loop i and 0 otherwise.

---

[2]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

Defining the vector of flows through the components in the system as $x = B^T q \in \mathbb{R}^m$, the vector field $f(\cdot)$ can be written as [4]

$$f(x) = -B(\lambda(x) + \mu(x)), \; \forall x \in \mathbb{R}^m, \tag{9.9}$$

where $\lambda(x) = [\lambda_1(x_1), \ldots, \lambda_m(x_m)]^T$; $\mu(x) = [\mu_1(x_1), \ldots, \mu_m(x_m)]^T$ and $\lambda_i(\cdot)$ is non-zero for pipe components and $\mu_i(\cdot)$ is non-zero for valve components.

The matrix J in (9.7) is given by

$$J = B\mathcal{J}B^T \tag{9.10}$$

where $\mathcal{J} = \mathrm{diag}(\mathcal{J}_1, \ldots, \mathcal{J}_m)$.

Output Regulation Problem

It is desired to regulate the pressure y_i across the ith end-user valve to a given reference value r_i with the use of a feedback controller having available only local information. The vector $r = (r_1, \ldots, r_n)$ of reference values takes values in a known compact set $\mathcal{R}$:

$$\mathcal{R} = \{ r \in \mathbb{R}^n \mid 0 < r_m \leq r_i \leq r_M \}. \tag{9.11}$$

For the purpose of asymptotic output regulation, a set of decentralized proportional-integral controllers is the focus of the work presented here. The controllers considered will be of the form

$$\dot{\xi}_i = -K_i(y_i(q_i) - r_i) \tag{9.12}$$

$$u_i = \xi_i - N_i(y_i(q_i) - r_i) \tag{9.13}$$

where $K_i, N_i > 0$ and $i = 1, 2, \ldots, n$.

3   Stability properties of closed loop system

In this section, the stability properties of the closed loop system will be examined. First, it is shown that global asymptotic output regulation can be proved by adding an additional assumption on the algebraic relations governing the components in the system. Secondly, semi-global exponential stability is shown.

Global Asymptotic Stability

If it is further assumed that the functions $\lambda_i(\cdot)$, which govern the behaviour of pipes, have derivatives $\frac{\partial}{\partial x_i}(\lambda_i(x_i))$ bounded away from zero, then it is possible to show global asymptotic stability of the closed loop system. This assumption is motivated by the fact that for small values, the flow through the pipes can be considered laminar [7].

First, define the proportional gain matrix $N = \mathrm{diag}(N_i)$. The following lemma will be instrumental in deriving the closed loop stability properties of the system.

Lemma 3.1: *Let the matrix $G(q) \in M(n, \mathbb{R})$ be given by*

$$G(q) \equiv NDy(q) - Df(B^T q), \tag{9.14}$$

then $G(q) > 0$.

Proof. Again, let $x = B^T q$ and recall that

$$f(x) = -B\left(\lambda(x) + \mu(x)\right) \tag{9.15}$$

where $\lambda(x) = (\lambda_1(x_1), \ldots, \lambda_m(x_m))^T$ and $\mu(x) = (\mu_1(x_1), \ldots, \mu_m(x_m))^T$. Then $-Df(B^T q)$ is given by

$$-Df(B^T q) = -\frac{\partial}{\partial x} f(x) \frac{\partial x}{\partial q} \tag{9.16}$$

$$= B\Lambda(x) B^T. \tag{9.17}$$

where $\Lambda(x) = \mathrm{diag}(\frac{\partial}{\partial x_i}(\lambda_i(x_i) + \mu_i(x_i)))$.

By Assumption 2.2, each chord in the graph $\mathcal{G}$ described by the network corresponds to a pipe in series with a user valve. Therefore, by rearranging the numbering of the components, such that the first n components are the pipes in the chords of $\mathcal{G}$, (9.17) can be rewritten as

$$-Df(B^T q) = \begin{pmatrix} I_n & F \end{pmatrix} \begin{bmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{bmatrix} \begin{pmatrix} I_n \\ F^T \end{pmatrix} \tag{9.18}$$

$$= \Lambda_1(x) + F\Lambda_2(x)F^T \tag{9.19}$$

where $\Lambda_1(x) = \mathrm{diag}(\frac{\partial}{\partial x_i}(\lambda_i(x_i)))$ for $i = 1, \ldots, n$ and $\Lambda_2(x) = \mathrm{diag}(\frac{\partial}{\partial x_i}(\lambda_i(x_i) + \mu_i(x_i)))$ for $i = n+1, \ldots, m$.

Since $\lambda_i(\cdot)$ are monotonically increasing functions with derivatives $\frac{\partial}{\partial x_i}(\lambda_i(x_i))$ bounded away from zero, the matrix $\Lambda_1(x)$ is positive definite for all x. Furthermore, since $\mu_i(\cdot)$ are monotonically increasing functions, the matrix $\Lambda_2(x)$ is positive semi definite for all x (recall that $\lambda_i(\cdot)$ is non-zero only for pipe components and $\mu_i(\cdot)$ is non-zero only for valve components). Then it follows that $-Df(B^T q)$ is positive definite.

The matrix $Dy(q)$ is given by

$$Dy(q) = \mathrm{diag}(\frac{\partial}{\partial q_i} y_i(q_i)). \tag{9.20}$$

Recall that $y_i(q_i) = \mu_i(q_i)$, and $\mu_i(\cdot)$ is a monotonically increasing function. As a consequence $Dy(q)$ is positive semi definite. Since N is diagonal and positive definite and $Dy(q)$ is positive semi definite diagonal, it follows that $NDy(q)$ is positive semi definite.

From the derivations above, it is concluded that $NDy(q) - Df(B^T q)$ is a positive definite matrix. $\qquad \square$

Since the functions $\mu_i(\cdot)$ are monotonically increasing and proper, they admit inverses $\mu_i^{-1}(\cdot)$. Now, let

$$q_i^* \equiv \mu_i^{-1}(r_i), \tag{9.21}$$

that is: q_i^* is the flow through the ith end-user valve which produce the reference output.

Furthermore, define

$$\tilde{q} \equiv q - q^*, \tag{9.22}$$

then the main result of this subsection can be stated.

Proposition 24. *The point $(\tilde{q}^T, \dot{q}^T) = 0$ is a globally asymptotically stable equilibrium point of the closed loop system given by (9.7), (9.8), (9.12) and (9.13).*

Proof. Define the variable

$$\tilde{y}_i(\tilde{q}_i) \equiv \mu_i(\tilde{q}_i + q_i^*) - r_i. \tag{9.23}$$

The following Lemma has been derived in [5]:

Lemma 3.2: The function $\tilde{y}_i(\tilde{q}_i)$ is monotonically increasing and zero at $\tilde{q}_i = 0$, and moreover

$$\tilde{y}_i(\tilde{q}_i)\tilde{q}_i > 0, \ \forall \ \tilde{q}_i \neq 0 . \tag{9.24}$$

The closed loop system defined by (9.7), (9.8), (9.12) and (9.13) is

$$J\dot{q} = f(B^T q) + \xi - N\tilde{y}(\tilde{q}) \tag{9.25}$$

$$\dot{\xi} = -K\tilde{y}(\tilde{q}) \tag{9.26}$$

From (9.25) the following can be derived

$$J\ddot{q} = Df(B^T q)\dot{q} + \dot{\xi} - NDy(q)\dot{q} \tag{9.27}$$

which can be rewritten as

$$J\ddot{q} = -G(q)\dot{q} - K\tilde{y}(\tilde{q}). \tag{9.28}$$

In the above $J > 0$, $G(q) > 0$ by Lemma 3.1 and $K\tilde{y}(\tilde{q})$ can be written as $\nabla W(\tilde{q})$, with $W(\tilde{q}) > 0$ given as

$$W(\tilde{q}) = \sum_{i=1}^{n} K_i \int_0^{\tilde{q}_i} \tilde{y}_i(s)ds. \tag{9.29}$$

Therefore, the structure of (9.28) is similar to that of a mechanical system in the standard Lagrangian form [8].

This motivates the choice of the Lyapunov function candidate $V : \mathbb{R}^{2n} \to \mathbb{R}$, which can be seen as an equivalent of the total energy function for a mechanical system:

$$V(\tilde{q}, \dot{q}) = \sum_{i=1}^{n} K_i \int_0^{\tilde{q}_i} \tilde{y}_i(s)ds + \frac{1}{2}\dot{q}^T J\dot{q} \tag{9.30}$$

which is positive definite and radially unbounded.

The time derivative of $V(\tilde{q}, \dot{q})$ is

$$\frac{d}{dt}V(\tilde{q}, \dot{q}) = \dot{q}^T J\ddot{q} + \dot{q}^T K\tilde{y}(\tilde{q}) \tag{9.31}$$

$$= -\dot{q}^T G(q)\dot{q}. \tag{9.32}$$

From Lemma 3.1 it then follows that $\frac{d}{dt}V(\tilde{q}, \dot{q}) < 0$ for every $\dot{q} \neq 0$ and consequently that all trajectories are bounded and $\dot{q} \to 0$ as $t \to \infty$.

From (9.28) it follows that $K\tilde{y}(\tilde{q}) \to 0$ as $t \to \infty$. Since $\tilde{y}_i(\cdot)$ is monotonically increasing and zero in zero, it is concluded that $\tilde{q} \to 0$ as $t \to \infty$. $\qquad\square$

From (9.21) and (9.23) it can be seen as $\tilde{y}_i(\tilde{q}_i) \to 0$ the output $\mu_i(q_i) \to r_i$.

Since Proposition 24 is independent of the number n of end-users in the system, it follows that end-users can be added to or removed from the system while maintaining asymptotic output regulation.

Semi-global Exponential Stability

In this section semi-global exponential output regulation of the closed loop system is shown. This result does not depend on the derivatives $\frac{\partial}{\partial x_i}(\lambda_i(x_i))$ being bounded away from zero. Rather, it reposes on the assumption that $\frac{\partial}{\partial x_i}(\mu_i(x_i))|_{x_i=q_i^*} \neq 0$.

First, some preliminaries will be instrumental.

Perform the change of coordinates

$$\tilde{\xi} \equiv \xi + f(B^T q^*), \tag{9.33}$$

so as to obtain

$$J\dot{\tilde{q}} = \tilde{f}(\tilde{q}) + \tilde{\xi} - N\tilde{y}(\tilde{q}) \tag{9.34}$$

$$\dot{\tilde{\xi}} = -K\tilde{y}(\tilde{q}), \tag{9.35}$$

where $\tilde{f}(\tilde{q}) = f(B^T(\tilde{q} + q^*)) - f(B^T q^*)$.

Let $\tilde{F} \in M(n, \mathbb{R})$ be a Hurwitz matrix and define further the new coordinate ([9])

$$\chi = \tilde{\xi} - \tilde{F}J\tilde{q}$$

which yields

$$J\dot{\tilde{q}} = \tilde{f}(\tilde{q}) + \chi + \tilde{F}J\tilde{q} - N\tilde{y}(\tilde{q}) \tag{9.36}$$

$$\dot{\chi} = -K\tilde{y}(\tilde{q}) - \tilde{F}\tilde{f}(\tilde{q}) - \tilde{F}\chi - \tilde{F}^2 J\tilde{q} + \tilde{F}N\tilde{y}(\tilde{q}). \tag{9.37}$$

Lemma 3.3: [10] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function in a convex neighborhood U of 0 in $\mathbb{R}^n$, with $f(0) = 0$. Then

$$f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \ldots, x_n) \tag{9.38}$$

for some suitable C^1 functions $g_i : \mathbb{R}^n \to \mathbb{R}$ defined in U, with $g_i(0) = \frac{\partial}{\partial x_i} f(0)$.

By Lemma 3.3 the map $\tilde{f}(\tilde{q})$ can be written as

$$\tilde{f}(\tilde{q}) = \hat{\phi}(\tilde{q})\tilde{q} \tag{9.39}$$

with $\hat{\phi}(\tilde{q})$ a continuously differentiable matrix.

Choosing $K = \tilde{F}N$, the closed-loop system can be written more simply as

$$\dot{\chi} = -\tilde{F}\chi - \phi^{(a)}(\tilde{q})\tilde{q} \tag{9.40}$$

$$J\dot{\tilde{q}} = \phi^{(b)}(\tilde{q})\tilde{q} + \chi - N\tilde{y}(\tilde{q}) \tag{9.41}$$

where

$$\phi^{(a)}(\tilde{q}) = \tilde{F}\hat{\phi}(\tilde{q}) + \tilde{F}^2 J \tag{9.42}$$

$$\phi^{(b)}(\tilde{q}) = \hat{\phi}(\tilde{q}) + \tilde{F}J. \tag{9.43}$$

Again by Lemma 3.3, the relation between $\tilde{q}_i$ and $\tilde{y}_i$ can be written as

$$\tilde{y}_i(\tilde{q}_i) = g_i(\tilde{q}_i)\tilde{q}_i. \tag{9.44}$$

Observe that, as $\frac{\partial}{\partial \tilde{q}_i}(\tilde{y}_i(\tilde{q}_i))|_{\tilde{q}_i=0} = \frac{\partial}{\partial q_i}(\mu_i(q_i))|_{q_i=q_i^*} \neq 0$, and the fact that $\tilde{y}_i(x_i) = 0$ if and only if $x_i = 0$, for every $\tilde{q}_i$ the function $g_i(\cdot)$ on the right-hand side is positive. Moreover, if $\tilde{q}_i$ range over a compact set, then by continuity of $g_i(\cdot)$ there exists $m > 0$ such that

$$g_i(\tilde{q}_i) \geq m > 0 , \quad i = 1, 2, \ldots, n. \tag{9.45}$$

Then the main result of this subsection can be stated.

Proposition 25. *Given system (9.7), (9.8), and a compact set of initial conditions $\mathcal{Q} \subset \mathbb{R}^n \times \mathbb{R}^n$, there exist diagonal positive definite matrices N and K in (9.12), (9.13), such that every trajectory $(q(t), \xi(t))$ of the closed-loop system (9.7), (9.8), (9.12), (9.13) with initial condition in $\mathcal{Q}$ is bounded and satisfies $\lim_{t \to +\infty} y_i(t) = r_i$ for $i = 1, 2, \ldots, n$.*

Proof. Let $P > 0$ be such that $\tilde{F}^T P + P\tilde{F} = -I$ and consider the Lyapunov function candidate $V : \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$V(\chi, \tilde{q}) = \frac{1}{2}\chi^T P \chi + \frac{1}{2}\tilde{q}^T J \tilde{q}$$

which is positive definite and radially unbounded. Compute the time derivative along the trajectories of the system, to obtain

$$\frac{d}{dt}V(\chi, \tilde{q}) = -\chi^T P \tilde{F}\chi - \chi^T P \phi^{(a)}(\tilde{q})\tilde{q} + $$
$$+ \tilde{q}^T \phi^{(b)}(\tilde{q})\tilde{q} + \tilde{q}^T \chi - \tilde{q}^T N \tilde{y}(\tilde{q}) . \tag{9.46}$$

Let S be a level set of $V(\chi, \tilde{q})$ containing the set of initial conditions of the system. Bearing in mind (9.45), the time derivative of $V(\chi, \tilde{q})$ can be written in compact form as

$$\frac{d}{dt}V(\chi, \tilde{q}) \leq -|\chi|^2 + |\chi|\||P\phi^{(a)}(\tilde{q})\|||\tilde{q}| + $$
$$+ |\tilde{q}|\||\phi^{(b)}(\tilde{q})\|||\tilde{q}| + |\tilde{q}||\chi| - m\tilde{q}^T N \tilde{q} . \tag{9.47}$$

Let Θ be a positive constant such that $\max\{\||P\phi^{(a)}(\tilde{q})\||, \||\phi^{(b)}(\tilde{q})\|| \mid \tilde{q} \in S\} \leq \Theta$, which exists by continuity of $\phi^{(a)}(\cdot)$ and $\phi^{(b)}(\cdot)$. Then

$$\frac{d}{dt}V(\chi, \tilde{q}) \leq -\frac{|\chi|^2}{2} + (\Theta^2 + \Theta + 1)|\tilde{q}|^2 - m\tilde{q}^T N \tilde{q} .$$

Let $N_1 = \ldots = N_n = m^{-1}\tilde{N}$, with N_i's the diagonal entries of N and $\tilde{N}$ to design, so that

$$\frac{d}{dt}V(\chi, \tilde{q}) \leq -\frac{|\chi|^2}{2} + (\Theta^2 + \Theta + 1 - \tilde{N})|\tilde{q}|^2 .$$

Set $\tilde{N}^* = \Theta^2 + \Theta + 1 + \frac{1}{2}\frac{\sigma_M(J)}{\sigma_M(P)}$, where $\sigma_M(\cdot)$ denotes the maximum eigenvalue of a (symmetric) matrix. Then, for all $\tilde{N} \geq \tilde{N}^*$

$$\frac{d}{dt}V(\chi, \tilde{q}) \leq -\frac{1}{\sigma_M(P)}V(\chi, \tilde{q}). \tag{9.48}$$

This shows that the trajectories $(\chi(t), \tilde{q}(t))$ of the system are bounded and converge to the origin. Bearing in mind the definition of $(\chi, \tilde{q})$, it also shows that $(q(t), \xi(t))$ are bounded and $q(t) \to q^*$, $\xi(t) \to -f(B^T q^*)$ as $t \to \infty$. By continuity of $\mu(\cdot)$, it is concluded that $\mu(q(t)) \to \mu(q^*) = r$ as $t \to \infty$, i.e. the thesis. $\qquad \square$

Remark 7: In view of the quadratic nature of the Lyapunov function $V(\chi, \tilde{q})$ the proof actually shows exponential convergence of $(\tilde{q}, \chi)$ to the origin. Bearing in mind (9.44) and the boundedness of the state $(\tilde{q}, \chi)$, also the regulation error $\tilde{y}_i(\tilde{q}_i) = \mu_i(q_i) - r_i$ converges exponentially to zero.

4   Numerical Results

The closed loop system has been tested using numerical simulations. A four end-user system have been used in the simulations, and the diagram of the corresponding hydraulic network is shown in Fig. 9.2. The end-users in the system are comprised of the connections $\{c_4, c_5, c_6\}$, $\{c_9, c_{10}, c_{11}\}$, $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$. The parameters used in the simulations are: $J_{11} = 1.0787, J_{12} = J_{13} = J_{14} = J_{21} = J_{31} = J_{41} = 0.4421, J_{22} = 1.1318, J_{23} = J_{24} = J_{32} = J_{42} = 0.7074, J_{33} = 1.4854, J_{34} = J_{43} = 1.061, J_{44} = 1.7507, p_2 = p_{13} = 0.0586, p_3 = p_6 = 0.6755, p_7 = p_{12} = p_{21} = p_{26} = 0.0352, p_8 = p_{11} = p_{17} = p_{20} = p_{22} = p_{25} = 0.4503, p_{16} = p_{27} = 0.0469, v_5 = v_{10} = v_{19} = v_{24} = 0.005, v_{14} = 0.0013$ and controller gain matrices $K = N = 2I_4$. A
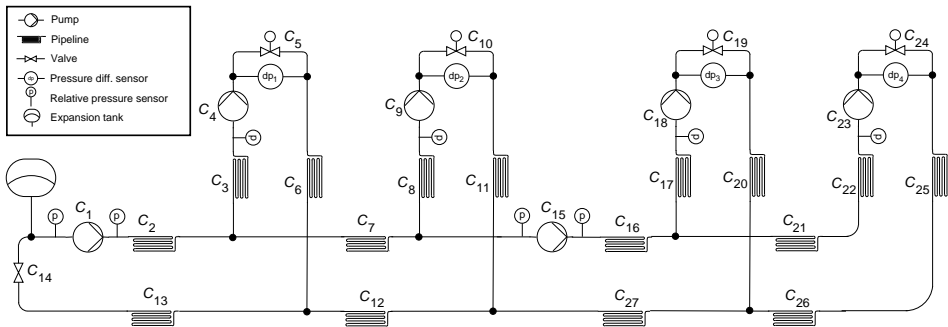
Figure 9.2: The hydraulic network diagram for the system with four end-users which has been used in the simulations.

scenario, where the end-user connections consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are removed from and later re-inserted into the system, has been simulated. This has been done by changing the valve parameters v_{19} and v_{24} to large values $(0.005 + 1.25 \cdot 10^3)$, thus reducing the flows through valves c_{19} and c_{24} to close to zero. The results of the simulations are shown in Fig. 9.3. In Fig. 9.3, it is seen that the closed loop system achieves asymptotic output regulation when all four end-users are present in the system as well as when only two of the end-users are present in the system.

5   Conclusion

An industrial case study involving a large-scale hydraulic network comprising a district heating system was presented. The problem of output regulation in the network was addressed. The results show that global asymptotic and semi-global exponential output regulation is achievable using a set of decentralized proportional-integral control actions. Furthermore, as the former result is global and independent on the number of end-users in the system, it is concluded that the property of asymptotic output regulation is maintained if end-users are arbitrarily added to or removed from the system. The results were supported by numerical simulations of a four end-user system.

The incorporation of positive constraints on the control signals is seen as a natural extension of the results presented here. Since the centrifugal pumps used in the network are only capable of delivering positive pressures, the explicit incorporation of this constraint in the stability analysis of the closed loop system will be necessary. This increases considerably the difficulty of the control problem and requires a deep independent study to be carried out in the future.

Acknowledgment

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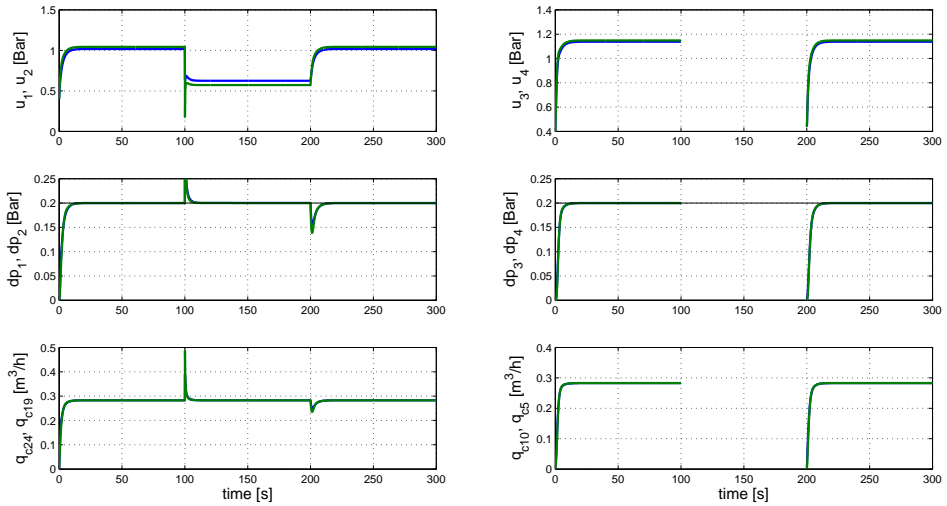
Figure 9.3: Result of a numerical simulation of the four end-user system in Fig. 9.2. The figure shows control inputs u_1, u_2, u_3, u_4, the controlled variable dp_1, dp_2, dp_3, dp_4, and the flow through valves $c_{24}, c_{19}, c_{10}, c_5$ obtained with the proportional-integral feedback control. At time 100 s, the end-user connections consisting of $\{c_{18}, c_{19}, c_{20}\}$ and $\{c_{23}, c_{24}, c_{25}\}$ are removed from the system. At time 200 s the end-user connections are re-inserted into the system. The solid line at 0.2 Bar in the two middle plots indicates the reference value.

Paper G

Output Regulation of Large-Scale Hydraulic Networks with Minimal Power Consumption

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Abstract

An industrial case study involving a large-scale hydraulic network is examined. The hydraulic network underlies a district heating system, with an arbitrary number of end-users. The problem of output regulation is addressed along with a optimization criterion for the control. The fact that the system is over-actuated is exploited for minimizing the steady state electrical power consumption of the pumps in the system, while maintaining output regulation. The proposed control actions are decentralized in order to make changes in the structure of the hydraulic network easy to implement.

1   Introduction

An industrial case study involving a large scale hydraulic system is the focus of the work presented here. The case study involves a new paradigm for the design of district heating systems in which the diameter of the pipes used in the network is reduced in order to lessen heat dispersion from the pipes, and thereby decrease the overall energy consumption of the system. This has the additional benefit that end-users arbitrarily can be added to and removed from the system ([1]). The case study has been proposed by one of the industrial partners in the ongoing research program *Plug & Play Process Control* ([2]), which considers automatic reconfiguration of the control system whenever components such as actuators, sensors or subsystems are added to or removed from the plant.

The reduced pipe diameter leads to larger pressure gradients across the pipes, which again leads to the danger of violating pressure constraints on the components in the network. In order to overcome this problem, a number of (pressure) boosting pumps are placed along the main pipe line. Furthermore, to accommodate the demands of the end-users in the system, so-called service pipe pumps are placed at the end-users. The multi-pump architecture means that the system is over actuated since the number of actuators in the system exceeds the number of states to be controlled. Since the system is over actuated, there is additional freedom in choosing the control signal. In this paper, this extra degree of freedom will be exploited to solve an optimization problem. The problem considered is to find the control signal which provide asymptotic output regulation, while using least possible electrical power in steady state.

The added flexibility in the system calls for a control structure which is able to handle structural changes in the network. The focus of the work presented here is a set of decentralized control actions, which use locally available information only. The results show that global asymptotic output regulation is achieved with the proposed controllers. Furthermore, global asymptotic convergence to the set of minimizers of the objective function is also proved. The decentralized nature of the control actions combined with the fact that the closed loop system is globally asymptotically stable, means that changes in the system structure will be easy to implement.

The layout of the paper is as follows. In Section 2, the component and network models of the system is introduced. The output regulation problem is introduced in Section 3 along with a set of controllers which accommodate the problem. In Section 4, the optimization problem is introduced along with a modified control to further accommodate the optimization problem. The stability properties of the closed loop system are examined in Section 5, where the main result of the exposition is given. The results of a numerical sim-

ulation performed on the closed loop system is given in Section 6. Finally, conclusions are drawn in Section 7 along with suggestions to future work.

Nomenclature: For a vector $x \in \mathbb{R}^n$, x_i denotes the ith element of x and $|x|$ denotes the Euclidean norm of x. Let $M(n, m; \mathbb{R})$ denote the set of $n \times m$ matrices with real entries, and $M(n; \mathbb{R}) = M(n, n; \mathbb{R})$. For a square matrix A, $A > 0$ means that A is positive definite. For a square matrix A, $A = \mathrm{diag}(x_i)$ means that A has x_i as entries on the main diagonal and zero elsewhere. For a matrix A, A_{ij} will be used to denote the entry in the ith row and jth column of A. Throughout the paper, C^1 denotes a continuously differentiable map. A continuous map is said to be *proper* if the inverse image of a compact set is compact. A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing if it is order preserving, i.e., for all x and y such that $x \leq y$ then $f(x) \leq f(y)$. Let I_n denote the n-dimensional identity matrix and $\mathbb{1}_n$ the vector in $\mathbb{R}^n$ consisting of ones. For a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, let $\nabla_x f(x, y) = (\partial f(x, y)/\partial x_1, \ldots, \partial f(x, y)/\partial x_n)$. Let $B_r(x)$ denote the open ball with radius r centred in x, that is: $B_r(x) = \{y \in \mathbb{R}^n \mid |y - x| < r\}$. For two sets X and Y, $X \subset Y$ means that X is a proper subset of Y. For a set $S \subseteq \mathbb{R}^n$ the set S_ϵ is given as $S_\epsilon = \bigcup_{x \in S} \overline{B}_\epsilon(x)$, where $\overline{B}_\epsilon(x)$ denotes the closure of $B_\epsilon(x)$. The Hausdorff metric $d(X, Y)$ between two sets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ is defined as $d(X, Y) = \inf\{\epsilon > 0 \mid X \subseteq Y_\epsilon \text{ and } Y \subseteq X_\epsilon\}$.

2   System Model

The system under consideration is a hydraulic network comprising a district heating system. The model has been derived in detail in [3] and will be recalled here, but in fewer details.

The hydraulic network consists of a number of connections between two-terminal components, which in this work are: valves, pipes and pumps. The kth system component is characterized by dual variables, the first of which is the pressure drop Δh_k across it

$$\Delta h_k = h_i - h_j, \tag{10.1}$$

where i, j are nodes in the network; h_i, h_j are the relative pressures at the nodes.

The second variable characterizing the component is the fluid flow q_k through it. The components have algebraic or dynamic expressions governing the relationships between the two variables.

Valves

The behaviour of valves in the network is governed by the following algebraic expression

$$h_i - h_j = \mu_k(q_k) \equiv \mu_k(v_k, q_k), \tag{10.2}$$

where v_k is the hydraulic resistance of the valve; $\mu_k(\cdot)$ is a C^1 and proper function, which for any fixed value of v_k is zero at $q_k = 0$, monotonically increasing and $\mu_k(v_k, \cdot) = 0$ for $v_k = 0$.

Pipes

The behaviour of pipes in the network is governed by the dynamic equation

$$\mathcal{J}_k \dot{q}_k = (h_i - h_j) - \lambda_k(q_k) \tag{10.3}$$

where $\lambda_k(q_k) \equiv \lambda(p_k, q_k)$; $\mathcal{J}_k$ and p_k are parameters of the pipe; $\lambda_k(\cdot)$ is a function with the same properties as $\mu_k(\cdot)$. Furthermore, $\frac{\partial}{\partial q_k} \lambda_k(q_k)$ is bounded away from zero.

Pumps

A (typically centrifugal) pump is a component which delivers a desired pressure difference Δh_k regardless of the value of the fluid flow through it. Thus, the behaviour of pumps in the network is governed by the following expression

$$h_i - h_j = -\Delta h_{p,k}, \tag{10.4}$$

where $\Delta h_{p,k}$ is a non-negative control input.

Component Model

A generalised component model can be derived using the following expression

$$\Delta h_k = \mathcal{J}_k \dot{q}_k + \lambda_k(q_k) + \mu_k(q_k) - \Delta h_{p,k}, \tag{10.5}$$

where $\mathcal{J}_k, p_k$ are non-zero for pipe components and zero for other components; v_k is non-zero for valve components and zero for other components; $\Delta h_{p,k}$ is non-zero for pump components and zero for other components.

The values of the parameters p_k and v_k are typically unknown, but they will be assumed to take values in a compact set of non-negative values. Likewise, the functions $\mu_k(q_k)$ and $\lambda_k(q_k)$ are not precisely known, only their properties of being C^1, monotone, zero in $q_k = 0$ and proper are guaranteed. The varying heating demand of the end-users, which is the main source of disturbances in the system, is modelled by a (end-user) valve with variable hydraulic resistance. In the network model, a distinction is to be made between end-user valves and the rest of the valves in the network. Two types of pumps are present in the network; the end-user pumps, which are mainly used to meet the demand at the end-users, and boosting pumps which are used to meet constraints on the relative pressures in the network ([4]).

Network Model

The network model has been derived using standard circuit theory ([3]). The hydraulic network consists of m components and n end-users ($m > n$). The network is associated with a graph $\mathcal{G}$ which has nodes coinciding with the terminals of the network components. The edges of the network are the components themselves. The graph satisfies the following:

Assumption 9. *([5]) $\mathcal{G}$ is a connected graph.*

By the use of graph theory, a set of n independent flow variables q_i have been identified. These flow variables have the property that their values can be set independently from other flows in the network. The independent flow variables coincide with the flows through the chords[1] of the graph ([3]). To each chord in the graph, a fundamental (flow) loop is associated, and along this loop Kirchhoffs voltage law holds. This means that the following equality applies

$$B\Delta h = 0, \tag{10.6}$$

where $B \in M(n, m; \mathbb{R})$ is called the fundamental loop matrix; Δh is a vector consisting of the pressure drops across the components in the network.

The entries of the fundamental loop matrix B are $-1, 1$ or 0, depending on the network topology. For the case study in question, the hydraulic network underlies a district heating system. Because of the latter, the following statements can be made regarding the network.

Assumption 10. *([5]) Each end-user valve is in series with a pipe and a pump, as seen in Fig. 10.1. Furthermore, each chord in $\mathcal{G}$ corresponds to a pipe in series with a user valve.*

Assumption 11. *([5]) There exists one and only one component called the heat source. It corresponds to a valve[2] of the network, and it lies in all the fundamental loops.*
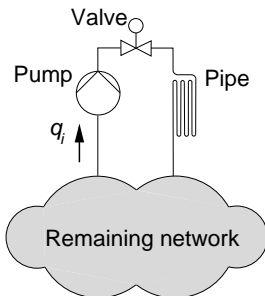
Figure 10.1: The series connection associated with each end-user ([3]).

Proposition 26. *([5]) Any hydraulic network satisfying Assumptions 9 and 10 admits the representation*

$$J\dot{q} = f(B^T q) + u \tag{10.7}$$

$$y_i(q_i) = \mu_i(q_i) , \ i = 1, \ldots, n, \tag{10.8}$$

where $q \in \mathbb{R}^n$ is the vector of independent flows; $u \in \mathbb{R}^n$ is a vector of independent inputs consisting of a linear combination of the delivered pump pressures; y_i is the measured

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[1]Let $\mathcal{T}$ denote the spanning tree of $\mathcal{G}$, i.e. a connected subgraph which contains all nodes of $\mathcal{G}$ but no cycles. Then the edges of $\mathcal{G}$ which are not included in $\mathcal{T}$ are the chords of $\mathcal{G}$ (see [3]).

[2]The valve models the pressure losses in the secondary side of the heat exchanger of the heat source.

pressure drop across the ith end-user valve; $J \in M(n; \mathbb{R})$, $J > 0$; $f(\cdot)$ is a C^1 vector field; $\mu_i(\cdot)$ is the fundamental law of the ith end-user valve. In (10.8), it is assumed that the first n components coincide with the end-user valves.

Under Assumptions 9-11, it is possible to select the orientation of the components in the network such that the entries of the fundamental loop matrix B are equal to 1 or 0, where B_{ij} is 1 if component j belongs to fundamental flow loop i and 0 otherwise.

Defining the vector of flows through the components in the system as $x = B^T q \in \mathbb{R}^m$, the vector field $f(\cdot)$ can be written as ([3])

$$f(x) = -B(\lambda(x) + \mu(x)), \; \forall x \in \mathbb{R}^m, \tag{10.9}$$

where $\lambda(x) = [\lambda_1(x_1), \ldots, \lambda_m(x_m)]^T$,
$\mu(x) = [\mu_1(x_1), \ldots, \mu_m(x_m)]^T$ and $\lambda_i(\cdot)$ is non-zero for pipe components and $\mu_i(\cdot)$ is non-zero for valve components.

The matrix J in (10.7) is given by

$$J = B \mathcal{J} B^T \tag{10.10}$$

where $\mathcal{J} = \operatorname{diag}(\mathcal{J}_1, \ldots, \mathcal{J}_m)$.

The input u to the system deserves a comment as well. Let $\Delta h_e \in \mathbb{R}^n$ and $\Delta h_b \in \mathbb{R}^k$ denote the vectors of pressures delivered by the end-user- and boosting pumps respectively. Then the input u can be written as

$$u = \Delta h_e + F \Delta h_b \tag{10.11}$$

$$= \begin{bmatrix} I_n & F \end{bmatrix} \begin{pmatrix} \Delta h_e \\ \Delta h_b \end{pmatrix} \tag{10.12}$$

$$= \bar{B} \begin{pmatrix} \Delta h_e \\ \Delta h_b \end{pmatrix} \tag{10.13}$$

where $F \in M(n, k; \mathbb{R})$ consisting of 1,0 is the sub-matrix of B mapping boosting pumps to the fundamental flow loops. That is; $F_{ij} \neq 0$ if and only if Δh_{bj} is present in the ith fundamental flow loop. Since $k \neq 0$ the system is over actuated.

Assumption 12. *For each $j = 1, 2, \ldots, k$ there exists at least one $i = 1, 2, \ldots, n$ such that $F_{ij} \neq 0$.*

The assumption above corresponds to the statement that each boosting pumps is present in minimum one fundamental flow loop.

3   Output regulation problem

It is desired to regulate the output y of the system to a piecewise constant vector r of reference values, which belongs to a known compact set $\mathcal{R}$

$$\mathcal{R} = \{r \in \mathbb{R}^n \mid 0 < r_m \leq r_i \leq r_M\} \tag{10.14}$$

In [6] the following set of controllers has been proposed for the purpose of output regulation

$$\dot{\xi}_i = -K_i(y_i - r_i) \tag{10.15}$$

$$u_i = \xi_i - N_i(y_i - r_i), \tag{10.16}$$

where $K, N > 0$, and $n = 1, 2, \ldots, n$.

Since the functions $\mu_i(\cdot)$ are monotonically increasing and proper, they admit inverses $\mu_i^{-1}(\cdot)$. Now, let

$$q_i^* \equiv \mu_i^{-1}(r_i), \qquad (10.17)$$

that is: q_i^* is the flow through the ith end-user valve which produce the reference output.

Furthermore, define the coordinate transformation

$$\tilde{q} \equiv q - q^*, \qquad (10.18)$$

then the following proposition follows

Proposition 27. *([6]) The point $(\tilde{q}^T, \dot{q}^T) = 0$ is a globally asymptotically stable equilibrium point of the closed loop system given by (10.7), (10.8), (10.15) and (10.16).*

Furthermore, the output regulation error $\tilde{y}_i(\tilde{q}_i)$, which is given as

$$\tilde{y}_i(\tilde{q}_i) \equiv y_i(q_i) - r_i = \mu_i(\tilde{q}_i + q_i^*) - r_i, \qquad (10.19)$$

is zero in steady state.

Let ξ^* denote the steady state value of ξ in the closed loop system given by (10.7), (10.8), (10.15) and (10.16).

The following lemma will prove instrumental later in the exposition.

Lemma 3.1: *If the vector r of reference values fulfils $r \in \mathbb{R}_+^n$, then $\xi^* \in \mathbb{R}_+^n$.*

Proof. In steady state, the following will be fulfilled

$$\mu_i(q_i^*) = r_i. \qquad (10.20)$$

Since $\mu_i(\cdot)$ is monotonically increasing and zero at zero, it follows that

$$r_i > 0 \Leftrightarrow q_i^* > 0, \qquad (10.21)$$

and consequently that $q^* \in \mathbb{R}_+^n$.

The closed loop system is given by

$$J\dot{q} = f(B^T q) + \xi - N(y - r) \qquad (10.22)$$
$$\dot{\xi} = -K(y - r) \qquad (10.23)$$

Since $y = r$ in steady state, from (10.22) it follows

$$\xi^* = -f(B^T q^*). \qquad (10.24)$$

Lemma 3.2: *([5]) Under Assumptions 9-11, $q \in \mathbb{R}_+^n$ implies $-f(B^T q) \in \mathbb{R}_+^n$.*

From Lemma 3.2 it follows that $\xi^* \in \mathbb{R}_+^n$.

This completes the proof. $\qquad \square$

4   Power-optimal control

From (10.11), it is seen that the control input u is a linear combination of the pump pressures delivered to the network. Therefore, it is necessary to find a mapping from u to the vectors Δh_e and Δh_b such that (10.11) is fulfilled. One possibility is the following map

$$
\begin{pmatrix} \Delta h_e \\ \Delta h_b \end{pmatrix} = \bar{B}^\dagger u,
\tag{10.25}
$$

where $\bar{B}^\dagger$ denotes the right inverse of $\bar{B}$.

Another possible mapping has been investigated in [7], where a map, which provides the property that $\Delta h_e \in \mathbb{R}^n$ and $\Delta h_b \in \mathbb{R}^k_+$ whenever $u \in \mathbb{R}^n_+$, is considered.

In the exposition presented here, an approach based on an optimality condition will be examined instead. The aim is to design the vectors Δh_e and Δh_b such that in steady state $\Delta h_e \in \mathbb{R}^n_+$ and $\Delta h_b \in \mathbb{R}^k_+$ (which is possible by Lemma 3.1) and the steady-state power consumption of the system is minimal.

Optimization Problem

In this subsection the objective function will be introduced along with the optimization problem. First, some preliminaries are given.

In steady-state $y = r$ and consequently from (10.11) and (10.16)

$$
\Delta h_e = \xi^* - F \Delta h_b
\tag{10.26}
$$

The equality (10.26) can also be written component-wise

$$
\Delta h_{ei} = \xi^*_i - F^T_i \Delta h_b
\tag{10.27}
$$

where

$$
F = \begin{pmatrix} F^T_1 \\ \vdots \\ F^T_n \end{pmatrix}
\tag{10.28}
$$

The electrical power consumption $P_e(\cdot)$ of a (centrifugal) pump is given by the expression

$$
P_e(q, \omega(\Delta h)) = -a_{t2}q^2\omega + a_{t1}q\omega^2 + a_{t0}\omega^3
\tag{10.29}
$$

where $a_{t2}, a_{t1}, a_{t0} > 0$ are parameters of the pump and ω is the rotational speed of the pump.

Furthermore, there exists the following relation between the pressure Δh delivered by the pump and ω

$$
\Delta h = -a_{h2}q^2 + a_{h1}q\omega + a_{h0}\omega^2
\tag{10.30}
$$

where a_{h2}, a_{h1}, a_{h0} are parameters of the pump and $a_{h2}, a_{h0} > 0$.

However, for the purpose of the exposition presented here, a simpler expression for the electrical power consumption of the pumps will be used.

The hydraulic power $P_h(\cdot)$ delivered by a pump is given by the following expression

$$
P_h(q, \Delta h) = q\Delta h,
\tag{10.31}
$$

for $q > 0$ and $\Delta h > 0$.

Let $a^{-1} \in (0,1)$ be the efficiency with which the pump turn the supplied electrical power into hydraulic power, then

$$P_h(q, \Delta h) = a^{-1} P_e(q, \Delta h) \Leftrightarrow \tag{10.32}$$

$$P_e(q, \Delta h) = aq\Delta h. \tag{10.33}$$

Usually, the parameter a will depend on q and Δh as seen in (10.29). However, in the following a will be considered constant. This is motivated by the fact that at steady state a will be a constant, and that only minimal power consumption in the steady state is considered here.

Let a_{ei}^{-1} and a_{bj}^{-1} denote the efficiency of the ith end-user pump and the jth boosting pump respectively.

Furthermore, let the vector q_b denote the vector of flows through the boosting pumps, then $q_b = F^T q$.

The electrical power consumption of the system $P(\cdot)$ will be the sum of the electrical power consumption of all the pumps present in the system

$$P(\Delta h_b, \Delta h_e, q) = \sum_{i=1}^{n} a_{ei} q_i \Delta h_{ei} + \sum_{j=1}^{k} a_{bj} q_{bj} \Delta h_{bj} \tag{10.34}$$

$$= q^T A_e \Delta h_e + q_b^T A_b \Delta h_b \tag{10.35}$$

$$= q^T A_e \Delta h_e + q^T F A_b \Delta h_b \tag{10.36}$$

where $A_e = \mathrm{diag}(a_{ei})$; $A_b = \mathrm{diag}(a_{bj})$.

It is now desired to find the vector Δh_b of boosting pump pressures which solve the following steady state optimization problem

$$\underset{\Delta h_b, \, \Delta h_e}{\mathrm{argmin}} \; P(\Delta h_b, \Delta h_e, q^*) \tag{10.37a}$$

subject to the constraints

$$\Delta h_e = \xi^* - F\Delta h_b \tag{10.37b}$$

$$\Delta h_e \geq 0 \tag{10.37c}$$

$$\Delta h_b \geq 0, \tag{10.37d}$$

where the equality constraint (10.37b) comes from (10.26). Note that since $q_i^* > 0$, $P(\Delta h_b, \Delta h_e, q^*) > 0$ for every $\Delta h_b > 0$ and $\Delta h_e > 0$.

The problem (10.37) is now rewritten to

$$\underset{\Delta h_b}{\mathrm{argmin}} \, P(\Delta h_b, q^*, \xi^*) \tag{10.38a}$$

subject to

$$\begin{bmatrix} I_k \\ -F \end{bmatrix} \Delta h_b + \begin{pmatrix} 0 \\ \xi^* \end{pmatrix} \geq 0 \tag{10.38b}$$

where $P(\Delta h_b, q, \xi) = q^T A_e \xi + q^T (FA_b - A_e F)\Delta h_b$. Subsequently, let $G = FA_b - A_e F$, $G \in M(n, k; \mathbb{R})$.

The constraint set (10.38b) is non-empty by Lemma 3.1. Again, note that the objective function is positive within the constraint set.

The vector q of free flow variables is generally assumed to be unknown. As a consequence it might be impossible to find a solution to the problem (10.38). However, it can be assumed that the pump parameters a_{ti}, a_{hi} in (10.29) and (10.30) are known. This can be exploited to calculate an estimate $\hat{q}$ of q using techniques described in [8].

Assumption 13. *The flow estimate $\hat{q}$ can be made accurate up to some unknown constant scalar $\alpha > 0$. That is, $\hat{q} = \alpha q$.*

Now, the problem (10.38) is rewritten to

$$\underset{\Delta h_b}{\arg\min} \, P(\Delta h_b, \hat{q}^*, \xi^*) \tag{10.39a}$$

subject to

$$\begin{bmatrix} I_k \\ -F \end{bmatrix} \Delta h_b + \begin{pmatrix} 0 \\ \xi^* \end{pmatrix} \geq 0 \tag{10.39b}$$

By Assumption 13 the problems (10.38) and (10.39) produce the same solution since

$$P(\Delta h_b, \hat{q}, \xi) = \hat{q}^T A_e \xi + \hat{q}^T G \Delta h_b \tag{10.40}$$

$$= \alpha \left(q^T A_e \xi + q^T G \Delta h_b \right) \tag{10.41}$$

$$= \alpha P(\Delta h_b, q, \xi). \tag{10.42}$$

It is now desired to redefine the optimization problem (10.39) to an unconstrained optimization problem, and then use the gradient of the objective function to find the vectors Δh_b and Δh_e of pump pressures. This is explored in the following subsection.

Dynamic optimization using penalty functions

In this section a method, in which the gradient of an objective function is used dynamically to update the vectors of pressure inputs, will be described.

First, a new convex objective function will be defined, with the property that it has a globally well defined minimum with respect to the vector Δh_b. Based on this new objective function, the vector Δh_b is updated using a dynamic expression depending on the gradient of the objective function.

This means that the controllers in (10.15) and (10.16) are replaced by the following

$$\dot{\xi}_i = -K_i(y_i - r_i) \tag{10.43}$$

$$\Delta \dot{h}_{bj} = -L_j \left(\frac{\partial}{\partial \Delta h_{bj}} \tilde{P}(\Delta h_b, \hat{q}, \xi) \right) \tag{10.44}$$

$$\Delta h_{ei} = \xi_i - N_i(y_i - r_i) - F_i^T \Delta h_b, \tag{10.45}$$

where $L_j > 0$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ and $\tilde{P}(\cdot)$ is the new objective function.

The objective function is modified to the following

$$\tilde{P}(\Delta h_b, \Delta h_e, \hat{q}) = \sum_{i=1}^{n}(P_i^e(\Delta h_{ei}, \hat{q}) + s_i^e(\Delta h_{ei})) +$$
$$+ \sum_{j=1}^{k}(P_j^b(\Delta h_{bj}, \hat{q}) + s_j^b(\Delta h_{bj})) \tag{10.46}$$

where $P_i^e(\cdot)$, $P_j^b(\cdot)$ denotes the electrical power consumption of the ith end-user pump and the jth boosting pump respectively; $s_i^e(\cdot)$, $s_j^b(\cdot)$ are additional terms which penalizes violation of inequality constraints (see for instance [9]).

A possible implementation of $s_l : \mathbb{R} \to \mathbb{R}$ is the following:

$$s_l(x) = \begin{cases} \kappa(x - \underline{x}_l)^2, & x \leq \underline{x}_l \\ 0, & x \geq \underline{x}_l \end{cases} \tag{10.47}$$

where the constant $0 \leq \underline{x}_l$ is the minimal allowed value of x and the gain $\kappa > 0$. This particular implementation of $s_l(\cdot)$ is C^1. Furthermore, for simplicity, the same κ is used for all $l = 1, 2, \ldots, n + k$.

Motivated by the fact that in steady state $y = r$ and consequently $u = \xi$ and that optimality is only considered for the steady state power consumption and not the power consumption during transients, using the identity in (10.45), the power function (10.46) is rewritten as

$$\tilde{P}(\Delta h_b, \hat{q}, \xi) = \hat{q}^T A_e \xi + \hat{q}^T G \Delta h_b +$$
$$+ \sum_{i=1}^{n} s_i^e(\xi_i - F_i^T \Delta h_b) + \sum_{j=1}^{k} s_j^b(\Delta h_{bj}) \tag{10.48}$$

where Δh_{ei} has been replaced by the expression $\xi_i - F_i^T \Delta h_b$.

Since $s_l(\cdot)$ is convex, $\tilde{P}(\cdot, \hat{q}, \xi)$ consists of a sum of functions which are convex and thus in itself is a convex function.

Furthermore, $\tilde{P}(\cdot, \hat{q}, \xi)$ is a sum of a C^∞ function and a number of C^1 functions; thus it is a C^1 function.

Define the set C by the inequalities

$$\Delta h_b \geq \Delta \underline{h}_b \tag{10.49a}$$

$$F \Delta h_b \leq \xi^* - \Delta \underline{h}_e. \tag{10.49b}$$

Note that C is compact. Furthermore, by Lemma 3.1 it is always possible to pick $\Delta \underline{h}_e, \Delta \underline{h}_b > 0$ such that C is a non-empty proper subset of the feasibility set of (10.38).

Lemma 4.1: The function $\tilde{P}(\cdot, \hat{q}, \xi)$ is radially unbounded; that is $\tilde{P}(x, \hat{q}, \xi) \to \infty$ as $|x| \to \infty$.

Proof. For $\Delta h_b \in C$ the functions $s_i^e(\cdot) = s_j^b(\cdot) = 0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.

As a consequence

$$\tilde{P}(\Delta h_b, \hat{q}, \xi) = P(\Delta h_b, \hat{q}, \xi) \tag{10.50}$$

$$= \hat{q}^T A_e \xi + \hat{q}^T G \Delta h_b \tag{10.51}$$

for every $\Delta h_b \in C$. For $\Delta h_b \notin C$, denote by $S(\Delta h_b, \xi)$

$$S(\Delta h_b, \xi) = \sum_{i=1}^{n} s_i^e(\xi_i - F_i^T \Delta h_b) + \sum_{j=1}^{k} s_j^b(\Delta h_{bj}) \tag{10.52}$$

and for $\Delta h_b \in bd(C)$ set $S(\Delta h_b, \xi) = 0$. By the definition of $s_l(\cdot)$ in (10.47) it can be seen that $S(\Delta h_b, \xi) \to \infty$ as $|\Delta h_b| \to \infty$ by Assumption 12.

Now, define $\tilde{P}(\cdot)$ as

$$\tilde{P}(\Delta h_b, \hat{q}, \xi) = P(\Delta h_b, \hat{q}, \xi) +$$
$$+ \begin{cases} 0 & \forall \Delta h_b \in C \\ S(\Delta h_b, \xi) & \forall \Delta h_b \in \mathbb{R}^k \setminus C \end{cases} . \tag{10.53}$$

Since $\hat{q}$ and ξ are bounded by Proposition 27 and $S(\Delta h_b, \xi)$ has quadratic growth for every $\Delta h_b \in \mathbb{R}^k \setminus C$ this means $\tilde{P}(\cdot, \hat{q}, \xi)$ is radially unbounded; i.e. as $|\Delta h_b| \to \infty$, $\tilde{P}(\Delta h_b, \hat{q}, \xi) \to \infty$, which completes the proof. $\qquad \square$

Now, the main result of the exposition can be derived.

5   Stability properties of the closed-loop system

In this section, the stability properties of the closed loop system are examined.

The closed loop system given by (10.7), (10.8), (10.11) and (10.43)-(10.45) is

$$J\dot{q} = f(B^T q) + \xi - N(y - r) \tag{10.54}$$

$$\dot{\xi} = -K(y - r) \tag{10.55}$$

$$\dot{\Delta h_b} = -L \left(\nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}, \xi) \right)^T \tag{10.56}$$

where $L = \text{diag}(L_j)$.

By Proposition 27 the system (10.54), (10.55) is globally asymptotically stable.

Let $\hat{q}^* = \alpha q^*$ and the set $\mathcal{M}$ denote the set of minimizers of $\tilde{P}(\Delta h_b, \hat{q}^*, \xi^*)$, that is $\mathcal{M} = \{x \in \mathbb{R}^k \mid \tilde{P}(x, \hat{q}^*, \xi^*) \leq \tilde{P}(y, \hat{q}^*, \xi^*), \ \forall y \in \mathbb{R}^k\}$.

The following two propositions will be instrumental in what follows

Proposition 28. *The set $\mathcal{M}$ is non-empty, convex and compact.*

For a proof see Appendix A.

Proposition 29. *Let κ denote the gain defined in (10.47), $\tilde{P}(\cdot, \hat{q}, \xi)$ be defined by (10.48), C denote the feasibility set of the problem (10.38) and let q^*, ξ^* be the steady state of the closed loop system (10.54)-(10.55). Then there exists $\kappa^* > 0$ such that for every $\kappa > \kappa^*$, $\mathcal{M} \subset C$ and $\tilde{P}(\cdot, \hat{q}^*, \xi^*) > 0$.*

For a proof of Proposition 29 see Appendix B.

Lemma 5.1: *For every $\kappa > \kappa^*$ defined in Proposition 29, the set $\mathcal{M}$ is globally asymptotically stable for the system (10.56) with $\hat{q} = \hat{q}^*$ and $\xi = \xi^*$.*

Proof. The following Lyapunov function candidate $W : \mathbb{R}^k \to \mathbb{R}$ for the system (10.56) with $\hat{q} = \hat{q}^*$ and $\xi = \xi^*$ is considered

$$W(\Delta h_b) = \tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) \tag{10.57}$$

which is positive definite for $\kappa > \kappa^*$ defined in Proposition 29 and radially unbounded.

The time derivative of $W(\cdot)$ is then bounded by

$$\frac{d}{dt} W(\Delta h_b) \leq -\min(L_j) \left| \nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) \right|^2 \tag{10.58}$$

where $j = 1, 2, \ldots, k$. The above proves global stability.

Furthermore, by the LaSalle invariance principle it follows that Δh_b converges to the largest invariant set of the system (10.56) where $\nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) = 0$.

Since $\tilde{P}(\Delta h_b, \hat{q}^*, \xi^*)$ is convex it follows that Δh_b is a minimizer and thus in $\mathcal{M}$ if and only if

$$\nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) = 0, \tag{10.59}$$

see for instance Appendix A.

This shows that all trajectories are bounded (by Proposition 28) and the set $\mathcal{M}$ is globally asymptotically stable for the system (10.56) with $\hat{q} = \hat{q}^*$ and $\xi = \xi^*$. That is

$$\lim_{t \to \infty} d(\Delta h_b(t), \mathcal{M}) = 0, \tag{10.60}$$

and

$$\nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) = 0, \ \forall \Delta h_b \in \mathcal{M}. \tag{10.61}$$

This proves the thesis. $\qquad\square$

Theorem 4. *Consider the system*

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z), \end{aligned} \tag{10.62}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $f(y, 0) = 0$, $\forall y \in Y$, $g(0) = 0$ and $Y \subset \mathbb{R}^n$ is non-empty, compact and connected and $f(x, z)$, $g(z)$ are locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$.

Suppose $Y \subset \mathbb{R}^n$ is a globally asymptotically stable set of $\dot{x} = f(x, 0)$ and the equilibrium $z = 0$ of $\dot{z} = g(z)$ is globally asymptotically stable. Suppose the integral curves of the composite system are defined for all $t \geq 0$ and bounded. Then, the state set $(x, z) \in (Y, 0)$ of (10.62) is globally asymptotically stable.

Proof. The proof follow along the lines of the proof of Theorem 10.3.1, Corollary 10.3.3 in [10]. Specifically, $\|x(t)\|$ should be replaced by $d(x(t), Y)$. $\qquad\square$

Now, note that the function $\tilde{P}(\cdot)$ can be rewritten as

$$
\begin{aligned}
\tilde{P}(\Delta h_b, \hat{q}, \xi) = {}& \hat{q}^T A_e \xi + \hat{q}^T G \Delta h_b + \\
& + \sum_{i=1}^{n} \kappa \rho_i(\xi_i, \Delta h_b)(\xi_i - F_i^T \Delta h_b - \Delta \underline{h}_{ei})^2 + \\
& + \sum_{j=1}^{k} \kappa \tau_j(\Delta h_{bj})(\Delta h_{bj} - \Delta \underline{h}_{bj})^2
\end{aligned}
\tag{10.63}
$$

where

$$
\rho_i(\xi_i, \Delta h_b) = \left\{ \begin{array}{ll} 0, & F_i^T \Delta h_b \leq \xi_i - \Delta \underline{h}_{ei} \\ 1, & F_i^T \Delta h_b > \xi_i - \Delta \underline{h}_{ei} \end{array} \right. ,
\tag{10.64}
$$

and

$$
\tau_j(\Delta h_{bj}) = \left\{ \begin{array}{ll} 0, & \Delta h_{bj} \geq \Delta \underline{h}_{bj} \\ 1, & \Delta h_{bj} < \Delta \underline{h}_{bj} \end{array} \right. .
\tag{10.65}
$$

Let, $R(\xi, \Delta h_b) = \text{diag}(\rho_i(\xi_i, \Delta h_b))$, $T(\Delta h_b) = \text{diag}(\tau_j(\Delta h_{bj}))$, $\Delta \underline{h}_e = (\Delta \underline{h}_{e1}, \ldots, \Delta \underline{h}_{en})$, and $\Delta \underline{h}_b = (\Delta \underline{h}_{b1}, \ldots, \Delta \underline{h}_{bk})$. Furthermore, let $b(\xi, \Delta h_b) = \xi - F \Delta h_b - \Delta \underline{h}_e$, then in compact form

$$
\begin{aligned}
\tilde{P}(\Delta h_b, \hat{q}, \xi) = {}& \hat{q}^T A_e \xi + \hat{q}^T G \Delta h_b + \\
& + \kappa b^T(\xi, \Delta h_b) R(\xi, \Delta h_b) b(\xi, \Delta h_b) + \\
& + \kappa (\Delta h_b - \Delta \underline{h}_b)^T T(\Delta h_b)(\Delta h_b - \Delta \underline{h}_b).
\end{aligned}
\tag{10.66}
$$

The remainder of the exposition will be restricted to the case $k \leq 2$. The following Lemma will be instrumental for the rest of the exposition

Lemma 5.2: Let $\nu(\Delta h_b, \xi)$ be the vector given by

$$
\nu(\Delta h_b, \xi) = \Delta h_b^T \left(T(\Delta h_b) + F^T R(\xi, \Delta h_b) F \right),
\tag{10.67}
$$

if $k \leq 2$ then $||\Delta h_b|| \to \infty \Rightarrow ||\nu(\Delta h_b, \xi)|| \to \infty$.

Proof. First, for arbitrary $k \in \mathbb{Z}_+$, define the following sets

$$
\mathcal{F}_j^b = \{i \mid F_{ij} \neq 0\}
\tag{10.68}
$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. That is $\mathcal{F}_j^b$ is the set of indices i such that the booster pump Δh_{bj} is in fundamental flow loop i. Recall that $\mathcal{F}_j^b$ is non-empty by Assumption 12.

Also, recall that $R(\xi, \Delta h_b) = \text{diag}(\rho_l(\xi_l, \Delta h_b))$, with

$$
\rho_l(\xi_l, \Delta h_b) = \left\{ \begin{array}{ll} 0, & F_l^T \Delta h_b \leq \xi_l - \Delta \underline{h}_{el} \\ 1, & F_l^T \Delta h_b > \xi_l - \Delta \underline{h}_{el} \end{array} \right. ,
\tag{10.69}
$$

where $l = 1, 2, \ldots, n$.

Then it follows that elements of the matrix $F^T R(\xi, \Delta h_b) F$ are given by

$$\left(F^T R(\xi, \Delta h_b) F \right)_{ij} = \sum_{l \in \mathcal{F}_i^b \cap \mathcal{F}_j^b} \rho_l(\xi_l, \Delta h_b), \tag{10.70}$$

if $\mathcal{F}_i^b \cap \mathcal{F}_j^b \neq \emptyset$ and

$$\left(F^T R(\xi, \Delta h_b) F \right)_{ij} = 0, \tag{10.71}$$

if $\mathcal{F}_i^b \cap \mathcal{F}_j^b = \emptyset$.
  Since

$$\mathcal{F}_i^b \cap \mathcal{F}_j^b \subseteq \mathcal{F}_i^b = \mathcal{F}_i^b \cap \mathcal{F}_i^b, \tag{10.72}$$

it follows that

$$\left(F^T R(\xi, \Delta h_b) F \right)_{ij} \leq \left(F^T R(\xi, \Delta h_b) F \right)_{ii} \leq n \tag{10.73}$$

for every $j \neq i$.
  Now, consider the case $k = 2$. In this case

$$T(\Delta h_b) + F^T R(\xi, \Delta h_b) F = \begin{pmatrix} \tau_1(\Delta h_{b1}) & 0 \\ 0 & \tau_2(\Delta h_{b2}) \end{pmatrix} + \begin{pmatrix} a & b \\ b & d \end{pmatrix} \tag{10.74}$$

with $a, d \geq b \geq 0$ (as a consequence of (10.73)), and recall that

$$\tau_j(\Delta h_{bj}) = \begin{cases} 0, & \Delta h_{bj} \geq \underline{\Delta h}_{bj} \\ 1, & \Delta h_{bj} < \underline{\Delta h}_{bj} \end{cases}. \tag{10.75}$$

If $||\Delta h_b|| \to \infty$ it follows that there exists at least one index j such that $|\Delta h_{bj}| \to \infty$.
  Assume, for instance, that $\Delta h_{b1} \to -\infty$, then it follows that $\tau_1(\Delta h_{b1}) = 1$ and consequently

$$\nu_1 = (1 + a)\Delta h_{b1} + b\Delta h_{b2} \tag{10.76}$$

from which it follows that either 1) $\nu_1 \to -\infty$, from which the thesis follows, or 2) $b \neq 0$ and there exists finite $c > 0$ such that

$$(1 + a)\Delta h_{b1} + b\Delta h_{b2} > -c \Leftrightarrow \tag{10.77}$$

$$\Delta h_{b2} > \frac{-c - (1 + a)\Delta h_{b1}}{b}. \tag{10.78}$$

For the case 2), since

$$\nu_2 = b\Delta h_{b1} + d\Delta h_{b2} \tag{10.79}$$

it follows

$$\nu_2 > \left(b - \frac{d(1 + a)}{b} \right) \Delta h_{b1} - \frac{cd}{b}. \tag{10.80}$$

Since $a, d \geq b > 0$ it follows that $\nu_2 \to \infty$ because $\Delta h_{b1} \to -\infty$ and $\frac{cd}{b}$ is bounded. For vectors $\Delta h_b \in \mathbb{R}^2_-$ or $\Delta h_b \in \mathbb{R}^2_+$ the arguments become easier since the entries of the matrix

$$T(\Delta h_b) + F^T R(\xi, \Delta h_b) F$$

are always non-negative. Similar arguments hold for the case $k = 1$.
  In conclusion $||\nu|| \to \infty$ when $||\Delta h_b|| \to \infty$ in the case $k \leq 2$.

$\square$

Proposition 30. *Suppose $k \leq 2$, then the trajectories of the system (10.56) are bounded for every $t \geq 0$.*

Proof. The outline of the proof is as follows: first a Lyapunov function candidate $\mathcal{P}(\cdot)$ is constructed, which is radially unbounded and bounded from below with respect to Δh_b. Next it is shown that $\mathcal{P}(\cdot)$ is decreasing along trajectories of the closed loop system (10.54)-(10.56) for sufficiently large value of $||\Delta h_b||$.

To that end define the function $\mathcal{P} : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$
\mathcal{P}(\Delta h_b, \hat{q}, \xi) \equiv \tilde{P}(\Delta h_b, \hat{q}, \xi) - \hat{q}^T G \Delta h_b + \\
+ 2\kappa \xi^T R(\xi, \Delta h_b) F \Delta h_b, \tag{10.81}
$$

which is radially unbounded in Δh_b (which can be proved in a similar way as Lemma 4.1). Furthermore, since $\mathcal{P}(\cdot)$ is convex with respect to Δh_b it follows that it is bounded from below (see for instance Appendix A). Then

$$
\mathcal{P}(\Delta h_b, \hat{q}, \xi) = \hat{q}^T A_e \xi + 2\kappa \xi^T R(\xi, \Delta h_b) F \Delta h_b + \\
+ \kappa b^T(\xi, \Delta h_b) R(\xi, \Delta h_b) b(\xi, \Delta h_b) + \\
+ \kappa (\Delta h_b - \Delta \underline{h}_b)^T T(\Delta h_b)(\Delta h_b - \Delta \underline{h}_b). \tag{10.82}
$$

The partial derivatives of $\mathcal{P}(\cdot)$ is given as

$$
\nabla_{\hat{q}} \mathcal{P}(\Delta h_b, \hat{q}, \xi) = \xi^T A_e \tag{10.83}
$$

$$
\nabla_{\xi} \mathcal{P}(\Delta h_b, \hat{q}, \xi) = \hat{q}^T A_e + 2\kappa(\xi - \Delta \underline{h}_e)^T R(\xi, \Delta h_b) \tag{10.84}
$$

$$
\nabla_{\Delta h_b} \mathcal{P}(\Delta h_b, \hat{q}, \xi) = 2\kappa \left(\Delta \underline{h}_e^T R(\xi, \Delta h_b) F + \right. \\
+ \Delta h_b^T \left(T(\Delta h_b) + F^T R(\xi, \Delta h_b) F \right) + \\
\left. - \Delta \underline{h}_b^T T(\Delta h_b) \right). \tag{10.85}
$$

The following expression is used in the Δh_b dynamics

$$
\nabla_{\Delta h_b} \tilde{P}(\Delta h_b, \hat{q}, \xi) = \hat{q}^T G + 2\kappa \left((\Delta \underline{h}_e - \xi)^T R(\xi, \Delta h_b) F + \right. \\
+ \Delta h_b^T \left(T(\Delta h_b) + F^T R(\xi, \Delta h_b) F \right) + \\
\left. - \Delta \underline{h}_b^T T(\Delta h_b) \right). \tag{10.86}
$$

Now, let $x = (\hat{q}^T, \xi^T)$, then by boundedness of $\hat{q}(t)$ and $\xi(t)$ for every $t \geq 0$, there exists constant $z > 0$ such that

$$
\nabla_x \mathcal{P}(\Delta h_b, x)\dot{x} < z. \tag{10.87}
$$

Furthermore, recall

$$
\nu(\Delta h_b, \xi) = \Delta h_b^T \left(T(\Delta h_b) + F^T R(\xi, \Delta h_b) F \right), \tag{10.88}
$$

then by boundedness of $\hat{q}(t)$ and $\xi(t)$ there exists constants $\gamma > 0$ and $z' > 0$ such that

$$
\nabla_{\Delta h_b} \mathcal{P}(\Delta h_b, x)\Delta \dot{h}_b < - 4\kappa^2 \min(L_j)||\nu(\Delta h_b, \xi)||^2 + \\
+ \gamma ||\nu(\Delta h_b, \xi)|| + z' \tag{10.89}
$$

and consequently

$$
\begin{aligned}
\frac{d}{dt}\mathcal{P}(\Delta h_b, x) < & -4\kappa^2 \min(L_j)||\nu(\Delta h_b, \xi)||^2+ \\
& + \gamma||\nu(\Delta h_b, \xi)|| + z + z'.
\end{aligned}
\tag{10.90}
$$

By applying Lemma 5.2 the proof is concluded. $\qquad\square$

Now, the main result of the paper can be stated.

Theorem 5. *Let $k \le 2$, $\tilde{q} = q - q^*$ and $\tilde{\xi} = \xi - \xi^*$. For $\kappa > \kappa^*$ defined in Proposition 29, the state set*

$$
M = \{(\Delta h_b, q, \xi) \in \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \mid \Delta h_b \in \mathcal{M} \wedge \tilde{q} = \tilde{\xi} = 0\}.
\tag{10.91}
$$

is globally asymptotically stable for the closed loop system (10.54)-(10.56). In particular

$$
\lim_{t \to \infty} d(\zeta(t), M) = 0,
\tag{10.92}
$$

and

$$
\dot{\zeta} = 0, \ \forall \zeta \in M,
\tag{10.93}
$$

where $\zeta = (\Delta h_b^T, q^T, \xi^T)$.

Proof. It follows from Proposition 27 that $(\tilde{q}^T, \tilde{\xi}^T) = 0$ is a globally asymptotically stable equilibrium point of the system (10.54)-(10.55). Then the thesis follows by letting $z = (\tilde{q}, \tilde{\xi})$, $x = \Delta h_b$ and $Y = \mathcal{M}$ in Theorem 4 and applying Lemma 5.1, and Proposition 30. $\qquad\square$

Since, by Theorem 5, the closed loop system is globally asymptotically stable independently on the number of end-users in the system, it is concluded that end-users can be arbitrarily added to or removed from the system while maintaining the stability properties of the closed loop system. However, for the result to hold the number of booster pumps in the system cannot be more than two.

6 Numerical Results

The proposed controllers have been tested by performing a simulation of the closed loop system. Specifically, a small laboratory scale system with four end-users and two booster pumps has been simulated. The hydraulic network diagram of the system is shown in Fig. 10.2. A scenario is simulated, where the system is in the initial condition $(\Delta h_b^T, q^T, \xi^T) = 0$ and the reference for all the outputs are 0.2 Bar.

The parameters used in the simulation are: $J_{11} = 1.0787$, $J_{12} = J_{13} = J_{14} = J_{21} = J_{31} = J_{41} = 0.4421$, $J_{22} = 1.1318$, $J_{23} = J_{24} = J_{32} = J_{42} = 0.7074$, $J_{33} = 1.4854$, $J_{34} = J_{43} = 1.061$, $J_{44} = 1.7507$; $p_2 = p_{13} = 0.0586$, $p_3 = p_6 = 0.6755$, $p_7 = p_{12} = p_{21} = p_{26} = 0.0352$, $p_8 = p_{11} = p_{17} = p_{20} = p_{22} = p_{25} = 0.4503$, $p_{16} = p_{27} = 0.0469$; $v_5 = v_{10} = v_{19} = v_{24} = 0.005$, $v_{14} = 0.0013$; $A_b = 2I_2$, $A_e = 3I_4$; $r = 0.2\mathbb{1}_4$; $K = N = 2I_4$, $L = 2I_2$; $\kappa = 2$; $\Delta \underline{h}_e = \Delta \underline{h}_b = 0.1$; $\alpha = 1$. Furthermore, the parameters for all pumps in the system are $a_{h2} = 0.0148$,
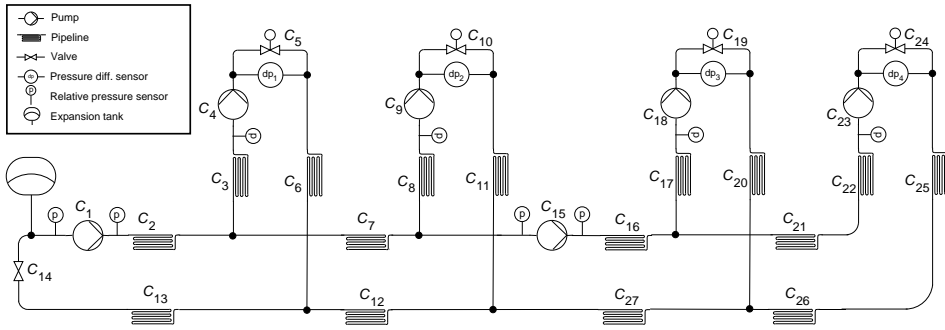
Figure 10.2: Hydraulic network diagram of the system used in the simulation.

$a_{h1} = -0.35379 \cdot 10^{-3}$, $a_{h0} = 6.0854 \cdot 10^{-6}$, $a_{t2} = 0.0039$, $a_{t1} = 0.10648 \cdot 10^{-3}$ and $a_{t0} = 0.96902 \cdot 10^{-6}$, these are used in calculating the power consumption of the pumps.

Results of a simulation of the closed loop system are shown in Fig. 10.3. The power consumption of the pumps have been calculated using the true power function given in (10.29).
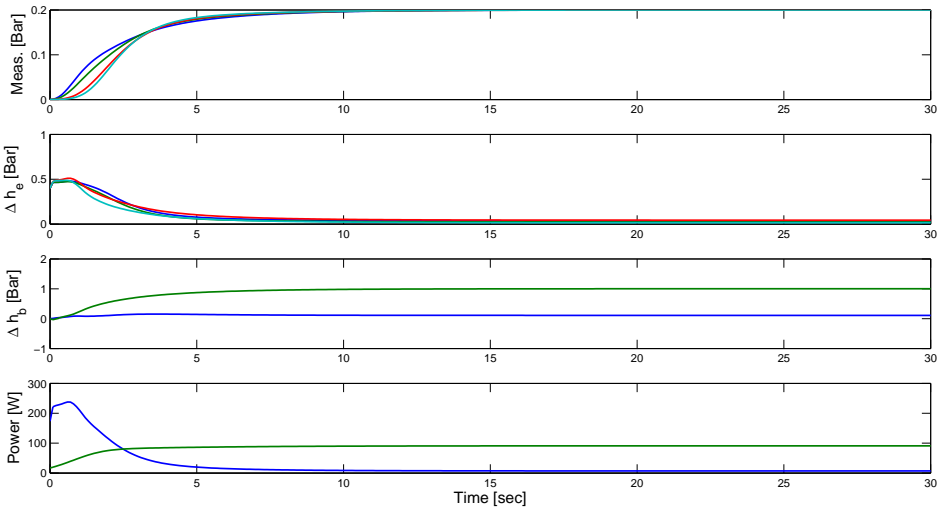

Figure 10.3: Result of simulation of closed loop system. The green line in the bottom graph shows the power consumption of the system, when using the map (10.94) from u to Δh_e, Δh_b.

In [5], the following map is used to calculate Δh_e and Δh_b

$$
\begin{aligned}
\Delta h_{c_1} &= 0.7 \min(u_1, u_2, u_3, u_4) \\
\Delta h_{c_{15}} &= 0.7(\min(u_3, u_4) - \Delta h_{c_1}) \\
\Delta h_{c_4} &= u_1 - \Delta h_{c_1} \\
\Delta h_{c_9} &= u_2 - \Delta h_{c_1} \\
\Delta h_{c_{18}} &= u_3 - \Delta h_{c_1} - \Delta h_{c_{15}} \\
\Delta h_{c_{23}} &= u_4 - \Delta h_{c_1} - \Delta h_{c_{15}},
\end{aligned}
\tag{10.94}
$$

where $\Delta h_b = (\Delta h_{c_1}, \Delta h_{c_{15}})$ and
$\Delta h_e = (\Delta h_{c_4}, \Delta h_{c_9}, \Delta h_{c_{18}}, \Delta h_{c_{23}})$. Notice that this map guarantees that if $u \in \mathbb{R}_+^4$ then $\Delta h_b \in \mathbb{R}_+^2$ and $\Delta h_e \in \mathbb{R}_+^4$.

In Fig. 10.3, bottom, a comparison has been made between the power consumption of the pumps in the system using the approach for calculating Δh_e and Δh_b presented in this paper, and the approach in (10.94). The relatively low power consumption (≈ 7 W in steady state) is due to the fact that a laboratory scale system is simulated.

In Fig. 10.3, it can be seen that in steady state, the control tends to leave as much of the actuation to the boosting pumps as possible. This is natural since $a_b < a_e$, which from the point of view of the controller means that the boosting pumps are more efficient than the end-user pumps, since $a_b^{-1} > a_e^{-1}$. However, this might not be the general case and some care should be taken into account when choosing the efficiency matrices A_e and A_b, since the actual power function is not bilinear, but the one given in (10.29).

7 Conclusion

A case study involving a large scale hydraulic system was examined. The problem of regulating the pressure drop across the so-called end-user valves in the system while minimizing the steady state power consumption of the actuators was addressed. The results show that the proposed controllers are able to provide global asymptotic output regulation, while converging to the set of minimizers of a desired objective function. In particular, the objective function describes the electrical power consumption of the actuators in the system.

The result is supported by numerical simulations performed on a small laboratory scale system with four end-users.

Since the result is global and independent on the number of end-users in the system, it shows that end-users can be arbitrarily added to or removed from the system while maintaining the stability properties.

A natural extension of the work presented here will be the generalisation of the result to an arbitrary number of booster pumps in the system. A further extension is an analysis of the stability properties of the closed loop system, when the control actions u_i are restricted to non-negative values. This is important since the centrifugal pumps used in the system are only able to provide non-negative actuation.

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A Proof of Proposition 28

This appendix presents a proof of Proposition 28. Let $\mathcal{P}(\cdot) = \tilde{P}(\cdot, \hat{q}^*, \xi^*)$.

Proof.

Definition A.1: *For a function $f : S \to \mathbb{R}$ and $c \in \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ define the set $f^{-1}(c)$ as the preimage of c under f, or $f^{-1}(c) = \{x \in S \mid f(x) = c\}$.*

Lemma A.1: *Let $c \in im(\mathcal{P})$ and $\mathcal{D} = \mathcal{P}^{-1}(c)$, then there exists $r \in \mathbb{N}$ such that $\mathcal{D} \subset B_r(0)$.*

Proof. The proof will be done by contradiction. Assume that for every $r \in \mathbb{N}$, there exists some $x^* \in \mathcal{D}$ such that $x^* \notin B_r(0)$. Take a sequence r_i, x_i^* such that $r_i \in \mathbb{N}$, then as $r_i \to \infty$, $|x_i^*| \to \infty$. On the other hand, $\mathcal{P}(x) \to \infty$ as $|x| \to \infty$, which is a contradiction. $\square$

Lemma A.1 shows that the set of minimizers of $\mathcal{P}(\cdot)$ is bounded. This can be seen by choosing $c = \min(\mathcal{P}(\cdot))$, which exists by the following theorem.

Theorem 6. *([11]) Let $F(X)$ be a convex function on a finite-dimensional space $\mathcal{E}$. If*

$$\forall X \neq 0 \ \lim_{t \to \infty} F(tX) = +\infty, \tag{10.95}$$

then there exists minimum of $F(X)$ on $\mathcal{E}$.

Next, the property of closedness of the set of minimizers is examined. First, the following helpful theorems will be given.

Definition A.2: Let C be a non-empty convex set in $\mathbb{R}^n$, and let $f : C \to \mathbb{R}$ be convex on C. Then,

$$\min_{x \in C} f(x) \tag{10.96}$$

is said to be a convex program.

Theorem 7. *([12]) Let x^* be a local minimum of a convex program. Then, x^* is also a global minimum.*

Theorem 8. *([12]) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at x^*. If x^* is a local minimum, then $\nabla f(x^*) = 0$.*

Theorem 9. *([12]) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at x^* and convex on $\mathbb{R}^n$. If $\nabla f(x^*) = 0$, then x^* is a global minimum of f on $\mathbb{R}^n$*

From the above, it is concluded that x^* is a global minimum of $\mathcal{P}$ if and only if $\nabla \mathcal{P}(x^*) = 0$. Since $\mathcal{P}(\cdot)$ is continuously differentiable, $\nabla \mathcal{P}(\cdot)$ is continuous. Because $\nabla \mathcal{P}(\cdot)$ is continuous, the preimage of a closed set under $\nabla \mathcal{P}(\cdot)$ is closed. In particular, the set $\{0\}$ is closed, which means that $\mathcal{M} = \nabla \mathcal{P}^{-1}(0)$ is also closed.

Lastly, a well known result is that the set of minimizers of a convex function defined over a convex set is convex.

Theorem 10. *Let $f : S \to \mathbb{R}$ be a convex function defined on a convex set $S \subseteq \mathbb{R}^n$. Then the set of all global minimizers of f is a convex set.*

Proof. Let $y^* \in \mathbb{R}$ denote the minimum of f over S, that is, $f(x) \geq y^*$ for every $x \in S$. Since f is a convex function it fulfils the inequality

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2) \tag{10.97}$$

for two points $x_1, x_2 \in S$. Note, $\alpha x_1 + (1 - \alpha)x_2$ belongs to S since S is a convex set.

Now, let two points $x_1^*, x_2^* \in S$ be such that $f(x_1^*) = f(x_2^*) = y^*$.

From convexity of f

$$f(\alpha x_1^* + (1 - \alpha)x_2^*) \leq \alpha f(x_1^*) + (1 - \alpha)f(x_2^*) = y^*. \tag{10.98}$$

However, since y^* is the minimum of f on S

$$f(\alpha x_1^* + (1 - \alpha)x_2^*) = y^*,$$ (10.99)

which completes the proof.

□

By Theorem 10 Proposition 28 is true.

□

B   Proof of Proposition 29

First, recall that $\mathcal{C}$ denotes the feasibility set of the original minimization problem (10.39), which is recalled here

$$\Delta h_b \geq 0$$ (10.100a)

$$F \Delta h_b \leq \xi^*,$$ (10.100b)

and that $\mathcal{C}$ is compact and non-empty by Lemma 3.1, and $P(\Delta h_b, \hat{q}^*, \xi^*) > 0$ for every $\Delta h_b \in \mathcal{C}$.

Also, by Lemma 3.1, the compact set $C \subset \mathcal{C}$ defined by

$$\Delta h_b \geq \Delta \underline{h}_b$$ (10.101a)

$$F \Delta h_b \leq \xi^* - \Delta \underline{h}_e,$$ (10.101b)

is non-empty for a proper choice of $\Delta \underline{h}_b > 0$ and $\Delta \underline{h}_e > 0$.

Furthermore, the objective function with the penalty terms included and steady state $\hat{q}$ and ξ is given as

$$\tilde{P}(\Delta h_b, \hat{q}^*, \xi^*) = P(\Delta h_b, \hat{q}^*, \xi^*) +$$
$$+ \begin{cases} 0 & \forall \Delta h_b \in C \\ S(\Delta h_b, \xi^*) & \forall \Delta h_b \in \mathbb{R}^k \setminus C \end{cases}$$ (10.102)

where $C \subset \mathcal{C}$ and $S(\Delta h_b, \xi^*) > 0$ for every $\Delta h_b \in \mathbb{R}^k \setminus C$.

Proof. Notice that since the same κ is used across all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ the function $S(\Delta h_b, \xi)$ in (10.52) can be written as

$$S(\Delta h_b, \xi) = \kappa \tilde{S}(\Delta h_b, \xi),$$ (10.103)

where

$$\tilde{S}(\Delta h_b, \xi) = \sum_{i=1}^{n} \tilde{s}_i^e(\xi_i - F_i^T \Delta h_b) + \sum_{j=1}^{k} \tilde{s}_j^b(\Delta h_{bj})$$ (10.104)

and

$$\tilde{s}_l(x) = \begin{cases} (x - \underline{x}_l)^2, & x \leq \underline{x}_l \\ 0, & x \geq \underline{x}_l \end{cases}.$$ (10.105)

Now, let Q be an open set such that

$$C \subset Q \subset \mathcal{C}.$$ (10.106)

Furthermore, let $m > 0$ be such that

$$P(x, \hat{q}^*, \xi^*) \leq m, \ \forall x \in C, \tag{10.107}$$

which exists by continuity of $P(\cdot, \hat{q}^*, \xi^*)$ and compactness of C.

The function $\tilde{S}(\cdot, \xi^*)$ fulfils

$$\tilde{S}(x, \xi^*) > 0, \ \forall x \in \mathbb{R}^k \setminus C, \tag{10.108}$$

and

$$\tilde{S}(x, \xi^*) = 0, \ \forall x \in C. \tag{10.109}$$

Therefore, for every $x \in C \setminus Q$ there exists κ^* such that

$$\tilde{P}(x, \hat{q}^*, \xi^*) = P(x, \hat{q}^*, \xi^*) + \kappa \tilde{S}(x, \xi^*) > m, \ \forall \kappa > \kappa^*, \tag{10.110}$$

again by continuity of $P(\cdot, \hat{q}^*, \xi^*)$ and compactness of $C \setminus Q$.

Let $\mathcal{M}$ denote the set of minimizers of $\tilde{P}(\cdot, \hat{q}^*, \xi^*)$, then from the above and convexity of $\tilde{P}(\cdot, \hat{q}^*, \xi^*)$ it follows that

$$\mathcal{M} \subset Q \subset \mathcal{C}. \tag{10.111}$$

Furthermore, since

$$P(x, \hat{q}^*, \xi^*) > 0, \ \forall x \in \mathcal{C}, \tag{10.112}$$

it follows that for every $\kappa > \kappa^*$

$$\tilde{P}(x, \hat{q}^*, \xi^*) = P(x, \hat{q}^*, \xi^*) + \kappa \tilde{S}(x, \xi^*) > 0, \ \forall x \in \mathbb{R}^k. \tag{10.113}$$

Then the thesis follows. $\qquad\qquad\square$