Flow Lines Under Perturbation within Section Cones

Wisniewski, Rafal

Publication date:
2005

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Wisniewski, R. (2005). *Flow Lines Under Perturbation within Section Cones*. Department of Mathematical Sciences, Aalborg University.

Ph.D. Thesis

Flow Lines under Perturbations within Section Cones

Rafał Wiśniewski

August 2005

Department of Mathematical Sciences
Aalborg University, Denmark

This thesis was set with the help of KOMA-Script and LaTeX.

To Dorde, Viktor, Christian and Benjamin

Summary

In this Ph.D. thesis we want to examine a closed smooth manifold M together with a certain partial order: In the set $\mathfrak{X}^r(M)$ of C^r vector fields on M, $r \geq 1$, we define a section cone - a convex subset of $\mathcal{K}$ characterized by the property that if p is a singular point for some vector field in $\mathcal{K}$ then this is the case for all members of $\mathcal{K}$. We say that a point q is greater than or equal to a point p if there exists a flow line from p to q corresponding to some vector field in $\mathcal{K}$. The partial order that – under a certain condition – arises from the transitive closure of that relation – gives rise to (the concept of) a *di-path* (directed path). That is a continuous map from the closed unit interval with the natural partial order inherited from the order of the real numbers to the manifold with the partial order defined as above, which furthermore preserves the partial orders. We examine di-paths between two critical points of minimal and of maximal index up to a particular homotopy relation.

We restrict the space of C^r vector fields to the set of Morse-Smale vector fields without closed orbits denoted by $\mathfrak{E}^r(M) \subset \mathfrak{X}^r(M)$. We define a *gradient-like section cone* as a convex subset of $\mathfrak{E}^r(M)$ consisting of fields whose singular points all coincide. Since Morse-Smale vector fields are structurally stable, there exists a reproducing cone $\mathcal{K}$ of vector fields in $\mathfrak{E}^r(M)$ containing $\xi \in \mathcal{K}$.

Another interesting class of section cones are the *Lyapunov section cones*. They are defined by the property that there is a single real function that is a Lyapunov function for all vector fields in $\mathcal{K}$. We show that such a cone induces a partial order relation on M. For two dimensional manifolds, the Lyapunov section cones are gradient-like. In the general case we refine Lyapunov section cones to Lyapunov-Smale section cones which are both Morse-Smale and Lyapunov. We show that such a section cone always exists.

The main result of this work relates the partial order induced by a Lyapunov-Smale section cone $\mathcal{K}$ with the partial order induced by just one of the vector fields in $\mathcal{K}$. Two flow-lines γ_0, γ_1 of a vector field $\xi \in \mathcal{K}$ joining two singular points p and q of minimal and maximal index, respectively, are said to be homotopic by ξ if there is a homotopy H such that H_t is a flow line of ξ and $H_0 = \gamma_0$, $H_1 = \gamma_1$. Two di-paths α_0, α_1 are di-homotopic if there exists a homotopy F so that F_t is a di-path, and $F_0 = \alpha_0$, $F_1 = \alpha_1$. We show that the classes of flow lines joining p and q up to homotopy by ξ are in one-to-one correspondence with the classes of flow lines connecting p and q up to homotopy by $\mathcal{K}$.

Resumé

Titel: *Variationer af vektorfelter og deres flowlinier indenfor snitkegler*

I denne Ph.D. afhandling undersøger vi en kompakt, glat mangfoldighed M uden rand som udstyres med en partiel orden på følgende måde: Vi definerer en snitkegle som en konveks delmængde $\mathcal{K}$ i mængden $\mathfrak{X}^r(M)$ af C^r vektorfelter. Mængden $\mathcal{K}$ karakteriseres yderligere ved betingelsen, at hvis p er et kritisk punkt for et vektorfelt i $\mathcal{K}$, så er det også et kritisk punkt for alle andre vektorfelter i $\mathcal{K}$. Vi siger så, at et punkt p er større eller lig et punkt q hvis der eksisterer en flowlinie af et vektorfelt tilhørende $\mathcal{K}$, som løber fra p til q. Den partielle orden som fremkommer - under visse betingelser - ved den transitive afslutning af denne relation bruges til definitionen af en di-sti (sti med retning/direction).

Vi undersøger Morse-Smale vektorfelter uden lukkede baner og betegner mængden af dem med $\mathfrak{E}^r(M) \subset \mathfrak{X}^r(M)$. En gradientlignende snitkegle er defineret som en snitkegle i $\mathfrak{E}^r(M)$. En snitkegle $\mathcal{K}$ kaldes reproducerende hvis dimensionen af $\mathcal{K}$ i ethvert punkt af M på nær de kritiske punkter svarer til dimensionen af mangfoldigheden M. Morse-Smale vektorfelter er strukturelt stabile. Derfor eksisterer for ethvert vektorfelt $\xi \in \mathfrak{E}^r(M)$ en reproducerende snitkegle $\mathcal{K}$ således at $\xi \in \mathcal{K}$.

En anden familie af snitkegler omtalt i afhandlingen består af Lyapunov snitkegler. En Lyapunov snitkegle har den egenskab, at der eksisterer én reel funktion som er en Lyapunov funktion for alle vektorfelter i keglen. Vi viser, at en Lyapunov snitkegle inducerer en partiel ordens relation på M. Hvis M har dimension to er enhver Lyapunov snitkegle gradientlignende. I det generelle tilfælde forfiner vi Lyapunov snitkegler til Lyapunov-Smale snitkegler, som er både Morse-Smale og Lyapunov. Vi viser eksistensen af sådanne Lyapunov-Smale snitkegler.

Afhandlingens hovedresultat sætter den partielle ordenen frembragt af en Lyapunov -Smale snitkegle $\mathcal{K}$ i forbindelse med den partielle orden frembragt af kun et af vektorfelterne tilhørende $\mathcal{K}$. To flowlinier γ_0 og γ_1 af et vektorfelt $\xi \in \mathcal{K}$, der løber fra et kritisk punkt p til et kritisk punkt q med henholdsvis minimal og maksimal indeks kaldes ξ-homotope hvis der eksisterer en homotopi H, således at H_t er en flowline af ξ, $H_0 = \gamma_0$ og $H_1 = \gamma_1$. To di-stier α_0, α_1 er di-homotope, hvis der eksisterer en homotopi F sledes at F_t er en di-sti for $t \in [0,1]$ og $F_0 = \alpha_0$, $F_1 = \alpha_1$. Vi viser at klassen af flowlinier som fødes i p og som dør i q op til ξ-homotopi korresponderer en-til-en til klassen af flowlinier fra p til q op til $\mathcal{K}$-homotopi.

Acknowledgement

Many people have helped me since I started working on this thesis. First of all, I wish to express my deepest thanks to my wife and children for continuing support and patience during the period of writing this thesis.

I am indebted to professor Martin Raussen for arousing my interest in mathematics and his motivating supervision of this Ph.D. work. I would like to thank him for the discussions we have had, especially those starting with my “do you have a minute” and which ended hours later.

My special thanks go to professor Robin Forman for his interest in this work during an extended stay at Rice University, Houston, TX, USA.

I express my thanks to professor Lisbeth Fajstrup and Ph.D. student John-Josef Leth for the discussions on all aspects of mathematics.

This work would not be possible without help from the Department of Control Engineering. I would also like to thank Department of Mathematical Sciences for hosting me during the last one and a half year of the Ph.D. study.

Aalborg, Denmark, August 2005 Rafał Wiśniewski

Contents

1 Introduction

This thesis comprises a part of a program initiated at the Department of Mathematical Sciences, Aalborg University, which aims at developing mathematical foundations of concurrency theory using ideas from geometry and topology.

Concurrency deals with scheduling computer resources in a situation where several tasks must be performed at the same time. This can be a true parallelism like in the case of several processors running concurrently, or also the particular case of a mono-processor machine where a unique processor is sharing its calculation time between several different tasks.

1.1 Partial Order and Concurrency

The execution of a computer program can be treated as a flow line of a certain vector field. The flow line is born at a point a, which is the start of a program, and dies at a point b, the end of the program. Due to different scheduling scenarios the execution of the same program may result in flow lines of vector fields close to each other. Small variations of a concurrent program do not change qualitatively its performance. To test a program it means to execute it and check if - for instance - it leads to a deadlock; that is a situation when two or more tasks access a computer resource at the same time and prevent each other from proceeding. Computational burden of such a validation might be huge. Therefore there is a wish to test only representative cases.

It is demonstrated in Fajstrup et al. [2005] that the execution of a program can be treated as a continuously increasing path in a po-space $(X, \leq)$, i.e a topological space X with a partial order relation $\leq$, which is a closed subset of $X \times X$ in the product topology; at least locally.

Definition 1.1.1 (Definition 3.7, Fajstrup et al. [2005]). *Let $(X, \leq)$ and $(Y, \precsim)$ be po-spaces. A continuous map $f : X \to Y$ is called a di-map (directed map) if and* *only if it preserves partial orders, that is,*

$$x_1 \leq x_2 \Rightarrow f(x_1) \precsim f(x_2), \ \textit{for all } x_1, x_2 \in X.$$

A model for a concurrent program is a di-map $\alpha : I \to X$ from the unit interval I with its natural order to a po-space $(X, \leq)$.

Definition 1.1.2 (Definition 4.2, Fajstrup et al. [2005]). *Let $(X, \leq)$ be a po-space and let $a, b \in X$. A di-path in X from a to b is a di-map α in X with $\alpha(0) = a$ and $\alpha(1) = b$. The set of all di-paths from a to b will be denoted by $\overrightarrow{P}_1(X; a, b)$.*

The equivalence of execution paths can be modelled geometrically by a homotopy relation on di-paths.

Definition 1.1.3 (Definition 4.2, Fajstrup et al. [2005]). *Let $(X, \leq)$ be a po-space and let $a, b \in X$.*

1. *A di-homotopy from a to b is a continuous map $H : I \times I \to X$ such that every map $H_s : I \to X, \ H_s(t) = H(s, t), \ s \in I$, is a di-path from a to b.*
2. *Two di-paths α, β in X from a to b are di-homotopic from a to b if and only if there is a di-homotopy $H : I \times I \to X$ from a to b with $H_0 = \alpha$ and $H_1 = \beta$.*

Di-homotopy from a to b is an equivalence relation. The equivalence classes - di-homotopy classes - constitute the di-homotopy set $\overrightarrow{\pi}_1(X; a, b)$. Now, we are able to state the aim of this research program as - classification of di-homotopy sets.

We attack the problem from the point of view of differential topology. In our case the topological space is a closed smooth manifold M^n (n is the dimension of the manifold M) with flow lines arising from a variety of vector fields on M. Let $\mathfrak{X}^r(M)$ denote the set of all C^r vector fields on M, and $\mathfrak{S}^r(M)$ be the subset of Morse-Smale vector fields. For a vector field $\xi \in \mathfrak{X}^r(M)$, let $\mathcal{C}r(\xi)$ denote the set of singular ("critical") points.

Definition 1.1.4 (Definition 6.1.1 in this report). *A C^r section cone $\mathcal{K}$ on a smooth manifold M is a subset of $\mathfrak{X}^r(M)$ that satisfies the following two conditions:*

1. *For every pair $\xi, \eta \in \mathcal{K}$, if $p \in \mathcal{C}r(\xi)$ then $p \in \mathcal{C}r(\eta)$. (All vector fields in the section cone $\mathcal{K}$ have the same singularities).*
2. *If ξ and η are in $\mathcal{K}$ and $\alpha, \beta > 0$ then $\alpha\xi + \beta\eta \in \mathcal{K}$.*

We define a counterpart of a di-path in the new geometric setup as concatenation of flow lines.

Definition 1.1.5 (Definition 6.2.3 in this report). *Suppose $\mathcal{K}$ is a C^r section cone on a closed smooth manifold. We call a piecewise C^r path $\sigma : I \to M$ a di-path if there exists a finite family of real numbers $0 = t_0 \leq t_1 \leq ... \leq t_k = 1$ and a family of vector of fields $\{\xi_1, ..., \xi_k\} \subset \mathcal{K}$ such that $\sigma|_{[t_i, t_{i+1}]}$ is the flow line of ξ_{i+1} from $\sigma(t_i)$ to $\sigma(t_{i+1})$ for $i \in \{0, ..., k-1\}$. The set of all di-paths of $\mathcal{K}$ from a singular point a to a singular point b is denoted by $\overline{P}(a, b; \mathcal{K})$.*

To characterize an equivalence of di-paths from Definition 1.1.5 we need a reformulation of di-homotopy.

Definition 1.1.6 (Definition 6.2.4 in this report). *Suppose $\mathcal{K}$ is a section cone on a closed smooth manifold M and $a,\ b$ are two singular points of $\mathcal{K}$.*

1. *A di-homotopy from a to b is a continuous map $H : I \times I \to M$ such that every map $H_s \in \overline{P}(a, b; \mathcal{K})$, $s \in I$.*
2. *Two di-paths $\gamma,\ \eta \in \overline{P}(a, b; \mathcal{K})$ are said to be di-homotopic if and only if there exists a di-homotopy $H : I \times I \to M$ with $H_0 = \gamma$ and $H_1 = \eta$.*

The set of equivalence classes of di-paths up to di-homotopy is denoted by $\pi(a, b; \mathcal{K})$.

To simplify the situation we suppose that there is only one singular point with index n, say a, and one singular point with index 0, say b. The aim of the thesis is to characterize the set $\pi(a, b; \mathcal{K})$. We focus on a particularly nice section cone - a Lyapunov-Smale section cone.

Definition 1.1.7 (Definition 6.1.11 in this report). *A section cone $\mathcal{K} \subset \mathfrak{S}^r(M)$ is Lyapunov-Smale if and only if there exists a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric on M such that for any $\xi \in \mathcal{K}$ we have*

1. *$\xi(f)(x) < 0$ for all $x \in M - \mathrm{Cr}(\mathcal{K})$,*

2. *there exist a constant $\kappa > 0$ and open neighborhoods $\{U_q\}_{q \in \mathrm{Cr}(\mathcal{K})}$ of the singular points such that*

$$-\xi(f)(x) \geq \kappa d(x,p)^2 \text{ for } p \in U_p, \text{ where } d \text{ is the Riemannian distance.}$$

For a C^r vector field η, $P(a,b;\eta)$ is the set of flow lines of η from the singular point a to the singular point b. The set of flow lines of the vector fields in a section cone $\mathcal{K}$, which are born in a and die in b are denoted by $P(a,b;\mathcal{K})$. We denote the flow line of the vector field ξ by $\phi_x^\xi(t)$, that is

$$\frac{d}{dt}\phi_x^\xi(t) = \xi\left(\phi_x^\xi(t)\right) \text{ with } \phi_x^\xi(0) = x.$$

Let $W(a,b;\xi) = \{x \in M| \lim_{t\to-\infty}\phi_x^\xi(t) = a \text{ and } \lim_{t\to+\infty}\phi_x^\xi(t) = b\}$. We define two notions of homotopy, by a vector field and by a section cone.

Definition 1.1.8 (Definition 7.1.1 in this report). *Let M be a closed smooth manifold. For $r \geq 1$, let $\xi \in \mathfrak{X}^r(M)$ and $\mathcal{K}$ be a C^r section cone on M.*

1. *Suppose $\gamma_0, \gamma_1 \in P(a,b;\xi)$. We say that γ_0 is homotopic to γ_1 by ξ and write $\gamma_0 \sim_\xi \gamma_1$ if and only if there is a path $\beta : I \to M$ such that $\beta(t) \in W(a,b;\xi)$, $\gamma_0(t) = \phi_{\beta(0)}^\xi(t)$ and $\gamma_1(t) = \phi_{\beta(1)}^\xi(t)$.*

2. *Suppose $\gamma_0, \gamma_1 \in P(a,b;\mathcal{K})$. We say that γ_0 is homotopic to γ_1 by $\mathcal{K}$ and write $\gamma_0 \sim_\mathcal{K} \gamma_1$ if and only if there exist a path $\sigma : I \to \mathcal{K}$ and a path $\beta : I \to M$ such that $\beta(t) \in W(a,b;\sigma(t))$, $\gamma_0(t) = \phi_{\beta(0)}^{\sigma(0)}(t)$ and $\gamma_1(t) = \phi_{\beta(1)}^{\sigma(1)}(t)$.*

The main result of the thesis is the following theorem.

Theorem 1.1.9 (Theorem 7.1.2 in this report). *Let M be a closed smooth manifold. Suppose $\mathcal{K}$ is a Lyapunov-Smale C^r section cone on M, $r \geq 5$, and $\xi \in \mathcal{K}$. Let a, b be the only singular points with indices 0 and n, respectively. Then there is a bijection $\Pi : P(a,b;\xi)/_{\sim_\xi} \to P(a,b;\mathcal{K})/_{\sim_\mathcal{K}}$.*

There are two steps remaining in the program of classification of $\pi(p, q; \mathcal{K})$. The first is to show that any di-path is homotopic by $\mathcal{K}$ to a flow line for some $\eta \in \mathcal{K}$. The second step is to establish results on detecting the connected components of the moduli space of the flow lines joining p and q corresponding to a particular vector field ξ. These subjects are not covered in this thesis and they are matters of further work.

This thesis is organized as follows. Chapters 2, 3, 4 are mainly reviews of the existing results. In Chapter 2 we introduce a notion of a cone in a vector space, which we later on to generalize to a section cone in the space of C^r vector fields on a closed manifold. Chapter 3 gives preliminaries of differential topology. The focus is on tubular neighborhoods, transversality and framings. The aim of Chapter 4 is to review Morse theory and to relate a framed connected manifold with the homotopy class of a relative attaching map. Geometric theory of dynamic systems is introduced in Chapter 5. It is mainly a review of the existing results with emphasis on perturbations of vector fields. We investigate Morse-Smale and gradient like vector fields. We analyze the dependence of the invariant manifolds under perturbations of vector fields. Chapters 6 and 7 comprise an original contribution of this thesis. The notion of a section cone is formulated in Chapter 6. The emphasis is on two classes of section cones: gradient-like and Lyapunov-Smale section cones. We show that a Lyapunov-Smale section cone on a compact manifold M induces a partial order relation on M. The main theorem of this thesis is formulated and proved in Chapter 7. It shows using the associated flow lines that the study of connected components of the space of flow lines of the vector fields in a Lyapunov-Smale section cone $\mathcal{K}$ can be reduced to the study of the connected components of flow lines of an arbitrary $\xi \in \mathcal{K}$.

2 Cones

We introduce the notion of a cone in a vector space V. A cone K is a convex subset of V characterized by the property that if x and $-x$ are in K then $x = 0$. A cone is the primary object we shall generalize to a section cone - a subset of the space of C^r vector fields on a manifold M. The reason for our interest in cones is that they define the set of admissible tangent vectors at each point of a di-path.

We use the following notation: If K is a subset of a vector space V then

$$-K = \{x \in V| -x \in K\}.$$

For $C,\ K \subset V$, $K - C$ is the subset $\{x \in V|\, x = k - c,\ k \in K,\ c \in C\}$. If U is a subset of V, we denote the interior of U by $\mathrm{int}(U)$ and the closure of U by $\mathrm{cl}(U)$.

2.1 Cones in Vector Spaces

We start with the definition of a cone in a vector space V.

Definition 2.1.1 (Barker [1981]). *Let V be a real vector space. A cone K in V is a subset of V satisfying*

1. *If $a, b \geq 0$ and $x,\ y \in K$, then $ax + by \in K$;*
2. *$K \cap (-K) = \{0\}$.*

The family of all cones in V together with the empty set is denoted by $\mathcal{D}(V)$.

It follows from the definition that a cone is a convex set containing 0, that is

$$\alpha x + (1-\alpha)y \in K \text{ for all } x, y \in K \text{ and } 0 \leq \alpha \leq 1.$$

If x and $-x$ are in K then $x = 0$. For finite dimensional vector space V the dimension of a cone K, $\dim(K)$ is by definition the dimension of the subspace $K - K$.

Definition 2.1.2 (Barker [1981]). *The cone $K \in \mathcal{D}(V)$ is reproducing if and only if $K - K = V$. We call K full if and only if $\mathrm{int}(K) \neq \emptyset$.*

In the case of finite dimensional V a cone is reproducing if and only if it is full, cf. Barker [1981].

Example 2.1.3. If $\{v_i\}_{i=1,...,n}$ is a basis and n is the dimension of V, then

$$\mathrm{span}^{+}\{v_1, ..., v_n\} \equiv \{w \in V |\, w = a_1 v_1 + ... + a_n v_n,\ a_i \geq 0\}$$

is a reproducing cone. In particular the quadrant $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0\}$ is a reproducing cone.

Example 2.1.4. Suppose a vector space V is endowed with a symmetric non-degenerate indefinite bilinear function $g : V \times V \to \mathbb{R}$ with index 1. The vector space V can be decomposed into the direct sum $V = V^+ \oplus V^-$, where V^+ is the subspace of maximal dimension such that g is positive definite in V^+, and V^- is the orthogonal complement with respect to the scalar product defined by g. In fact g restricted to V^- is negative definite, and $\dim V^- = 1$. We choose a vector $\xi \in V^-$ and construct the Lorenz cone

$$L \equiv L(\xi) = \{\lambda(\xi + v) |\, \lambda \geq 0,\ v \in V^+ \text{ and } -g(\xi, \xi) \geq g(v, v)\}.$$

We use the observation that $g(\lambda(\xi + v), \lambda(\xi + v)) = \lambda^2(g(\xi, \xi) + g(v, v))$ to conclude that if $\xi \in V^-$ then L is a (reproducing) cone. Analogously we may use the inner product h on V to define a cone $K(\eta)$ for $\eta \in V$

$$K(\eta) = \{\lambda(\eta + v) |\, \lambda \geq 0,\ h(v, \eta) = 0 \text{ and } h(\eta, \eta) \geq h(v, v)\}.$$

In particular we may choose h such that $L(\xi) = K(\xi)$.

Proposition 2.1.5. *Suppose A is a subset of a real vector space V. If (i) A is convex and (ii) $v \in A$ implies $-v \notin A$. Then*

$$cA = \{aw \in V |\, a \in \mathbb{R}_+, w \in A\}$$

is a cone.

Proof. Suppose the conditions (i) and (ii) are satisfied. Let $a \geq 0,\ b \geq 0$ and $x, y \in cA$. We show that $ax + by \in cA$. If a or b is zero the conclusion follows. We assume that $a > 0,\ b > 0$ then

$$ax + by = (a + b)(\alpha x + (1 - \alpha)y), \text{ where } \alpha = \frac{a}{a + b} \in (0, 1].$$

thus $ax + by \in cA$ since A is convex.

We show that if $x \in A$ and $c \leq 0$ then $cx \notin A$, which implies that cA is a cone. Due to (ii) $0 \notin A$. It is enough to assume $c < 0$. Let $x \in A$ and assume that $cx \in A$. Then

$$\alpha x + (1 - \alpha)cx \in A, \text{ for } \alpha \in [0, 1].$$

Since $\frac{-c}{1-c} \in (0, 1)$ there exists $\epsilon > 0$ such that both

$$\alpha_1 = \frac{-c + \epsilon}{1 - c},\ \alpha_2 = \frac{-c - \epsilon}{1 - c}$$

are in the interval $(0, 1)$. We see that

$$\alpha_1 x + (1 - \alpha_1)cx = \epsilon x \text{ and } \alpha_2 x + (1 - \alpha_2)cx = -\epsilon x$$

are in A, which is a contradiction. $\square$

Definition 2.1.6 (Barker [1981]). *Let K be a closed cone in a finite dimensional real vector space V (closed as a subset in V). A subset $F \subset K$ is a face of K if and only if*

1. $F \in \mathcal{D}(V)$,
2. $x \in F,\ y \in K,\ x - y \in K$ *imply* $y \in F$.

The collection of all faces of K is denoted $\mathcal{F}(K)$. The trivial faces are $\{0\}$ and K. Since $F \in \mathcal{F}(K)$ is a cone in V it has a dimension $\dim(F)$, defined as $\dim(F) = \dim(F - F)$. Each non-trivial face is a non-reproducing cone since $\dim(F) < \dim(K) = \dim(V)$.

If $\dim(F) = 1$, F is called an extreme ray of K. One can show that K is the convex hull of its extreme rays, cf. Barker [1972].

Definition 2.1.7 (Barker [1981]). *A polyhedral cone is a cone which has a finite set of extreme rays.*

Let K be a cone in V. If $x \in K$, we write $x \geq 0$. Then $x \geq y$ means $x - y \geq 0$, and K defines a partial order on V. Let V^* and $Hom(V) \equiv Hom(V, V)$ be the dual space of V and the space of linear maps $V \to V$, respectively. Set

$$\begin{aligned} K^* &= \{f \in V^* | \, f(x) \geq 0 \, \forall x \in K\} \\ \Pi(K) &= \{f \in Hom(V) | \, f(K) \subset K\}. \end{aligned}$$

Then K^* and $\Pi(K)$ are cones in V^* and $Hom(V)$, respectively. If K is a closed full cone so are K^* and $\Pi(K)$.

Proposition 2.1.8. *If $A : V \to W$ is a linear map of vector spaces, $K \in \mathcal{D}(V)$, and $K \cap \ker(A) = \{0\}$, then $AK \in \mathcal{D}(W)$.*
If V is finite dimensional and $(K-K) \cap \ker(A) = \{0\}$, then $\dim(K) = \dim(AK)$.

Proof. The property 1 of Definition 2.1.1 for AK follows immediately from the assumption that A is linear. To prove property 2, assume $y, -y \in AK$. Then there exist $x_1, x_2 \in K$ such that $Ax_1 = y$ and $Ax_2 = -y$. Hence, $A(x_1 + x_2) = 0$, and we see that $x_1 + x_2 \in \ker(A) \cap K = \{0\}$. Thus, $x_1 = -x_2$ and so $x_1 = x_2 = 0$ by property 2 applied to K. Therefore, $y = 0$.

If V is finite dimensional, then $\dim(\ker(B)) + \dim(AK - AK) = \dim(K - K)$ where $B : K - K \to AK - AK$ denotes the restriction of A to $K - K$. Since $(K - K) \cap \ker(A) = \ker(B) = \{0\}$ we have $\dim(K) = \dim(AK)$. $\square$

Immediately we have the following corollary.

Corollary 2.1.9. *If $A : V \to W$ is an injective linear map of vector spaces and $K \in \mathcal{D}(V)$ then $AK \in \mathcal{D}(W)$.*

Proposition 2.1.8 does not give sufficient conditions for AK to be a cone. Consider a simple example; $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the projection on the first factor, and K is the polyhedral cone $\mathbb{R}^2_+$. It is seen that $K \cap \ker(A) = \{0\} \times \mathbb{R}_+$ and $AK = \mathbb{R}_+ \in \mathcal{D}(\mathbb{R})$.

Theorem 2.1.10. *Let $A : V \to W$ be a linear map of finite dimensional vector spaces and $K \in \mathcal{D}(V)$. $AK \in \mathcal{D}(W)$ if and only if $K \cap \ker(A) \in \mathcal{F}(K)$.*

Proof. Assume that $K \cap \ker(A) = F \in \mathcal{F}(K)$. Property 1 of Definition 2.1.1 for AK follows immediately from the assumption that A is linear. To prove property 2, assume $y, -y \in AK$. Then there exist $x_1, x_2 \in K$ such that $Ax_1 = y$ and $Ax_2 = -y$. Hence, $A(x_1 + x_2) = 0$, and we see that $x_1 + x_2 \in \ker(A) \cap K = F$. Since $(x_1 + x_2) - x_1 = x_2 \in K$ we have $x_1 \in F \subset \ker(A)$. Therefore, $y = 0$.

Now assume that $AK \in \mathcal{D}(W)$. Since $\ker(A)$ is a subspace of V it is closed under vector addition and scalar multiplication, and hence, $K \cap \ker(A) \in \mathcal{D}(V)$. Let $x \in K \cap \ker(A)$, $y \in K$, and assume that $x - y \in K$. Then $Ay \in AK$ and $A(x - y) = -Ay \in AK$. Thus, $Ay = 0$ and $y \in K \cap \ker(A)$. $\square$

3 Elements of Differential Topology

The aim of this chapter is to review elements of differential topology. It consists of standard results on existence of a tubular neighborhood, transversality, framed cobordism and stable framings. These comprise a foundation for the work on perturbations of flow lines in this thesis. Our focus in this chapter is not distributed evenly. Subjects more fundamental - not necessarily more complex - for this thesis are covered more thoroughly.

By an n-dimensional manifold M^n we understand a topological space that is locally homeomorphic to $\mathbb{R}^n$, Hausdorff and second countable with C^r differentiable structure. Particularly, a manifold in this thesis is a paracompact space. A vector bundle over a manifold X with total space E and projection $p : E \to X$ is denoted by $\mu = (p, E, X)$. A map $f : X_0 \to X$ induces a pullback $f^*\mu = (p_0, E_0, X_0)$, where

$$E_0 = \{(x, y) \in X_0 \times E |\ f(x) = p(y)\} \text{ and } p_0(x, y) = x.$$

The tangent space to a C^r manifold M is indicated by $(\pi, T(M), M)$ or shortly $T(M)$.

We study an n-dimensional submanifold M of a C^r, $(n + k)$-dimensional manifold N with a Riemannian structure g. The geometric normal bundle of a submanifold $i : M \hookrightarrow N$ is identified with the subbundle of $T_M(N) \equiv i^*T(N)$ consisting of the tangent vectors in $T_p(N)$, $p \in M$, which are perpendicular, with respect to the Riemannian metric on N, to $T_p(M)$. Alternatively, we might define the algebraic normal bundle of M in N as the quotient bundle $T_M(N)/T(M)$. In this section we shall not distinguish between them, and denote both of them by $\nu(M, N)$.

3.1 Tubular Neighborhoods

Denote by ν_0 the zero section of $\nu(M, N)$. We refer in the sequel to the following theorem.

Theorem 3.1.1 (Theorem III.2.2 in Kosinski [1993]). *Suppose that M, N are C^r $(r \geq 3)$ manifolds and M is a closed subset of N, $\partial N = \emptyset$. Then there is a neighborhood of ν_0 on which the exponential map is a C^{r-2}-embedding.*

A Riemannian metric h on a C^r vector bundle $\mu = (\pi, E, B)$ provides a way of shrinking it. Let ϵ be a C^r positive function on B. Consider a map $F : E \to E$ given by

$$F(v) = \epsilon(\pi(v))\frac{v}{(1 + h(v,v))^{1/2}}.$$

Then F maps the fiber over p onto the open disk in E_p centered at 0 and diameter $\epsilon(p)$. Thus $F(E)$ is an open disk bundle. The map F is a C^r diffeomorphism, hence the vector bundle structure on E induces a C^r disk bundle structure on $F(E)$. This operation will be called ϵ-shrinking of E.

Let X be a C^r manifold and $f : X \to N$ be a C^r embedding, then the normal bundle to f is defined by $\nu_f = f^*\nu(f(X), N)$. Suppose also that f embeds X as a closed subset of N. Then by Theorem 3.1.1 there is a neighborhood U of the zero section in E, where E is the total space of the vector bundle ν_f, and an embedding $h : U \to N$. We apply ϵ-shrinking $F : E \to E$ such that $F(E) \subset U$. Hence the composition $h \circ F$ gives an embedding of E in N. We summarize this into a corollary.

Corollary 3.1.2 (Corollary III.2.3 in Kosinski [1993]). *Let $f : X \to N$ be a C^r embedding, $r \geq 3$, and suppose $f(X)$ is a closed subset of N. Denote the total space of ν_f by E. Then f extends to a C^{r-2} embedding $\bar{f} : E \to N$. If $\partial X = \emptyset$, then $\bar{f}(E)$ is an open neighborhood of $f(X)$ in N.*

We recall that a tubular neighborhood of X in N is a neighborhood U of $f(X)$ in N such that there exists a vector bundle $\mu = (\pi, E, X)$ and there is an embedding

$e : E \to U$ making the following diagram commute

$$\begin{tikzcd} E \arrow[dr, "e"] & \\ X \arrow[u, "\mu_0"] \arrow[r, "f"] & U \subset N, \end{tikzcd}$$

where μ_0 is the zero section of μ. If the subset U is open then we shall call it an open tubular neighborhood. In particular, it follows from the corollary that if X is a closed manifold then X possesses an open tubular neighborhood in N with the vector bundle structure that of the normal bundle.

The same holds for a certain class of manifolds with boundary. Recall that in the definition of an n-dimensional manifold with boundary we allow homeomorphisms onto open subsets of either $\mathbb{R}^n$ or $\overline{\mathbb{R}}_+^n \equiv \{(x_1, ..., x_n) \in \mathbb{R}^n | \, x_n \geq 0\}$. In the next definition we make use of the subset

$$\overline{\mathbb{R}}_+^{m,n} \equiv \{(x_1, ..., x_n) \in \overline{\mathbb{R}}_+^n | \, x_1, ..., x_{n-m} = 0\}, \text{ with } n > m.$$

Definition 3.1.3 (Definition II.2.2 in Kosinski [1993]). *A submanifold $M \subset N$ is neat if it is a closed subset of N and satisfies*

1. $M \cap \partial N = \partial M$;
2. *At every point $p \in \partial M$ there is a coordinate chart $\psi : U \to \overline{\mathbb{R}}_+^n$, such that $\psi^{-1}(\overline{\mathbb{R}}_+^{m,n}) = U \cap \partial M$.*

The last condition in the definition says that ∂M meets ∂N like $\overline{\mathbb{R}}_+^{m,n}$ meets $\overline{\mathbb{R}}_+^n$.

Definition 3.1.4 (Definition III.4.1 in Kosinski [1993]). *Let U be a tubular neighborhood of a neat submanifold M of the manifold N. We say that U is neat if $U \cap \partial N$ is a tubular neighborhood of ∂M in ∂N.*

Theorem 3.1.5 (Theorem III.4.2 in Kosinski [1993]). *If M is a neat submanifold of N, then it has a neat open tubular neighborhood.*

To conclude the subject of tubular neighborhoods we shall state another theorem which has a consequence for the existence of a framing defined in the next section.

The boundary of a manifold cannot have a tubular neighborhood. However, it almost has a tubular neighborhood in the following sense. A collar on a manifold M with boundary is an embedding

$$h : \partial M \times [0, \infty) \to M,$$

such that $h(x, 0) = x$.

Theorem 3.1.6 (Collaring Theorem, 4.6.1 in Hirsch [1976]). *Suppose M is a manifold with boundary. Then ∂M has a collar.*

3.2 Framings

We assume in this section that M is a smooth closed manifold, a submanifold of $N^{(n+k)}$. A trivialization of the normal bundle $\nu(M, N)$, i.e. a bundle isomorphism

$$\begin{tikzcd} \nu(M,N) \arrow[rr, "\phi"] \arrow[dr] & & M \times \mathbb{R}^k, \arrow[dl] \\ & M & \end{tikzcd}$$

is called a framing of M in N. We shall focus on the question of when a framing of $\nu(M, N)$ exists and how to construct it.

For an n-plane vector bundle we have the following theorem.

Theorem 3.2.1 (2.2 in Milnor and Stasheff [1974]). *An n-dimensional vector bundle ξ is trivial if and only if ξ admits n sections $s_1, ..., s_n$ which are nowhere linearly dependent.*

We use the theorem above to formulate an equivalent definition of a framing.

Definition 3.2.2. *A framing of a submanifold $M \subset N$ is a smooth map σ which assigns to each $p \in M$ a basis*

$$\sigma(p) = (\sigma^1(p), ..., \sigma^k(p))$$

for the normal space $\nu_p(M, N)$. The pair (M, σ) is called a framed submanifold of N.

We will use the following version of the Tubular Neighborhood Theorem.

Theorem 3.2.3 (Product Neighborhood Theorem in Milnor [1997]). *Let (M, σ) be a framed submanifold of N. There is a neighborhood U of M in N diffeomorphic to the product $M \times \mathbb{R}^k$. Furthermore, the diffeomorphism can be chosen so that each $x \in M$ corresponds to $(x, 0) \in M \times \mathbb{R}^k$ and so that each normal frame $\sigma(x)$ corresponds to the standard basis for $\mathbb{R}^k$.*

The diffeomorphism in the Product Neighborhood Theorem is given by the composition $M \times \mathbb{R}^k \xrightarrow{\mathrm{id}\times\phi} M \times U_\epsilon \xrightarrow{\psi} U$, where ϕ is a diffeomorphism taking $\mathbb{R}^k$ onto a sufficiently small ϵ-neighborhood U_ϵ of 0 in $\mathbb{R}^k$, and ψ is defined by

$$\psi(x, t_1, ..., t_k) = \exp_x(t_1\sigma^1(x) + ... + t_k\sigma^k(x)),$$

$\exp$ is the exponential map corresponding to the Riemannian metric g on N.

Not every compact manifold has a framing, therefore below we give examples of framed manifolds.

Theorem 3.2.4 (Covering Homotopy Theorem 4.1.5 in Hirsch [1976]). *Assume ξ is a C^r $(0 \leq r \leq \infty)$ vector bundle over $B \times I$, with B a C^r manifold. Let $(\xi|_{B\times 0}) = \eta = (p, E, B)$ and $\eta \times I = (p \times \mathrm{id}_I, E \times I, B \times I))$. Then ξ is C^r isomorphic to the vector bundle $\eta \times I$*

A corollary of the covering homotopy theorem is that every vector bundle over a contractible paracompact space is trivial, see Corollary 3.2.5 below.

Corollary 3.2.5. *Suppose B is a C^r manifold and ξ is a C^r vector bundle over M. Let $H : B \times I \to M$ be a C^r homotopy, $H_0 = f$ and $H_1 = g$. Then the pullbacks $f^*\xi$ and $g^*\xi$ are C^r isomorphic. In particular, if g is constant then $f^*\xi$ is trivial.*

Proof. $H^*\xi$ is isomorphic to $H^*\xi|_{B\times 0} \times I$. But $H^*\xi|_{B\times 0} = H_0^*\xi = f^*\xi$. By replacing t by $1 - t$ in the homotopy H_t, $H^*\xi$ is isomorphic to $g^*\xi \times I$. We conclude that $f^*\xi$ and $g^*\xi$ are also isomorphic. □

Another framing can arise when a 1-dimensional normal bundle is orientable.

Theorem 3.2.6 (4.4.3 in Hirsch [1976]). *An orientable C^r 1-dimensional vector bundle is trivial.*

Proof. Let $\xi = (p, E, B)$ be an orientable 1-dimensional vector bundle, e_1 be the standard orientation on $\mathbb{R}$, and ω be the orientation of ξ. Suppose that

$$\Phi = \{\phi_\alpha : \xi|_{U_\alpha} \to U_\alpha \times \mathbb{R}\}_{\alpha\in\Lambda}$$

is an oriented atlas belonging to ω that is

$$\phi_\alpha(x) : (E_x, \omega_x) \to (\mathbb{R}, e_1)$$

is orientation preserving for all $\alpha \in \Lambda$ and $x \in U_\alpha$. Let $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ be partition of unity subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$. Then $\psi : E \to B \times \mathbb{R}$, defined by

$$\psi(y) = \sum_{\alpha\in\Lambda} \lambda_\alpha(x)\phi_\alpha(y)$$

gives the trivialization of ξ. $\square$

We shall use Theorem 3.2.6 in connection with the remark that every vector bundle over a simply connected space is orientable. More generally, a vector bundle ξ over a CW-complex B is orientable if and only if the Stiefel-Whitney class $w_1(\xi) \in H^1(B; \mathbb{Z}_2)$ is zero.

3.3 Framed Cobordism and the Pontrjagin-Thom Construction

We will now consider two closed framed submanifolds (M_1, σ_1) and (M_2, σ_2) of a manifold N.

Definition 3.3.1 (Ch.7 of Milnor [1997]). *Two framed submanifolds (M_1, σ_1) and (M_2, σ_2) of N are framed cobordant if the subsets*

$$M_1 \times [0, \epsilon) \cup M_2 \times (1 - \epsilon, 1]$$

of $N \times [0, 1]$ can be extended to a compact manifold $X \subset N \times [0, 1]$ so that

1. $\partial X = M_1 \times \{0\} \cup M_2 \times \{1\}$;
2. X *does not intersect* $N \times \{0\} \cup N \times \{1\}$ *except at points of* ∂X;
3. *There exists a framing* κ *of* X *in* $N \times I$, *so that*

$$\begin{aligned} \kappa^i(x,t) &= (\sigma_1^i(x), 0) \text{ for } (x,t) \in M_1 \times [0, \epsilon) \\ \kappa^i(x,t) &= (\sigma_2^i(x), 0) \text{ for } (x,t) \in M_2 \times (1-\epsilon, 1]. \end{aligned}$$

The set of cobordism classes of n *dimensional framed submanifolds of* N *is denoted by* $\Omega^{\mathrm{fr}}_{n,N}$.

The conditions 1. and 2. alone say that the manifolds M_1 and M_2 are cobordant. The relation of being (framed) cobordant will be called (framed) cobordism. Both cobordism and framed cobordism are equivalence relations.

Let N be a $n+k$-dimensional manifold. We consider a smooth map $f : N \to S^k$ with a regular value $y \in S^k$, and a submanifold $f^{-1}(y)$ of N. The differential

$$df_x : T_x(N) \to T_y(S^k), \text{ where } x \in f^{-1}(y)$$

has the subspace $T_x(f^{-1}(y))$ of $T_x(N)$ as its kernel. Therefore, the orthogonal complement $\nu_x(f^{-1}(y), N)$ of $T_x(f^{-1}(y))$ maps isomorphically onto $T_y(S^k)$. Having chosen a positively oriented basis $\omega = (\omega^1, ..., \omega^k)$ of $T_y(S^k)$ we have a unique basis $\sigma_x = \{\sigma_x^1, ..., \sigma_x^k\}$ on $\nu_x(f^{-1}(y), N)$, such that $df_x \sigma_x^i = \omega^i$. This gives k smooth linearly independent sections $\sigma_i : x \to \sigma_x^i$. We adopt here the notation $\sigma = f^* \omega$.

Definition 3.3.2 (Ch.7 of Milnor [1997]). *The framed manifold* $(f^{-1}(y), f^*(\omega))$ *will be called the Pontrjagin manifold associated with* f.

In the definition of the Pontrjagin manifold we have made a choice of the basis ω, however if we choose some other positively oriented basis ω' the Pontrjagin manifolds $(f^{-1}(y), f^*(\omega))$ and $(f^{-1}(y), f^*(\omega'))$ are framed cobordant. This follows from the observation that the space of matrices with positive determinant is connected. A path joining ω with ω' gives the desired framing of $f^{-1}(y) \times I$.

We have the following fundamental result making a connection between homotopy and cobordism classes.

Theorem 3.3.3 (Theorem 7.B in Milnor [1997]). *Two mappings from N to S^k are smoothly homotopic if and only if the associated Pontrjagin manifolds are framed cobordant.*

We shall now formulate the inverse to the construction of the Pontrjagin manifold above. Given a closed framed submanifold (M, σ) of N we shall generate a function $f : N \to S^k$ with a regular value y, such that its associated Pontrjagin manifold is (M, σ).

We use Product Neighborhood Theorem to find a diffeomorphism

$$\psi : M \times \mathbb{R}^k \to U \subset N.$$

We denote the standard basis of $T_0(\mathbb{R}^k) \simeq \mathbb{R}^k$ by $\omega = [e^1, ..., e^k]$ and define the projection $\pi : U \to \mathbb{R}^k$ given by

$$\pi \circ \psi(x, y) = y.$$

The value 0 is regular, and $\pi^{-1}(0) = M$, also $\sigma = \pi^* \omega$. We choose a smooth map $\phi : \mathbb{R}^k \to S^k$ satisfying

1. ϕ maps the open ball $D^k = \{x \in \mathbb{R}^k | \ ||x|| < 1\}$ diffeomorphically onto $S^k - \{s_0\}$,
2. ϕ maps every $x \in \mathbb{R}^k - D^k$ onto a base point s_0,
3. $d\phi_0$ is orientation preserving.

An example of such a map is given in Milnor [1997], p. 48. We define a collapse map $f : N \to S^k$, which gives the desired associated Pontrjagin manifold, by

$$f(q) = \begin{cases} \phi \circ \pi(q) & \text{for} \quad q \in U \\ s_0 & \text{for} \quad q \notin U \ . \end{cases} \tag{3.1}$$

The map f is smooth, and $\phi(0)$ is a regular value of f. The preimage

$$f^{-1}(\phi(0)) = \pi^{-1}(0) = M,$$

furthermore, $df_x \sigma^i(x) = d\phi_0 \omega^i$, which are the basis of $T_{\phi(0)}(S^k)$. We have arrived at the following theorem.

Theorem 3.3.4 (Theorem 7.C in Milnor [1997]). *Any compact framed submanifold (M, σ) of co-dimension k in N occurs as the Pontrjagin manifold for some smooth map $f : N \to S^k$.*

The method of translating between framed cobordism and the maps f is called the Pontrjagin-Thom construction.

Corollary 3.3.5. *Let $[N, S^k] = C^\infty(N, S^k)/\simeq$, where $\simeq$ is the homotopy relation. The Pontrjagin-Thom construction induces a bijection $\Omega^{\mathrm{fr}}_{n,N} \to [N, S^k]$.*

3.4 Stably Framed Manifold Group

Suppose that the manifold N in Corollary 3.3.5 is a sphere S^{n+k}. The function $\pi_{n+k}(S^k) \to [S^{n+k}, S^k]$ obtained by forgetting base points is a bijection, cf. Davis and Kirk [2001]. Since $\pi_{n+k}(S^k)$ is an abelian group, the set of framed cobordism classes $\Omega^k_n \equiv \Omega^{\mathrm{fr}}_{n,S^{n+k}}$ inherits an abelian group structure. In fact this group structure is given by taking the disjoint union

$$[V_0] + [V_1] := V_0 \bigsqcup V_1 \subset S^{n+k} \# S^{n+k} \cong S^{n+k},$$

where $\#$ stands for the connected sum. Inverses are obtained by changing the orientation of the framing, $-(M, \sigma) \equiv (M, \sigma^-)$.

The embedding in the equator $S^{n+k} \subset S^{n+k+1}$ defines the homomorphisms $\Omega^k_n \to \Omega^{k+1}_n$. The resulting directed system

$$\Omega^0_n \to \Omega^1_n \to \ldots \to \Omega^k_n \to \ldots$$

of abelian groups possesses a direct limit.

Definition 3.4.1. *The stably framed n-manifold group is the direct limit*

$$\Omega^{\mathrm{fr}}_n = \varinjlim \Omega^k_n.$$

Recall the definition of the k-stable homotopy group of a based space X, cf. Hatcher [2002], as $\pi^S_k(X) = \varinjlim \pi_{k+l}(S^l X)$, where S is the reduced suspension. In particular the stable k-stem is $\pi^S_k = \pi^S_k(S^0)$. By the Freudenthal suspension theorem $\pi^S_k = \pi_{k+l}(S^l)$ for $l \geq k+2$. As conclusion of the discussions above we have the following theorem.

Theorem 3.4.2 (Pontrjagin-Thom). *The Pontrjagin-Thom construction defines an isomorphism from π_k^S to Ω_k^{fr}.*

3.5 Stable Tangential Framings

We shall denote a trivial line bundle over a space by ε. Consider a submanifold M^n of S^{n+k}. The inclusion $S^{n+k} \subset \mathbb{R}^{n+k+1}$ has a trivial 1-dimensional normal bundle. This implies that

$$T(S^{n+k}) \oplus \varepsilon \cong \varepsilon^{n+k+1},$$

and a framing of the normal bundle $\nu(M, S^{n+k}) \cong \varepsilon^k$ induces a trivialization

$$T(M) \oplus \varepsilon^{k+1} \cong T(M) \oplus \nu(M, S^{n+k}) \oplus \varepsilon = T_M(S^{n+k}) \oplus \varepsilon \cong \varepsilon^{n+k+1}.$$

Conversely, a trivialization $T(M) \oplus \varepsilon \cong \varepsilon^{n+1}$ induces an isomorphism

$$\nu(M, S^{n+k}) \oplus \varepsilon^{n+1} \cong \varepsilon^{n+k+1}.$$

Similarly, if the sphere S^{n+k} is substituted by an $(n+k)$-dimensional manifold N with $\nu(M, N) \cong \varepsilon^k$ and $\nu(N, \mathbb{R}^l) \cong \varepsilon^{l-n-k}$, then we have $T(M) \oplus \varepsilon^{l-n} \cong \varepsilon^l$.

Definition 3.5.1 (Definition 8.12 in Davis and Kirk [2001]). *A stable tangential framing of an n-dimensional manifold M is an equivalence class of trivializations of*

$$T(M) \oplus \varepsilon^k.$$

Two trivializations

$$\phi_1 : T(M) \oplus \varepsilon^{k_1} \cong \varepsilon^{n+k_1} \text{ and } \phi_2 : T(M) \oplus \varepsilon^{k_2} \cong \varepsilon^{n+k_2}$$

are equivalent if $\exists N > \max(k_1, k_2)$ such that the trivializations

$$\begin{aligned} \phi_1 \oplus \mathrm{id} \quad &: \quad T(M) \oplus \varepsilon^{k_1} \oplus \varepsilon^{N-k_1} \cong \varepsilon^{n+N}, \\ \phi_2 \oplus \mathrm{id} \quad &: \quad T(M) \oplus \varepsilon^{k_2} \oplus \varepsilon^{N-k_2} \cong \varepsilon^{n+N} \end{aligned}$$

are homotopic. Similarly, a stable normal framing of a submanifold M of S^l, $l \in \mathbb{N}$, is an equivalence class of trivializations of $\nu(M, S^l) \oplus \varepsilon^k$.

Theorem 3.5.2 (Theorem 8.13 in Davis and Kirk [2001]). *There is a bijection between stable tangential framings and stable normal framings of a manifold M.*

In the next chapter we shall use the following proposition.

Proposition 3.5.3. *Suppose M is a manifold with boundary. If $\mathrm{int}(M)$ has a stable tangential framing so does M.*

Proof. In the proof we shall construct a diffeomorphism $\phi : M \to M'$, such that $M' \subset \mathrm{int}(M)$.

By the collaring theorem we have an embedding $h : \partial M \times [0,\infty) \to M$ such that $h(x,0) = x$. Denote the image $h(\partial M \times [0,\infty))$ by V. For a pair of real numbers $a < b$ construct a diffeomorphism $\psi : [0,\infty) \to [a,\infty)$ such that the restriction $\psi|_{[b,\infty)} = \mathrm{id}$. The composition $f : V \to M$, $f = h \circ (id \times \psi) \circ h^{-1}$ is a diffeomorphism onto its image. Choosing $c > b$ and denoting $h(\partial M \times [0,c])$ by W we define the map ϕ

$$\phi(q) = \begin{cases} f(q) & \text{for} \quad q \in W \\ \mathrm{id} & \text{for} \quad q \notin W. \end{cases}$$

Denote $M' \equiv \phi(M)$ and observe that $T(M) \cong \phi^* T(M')$. Furthermore, since $M' \subset \mathrm{int}(M)$ and $\dim(M') = \dim(M)$ we have that $T(M') \cong T_{M'}(\mathrm{int}(M))$. But $\phi^* T_{M'}(\mathrm{int}(M))$ has a stable tangential framing so does $T(M)$. □

A stably framed nullcobordism for a stably framed manifold (M_1, ϕ_1) with a stable framing $\phi_1 : T(M_1) \oplus \varepsilon^{k_1} \cong \varepsilon^{n+k_1}$ is (i) a compact manifold (X, Φ) with a stable framing $\Phi : T(X) \oplus \varepsilon^{k_1+k_2} \cong \varepsilon^{n+1+k_1+k_2}$ and (ii) a bundle isomorphism $\theta : T(M_1) \oplus \varepsilon^{k_1+1+k_2} \cong T_{M_1}(X) \oplus \varepsilon^{k_1+k_2}$ coming from an orientation preserving diffeomorphism $\vartheta : M_1 \to \partial X$, so that $\Phi \circ \theta = \phi_1 \oplus \mathrm{id}_{\varepsilon^{k_2+1}}$.

We define a notion of a stably framed cobordism from a stably framed manifold (M_1, ϕ_1) to another stably frame manifold (M_2, ϕ_2) to be a stably framed nullcobordism for the disjoint union of $(M_1^-, \phi_1^-))$ and (M_2, ϕ_2), where M_1^- is obtained form M_1 by reversing the orientation and ϕ_1^- is the composition:

$$\phi_1^- : T(M_1) \oplus \varepsilon^{k_1} \xrightarrow{\phi_1} \varepsilon^{k_1+n} = \varepsilon \oplus \varepsilon^{k_1+n-1} \xrightarrow{f} \varepsilon \oplus \varepsilon^{k_1+n-1} = \varepsilon^{k_1+n},$$

where $f = -\mathrm{id}_\varepsilon \oplus \mathrm{id}_{\varepsilon^{k_1+n-1}}$.

Theorem 3.5.4 (Corollary 8.15 in Davis and Kirk [2001]). *The stable k-stem π_k^S is isomorphic to the group of stably tangentially framed cobordism classes of stably tangentially framed k-dimensional smooth, oriented, compact manifolds without boundary.*

3.6 The J-homomorphism

Consider $S^k \subset \mathbb{R}^{k+n}$. Denote the framing of $\nu(S^k, \mathbb{R}^{k+1})$ by η and the canonical basis of $\mathbb{R}^{k+n}$ by $\{e_l\}_{l\in\{1,...,k+n\}}$. Then $\sigma = (\eta, e_{k+2}, ..., e_{k+n})$ is a framing of $\nu(S^k, \mathbb{R}^{k+n})$. The framing gives rise to the trivialization of the normal bundle

$$\phi_\sigma : \nu(S^k, \mathbb{R}^{k+n}) \xrightarrow{\cong} S^k \times \mathbb{R}^n.$$

Given a smooth map $\gamma : S^k \to O(n)$ we get a new framing σ' induced by the composition $(\mathrm{pr}_1, (\gamma \circ \mathrm{pr}_1) \cdot \mathrm{pr}_2)$, where pr_i is the projection of $S^k \times \mathbb{R}^n$ on the ith factor. The Pontryagin-Thom construction applied to σ' defines a map $S^{n+k} \to S^n$. This construction induces a map on homotopy groups

$$J : \pi_k(O(n)) \to \pi_{k+n}(S^n),$$

which is a homomorphism, cf. Sec. IX.6 in Kosinski [1993]. We shall call it J-homomorphism.

The inclusion $i : O(n-1) \hookrightarrow 0(n)$ given by

$$A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$$

induces the map $i_* : \pi_k(O(n-1)) \to \pi_k(O(n))$. By Freudethal suspension theorem $\pi_{k+n}(S^n)$ is independent of n for $n > k+1$. The same is true for $\pi_k(O(n))$ as we have a fibration $O(n-1) \hookrightarrow O(n) \to S^{n-1}$, thus the sequence

$$\cdots \longrightarrow \pi_k(S^{n-1}) \longrightarrow \pi_k(O(n)) \longrightarrow \pi_k(O(n-1)) \longrightarrow \pi_{k+1}(S^{n-1}) \longrightarrow \cdots$$

is exact, cf. Corollary 6.44 in Davis and Kirk [2001], furthermore $\pi_k(S^{n-1}) \approx \pi_{k+1}(S^{n-1}) \approx 0$ for $n > k+1$. Since the following diagram commutes

$$\begin{tikzcd}
\pi_k(O(n)) \arrow[r, "i_*"] \arrow[d, "J"] & \pi_k(O(n+1)) \arrow[d, "J"] \\
\pi_{n+k}(S^n) \arrow[r, "S"] & \pi_{n+k+1}(S^{n+1}),
\end{tikzcd}$$

where S is the homomorphism induced by the suspension, the J-homomorphism induces a stable J-homomorphism, cf. Sec. 8.2 in Davis and Kirk [2001].

3.7 Comments on Transversality

We shall finish this section by making a link between a regular value of a map and transversality.

Lemma 3.7.1. *Let M, U, V be smooth manifolds. The map $f : M \to U \times V$ is transverse to $\{x\} \times V$ if and only if the composite map*

$$M \xrightarrow{f} U \times V \xrightarrow{\pi} U,$$

where π is the projection, has x as a regular value.

Proof. Suppose x is a regular value of $\pi \circ f$. Let $p \in f^{-1} \circ \pi^{-1}(x)$, then $f(p) = (x, y)$ for some $y \in V$. Pick $(a, b) \in T_{(x,y)}(U \times V)$. The differential $d\pi_{(x,y)} df_p$ is surjective, hence there exists $c \in T_p(M)$ so that $d\pi_{(x,y)} df_p(c) = a = d\pi_{(x,y)}(a, b)$. We have that $df_p(c) = (a, \beta)$, for some $(0, \beta) \in T_{(x,y)}(\{x\} \times V)$. We conclude that $df_p(c) + (0, b - \beta) = (a, b)$. This proofs the necessity. The sufficiency is proved along the same lines. $\square$

Suppose we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, hook, "i_1"] \arrow[d, hook, "i_2"] & M_1 \arrow[d, hook, "j_1"] \\
M_2 \arrow[r, hook, "j_2"] & N,
\end{tikzcd}$$

where $i_1,\ i_2$ and $j_1,\ j_2$ are inclusions. If the intersection $A = M_1 \cap M_2$ is transversal in N, denoted by $A = M_1 \pitchfork M_2$, then there is a short exact sequence of vector bundles

$$0 \longrightarrow T(A) \xrightarrow{(di_1,-di_2)} T_A(M_1) \oplus T_A(M_2) \xrightarrow{dj_1+dj_2} T_A(N) \longrightarrow 0. \tag{3.2}$$

Lemma 3.7.2. *Suppose $A = M_1 \pitchfork M_2$ and the diagram above commutes then the following are true*

1. $\nu(A, M_2) \cong \nu(M_1, N)|_A$,
2. $\nu(A, N) \cong \nu(A, M_1) \oplus \nu(A, M_2)$,
3. $\nu(A, N) \cong \nu(M_1, N)|_A \oplus \nu(M_2, N)|_A$.

Proof. The exact sequence (3.2) can be written

$$0 \longrightarrow T(A) \longrightarrow T(A) \oplus \nu(A, M_1) \oplus T(A) \oplus \nu(A, M_2) \longrightarrow T_A(N) \longrightarrow 0$$

which implies the exact sequence

$$0 \longrightarrow T(A) \oplus \nu(A, M_1) \oplus \nu(A, M_2) \longrightarrow T(A) \oplus \nu(A, N) \longrightarrow 0,$$

which gives the isomorphism in 2. The isomorphism in 1. follows from

$$\begin{tikzcd}
0 \arrow[r] & T_A(M_1) \oplus \nu(A, M_2) \arrow[r] \arrow[d, dashed, "\cong"] & T_A(N) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_A(M_1) \oplus \nu(M_1, N)|_A \arrow[r] & T_A(N) \arrow[r] & 0.
\end{tikzcd}$$

The last isomorphism is a consequence of property 1 and

$$\nu(A, N) = \nu(A, M_1) \oplus \nu(M_1, N)|_A.$$

□

4 Stable, Unstable and Connecting Manifolds

In this chapter we study framed connecting manifolds. The aim is two fold, we shall review Morse Theory as exposed in Milnor [1965]. Secondly we shall present the proof of Theorem 3.3 in Franks [1979]. The theorem says that framed connected manifolds are, by the Pontrjagin-Thom construction, in one to one correspondence with homotopy classes of relative attaching maps.

4.1 Elements of Morse Theory

Let M be a closed smooth manifold, g_0 a Riemannian metric on M, and f be a Morse function on M. It follows that there are finitely many critical points and all of them are nondegenerate.

We shall consider the gradient flow, that is the flow line through $x \in M$

$$\gamma_x : (a, b) \to M$$

that satisfies the differential equation

$$\frac{d\gamma_x}{dt} = -\nabla_{\gamma_x}(f),$$

where $g_0(\nabla f, \zeta) = df(\zeta)$ for any smooth vector field ζ on M, and $\gamma_x(0) = x$.

We shall study the relation between the structure of the critical set of f,

$$\mathcal{C}r(f) = \{x \in M |\, df(x) = 0\}$$

and the topology of M.

Lemma 4.1.1. *Let M^n be a closed smooth manifold, $f : M \to \mathbb{R}$ be a Morse function and $\mathcal{C}r(f)$ the set of the critical points of f. Then there is a metric g and a family of open neighborhoods $\{U_p\}_{p\in\mathcal{C}r(f)}$ of the critical points such that in a local coordinate system $(x_1, ..., x_n)$ of U_p*

$$f(x_1, ..., x_n) = f(p) - \sum_{i=1}^{\lambda_p} x_i^2 + \sum_{j=\lambda_p+1}^{n} x_j^2, \text{ and} \tag{4.1}$$

$$\nabla f(x_1, ..., x_n) = -2\sum_{i=1}^{\lambda_p} x_i \frac{\partial}{\partial x_i} + 2\sum_{j=\lambda_p+1}^{n} x_j \frac{\partial}{\partial x_j}, \tag{4.2}$$

where λ_p is the index of a critical point p.

Proof. The Morse Lemma (Lemma 2.2 in Milnor [1997]) provides $\{U_p\}_{p\in\mathcal{C}r(f)}$ and the first statement of the lemma. Shrink the U_p's so that they are disjoint. We choose the standard Euclidean metric on U_p. Inside each U_p we consider an open set V_p containing p. On the open set $V = M - \bigcup_{p\in\mathcal{C}r(f)} \mathrm{cl}(V_p)$ we select an arbitrary metric g_0. Using a smooth partition of unity subordinate to $\{U_p\}_{p\in\mathcal{C}r(f)}$ and V we obtain the desired metric g. □

Definition 4.1.2 (Definition 6.30 in Banyaga and Hurtubise [2004]). *A gradient vector field ∇f of a Morse function f is said to be in standard form near a critical point p if and only if there exists a smooth coordinate chart around p such that in the local coordinates determined by the chart we have Equations (4.1), (4.2).*

If the gradient vector field is in the standard form near every critical point, then we shall say that the Riemannian metric g is compatible with the Morse charts for the function f.

We shall assume for the rest of this chapter that the Riemannian metric g on M is compatible with the Morse charts for f.

Let a be a critical point for f. We define two subsets of M:

$$W^s(a) = \{x \in M | \lim_{t\to+\infty} \gamma_x(t) = a\}, \tag{4.3}$$

$$W^u(a) = \{x \in M | \lim_{t\to-\infty} \gamma_x(t) = a\}. \tag{4.4}$$

Theorem 4.1.3 (3.9 in Milnor [1965]). *Let $a \in \mathcal{C}r(f)$, and the index of a be λ. The sets $W^s(a)$ and $W^u(a)$ on (M, g) are diffeomorphic to open disks of dimension $n - \lambda$ and λ respectively.*

The subset $W^s(a)$ is called the stable manifold of a, and the subset $W^u(a)$ is called the unstable manifold of a.

Proof. We prove the theorem for $W^u(a)$. The proof for $W^s(a)$ is analogous. Following Lemma 4.1.1 there exists a chart $\phi : U \to \mathbb{R}^n$, so that $a \in U$, $\phi(a) = 0$ and in the local coordinates the Morse function f and its gradient are of the forms (4.1) and (4.2). Consider a ball $W_0 \subset \phi(U)$ of dimension λ, centered at 0 and sufficiently small radius r_0: $W_0 = \{x \in \phi(U) |\ x_{\lambda+1} = ... = x_n = 0,\ \|x\| < r_0\}$. We see that $\phi^{-1}(W_0) \subset W^u(a)$, since the gradient flow line starting at a point $x \in W_0$ satisfies the following differential equation

$$\frac{d}{dt}\gamma_i(t) = 2\gamma_i(t),$$

or explicitly

$$\gamma_i(t) = \gamma_i(0)e^{2t}.$$

The manifold M is compact thus the gradient vector field generates a 1-parameter group $\Phi_t : M \to M,\ t \in \mathbb{R}$, of diffeomorphisms and the map $\Phi : \mathbb{R} \times M \to M$,

$$\Phi(t, y) = \gamma_y(t)$$

is smooth. Every element of $W^u(a)$ when flown backward in time converges to the point a hence after some finite time ends up in the set $\phi^{-1}(W_0)$. Hence we conclude that $W^u(a) = \bigcup_{t \geq 0} \Phi(t, \phi^{-1}(W_0))$.

In the next step we stretch $\phi^{-1}(W_0)$ to the whole $W^u(a)$. For this we shall use a smooth monotonic function $\psi : [0, r_0^2) \to \mathbb{R}$, where r_0 is the radius of the ball W_0, with $\psi(0) = 0$ and $\lim_{t \to r_0} \psi(t) = +\infty$. The map $S : W_0 \to W^u(a)$, defined by $S(x) = \Phi(\psi(||x||^2), \phi^{-1}(x))$ is the desired diffeomorphism of the λ-ball onto the unstable manifold $W^u(a)$. $\square$

The proof of Theorem 4.1.3 relies on the metric g compatible with the Morse charts for f, nevertheless the theorem is true for an arbitrary Riemannian metric on the manifold M, see Theorem 4.2: Stable/Unstable Manifold Theorem for a Morse Function in Banyaga and Hurtubise [2004]. As a matter of fact it is valid for all vector fields with hyperbolic singular points, in which case $W^u(a)$ is an injectively immersed λ-open disk. We will postpone the discussion on stable manifolds for vector fields until the next chapter.

In the following we shall assume that the function f is Morse-Smale, that is all stable and unstable manifolds intersect transversally. On the set $\mathcal{C}r(f)$ of the critical points of a Morse-Smale function f we define a partial order relation by $a \succeq b$ if and only if $W^u(a) \cap W^s(b) \neq \emptyset$ (there is a gradient flow line γ with $\lim_{t\to-\infty}\gamma(t) = a$ and $\lim_{t\to+\infty}\gamma(t) = b$).

Definition 4.1.4. *Suppose a is a critical point with index λ_a. Let $r < \lambda_a$ be the largest integer for which there exists a critical point b with the index $\lambda_b = r$ and $a \succeq b$. Then the points a and b are called successive.*

For $a, b \in \mathcal{C}r(f)$ we shall use the notation

$$W(a,b) = W^u(a) \pitchfork W^s(b).$$

If $a \succ b$ the intersection $W(a,b)$ is nonempty and due to transversality it is a manifold of dimension corresponding to the relative index of a and b, that is $\lambda_a - \lambda_b$.

Let $\tau \in \mathbb{R}$ be a regular value of f such that $f(a) > \tau > f(b)$. Such a value τ exists since the function f is strictly decreasing along the gradient flow line, which does not contain a critical point. We will consider the preimage

$$V_\tau = f^{-1}(\tau).$$

Lemma 4.1.5. *Suppose that a and b are successive points and let τ be such that $f(a) > \tau > f(b)$. The intersections $S^u(a) = W^u(a) \cap V_\tau$ and $S^s(b) = W^s(b) \cap V_\tau$ are diffeomorphic to spheres of dimension $\lambda_a - 1$ and $n - \lambda_b - 1$, respectively.*

Proof. Away from the critical set $\mathcal{C}r(f)$, we consider the vector field

$$X(y) = \frac{\nabla_y f}{|\nabla_y f|^2},$$

and a flow line η_p of X with initial condition $\eta_p(0) = p \in (M - \mathcal{C}r(f))$. It follows that

$$\frac{d}{dt} f(\eta_p(t)) = X_{\eta_p(t)} \left(f_{\eta_p(t)} \right) = g_{\eta_p(t)} \left(\nabla f, X \right) = 1.$$

Let $S_0 = \partial \left(\mathrm{cl}(W_0) \right)$, where W_0 is the disk of radius r_0 as defined in the proof of Theorem 4.1.3.

For all $x, y \in S_0$, $f(x) = f(y) = f(a) - r_0$, also

$$f\left(\eta_x(\tau - f(a) + r_0)\right) \quad = \quad f\left(\eta_y(\tau - f(a) + r_0)\right) = \tau.$$

We define the map $R : \partial W_0 \to S^u(a)$ by

$$R(x) = \eta_x(\tau - f(a) + r_0).$$

The map R is a diffeomorphism. This proves that $S^u(a)$ is diffeomorphic to a sphere of dimension $\lambda_a - 1$. The proof for $S^s(b)$ is analogous. □

Lemma 4.1.6. *Suppose that a and b are successive points and let τ be such that $f(a) > \tau > f(b)$. Then the stable sphere $S^s(b)$ and the unstable sphere $S^u(a)$ intersect transversally in the level manifold V_τ. More generally, if $a \succeq b$ (not necessary successive) and τ is a regular value then the manifolds $W^u(a) \cap f^{-1}(\tau)$ and $W^s(b) \cap f^{-1}(\tau)$ intersects transversally.*

Proof. The stable and unstable manifolds intersect transversely, which results in the exact sequence on $W^u(a) \pitchfork W^s(b)$, see (4.5). We use the following shorthand notations $W_{a,b} \equiv W(a,b)$, $W_a^u \equiv W^u(a)$ and $W_b^s \equiv W^s(b)$,

$$0 \longrightarrow T(W_{a,b}) \xrightarrow{(di_u, -di_s)} T_{W_{a,b}}\left(W_a^u\right) \oplus T_{W_{a,b}}\left(W_b^s\right) \xrightarrow{dj_u + dj_s} T_{W_{a,b}}(M) \longrightarrow 0, \tag{4.5}$$

where $i_\rho : W(a,b) \to W^\rho(a)$, and $j_\rho : W^\rho(b) \to M$ are the inclusions.

Since τ is a regular value, we have that $T_{S^u(a)}(W^u(a)) \cong T(S^u(a)) \oplus \varepsilon$. Let $\mathcal{R} \equiv \mathrm{span}\{\nabla_p f\}$ and use the notation $S_a^u \equiv S^u(a)$, and $S_b^s \equiv S^s(b)$. From (4.5), for $p \in W(a,b) \cap V_\tau$ we get the short exact sequence

$$0 \longrightarrow (T_p(S_a^u) \cap T_p(S_b^s)) \oplus \mathcal{R} \xrightarrow{\alpha} T_p(S_a^u) \oplus \mathcal{R} \oplus T_p(S_b^s) \oplus \mathcal{R} \xrightarrow{\beta} T_p(V_{t_1}) \oplus \mathcal{R} \longrightarrow 0,$$

where $\alpha : (v, r) \mapsto (v, r, -v, -r)$, and $\beta : (v, r, w, s) \mapsto (v + w, r + s)$. The map β is surjective, so is the map $\tilde{\beta} : T_p(S^u(a)) \oplus T_p(S^s(b)) \to T_p(V_\tau)$ given by $(v, w) \mapsto v + w$. Thus $S^u(a)$ and $S^s(b)$ intersect transversally.

In the proof above we have only used the property that τ is a regular value of $f|_{W^u(a)}$. Observe also that $T_{W^u(a)\cap V_\tau}(W^u(a)) \cong T(W^u(a) \cap V_\tau) \oplus \varepsilon$. Hence the second statement of the lemma follows.□

4.2 Connecting Manifolds

Definition 4.2.1. *Let a and b be successive critical points of a Morse-Smale function f. The intersection of the stable and the unstable spheres, $N(a, b) = S^u(a) \cap S^s(b)$, will be called their connecting manifold.*

The connecting manifold is a compact submanifold of $S^u(a)$, it has also a framing, as it is shown in the next proposition.

Proposition 4.2.2. *If $N(a, b)$ is the connecting manifold of two (not necessary successive) critical points a and b of a Morse-Smale function f, then $N(a, b)$ is a framed manifold in $S^u(a)$.*

Proof. The stable manifold $W^s(b)$ is diffeomorphic to a disk thus it is contractible. It follows that the normal bundle $\nu(W^s(b), M)$ is trivial.

The intersection of $S^u(a)$ and $W^s(b)$ in M is transversal and inclusions induce the following commutative diagram

$$\begin{tikzcd} N(a,b) \arrow[r, hook] \arrow[d, hook] & W^s(b) \arrow[d, hook] \\ S^u(a) \arrow[r, hook] & M. \end{tikzcd}$$

Then Lemma 3.7.2 gives the isomorphism

$$\nu(N(a, b), S^u(a)) \cong \nu(W^s(b), M)|_{N(a,b)}.$$

□

We shall study a CW-complex structure of the manifold M associated to the Morse function f. Our aim is to describe the relative attaching maps.

Let D^m_r denote the open ball of radius r with center 0 in $\mathbb{R}^m$, $D^m \equiv D^m_1$, and S^{m-1} the boundary of $\mathrm{cl}(D^m)$ in $\mathbb{R}^m$. Suppose $p \in M$ is a critical point with index λ, and critical value c. For some small $\epsilon > 0$, $V_{-\epsilon} = f^{-1}(c-\epsilon)$ and $V_\epsilon = f^{-1}(c+\epsilon)$ are level manifolds such that c is the only critical value in the interval $[c-\epsilon, c+\epsilon]$. Already classical Morse theory, Theorem 3.2 in Milnor [1973] states that the set $M^{c+\epsilon} = f^{-1}(-\infty, c+\epsilon]$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached. However, in order to extract an explicit form of the attaching map we shall move along the lines of Ch. 3 in Milnor [1965].

The manifolds $V_{-\epsilon}$ and V_ϵ are cobordant, since they comprise the two components of the boundary of $W = f^{-1}([c-\epsilon, c+\epsilon])$. We shall denote this cobordism by $(W; V_{-\epsilon}, V_\epsilon)$, and call it an elementary cobordism.

There is a local coordinate system $\theta : U \to D^n_{2\sqrt{\epsilon}}$ such that the function and the gradient vector field are locally given by the normal forms (4.1) and (4.2), respectively.

Definition 4.2.3 (Definition 3.9 in Milnor [1965]). *The characteristic embedding $\phi : S^{\lambda-1} \times D^{n-\lambda} \to V_{-\epsilon}$ is given by $\phi(u, \alpha v) = \theta^{-1}(\sqrt{\epsilon} u \cosh(\alpha), \sqrt{\epsilon} v \sinh(\alpha))$ for $u \in S^{\lambda-1},\ v \in S^{n-\lambda-1}$, and $0 < \alpha < 1$.*

Definition 4.2.4 (Definition 3.13 in Milnor [1965]). *Given a manifold V' of dimension $n-1$ and an embedding $\phi' : S^{\lambda-1} \times D^{n-\lambda} \to V'$, $\chi(V', \phi')$ is the quotient manifold obtained from the disjoint union $\big(V' - \phi'(S^{\lambda-1} \times 0)\big) \sqcup \big(D^\lambda \times S^{n-\lambda-1}\big)$ by identifying $\phi'(u, \alpha v)$ with $(\alpha u, v)$ for each $u \in S^{\lambda-1},\ v \in S^{n-\lambda-1}$, and $0 \le \alpha < 1$. If V'' is any manifold diffeomorphic to $\chi(V', \phi')$ then we say that V'' is obtained from V' by surgery of type $(\lambda, n-\lambda)$.*

We define a manifold L_λ by

$$L_\lambda = \{(x,y) \in \mathbb{R}^\lambda \times \mathbb{R}^{n-\lambda} |\, -1 \le -|x|^2 + |y|^2 \le 1 \text{ and } |x||y| < \sinh(1)\cosh(1)\}.$$

The boundary of L_λ has two components. One on $-|x|^2 + |y|^2 = -1$ diffeomorphic to $S^{\lambda-1} \times D^{n-\lambda}$ via the map

$$S^{\lambda-1} \times D^{n-\lambda} \to \partial L_\lambda,\ (u, \alpha v) \mapsto (u \cosh(\alpha), v \sinh(\alpha)),$$

and the second on $-|x|^2+|y|^2=1$ diffeomorphic to $D^\lambda \times S^{n-\lambda-1}$ by

$$D^\lambda \times S^{n-\lambda-1} \to \partial L_\lambda,\ (\alpha u, v) \mapsto (u\sinh(\alpha), v\cosh(\alpha)).$$

We construct a manifold $\omega(V_{-\epsilon},\phi)$ as follows. Let $\bar{D}^1 \equiv [-1,1]$ and

$$W' = \Big(V_{-\epsilon} - \phi(S^{\lambda-1}\times 0)\Big) \times \bar{D}^1 \sqcup L_\lambda,$$

and $\sim$ denotes the following equivalence relation. For each $u \in S^{\lambda-1}$, $v \in S^{n-\lambda-1}$, $0<\alpha<1$, and $c \in \bar{D}^1$, $(\phi(u,\alpha v), c) \in \big(V_\epsilon - \phi(S^{\lambda-1}\times 0)\big) \times \bar{D}^1$ is identified with the unique point $(x,y)\in L_\lambda$ such that

1. $-|x|^2+|y|^2=c$
2. (x,y) lies on the flow line of the gradient vector field (4.2), which passes through the point $(u\cosh(\alpha), v\sinh(\alpha))$.

Then the manifold $\omega(V_{-\epsilon},\phi)$ is

$$\omega(V_{-\epsilon},\phi) = W'/\sim .$$

The boundary of the manifold $\omega(V_{-\epsilon},\phi)$ has two components: $V_{-\epsilon}$ (corresponding to the value $c=-1$) and a component (corresponding to $c=1$) identified with $\chi(V_{-\epsilon},\phi)$ by the map $g:\chi(V_{-\epsilon},\phi)\to\partial\omega(V_{-\epsilon},\phi)$ defined by

$$\begin{cases} g(z) = (z,1) & \text{for } z \in V_\epsilon - \phi(S^{\lambda-1}\times 0) \\ g(\alpha u, v) = (u\sinh(\alpha), v\cosh(\alpha)) & \text{for } (\alpha u, v) \in D^\lambda \times S^{n-\lambda-1}. \end{cases}$$

Theorem 4.2.5 (Theorem 3.13 in Milnor [1965]). *Suppose $(W;V_{-\epsilon},V_\epsilon)$ is an elementary cobordism, and $\phi: S^{\lambda-1}\times D^{n-\lambda}\to V_{-\epsilon}$ is the characteristic embedding. Then there is a diffeomorphism $k:(\omega(V_{-\epsilon},\phi);V_{-\epsilon},\chi(V_{-\epsilon},\phi))\to(W;V_{-\epsilon},V_\epsilon)$.*

Theorem 4.2.6 (Theorem 3.14 in Milnor [1965]). *Suppose $(W;V_{-\epsilon},V_\epsilon)$ is an elementary cobordism and $\hat{W}^u(p)=W^u(p)\cap W$. Then there is a deformation retract $r:W\to V_{-\epsilon}\cup\hat{W}^u(p)$.*

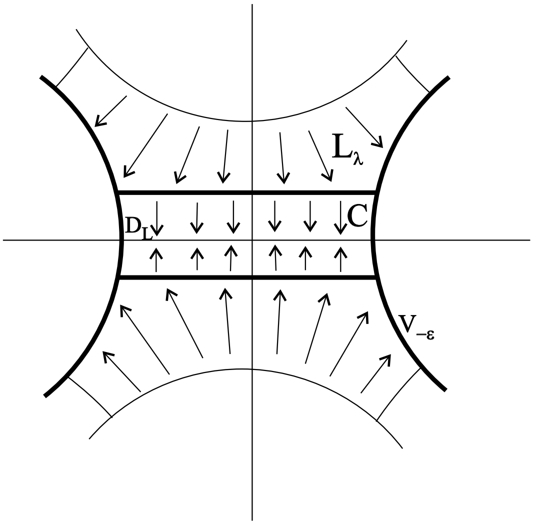

Figure 4.1: $V_{-\epsilon} \cup D_L$ is a deformation retract of $\omega(V_{-\epsilon}, \phi)$.

Proof (sketch). Let D_L and C be the following sets

$$D_L = \{(x, y) \in L_\lambda |\ y = 0\},$$

and its collar neighborhood

$$C = \{(x, y) \in L_\lambda |\ |y| \leq 1/10\}.$$

Since the set D_L is diffeomorphic to $\hat{W}^u(p)$, cf. Theorem 4.1.3, it is enough to show that $V_{-\epsilon} \cup D_L$ is a deformation retract of $\omega(V_{-\epsilon}, \phi)$. For $t \in [0, 1]$ we define deformation retractions r'_t from $\omega(V_{-\epsilon}, \phi)$ to $V \cup C$ and r''_t from $V \cup C$ to $V \cup D_L$. The composition of these maps gives the desired retraction. The sketch of the situation is drawn in Figure 4.1. For details see Milnor [1965]. □

Suppose a and b are two successive critical points of f. We define the composition

$$h : D^\lambda \times S^{n-\lambda-1} \xhookrightarrow{j_1} \chi(V_{-\epsilon}, \phi) \xrightarrow[\approx]{k|_{\chi(V_{-\epsilon},\phi)}} V_\epsilon \xhookrightarrow{j_2} W \xrightarrow{r} V_{-\epsilon} \cup \hat{W}^u(p) \xrightarrow{c} S^\lambda_,$$

where j_1, j_2 are the inclusion and c collapses $\left\{\left(V_{-\epsilon} \cup \hat{W}^u(p)\right) - \mathrm{int}(\hat{W}^u(p))\right\}$ to a point. Analyzing the retractions r'_t and r''_t in the proof of Theorem 4.2.6 one sees that the map h collapses the sphere $S^{n-\lambda-1}$ to a point $\{*\}$, and for a sufficiently small $\delta > 0$, the restriction of h to $D^{\lambda}_{\delta} \times \{*\}$ is a diffeomorphism onto its image. We shall denote this image by D_h.

Let $M(f)$ be a CW-complex associated to f, and $M(f)^k$ be its k-skeleton. The relative attaching map ϕ_{ab} of a λ_a-cell e^{λ_a} to a λ_b-cell e^{λ_b} in $M(f)$ is defined by the following composition, cf. Sec. IV.9 in Bredon [1993],

$$\phi_{ab}:\ S^{\lambda_a-1} = \partial e^{\lambda_a} \xrightarrow{\gamma_{ab}} M(f)^{\lambda_b} \longrightarrow M(f)^{\lambda_b}/\left(M(f)^{\lambda_b} - \mathrm{int}(e^{\lambda_b})\right) = S^{\lambda_b}_{,}$$

where γ_{ab} is the attaching map of the λ_a-cell to the CW-complex $M(f)^{\lambda_b}$. The map ϕ_{ab} is homotopic to the composition

$$S^{\lambda_a-1} \cong S^u(a) \xhookrightarrow{i} V_\epsilon \xrightarrow{c \circ r \circ j} S^{\lambda_b},$$

where i is the inclusion of the stable sphere in the level manifold V_ϵ.

We have the commutative diagram

$$\begin{tikzcd}
S^u(a) \arrow[r, hook, "i"] & V_\epsilon \arrow[r, "c \circ r \circ j_2"] & S^{\lambda_b}. \\
& D^{\lambda_b}_{\delta} \times S^{n-\lambda_b-1}. \arrow[u, hook, "k \circ j_1"] \arrow[ur, "h"] &
\end{tikzcd}$$

The image of $\{0\} \times S^{n-\lambda_b-1}$ by the embedding $k \circ j_1$ is $S^s(b)$. Furthermore, $h(0,q) = v$ for some $v \in S^{\lambda_b}$ and for all $q \in S^{n-\lambda_b-1}$. Denote the composition of the upper maps in the diagram by $\beta_{ab} = c \circ r \circ j_2 \circ i$. It follows that

$$\beta_{ab}^{-1}(v) = S^u(a) \cap S^u(b),$$

and we have a situation

$$S^u(a) \cap S^u(b) \to D^{\lambda_b}_{\delta} \times S^{n-\lambda_b-1} \to D_h \approx D^{\lambda_b}_{\delta}$$

as in Lemma 3.7.1. We conclude that since $S^u(a)$ and $S^u(b)$ intersect transversally, v is the regular point of β_{ab}.

We shall consider the attaching map ϕ_{ab} as an element in the homotopy group $\pi_{\lambda_a-1}(S^{\lambda_b})$ and see that the connecting manifold is the Pontryagin manifold associated to β_{ab}. Corollary 3.3.5 proves Connecting Manifold Theorem due to J.M. Franks, cf. Theorem 3.3 in Franks [1979]. The formulation of Connecting Manifold Theorem used in this chapter is not Frank's original one, but taken form Banyaga and Hurtubise [2004]. The second author of Banyaga and Hurtubise [2004] made us aware that an isomorphism between $\nu(N(a,b), S^u(a))$ and $\nu(W^s(b), M)|_{N(a,b)}$ in the proof of Proposition 4.2.2 gives an ambiguity of the sign ± 1 in the formulation of Connecting Manifold Theorem.

Theorem 4.2.7 (Theorem 6.40 in Banyaga and Hurtubise [2004]). *Suppose that $f : M \to \mathbb{R}$ is a Morse-Smale function on a finite dimensional compact smooth Riemannian manifold (M, g), and assume that the metric g is compatible with the Morse charts for f. Suppose that a, b are successive critical points and let $(N(a,b), \sigma)$ be a framed connecting manifold. Let $M(f)$ be the CW-complex associated to f and let γ_{ab} be the relative attaching map of the cell in $M(f)$ corresponding to a to the cell corresponding to b. Then the Thom-Pontryagin construction applied to the framed submanifold $(N(a,b), \sigma)$ produces a map that is homotopic to γ_{ab} up to precomposing with a representative of $\pm 1 \in \pi_j(S^j)$ where $j = \lambda_a - 1$.*

We follow Remark 6.41 in Banyaga and Hurtubise [2004] and state that the sign $\pm 1 \in \pi_j(S^j)$ depends on the homotopy class of the framing of the connecting manifold $(N(a,b),\ \sigma)$. An orientation of $W^s(b)$ and an orientation on M will determine a homotopy class for the framing of $N(a,b)$. So, the sign $\pm 1 \in \pi_j(S^j)$ is determined by the orientation chosen for $W^s(b)$ when M is oriented.

5 Morse-Smale Vector Fields

We shall review the main results of the geometric theory of dynamical systems. In the exposition we have extensively used Palis and de Melo [1982]. We study properties of the Morse-Smale vector fields, i.e. vector fields with hyperbolic singular elements whose stable and unstable manifolds intersect transversally. We bring in the notion of structural stability. Morse-Smale vector fields form a nonempty subset whose elements are structurally stable. We spend some time in this chapter investigating gradient-like vector fields which are the building blocks of section cones defined in the next chapter. To any Morse-Smale vector field ξ we associate a Lyapunov function that is a function, which decreases along the orbits of ξ apart from its singular elements. The main contribution of this part is the analysis of dependence of the invariant manifolds on small perturbations of vector fields. We show that a local stable manifold depends continuously on a perturbation.

5.1 The C^r Topology

We shall recall a notion of the derivative of a map in a Banach space. We follow Sec. I.3 in Lang [1999]. Let E and F be two Banach spaces and W open in E. Let $f : W \to F$ be a continuous map. We say that f is differentiable at a point x_0 if there exists a continuous linear map L_{x_0} of E into F such that, if we let

$$f(x_0 + h) = f(x_0) + L_{x_0}h + \alpha_{x_0}(h) \tag{5.1}$$

for small h, then α_{x_0} is tangent to 0; that is

$$\|\alpha_{x_0}(x)\| \leq \|x\|\beta(x) \text{ with } \lim_{\|x\|\to 0} \beta(x) = 0.$$

We say that L_{x_0} is the derivative of f at x_0. We denote the derivative by $df(x_0)$. If f is differentiable at every point of W, then df is considered as a map

$$df : U \to L(E, F).$$

Definition 5.1.1 (Def. 5.28, Banyaga and Hurtubise [2004], Hirsch [1976]). *For $0 \leq r \leq \infty$, let $C^r(M,N)$ denote the space of C^r maps between two C^r manifolds M and N. Let $f \in C^r(M,N)$, and let (ϕ, U) and (ψ, V) be charts on M and N respectively. Let $K \subset U$ be a compact set such that $f(K) \subset V$, and let $0 \leq \epsilon \leq \infty$. Define the subbasis element*

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \epsilon)$$

to be the set of C^r maps $g : M \to N$ such that $g(K) \subset V$ and

$$\|d^k(\psi \circ f \circ \phi^{-1})(x) - d^k(\psi \circ g \circ \phi^{-1})(x)\| < \epsilon$$

for all $x \in \phi(K)$ and $k = 0, ..., r$. The C^r topology on $C^r(M,N)$ is defined to be the topology generated by the subbasis elements $\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \epsilon)$. The C^∞ topology on $C^\infty(M,N)$ is defined to be the union of topologies induced by the inclusions $C^\infty(M,N) \to C^r(M,N)$ for all $0 \leq r \leq \infty$.

By Theorem 2.4.4 in Hirsch [1976], $C^r(M,N)$, $0 \leq r \leq \infty$, with the C^r topology arises from a complete metric. In the following we shall construct a metric for the space $C^r(M^n, \mathbb{R}^s)$ with M a compact C^r manifold such that this metric generates the topology, which coincides with the C^r topology.

The space $C^r(M, \mathbb{R}^s)$ has a canonical vector space structure:
For $f,\ g \in C^r(M, \mathbb{R}^s)$ and a real λ we define

$$(f+g)(p) = f(p) + g(p),\ (\lambda f)(p) = \lambda f(p) \ \text{ for all } p \in M.$$

We shall take a finite open cover $\{V_i\}_{i=1,...,k}$ of M such that each V_i is contained in the domain of a local chart (ψ_i, U_i) with $\psi_i(U_i) = D_2^n$ and $\psi_i(V_i) = D_1^n$, where D_r^n denotes the open ball of radius r with center 0 in $\mathbb{R}^n$. We shall use the notation

$$f^i = f \circ \psi_i^{-1} :\ D_2^n \to \mathbb{R}^s,$$

and define a norm

$$||f||_r = \max_i \sup\{||f^i(u)||,\ ||df^i(u)||,\ ...,\ ||d^r f^i(u)||\ |\ u \in D_1^n\}.$$

In the proposition below we see that that the norm $\|\cdot\|_r$ generates the C^r topology on $C^r(M, \mathbb{R}^s)$.

Proposition 5.1.2. *Let M^n be a compact manifold. The norm $\|\cdot\|_r$ on $C^r(M,\mathbb{R}^s)$ generates the C^r topology of it.*

Proof. We show that for any map $f \in C^r(M,\mathbb{R}^s)$ the ball

$$B_\epsilon(f) = \{g \in C^r(M,\mathbb{R}^s) |\ \|g-f\|_r < \epsilon\}$$

is open in the C^r topology. Let $\{(\psi_i, U_i)\}_{i\in\{1,\ldots,k\}}$ be the family of coordinate charts and $\{V_i\}_{i\in\{1,\ldots,k\}}$ be the cover of M as in the definition of the $\|\cdot\|_r$ norm. We see that $B_\epsilon(f) = \bigcap_{i=1}^{k} \mathcal{N}(f;(\psi_i,U_i),\mathbb{R}^s,\mathrm{cl}(V_i),\epsilon)$.

Now we prove that the topology generated by the $\|\cdot\|_r$ norm, denoted in the sequel by $\mathcal{T}$, is finer than the C^r topology. We show that for any $f \in C^r(M,\mathbb{R}^s)$, any $i \in \{1,...,k\}$, any open set B in $\mathbb{R}^s$, any compact subset K of $\mathrm{cl}(V_i)$ and $\epsilon > 0$, the set $\mathcal{N}(f;(\psi_i,U_i),B,K,\epsilon)$ is open in $\mathcal{T}$. Pick $g \in \mathcal{N}(f;(\psi_i,U_i),B,K,\epsilon)$. Because $g(K)$ is compact we can choose δ_1 such that the δ_1-neighborhood of $g(K)$ is contained in B. Hence, $h(K) \subset B$ for any $h \in B_{\delta_1}(g)$. For $0 < \delta \leq \min\{\delta_1,\epsilon\}$ we have $B_\delta(g) \subset \mathcal{N}(f;(\psi_i,U_i),B,K,\epsilon)$.

Let (θ, W) be a chart on M, K be a compact subset of W, B as before be any open set in $\mathbb{R}^s$ and $\epsilon > 0$. Suppose K intersects l of the elements of the family $\{V_i\}_{i\in\{1,\ldots,k\}}$, say $V_1, V_2, ..., V_l$. Let

$$b_i^j = \sup\{\|d^j(\psi\circ\theta^{-1})(x)\| \mid x \in V_i \cap K\}, i = 1,...,l$$

and

$$b = \min\{b_i^j \mid i \in \{1,...,l\},\ j \in \{1,...,r\}\}.$$

If $\delta \leq \epsilon/b$ then $\bigcap_{i=1}^{l} \mathcal{N}(g;(\psi_i,U_i),B,K\cap\mathrm{cl}(V_i),\delta) \subset \mathcal{N}(f;(\theta,W),B,K,\epsilon)$. Since the sets $\mathcal{N}(g;(\psi_i,U_i),B,K\cap\mathrm{cl}(V_i),\delta)$ are open in $\mathcal{T}$ and they contain g, the set $\mathcal{N}(f;(\theta,W),B,K,\epsilon)$ is open in $\mathcal{T}$. □

Proposition 5.1.3 (Proposition 2.1, Palis and de Melo [1982]). *The vector space $C^r(M,\mathbb{R}^s)$ with the norm $||\cdot||_r$ is Banach.*

Proposition 5.1.4 (Proposition 2.3, Palis and de Melo [1982]). *$C^r(M,\mathbb{R}^s)$ is separable; that is, it has a countable base of open sets.*

Proposition 5.1.5 (Proposition 2.2.2, Hirsch [1976]). *The subset of maps of class $C^r(M, \mathbb{R}^s)$ ($1 \leq r \leq \infty$) is dense in $C^0(M, \mathbb{R}^s)$.*

Proposition 5.1.6 (Proposition 2.2.4, Hirsch [1976]). *The subset of maps of class $C^\infty(M, \mathbb{R}^s)$ is dense in $C^r(M, \mathbb{R}^s)$.*

Proposition 5.1.7. *For $r \geq 0$, let M be a compact C^r manifold. The evaluation map*

$$e : M \times C^r(M, \mathbb{R}^s) \to \mathbb{R}^s$$

defined by the equation

$$e(p, \xi) = \xi(p)$$

is C^r (with C^r topology imposed on $C^r(M, \mathbb{R}^s)$).

Proof. We shall proof the proposition by induction. We start by showing that e is continuous. Denote the C^r maps $M \to \mathbb{R}^s$ with the C^0 topology by $C_0^r(M, \mathbb{R}^s)$. The compact-open topology on $C^r(M, \mathbb{R}^s)$ coincides with the topology of compact convergence thus with the topology generated by $\|\cdot\|_0$. We conclude that the evaluation map $\bar{e} : M \times C^0(M, \mathbb{R}^s) \to \mathbb{R}^s$ is continuous, cf. Theorem 46.10 in Munkres [2000], so is its restriction to $M \times C_0^r(M, \mathbb{R}^s)$. By the standard $\delta - \epsilon$ argument we see that the inclusion $i : C^r(M, \mathbb{R}^s) \to C_0^r(M, \mathbb{R}^s)$ is continuous. We see that the evaluation map e factors through $\bar{e}|_{M \times C_0^r(M, \mathbb{R}^s)} \circ (\mathrm{id} \times i)$, thus it is continuous.

Now suppose that the evaluation map e is C^{r-1}. We shall show that it is of class C^r. The partial derivative with respect to the first variable is

$$d_1 e_{(x,f)} = e(x, df).$$

It follows that $d_1 e$ is a composition of two C^{r-1} maps hence it is C^{r-1}. The derivative with respect to the second variable is

$$d_2 e_{(x,f)}(h) = e(x, h),$$

which is C^{r-1} by the assumption. Thus e is indeed of class C^r. $\square$

Suppose that $j : M \hookrightarrow \mathbb{R}^s$ is an inclusion map, then the differential

$$dj : T(M) \hookrightarrow M \times \mathbb{R}^s$$

shows that the space $\mathfrak{X}^r(M)$ of C^r vector fields on M is a subspace of $C^r(M, \mathbb{R}^s)$. The tangent bundle $T(M)$ is closed in $M \times \mathbb{R}^s$ since $T(M) = \pi^{-1}(\nu_0)$, where $\pi : T(M) \oplus \nu(M, \mathbb{R}^s) \to \nu(M, \mathbb{R}^s)$ is the projection and ν_0 is the zero section in $\nu(M, \mathbb{R}^s)$. The evaluation map $e : M \times C^r(M, \mathbb{R}^s) \to \mathbb{R}^s$ is continuous, thus the space $\mathfrak{X}^r(M)$ is closed in $C^r(M, \mathbb{R}^s)$, i.e. $\mathfrak{X}^r(M)$ is the preimage of $T(M)$ under the induced map

$$M \times C^r(M, \mathbb{R}^s) \to M \times \mathbb{R}^s.$$

We define a norm on the space $\mathfrak{X}^r(M)$ by

$$\|\xi\|_r^{\sim} = \|\xi - 0_M\|_r,$$

where 0_M is the zero section of $T(M)$. The space $\mathfrak{X}^r(M)$ with the norm $\|\cdot\|_r^{\sim}$ is a Banach space. To suppress the notation we shall write $\|\cdot\|_r$ instead of $\|\cdot\|_r^{\sim}$ when dealing with $\mathfrak{X}^r(M)$.

Definition 5.1.8 (Sec. 1.2 in Palis and de Melo [1982]). *A subset U of a topological space X is called residual if and only if it is a countable intersection of open dense subsets of X, that is $U = \bigcap_{j=1}^{\infty} G_j$ with $G_j \subset X$ open and dense in X for all $j \in \mathbb{N}$. A subset of a topological space X is called generic if and only if it contains a residual set. A topological space X is called Baire if and only if every generic subset is dense.*

Baire's Category Theorem (Theorem 17.1 in Bredon [1993]) says that if X is either a complete metric space or a locally compact Hausdorff space then the intersection of countably many open dense sets is dense. So we conclude that both $C^r(M, \mathbb{R}^s)$ and $\mathfrak{X}^r(M)$ are Baire spaces.

5.2 Vector Fields on a Closed Manifold

An integral curve of a vector field $\xi \in \mathfrak{X}^r(M)$ through a point $p \in M$ is a C^{r+1} map $\alpha : (-\epsilon, \epsilon) \to M$, with a real number $\epsilon > 0$, such that $\alpha(0) = p$ and

$\frac{d}{dt}\alpha(t) = \xi(\alpha(t))$ for all $t \in (-\epsilon, \epsilon)$. The image of an integral curve is an orbit. The set of singularities of a vector field ξ is denoted by $\mathcal{C}r(\xi) = \{p \in M |\ \xi(p) = 0\}$. The theorems on existence, uniqueness and differentiability of solutions of ordinary differential equations in $\mathbb{R}^n$ extend to vector fields on M.

Theorem 5.2.1 (Proposition 1.1, Palis and de Melo [1982]). *Let E be a Banach space and $F : E \times M \to TM$ a C^r map ($r \geq 1$), such that $F_\lambda = F(\lambda, \cdot)$ is a section for any $\lambda \in E$. For every $\lambda_0 \in E$ and $p_0 \in M$ there exist an open neighborhood $W \subset E$ of λ_0 and an open neighborhood $V \subset M$ of p_0, a real number $\epsilon > 0$ and a C^r map $\Phi : (-\epsilon, \epsilon) \times V \times W \to M$ such that*

$$\begin{aligned} \Phi(0, \lambda, p) &= p \text{ and} \\ \frac{\partial}{\partial t}\Phi(t, p, \lambda) &= F(\lambda, \Phi(t, p, \lambda)). \end{aligned} \tag{5.2}$$

for all $t \in (-\epsilon, \epsilon),\ p \in V,\ \lambda \in W$. Moreover, if $\alpha : (-\epsilon, \epsilon) \to M$ is an integral curve of the vector field F_λ with $\alpha(0) = p$ then

$$\alpha = \Phi(\cdot, p, \lambda). \tag{5.3}$$

Proposition 5.2.2. *If the manifold M is compact and a map F satisfies the assumptions of Theorem 5.2.1 then for any $\lambda_0 \in E$ and $p_0 \in M$ there exists an open neighborhood $W \subset E$ of λ_0 and a C^r map $\Phi : \mathbb{R} \times M \times W \to M$ such that (5.2) and (5.3) are satisfied.*

Proof. This proof is motivated by the proof to Lemma 2.4 in Milnor [1973]. We fix λ_0. For any $p \in M$ we have open neighborhoods W_p of λ_0 in E and V_p of p in M and $\epsilon_p > 0$ and the maps $\Phi_p : (-\epsilon_p, \epsilon_p) \times V_p \times W_p \to M$ of Theorem 5.2.1. Since M is compact there is finite number of neighborhoods V_p covering M. Let $\epsilon > 0$ denote the smallest of the numbers $\{\epsilon_{p_i}\}_{i \in \{1 \dots N\}}$, and $W = \bigcap_{i=1}^N W_{p_i}$. We shall not keep track of the subscript p of the function Φ_p, and use the notation $\Phi_t^\lambda(x) = \Phi(t, x, \lambda)$. We see that Φ is defined for all $t \in (-\epsilon, \epsilon)$, $x \in M$ and $\lambda \in W$. It remains to define Φ for $|t| \geq \epsilon$. We express $t = n\epsilon/2 + r$, where $n \in \mathbb{N}$, and $|r| < \epsilon/2$. We define

$$\Phi(t, x, \lambda) = \Phi^\lambda_{\epsilon/2} \circ \dots \circ \Phi^\lambda_{\epsilon/2} \circ \Phi^\lambda_r(x).$$

Φ is well defined by (5.3), and it is C^r as composition of C^r maps. $\square$

5.3 Stability Theory

We shall briefly introduce the Lyapunov stability theory. The aim is to provide necessary conditions for a dynamic system to be stable. Later on we shall associate a function to a Morse-Smale vector field ξ, which is non-increasing along the the flow lines of ξ. We denote a flow line of ξ by $\phi_x^\xi(t)$, that is

$$\frac{d}{dt}\phi_x^\xi(t) = \xi\left(\phi_x^\xi(t)\right) \text{ with } \phi_x^\xi(0) = x.$$

Definition 5.3.1 (Definition 2.1.24 in Abraham and Marsden [1977]). *Let a be a critical point of $\xi \in \mathfrak{X}^r(M)$. Then*

1. *The point a is stable if for any neighborhood U of a, there is a neighborhood V of a such that if $x \in V$ then $\bigcup_{t\geq 0}\phi_t^\xi(x) \subset U$.*
2. *The point a is asymptotically stable if it is stable and there is a neighborhood V' of a such that if $x \in V'$, then*

$$\lim_{t\to+\infty}\phi_t^\xi(x) = a.$$

Locally in $\mathbb{R}^n$ we can formulate the following sufficient conditions for stability.

Theorem 5.3.2 (Theorem 4.1 in Khalil [2002]). *Let 0 be a singular point of a vector field $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$ ($r \geq 1$). If there exist an open neighborhood U of 0 and a C^1 function $f : U \to \mathbb{R}$ such that $f(0) = 0$, $f(x) > 0$ for $x \in U - \{0\}$, and $-\xi(f)(x) \geq 0$ for $x \in U$. Then 0 is stable. Moreover, if $-\xi(f)(x) > 0$ for $x \in D - \{0\}$ then 0 is asymptotically stable.*

Corollary 5.3.3 (Theorem 4.2 in Khalil [2002]). *Let 0 be a singular point of a $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$. If there is a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(0) = 0$, $f(x) > 0$ for $x \neq 0$, $-\xi(f)(x) > 0$ for $x \in \mathbb{R}^n$ and $f(x) \to +\infty$ as $\|x\| \to +\infty$. Then 0 is asymptotically stable on $\mathbb{R}^n$, i.e. for any $x \in \mathbb{R}^n$, $\lim_{t\to+\infty}\phi_t^\xi(x) = 0$.*

The singular point 0 of a linear vector field $L \in \mathcal{L}(\mathbb{R}^n)$ is asymptotically stable if and only if all the eigenvalues of L have negative real part. In the next theorem we shall relate asymptotic stability to the solution of a certain equation.

Theorem 5.3.4 (Lyapunov Stability Theorem 3.2 in Datta [1999]). *The singular point* 0 *of a linear vector field* $L \in \mathcal{L}(\mathbb{R}^n)$ *is asymptotically stable if and only if, for any selfadjoint positive definite matrix* Q *there exists a unique selfadjoint positive definite matrix* P *satisfying the Lyapunov equation*

$$L^{\mathrm{T}}P + PL = -Q.$$

Proof. We show that if L is asymptotically stable then there is a unique solution of the Lyapunov equation. We shall show that the following selfadjoint matrix

$$P = \int_0^\infty e^{L^{\mathrm{T}}t} Q e^{Lt} dt$$

is indeed a solution of the Lyapunov equation. Substitute P into the Lyapunov equation then

$$\begin{aligned} L^{\mathrm{T}}P + PL &= \int_0^\infty L^{\mathrm{T}} e^{L^{\mathrm{T}}t} Q e^{Lt} dt + \int_0^\infty e^{L^{\mathrm{T}}t} Q e^{Lt} L dt \\ &= \int_0^\infty \frac{d}{dt} e^{L^{\mathrm{T}}t} Q e^{Lt} dt = e^{L^{\mathrm{T}}t} Q e^{Lt} \Big|_0^\infty . \end{aligned}$$

But L is asymptotically stable thus $e^{Lt} \to 0$ as $t \to +\infty$. Thus $L^{\mathrm{T}}P + PL = -Q$. To prove that P is unique, assume that there are two solutions P_1 and P_2 to the Lyapunov equation. Then

$$L^{\mathrm{T}}(P_1 - P_2) + (P_1 - P_2)L = 0,$$

which implies that

$$e^{L^{\mathrm{T}}t} \left(L^{\mathrm{T}}(P_1 - P_2) + (P_1 - P_2)L \right) e^{Lt} = 0$$

or

$$\frac{d}{dt} \left[e^{L^{\mathrm{T}}t} (P_1 - P_2) e^{Lt} \right] = 0.$$

It follows that the matrix $e^{L^{\mathrm{T}}t}(P_1 - P_2)e^{Lt}$ is constant for all t. Evaluating this expression for $t = 0$ and $t = +\infty$, we conclude that $P_1 - P_2 = 0$. It is positive definite, since

$$x^{\mathrm{T}}Px = \int_0^{\infty} x^{\mathrm{T}}e^{L^{\mathrm{T}}t}Qe^{Lt}x dt$$

and e^A is nonsingular and Q positive definite, thus $x^{\mathrm{T}}Px > 0$ for $x \neq 0$.

We prove the converse. Pick any selfadjoint positive definite matrix Q, then there is a solution P to the Lyapunov equation, and we can define a function $f : \mathbb{R}^n \to \mathbb{R}$ by $x \mapsto x^{\mathrm{T}}Px$ then

$$-L(f)(x) = -x^{\mathrm{T}}(L^{\mathrm{T}}P + PL)x = x^{\mathrm{T}}Qx > 0$$

for $x \neq 0$. By Corollary 5.3.3, the singular point 0 is asymptotically stable. □

Suppose that $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$, $r \geq 1$. By Taylor expansion , cf. Sec. XIII.6 in Lang [1999], ξ may be considered as a perturbation of a linear ordinary differential equation of the form

$$\begin{aligned} \frac{d}{dt}\phi_x^{\xi}(t) &= \xi \circ \phi_x^{\xi}(t) = L\phi_x^{\xi}(t) + \eta \circ \phi_x^{\xi}(t), \\ \phi_x^{\xi}(0) &= x \end{aligned} \tag{5.4}$$

in some open neighborhood U of 0 in $\mathbb{R}^n$, where $L \in \mathcal{L}(\mathbb{R}^n)$ and $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a C^{r-1} map that satisfies

$$\begin{aligned} \eta(0) &= 0 \\ \|\eta(x) - \eta(y)\| &\leq \delta(\epsilon)\|x - y\| \text{ for } \|x\|, \|y\| < \epsilon \end{aligned} \tag{5.5}$$

with the function $\delta : [0, \infty) \to [0, \infty)$ continuous and monotonically increasing.

In the next corollary we relate asymptotic stability of a vector field to asymptotic stability of its linearization.

Corollary 5.3.5. *Let 0 be a singular point of a vector field $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$, $r \geq 1$. Suppose $L = d\xi(0)$. If L is asymptotically stable, then the point 0 is asymptotically stable for ξ.*

Proof. The linear system L is asymptotically stable thus for any selfadjoint positive definite Q there is a unique solution P to the Lyapunov equation. Define a map $f : \mathbb{R}^n \to \mathbb{R}$ by $x \mapsto x^{\mathrm{T}} P x$. By the Taylor expansion of ξ we have

$$-\xi(f)(x) = x^{\mathrm{T}} Q x - 2x^{\mathrm{T}} P \eta(x).$$

The matrix Q is selfadjoint positive definite, therefore by the Spectral Theorem, $x^{\mathrm{T}} Q x \geq c\|x\|$ where c is the smallest eigenvalue of Q. Furthermore, we use the estimate

$$|x^{\mathrm{T}} P \eta(x)| \leq \|x\| \; \|P\| \; \|\eta(x)\| \leq \delta(\epsilon)\|P\|\|x\|^2,$$

where δ is continuous and monotonically nondecreasing as in (5.5). Therefore we can choose ϵ such that $\delta(\epsilon) < d$, where d is an arbitrary real number. For $\|x\| < \epsilon$ we have

$$-\xi(f)(x) = x^{\mathrm{T}} Q x - 2x^{\mathrm{T}} P \eta(x) \geq c\|x\|^2 - 2|x^{\mathrm{T}} P \eta(x)| \geq (c - 2\delta(\epsilon)\|P\|)\, \|x\|^2.$$

We shrink ϵ such that $\kappa \equiv c - 2\delta(\epsilon)\|P\| > 0$ and get

$$-\xi(f)(x) \geq \kappa\|x\|^2, \; \forall \; \|x\| < \epsilon.$$

Thus by Theorem 5.3.2, the singular point 0 of ξ is asymptotically stable. □

The result below relates the spectrum of P with the spectrum of L. First we define an inertia.

Definition 5.3.6. *The inertia of a matrix $L \in \mathcal{L}(\mathbb{R}^n)$ of order n, denoted by* $\mathrm{In}(L)$, *is the triplet*

$$(\pi(L), \nu(L), \delta(L)),$$

where $\pi(L), \nu(L)$ and $\delta(L)$ are, respectively, the number of eigenvalues of L with positive, negative, and zero real parts, counting multiplicities.

Note that $\pi(L)+\nu(L)+\delta(L) = n$, and L is stable if and only if $\mathrm{In}(L) = (0, n, 0)$. Below we state the Main Inerta Theorem.

Theorem 5.3.7 (Theorem 2.5 in Stykel [2002]). . *A necessary and sufficient condition that there exists a selfadjoint matrix P such that*

$$L^{\mathrm{T}}P + PL = -Q, \text{ where } Q \text{ is selfadjoint and positive definite,}$$

is that $\delta(L) = 0$. Furthermore, we have $\pi(L) = \nu(P)$ and $\pi(P) = \nu(L)$.

Suppose $\xi \in \mathfrak{X}^r(M)$ and a is a singular point of ξ. Consider a local chart (ψ, U) with $a \in U$ and $\phi(a) = 0$. Then ξ in the local coordinates is $\hat{\xi} = d\psi\xi \circ \psi^{-1}$. Notice that $\mathrm{In}(d\hat{\xi}_0)$ is independent of the local chart.

Definition 5.3.8. *Suppose $\xi \in \mathfrak{X}^r(M)$. A singular point $a \in M$ is called hyperbolic if and only if $d\hat{\xi}_0 : \mathbb{R}^n \to \mathbb{R}^n$ is hyperbolic, i.e. $\delta(d\hat{\xi}_0) = 0$ ($d\hat{\xi}_0$ does not have any complex eigenvalues whose real part is zero).*

Proposition 5.3.9. *Suppose $\xi \in \mathfrak{X}^r(M)$, $r \geq 1$ and a is a hyperbolic singular point of ξ. Then there exist an open neighborhood $U \subset M$ of a, a C^r function $f : U \to \mathbb{R}$ and a real number $\kappa > 0$ such that $-\xi(f)(x) \geq \kappa d(x, a)^2$, where d is the distance introduced by the Riemannian metric on M.*

As in the proof of Corollary 5.3.5

Proof. Use the exponential map to get the normal coordinates, $\psi : V \to T_a(M)$. We represent ξ in local coordinates $\hat{\xi} = d\psi\xi \circ \psi^{-1}$. By the Taylor expansion we have $\hat{\xi} = L + \eta$ with $\|\eta(x)\| \leq \delta(\epsilon)\|x\|$ for $\|x\| < \epsilon$. Pick a selfadjoint and positive definite Q then there is a unique selfadjoint hyperbolic P solving the Lyapunov equation

$$L^{\mathrm{T}}P + PL = -Q.$$

Define the function $\hat{f} : \psi(V) \to \mathbb{R}$ by $x \mapsto x^{\mathrm{T}}Px$. Then following the proof of Corollary 5.3.5 there is a neighborhood $V' \subset \psi(V)$ of 0 and a constant κ such that

$$-\hat{\xi}(\hat{f})(x) = x^{\mathrm{T}}Qx - 2x^{\mathrm{T}}P\eta(x) \geq \kappa\|x\|^2.$$

The desired function is $f = \hat{f} \circ \psi^{-1}$ defined on $U = \psi^{-1}(V')$. Then

$$df \circ \xi(p) = df d\psi \circ d\psi^{-1}\xi(p) = -d\hat{f} \circ \hat{\xi}(x) \geq \kappa\|x\|^2 = \kappa d(a, p)^2,$$

where $x = \psi(p)$. $\square$

5.4 Invariant Manifolds and Their Perturbations

We study invariant manifolds of a C^r vector field on a closed C^r manifold M, $r \geq 1$. We will restrict out attention to the stable manifolds. The results on unstable manifolds are analogous. It will be shown that a stable manifold depends continuously on perturbations of a vector field in a sense specified later in this section.

The stable manifold of ξ at a singular point a is defined by

$$W_a^s(\xi) = \{x \in M | \lim_{t\to+\infty} \phi_x^\xi(t) = a\},$$

and the unstable manifold of ξ at a is

$$W_a^u(\xi) = \{x \in M | \lim_{t\to-\infty} \phi_x^\xi(t) = a\}.$$

At this stage it is not clear that the sets $W_a^s(\xi)$ and $W_a^u(\xi)$ are manifolds. This is indeed the case if the singular points are hyperbolic. This is shown in Theorem 5.4.1 and Corollary 5.4.2.

Significance of the notion of a hyperbolic singular point stems from the following observation. If $L \in \mathcal{L}(\mathbb{R}^n)$ and it is hyperbolic then there is a direct sum decomposition

$$\mathbb{R}^n = E^s \oplus E^u,$$

where E^s and E^u are invariant subspaces for L. Moreover the eigenvalues of $L^s \equiv L|_{E^s}$ have negative real part and the eigenvalues of $L^u \equiv L|_{E^u}$ have positive real part, cf. Ch. 7 in Hirsch and Smale [1974]. In particular $W_0^s(L) = E^s$ and $W_0^u(L) = E^u$. We shall use the projections

$$P^s : \mathbb{R}^n \to E^s \text{ and } P^u : \mathbb{R}^n \to E^u. \tag{5.6}$$

For a hyperbolic L there are $c_0, \alpha > 0$ such that

$$\begin{array}{rcccc} \|e^{Lt}P^s\| & \leq & c_0 e^{-\alpha t} & \text{for} & t \geq 0 \\ \|e^{Lt}P^u\| & \leq & c_0 e^{\alpha t} & \text{for} & t \leq 0. \end{array} \tag{5.7}$$

We define the local stable and unstable manifold

$$
\begin{aligned}
W_0^s(\xi, U) &= \{x \in U : \phi_x(t) \text{ is defined and contained in } U \text{ for all } t \geq 0, \text{ and} \\
&\qquad \lim_{t \to +\infty} \phi_x^\xi(t) = 0\}, \\
W_0^u(\xi, U) &= \{x \in U : \phi_x(t) \text{ is defined and contained in } U \text{ for all } t \leq 0, \text{ and} \\
&\qquad \lim_{t \to -\infty} \phi_x^\xi(t) = 0\}.
\end{aligned}
$$

Theorem 5.4.1 (Local Stable Manifold Theorem, Theorem 6.3.1, Jost [2002]). *Let $\phi_x(t)$ satisfy the differential equation (5.4) with a hyperbolic linear operator L and $\eta \in \mathfrak{X}^r(\mathbb{R}^n)$ obeying inequality (5.5). Then there is an open neighborhood U of 0 such that $W_0^s(\xi, U)$ is a Lipschitz graph over $U \cap E^s$, which is tangent to E^s at 0. If η is of class C^k in U, so is $W_0^s(\xi, U)$.*

Let $\xi \in \mathfrak{X}^r(\mathbb{R}^n)$ and suppose 0 is a hyperbolic singular point. Denote $L = d\xi_0$ and let E^s, E^u be the stable and unstable subspaces of $\mathbb{R}^n$ for L. Furthermore, let $D_r^s,\ D_r^u$ be centered at 0 open balls respectively in E^s and E^u, both with the radius r. By the Local Stable/Unstable Manifold Theorem there are two maps $\alpha^s : D_r^s \to E^u$, and $\alpha^u : D_r^u \to E^s$, such that $W_a^s(\xi, D_r^s \oplus D_r^u)$ and $W_a^u(\xi, D_r^s \oplus D_r^u)$ are graphs of α^s and α^u, respectively. The differentials, $d\alpha^s(0) = d\alpha^u(0) = 0$.

We define a map $\alpha : D_r^s \oplus D_r^u \to E^s \oplus E^u$ by

$$
\alpha(x_s, x_u) = (x_s - \alpha^u(x), x_u - \alpha^s(x)). \tag{5.8}
$$

The map α is C^r and $d\alpha(0) = \mathrm{id}_{\mathbb{R}^n}$. Thus α is a diffeomorphism when restricted to some open neighborhood of 0 in $\mathbb{R}^n$. If we represent ξ in the local coordinates determined by α, $\hat{\xi} = d\alpha\xi \circ \alpha^{-1}$ then the local stable manifold of $\hat{\xi}$ is an open neighborhood of the origin in E^s and the local unstable manifold is an open neighborhood of the origin in E^u.

Suppose now that $\xi \in \mathfrak{X}^r(M)$ and a is a hyperbolic singular point, we formulate a global version of Theorem 5.4.1.

Corollary 5.4.2 (Global Stable Manifold Theorem for Vector Fields). *Suppose $\xi \in \mathfrak{X}^r(M)$, $r \geq 1$, a is a hyperbolic critical point and λ is the index of ξ. Then $W_a^s(\xi)$ is the surjective image of a C^r injective immersion*

$$
\alpha^s : \mathbb{R}^\lambda \to W_a^s(\xi) \subset M.
$$

Hence, $W^s_a(\xi)$ is an injectively immersed open disk in M. Furthermore,

$$T_a(W^s_a(\xi)) = T_a(M)^s.$$

In the proof of the corollary we shall use the following lemma.

Lemma 5.4.3. *For $r \geq 1$, let ξ be a C^r vector field on $\mathbb{R}^n$ such that $\xi(0) = 0$ and $L = d\xi_0$. Suppose L is asymptotically stable. Then there is a neighborhood U of 0 and an extension ξ' of $\xi|_U$ to $\mathbb{R}^n$ such that $W^s_0(\xi') = \mathbb{R}^n$.*

Proof. For each $r > 0$ there is a smooth bump function $\rho : \mathbb{R}^n \to [0,1]$ with the properties: $\rho(x) = 1$ for $\|x\| < r/2$, $\rho(x) = 0$ for $\|x\| > r$. By Taylor expansion, ξ has the form (5.4) and (5.5). We define the vector field $\xi' = L + \theta$, where $\theta(x) = \rho(x)\eta(x)$. It is equal to ξ on the open ball $D^n_{r/2}$ and it coincides with L for $\|x\| > r$.

Since L is asymptotically stable, by Theorem 5.3.4 for any choice of a selfadjoint positive definite matrix Q there is a selfadjoint positive definite matrix P such that $L^{\mathrm{T}}P + PL = -Q$. Let c be the smallest eigenvalue of Q. Pick r such that $\kappa \equiv c - 2\delta(r)\|P\| > 0$. We define a function $\nu(x) = x^{\mathrm{T}}Px$ and see that

$$-\xi'(\nu)(x) = x^{\mathrm{T}}Qx - 2\rho(x)x^{\mathrm{T}}P\eta(x) \geq \kappa\|x\|^2$$

for $x \in \mathbb{R}^n - \{0\}$, thus by Corollary 5.3.3, the system ξ' is asymptotically stable on $\mathbb{R}^n$. $\square$

Proof (of Corollary 5.4.2). The argument below follows the proof of Theorem 4.15 in Banyaga and Hurtubise [2004]. It is modified it to deal with the stable manifolds for vector fields.

Consider an open neighborhood $U' \subset M$ of the point a and apply the exponential map to get a coordinate chart $\psi : U' \to U \subset T_a(M)$ with U an open neighborhood of 0. We represent the vector field ξ in the local coordinates $\bar{\xi} = d\psi\xi \circ (\psi)^{-1}$. Denote the stable and unstable subspaces of $T_a(M)$ for $\bar{\xi}$ by respectively $T_a(M)^s$ and $T_a(M)^u$. Note that the fact that ξ is of class C^1 implies that inequality (5.5) holds locally. By Theorem 5.4.1 we can shrink U' such that $\psi(W^s_a(\xi, U'))$ is a graph of a function $g : U \cap T_a(M)^s \to T_a(M)^u$. Since ψ is the inverse of the exponential map, we have $d_a\psi = \mathrm{id}_{T_a(M)}$. Moreover, $T_a(\psi(W^s_a(\xi, U'))) = T_a(M)^s$ thus $T_a(W^s_a(\xi)) = T_a(M)^s$.

On an open neighborhood V of a in $W^s_a(\xi)$, the composition $\psi' = P^s \circ \psi|_{W^s_a(\xi,U')}$ is a coordinate chart $\psi' : V \to T_a(M)^s$. We shall denote the image $\psi'(V)$ by W. We are ready to define a differentiable structure on $W^s_a(\xi)$ by

$$W^s_a(\xi) = \bigcup_{\substack{k \in \mathbb{Z} \\ k \le 0}} \Phi(k, V), \text{ where } \Phi \text{ is the flow of } \xi, \text{ that is } \Phi(t, x) = \phi^\xi_x(t).$$

For $k \in \{0, 1, 2, ...\}$ we define $V_k = \Phi(-k, V)$ and $\psi_k : V_k \to T_a(M)^s$, $\psi_k(x) = \psi' \circ \Phi(k, x)$. The atlas (V_k, ψ_k) makes the inclusion $W^s_a(\xi) \hookrightarrow M$ an immersion.

We represent the restriction of the vector field ξ to V in the local coordinates

$$\xi' = d\psi'\xi \circ (\psi')^{-1}.$$

By Lemma 5.4.3 we can extend the vector field $\xi' \in \mathfrak{X}^r(W)$ to a stable vector field $\xi'' \in \mathfrak{X}^r(T_a(M)^s)$ and define a map $\alpha^s : T^s_a(M) \to W^s_a(\xi)$ by

$$\alpha^s(x) = \phi^\xi_{-t} \circ (\psi')^{-1} \circ \phi^{\xi''}_t(x),$$

where t is any positive real such that $\phi^{\xi'}_t(x) \in V$. The map α^s is well defined since for $\tau \ge t$ we have

$$\begin{aligned} \phi^\xi_{-\tau} \circ (\psi')^{-1} \circ \phi^{\xi''}_\tau(x) &= \phi^\xi_{-\tau} \circ (\psi')^{-1} \circ \phi^{\xi''}_{\tau-t} \circ \phi^{\xi''}_t(x) \\ &= \phi^\xi_{-\tau} \circ \phi^\xi_{\tau-t} \circ (\psi')^{-1} \circ \phi^{\xi''}_t(x) \\ &= \phi^\xi_{-t} \circ (\psi')^{-1} \circ \phi^{\xi''}_t(x). \end{aligned}$$

By the flow properties the map α^s is C^r, it is injective and also surjective since for any $q \in W^s_a(\xi)$ there is $t > 0$ such that $\phi^\xi_t \in V$. The differential $d\alpha^s$ is a composition of injective maps thus α^s is an injective immersion of $T_a(M)^s$ onto $W^s_a(\xi)$. $\square$

We shall remark that there is an ”unstable” counterpart of the global and local stable manifold theorems. A version of Theorem 5.4.1 for a C^r diffeomorphism on M can be found in Banyaga and Hurtubise [2004] and Palis and de Melo [1982].

In the remaining of this section we will discuss perturbations of vector fields.

Definition 5.4.4. *Suppose U is an open subset in $\mathbb{R}^n$, $\xi, \vartheta \in \mathfrak{X}^r(U)$, and p is a singular point for both ξ and ϑ. We say that ξ is locally topologically conjugate to ϑ at p if there are two neighborhoods V and V' of p in U and a homeomorphism $h : V \to V'$ such that*

$$h(\phi_t^{\xi}(x)) = \phi_t^{\vartheta}(h(x))$$

for $x \in V$ and $t \in \mathbb{R}$ and both sides of the equation are defined.

Suppose $U \subset \mathbb{R}^n$ is an open neighborhood of 0 with compact closure $\mathrm{cl}(U)$. According to the Grobman-Hartman Theorem any hyperbolic $\xi \in \mathfrak{X}^r(U)$ with $\xi(0) = 0$ is locally topologically conjugate to its linearization $d\xi_0$. We modify the Grobman-Hartman Theorem to deal with small perturbations. For an open neighborhood $V \subset U$ and $\vartheta \in \mathfrak{X}^r(U)$ with a hyperbolic critical point 0 we define a set $B_\delta(\vartheta, V) \subset \mathfrak{X}^r(U)$ by

$$\begin{aligned} B_\delta(\vartheta, V) \;\; = \;\; & \{\xi \in \mathfrak{X}^r(U) | \, \|\xi' - \vartheta'\|_1 < \delta \text{ where } \xi' = \xi|_V, \vartheta' = \vartheta|_V \in \mathfrak{X}^r(V) \\ & \text{and } \xi(0) = 0\}, \end{aligned}$$

where for $\xi \in \mathfrak{X}^r(V)$ we have $||\xi||_1 = \sup\{||\xi(x)||, ||d\xi(x)|| \mid x \in V\}$. Note that $B_\delta(\vartheta, V)$ is open in the space $\{\xi \in \mathfrak{X}^r(U) | \, \xi(0) = 0\}$ with the topology generated by the norm $\| \cdot \|_1$.

Proposition 5.4.5. *Let U be an open subset of $\mathbb{R}^n$ with compact closure. Suppose $\vartheta \in \mathfrak{X}^r(U)$, $1 \leq r < \infty$, and 0 is a hyperbolic singular point. Denote $L = d\vartheta_0$. Then there is a neighborhood $V \subset U$ of 0, a real $\delta > 0$ and a continuous map*

$$h : V \times B_\delta(\vartheta, U) \to U$$

such that for any $\xi \in B_\delta(\vartheta, U)$,

1. *$h_\xi : V \to U$ is a homeomorphism onto its image,*
2. *$h_\xi(\phi_t^{\xi}(x)) = \phi_t^{L}(h_\xi(x))$.*

In the proof of this proposition and Proposition 5.4.7 we use the following version of the Banach fixed point theorem, cf. Lemma 10.2 in Jost [1998].

Lemma 5.4.6. *Let C be a closed set in a Banach space A and P be an open set in a Banach space B. For $r \geq 0$, let $T : P \times C \to C$ be a C^r map that satisfies*

$$\|T_x(y_1) - T_x(y_2)\| \leq \lambda \|y_1 - y_2\| \text{ for all } y_1, y_2 \in C \text{ and } x \in P$$

with $0 \leq \lambda < 1$.

Then there is a unique C^r map $Y : P \to C$ such that

$$T_x(Y_x) = Y_x.$$

Proof. By Lemma 10.2 in Jost [1998], the lemma is true for $r = 0$. We prove it for $r \geq 1$. Suppose $y = Y(x)$ and $y_0 = Y(x_0)$. Since T is of class C^r in particular it is differentiable. We use Eq.(5.1) to write

$$T(x,y) - T(x_0,y_0) = A_{(x_0,y_0)}(x-x_0) + B_{(x_0,y_0)}(y-y_0) + \alpha_{(x_0,y_0)}(x-x_0, y-y_0), \tag{5.9}$$

where $\alpha_{(x_0,y_0)}$ is tangent to 0. Since y and y_0 are assumed to be fixed points, the above equation yields

$$y - y_0 = A_{(x_0,y_0)}(x-x_0) + B_{(x_0,y_0)}(y-y_0) + \alpha_{(x_0,y_0)}(x-x_0, y-y_0). \tag{5.10}$$

From (5.10) we get

$$\left(\mathrm{id_A} - B_{(x_0,y_0)}\right)^{-1}(y - y_0) = A_{(x_0,y_0)}(x-x_0) + \alpha_{(x_0,y_0)}(x-x_0, y-y_0).$$

Since the map T_{x_0} is a contraction for all $x_0 \in P$, the norm of the linear operator $B_{(x_0,y_0)}$ is less than 1 for all $(x_0,y_0) \in P \times C$. This is seen by substituting in Eq. (5.9) x for x_0 and $y - y_0$ for hv with $h \in \mathbb{R}$ and v such that

$$\|B_{(x_0,y_0)}v\| = \|B_{(x_0,y_0)}\|.$$

This yields

$$\lambda|h| \geq \|B_{(x_0,y_0)}\||h| - \|\alpha_{(x_0,y_0)}(0,hv)\| \geq (\|B_{(x_0,y_0)}\| - \beta(hv))|h| \ \ \forall h \in \mathbb{R}$$

with $\lim_{\|z\|\to 0}\beta(z) = 0$. Thus $\|B_{(x_0,y_0)}\| \le \lambda < 1$. It follows that the matrix $\left(\mathrm{id}_{\mathrm{A}} - B_{(x_0,y_0)}\right)$ is nonsingular and

$$\begin{aligned} y - y_0 &= \left(\mathrm{id}_{\mathrm{A}} - B_{(x_0,y_0)}\right)^{-1} A_{(x_0,y_0)}(x - x_0) \\ &+ \left(\mathrm{id}_{\mathrm{A}} - B_{(x_0,y_0)}\right)^{-1} \alpha_{(x_0,y_0)}(x - x_0, y - y_0). \end{aligned}$$

We conclude that the derivative $dY(x) = \left(\mathrm{id}_{\mathrm{A}} - B_{(x,y(x))}\right)^{-1} A_{(x,y(x))}$ is of class C^{r-1}. □

Proof (of proposition). The theorem follows from the proof of Grobman-Hartman theorem, cf. Chicone and Swanson [2000], and Lemma 5.4.6. □

Proposition 5.4.7. *Suppose $\vartheta \in \mathfrak{X}^r(\mathbb{R}^n)$, $1 \le r < \infty$, and 0 is a hyperbolic singular point of ϑ. Let $L = d\vartheta_0$ and $E^s,\ E^u$ be stable and unstable subspaces of $\mathbb{R}^n$ for L. Then there are (1) two open neighborhoods U, V of 0 in $\mathbb{R}^n$ with $V \subset U$ and $\mathrm{cl}(U)$ compact, (2) a real $\delta > 0$ and (3) a C^r map*

$$\beta^s : (V \cap E^s) \times B_\delta(\vartheta, U) \to E^u,$$

such that for any $\xi \in B_\delta(\vartheta, V)$, $W_0^s(\xi, V)$ is the graph of $\beta_\xi^s : V \cap E^s \to E^u$, where $\beta_\xi^s(x) = \beta^s(x, \xi)$.

Proof. The proposition follows from the proof of Theorem 6.3.1 in Jost [2002]. The difference lies in extending it to cope with small perturbations in $B_\delta(\vartheta)$. Here we give a sketch of the proof.

Consider $c_0,\ \alpha$ as in the inequalities (5.7) on page 50. For $0 < \lambda < \alpha$ we define the following Banach space

$$M_\lambda = \left\{ Y : \mathbb{R}^+ \to \mathbb{R}^n \mid \|Y\|_{\exp,\lambda} \equiv \sup_{t \ge 0} e^{\lambda t} \|Y(t)\| < \infty \right\}.$$

We choose $\epsilon > 0$ and consider a closed subset

$$M_\lambda(\epsilon) = \{ Y \in M_\lambda \mid \|Y\|_{\exp,\lambda} \le \epsilon \} \subset M_\lambda.$$

Thus $M_\lambda(\epsilon)$ is the set of those Y for which $\|Y(0)\| \le \epsilon$ and that are exponentially decreasing.

In the following we keep λ fixed. Pick $\delta > 0$ and $\xi \in B_\delta(\vartheta, D_{2\epsilon})$. We define $\theta \equiv \xi - \vartheta$ and consider the following ordinary differential equation

$$\frac{d}{dt}\phi_x^\xi(t) = \xi(\phi_x^\xi(t)) = \vartheta(\phi_x^\xi(t)) + \theta(\phi_x^\xi(t)) = L\phi_x^\xi(t) + \omega(\phi_x^\xi(t)) + \theta(\phi_x^\xi(t)), \tag{5.11}$$

with $\phi_x^\xi(0) = x$. We think about L and ω as fixed (obtained from the Taylor expansion of ϑ) and θ plays the role of a perturbation. Thus the vector field $\omega : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\omega(0) = 0$ and

$$\|\omega(x) - \omega(y)\| \leq \delta'(\epsilon)\|x - y\| \ \text{ for } \ \|x\|, \|y\| < \epsilon$$

with the function $\delta' : [0, \infty) \to [0, \infty)$ continuous and monotonically increasing. If $\|\theta|_{D_{2\epsilon}}\|_1 < \delta''$ then $\sup\{\|d\theta(u)\| \mid \|u\| < 2\epsilon\} < \delta''$, by the definition of $\| \cdot \|_1$ norm . We use the Mean Value Theorem to show that

$$\|\theta(x) - \theta(y)\| \quad \leq \quad \delta''\|x - y\| \ \text{ for } \|x\|, \|y\| < \epsilon.$$

Denoting $\eta = \omega + \theta$ and $\delta(\epsilon) = \delta'(\epsilon) + \delta''$ we see that

$$\|\eta(x) - \eta(y)\| \leq \delta(\epsilon)\|x - y\| \ \text{ for } \ \|x\|, \|y\| < \epsilon.$$

Our setup is like the one in the proof of Theorem 6.3.1 in Jost [2002]. The only difference is that $\delta(0)$ is no longer 0 but δ''. However, we can permit the perturbations $\|\theta|_{D_{2\epsilon}}\|_1 < \delta''$ to be as small as desired.

Let $D_\epsilon^s \subset E^s$ be an open disk of radius ϵ centered at 0. For sufficiently small ϵ and δ we can define an operator $T : D_\epsilon^s \times M_\lambda(\epsilon) \times B_\delta(\omega, D_{2\epsilon}) \to M_\lambda(\epsilon)$ by

$$T(x, Y, \eta)(t) = e^{Lt}x + \int_0^t e^{L(t-s)}P^s\eta(Y(s))ds - \int_t^\infty e^{L(t-s)}P^u\eta(Y(s))ds,$$

where P^s and P^u have been defined in Equation (5.6) on page 50.

The form of the operator is such that if $\tilde{Y}$ is bounded and $\tilde{Y}(t) = T(x, \tilde{Y}, \eta)(t)$ then it is a solution of $\frac{d}{dt}Y(t) = \xi(Y(t))$.

Following the equations (6.3.20) to (6.3.21) on page 298 in Jost [2002] we observe that

$$\|T(x, Y, \eta)(t)\| \leq c_0 e^{-\alpha t}\|x\| + c_1\delta(\epsilon)e^{-\lambda t}\|Y\|_{\exp,\lambda},$$

where c_0 and c_1 are positive real numbers, furthermore

$$\|T(x, Y_1, \eta)(t) - T(x, Y_2, \eta)(t)\| \leq 2c_1\delta(\epsilon)e^{-\lambda t}\|Y_1 - Y_2\|_{\exp,\lambda}.$$

We shrink ϵ and δ'' such that $2c_1\delta(\epsilon) < 1/2$ and pick $0 < \epsilon' \leq \frac{\epsilon}{2c_0}$. Then we have that for any $Y_1, Y_2 \in M_\lambda(\epsilon)$

$$\|T(x, Y_1, \eta)\|_{\exp,\lambda} \leq \epsilon$$

and

$$\|T(x, Y_1, \eta) - T(x, Y_2, \eta)\|_{\exp,\lambda} \leq \frac{1}{2}\|Y_1 - Y_2\|_{\exp,\lambda}$$

for all $x \in D^s_{\epsilon'}$(open disk in E^s centered at 0 and radius ϵ') and $\eta \in B_\delta(\omega, D_{2\epsilon})$. In conclusion the operator T is such that $T(x, \cdot, \eta)(M_\lambda(\epsilon)) \subset M_\lambda(\epsilon)$ for all $x \in D^s_{\epsilon'}$ and $\eta \in B_\delta(\omega, D_{2\epsilon})$ and has a contraction constant equal to $\frac{1}{2}$. Therefore, by applying the Banach Fixed Point Theorem, Lemma 5.4.6, we get a unique solution $Y_{(x,\eta)} \in M_\lambda(\epsilon)$ to the equation

$$Y(t) = T(x, Y, \eta)(t) \text{ for any } x \in D^s_{\epsilon'} \text{ and } \eta \in B_\delta(\omega, D_{2\epsilon}). \tag{5.12}$$

Observe that T is C^r, so does the map $D^s_{\epsilon'} \times B_\delta(\omega, D_{2\epsilon}) \to M_\lambda(\epsilon)$ taking (x, η) to the solution $Y_{(x,\eta)}$.

Notice that $T(0, 0, \eta) = 0$. Since $Y_{(x,\eta)} \in M_\lambda(\epsilon)$ is decaying exponentially, $\lim_{t\to\infty} Y_{(x,\eta)}(t) = 0$, therefore

$$Y_{(x,\eta)}(0) \in W^s_0(\xi) = W^s_0(L + \eta).$$

It is shown in Jost [2002] that for any open neighborhood V' of 0 with

$$V' \subset (D_{2\epsilon} \cap (P^s)^{-1}(D^s_{\epsilon'}))$$

and for any $\eta \in B_\delta(\omega, D_{2\epsilon})$ we have a map

$$g_\eta : V' \cap E^s \to W^s_0(L + \eta),\ x \mapsto Y_{(x,\eta)}(0),$$

satisfying

1. g_η is a bijection between $E^s \cap V'$ and its image in $W_0^s(\xi)$;
2. $\|Y_{(x_1,\eta)}(0) - Y_{(x_2,\eta)}(0)\| \le 2c_0\|x_1 - x_2\|$ (g_η is Lipschitz);
3. There exists an open neighborhood $V_\eta \subset \mathbb{R}^n$ such that $V_\eta \cap E^s = V' \cap E^s$ and the image $g_\eta(V_\eta \cap E^s) = W_0^s(\eta, V_\eta)$.

Let $V = \bigcup_{\eta \in B_\delta(\omega, D_{2\epsilon})} V_\eta$, then V is an open neighborhood of 0 and

$$g_\eta(V \cap E^s) = W_0^s(\eta, V_\eta) = W_0^s(\eta, V) \text{ for all } \eta \in B_\delta(\omega, D_{2\epsilon}).$$

The last equality follows from the following observation: If $x, y \in W_0^s(\xi, V)$ and $P^s(x) = P^s(y)$ then $x = y$, cf. Lemma 2.6.3 in Palis and de Melo [1982]. We define a map $g : (V \cap E^s) \times B_\delta(\omega, D_{2\epsilon}) \to E^s \times E^u$ by $g(x, \eta) \equiv g_\eta(x) = Y_{(x,\eta)}(0)$, and the desired map $\beta^s : (V \cap E^s) \times B_{\delta''}(\vartheta, D_{2\epsilon}) \to E^u$ is given by

$$(x, \xi) \mapsto P^u g(x, \xi - L).$$

□

Let U be an open subset of $\mathbb{R}^n$ with compact closure. Suppose $\vartheta \in \mathfrak{X}(U)$ and 0 is a hyperbolic singular point of ϑ. Without loss of generality we may assume that the stable manifold of ϑ is a neighborhood of the origin in E^s if not apply the diffeomorphism α in Equation (5.8) and consider $d\alpha\vartheta \circ \alpha^{-1}$.

For any differential equation of the form $\dot{x} = \vartheta(x) = Lx + \eta(x)$ where $\eta(0) = d\eta(0) = 0$ and L is hyperbolic, there exist two matrices P and Q, Q is selfadjoint positive definite and P is selfadjoint nonsingular, such that $L^{\mathrm{T}}P + PL = -Q$ and $\pi(L) = \nu(P)$. We define a function $f(x) = x^{\mathrm{T}}Px$. By the Lyapunov arguments, cf. Proposition 5.3.9, there exists a sufficiently small neighborhood $V \subset U$ of 0 and a constant $\kappa > 0$ such that

$$-\vartheta(f)(x) \ge \kappa\|x\|^2 \text{ for } x \in V. \tag{5.13}$$

We will consider the preimage $V_\tau = f^{-1}(\tau)$ for a regular value τ. By Equation (5.13) the intersection $S_0^s(\vartheta, V) \equiv V_\tau \cap W_0^s(\vartheta, V)$ is transversal, thus $S_0^s(\vartheta, V)$ is a manifold. By assumption $S_0^s(\vartheta, V) \subset S_0^s(\vartheta, V) \subset E^s$ therefore all eigenvalues of $P|_{E^s}$ are positive definite, hence $V'_\tau \equiv f|_{E^s}^{-1}(\tau)$ is a sphere. We conclude that $S_0^s(\vartheta, V) = V'_\tau \cap \operatorname{cl}(W_0^s(\vartheta, V))$ is a closed manifold. For sufficiently small τ the manifold $S_0^s(\vartheta, V)$ is nonempty, and its dimension is equal to $\nu(L) - 1$, where $\nu(L)$ is the index of L.

Proposition 5.4.8. *Let $\vartheta \in \mathfrak{X}^r(\mathbb{R}^n)$, $r \geq 1$, and 0 is a hyperbolic singular point of ϑ. Let U, V be the sets as in Proposition 5.4.7. Suppose there is a Morse function $f : U \to \mathbb{R}$ with*

$$\vartheta(f)(x) \geq \kappa\|x\|^2 \text{ for all } x \in U,$$

there is a regular value τ and there is a neighborhood V of 0 so that the intersection $S_0^s(\vartheta, V) \equiv f^{-1}(\tau) \pitchfork W_0^s(\vartheta, V)$ is nonempty closed manifold. Then for any open neighborhood $\mathcal{N}$ of $S_0^s(\vartheta, V)$ in $f^{-1}(\tau) \cap V$ there is $\delta > 0$ such that

$$\bigcup_{\xi \in B_\delta(\vartheta, U)} S_0^s(\xi, V)) \subset \mathcal{N}.$$

Proof. Define a map

$$\beta : (V \cap E^s) \times B_\delta(\vartheta, U) \to E^s \times E^u \text{ by } \beta(x, \xi) = (x, \beta^s(x, \xi)).$$

Without loss of generality we may assume that the given neighborhood $\mathcal{N}$ has compact closure. Pick a tubular neighborhood B of $S_0^s(\vartheta, V)$ in V such that $\mathcal{N}' \equiv B \cap f^{-1}(\tau) \subset \mathcal{N}$. Define the following set

$$K = \mathrm{cl}(P^s\mathcal{N}') \subset E^s.$$

The set K is compact. For any $x \in K$ there is a neighborhood V_x of x and $\delta_x > 0$ such that $\beta(V_x, B_{\delta_x}(\vartheta)) \subset B$. Since the set K is compact there is a finite number of $\{V_{x_i}\}_{i \in \{1,\ldots,N\}}$ covering K.

If $\delta = \min\{\delta_{x_i} | \; i = 1, ..., N\}$ then $\beta(K, B_\delta(\vartheta, U)) \subset B$. We observe that

$$\begin{aligned}
\bigcup_{\xi \in B_\delta(\vartheta, U)} S_0^s(\xi, V)) &= \bigcup_{\xi \in B_\delta(\vartheta, U)} S_0^s(\xi, V) \cap \mathcal{N}' \\
&= \bigcup_{\xi \in B_\delta(\vartheta, U)} f^{-1}(\tau) \cap W_0^s(\xi, V) \cap \mathcal{N}' \\
&= \bigcup_{\xi \in B_\delta(\vartheta, U)} f^{-1}(\tau) \cap \beta(P^s\mathcal{N}', \xi) \\
&\subset f^{-1}(\tau) \cap \beta(P^s\mathcal{N}', B_\delta(\vartheta, U)) \\
&\subset f^{-1}(\tau) \cap \beta(K, B_\delta(\vartheta, U)).
\end{aligned}$$

Thus

$$\bigcup_{\xi \in B_\delta(\vartheta, U)} S_0^s(\xi, V)) \subset f^{-1}(\tau) \cap \beta(K, B_\delta(\vartheta, U)) \subset f^{-1}(\tau) \cap B = \mathcal{N}' \subset \mathcal{N}.$$

□

We remark that the similar results can be formulated for the unstable manifolds.

5.5 Structural Stability

We shall define Morse-Smale vector fields and see that they are structurally stable. A Morse-Smale vector field has a finite number of hyperbolic singular points and a finite number of closed orbits that are hyperbolic. Furthermore, all the stable and the unstable manifolds intersect transversally. We do not focus the attention on the closed orbits, since they do not give rise to the partial order discussed in the Introduction. We refer instead to Ch. 3 in Palis and de Melo [1982] for details. In the next section we shall introduce the primary object of our study: an essential gradient-like vector field, which is a Morse-Smale vector field that does not have any closed orbits.

Definition 5.5.1 (Sec. 3.1 of Palis and de Melo [1982]). *Let γ be a closed orbit of a vector field $\xi \in \mathfrak{X}^r(M)$ and $x \in \gamma$. Let Σ be a section transversal to ξ through the point x. We say γ is a hyperbolic closed orbit of ξ if p is a hyperbolic fixed point of the Poincaré map $P : V \to \Sigma$, where V is an open neighborhood of x and P is a diffeomorphism onto its image.*

Definition 5.5.2 (α- and ω-limit sets). *If $\xi \in \mathfrak{X}^r(M)$ and $x \in M$, then the α- and ω-limit sets for ξ are*

$$\begin{aligned} \alpha(x) &= \bigcap_{\tau \leq 0} \bigcup_{t \leq \tau} \phi_t^\xi(x) \ \textit{ and} \\ \omega(x) &= \bigcap_{\tau \geq 0} \bigcup_{t \geq \tau} \phi_t^\xi(x). \end{aligned}$$

If γ is a hyperbolic closed orbit of a vector field $\xi \in \mathfrak{X}^r(M)$ we define stable and unstable manifolds of γ by

$$\begin{aligned} W^s_\gamma(\xi) &= \{x \in M| \, \omega(y) = \gamma\} \\ W^u_\gamma(\xi) &= \{x \in M| \, \alpha(y) = \gamma\}. \end{aligned}$$

The sets $W^s_\gamma(\xi)$ and $W^u_\gamma(\xi)$ are immersed manifolds of M of class C^r, cf. Proposition 3.1.5 and the following Corollary in Palis and de Melo [1982].

Definition 5.5.3 (Sec. 4.1 in Palis and de Melo [1982]). *Let $\xi \in \mathfrak{X}^r(M)$. We say that $p \in M$ is a wandering point for ξ if there exists a neighborhood V of p and a number t_0 such that $\phi^\xi_t(V) \cap V = \emptyset$ for $|t| > t_0$. Otherwise we say that p is nonwandering.*

The set of nonwandering points of ξ will be denoted by $\Omega(\xi)$.

Definition 5.5.4 (Morse-Smale Vector Field, Sec. 4.1, Palis and de Melo [1982]). *A vector field $\xi \in \mathfrak{X}^r(M)$ will be called Morse-Smale provided it satisfies the following five conditions:*

1. *ξ has a finite number of singular points, say $\beta_1, ..., \beta_k$, each hyperbolic,*
2. *ξ has a finite number of closed orbits (periodic solutions), say $\beta_{k+1}, ..., \beta_N$, each hyperbolic;*
3. *For any $x \in M,\ \alpha(x) = \beta_i$ and $\omega(x) = \beta_j$ for some i and j;*
4. *$\Omega(\xi) = \{\beta_1, ..., \beta_N\}$;*
5. *The stable and unstable manifolds associated with the β_i have transversal intersection.*

The set of all Morse-Smale vector fields on M is denoted by $\mathfrak{S}^r(M)$.

The sets $\beta_1, ..., \beta_N$ will be called the singular elements of the vector field ξ. The set of the singular elements of ξ will be denoted by $\mathcal{C}r(\xi)$. As for Morse-Smale functions we can define a partial order relation on the singular elements of a Morse-Smale vector field.

Proposition 5.5.5 (Smale [1960]). *Let ξ be a Morse-Smale vector field on M. Let $\beta_i \succ \beta_j$ mean that there is a trajectory not equal to β_i nor β_j whose α-limit set is β_i and whose ω-limit set is β_j. Then $\succ$ satisfies:*

1. *It is never true that $\beta_j \succ \beta_i$;*
2. *If $\beta_i \succ \beta_j$ and $\beta_j \succ \beta_l$ then $\beta_i \succ \beta_l$;*
3. *If $\beta_i \succ \beta_j$ then $\dim(W^u_{\beta_i}) \geq \dim(W^u_{\beta_j})$ and equality can only happen if β_j is a closed orbit.*

We use this proposition to define a partial order relation on the singular elements: $\beta_i \succeq \beta_j$ if and only if $\beta_i = \beta_j$ or $\beta_i \succ \beta_j$.

We shall conclude the section by stating the results on structural stability of Morse-Smale vector fields.

Definition 5.5.6. *Two vector fields $\xi, \eta \in \mathfrak{X}^r(M)$ are topologically equivalent if there exists a homeomorphism $h : M \to M$ such that*

1. *$h \circ \Phi^\xi(\mathbb{R}, x) = \Phi^\eta(\mathbb{R}, h(x))$ for each $x \in M$,*
2. *h preserves the orientation, that is if $x \in M$ and $\delta > 0$ there exists $\epsilon > 0$ such that, for $0 < t < \delta$, $h \circ \Phi^\xi(x, t) = \Phi^\eta(h(x), \tau)$ for some $0 < \tau < \epsilon$.*

We say that h is a topological equivalence, and use a notation $\xi \sim \eta$ to denote that ξ and η are topologically equivalent.

The first condition of the definition states that the homeomorphism h takes orbits into orbits. The second states that a stable manifold of ξ goes to a stable manifold of η. Specifically, for a pair of topologically equivalent vector fields ξ and η via a homeomorphism $h : M \to M$ and $p \in \mathcal{C}r(\xi)$ we have $W^s_\xi(p) = h(W^s_\eta(h(p)))$.

We will be interested in behavior of a vector field whose orbits do not change qualitatively under small perturbations.

Definition 5.5.7. *A vector field $\xi \in \mathfrak{X}^r(M)$ is structurally stable if there exists an open neighborhood U of ξ in $\mathfrak{X}^r(M)$ such that every $\eta \in U$ is topologically equivalent to ξ.*

We have the following result.

Theorem 5.5.8 (Theorem 4.1, Palis and Smale [1970]). *If $\xi \in \mathfrak{X}^r(M)$ is a Morse-Smale vector field then X is structurally stable.*

If the dimension of the manifold M is 2 then the subset consisting of Morse-Smale vector fields, $\mathfrak{S}^1(M)$, is dense in $\mathfrak{X}^1(M)$, and if in addition M is orientable the set $\mathfrak{S}^r(M)$ is dense in $\mathfrak{X}^r(M)$ for $r \geq 1$, c.f. Palis and de Melo [1982], Ch.4.

5.6 Gradient-like Vector Fields

Slightly confusingly the literature provides two definitions of a gradient-like vector field.

Definition 5.6.1 (Essential Gradient-like Vector Field). *A Morse-Smale vector field $\xi \in \mathfrak{S}^r(M)$ for which the only singular elements are singular points (there are no closed orbits) is called an essential gradient-like vector field. We denote the set of all essential gradient-like vector fields by $\mathfrak{E}^r(M)$.*

Another definition of a gradient-like vector field was introduced by Smale in Smale [1961].

Definition 5.6.2 (Gradient-like Vector Field). *A C^∞ vector field X on a smooth compact manifold M^n (with or without boundary) is called gradient-like if it satisfies the following conditions:*

1. *At each singular point $p_i,\ i = 1, ..., N$, of X, there is an open neighborhood U_i and a C^∞ function f_i on U_i such that X is the gradient of f_i in some Riemannian structure on U_i. Furthermore p_i is a non-degenerated critical point of f_i.*
2. *If $x \in \partial M$, X is transversal at x (X has no singular points on ∂M).*
3. *The set $\Omega(X)$ of nonwandering points of X is equal to $\{p_1, ..., p_N\}$.*
4. *The stable and unstable manifolds of the singularities $p_i,\ p_j,\ i \neq j$, intersect transversally.*

It is not true that all singularities of Morse-Smale flows are of standard form, the first condition of Definition 5.6.2, therefore the two definitions of a gradient-like vector field are not equivalent. In fact, the gradient-like vector field of Definition 5.6.2 is a gradient vector field of a Morse-Smale function if we are allowed to change the Riemannian metric on M.

Theorem 5.6.3 (Theorem B in Smale [1961]). *Let X be a gradient-like vector field on M, V_1 be the connected component of ∂M at which X is oriented in, and V_2 the connected component of ∂M at which X is oriented out. Then there is a C^∞-function f on M, which has the following properties:*

1. *The critical points of f coincide with the singularities of X. For $i = 1, ..., N$, f coincides with the function f_i of Condition (1) in Definition 5.6.2 plus a constant in some some neighbourhood of p_i.*

2. *If X is nonzero at $x \in M$, then it is transversal to the level hypersurface of f at x.*

3. *If p_i is a critical point $f(p_i) = \lambda_{p_i}$, where λ_{p_i} is the index of X at p_i.*

4. *The function f has value $-\frac{1}{2}$ on V_1 and $n + \frac{1}{2}$ on V_2.*

The theorem shows that a gradient-like vector field is the gradient of a function in some Riemannian structure.

Corollary 5.6.4. *There is a Riemannian metric on M such that $\nabla f = X$.*

Proof. Away from the singular points of X construct a Riemannian metric g such that $\nabla_g f = X$. Since M is compact it is enough to show that for each $p \in M$ there exist an open neighborhood U of p and a Riemannian metric g_0 on U such that $\nabla_{g_0} f = X$. There are local coordinates in U such that the vector field X is written $X = \frac{\partial}{\partial x_1}$, and $df = v_1 dx_1 + ... + v_n dx_n$. Since $df(X) \neq 0$ without loss of generalization we may assume that $df(X) = v_1 = 1$. Let

$$\mathcal{S} = \{A \in GL(n, \mathbb{R}) |\ A = A^{\mathrm{T}}\}$$

and pick $a > 0$; consider the smooth map

$$\phi : U \to \mathcal{S},\ q \mapsto A(q) = \begin{bmatrix} 1 & v_2(q) & v_3(q) & \dots & v_n(q) \\ v_2(q) & a & 0 & \dots & 0 \\ v_3(q) & 0 & a & \dots & 0 \\ \dots & & & & \\ v_n(q) & 0 & 0 & \dots & a \end{bmatrix}.$$

If $a > \max_{q \in \mathrm{cl}(U)} \sum_{i=2}^{n} v_i^2(q)$ then $A(q)$ is positive definite for each $q \in U$, and the map ϕ defines the desired Riemannian metric g_0 on U.

As in the proof of Lemma 4.1.1 use a smooth partition of unity to extend the Riemannian metric to the whole M, such that it coincides with the Riemannian structure in Condition (1) of Definition 5.6.2 on a neighborhood of each critical point. □

The next proposition shows that an essential gradient-like vector field can be connected by a curve in $\mathfrak{X}^r(M)$ with a gradient-like vector field.

Proposition 5.6.5 (Lemma 2 in Newhouse and Peixoto [1976]). *Let M be a compact smooth manifold and X be an essential gradient-like vector field on M. For $k \geq 1$ and $r \geq 2$ there is a curve $\sigma \in C^k(I, \mathfrak{X}^r(M))$ with $\sigma(0) = X$, σ_t is a Morse Smale vector field for $t \in I$, and $\sigma_1 = \nabla_g f$ for some Morse function f and some Riemannian metric g on M.*

Using compactness of I and structural stability of the Morse-Smale vector fields we see that the vector fields σ_1 and $X = \sigma_0$ are topologically equivalent. This remark can be used to translate the result on the gradient vector fields of Morse-Smale functions to essential gradient-like vector fields.

For $a,\ b \in \mathcal{C}r(\xi)$ we shall denote $W(a, b; \xi) = W_a^u(\xi) \cap W_a^s(\xi)$.

Proposition 5.6.6. *Let a, b be singularities of a gradient-like vector field ξ. Then* $\mathrm{cl}(W(a, b; \xi))$ *is compact and* $\mathrm{cl}(W(a, b; \xi)) = \bigcup_{a \succeq a' \succeq b' \succeq b} W(a', b'; \xi)$.

Proof. By Proposition 5.6.5, for any essential gradient-like vector field η there is a Morse function $f : M \to \mathbb{R}^n$ and a Riemannian metric g on M making f Morse-Smale such that η and $\nabla_g f$ are topologically equivalent. That is, there exists a homeomorphism $h : M \to M$ taking orbits of η into orbits of $\nabla_g f$ preserving their orientation. Thus for any $p, q \in \mathcal{C}r(\eta)$ we have that

$$h(W(p, q; \eta)) = W(h(p), h(q); \nabla_g f)).$$

By Corollary 6.28 in Banyaga and Hurtubise [2004] the proposition is already true for $\xi = \nabla_g f$. Therefore it is also true for the essential gradient-like vector field η.
$\square$

5.7 Lyapunov Functions

This section is based on Meyer [1968]. We consider a closed smooth manifold M and a smooth function $f : M \to \mathbb{R}$ with the set of critical points denoted as usual by $\mathcal{C}r(f)$. Let Δ_i denote the set of points in $\mathcal{C}r(f)$, where the Hessian of f has nullity i.

Definition 5.7.1 (Lyapunov function for M, Meyer [1968]). *A smooth function $f : M \to \mathbb{R}$ will be called Lyapunov function for M provided the following conditions are satisfied:*

1. $\mathcal{C}r(f) = \Delta_0 \cup \Delta_1$*;*
2. Δ_1 *is the disjoint union of a finite number of embedded circles in M, say $\delta_1, ..., \delta_l$, such that the index of f is constant on each circle;*
3. *For $i = 1, ..., l$ there exists a neighborhood V_i of δ_i and a diffeomorphism ψ_i such that ψ_i maps V_i into the product of D^{n-1} and S^1 if V_i is orientable or into twisted product of D^{n-1} and S^1 if V_i is nonorientable with the property that $f \circ \psi_i^{-1}(x) = f(\delta_i) + Q(x)$, where Q is a nonsingular quadratic form in $x_1, ..., x_{n-1}$, the coordinates in D^{n-1}, and it is periodic in x_n, the coordinate on S^1.*

Notice also that if $\delta_{l+1}, ..., \delta_N \in \Delta_0$ then by the Morse Lemma, cf. Lemma 2.2 in Milnor [1973], there is a family of coordinate systems $\{(V_i, \psi_i)\}_{i \in \{l+1,...,N\}}$ such that

$$f \circ \psi_i^{-1}(y) = f(\delta_i) + Q(y),$$

where Q is a nonsingular quadratic form in x whose index is equal to the index of f.

Definition 5.7.2 (Lyapunov function for a vector field, Meyer [1968]). *Let $X \in \mathfrak{X}^\infty(M)$. Then a Lyapunov function f for M will be called a Lyapunov function for ξ provided*

1. *$\xi(f)(x) < 0$ for all $x \in M - \mathcal{C}r(f)$;*
2. *If p is a singular point of ξ then $p \notin \Delta_1$;*
3. *There exists a real number $\kappa > 0$ such that on each V_i, $i = 1, ..., N$, we have*
$$-\xi(f)(x) \geq \kappa d(y, \delta_i)^2 \text{ for } y \in V_i,$$
where d is the distance induced by some Riemannian metrics g on M.

The next theorem shows that Morse-Smale vector fields admit Lyapunov functions. This result will be extensively used in the proof of the Central Vector Field Theorem in Ch. 7.

Theorem 5.7.3 (Meyer [1968]). *If $\xi \in \mathfrak{X}^\infty(M)$ is Morse-Smale then there exists a Lyapunov function for ξ. Furthermore the Lyapunov function f can be chosen in the following way. Let $\{p_i\}_{i\in\{1,...,l\}}$ be the sequence of singular points of ξ and $\{r_i\}_{i\in\{1,...,l\}}$ be a sequence of real numbers so that if $p_i \succ p_j$ then $r_i > r_j$. Then f can be chose such that $f(p_i) = r_i$.*

Proof (sketch of). We shall only give an outline of the proof for a gradient-like vector field. For details we refer to Meyer [1968]. Suppose $\{p_1, ..., p_k\} = \Delta_0 = \mathcal{C}r(\xi)$. By Proposition 5.5.5 we can find k real numbers r_i such that $r_i > r_j$ whenever $p_i \succ p_j$. First step is to define the function f on the p_i by $f(p_i) = r_i$. The next step is to define f in a neighborhood of each singular point p_i. Using a local coordinate chart (V_i, ψ_i) of p_i, the vector field ξ has the form

$$\hat{\xi} = d\psi_i \xi \circ \psi_i^{-1} = L_i + \eta_i,$$

where L_i hyperbolic and $\eta_i(0) = d\eta_i(0) = 0$. By Theorem 5.3.7, there are nonsingular symmetric matrices P_i and Q_i with Q_i positive definite such that

$$L_i^{\mathrm{T}} P_i + P_i L_i = -Q_i$$

and $\pi(L_i) = \nu(P_i)$, $\pi(P_i) = \nu(L_i)$.

We define f in V_i by $f(x) = r_i + x^{\mathrm{T}}Qx$ then by Proposition 5.3.9 there exists a real number $\kappa_i > 0$ such that in a neighborhood $U_i \subset V_i$ of 0 we have

$$-\xi(f)(x) \geq \kappa d(y, p_i)^2.$$

We may assume that all the U_i are sufficiently small so that they do not overlap. In conclusion we have defined Lyapunov function for ξ in open neighborhoods of the singular points of ξ. The extension of this function to M can be accomplished by the same procedure as in the proof of Theorem B in Smale [1961]. □

The theorem above has a partial converse.

Proposition 5.7.4 (Meyer [1968]). *Let $\xi \in \mathfrak{X}^\infty(M)$. If there exists a Lyapunov function for ξ then ξ satisfies the conditions 1), 2), 3) and 4) in Definition 5.5.4 of a Morse-Smale vector field. Moreover, the vector field ξ can be approximated arbitrary closely in the C^r topology by a Morse-Smale vector field.*

Corollary 5.7.5 (Lyapunov function Meyer [1968]). *If M is compact and two dimensional then a necessary and sufficient condition for ξ to be structurally stable is the existence of a Lyapunov function for ξ.*

6 Section Cones

We formulate the definition of a section cone. We keep the conventions from the previous chapter: $M = M^n$ is a closed smooth manifold, $\mathfrak{S}^r(M)$ denotes the set of Morse-Smale C^r vector fields on M, $\mathfrak{E}^r(M)$ stands for the set of essential gradient-like C^r vector fields on M. All maps are continuous. By a path on a topological space Y we mean a map $I \to Y$.

A section cone is a convex subset of $\mathcal{K}$ characterized by the property that if p is a singular point for some vector field in $\mathcal{K}$ then this is the case for all members of $\mathcal{K}$. A section cone induces a di-path. A di-path is a curve which is a finite concatenation of integral arcs of the vector fields within the section cone. We define a relation $\succeq_{\mathcal{K}}$ on M by $p \succeq_{\mathcal{K}} q$ if and only if there is a di-path from p to q. We ask the question whether this relation is a partial order relation. For this we define a Lyapunov section cone. It is defined by the property, that there is a single real function that is a Lyapunov function for all vector fields in this section cone. We show that a Lyapunov section cone gives rise to a relation $\succeq_{\mathcal{K}}$ satisfying the properties of a partial order relation. An important feature of a section cone $\mathcal{K}$ is that there exists a path in $\mathcal{K}$ joining any two vector fields $\xi, \eta \in \mathcal{K}$. We introduce a notion of a stable and unstable manifold for a path $\sigma \in C^r(I, K)$. If $\mathcal{K}$ is a subset of $\mathfrak{E}^r(M)$ then all the stable and the unstable manifolds of an element $\sigma \in C^r(I, K)$ intersect transversally.

6.1 Construction of a Section Cone

We use the notation $\mathbb{R}^+ = \{x \in \mathbb{R} |\ x \geq 0\}$ and $\mathbb{R}^+_* = \mathbb{R}_+ - \{0\}$. Recall that the set of singularities of a vector field ξ is denoted by $\mathcal{C}r(\xi) = \{p \in M |\ \xi(p) = 0\}$.

Definition 6.1.1 (Section Cone). *Let M be a smooth manifold. A C^r section cone $\mathcal{K}$ on M is a subset of $\mathfrak{X}^r(M)$ satisfying the following two conditions:*

1. *For every pair $\xi, \eta \in \mathcal{K}$, if $p \in \mathcal{C}r(\xi)$ then $p \in \mathcal{C}r(\eta)$.*
2. *If ξ and η are in $\mathcal{K}$ and $a, b \in \mathbb{R}^+_*$ then $a\xi + b\eta \in \mathcal{K}$.*

We shall denote the set of all section cones on a manifold M by $\mathcal{D}(M)$.

The first condition says that all vector fields in a section cone have the same singularities. Also if the zero section 0_M is in $\mathcal{K}$ then $\mathcal{K} = 0_M$. The second condition imposes convexity on the subset $\mathcal{K}$. Particularly, if $\xi \in \mathcal{K}$ then $a\xi \in \mathcal{K}$ for $a > 0$.

The condition 1. allows to speak about singular points of a section cone.

Definition 6.1.2. *A point p is a singular point of a section cone $\mathcal{K}$ if $p \in \mathcal{C}r(\xi)$ for some, thus for all, $\xi \in \mathcal{K}$. We denote the set of singular points of $\mathcal{K}$ by $\mathcal{C}r(\mathcal{K})$.*

Note that a section cone without singular points can only be constructed on a manifold with zero Euler characteristic.

We shall use the notation $\mathcal{K}(p) \equiv \{s(p)|\ s \in \mathcal{K}\} \subset T_p(M)$. In particular, $p \in \mathrm{C}r(\mathcal{K})$ if and only if $\mathcal{K}(p) = \{0\}$.

Proposition 6.1.3. *Let $\mathcal{K}$ be a section cone. If $\xi, \eta \in \mathcal{K}$ and $\xi(p) = -\eta(p)$ for some $p \in M$ then $p \in \mathcal{C}r(\xi)$. As a consequence, for each $x \in M$*

$$\mathcal{K}(x) \cup \{0\} \in \mathcal{D}(T_x(M)).$$

Proof. Since $\xi, \eta \in \mathcal{K}$, $\vartheta = \xi + \eta \in \mathcal{K}$. In particular

$$\vartheta(p) = \xi(p) + \eta(p) = -\eta(p) + \eta(p) = 0.$$

We conclude that $p \in \mathcal{C}r(\vartheta)$. By condition 1. the point p is also a singular point of ξ. $\square$

To suppress the notation, the set of all cones without the tip (without the point 0) in a vector space V is denoted by $\breve{\mathcal{D}}(V)$. In other words $K \in \breve{\mathcal{D}}(V)$ if $K \cup \{0\} \in \mathcal{D}(V)$.

Since $\mathfrak{X}^r(M)$ has a structure of a real vector space we may consider a cone in $\mathcal{D}(\mathfrak{X}^r(M))$. Proposition 6.1.3 implies that if $\mathcal{K}$ is a section cone then $\mathcal{K} \in \breve{\mathcal{D}}(\mathfrak{X}^r(M))$, that is $\mathcal{K} \cup \{0_M\} \in \mathcal{D}(\mathfrak{X}^r(M))$, where 0_M is the zero section in the tangent bundle.

Definition 6.1.4. *A C^r section cone $\mathcal{K}$, $r \geq 1$, on a closed smooth manifold M is Morse-Smale if and only if $\mathcal{K} \subset \mathfrak{S}^r(M)$. It is gradient-like if and only if the section cone $\mathcal{K} \subset \mathfrak{E}^r(M)$.*

Proposition 6.1.5. *Let $\mathcal{K}$ be a Morse-Smale section cone. If there is an essential gradient-like vector field $\xi \in \mathcal{K}$, then $\mathcal{K}$ is gradient-like, i.e. every vector field in $\mathcal{K}$ is an essential gradient-like vector field.*

Proof. For any vector field $\eta \in \mathcal{K}$ we define $\sigma_\eta : I \to \mathfrak{X}^r(M)$ by $t \mapsto t\eta + (1-t)\xi$. By the second condition of Definition 6.1.1, for all $t \in I$, $\sigma_\eta(t) \in \mathcal{K} \subset \mathfrak{S}^r(M)$. Since the Morse-Smale vector fields are structurally stable and the unit interval is compact, all the vector fields on the path σ are topologically equivalent. In particular ξ and η are topologically equivalent, hence η is essential gradient-like. $\square$

If $\mathcal{K}$ is a Morse-Smale section cone then the singular points are isolated. Moreover, any two $\xi, \eta \in \mathcal{K}$ are topologically equivalent, i.e. there exists a homeomorphism $h : M \to M$ taking orbits of ξ to orbits of η, cf. Section 5.5, and the restriction $h|_{\mathcal{C}r(\xi)}$ is a permutation. Another consequence is that the indices of ξ and η at the same singular point p are the same. This will be shown in Proposition 6.1.6. Therefore it makes sense to define the index of $\mathcal{K}$ at a singular point p as the index of some (thus all) vector field in $\mathcal{K}$ at p.

Define a set $\mathfrak{X}^r_p(M) = \{\xi \in \mathfrak{X}^r(M) |\ \xi(p) = 0$ and p is hyperbolic$\}$. The index function on $\mathfrak{X}^r_p(M)$ is

$$\mathrm{Ind}_p : \mathfrak{X}^r_p(M) \to \mathbb{N},\ \xi \mapsto \mathrm{index}_\xi(p),$$

where $\mathrm{index}_\xi(p)$ is the index of the vector field ξ at the singular point p. Recall that the index of a vector field ξ at a singular point p is the index of the linear map $L = d\psi\xi_0$ (the number of eigenvalues of L with negative real parts) for some thus any coordinate chart (U, ψ) with $p \in U$.

Proposition 6.1.6. *Suppose $\mathcal{K}$ is a Morse-Smale section cone and $p \in \mathcal{C}r(\mathcal{K})$. The index function on $\mathcal{K}$ at p, $\mathrm{Ind}^{\mathcal{K}}_p \equiv \mathrm{Ind}_p|_{\mathcal{K}}$, is continuous and thus constant.*

In the proof we make use of the following proposition.

Proposition 6.1.7 (Proposition 2.2.18, Palis and de Melo [1982]). *The eigenvalues of an operator $L \in \mathcal{L}(\mathbb{R}^n)$ depend continuously on L.*

Proof (of Proposition 6.1.6). We shall denote the map in Proposition 6.1.7 by θ

$$\theta : \mathcal{L}(\mathbb{R}^n) \to \mathbb{C}^n/S_n,\ L \mapsto [(\lambda_1, ..., \lambda_n)],$$

where S_n is the symmetric group of degree n, and $\lambda_1, ..., \lambda_n$ are eigenvalues of L, possibly with multiplicities.

Let $(\psi, V))$ be a coordinate neighborhood of the point p with $U \equiv \psi(V)$. Consider the set $\mathfrak{X}^r(U)$ of C^r vector fields on U, that is C^r maps $U \to \mathbb{R}^n$. Define the composition $\kappa = \theta \circ \mathrm{e_p} \circ d$, where $d : \mathfrak{X}^r(U) \to C^r(U, \mathcal{L}(\mathbb{R}^n))$ is the derivative, $\mathrm{e} : U \times C^r(U, \mathcal{L}(\mathbb{R}^n)) \to \mathcal{L}(\mathbb{R}^n)$ is the evaluation map and $e_p(\cdot) = e(p, \cdot)$. By Proposition 6.1.7, θ is continuous, so is the composition κ. We denote by $\varrho : (\mathbb{C} - i\mathbb{R})^n/S_n \to \mathbb{N}$ the map assigning the number of complex numbers with negative real part in the n-tuple of complex numbers. The map ϱ is a continuous discrete valued map. The local representation of Ind_p is the composition $\mathrm{Ind}_p = \varrho \circ \kappa$. Thus we conclude that Ind_p is continuous.

A topological space is connected if and only if every discrete valued map defined on it is constant. The space $\mathfrak{X}^r_p(M)$ has $n+1$ connected component corresponding to index 0 to n. On the other hand any pair $\xi,\ \eta \in \mathcal{K}$ can be connected by a path in $\mathcal{K}$. Hence $\mathcal{K}$ is a subset of one and only one connected component of $\mathfrak{X}^r_p(M)$. Therefore $\mathrm{Ind}_p^{\mathcal{K}}$ is constant. $\square$

Proposition 6.1.6 says that the index of a Morse-Smale section cone at a singular point $p \in \mathcal{C}r(\mathcal{K})$, agreeing with the index of one of its vector fields, is well-defined.

Definition 6.1.8. *Suppose $\mathcal{K}$ is a Morse-Smale section cone and p is a singular point. Then the index of $\mathcal{K}$, $\mathrm{index}_{\mathcal{K}}(p)$ = $\mathrm{index}_{\xi}(p)$, for some (thus all) $\xi \in \mathcal{K}$.*

The objective of this chapter is to introduce a section cone which induces a partial order relation on M. The candidates are those section cones which do not allow closed orbits. For this we define a Lyapunov section cone.

Definition 6.1.9. *A C^r section cone $\mathcal{K}$, $r \geq 1$, on a smooth manifold M is Lyapunov if and only if there exists a C^r Morse function $f : M \to \mathbb{R}$ and a Riemannian metric on M such that for any $\xi \in \mathcal{K}$ we have*

1. *$\xi(f)(x) < 0$ for all $x \in M - \mathrm{C}r(\mathcal{K})$,*
2. *there exist a constant $\kappa > 0$ and open neighborhoods $\{U_q\}_{q \in \mathrm{C}r(\mathcal{K})}$ of the singular points such that*

   $$-\xi(f)(x) \geq \kappa d(x,p)^2 \text{ for } p \in U_p, \text{ where } d \text{ is the Riemannian distance.}$$

Proposition 6.1.10. *If M is two dimensional compact manifold then any Lyapunov section cone on M is a Morse-Smale section cone.*

Proof. Corollary 5.7.5 says that the vector fields in a Lyapunov cone are structurally stable. By Peixoto's Theorem, cf. Peixoto [1962], a vector field on a compact 2-manifold is C^r structurally stable if it is Morse-Smale. □

In general for dimension greater than 2 the above corollary is not valid. If $\mathcal{K}$ is a Lyapunov cone and $\xi \in \mathcal{K}$, then by Proposition 5.7.4, ξ satisfies conditions 1) to 4) of Definition 5.5.4 of a Morse-Smale vector field. In particular all singular points of $\mathcal{K}$ are hyperbolic.

Definition 6.1.11. *A Lyapunov-Smale section cone is a Lyapunov section cone which is Morse-Smale.*

In particular a Lyapunov-Smale section cone is a gradient-like section cone.

Proposition 6.1.12. *Suppose $\mathcal{K}$ is a Lyapunov section cone. For each singular point p, we have*

$$\left(\bigcup_{X \in \mathcal{K}} W_p^s(X)\right) \cap \left(\bigcup_{X \in \mathcal{K}} W_p^u(X)\right) = \{p\}$$

Proof. Suppose η and ξ are vector fields in $\mathcal{K}$ and there exists $q \in W_p^s(\xi) \cap W_p^u(\eta)$. Since the stable and unstable manifolds are invariant sets

$$\forall t \in \mathbb{R},\ \phi_t^{\xi}(q), \phi_t^{\eta}(q) \in W_p^s(\xi) \cap W_p^u(\eta). \tag{6.1}$$

On the other hand the section cone $\mathcal{K}$ is Lyapunov, hence $\eta(f)(x),\ \xi(f)(x) < 0$ for $x \in M - \mathrm{C}r(\mathcal{K})$, or in other words the function f is nonincreasing along the trajectories of ξ and η. Then

$$f(q) \leq f(p) \leq f(q).$$

Hence $f(p) = f(q)$ and because of (6.1) $p = q$. $\square$

Definition 6.1.13. *A gradient section cone $\mathcal{K}$ is a Morse-Smale section cone, which satisfies the following: There is a Riemannian metric g and a Morse-Smale function f such that $\nabla_g f \in \mathcal{K}$.*

Due to Proposition 6.1.5 a gradient section cone is gradient-like.

Definition 6.1.14. *A section cone $\mathcal{K}$ on M is reproducing if the cone $\mathrm{cl}(\mathcal{K}(x))$ is reproducing for all $x \in (M - \mathcal{C}r(\mathcal{K}))$.*

Next we shall give some examples of sections cones on M and cones in $\mathfrak{X}^r(M)$, $1 \leq r < \infty$.

Example 6.1.15. Let g be a Riemannian metric on M. We pick $\eta \in \mathfrak{X}^r(M)$ and define the set $\mathcal{K}(\eta) \subset \mathfrak{X}^r(M)$ by

$$\mathcal{K}(\eta) = \{\alpha(\eta + \xi) \in \mathfrak{X}^r(M) |\ \xi \in \mathfrak{X}^r(M), \alpha > 0,\ g(\xi, \eta) = 0, g(\eta, \eta) \geq g(\xi, \xi)\}.$$

Note that for $\eta + \xi \in \mathcal{K}(\eta)$ we have $\eta(p) = 0$ for some $p \in M$ if and only if $(\eta + \xi)(p) = 0$. Furthermore, if $\vartheta_i = \alpha_i(\eta + \xi_i) \in \mathcal{K}(\eta)$ for $\alpha_i > 0$ and $i \in \{1, 2\}$ then

$$\|\alpha_1 \xi_1 + \alpha_2 \xi_2\|^2 \leq (\alpha_1 + \alpha_2)^2 \|\eta\|^2, \ \text{ where } \|\cdot\|^2 \equiv g(\cdot, \cdot).$$

Hence $\vartheta_1 + \vartheta_2 = (\alpha_1 + \alpha_2)\eta + \alpha_1 \xi_1 + \alpha_2 \xi_2 \in \mathcal{K}(\eta)$, and $\mathcal{K}(\eta)$ is a section cone.

Example 6.1.16. Let $\|\cdot\|_r$ be the norm on $\mathfrak{X}^r(M)$ discussed in Section 5.1. For a pair of real numbers $0 < \delta < \ \epsilon$ suppose that $\eta \in \mathfrak{E}^r(M)$ with $||\eta||_r > \epsilon$ and consider an open ball $\mathcal{B}(\eta, \delta)$ centered at η and radius δ

$$\mathcal{B}(\eta, \delta) = \{\xi \in \mathfrak{X}^r(M) |\ ||\eta - \xi||_r < \delta\}.$$

Define the set

$$\mathcal{C}(\eta, \delta) = \{a\xi \in \mathfrak{X}^r(M) |\ \xi \in \mathcal{B}(\eta, \delta), a \geq 0\}.$$

We shall show that $\mathcal{C}(\eta, \delta)$ is a cone in $\mathfrak{X}^r(M)$. If ξ and $-\xi$ are in $\mathcal{C}(\eta, \delta)$ then for some $a, a' \in \mathbb{R}^+_*$ we have that $a\xi, -a'\xi \in \mathcal{B}(\eta, \delta)$ and

$$\|\eta - a\xi\|_r < \delta \ \text{ and } \|\eta + a'\xi\|_r < \delta.$$

It follows that

$$\|a'\eta - aa'\xi\|_r < a'\delta \ \text{ and } \|a\eta + aa'\xi\|_r < a\delta,$$

thus

$$(a + a')\|\eta\|_r < (a + a')\delta,$$

and hence $\|\eta\|_r < \delta$, which is a contradiction. By Proposition 2.1.5 we conclude that $\mathcal{C}(\eta, \delta)$ is a cone in $\mathfrak{X}^r(M)$.

Example 6.1.17. Suppose $f : M \to \mathbb{R}$ is a smooth Morse-Smale function on a closed smooth manifold M with Riemannian metric g. We define the subset $\Delta(f) \subset \mathfrak{X}^r(M)$ as follows: a vector field $\xi \in \Delta(f)$ if and only if

1. $\mathcal{C}r(\xi) = \mathcal{C}r(f)$,
2. $\xi(f)(x) < 0$ for all $x \in M - \mathrm{C}r(f)$,
3. there exist a constant $\kappa > 0$ and open neighborhoods $\{U_p\}_{p \in \mathrm{C}r(f)}$ of the singular points such that

$$-\xi(f)(x) \geq \kappa d(x, p)^2 \text{ for } x \in U_p, \text{ where } d \text{ is the Riemannian distance.}$$

Proposition 6.1.18. *The set $\Delta(f)$ is a Lyapunov section cone.*

Proof. The singular points of the vector fields in $\Delta(f)$ coincide. For a pair of vector fields $\xi_1, \xi_2 \in \mathcal{K}$ and functions $a_1, a_2 \in \mathbb{R}^+_*$ we have

$$df(a_1\xi_1 + a_2\xi_2) = a_1 df(\xi_1) + a_2 df(\xi_2).$$

Furthermore, there are open neighborhoods U_p^1 and U_p^2 of $p \in \mathcal{C}r(\mathcal{K})$ and constants $\kappa_1,\ \kappa_2 > 0$ such that

$$-\xi_i(f)(x) > \kappa_i d(x,p) \text{ for } x \in U_p,\ i = 1,2.$$

Choosing an open neighborhood U_p of p with $U_p \subset U_p^1 \cap U_p^2$ and defining

$$\kappa = \min\{a_1\kappa_1, a_2\kappa_2\}$$

gives $-(a_1\xi_1 + a_2\xi_2)(f)(x) > \kappa d(x,p)$ for $p \in U_p$. □

Suppose that the Riemannian metric on M is compatible with the Morse charts for f, cf. Definition 4.1.2, then $-\nabla f \in \Delta(f)$ and hence $\Delta(f)$ is nonempty. This is also true for an arbitrary Riemannian metric.

Lemma 6.1.19. *If f be a Morse function on a closed Riemannian manifold M then $\Delta(f)$ is nonempty.*

Proof. By Lemma 3.2 in Milnor [1965] for every Morse function f on a closed manifold there exists a gradient-like vector field ξ, cf. Definition 5.6.2. Hence $-\xi \in \Delta(f)$. □

Example 6.1.20. Let M be a closed smooth manifold with a Riemannian metric compatible with the Morse charts for the Morse-Smale function f. Suppose $\delta > 0$ is sufficiently small so that $\mathcal{B}(\nabla f, \delta) \subset \mathfrak{E}^r(M)$ and $\delta < \|\nabla f\|_r$. Such a δ exists because Morse-Smale vector fields are structurally stable. Define the set

$$\mathcal{K}(f,\delta) = \mathcal{C}(-\nabla f, \delta) \cap \Delta(f).$$

Since re-scaling leaves the orbits unchanged, $\mathcal{K}(f,\delta) \subset \mathcal{C}(-\nabla f,\delta) \subset \mathfrak{E}^r(M)$. Both $\mathcal{C}(-\nabla f,\delta)$ and $\Delta(f)$ are convex sets thus $\mathcal{K}(f,\delta)$ is a convex set. All singular points of the vector fields in $\mathcal{K}(f,\delta)$ coincide because $\mathcal{K}(f,\delta) \subset \Delta(f)$. We conclude that $\mathcal{K}(f,\delta)$ is Lyapunov-Smale section cone.

Proposition 6.1.21. *Let M be a closed smooth manifold and $f : M \to \mathbb{R}$ be a smooth Morse-Smale function. Suppose a Riemannian metric g on M is compatible with the Morse charts for the Morse-Smale function f. Then for $1 \le r < \infty$ the set $\mathcal{K}(f,\delta)$ with $\delta < \|\nabla f\|_r$ is a reproducing Lyapunov-Smale C^r section cone on M.*

Proof. To prove that for all $q \in M - \mathcal{C}r(f)$ the cone $\mathcal{K}(f, \delta)(q)$ is reproducing, it is enough to show that for each $v \in T_q(M)$ there exists $\xi \in \mathcal{K}(f, \delta)$ such that $v \in \operatorname{span}\{\xi(q) - \nabla f(q)\}$.

Pick a nonzero vector $v \in T_q(M)$. Consider a local coordinate neighborhood (U, ϕ) of q. We use the local trivialization $d\phi : T(U) \to \phi(U) \times \mathbb{R}^n$ to define a constant vector field ϑ_U on $\phi(U)$ by

$$\vartheta_U : x \mapsto (x, d\phi_q(v)).$$

Then $(d\phi)^{-1}(\vartheta_U) \in \mathfrak{X}^r(U)$. Since all critical points of a Morse function are isolated, there exists a smooth bump function $h : M \to \mathbb{R}^+_*$ with compact support such that there are no critical points of f in $\operatorname{supp}(h)$ and $\operatorname{supp}(h) \subset U$. We extend $(d\phi)^{-1}(\vartheta_U)$ to the whole M:

$$\vartheta(x) = \begin{cases} h(x)(d\phi)^{-1}(\vartheta_U(x)) & \text{for} \quad x \in U \\ 0 & \text{for} \quad x \in M - U \end{cases}$$

For a constant $\alpha \in R^+_*$ we define a vector field $\xi_\alpha = -\nabla f + \alpha\vartheta$. It is possible to choose $\hat{\alpha} > 0$ such that

1. $\hat{\alpha}\|\vartheta\|_r < \delta$ and
2. $\hat{\alpha} \min\{g(\nabla f, \vartheta)(x) | \, x \in \operatorname{supp}(h)\} < g(\nabla f, \nabla f)$.

Condition 1. says that $\xi_{\hat{\alpha}} \in \mathcal{K}(f, \delta)$. Condition 2. implies that $\xi_{\hat{\alpha}}(f)(x) < 0$ for all $x \in M - \mathcal{C}r(\mathcal{K}(f, \delta))$ since

$$df \circ \xi_{\hat{\alpha}} = g(\nabla f, \xi_{\hat{\alpha}}) = -g(\nabla f, \nabla f) + \hat{\alpha} g(\nabla f, \vartheta) < 0.$$

We conclude that $\xi_{\hat{\alpha}} \in \Delta(f) \cap \mathcal{K}(-\nabla f, \delta)$ and $\xi_{\hat{\alpha}}(q) + \nabla f(q) = \hat{\alpha} v$. $\square$

The last proposition shows that there are nonempty reproducing gradient section cones.

Corollary 6.1.22. *There exists a reproducing gradient section cone.*

6.2 Partial Orders

A Lyapunov C^r section cone $\mathcal{K}$ on a closed smooth manifold M will be used in this section to define a partial order on M. We keep the notation from Chapter 5 and denote the flow line of a vector field $\xi \in \mathfrak{X}^r(M)$ passing through $x \in M$ by $\phi_x^\xi(t)$, that is

$$\frac{d}{dt}\phi_x^\xi(t) = \xi\left(\phi_x^\xi(t)\right) \text{ with } \phi_x^\xi(0) = x.$$

We start by introducing the notion of an integral arc from a point p to a point q on M. This will be a segment of the flow line ϕ_x^ξ for a $\xi \in \mathfrak{X}^r(M)$ and an $x \in M$ joining p with q. Let $\alpha \in \mathbb{R}_*^+$ and $\xi \in \mathfrak{X}^r(M)$. The orbits of ξ and of $\alpha\xi$ coincide.

Definition 6.2.1. *$\gamma : I \to M$ is an integral arc of a vector field ξ if there exists an $\alpha \in \mathbb{R}_*^+$ and an $x \in M$ such that $\phi_x^{\alpha\xi}(t) = \gamma(t)$ for all $0 < t < 1$.*

The definition allows to re-parameterize flow lines. Given a flow line ϕ_x^ξ for a vector field ξ and an $x \in M$ let $\phi_x^\xi(t_1) = p$ and $\phi_x^\xi(t_2) = q$ with $t_1 < t_2$. We are allowed to re-parameterize the flow ϕ_x^ξ by a function $\beta : \mathbb{R} \to \mathbb{R}$, $\beta = (t_2 - t_1)t + t_1$. Then $\phi_x^{(t_2-t_1)\xi}(t) = \phi_x^\xi(\beta(t))$ with $\phi_x^{(t_2-t_1)\xi}(0) = p$ and $\phi_x^{(t_2-t_1)\xi}(1) = q$. We notice that if ξ is in a section cone $\mathcal{K}$ and $\alpha \in R_*^+$, then also $\alpha\xi \in \mathcal{K}$.

We will study paths consisting of a concatenation of a finite number of integral arcs of vector fields in $\mathcal{K}$.

Definition 6.2.2. *If γ is an integral arc from x_0 to x_1, and μ is an integral arc from x_1 to x_2, then the product $\gamma * \mu$ is the path σ defined by the equation*

$$\sigma(t) = \begin{cases} \gamma(2t) & \textit{for} \quad t \in [0, \frac{1}{2}], \\ \mu(2t-1) & \textit{for} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

The function σ is well defined, continuous (by the pasting lemma, cf. Theorem 18.3 in Munkres [2000]) and piecewise C^r.

Definition 6.2.3. *Suppose $\mathcal{K}$ is a C^r section cone. We call a piecewise C^r path $\sigma : I \to M$ a di-path of $\mathcal{K}$ if there exists a finite set of integral arcs $\{\gamma_1, ..., \gamma_k\}$, for $i = 1, ..., k$ where γ_i is an integral arc of the vector field ξ_i satisfying*

1. *$\{\xi_1,...,\xi_k\} \subset \mathcal{K}$ and*
2. *$\sigma = \gamma_1 * ... * \gamma_k$.*

If all γ_i, $i \in \{1, ..., k\}$ is an integral arc of the same vector field, i.e. $\xi_1 = ... = \xi_k$, then we shall call σ a simple di-path. Otherwise we shall call it a shattered di-path. The set of all di-paths of $\mathcal{K}$ from x to y is denoted by $\overline{P}(x, y; \mathcal{K})$.

Given a vector field $\xi \in \mathfrak{X}^r(M)$, we define the section cone associated to ξ by

$$\underline{\xi} = \{\alpha\xi \in \mathfrak{X}^r(M) |\ \alpha \in C^r(M, \mathbb{R}_*^+)\}. \tag{6.2}$$

In particular all di-paths of $\overline{P}(x, y; \underline{\xi})$ are simple. Thus a simple di-path is either a broken or an unbroken flow line. We want to investigate deformations of di-paths by appropriate homotopies.

Definition 6.2.4. *Suppose $\mathcal{K}$ is a section cone on M and a, b are two singular points of $\mathcal{K}$.*

1. *A di-homotopy by $\mathcal{K}$ from a to b is a continuous map $H : I \times I \to M$ such that $H_s \in \overline{P}(a, b; \mathcal{K})$ for all $s \in I$.*
2. *Two di-paths γ, $\eta \in \overline{P}(a, b; \mathcal{K})$ are said to be di-homotopic by $\mathcal{K}$ if and only if there exists a di-homotopy $H : I \times I \to M$ with $H_0 = \gamma$ and $H_1 = \eta$.*

The set of equivalence classes of di-paths up to di-homotopy by $\mathcal{K}$ is denoted by $\pi(a, b; \mathcal{K})$.

Definition 6.2.5. *Suppose $\mathcal{K}$ is a section cone on the manifold M. For a pair of points x and y in M we say that $x \succeq_{\mathcal{K}} y$ if and only if there exists a di-path $\sigma : I \to M$ of $\mathcal{K}$ with $\sigma(0) = x$ and $\sigma(1) = y$.*

Theorem 6.2.6. *Let $\mathcal{K}$ be a Lyapunov-Smale cone. Then the relation $\succeq_{\mathcal{K}}$ is a partial order relation.*

Proof. For reflexivity notice that for any vector field ξ and any point $x \in M$, $\phi_x^\xi(0) = x$. Also transitivity follows directly from the definition of the di-path.

We show antisymmetry. Suppose $x \succeq_{\mathcal{K}} y$ and $y \succeq_{\mathcal{K}} x$. Let f be the Lyapunov function associated to the Lyapunov section cone. Then

$$\begin{aligned} x \succeq_{\mathcal{K}} y \quad &\Rightarrow \quad f(x) \geq f(y) \\ y \succeq_{\mathcal{K}} x \quad &\Rightarrow \quad f(x) \leq f(y), \end{aligned}$$

thus $f(x) = f(y)$. Assume $x \neq y$ and let σ be a di-path joining x with y then if $x \neq y$ there exists an integral arc γ of a $\xi \in \mathcal{K}$ such that $\gamma(I) \subset \sigma(I) \subset f^{-1}(x)$. However, away from the singular points, f is strictly decreasing along the flow lines of the vector fields in $\mathcal{K}$. This is a contradiction. $\square$

6.3 Invariant Manifolds of Paths in Section Cones

If a C^r section cone $\mathcal{K}$ on a smooth closed manifold M is a Lyapunov or Morse-Smale section cone then all its singular points are hyperbolic. Let $p \in \mathcal{C}r(\mathcal{K})$ and recall that for any $\xi \in \mathcal{K}$, $W_p^s(\xi)$ is an injectively immersed open disk in M. We consider a C^r ($r \geq 1$) path $\sigma \in C^r(I, \mathcal{K})$ and define its stable manifold. We will show that σ gives rise to a notion of stable and unstable manifolds on $I \times M$. If the section cone is Morse-Smale then the stable and unstable manifolds intersect transversally.

Suppose $\sigma : I \to \mathfrak{X}^r(M)$ is a C^r map. We define a map $s : I \times M \to T(M)$ by $s = e|_{\mathfrak{X}^r(M)} \circ (\sigma \times \mathrm{id}_M)$, where e is the evaluation map in Proposition 5.1.7. The map s is a composition of C^r maps, thus it is of class C^r. Observe that s_t is a vector field on M for all $t \in I$.

Definition 6.3.1. *Let $\mathcal{K}$ be a Lyapunov or Morse-Smale C^r section cone, $r \geq 1$. Let $\sigma \in C^r(I, \mathcal{K})$ and $p \in \mathcal{C}r(\mathcal{K})$. The stable manifold $W_p^s(\sigma) \subset I \times M$ of the path σ is defined by*

$$W_p^s(\sigma) = \bigcup_{t \in I} \{t\} \times W_p^s(\sigma(t)).$$

Likewise, we define the unstable manifold $W_p^u(\sigma)$ of σ.

In the next theorem we confirm that the set $W_p^s(\sigma)$ is an immersed submanifold of $I \times M$.

Theorem 6.3.2. *Let M be a closed smooth manifold. For $1 \leq r < \infty$, let p be a singular point of a Lyapunov or Morse-Smale section cone $\mathcal{K} \subset \mathfrak{X}^r(M)$ (p is hyperbolic) with index λ and $\sigma \in C^r(I, \mathcal{K})$. Then the set $W_p^s(\sigma)$ is an immersed C^r submanifold of dimension $\lambda + 1$.*

Proof. We follow the proof of Corollary 5.4.2 we consider an open neighborhood $U' \subset M$ of the point p and apply the exponential map to get a coordinate chart $\psi : U' \to U \subset T_p(U)$, where U is an open neighborhood of 0. Pick $\tau \in I$. We represent the vector field $\sigma(\tau)$ in local coordinates $\bar{\sigma}(\tau) = d\psi\sigma(\tau) \circ \psi^{-1}$. Denote the stable and unstable subspaces for $\bar{\sigma}(\tau)$ by respectively E_τ^s and E_τ^u, and let $U_\tau^s = U \cap E_\tau^s$. Then by Proposition 5.4.7 there is a real number $\delta > 0$ and a C^r map

$$g^\tau : I_\delta^\tau \times U_\tau^s \to E_\tau^u, \text{ where } I_\delta^\tau = I \cap (\tau - \delta, \tau + \delta),$$

such that for any $t \in I_\delta^\tau$, $\psi(W_0^s(\sigma(t), U'))$ is the graph of g_t^τ, where $g_t^\tau(x) = g^\tau(t, x)$. We define a map

$$h^\tau : I_\delta^\tau \times U_\tau^s \to I_\delta^\tau \times U_\tau^s \times E_\tau^u \text{ by } h^\tau(t, x) = (t, x, g^\tau(x, t)).$$

We conclude that $f^\tau \equiv (\mathrm{id}_I \times \psi) \circ (h^\tau) : I_\delta^\tau \times U_\tau^s \to W_p^s(\sigma)$ is a homeomorphism onto its image. Its inverse is used for defining a coordinate map $\psi_\tau : V_\tau \to I_\delta^\tau \times E_\tau^s$ on an open neighborhood $V_\tau \subset \mathrm{im} f^\tau$.

Since the interval I is compact there is a finite number of τ_i, $i = 1, ..., l$ such that $I_{\delta_i}^{\tau_i}$ covers I. We define the differentiable structure on $W_p^s(\sigma)$. We see that $W_p^s(\sigma)$ is an extension of V_{τ_i}'s using the flows

$$W_p^s(\sigma) = \bigcup_{i=1}^{l} \bigcup_{\substack{k \in \mathbb{Z} \\ k \leq 0}} \Psi^{\tau_i}(k, V_{\tau_i}),$$

where $\Psi^{\tau_i} : \mathbb{R} \times I_{\delta_i}^{\tau_i} \times M \to I_{\delta_i}^{\delta_i} \times M$ given by $\Psi_t^{\tau_i}(x) = (\pi_1(x), \phi_t^{\sigma(\pi_1(x))}(\pi_2(x)))$ is a C^r diffeomorphism, cf. Proposition 5.2.2.

The family $\{\Psi^{\tau_i}\}_{\tau_i \in \{\tau_1, \ldots, \tau_l\}}$ defines the map $\Psi : \mathbb{R} \times I \times M \to I \times M$ by: If $\tau \in I_{\delta_i}^{\tau_i}$ then $\Psi(t, \tau, x) = \Psi^{\tau_i}(t, \tau, x)$. This map is well defined by uniqueness of solutions of differential equations. It is C^r by the pasting lemma, cf. Theorem

18.3 in Munkres [2000]. It can be shown that $\Psi_t : I \times M \to I \times M$ given by $\Psi_t(\tau, x) = \Psi(t, \tau, x)$ is a C^r diffeomorphis for each $t \in I$.

For $k \in \{0, 1, 2, ...\}$ we define

$$V^k_{\tau_i} = \Psi(k, V_{\tau_i}) \text{ and } \psi^k_{\tau_i} : V^k_{\tau_i} \to I^{\tau_i}_{\delta_i} \times E^s_{\tau_i}, \text{ by } \psi^k_{\tau_i}(x) = \psi_{\tau_i} \circ \Psi(-k, x). \tag{6.3}$$

The atlas $(V^k_{\tau_i}, \psi^k_{\tau_i})$ makes the inclusion $W^s_p(\sigma) \hookrightarrow I \times M$ an immersion. □

We shall use the two projections $\pi_1 : I \times M \to I$ and $\pi_2 : I \times M \to M$.

Corollary 6.3.3. *For any coordinate chart $(V^k_{\tau_i}, \psi^k_{\tau_i})$ on $W^s_p(\sigma)$ the following diagram commutes:*

$$\begin{tikzcd}
V^k_{\tau_i} \arrow[r, "\psi^k_{\tau_i}"] \arrow[d, "\pi_1"] & I_{\delta_i} \times E^{\tau_i}_s \arrow[d, "p_1"] \\
I_{\delta_i} \arrow[r, "\mathrm{id}"] & I_{\delta_i},
\end{tikzcd}$$

(with p_1 the projection on the first factor).

Proof. The corollary follows from the observation that the diagram

$$\begin{tikzcd}
V^k_{\tau_i} \arrow[r, "\Psi^{\tau_i}(-k,\cdot)"] \arrow[d, "\pi_1"] & V_{\tau_i} \arrow[r, "\psi_{\tau_i}"] \arrow[d, "\pi_1"] & I_{\delta_i} \times E^{\tau_i}_s \arrow[d, "p_1"] \\
I_{\delta_i} \arrow[r, "\mathrm{id}"] & I_{\delta_i} \arrow[r, "\mathrm{id}"] & I_{\delta_i}
\end{tikzcd}$$

commutes. □

The next corollary shows that the stable and unstable manifolds of a Morse-Smale section cone just like for the Morse-Smale vector fields intersect transversally.

Proposition 6.3.4. *Let M be a smooth closed manifold. If $\mathcal{K}$ is a Morse-Smale C^r section cone on M, $r \geq 1$. Then for all $p, q \in \mathcal{C}r(\mathcal{K})$ the intersection $W^u_p(\sigma) \cap W^s_q(\sigma)$ is transversal.*

Lemma 6.3.5. *The restriction* $\pi' = \pi_1|_{W_q^s(\sigma)} : \bigsqcup_{t\in I}\{t\} \times W_q^s(\sigma(t)) \to I$ *given by* $(t, x) \mapsto t$ *is a submersion (no critical points).*

Proof. We represent π' in the local coordinates of a chart (V_τ^k, ψ_τ^k), cf. (6.3), and conclude by Corollary 6.3.3 that

$$\pi' \circ (\psi_\tau^k)^{-1} = p_1.$$

The map p_1 has no critical points, neither does π'. □

Proof (of 6.3.4). For any $t \in I$ the intersection $W(p, q; \sigma(t)) = W_p^u(\sigma(t)) \cap W_q^s(\sigma(t))$ is transversal. Therefore $T_x(M) = T_x(W_p^u(\sigma(t)) + T_x(W_q^s(\sigma(t))$. We observe that $d\pi_1|_{T(W_q^s(\sigma))}$ is nonsingular, then by the dimension argument

$$T_{(t,x)}(I \times M) = T_{(t,x)}(W_p^u(\sigma)) + T_{(t,x)}(W_q^s(\sigma)).$$

□

In the following we want to introduce a new object, a certain manifold, which will substitute for the stable and unstable manifolds of σ. To ease subsequent arguments we want it to be compact. More importantly we wish to represent a flow line by a single element. We have met similar objects in the chapter on Morse Theory where we studied intersections of stable manifolds with the preimage of a regular point of a Morse function.

Let $\mathcal{K}$ be a Lyapunov section cone, and let $f : M \to \mathbb{R}$ be the Morse function from Definition 6.1.9 and p be a singular point of $\mathcal{K}$. All critical points of a Morse function are isolated therefore there is an open neighborhood V of $f(p)$ in $\mathbb{R}$ such that

$$\forall_{v\in V-\{p\}}\ v \text{ is a regular value for } f. \tag{6.4}$$

Pick a $c \in V - \{p\}$. Since $f(\sigma_t)(x) < 0$ for all $x \in M - \mathcal{C}r(\mathcal{K})$ and $t \in I$ we have that $W_p^s(\sigma_t) \pitchfork f^{-1}(c)$ in M. Thus $W_p^s(\sigma)$ intersects $I \times f^{-1}(c)$ transversally in $I \times M$. We conclude that $S_p^s(\sigma) \equiv W_p^s(\sigma) \pitchfork I \times f^{-1}(c)$ is a compact manifold with boundary given by

$$\begin{aligned} \partial S_p^s(\sigma) &= (\partial W_p^s(\sigma) \cap I \times f^{-1}(c)) \cup (W_p^s(\sigma) \cap \partial I \times f^{-1}(c)) \\ &= \{0\} \times S_p^s(\sigma(0)) \cup \{1\} \times S_p^s(\sigma(1)). \end{aligned}$$

We shall recall a definition of a manifold triad.

Definition 6.3.6 (Definition 1.3 in Milnor [1965]). *$(W; V_0, V_1)$ is a C^r manifold triad if W is a compact C^r manifold and ∂W is the disjoint union of two connected submanifolds V_0 and V_1.*

We observe that $(S_p^s(\sigma); \{0\} \times S_p^s(\sigma(0)), \{1\} \times S_p^s(\sigma(1)))$ is a C^r manifold triad.

Definition 6.3.7 (Definition 3.4 in Milnor [1965]). *A triad $(W; V_0, V_1)$ is said to be a product cobordism if it is C^r diffeomorphic to the triad*

$$([0,1] \times V_0; \{0\} \times V_0, \{1\} \times V_0).$$

Proposition 6.3.8. *Let p be a singular point of $\mathcal{K}$. Then the manifold triad*

$$(S_p^s(\sigma); \{0\} \times S_p^s(\sigma(0)), \{1\} \times S_p^s(\sigma(1)))$$

is a product cobordism.

Lemma 6.3.9. *The projection map $\pi_1 : I \times M \to I$ restricted to $S_p^s(\sigma)$ is a submersion.*

Proof. The normal bundle $\nu(S_p^s(\sigma), W_p^s(\sigma))$ is one dimensional and orientable, thus trivial, cf. Theorem 3.2.6. There is a vector bundle isomorphism

$$\phi : \nu(S_p^s(\sigma), W_p^s(\sigma)) \to S_p^s(\sigma) \times \mathbb{R}.$$

By Product Neighborhood Theorem 3.2.3 there is a neighborhood U of $S_p^s(\sigma)$ in $W_p^s(\sigma)$ and a diffeomorphism

$$\varrho : S_p^s(\sigma) \times \mathbb{R} \to U \subset W_p^s(\sigma)$$

such that $\phi|_{S_p^s(\sigma) \times \{0\}}$ is the identity map.

We fix $x \in S_p^s(\sigma)$ and see that $\pi_1 \circ \varrho(x, s)$ is a constant map for each $s \in \mathbb{R}$. Thus each fibre of $\nu(S_p^s(\sigma), W_p^s(\sigma))$ goes to 0 under $d\pi_1$, that is $d\pi_1 d\varrho(\nu_0 \oplus \epsilon) = 0$, where ν_0 is the zero section of $T(S_p^s(\sigma) \times \mathbb{R})$ and ϵ trivial line bundle. On the other hand by Lemma 6.3.5, $d\pi_1|_{W_p^s(\sigma)}$ is a submersion, and so is $d\pi_1|_{\varrho(S_p^s(\sigma) \times \mathbb{R})}$. We have observed that $d\pi_1 d\varrho(\nu_0 \oplus \epsilon) = 0$, therefore $d\pi_1|_{\varrho(S_p^s(\sigma) \times 0)} = d\pi_1|_{S_p^s(\sigma)}$ is a submersion. $\square$

Proof (of proposition). Recall that a Morse function on a manifold triad $(W; V_0, V_1)$ is a C^r function $f : W \to [a, b]$ such that

1. $f^{-1}(a) = V_0$, $f^{-1}(b) = V_1$;
2. All the critical point of f lie in $W - \partial W$ and are non-degenerate.

We conclude that $\pi|_{S_p^s(\sigma)}$ is a Morse function on the manifold triad $(S_p^s(\sigma); \{0\} \times S_p^s(\sigma(0)), \{1\} \times S_p^s(\sigma(1)))$ with no critical points. By Theorem 3.4 in Milnor [1965] this manifold triad is a product cobordism. $\square$

Corollary 6.3.10. *The diffeomorphism $\Psi : I \times S_p^s(\sigma(0)) \to S_p^s(\sigma)$ in Proposition 6.3.8 is such that, for any $(t, x) \in I \times S_p^s(\sigma(0))$ we have $t = \pi_1 \circ \Psi(t, x)$.*

Proof. The corollary follows from the proof of Theorem 3.4 in Milnor [1965]. $\square$

Let $\mathcal{K}$ be a Lyapunov section cone on M and $f : M \to \mathbb{R}$ be the Morse function from Definition 6.1.9. Suppose $c, c' \in \mathbb{R}$, $c > c'$ are two regular values of the function f and $f^{-1}([c', c])$ does not contain any critical points. We define $\theta \in C^r(I, \mathfrak{X}^r(M - \mathcal{C}r(\mathcal{K})))$ by $\theta(t) = \sigma_t/\sigma_t(f)$, then $f(\theta_t) = 1$ for all $t \in I$. Furthermore, we define a map

$$\Theta : I \times (M - \mathcal{C}r(\mathcal{K})) \to I \times (M - \mathcal{C}r(\mathcal{K})) \text{ by } \Theta(t, x) = (t, \phi_{c-c'}^{\theta(t)}(x)).$$

The map Θ is a diffeomorphism which takes $S_p^s(\sigma)' \equiv W_p^s(\sigma) \pitchfork I \times f^{-1}(c')$ onto $S_p^s(\sigma) \equiv W_p^s(\sigma) \pitchfork I \times f^{-1}(c)$.

Proposition 6.3.11. *For $1 \leq r < \infty$, let $\mathcal{K}$ be a Lyapunov C^r section cone on a smooth manifold M and $f : M \to \mathbb{R}$ be the C^r Morse function from Definition 6.1.9. Let p be a singular point of $\mathcal{K}$. Suppose $c \in V - \{p\} \subset \mathbb{R}$, where V is an open neighborhood of $f(p)$ in $\mathbb{R}$ as defined in (6.4) (c is a regular value of f). Then the normal bundle $\nu(S_p^s(\sigma), I \times f^{-1}(c))$ is trivial.*

Proof. The intersection of $I \times f^{-1}(c)$ and $W_p^s(\sigma)$ in $I \times M$ is transversal and inclusions induce the following commutative diagram

$$\begin{tikzcd} S_p^s(\sigma) \arrow[r, hook] \arrow[d, hook] & W_p^s(\sigma) \arrow[d, hook] \\ I \times f^{-1}(c) \arrow[r, hook] & I \times M. \end{tikzcd}$$

Then Lemma 3.7.2 shows that the normal bundles

$$\nu(S^s_p(\sigma), I \times f^{-1}(c)) \text{ and } \nu(W^s_p(\sigma), I \times M)|_{S^s_p(\sigma)}$$

are isomorphic. We note that the latter is trivial since $W^s_p(\sigma)$ is contractible. $\square$

7 The Central Vector Field Theorem

We formulate and prove the main theorem of this thesis. If $\mathcal{K}$ is a Lyapunov-Smale section cone on a closed smooth manifold M^n, a is a singular point of the section cone $\mathcal{K}$ of index 0 and b is a singular point of $\mathcal{K}$ of index n. Then the study of the connected components of the space of flow lines of the vector fields within the section cone $\mathcal{K}$ can be reduced to the study of the connecting components of $W(a,b;\xi)$ for an arbitrary $\xi \in \mathcal{K}$.

7.1 Problem Formulation

For $\eta \in \mathfrak{E}^r(M)$, $r \geq 1$, $P(a,b;\eta)$ is the set of flow lines of η from the singular point a to the singular point b. The set of flow lines of the vector fields in a C^r section cone $\mathcal{K}$ born in a and dying in b is denoted by $P(a,b;\mathcal{K})$.

Definition 7.1.1. *Let M be a closed smooth manifold. For $r \geq 1$, let $\xi \in \mathfrak{X}^r(M)$ and $\mathcal{K}$ be a C^r section cone on M.*

1. $\gamma_0,\ \gamma_1 \in P(a,b;\xi)$. *We say that γ_0 is homotopic to γ_1 by ξ, write $\gamma_0 \sim_\xi \gamma_1$, if and only if there is a path $\beta : I \to M$ such that $\beta(t) \in W(a,b;\xi)$, $\gamma_0(t) = \phi^\xi_{\beta(0)}(t)$ and $\gamma_1(t) = \phi^\xi_{\beta(1)}(t)$.*

2. *Suppose $\gamma_0,\ \gamma_1 \in P(a,b;\mathcal{K})$. We say that γ_0 is homotopic to γ_1 by $\mathcal{K}$ and write $\gamma_0 \sim_{\mathcal{K}} \gamma_1$ if and only if there exist a path $\sigma : I \to \mathcal{K}$ and a path $\beta : I \to M$ such that $\beta(t) \in W(a,b;\sigma(t))$, $\gamma_0(t) = \phi^{\sigma(0)}_{\beta(0)}(t)$ and $\gamma_1(t) = \phi^{\sigma(1)}_{\beta(1)}(t)$.*

The relations $\sim_{\mathcal{K}}$ and $\sim_\xi$ are equivalence relations.

Theorem 7.1.2. *Let M be a closed smooth manifold. Suppose $\mathcal{K}$ is a Lyapunov-Smale C^r section cone on M, $r \geq 5$, and $\xi \in \mathcal{K}$. Let a, b be singular points with indices 0 and n, respectively. Then there is a bijection*

$$\Pi : P(a, b; \xi)/_{\sim_\xi} \to P(a, b; \mathcal{K})/_{\sim_\mathcal{K}}.$$

We shall present two proofs for the surjectivity of Π. The reason is that two entirely different techniques were developed for this purpose and both are used in the proof of injectivity for the map Π. The first proof relies on the properties of $S^s_p(\sigma),\ \sigma \in C^r(I, \mathcal{K})$, as developed in Section 6.3. For the second proof we will introduce the subject of stability of a one-parameter family of diffeomorphisms. Our exposition of this subject follows Newhouse et al. [1983]. We shall introduce the notion of a proper selfconjugacy of a vector field ξ that takes a connected component of $W(a, b; \xi)$ to itself. We will show that if $\mathcal{K}$ is a Lyapunov-Smale section cone, then a path $\sigma \in C^r(I, \mathcal{K})$, $r \geq 5$, induces a proper selfconjugacy.

7.2 A First Proof for the Surjectivity of Π

We recall that $W(a, b; \xi) = W^u_a(\xi) \cap W^s_b(\xi)$. For $0 \leq r \leq \infty$, let $\|\cdot\|_r$ be the norm on $\mathfrak{X}^r(M)$ discussed in Section 5.1. We define the open ball $B^{\mathcal{K}}_\delta(\eta) \subset \mathcal{K}$ centered at $\eta \in \mathcal{K}$ and the radius δ by

$$B^{\mathcal{K}}_\delta(\eta) = \{\xi \in \mathcal{K} : \|\xi - \eta\|_r < \delta\}.$$

The first proposition shows that a perturbation of a vector field within a section cone and a small perturbation of the initial values do not change the points where the flow lines are born and die.

Proposition 7.2.1. *Let M^n be a smooth Riemannian manifold. Suppose $\mathcal{K}$ is a gradient-like or Lyapunov C^r section cone on M, $r \geq 1$, with a singular point a of index 0. Let $\eta \in \mathcal{K}$. If $y \in W^u_a(\eta)$ then there is $\delta > 0$ and an open neighborhood U of y such that for any $\xi \in B^{\mathcal{K}}_\delta(\eta)$ and $x \in U$ we have that $x \in W^u_a(\xi)$. In particular, if b is a singular point of $\mathcal{K}$ of index n and $y \in W(a, b; \eta)$ then there is $\delta' > 0$ and an open neighborhood U' of y such that for any $\xi \in B^{\mathcal{K}}_{\delta'}(\eta)$ and $x \in U'$, $x \in W(a, b; \xi)$.*

Lemma 7.2.2. *Let M^n be a smooth Riemannian manifold. If $\eta \in \mathfrak{E}^r(M)$, $r \geq 1$, then there is a Morse function $f : M \to \mathbb{R}$ and a real number $\delta > 0$ such that f is a Lyapunov function for any $\xi \in B_\delta^{\mathcal{K}}(\eta)$.*

Proof. Any Morse-Smale, hence also gradient-like vector field, has an associated Lyapunov function, cf. 5.7.3. Suppose f is such a function for η. For each critical point p there exist an open neighborhood N of p and constants $\kappa > 0$ such that

$$-\eta(f)(x) \geq \kappa d(x,p),$$

where d is the Riemannian metric. On the other hand, for each $p \in \mathcal{C}r(\mathcal{K})$ there is a coordinate chart (U, ψ) such that

$$\hat{f} \equiv f \circ \psi^{-1} = f(p) + Q(x,x),$$

where Q is a nonsingular quadratic form whose index is the same as the index of the Hessian of f at p.

We want to show that there exists a continuous monotonically increasing function $c : [0,\infty) \to [0,\infty)$ such that

$$|(\eta - \xi)(f)(x)| \leq c(\delta) d(x,p) \text{ whenever } x \in U \text{ and } \|\eta - \xi\|_r < \delta.$$

We denote the vector fields η and ξ in the local coordinates of ψ by $\hat{\eta}$ and $\hat{\xi}$, respectively. We get

$$(\hat{\eta} - \hat{\xi})(\hat{f})(y) = d\hat{f}_y(\hat{\eta}(y) - \hat{\xi}(y)) = Q(y, (\hat{\eta}(y) - \hat{\xi}(y))).$$

Thus $|(\hat{\eta} - \hat{\xi})(\hat{f})(y)| < \delta \|Q\| \, \|y\|$ for $y \in \psi(U)$. On the open set $N \cap U$ we have

$$-\xi(f)(x) = -\eta(f)(x) + (\eta - \xi)(f) \geq \kappa d(x,p) - c(\delta) d(x,p).$$

For sufficiently small δ, $\kappa' \equiv \kappa - c(\delta) > 0$ and $-\xi(f)(x) \geq \kappa' d(x,p)$.

Denote by $\{N_q\}_{q \in \mathcal{C}r(\mathcal{K})}$ open neighborhoods of the singular points of η for which the above inequality is valid for some $\delta > 0$. Let $K = M - \bigcup_{q \in \mathcal{C}r(\mathcal{K})} N_q$. By compactness of K, the function $g : K \to \mathbb{R}^+$, $x \mapsto -df(\eta)(x)$ has a minimum, say $r > 0$. Suppose also that $e = \|df\|_0$ on K, where $\|\cdot\|_0$ stands for C^0 norm. If $\delta' = \min(\delta, r/e)$ then for any $\xi \in B_{\delta'}^{\mathcal{K}}(\eta)$

$$-df(\xi)(x) = -df(\eta)(x) + df(\eta - \xi)(x) > 0 \text{ for } x \in M - \mathcal{C}r(\eta).$$

We conclude that f is a Lyapunov function for all $\xi \in B_{\delta'}^{\mathcal{K}}(\eta)$. $\square$

Proof (of proposition). By the lemma there is a Morse function f and a constant δ such that f is a Lyapunov function for any $\xi \in B_\delta^{\mathcal{K}}(\eta)$. Hence, for each $p \in \mathcal{C}r(\eta)$ there exist a real number $\kappa_p > 0$ and an open neighborhood N_p of p for which

$$-\xi(f)(x) \geq \kappa_p d(x, p).$$

In particular, if $\xi \in B_\delta^{\mathcal{K}}(\eta)$, then the singular point a is asymptotically stable for $-\xi$ in N_a and b is asymptotically stable for ξ in N_b .

On the other hand by Proposition 5.2.2 there exists a C^r map $\Phi : \mathbb{R} \times M \times B_r^{\mathcal{K}}(\eta) \to M$ such that $\Phi(\cdot, x, \xi) = \phi_x^\xi(\cdot)$ is an integral curve of the vector field ξ. Pick τ such that $\phi_y^\eta(-\tau) \in N_a$ and $\phi_y^\eta(\tau) \in N_b$. By continuity of Φ it is possible to choose δ'' and an open neighborhood U of y such that $\phi_x^\xi(-\tau) \in N_a$ whenever $x \in U$ and $\xi \in B_{\delta''}(\eta)$.

Let $\delta = \min(\delta', \delta'')$. For any $\xi \in B_\delta(\eta)$ and $x \in U$ we have

$$\lim_{t \to -\infty} \phi_t^\xi(x) = a.$$

□

Definition 7.2.3. *A nondecreasing surjective map $\alpha : (I, \{0\}, \{1\}) \to (I, \{0\}, \{1\})$ is called a re-scaling of the unit interval.*

Let M^n be a closed smooth manifold, $\mathcal{K}$ be a gradient-like C^r section cone and let $\sigma \in C^0(I, \mathcal{K})$. Suppose that $\mathcal{K}$ has one singular point a with index 0, one singular point b with index n, k singular points with index 1, say $p_1, ..., p_k$ and l singular points with index $n-1$, say $q_1, ..., q_l$. For $x \in M$ with $x \in W(a, b, \sigma(0))$ we define the subset B_x^σ of $C^0(I, M)$ by the following conditions: A path β is in B_x^σ if and only if

1. $\beta(0) = x$;
2. $\beta(I) \subset M - \mathcal{C}r(\mathcal{K})$;
3. There exists a re-scaling α of the unit interval such that for each $\tau \in I$

$$\lim_{t \to \infty} \left(\phi_{-t}^{\sigma\circ\alpha(\tau)}(\beta(\tau)), \phi_t^{\sigma\circ\alpha(\tau)}(\beta(\tau))\right) \in D,$$

where $D = \{(a, b), (p_1, b), ..., (p_k, b), (a, q_1), ..., (a, q_l)\}$.

The re-scaling α of the unit interval from condition 3. will be called an associated re-scaling for β.

The set B_x^σ consists of paths of initial conditions for σ such that the α-limit sets are singular points of $\mathcal{K}$ of index 0 or 1, and ω-limit sets are singular points of index $n-1$ or n. The next proposition demonstrates that the set B_x^σ is nonempty.

Proposition 7.2.4. *Let M^n be a closed smooth manifold, $\mathcal{K}$ be a gradient-like C^r section cone, $r \geq 1$, and let $\sigma \in C^r(I, M)$. Suppose that $\mathcal{K}$ has only one singular point a of index 0 and one singular point b of index n. Then for any $x \in W(a, b; \sigma(0))$, the set B_x^σ is nonempty.*

In the proof of the proposition we shall make the use of the following lemma.

Lemma 7.2.5. *Let M^n be a C^r manifold ($r \geq 0$) and $\{N_i\}_{i\in\mathbb{N}}$ be a collection of immersed submanifolds of $I \times M$ of co-dimension 2 or more. Let $\pi_1 : I \times M \to I$ be the projection on the first factor. Then for any open subset U of M there is a path $\upsilon \in C^0(I, I \times M - \bigcup_{i=1}^k N_i)$ such that $\upsilon(0) \in \{0\} \times U$ and the following diagram commutes*

$$\begin{tikzcd} I \arrow[r, "\upsilon"] \arrow[dr, "\mathrm{id}"'] & I \times M \arrow[d, "\pi_1"] \\ & I. \end{tikzcd} \tag{7.1}$$

Proof. For some sufficiently large $s \in \mathbb{N}$ we have an embedding $f : M \to \mathbb{R}^s$ and the embedding $\mathrm{id}_I \times f : I \times M \to \mathbb{R}^{s+1}$. Consider Hausdorff dimension of a subset $F \subset \mathbb{R}^{s+1}$, cf. Falconer [1986],

$$\dim_H F = \inf\{v \in \mathbb{R}^+ | \, \mathcal{H}^v(F) = 0\},$$

where $\mathcal{H}^v(F)$ is the v-dimensional Hausdorff measure of F. We observe that for $i \in \mathbb{N}$, $\dim_H N_i \leq n-1$. We define the map $p : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R} \times \mathbb{R}^s$ by $p(x, y) = (0, y)$. The map p is Lipschitz. Let $N = \bigcup_{i\in\mathbb{N}} N_i$. By Lemma 1.8 in Falconer [1986], $\dim_H(p(N_i)) \leq \dim_H N_i$ and hence $\dim_H (p(N)) \leq n-1$. We have

$$\begin{aligned} n &= \dim_H (\{0\} \times U) = \dim ((\{0\} \times U - p(N)) \cup p(N)) \\ &= \max\{\dim(\{0\} \times U - p(N)), \dim N\} \end{aligned}$$

Thus $\dim((\{0\} \times U - p(N))) = n$. It follows that the set $\{0\} \times U - p(N)$ is nonempty. Therefore for any $y \in U - \pi_2(N)$, where $\pi_2 : I \times M \to M$ is the projection on the second factor, we have that $I \times \{y\} \cap N = \emptyset$. Now, take the constant path $\upsilon(t) = (t, y)$. $\square$

Proof (of proposition 7.2.4). By Proposition 7.2.1 there is an open neighborhood $U' \subset M$ of x such that if $z \in U'$ then $z \in W(a, b; \sigma(0))$. We choose a path connected open neighborhood $U \subset U'$ of x.

We define a collection $\{N_i\}_{i \in \{1,\ldots,k\}}$ consisting of

1. $W_p^s(\sigma)$ with p a singular point of index less than $n-1$,
2. $W_q^u(\sigma)$ with q a singular point of index greater than 1,
3. $W_q^u(\sigma) \pitchfork W_p^s(\sigma)$ where q, p are singular points with relative index $\mathrm{Ind}_{\mathrm{q}}^{\mathcal{K}} - \mathrm{Ind}_{\mathrm{p}}^{\mathcal{K}} = n - 2$,
4. $\mathcal{C}r(\mathcal{K})$.

Then the co-dimension of each N_i is 2 or more. Lemma 7.2.5 applies and for $x \in M$ and the open neighborhood $U \subset M$ of x there is a path $\upsilon : I \to I \times M - \bigcup_{i=1}^k N_i$ with $(0, y) \equiv \beta'(0) \in \{0\} \times U$ that satisfies (7.1). Define $\beta' = \pi_2 \circ \upsilon$, where $\pi_2 : I \times M \to M$ is the projection. Pick a path $\beta'' : I \to U$ with $\beta''(0) = x$ and $\beta''(1) = y$. Then the desired path $\beta = \beta'' * \beta'$, that is

$$\beta(t) = \begin{cases} \beta''(2t) & \text{for} \quad t \in [0, 1/2] \\ \beta'(2t-1) & \text{for} \quad t \in [1/2, 1] \end{cases}$$

and the associated re-scaling α for β is

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad t \in [0, 1/2] \\ 2t - 1 & \text{for} \quad t \in [1/2, 1]. \end{cases}$$

$\square$

Corollary 7.2.6. *Let M, $\mathcal{K}$ and σ be as in Proposition 7.2.4. Suppose λ is a re-scaling of the unit interval and $\sigma' = \sigma \circ \lambda$. Then for any $x \in W(a, b, \sigma'(0))$ the set $B_x^{\sigma'}$ is nonempty.*

Proof. We have seen in the proof of Proposition 7.2.4 that there is a path $\upsilon : I \to I \times M - \bigcup_{i=1}^{k} N_i$. We define the desired path β by

$$\beta(t) = \begin{cases} \beta''(2t) & \text{for} \quad t \in [0, 1/2] \\ \pi_2 \circ \upsilon \circ \lambda(2t-1) & \text{for} \quad t \in [1/2, 1]. \end{cases}$$

For the re-scaling of the unit iterval

$$\alpha(t) = \begin{cases} 0 & \text{for} \quad t \in [0, 1/2] \\ 2t-1 & \text{for} \quad t \in [1/2, 1] \end{cases}$$

β satisfies condition 3. of the definition of the set $B_x^{\sigma'}$. $\square$

We are ready to proof the surjectivity of the map Π in Theorem 7.1.2.

Proposition 7.2.7. *Let M^n be a closed smooth manifold, $\mathcal{K}$ be a gradient-like C^r section cone, $r \geq 1$. Suppose that $\mathcal{K}$ has only one singular point a of index 0 and one singular point b of index n. If $\xi \in \mathcal{K}$ then for any $\eta \in \mathcal{K}$ and any $\gamma_0 \in P(a, b; \eta)$ there is some $\gamma_1 \in P(a, b; \xi)$ such that $\gamma_0 \sim_{\mathcal{K}} \gamma_1$. That is, the following composition*

$$P(a, b; \xi) \hookrightarrow P(a, b; \mathcal{K}) \to P(a, b; \mathcal{K}) / \sim_{\mathcal{K}}$$

is surjective.

Proof. Since $\xi, \eta \in \mathcal{K}$ we can define a path $\sigma \in C^r(I, \mathcal{K})$ by $\sigma(t) = t\xi + (1-t)\eta$. We have that $\sigma(0) = \eta$ and $\sigma(1) = \xi$. Since $\gamma_0 \in P(a, b; \eta)$ there is a point $x \in W(a, b; \eta)$ such that $\gamma_0(t) = \phi_x^{\eta}(t)$ for all $t \in \mathbb{R}$. Our aim is to show that there is a path $\beta : (I, \{0\}) \to (M, \{x\})$ and a re-scaling α of the unit interval such that $\beta(t) \in W(a, b; \sigma \circ \alpha(t))$ for all $t \in I$. For this we define a set

$$\begin{aligned} A \quad = \quad & \{\tau \in I | \text{ there exist } \beta \in B_x^{\sigma} \text{ with a re-scaling } \alpha \text{ for } \beta \text{ such that} \\ & \beta(t) \in W(a, b; \sigma \circ \alpha(t)) \text{ for } 0 \leq \alpha(t) \leq \tau\}. \end{aligned}$$

We want to show that the set A is equal the whole interval I. This will be proven by the following five claims.

Claim (A). The set A is open.

Proof (A). Assume $\tau \in A$ then there is a $\beta_1 \in B_x^\sigma$ with an associated re-scaling α_1 for β_1 such that

$$\beta_1(t) \in W(a,b;\sigma \circ \alpha_1(t)) \text{ for } 0 \le \alpha_1(t) \le \tau.$$

Let $\nu = \min \alpha_1^{-1}(\tau)$. By Lemma 7.2.1, there is $\epsilon > 0$ such that for all $\tau - \epsilon < \rho < \tau + \epsilon$ we have $\beta_1(\nu) \in W(a,b;\sigma(\rho))$. Define a constant path

$$\beta_2 : [\nu, \nu+\epsilon) \to M,\ \beta_2(t) = \beta_1(\nu).$$

Define $\varsigma : I \to \mathcal{K}$ by $\varsigma(t) = \sigma((1-\tau)t + \tau)$. Choose $\beta_3 \in B^\varsigma_{\beta_1(\nu)} \neq \emptyset$ and pick a re-scaling α_3 for β_3.

The desired path β is obtained by extending $\beta_1|_{[0,\tau]}$ by a constant path β_2 and β_3. 0n β_2 we change the vector fields according to σ from $\sigma(\nu)$ to $\sigma(\nu+\epsilon)$, and on β_3 from $\sigma(\nu+\epsilon)$ to $\sigma(1)$. Explicitly, the path β is defined by

$$\beta(t) = \begin{cases} \beta_1(t) & \text{for} \quad t \in [0,\nu] \\ \beta_2(t) & \text{for} \quad t \in [\nu, \nu+\epsilon] \\ \beta_3((t-\nu-\epsilon)/(1-\nu-\epsilon)) & \text{for} \quad t \in [\nu+\epsilon, 1], \end{cases}$$

and the associated re-scaling α for β is given by

$$\alpha(t) = \begin{cases} \alpha_1(t) & \text{for} \quad t \in [0,\nu] \\ t + \tau - \nu & \text{for} \quad t \in [\nu, \nu+\epsilon] \\ (1-\nu-\epsilon)\alpha_3((t-\nu-\epsilon)/(1-\nu-\epsilon)) + \nu + \epsilon & \text{for} \quad t \in [\nu+\epsilon, 1]. \end{cases}$$

$\square$

It follows that the set A is of the form $[0,T)$ or I.

Claim (B). Let $\vartheta \in C^0(I,\mathcal{K})$. Suppose that there is a path $\beta : I \to M$ such that $\forall_{t<T}\ \beta(t) \in W(a,b;\vartheta(t))$ and $\beta(T) \in W_p^s(\vartheta(T))$ for some $p \in \mathcal{C}r(\mathcal{K})$. Then for any neighborhood U of p there is a path β' satisfying $\beta'(0) = \beta(0)$, $\forall_{t<T}\ \beta'(t) \in W(a,b;\vartheta(t))$ and $\beta'(T) \in W_p^s(\vartheta(T)) \cap U$.

Proof (B). Pick τ such that $\phi_\tau^{\vartheta(T)}(\beta(T)) \in U$. If $\beta_1(t) = \phi_\tau^{\vartheta(t)}(\beta(t))$ then for any $t \in [0,T]$ the orbits of $\beta_1(t)$ and $\beta(t)$ coincide. Furthermore $\beta_1(T) \in U$. Let β_2 be the integral arc of the vector field $\vartheta(0)$ from $\beta(0)$ to $\beta_1(0)$. Then the desired path β' is the concatenation of β_2 with β_1. $\square$

Claim (C). Let $\vartheta \in C^0(I, \mathcal{K})$. Suppose that there is a smooth function $f : M \to \mathbb{R}$, which is a Lyapunov function for each $\vartheta(t)$, $t \in I$. Assume there are real numbers $r_1, r_2 > 0$ so that $H \equiv \{x \in M | \, r_1 \leq f(x) \leq r_2\}$ contains no singular points for $\mathcal{K}$. For any path $\beta : I \to H$ there is a path $\beta' : I \to f^{-1}(r_2)$ such that the orbits of $\beta(t)$ and $\beta'(t)$ coincides.

Proof (C). We define

$$\vartheta' \in C^0(I, \mathfrak{X}^r(H)) \text{ by } \tau \mapsto \frac{\vartheta(\tau)}{\vartheta(\tau)(f)}.$$

If $\beta'(t) = \phi^{\vartheta'(\tau)}_{r_2 - f(\beta(t))}(\beta(t))$ then $f(\beta'(t)) = r_2$ for all $t \in I$. The conclusion follows from the observation that the orbits of the vector fields ϑ and ϑ' coincide on H. $\square$

Claim (D). If $A = [0, T_0)$ then $T_0 \in A$. Hence $A = I$.

Proof (D). Let $T = \min \alpha^{-1}(T_0)$. Choose any $\beta \in B^\sigma_x$ and any associated re-scaling α for β such that $\forall_{t<T}\, \beta(t) \in W(a, b; \sigma \circ \alpha(t))$ and $\beta(T) \in W^s_p(\sigma \circ \alpha(T))$ or $\beta(T) \in W^u_p(\sigma \circ \alpha(T))$ for some $p \in \mathcal{C}r(\mathcal{K}) - \{a, b\}$. We will prove the claim by extending the path $\beta|_{[0,T-\delta]}$, for some small $\delta > 0$, to a path $\beta' \in B^\sigma_x$ with a re-scaling α' for β' such that $\forall_{t \leq T}\, \beta'(t) \in W(a, b; \sigma \circ \alpha'(t))$.

Without loss of generality we suppose that $\beta(T) \in W^s_p(\sigma \circ \alpha(T))$ and the index of p is $n - 1$. We make an observation that, since $\beta(T) \in W^u_a(\sigma \circ \alpha(T))$ and α is continuous there is, by Proposition 7.2.1, a real number $\delta > 0$ such that for any $T - \delta < t < T + \delta$ we have $\beta(t) \in W^u_a(\sigma \circ \alpha(t))$.

Let $f : M \to \mathbb{R}$ be a Lyapunov function for $\sigma \circ \alpha(T)$. We shall postpone the discussion on the choice of f for a while. By Lemma 7.2.2 there is a real number $c_1 > 0$ such that f is a Lyapunov function for $\sigma(t)$ whenever $-c_1 \leq t - T_0 \leq c_1$.

Let σ'' be the restriction of σ to the segment $J \equiv [T_0 - c_1, T_0 + c_1]$. Let V be an open neighborhood of $f(p)$ in $\mathbb{R}$ such that $\forall_{v \in V - \{p\}}\, v$ is a regular value for f. Pick $\lambda \in V - \{p\}$ and consider the manifolds $S^s_p(\sigma'') = W^s_p(\sigma'') \cap I \times f^{-1}(\lambda)$ and $S^u_p(\sigma'') = W^u_p(\sigma'') \cap I \times f^{-1}(\lambda)$, cf. Section 6.3. Since $S^s_p(\sigma'')$ and $S^u_p(\sigma'')$ are compact and disjoint they have disjoint closed tubular neighborhood both with radius ϵ, say $\mathcal{N}^s_\epsilon$ and $\mathcal{N}^u_\epsilon$, respectively.

By Claim (B) we may assume that there is $c_2 > 0$ such that $\beta((T-c_2, T+c_2)) \subset (V - \{0\})$. Then by Lemma 7.2.2 and Claim (C) we may additionally assume that there is a real number c_3 with $0 < c_3 < c_2$ such that $\beta([T-c_3, T+c_3]) \subset f^{-1}(\lambda)$. Pick a real number δ with $0 < \delta < c_3$ such that $(T_0 - \delta_0, \beta(T-\delta)) \in \mathcal{N}^s_\epsilon$, where $\delta_0 = T_0 - \alpha(T-\delta)$. This is possible since $(T_0, \beta(T)) \in S^s_p(\sigma(T_0)) \subset \mathcal{N}^s_\epsilon$.

The one-dimensional vector bundle $\nu(S^s_p(\sigma''), J \times S^{n-1})$ is trivial by Proposition 6.3.11. Therefore, there is an embedding $e : S^s_p(\sigma'') \times [-\epsilon, \epsilon] \to \mathcal{N}^s_\epsilon$, cf. Theorem 3.2.3. Let $(v, r) = e^{-1}(T_0 - \delta_0, \beta(T-\delta))$.

Claim (D.1). There exists a path $\gamma : [T_0 - \delta_0, T_0] \to \mathcal{N}^s_\epsilon$ with $\gamma(T_0 - \delta_0) = (T_0 - \delta_0, \beta(T-\delta))$ such that

1. $\pi_1 \circ \gamma = \mathrm{id}_{[T_0 - \delta_0, T_0]}$, where $I \times S^{n-1} \to I$ is the projection on the first factor;
2. $\mathrm{im}(\gamma) \cap S^S_p(\sigma'') = \emptyset$.

Proof (D.1). By Corollary 6.3.10, there is a diffeomorphism

$$\Psi : J \times S_p\left(\sigma''(T' - c_1)\right) \to S^s_p(\sigma'')$$

such that, for any $(t, x) \in J \times S^s_p(\sigma''(T' - c_1))$ we have $t = \pi_1 \circ \Psi(t, x)$. Let $(\tau, z) = \Psi^{-1}(v)$ and note that $\tau = T_0 - \delta_0$. We define

$$\omega : [T_0 - \delta_0, T_0] \to J \times S_p\left(\sigma''(T' - c_1)\right) \ \text{ by } \ \omega(t) = (t, z).$$

Then the desired path γ is

$$\gamma(t) = e^{-1}\left(\Psi \circ \omega(t), r\right).$$

□

We are ready to define the desired path $\beta' \in B^\sigma_x$. We follow $\beta_1 \equiv \beta|_{[0, T-\delta]}$ in the interval $[0, T-\delta]$ simultaneously changing the vector field according to $\sigma \circ \alpha$ from $\sigma \circ \alpha(0)$ to $\sigma \circ \alpha(T-\delta)$. Then, we follow $\beta_2 \equiv \pi_2 \circ \gamma$, where $\pi_2 : I \times M \to M$ is the projection on the second factor. We change the vector field linearly from

$\sigma(T_0 - \delta_0)$ to $\sigma(T_0)$. Final step is to extend the resulting path to the whole interval I by a path β_3. For this we define $\varsigma : I \to \mathcal{K}$ by $\varsigma(t) = \sigma((1-T)t + T)$ and pick $\beta_3 \in B^{\varsigma}_{\beta_2(T_0)}$ and an associated re-scaling α_3 for β_3. The desired path β' is

$$\beta'(t) = \begin{cases} \beta(t) & \text{for} \quad t \in [0, T-\delta], \\ \beta_2 \circ \alpha(t) & \text{for} \quad t \in [T-\delta, T], \\ \beta_3(1/(1-T)(t-T)) & \text{for} \quad t \in [T, 1]. \end{cases}$$

The associated re-scaling α' for β' is

$$\alpha'(t) = \begin{cases} \alpha(t) & \text{for} \quad t \in [0, T-\delta], \\ t & \text{for} \quad t \in [T-\delta, T], \\ \alpha_3(1/(1-T)(t-T)) & \text{for} \quad t \in [T, 1]. \end{cases}$$

We observe that $\beta'(t) \notin W^s_p(\sigma \circ \alpha'(t))$ for $t \in [0, T]$. It remains to show that $\beta'(t) \in W(a, b; \sigma \circ \alpha'(t))$ for $t \in [0, T]$.

At this point we shall proceed our postponed discussion on the choice of a Lyapunov function. Let $\mathcal{C}r^{n-1}(\mathcal{K})$ be the set of singular points of index $n-1$. By Theorem 5.7.3 there is a Lyapunov function for $\sigma(T_0)$ such that for any $q \in (\mathcal{C}r^{n-1}(\mathcal{K}) - \{p\})$ we have $f(p) < f(q)$. By Lemma 7.2.2 there is a positive real $\delta' > 0$ such that the function f is Lyapunov for each $\sigma(t)$ with $|t - T_0| < \delta'$. We can shrink the neighborhood V of p so that

$$f(x) < \min\{f(q) |\, q \in \mathcal{C}r^{n-1}(\mathcal{K}) - \{p\}\} \text{ for all } x \in V$$

and pick $0 < \delta_0 < \min\{\delta', \min \alpha^{-1}(c_3)\}$. Then by the property

$$df(\sigma(t))(y) < 0 \text{ for } |t - T_0| < \delta_0 \text{ and } y \in M - \mathcal{C}r(\mathcal{K})$$

we conclude that $\phi^{\sigma \circ \alpha'(t)}_{\beta'(t)}$ dies at b for $t \in [0, T]$, since b is the only singular point of index n. We conclude that $\beta'(t) \in W(a, b; \sigma \circ \alpha'(t))$ for all $t \in [0, T]$. $\square$

This ends the proof of the proposition. $\square$

7.3 Stability of One-Parameter Families of Diffeomorphisms

We study a one-parameter family $\{\kappa_t\}_{t\in I}$ of diffeomorphisms starting at a Morse-Smale diffeomorphism. For a bifurcation point $t_0 \in I$ the diffeomorphism κ_{t_0} ceases to be Morse-Smale, that is κ_{t_0} have a nonhyperbolic singular point or its stable and unstable manifolds do not intersect transversally. We shall recall that if p is a hyperbolic singularity for a diffeomorphism κ_t, then the stable manifold $W_p^s(\kappa_t)$ is an injectively immersed open disk in M. The same is true for the strong stale and strong unstable manifolds at a nonhyperbolic singular point, cf. Appendix III, Shub [1986].

Let M^n be a closed smooth manifold. We follow Newhouse et al. [1983] and consider the set of C^r diffeomorphisms on M denoted by $\mathrm{Diff}^r(M)$.

Definition 7.3.1 (Newhouse et al. [1983]). *We call a C^r map $\kappa : I \times M \to I \times M$ an arc of diffeomorphisms on M if and only if $\kappa(t,x) = (t, \kappa_t(x))$, where $x \mapsto \kappa_t(x)$ is a C^r diffeomorphism for each $t \in I$. The space of arcs of C^r diffeomorphisms on M will be denoted by $\mathcal{P}^r(M)$.*

We give $\mathrm{Diff}^r(M)$ and $\mathcal{P}^r(M)$ the C^r topology.

Suppose an arc $\kappa \in \mathcal{P}^r(M)$ with $\kappa_0 \in \mathcal{MS}^r$, where $\mathcal{MS}^r$ is the set of Morse-Smale C^r diffeomorphisms on M. Let $b = b(\kappa) = \inf\{t \in I | \ \kappa(t) \notin \mathcal{MS}^r\}$. As for diffeomorphisms also for arcs (of diffeomorphisms) we can introduce a notion of conjugacy.

Definition 7.3.2 (Newhouse et al. [1983]). *If $\kappa, \kappa' \in \mathcal{P}^r(M)$, then we say that (h, H) is a conjugacy if $h : I \to I$ is a homeomorphism with $h(b(\kappa)) = b(\kappa')$, $H : I \times M \to M$ is a map with H_t being a conjugacy between κ_t and $\kappa'_{h(t)}$ for all t in some neighborhood of $[0, b(\kappa)]$.*

The definition of conjugacies gives rise to the concept of structural stability for arcs of diffeomorphisms.

Definition 7.3.3 (Newhouse et al. [1983]). *An arc $\kappa \in \mathcal{P}^r(M)$ is stable if there is an open neighborhood U of κ in $\mathcal{P}^r(M)$ such that any $\kappa' \in U$ is conjugate to κ.*

The necessary and sufficient conditions for structural stability of arcs of diffeomorphisms have been formulated and proven in Newhouse et al. [1983].

Definition 7.3.4 (Newhouse et al. [1983]). *Let $r \geq 5$, the subset $\mathcal{S}^r(M) \subset \mathcal{P}^r(M)$ is the set of arcs κ that satisfy:*

1. *The limit set of each κ_t has finitely many orbits, $t \in I$;*
2. *κ has only finitely many bifurcation values, say b_1 to b_s in $(0, 1)$;*
3. *All stable, strong stable, unstable, and strong unstable manifolds intersects transversally;*
4. *For each $i \in \{1, ..., s\}$, κ_{b_i} has no cycles and has exactly one non-hyperbolic periodic orbit which is either a noncritical saddle-node, cf. Sec. 3, Newhouse et al. [1983], or a flip, cf. Sec. 4, Newhouse et al. [1983]; this non-hyperbolic orbit unfolds generically.*

We shall not explain the meaning of Definition 7.3.4, instead we refer to Newhouse et al. [1983] for details and remark merely that any arc of diffeomorphisms κ, such that $\kappa_t \in \mathcal{MS}^r$ for $t \in I$ belongs to the set $\mathcal{S}^r(M)$.

Theorem 7.3.5 (Theorem 4.4 in Newhouse et al. [1983]). *For $r \geq 5$, the arcs in $\mathcal{S}^r(M)$ are stable.*

For $r \geq 5$ we define the subset $\mathcal{R}^r(M) \subset \mathcal{S}^r(M)$ of arcs κ that satisfy $\kappa_t \in \mathcal{MS}^r$.

Corollary 7.3.6. *Let $G : I \to \mathcal{R}^r(M)$, $r \geq 5$, be a map. Then there is a conjugacy between the arc G_0 and G_1. In particular there exist a homeomorphism $h : I \to I$ and a map $H : I \times M \to M$, where H_t is a conjugacy between $G_0(t)$ and $G_1(h(t))$ for $t \in I$.*

Proof. The conclusion follows from compactness of the unit interval. We cover I be finite number of open intervals $\{U_i\}_{i\in\{1,\ldots,l\}}$ and propagate the conjugacy from the neighborhood of 0 to 1. $\square$

We shall relate the results on conjugacy of arcs of diffeomorphism with arcs of vector fields, which is the primary object of the study in this thesis. Below, we show that the Stable Manifold Theorem for Vector Fields follows from for the Stable Manifold Theorem for Diffeomorphisms.

Let M be a compact smooth manifold. A vector field $\xi \in \mathfrak{X}^r(M)$, $r \geq 1$, determines a one-parameter family of C^r diffeomorphisms $\phi_t : M \to M$ for $t \in \mathbb{R}$ given by

$$\phi_t^\xi(x) = \phi_x^\xi(t).$$

Suppose Ψ is a C^r diffeomorphism and a is a fixed point, then we define the stable manifold for Ψ at a by

$$W_a^s(\Psi) \quad = \quad \{x \in M | \lim_{n\to+\infty} \Psi^n(x) = a\}.$$

Since $(\phi_t^\xi)^n = \phi_{nt}^\xi$ for all $n \in \mathbb{N}$, for any singular point a of ξ and any fixed $t > 0$ we have

$$W_a^s(\phi_t^\xi) = \{x \in M | \lim_{n\to+\infty} \phi_{tn}^\xi(x) = a\} = \{x \in M | \lim_{\lambda\to+\infty} \phi_x^\xi(\lambda) = a\} = W_a^s(\xi).$$

7.3.1 A Second Proof for the Surjectivity of Π

As an application of Corollary 7.3.6 we will give an alternative and more elegant proof of Proposition 7.2.7 for class C^r with $r \geq 5$. Additionally we release the assumption that the singular points are of maximal and minimal indices.

Proposition 7.3.7. *Let M^n be a closed smooth manifold, $\mathcal{K}$ be a gradient-like C^r section cone, $r \geq 5$. Suppose that a, b are singular points of $\mathcal{K}$ with $a \succeq_\mathcal{K} b$. If $\xi \in \mathcal{K}$ then for any $\eta \in \mathcal{K}$ and any $\gamma_0 \in P(a, b; \eta)$ there is some $\gamma_1 \in P(a, b; \xi)$ such that $\gamma_0 \sim_\mathcal{K} \gamma_1$. That is, the following composition*

$$P(a, b; \xi) \hookrightarrow P(a, b; \mathcal{K}) \to P(a, b; \mathcal{K})/ \sim_\mathcal{K}$$

is surjective.

Proof. Since $\xi, \eta \in \mathcal{K}$ we can define a path $\sigma \in C^r(I, \mathcal{K})$ by $\sigma(t) = t\xi + (1-t)\eta$ with $\sigma(0) = \eta$ and $\sigma(1) = \xi$. Let c_η be a constant path $c_\eta(t) = \eta$. We define a C^r map $g : I \times I \to \mathcal{K}$ by $g(s,t) = (1-s)c_\eta(t) + s\sigma(t)$. The map g gives rise to a map $G : I \to \mathcal{R}^r(M)$. Pick $\tau > 0$ then $G(s)(t) \equiv G^\tau(s)(t) = \left(t, \phi_\tau^{g(s,t)}\right)$. We shall use the notation $G_s(t) = G(s)(t)$. We note that $G_0(t) = (t, \phi_\tau^\eta)$ and $G_1(t) = \left(t, \phi_\tau^{\sigma(t)}\right)$.

By Corollary 7.3.6 there exists a homeomorphism $h : I \to I$ and there is a map $H : I \times M \to M$ with $H_t : M \to M$, where $H_t(z) = H(t,z)$, is a conjugacy between $G_0(t)$ and $G_1(h(t))$ for all $t \in I$.

Since $\gamma_0 \in P(a,b;\eta)$ there is a point $x \in W(a,b;\eta)$ such that $\gamma_0(t) = \phi_x^\eta(t)$ for all $t \in \mathbb{R}$. Let c_x be a constant path in M given by $c_x : t \mapsto x$. Then $c_x(t) \in W(a,b,c_\eta(t))$.

We define a path $\beta : I \to M$ by $\beta(t) = H_0^{-1} \circ H_t \circ c_x \circ h(t)$ if $h(0) = 0$. If $h(0) = 1$ then we define β by $\beta(t) = \beta'(1-t)$ where $\beta'(t) = H_1^{-1} \circ H_t \circ c_x \circ h(t)$. We observe that $\beta(t) \in W(a,b;\sigma(t))$, $\beta(0) = x$, $\sigma(0) = \xi$, and $\sigma(1) = \eta$. If $\gamma_1(t) = \phi_{\beta(1)}^\xi(t)$ then $\gamma_1 \in P(a,b;\xi)$ and $\gamma_0 \sim_{\mathcal{K}} \gamma_1$. $\square$

7.4 A Proof for the Injectivity of Π

We consider a closed smooth manifold M^n. By the discussion in Section 5.1, $\mathfrak{X}^r(M)$, $0 \le r \le \infty$, with the C^r topology arises from a complete metric. Pick such a metric and denote it by $d_r(\cdot,\cdot)$. Furthermore, on the space $C^0(I, \mathfrak{X}^r(M))$ we impose the topology of compact convergence.

Let $\mathcal{K}$ be a C^r section cone on M, $r \ge 1$. Let a, b be two singular points of $\mathcal{K}$ of index 0 and n, respectively. In the proof for the injectivity of the map Π we will need an approximation of elements in $C^0(I,\mathcal{K})$ by elements in $C^s(I,\mathcal{K})$ ($0 \le s \le r$). This is resolved by the following proposition.

Proposition 7.4.1. *Let M^n be a closed smooth manifold, $\mathcal{K}$ be a C^r section cone on M with $r \ge 1$. Let a, b be two singular points of $\mathcal{K}$ of index 0 and n, respectively. Suppose $\sigma \in C^0(I,\mathcal{K})$ and $\beta \in C^0(I,M)$ such that $\beta(t) \in W(a,b;\sigma(t))$, $t \in I$. Then there is $\sigma' \in C^s(I,\mathcal{K})$, $0 \le s \le r$, such that $\sigma'(0) = \sigma(0)$, $\sigma'(1) = \sigma(1)$ and $\beta(t) \in W(a,b;\sigma'(t))$ for each $t \in I$.*

Lemma 7.4.2. *Let $\mathcal{K}$ be a C^r section cone, $r \geq 0$, on a smooth closed manifold M. Then the set $C^s(I, \mathcal{K})$, $0 \leq s \leq r$, is dense in $C^0(I, \mathcal{K})$.*

Proof. For every $\sigma \in C^0(I, \mathcal{K})$ and every $\epsilon > 0$ we shall find a path $\varsigma \in C^s(I, \mathcal{K})$ such that $\sup_{t \in I} d_r(\varsigma(t), \sigma(t)) < \epsilon$. By compactness of I and continuity of σ we cover I by a family of open neighborhoods $\{V_i\}_{i \in \{1,\ldots,l\}}$ of $t_i \in I$ such that $t_0 = 0$, $t_1 = 1$ and $d_r(\sigma(t), \sigma(t_i)) < \epsilon$ for $t \in V_i$. We define constant paths $c_i \in C^s(I, \mathcal{K})$ by $c_i(t) = \sigma(t_i)$. We use a smooth partition of unity $\{\lambda\}_{i \in \{1,\ldots,l\}}$ subordinate to $\{V_i\}_{i \in \{1,\ldots,l\}}$. Define

$$\varsigma(t) = \sum_i^l \lambda_i(t) c_i(t),$$

then

$$\begin{aligned} d_r(\varsigma(t), \sigma(t)) &= d_r\left(\sum_i^l \lambda_i(t)\sigma(t), \sum_i^l \lambda_i(t) c_i(t) \right) \leq \sum_i^l \lambda_i(t) d_r(\sigma(t), c_i(t)) \\ &< \sum_i^l \lambda_i(t)\epsilon = \epsilon \end{aligned}$$

as desired. □

Proof (of Proposition 7.4.1). By Proposition 7.2.1 for each $t \in I$, there is an open neighborhood U_t of $\beta(t)$ and a ball $B^{\mathcal{K}}_{\delta_t}(\sigma(t))$ such that for any $x \in U_t$ and any $\theta \in B^{\mathcal{K}}_{\delta_t}(\sigma(t))$ we have that $x \in P(a, b; \theta)$. Let $V_t \subset U_t$ be an open neighborhood of t such that for each $\tau \in V_t$, $\|\sigma(\tau) - \sigma(t)\|_r < \delta_t/2$. The interval I is compact and we get a finite family of such V_t's, say $\{V_i\}_{i \in \{t_1,\ldots,t_l\}}$ covering I. Let $\delta = \min\{\delta_{t_1}/2, \ldots, \delta_{t_l}/2\}$. By Lemma 7.4.2 we can find $\sigma' \in C^s(I, \mathcal{K})$ such that $\sup_{t \in I} d_r(\sigma'(t), \sigma(t)) < \delta$. Then for $t \in V_i$

$$d_r(\sigma'(t), \sigma(t_i)) \leq d_r(\sigma'(t), \sigma(t)) + d_r(\sigma(t), \sigma(t_i)) < \delta + \delta_i/2 \leq \delta.$$

Therefore $\beta(t) \in P(a, b; \sigma'(t))$ for all $t \in I$. □

In our preparation for the proof of injectivity of Π we consider an essential gradient-like vector field ξ and study connected components of $W(a, b; \xi)$. Suppose ξ has l singular points of index 1, say $p_1, \ldots, p_l$, and $k - l$ singular points of index $n-1$, say $p_{l+1}, ..., p_k$. We consider a selfconjugacy of the vector field ξ, that is a homeomorphism $H : M \to M$ such that $H \circ \phi_t^\xi(x) = \phi_t^\xi \circ H(x)$. Note that H preserves the stable and unstable manifolds of $p_i,\ 1 \le i \le k$. We shall assume that $H(p_i) = p_i$ for $i = 1, ..., k$.

Pick a singular point p of ξ of index $n-1$. By the discussion in Section 5.4, cf. Equation (5.8), there is a local coordinate system (V, ψ) of p

$$\psi : V \to \mathbb{R}^n = E^s \oplus E^u,\ \psi(p) = 0, \tag{7.2}$$

where E^s and E^u are the stable and unstable subspaces of $\mathbb{R}^n$ for $d\hat{\xi}_0$ where $\hat{\xi} = d\psi\xi \circ \psi^{-1}$. The local stable manifold of $\hat{\xi}$ is an open neighborhood of the origin in E^s and the local unstable manifold is an open neighborhood of the origin in E^u. We restrict the homeomorphism H to a sufficiently small neighborhood V' of p such that $H(V') \subset V$ with $W \equiv \psi(V')$ convex. We define $\hat{H} = \psi \circ H \circ \psi^{-1}|_W$ and see that $\hat{H}$ is a homeomorphism onto its image. Finally we note that since the index of p is $n-1$, the stable subspace E^s is a point in the Grassmann manifold $G_{n-1}(\mathbb{R}^n)$.

Let $V_{n-1}^0(\mathbb{R}^n)$ be the Stiefel manifold of orthonormal $(n-1)$-tuples of vectors in $\mathbb{R}^n$. A principal bundle

$$O(\mathbb{R}^{n-1}) \longrightarrow V_{n-1}^0(\mathbb{R}^n) \xrightarrow{P} G_{n-1}(\mathbb{R}^n)$$

over the Grassmann manifold $G_{n-1}(\mathbb{R}^n)$ is defined by the mapping P taking $X \in V_{n-1}^0$ to the hyperplane spanned by X.

Fix an orientation ω on $\mathbb{R}^n$. There is a correspondence $X \mapsto X^\perp$, which assigns to each $(n-1)$-tuple of orthonormal vectors in $\mathbb{R}^n$ the n-th orthonormal vector making the n-tuple positively oriented, i.e. $[X \oplus X^\perp] = \omega$. This correspondance defines a homeomorphism between $V_{n-1}^0(\mathbb{R}^n)$ and $V_1^0(\mathbb{R}^n) \cong S^{n-1}$ and a homeomorphism between $G_{n-1}(\mathbb{R}^n)$ and $G_1(\mathbb{R}^n) \cong \mathbb{R}\mathrm{P}^{n-1}$. In conclusion, the following diagram commutes

$$\begin{tikzcd}
V_{n-1}^0(\mathbb{R}^n) \arrow[r, "\cong"] \arrow[d, "P"] & S^{n-1} \arrow[d, "\tilde{P}"] \\
G_{n-1}(\mathbb{R}^n) \arrow[r, "\cong"] & \mathbb{R}\mathrm{P}^{n-1},
\end{tikzcd}$$

where $\tilde{P}$ is the quotient map corresponding to the antipodal $\mathbb{Z}_2$ action.

Fix an $(n-1)$-tuple X which spans the stable subspace E^s. The complement of the subspace $E^s = P(X)$ in $\mathbb{R}^n$ has two connected components:

$$E^- = \{x \in \mathbb{R}^n | \left\langle x, X^\perp \right\rangle < 0\} \text{ and } E^+ = \{x \in \mathbb{R}^n | \left\langle x, X^\perp \right\rangle > 0\}. \quad (7.3)$$

Let $W^+ = E^+ \cap W$ and $W^- = E^- \cap W$.

Since the map $\hat{H}$ takes a stable (unstable) manifold to itself we have either $\hat{H}(W^+) \subset E^+$ or $\hat{H}(W^+) \subset E^-$.

We can also apply the above construction for a singular point of index 1. The subspace E^s is then replaced by E^u and an $(n-1)$-tuple X spans now E^u instead. The rest of the construction remains unchanged.

Definition 7.4.3. *Let M^n be a closed smooth manifold, $\xi \in \mathfrak{E}^r(M)$, $r \geq 1$, and $H : M \to M$ be a selfconjugacy of ξ. Let p be a singular point of ξ of index $n-1$ (or 1). Suppose a triple (ψ, W, X) is as follows*

1. *ψ is a coordinate chart at p*

$$\psi : V \to \mathbb{R}^n = E^s \oplus E^u, \ \psi(p) = 0$$

with $\psi(W^s_p(\xi, V)) \subset E^s$ and $\psi(W^u_p(\xi, V)) \subset E^u$

2. *W is a convex neighborhood of 0 in $\mathbb{R}^n$ such that $H \circ \psi^{-1}(W) \subset V$,*

3. *X is an $(n-1)$-tuple which spans E^s (E^u).*

Let $\hat{H} = \psi \circ H \circ \psi^{-1}|_W$. If $\hat{H}(E^+ \cap W) \subset E^+$, where E^+ as in (7.3) then H will be called proper at p.

The next proposition shows that if H is proper at p for a triple (ψ, W, X) then it is proper at p for any other triple satisfying conditions 1., 2. and 3. of Definition 7.4.3.

Proposition 7.4.4. *Let M^n be a closed smooth manifold, $\xi \in \mathfrak{E}^r(M)$, $r \geq 1$, and $H : M \to M$ be a selfconjugacy of ξ. Let p be a singular point of ξ of index $n-1$ or 1. If H is proper at p with respect to a triple (ψ_1, W_1, X_1) then it is proper with respect to any other triple (ψ_2, W_2, X_2) which satisfies conditions 1., 2. and 3. of Definition 7.4.3.*

Proof. We prove the proposition by contradiction. Let $x \in \psi_1^{-1}(W_1) \cap \psi_2^{-1}(W_2)$ and $x_i = \psi_i(x)$, $i = 1, 2$. Without loss of generality we may assume that $x_2 \in E_2^+$. Suppose that $\hat{H}_2(x_2) \in E_2^-$. The composition $\psi_1 \circ \psi_2^{-1}$ is a homeomorphism, which takes x_2 to x_1 and takes $\hat{H}_2(x_2)$ to $\hat{H}_1(x_1)$. But x_1 and $\hat{H}_1(x_1)$ lie in the same component of $\mathbb{R}^n - P(X_1)$, which is a contradiction. $\square$

Definition 7.4.5. *Let M^n be a closed smooth manifold and $\xi \in \mathfrak{E}^r(M)$, $r \geq 1$. We say a selfconjugacy $H : M \to M$ of ξ is proper if and only if it is proper at each singular point of ξ with index 1 and $n-1$.*

Proposition 7.4.6. *Let M^n be a closed smooth manifold and $\xi \in \mathfrak{E}^r(M)$, $r \geq 1$. Suppose a, b are singular points of indices 0 and n respectively. If a selfconjugacy $H : M \to M$ of ξ is proper, then H maps every connected component U_α of $W(a, b; \xi)$ into itself.*

We define a collection $\{N_i\}_{i \in \{1,\ldots,N\}}$ consisting of

1. singular points of ξ,
2. stable manifolds $W_p^s(\xi)$ of co-dimension more than 1,
3. unstable manifolds $W_q^u(\xi)$ of co-dimension more than 1 and
4. connecting manifolds $W(p, q; \xi)$ of co-dimension more than 1.

Let $N = \bigcup_{i=1}^k N_i$.

Lemma 7.4.7. *For any pair of points $x, y \in W(a, b; \xi)$ there is a path*

$$\beta : I \to (M - N)$$

such that $\beta(0) = x$, $\beta(1) = y$.

Proof. By Thom's Transversality Theorem, cf. Theorem 3.2.1 Hirsch and Smale [1974], the set of maps $\gamma \in C^r(I, M)$ that are transversal to N_i is dense. Therefore the set of maps in $C^r(I, M)$ that are transversal to all N_i is dense. Note that if $\gamma \in C^r(I, M)$ and $\gamma \pitchfork N_i$ then $\gamma(I) \cap N_i = \emptyset$.

Since the points x, y are in $W(a,b;\xi)$, there are open neighborhoods U_x of x and U_y of y such that for any $z \in U_x \cup U_y$, $z \in W(a,b;\xi)$. Furthermore, there is $\gamma' \in C^r(I, M-N)$ such that $\gamma'(0) \in U_x$, $\gamma'(1) \in U_y$. Connect x with $\gamma'(0)$ by a path β' in U_x, and $\gamma'(1)$ with y by a path β'' in U_y. Concatenation of β', γ' and β'' gives the desired path β. $\square$

Proof (of Proposition 7.4.6). For M of dimension 2 the proposition is trivially true. We assume in the following that $\dim(M) \geq 3$.

We shall show that for any $x \in U_\alpha$ there is a path $\beta : I \to U_\alpha$ such that $\beta(0) = x$ and $\beta(1) = H(x)$. Using Lemma 7.4 pick a path $\gamma : I \to M$ connecting x with $H(x)$ such that $\mathrm{im}(\gamma) \subset (M - N)$. Suppose γ leaves U_α. Define τ by

$$\tau = \inf\{t \in I |\, \gamma(t) \in W^s_p(\xi), \mathrm{index}(p) = n-1 \text{ or } \gamma(t) \in W^u_q(\xi), \mathrm{index}(q) = 1\}.$$

Without loss of generality we suppose that $\gamma(\tau) \in W^s_p(\xi)$. Pick a triple (ψ, W, X) which satisfies conditions 1., 2., and 3. of Definition 7.4.3. We may suppose that $\gamma(\tau)$ belongs to an open neighborhood $\psi^{-1}(W)$ of p (if not concatenate γ with a flow line as in Claim (B) in the proof of Proposition 7.2.7). We shall show below that for a sufficiently small real number $\delta > 0$ there exists a path $\gamma' : I \to W(a,b;\xi)$ with $\gamma'(0) = \gamma(\tau - \delta)$ and $\gamma'(1) = H \circ \gamma(\tau - \delta)$. Then the desired path β joining x with $H(x)$ is a concatenation of $\gamma|_{[0,\tau-\delta]}$, γ' and $H \circ \gamma|_{[0,\tau-\delta]}$, that is

$$\beta(t) = \begin{cases} \gamma(t) & \text{for} \quad t \in [0, \tau - \delta], \\ \gamma'(t + 2(\tau - \delta)(t-1)) & \text{for} \quad t \in [\tau - \delta, 1 - \tau + \delta], \\ H \circ \gamma(1 - t) & \text{for} \quad t \in [1 - \tau + \delta, 1]. \end{cases}$$

We construct γ'. Observe that $\gamma(\tau) \in W^u_a(\xi)$ therefore there is an open neighborhood $\mathcal{N}$ of $\gamma(\tau)$ such that $x \in W^u_a(\xi)$ whenever $x \in \mathcal{N}$. Pick δ such that $\gamma(\tau - \delta) \in \mathcal{N}$. Since H is proper $\gamma(\tau-\delta)$ and $H \circ \gamma(\tau - \delta)$ are both in E^+ (or E^-). The complement of E^u in E^+ has only one connected component, therefore there is a path $\gamma' : I \to (E^+ - E^u)$ with $\gamma'(0) = \gamma(\tau - \delta)$ and $\gamma'(1) = H \circ \gamma(\tau - \delta)$. $\square$

Proposition 7.4.6 says that the study of a selfconjugacy can be reduced to a local analysis of the induced maps $\hat{H}$ for each singular point of ξ of index 1 or $n-1$.

We shall study a selfconjugacy originated from the following situation. Let $\mathcal{K}$ be a gradient-like section cone. We regard a path $\sigma : I \to \mathcal{K}$ with $\sigma(0) = \sigma(1) = \xi$

and we suppose that there is a map $G : I \times M \to M$ with G_t a conjugacy between $\sigma(0)$ and $\sigma(t)$. In particular $H = G(1)$ is a selfconjugacy for ξ. The singular points of $\mathcal{K}$ are isolated and G is continuous therefore $H(p_i) = p_i$ for each $i \in \{1, ..., k\}$.

Definition 7.4.8. *Let M be a closed smooth manifold, $\mathcal{K}$ be a C^r section cone, $r \geq 1$, on M. Let $\sigma : I \to \mathcal{K}$ be a path in a section cone $\mathcal{K}$. We say that a map $G : I \times M \to M$ is an arc of conjugacies for σ if G_t is a conjugancy between $\sigma(0)$ and $\sigma(t)$ for all $t \in I$.*

Proposition 7.4.9. *Let $\mathcal{K}$ be a Lyapunov-Smale C^r section cone, $r \geq 1$, and $\xi \in \mathcal{K}$. Suppose $\sigma : (I, \partial I) \to (\mathcal{K}, \xi)$ is a loop in $\mathcal{K}$ and $G : I \times M \to M$ is an arc of conjugancies for σ. Then $G(1)$ is a proper selfconjugacy.*

Lemma 7.4.10. *Suppose $\mathcal{K}$ is a Lyapunov C^r section cone, $r \geq 1$, and p is a singular point of $\mathcal{K}$. Let $\xi, \eta \in \mathcal{K}$ and (ψ, V) be a local coordinate chart with $p \in V$ and $\psi(p) = 0$. Let $L_\xi, L_\eta \in \mathfrak{X}^r(\psi(V))$ be linear approximations of $d\psi\xi \circ \psi^{-1}$ and $d\psi\eta \circ \psi^{-1}$, respectively. Then $W_0^s(L_\xi) \cap W_0^u(L_\eta) = \{0\}$.*

Proof. Since $\mathcal{K}$ is a Lyapunov section cone there exists a function $f : M \to \mathbb{R}$ which is a Lyapunov function for both ξ and η. Moreover, by Lemma 7.2.2 for sufficiently small open neighborhood U of 0 the function $\hat{f} = f \circ \psi^{-1}|_U$ is a Lyapunov function for L_ξ and L_η. Then by Proposition 6.1.12

$$W_0^s(L_\xi, U) \cap W_0^u(L_\eta, U) = \{0\},$$

but $W_0^s(L_\xi)$ and $W_0^u(L_\eta)$ are both linear subspaces of $\mathbb{R}^n$, thus also

$$W_0^s(L_\xi) \cap W_0^u(L_\eta) = \{0\}.$$

□

Proof (of Proposition 7.4.9). We shall show that $G(1)$ is proper at each singular point of index 1 and $n-1$. We present the proof for a singular point of index $n-1$. The proof for a singular point of index 1 is analogous.

Let p be a singular point of the vector field $\sigma(0) = \sigma(1)$ with index $n-1$. Pick a triple (ψ, W, X) satisfying conditions 1., 2. and 3. of Definition 7.4.3. Let $\hat{\sigma}$ be the local representation of σ, that is $\hat{\sigma}(t) = d\psi\sigma(t) \circ \psi^{-1}$ for $t \in I$. Define a path

$\varsigma : I \to \mathfrak{X}^r(\mathbb{R}^n)$ consisting of linear vector fields defined by $\varsigma_t : x \mapsto d\hat{\sigma}(t)_0 x$. We observe that $W_0^s(\varsigma(0)) = W_0^s(\varsigma(1))$.

Let $f : M \to \mathbb{R}$ be the Morse function for $\mathcal{K}$ from Definition 6.1.9. Without loss of generality we assume that p is the only critical point of f in $\psi^{-1}(W)$. Pick a regular value c of f with $c < f(p)$ sufficiently closed to $f(p)$ so that

$$f^{-1}(c) \cap W_p^u(\sigma(t)) \subset \psi^{-1}(W) \text{ for all } t \in I. \tag{7.4}$$

By Proposition 5.4.8 and compactness of I such a c exists.

Let $\hat{f}$ be the local representation of f, $\hat{f} = f \circ \psi^{-1}$. For sufficiently small W $\hat{f}$ is a Lyapunov function for each $\varsigma(t)$, $t \in I$. Recall Definition 6.3.1 and consider $W_0^u(\varsigma)$, then $S_0^u(\varsigma) = W_0^u(\varsigma) \pitchfork I \times \hat{f}^{-1}(c)$ is nonempty by (7.4). We apply Proposition 6.3.8 to conclude that $S_0^u(\varsigma)$ and $I \times S_0^u(\varsigma(0))$ are diffeomorphic. Note that $S_0^u(\varsigma(0)) \cong S^0 = \{-,+\}$. We define the following composition

$$g : I \times \{-,+\} \xrightarrow{\cong} S_0^u(\varsigma) \xhookrightarrow{j_1} W_0^u(\varsigma) \xhookrightarrow{j_2} \textstyle\bigsqcup_{t\in I}\{t\} \times (W_0^u(\varsigma(t)) + W_0^s(\varsigma(0)))$$

$$\xhookrightarrow{j_3} I \times \mathbb{R}^n \xrightarrow{\mathrm{id}\times Q} I \times (\mathbb{R}^n / W^s(\varsigma(0))) \;\cong\; I \times \mathbb{R},$$

where $j_1,\ j_2$ and j_3 are the inclusions and $Q : \mathbb{R}^n \to \mathbb{R}^n / W^s(\varsigma(0))$ is the quotient map.

Suppose that $G(1)$ is not a proper at p. Then $g(1,-)$ and $g(0,-)$ are in two different connected components of the complement of $W^s(\varsigma(0))$ in $\mathbb{R}^n$. It follows that $g(0,-)g(1,-) < 0$. Since g is continuous there is $\tau \in I$ such that $g(\tau) = 0$. But this implies that $S_0^u(\varsigma(\tau)) \subset W^s(\varsigma(0))$. The section cone $\mathcal{K}$ is Lyapunov therefore by Lemma 7.4.10, for any $t \in I$, $W_0^u(\varsigma(t)) \cap W_0^s(\varsigma(0)) = \{0\}$. This is a contradiction. $\square$

We are ready to prove injectivity of the map Π.

Proposition 7.4.11. *Let M be a closed smooth manifold and $\mathcal{K}$ be a Lyapunov-Smale C^r section cone on M, $r \geq 5$. Let a, b be singular points of $\mathcal{K}$ with indices 0 and n, respectively. Suppose $\xi \in \mathcal{K}$. If $\gamma_1, \gamma_2 \in P(a, b; \xi)$ and $\gamma_1 \sim_{\mathcal{K}} \gamma_2$ then $\gamma_1 \sim_\xi \gamma_2$.*

Proof. Suppose there is a map $\sigma \in C^0(I, \mathcal{K})$ such that $\sigma(0) = \sigma(1) = \xi$ and a path $\beta : I \to M$ with $\beta(t) \in W(a, b; \sigma(t))$ for $t \in I$. We will show that there is a path $\beta' : I \to M$ with $\beta'(0) = \beta(0)$ and $\beta'(1) = \beta(1)$ such that $\beta'(t) \in W(a, b; \xi)$.

By Proposition 7.4.1 there is $\sigma' \in C^r(I, \mathcal{K})$ such that $\sigma'(0) = \sigma'(1) = \xi$ and $\beta(t) \in W(a, b; \sigma'(t))$ for $t \in I$.

Let c_ξ be a constant path $c_\xi(t) = \xi$. We define a C^r map $g : I \times I \to \mathcal{K}$ by $g(s, t) = (1 - s)\sigma'(t) + sc_\xi(t)$. We use the same argument as in the proof of Proposition 7.3.7. The map g gives rise to a map $G : I \to \mathcal{R}^r(M)$, where $\mathcal{R}^r(M)$ is the set of arcs of Morse-Smale diffeomorphisms on M. Pick $\tau > 0$ then $G(s)(t) \equiv G^\tau(s)(t) = \left(t, \phi_\tau^{g(s,t)}\right)$. We shall use the notation $G_s(t) = G(s)(t)$. Note that $G_0(t) = \left(t, \phi_\tau^{\sigma'(t)}\right)$ and $G_1(t) = (t, \phi_\tau^\xi)$.

By Corollary 7.3.6 there exist a homeomorphism $h : I \to I$ and a map $H : I \times M \to M$, where H_t is a conjugacy between $G_0(t)$ and $G_1(h(t))$ for $t \in I$. We observe that H_0 and H_1 are both selfconjugacy of Φ_τ^ξ. Since $\mathcal{K}$ is a Lyapunov-Smale section cone the selfconjugacies H_0 and H_1 are proper.

The homeomorphism h takes 0 to 0 or 0 to 1. Without loss of generality we assume that $h(0) = 1$. We define a path $\gamma : I \to M$ by $\gamma(t) = H_1^{-1} \circ H_{(1-t)} \circ \beta \circ h^{-1}(1 - t)$. Since $H_t(\cdot) \equiv H(t, \cdot)$ is a conjugacy between $G_0(t)$ and $G_1(h(t))$ (orbits go to orbits) we have that $H_\tau \circ \beta \circ h^{-1}(\tau) \in W(a, b; \xi)$ for each $\tau \in I$. Thereby, $\gamma(t) \in W(a, b; \xi)$. We see that $\gamma(0) = \beta(0)$. Moreover, the points $\gamma(1)$ and $\beta(1)$ are in the same connected component of $W(a, b; \xi)$. This can be deduced form the fact that $H_1^{-1} \circ H_0$ is a proper conjugacy and the use of Proposition 7.4.6. It follows that there is a path $\gamma' : I \to M$ connecting $\gamma(1)$ with $\beta(1)$ such that $\gamma'(t) \in W(a, b; \xi)$ for all $t \in I$. The desired path β' is then the concatenation of γ and γ':

$$\beta'(t) = \begin{cases} \gamma(2t) & \text{for} \quad t \in [0, \frac{1}{2}], \\ \gamma'(2t - 1) & \text{for} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

□

Propositions 7.3.7 and 7.4.11 prove Theorem 7.1.2. A consequence of Theorem 7.1.2 is the following corollary.

Corollary 7.4.12. *Let M be a closed smooth manifold and $\mathcal{K}$ be a Lyapunov-Smale C^r section cone on M, $r \geq 5$. Let a, b be singular points of $\mathcal{K}$ with indices* 0

and n, respectively. If $\xi, \eta \in \mathcal{K}$ then there is a bijection $\Theta : P(a,b;\xi)/_{\sim_\xi} \to P(a,b;\eta)/_{\sim_\eta}$.

Let $p, q \in \mathcal{C}r(\mathcal{K})$. Recall that $p \succ_\xi q$ means that there is an orbit of ξ not equal to p nor q whose α-limit set is p and whose ω limit set is q.

Corollary 7.4.13. *Let M be a closed smooth manifold and $\mathcal{K}$ be a Lyapunov-Smale C^r section cone on M, $r \geq 5$. Let a, b be singular points of $\mathcal{K}$ with indices 0 and n, respectively. If $\xi, \eta \in \mathcal{K}$ then*

$$a \succ_\xi b \Leftrightarrow a \succ_\eta b.$$

7.5 The Central Vector Field Theorem for Di-paths

This section differs from the rest of the thesis in the sense that we shall present a conjecture here whose proof is left for further work.

To complete the program started by this thesis we need to establish results on detecting the connected components of $W(a,b;\xi)$ for an essential gradient-like vector field ξ. Due to Proposition 5.6.5 (up to homeomorphism) it is enough to consider a gradient vector field. The second task is to extend the Central Vector Theorem to deal with genuine di-paths.

Conjecture 7.5.1. Let M be a closed smooth manifold and $\mathcal{K}$ be a Lyapunov-Smale C^r section cone on M, $r \geq 1$. Let a, b be singular points of $\mathcal{K}$ with indices 0 and n, respectively. If $\xi \in \mathcal{K}$ then there is a bijection $\tilde{\Pi} : \pi(a,b;\xi) \to \pi(a,b;\mathcal{K})$.

The proof of the conjecture follows from the Central Vector Field Theorem if we could demonstrate that any shattered di-path (Definition 6.2.3) is di-homotopic by $\mathcal{K}$ to an unbroken flow line for some $\eta \in \mathcal{K}$. Conjecture 7.5.2 below says that any shattered di-path from the singular point a to b is di-homotopic by $\mathcal{K}$ to a simple one. Thus to prove Conjecture 7.5.1 it remains to show that any broken flow line of a gradient-like vector field η is homotopic by η to an unbroken flow line, see Conjecture 7.5.3.

Conjecture 7.5.2. Let M be a closed smooth manifold and $\mathcal{K}$ be a Lyapunov-Smale C^r section cone on M, $r \geq 1$. Let a, b be singular points of $\mathcal{K}$ with indices 0 and n, respectively. Every shattered di-path from a to b is di-homotopic by $\mathcal{K}$ to a simple di-path.

Conjecture 7.5.3. Let M be a closed smooth manifold and $\eta \in \mathfrak{E}^r(M)$. Suppose a, b are singular points of η with $a \succ b$. If γ be a broken flow line of η born in a and which dies in b then there is a flow line (unbroken) $\gamma' \in P(a, b; \eta)$ di-homotopic to γ by $\underline{\eta}$, cf. Equation 6.2.

Bibliography

R. Abraham and J. Marsden. *Foundations of Mechanics*. Westview Press, 1977.

A. Banyaga and D. Hurtubise. *Lectures on Morse Homology*. Kluwer, 2004.

G. Barker. Matricial norms over cones. *Linear Algebra and Its Applications*, 6: 175–182, 1972.

G. Barker. Theory of cones. *Linear Algebra and Its Applications*, 39:263–291, 1981.

G. Bredon. *Algebraic Topology*. Springer-Verlag, 1993.

C. Chicone and R. Swanson. Linearization via the Lie derivative. *Electronic Journal of Differential Equations, Monograph 02, http://ejde.math.txstate.edu*, 2000.

B. Datta. Stability and inertia. *Linear Algebra and Its Applications*, 302-303: 175–182, 1999.

J. Davis and P. Kirk. *Lecture Notes in Algebraic Topology*. AMS, 2001.

L. Fajstrup, E. Goubault, and M. Raussen. *Algebraic Topology and Concurrency*. Theoretical Computer Science, 2005.

K. Falconer. *The Geometry of Fractal Sets*. Cambridge University Press, 1986.

J. Franks. Morse-Smale flows and homotopy theory. *Topology*, 18:199–215, 1979.

A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

M. W. Hirsch. *Differential Topology*. Springer-Verlag, 1976.

M. W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic Press, 1974.

J. Jost. *Post Modern Analysis*. Springer, 1998.

J. Jost. *Riemannian Geometry and Geometric Analysis*. Springer, 2002.

H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 2002.

A. Kosinski. *Differential Manifolds*. Academic Press, 1993.

S. Lang. *Fundamentals of Differential Geometry*. Springer-Verlag, 1999.

K. R. Meyer. Energy functions for Morse-Smale systems. *Americal Journal of Mathematics*, 90(4):1031–1040, 1968.

J. Milnor. *Lectures on the h-Cobordism Theory*. Princeton University Press, 1965.

J. Milnor. *Morse Theory*. Annals of Mathematics Studies. Princeton Iniversity Press, 1973.

J. Milnor. *Topology from the Differentiable Viewpoint*. Princeton University Press, 1997.

J. Milnor and J. Stasheff. *Characteristic Classes*. Princeton University Press, 1974.

J. Munkres. *Topology*. Prentice-Hall, 2000.

S. Newhouse and M. Peixoto. There is a simple arc joining any two Morse-Smale flows. *Asterisque. Trois etudes en dynamique qualitative*, 31:16–41, 1976.

S. E. Newhouse, J. Palis, and F. Takens. Bifurcations and stability of family of diffeomorphisms. *Publications mathématiques de l'I.H.É.S.*, (57):5–71, 1983.

J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems*. Springer-Verlag, 1982.

J. Palis and S. Smale. Structural stability theorems. In *Global Analysis. Proc. Symp. in Pure Math. vol. XIV. Americal Math. Soc.*, pages 223–231, 1970.

M. Peixoto. Structural stability on two-dimensional manifolds. *Topology*, 1:101–120, 1962.

M. Shub. *Global Stability of Dynamical Systems*. Springer-Verlag, 1986.

S. Smale. Morse inequalities for dynamical systems. *Bulletin of the American Mathematical Society*, 66:43–49, 1960.

S. Smale. On gradient dynamical systems. *Annals of Mathematics*, 74:199–206, 1961.

T. Stykel. Stability and inertia theorems for generalized Lyapunov equations. *Linear Algebra and Its Applications*, 355:297–314, 2002.