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BY
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Dissertation submitted

DANMARKS FRIE FORSKNINGSFOND

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English summary

This PhD project investigates fourth grade students' understanding and development of the concept of fractions in a Danish school system setting. There is international consensus about the importance of understanding fractions for students' further mathematical development, but fraction proficiency has proven to be particularly difficult for some students. In Denmark, there has been limited focus on the topic of fractions, and no quantitative studies have been conducted based on student development of fractions over time. The present PhD project seeks to remedy this knowledge gap in the Danish context.

The dissertation is based on five articles that shed light on various aspects of the development of the concept of fractions, methodologically, empirically, and theoretically. It seeks to answer the overarching research question:

How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in the fourth grade?

Methodically I have addressed the first part of the research question, 'How can we investigate students' difficulties?' through the development of a measuring instrument analysed in Study 1, reported in Paper 1. The empirical foundations for this study consist of data collected in the form of student responses to different fraction tasks and expert evaluations of the measuring instrument's content. Afterwards, different statistical analyses have been carried out to investigate the measuring tool's accuracy, for example, a Rasch analysis.

The enquiry into the second part of the question, 'How are students' difficulties explained?', is therefore primarily based on quantitative data collected through the measuring instrument that has been developed. Where the student responses to selected tasks are examined in further detail, that is, the connection between the answers in fraction comparison tasks and previous answers to natural numbers arithmetic, the theoretical analysis was not based on quantitative data. However, the curiosity for the theoretical study 3 arose from the observed answers in Study 2, which is reported in Paper 2. Although this dissertation's studies are primarily based on the collected quantitative data, it is important to emphasise that various qualitative data were collected throughout the PhD project from teacher training courses and interviews with students through a fraction intervention instruction phase among others.

The five articles (studies) that are part of of this dissertation shed light on:

I. How to collect data through a quantitative measuring tool.
II. How the answers in four arithmetic operations are related to the answers in fraction comparison tasks.
III. How two different conceptions of equivalence influence the understanding of fractions.
IV. How natural number bias can distract in the fraction-learning process.
V. How high-performing and low-performing students differ in their development of fraction proficiency throughout the fourth grade.

The main conclusions can be summarised as follows: the newly developed measuring instrument measures within acceptable accuracy (Paper 1). The pattern between answers to four arithmetic tasks and answers to fraction comparison tasks differ, and there is a significant relationship between correct answers to division or division tasks and correct answers to fractional comparison tasks. However, these patterns differ depending on whether the fraction comparison task contains equal fractions or non-equal fractions. In addition, when the two compared fractions were equivalent, the pattern differed, and the comparison of equivalent fractions appeared to be more difficult (Paper 2). The theoretical Study 3 detects two understandings of equivalence: proportional and unity equivalence. Both conceptions of equivalence are important and appear differently in the understanding of fractions (Paper 3). For further exploration into the different answers to fraction tasks, the students' different answers were coded based on whether the answers could be explained as based in a natural number bias or not. The patterns between the different natural number bias aspects were then analysed. I found that the different types did not seem to be related to each other in the beginning of the fourth grade (Paper 4). Instruction on multiplicative principles seems to support the high-performing students' development of fraction proficiency; however, the same development was not found in the low-performing student group (Paper 5).

These results provide directions for different points of focus in the classroom.

a) It is of central importance that students be given the opportunity to develop the two understandings of equivalence; especially because these are related to the development of equivalence within, for example, algebra and percentages. Equivalence can thus support a conceptual understanding of these more advanced mathematical concepts as it helps to create coherence between concepts.
b) Students must be given opportunities to recognise the differences between natural numbers and rational numbers in different contexts in order to understand the differences between natural numbers and fractions and overcome the tendency of distraction from natural numbers.

c) Students with mathematical difficulties must be supported in developing connections between different mathematical subjects.

These results suggest that students' development of their concept of numbers is integrated with their understanding of integers and, at the same time, that students must develop a conceptual change in their understanding of numbers in order to accomplish the multifaceted fraction concepts. This means that students need to recognise how fractions (rational numbers) differ from natural numbers through, for example, density – that is, one can no longer count one's way to the next number in the series. One can therefore see fraction concept development as an integrated conceptual change of the concept of numbers.

Keywords: fractions, learning, development of the concept of fraction, equivalence, fourth grade

Dansk resume

Denne afhandling undersøger elevers forståelse og udvikling af brøkbegrebet i 4. klasse i det danske skolesystem. Internationalt er der generelt konsensus om vigtigheden af brøkforståelsen for elevernes videre matematiske udvikling, og at netop udviklingen af brøkbegrebet har vist sig at være særligt vanskelig for elever at lære. Men inden for dansk kontekst har der været en begrænset opmærksomhed på området og ingen kvalitative studier med afsæt i elevernes begrebsudvikling af brøker. Dette videnshul inden for den danske kontekst søger afhandlingen at råde bod på.

Afhandlingen bygger på fem artikler, der belyser forskellige aspekter i udviklingen af brøkbegrebet både metodisk, empirisk og teoretisk. Gennem afhandlingen søges at besvare følgende forskningsspørgsmål:

Hvordan kan vi undersøge og forklare elevers vanskeligheder ved udviklingen af det komplekse brøkbegreb i 4. klasse?

Metodisk har jeg adresseret den første del af forskningsspørgsmålet, "hvordan kan vi undersøge elevers vanskeligheder?", gennem udviklingen af måleinstrumentet beskrevet i Studie 2, som afrapporteres i Artikel 2. Det empiriske fundament for denne undersøgelse består af indsamlet data fra elevbesvarelser på opgaver i måleinstrumentet og evaluering fra eksperter af måleinstrumentets opgavers indhold. Efterfølgende er der lavet statistiske analyser for yderligere at undersøge måleinstrumentets nøjagtighed fx via en Rasch analyse.

Undersøgelsen af anden del af spørgsmålet, "hvordan forklares elevers vanskeligheder?", bygger derfor primært metodisk på kvantitative dataindsamlinger gennem det udviklede måleinstrument. Her bruges data til at undersøge elevbesvarelserne; fx sammenhængen mellem svarene på brøkopgaver og tidligere løste regneopgaver med naturlige tal. Det tredje studie bygger på en teoretisk analyse af ækvivalensbegrebet, men nysgerrigheden for netop en teoretisk undersøgelse udsprang af forundringen over de observerede svar i Studie 2, som er afrapporteret i Artikel 2. Selv om afhandlingens studier primært bygger på de indsamlede kvantitative data, blev der gennem projektet foretaget forskellige kvalitative dataindsamlinger; fx gennem observationer af lærerkurser og interview af elever gennem interventionsfasen.

Afhandlingen består ud over denne kappe af fem artikler (studier) der belyser:

I. Hvordan man kan indsamle data gennem et kvantitativt måleredskab.
II. Hvordan svar inden for hver af de fire regneoperationer hænger sammen med svarene på opgaver omhandlende sammenligning af brøker.
III. Hvordan to forskellige ækvivalensforståelser: proportional- og enhedsækvivalens influerer på brøkforståelsen.
IV. Hvordan naturlige tal kan distrahere i udviklingen af brøkbegrebet.
V. Hvordan højt præsterende og lavt præsterende elever adskiller sig i deres udvikling af brøkbegrebet gennem 4. klasse.

Hovedkonklusionerne kan opsummeres som følger: Det udviklede målingsinstrument måler inden for en acceptabel nøjagtighed (Artikel 1). Sammenhængene mellem de fire svar i de fire regnearter og brøksammenligningsopgaver afviger fra hinanden. Der er en signifikant sammenhæng mellem svar på multiplikations- og divisionsopgaver og svar på brøksammenligningsopgaver afhængig af, om brøkopgaven indeholder ækvivalente brøker eller ikke (Artikel 2). Ud fra en teoretisk undersøgelse i Studie 3 kan man finde, at der er to forståelser af ækvivalens: proportional- og enhedsækvivalens. Begge forståelser er vigtige og optræder forskelligt i forståelsen af brøker (Artikel 3). For at undersøge og forklare de forskellige svar og mønstre fundet i brøkopgaverne er en analyse af de forskellig naturlige tal distraktorer (natural number bias) blevet udført. Jeg fandt, at de forskellige naturlige tal distraktorer ikke ser ud til at hænge sammen i starten af 4. klasse (Artikel 4). Højt præsterende elever udvikler deres brøkbegreb, når de modtager undervisning i multiplikative principper, men den samme udvikling er ikke fundet hos de lavt præsterende elever (Artikel 5).

Disse resultater influerer og giver anvisninger til forskellige fokusområder i klasserummet.

a) Det er centralt, at eleverne får mulighed for at udvikle de to forståelser af ækvivalens – særligt fordi det hænger sammen med udviklingen af ækvivalens inden for fx algebra og procent. Ækvivalens kan dermed støtte en konceptuel forståelse af disse begreber, da det er med til at skabe sammenhæng mellem begreber.
b) Eleverne skal gives mulighed for at udvikle en forståelse af forskellene mellem naturlige tal og rationale tal i forskellige kontekster og dermed forstå forskellen mellem naturlige tal og brøker. Med andre ord skal de overkomme tendensen til distraktorerne fra de naturlige tal.
c) Elever med matematikvanskeligheder skal støttes i at udvikle sammenhænge mellem forskellige matematiske emner.

Resultaterne tyder på, at elevernes udvikling af deres talbegreber på den ene side er integreret med deres heltalsforståelser, og på den anden side skal de samtidigt skabe en konceptuel forandring af deres talforståelse for at udvikle det komplekse

brøkbegreb. Det betyder, at eleverne skal lære, hvordan brøker (rationale tal) adskiller sig fra de naturlige tal gennem for eksempel *densitet*. Dvs. at man ikke længere kan tælle sig frem til det næste tal i rækken. Man kan derfor se det som en *integreret konceptuel forandring* af talbegrebet, når brøkbegrebet udvikles.

Emneord: brøker, læring, udvikling af brøkbegrebet, ækvivalens, fjerde klasse

Acknowledgements

When I started on this journey three years ago, little did I know what the future would bring. It has been a journey of new people, new insights, and new experiences.

My interest in fractions started when I began my work as a math teacher for an eighth-grade class at Randers Realskole. I observed how some students were struggling with their understanding of the multifaceted concept of fractions. My interest in this topic was further stimulated when I started as a mathematics teacher at a smaller country school where I spent many afternoons helping students with their math homework, and later when I was a special teacher in mathematics in Time to Learn, where I had private students from all over the country. Every student tells their own story about their difficulties with mathematics; especially fractions. I am grateful for all their stories and all that I have learned from these shared students' experiences with fractions. I also want to thank all the students and teachers for your time and help during this PhD project – you made this project possible. I am grateful to the Independent Research Fund Denmark for supporting this research and making this journey possible.

Over the last three years, I have been affiliated with two organisations; Aalborg University and VIA University College. This has given me two sets of colleagues. Unfortunately, the last year was influenced by the Corona pandemic, and therefore I was not able to be physically present in the research group at Aalborg University in Copenhagen. For this, I am very sorry because you always made my days in the lab more interesting and rewarding – not to mention the laughs during lunch break. Thanks to Camilla Finsterbach for all our PhD talks over the phone; these made my long days containing breaks with laughter. I want to express my special gratitude to my supervisor Lars Birch Andreasen for all your support, your feedback, your advice, and the patience you have given me. You took over when Morten Misfeldt left Aalborg University, and you made my PhD project stable through this transition and during the uncertain times of the pandemic.

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Content

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Chapter 1: Introduction

The subject of this PhD project is fractions: how students understand fractions, and how fractions are taught to students in elementary school in Denmark.

The concept of rational numbers has proven to be a critical point in students' development of more advanced mathematical thinking (Bailey et al., 2012; Siegler et al., 2011, 2012; Siegler & Pyke, 2013). In particular, the rational number notation known as fractions is associated with algebra readiness and algebra ability (Booth et al., 2014; DeWolf et al., 2015b, 2016; Siegler et al., 2013). Unfortunately, many students have difficulty in developing an understanding of fractions (Tian & Siegler, 2017; Torbeyns et al., 2015), and these difficulties often persist as students advance through their education (Fazio et al., 2016; Schneider & Siegler, 2010).

In a Danish context, data obtained from a test question on a 2019 final examination in mathematics presented to students after 10 years of compulsory education revealed that only 42% of the students could successfully identify the fraction that would result from $\frac{5}{6}$ added to $\frac{1}{3}$ (Winsløw, 2019b). An ongoing study of the 13% of Danish youth who are not in employment, education, or training indicated that 70–88% of them had not completed the mathematics section of the compulsory school leaving exam, whereas only 40–50% had not yet completed the Danish language part of the exam (Görlich et al., 2015). This gap in the mathematics performance between students who continue in the educational system and the group who leave compulsory school without further education arises early in primary school and increases throughout the course of schooling (Gustafsson et al., 2015). Moreover, there is a clear connection between Danish students' mathematics grades in school and their ability to enter and complete secondary education; especially among young men in vocational schools (Hvidtfeldt & Tranæs, 2013). Mathematics can be considered one gatekeeper to further success in the Danish school system, and thus, success in later life.

International studies have found that rational numbers have especially proven to be the gatekeeper to more advanced mathematics (Booth & Newton, 2012; Siegler et al., 2013) and developing an understanding of fractions is particularly challenging for many students (Fuchs, Schumacher, et al., 2016; Hwang et al., 2019; Lortie-Forgues et al., 2015; Tian & Siegler, 2017). In particular, this subject is difficult for students with mathematical disabilities or difficulties (Hecht & Vagi, 2010; Mazzocco et al., 2013). Based on these findings, it should be essential for mathematical education and research to continue to explore how mathematics, and especially rational numbers, are taught in order to ensure that every student is given the best opportunity to learn mathematics.

Many international studies have been conducted on this topic over the last 40 years (Lortie-Forgues et al., 2015), but few studies exist in this area in the Danish context (e.g., Putra & Winsløw, 2018; Winsløw, 2019a), and the studies that do exist have focussed on the teachers' content knowledge (Putra & Winsløw, 2018, 2019) and the learning environment in the classroom (Larsen et al., 2006). Despite this research on the topic of fractions, students continue to show considerable difficulties.

Given the extensive international research on the subject and the importance of the topic, it is reasonable to investigate students' fraction-related difficulties further for at least three reasons:

- Students are still struggling with fractions.
- In the Danish context, little research has been done on the topic.
- Fractions are an important gatekeeper in students' mathematical development.

The purpose of this PhD project is to investigate and understand more about students' difficulties when developing their understanding of fractions. The next sections of the introduction will offer a definition of students with mathematical difficulties. It will then define fractions, and the overarching problem of this project will be explored. This will be followed by a short presentation of the current PhD project and end with an overview of the whole dissertation.

1.1 Students with mathematical learning difficulties

In this dissertation, the terms 'students with mathematical difficulties', 'struggling learners in mathematics' and 'low-performing students in mathematics' are used in different contexts. Therefore, it is important to define these terms. In the Danish research field and school culture, the term *elever i matematikvanskeligheder* ('students *in* mathematical difficulties') is emphasised rather than *elever med matematikvanskeligheder* ('students *with* mathematical difficulties'). The preposition *with* indicates something that one is stuck with or has to live with, whereas *in* indicates that the situation may change (Lindenskov, 2010). However, this distinction is not made internationally. Instead, the term 'disability' or 'difficulty' is a way to illustrate this difference. According to Mazzocco (2007), mathematical learning disabilities suggests a biologically based disorder, whereas mathematical learning difficulties is a broader term referring to children who show poor mathematical achievement that may be explained by several causes and circumstances (e.g., psychological reasons such as anxiety or sociological reasons such as family background). Therefore, it does not only refer to a presumed biological explanation.

Previously, the terms *mathematical learning difficulty* and *mathematical learning disability* have not been clearly defined, which has led to the use of different criteria for defining students who struggle with learning mathematics (Jitendra et al., 2018).

The traditional definition of learning difficulties has often been based on the discrepancy hypothesis, meaning that a student with learning difficulties in mathematics is achieving far below expectations (Lunde, 2012).

However, there are multiple examples of other definitions; for example, students with mathematical difficulties could be identified as those scoring < 25th percentile on a mathematics test (Dennis et al., 2016; Lunde, 2012; Shin & Bryant, 2015). Another definition could be students considered by their classroom teachers to have difficulties in mathematics (e.g., Gresham & MacMillan, 1997). Mazzocco and Räsänen (2013) found that mathematical learning difficulties were used synonymously with *developmental dyscalculia*, but at the same time, learning difficulties were distinct from *developmental dyscalculia* when it referred to a larger group of students with mathematical difficulties.

Overall, there are no consistent criteria to determine or judge whether learning difficulties are present in mathematics; therefore, the way the term is used varies. The term 'difficulty' implies a lower-than-average performance. Consequently, cut-off scores were used. The cut-off score is a way to operationalise mathematical difficulties in quantitative studies. However, the term *mathematical learning difficulties* has been defined by some researchers as students with poor achievement in mathematics from any number of causes (Mazzocco, 2007). In this project, the sub-score in the national test score for third grade is used in Study 5, and the cut-off score is scoring < 25th percentile. This is discussed further in Chapter 9.2.2.

The National Council of Teachers of Mathematics (2020) has developed the following definition of students with learning difficulties in mathematics: 'Students who struggle with learning mathematics regardless of their motivation, past instruction, and mathematical knowledge prior to starting school' (p. 1). When I use the term mathematical learning difficulties in this dissertation, it refers to this definition. However, in the last study (Study 5), in which I use the cut-off score of < 25th from the national test, I use the term low-performing students for this sub group. I need to emphasise that this term is not equal to 'students with learning difficulties in mathematics', but the group will most likely contain students with mathematical difficulties. Therefore, the classes can be seen as regular representations of an ordinary school class with an average population of fourth grade students which most likely include both low- and high performing students (see Chapter 6.2.1). The 25 percent cut-off was detected across schools and classes from the total of participants (N = 398).

1.2 Fractions

The mathematical topic of fractions has been shown to be a stumbling block for many students in general (e.g., Booth & Newton, 2012; Braithwaite et al., 2019; Hecht & Vagi, 2012) and for students with mathematical learning difficulties in particular (e.g., Mazzocco et al., 2013; Roesslein & Codding, 2019). The concept of fractions has a multifaceted structure that involves not only the ability to look at the notation as a rational number but also to see it as a proportional relation or operation division (e.g., Lamon, 2012). In this introduction, it is important to emphasise that a *fraction* cannot be explained by a unique mathematical definition, unlike the term *rational number*. Although the two terms are connected, they are not synonymous. All rational numbers can be expressed in the symbolic fraction form, for example, $.25 = \frac{1}{4}$. However, not all written numbers using the symbolic fraction notation are rational numbers, for example, $\frac{1}{\sqrt{2}} \notin Q$. Mathematically, the notation of a fraction is defined as $\frac{a}{b}$. In the context of this dissertation, the term 'fraction' refers to a rational number. This means that in $\frac{a}{b}$ both a and b are integers and $b \neq 0$.

Various researchers have made distinctions regarding the term *fraction*. Thompson and Saldanha (2003) distinguish between a *fraction* as a 'personally knowable system of ideas' and a *rational number* as a 'formal system developed by mathematicians'. They made this distinction because the mathematical formal system of rational numbers is abstract, which means that elementary school students are often unable to fully understand and comprehend the system. The term *fraction* and its notation system are further described and defined in Chapter 3.

1.3 Presentation of the PhD project

This project uses an enquiry-based approach grounded in the methodology of pragmatism (Brinkmann, 2011; Buch & Elkjaer, 2020; Elkjaer, 2000; Pedanik, 2019). The theoretical framework primarily stems from Dewey's later studies (Dewey, [1933]1986, [1938]1986), and this methodology is further elaborated on in Chapter 2. Therefore, each of my five studies included in the project must be viewed as an enquiry process in which I investigate why students have difficulties with learning fractions. In the enquiry process described by Dewey, it is important that the enquiry starts from an experienced problem. Therefore, I briefly describe my first encounter with the complex field of teaching fractions.

My curiosity and interest in studying students' difficulties with developing an understanding of fractions were sparked when I started working as an elementary school teacher in 2004. My first experience with students' problems with fractions occurred in a grade 8 classroom, where several students thought that when adding two fractions with no common denominator, they should simply multiply the denominators and then add the numerators (e.g., $\frac{1}{3} + \frac{1}{4} = \frac{2}{12}$). They expressed strong faith in this incorrect method and argued that it was how their previous teacher had taught them to add two fractions. Because of my position as a new mathematics teacher, it took a long time before they listened to my arguments. I was younger and had not been teaching for long, and I had to earn their respect. I was certain that their prior teacher had not instructed them to add two fractions this way. However, it astonished me how they could not see that $\frac{2}{12}$ was equal to $\frac{1}{6}$ and that the result of adding the two fractions was smaller than the sum of both fractions. This argument, which was logical from my perspective, had no effect. I ended up showing them the right procedure with the knowledge that they had not gained any conceptual understanding of adding fractions while doing so.

The above experience as a teacher was my starting point in the complex field of teaching and learning fractions. It can be recognised as the starting point of my enquiry process, which later led to my journey as a PhD student. However, this enquiry does not follow a linear process but rather resembles organic circles (Buch & Elkjaer, 2020; Elkjaer, 2000). Nevertheless, time is linear, and therefore the project exists simultaneously as a linear time-managing process (see Fig. 1) (i.e., collecting different datasets, conducting the intervention, etc.) as well as a circular enquiry process of exploring and questioning, which has led me to novel insights and questions. Furthermore, studies overlap, take longer than expected, or branch into new directions.

Looking at the linear structuring of this project, it is based on the four following phases: 1) My observation of the problem as an elementary school teacher of mathematics (the first experience phase). 2) My initiation as a researcher investigating the field, starting with the first literature review (the initial phase) and developing materials for the project (intervention and measuring tool). 3) The first data collection and investigation in the field (first data collection phase). 4) Implementation of an intervention in the field and different data collection methods during this period (intervention phase). 5) Writing and finishing the dissertation (completion phase). This gave two independent data collections in the third phase (the first data collection) and the fourth phase (intervention phase).

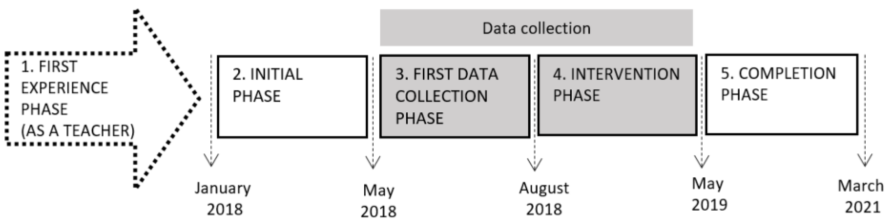

Fig. 1 Timeline over the different phases of the project

The current PhD project is an independent project funded by the Independent Research Fund Denmark. I therefore took part in two research groups, 'IT and Learning and Design' at Aalborg University and 'Program for Science and Matehmatics' at VIA University College. The research group at VIA University College also assisted me with organisations and discussions during my PhD. In particular, the research project group connected to the 'Teaching Routines and

Content Knowledge Project' (TRACK) was established by the researchers at VIA University College. Through TRACK, I received support in terms of my communication with the schools, a graphic designer as well as teachers and students connected to the project. In addition, the research group made it possible for national and international experts to help with developing the study instruments as well as facilitating aid from a contact expert teacher who evaluated the intervention material. However, this PhD project was an independent research project and was centred on a separate enquiry process related to fractions. Originally, the PhD project was designed as a quasi-experimental design with a control group. However, it changed for several reasons during the three-year period, which will be further discussed in the last chapter of the dissertation.

1.4 Aim and research questions

The aim of this PhD project was to explore the concept of fractions and how student's learning was supported and developed. This led to the overall research question in this dissertation and the starting point for the multifaceted enquiry process:

How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in the fourth grade?

As previously mentioned, this project uses enquiry-based research defined by pragmatism (outlined in CHAPTER 2: METHODOLOGY). The research question is connected to an enquiry process into the observed problem of why many students have difficulty in understanding fractions. In addition, it is a process of questioning, exploring, and understanding the problem in a continuous manner (see Fig 2). The knowledge developed during this project is organised into five papers that each contain a separate study which is related to and informs the overarching research question. This means that each of the five studies is reported in a separate paper.

It is important to emphasise that the five studies overlap and, at the same time, explore new corners of the problem (outlined in Chapter 8). The descriptions below give a brief introduction and overview of how the studies were connected and generated during the process.

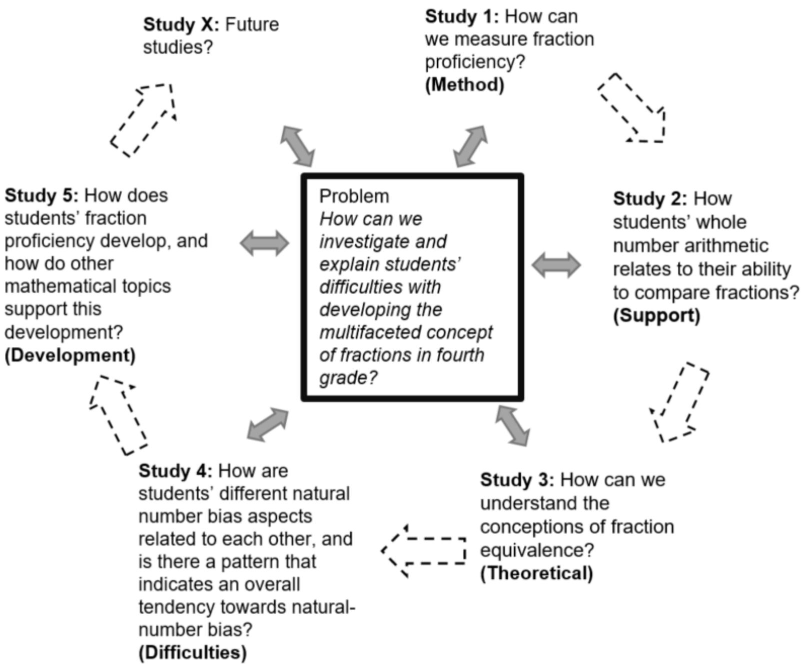

Fig. 2 The five studies informing the project. Each study is reported in a separate paper

Study 1: How can we measure fraction proficiency? (Paper 1)
Originally, it was planned that Study 1 would be finished during the first data collection phase (phase 3), but the development of the measurement instrument was more complex than anticipated, and I needed more time. The development included finding and analysing existing fraction measurement tools and designing and validating the accuracy of the measurement. Consequently, this study continued into the intervention phase, which was not ideal but forced by reality. As a result, Study 1 includes data from both phases 3 and 4. Retrospectively, this might be a lifelong study of how we can gather information/data about the observed problem of some students' difficulties with fractions and create new meaning from these data. The developed measurement tool must continue to be developed in the process of creating meaning from new data, representing a never-ending process.

Study 2: How does students' whole number arithmetic relate to their ability to compare fractions? (Paper 2)
Study 2 was conducted using the data collected during the first data collection phase. I made observations and identified patterns in the students' answers when comparing fractions, which piqued my curiosity. The students' answers in the developing and pilot testing of the measurement tool in Study 1 showed that I needed to investigate

equivalence and further answers that produced patterns in the dataset. How could I explain that the students showed greater difficulties in comparing $\frac{1}{4}$ with $\frac{2}{4}$ than $\frac{5}{11}$ with $\frac{3}{5}$? Could it be connected to their knowledge of the four arithmetic operations?

Study 3: How can we understand the concept of fraction equivalence? (Paper 3)
My curiosity about the difficulties of comparing equal fractions led to my search for knowledge about equivalence. I soon began Study 3, which was a theoretical study. In it, I asked the following question: Why is equivalence important in more advanced mathematics, and how can equivalence be seen in two different conceptions? The quest to explain and make sense of why $\frac{1}{4}$ compared with $\frac{2}{8}$ had shown to be more difficult than $\frac{5}{11}$ compared with $\frac{3}{5}$ continued into Study 4.

Study 4: How are students' different natural number bias aspects related to each other, and is there a pattern that indicates an overall tendency towards natural number bias?
In this study, I looked deeper into natural number bias to explain comparison difficulties. Natural number bias can be explained as the tendency to use natural numbers reasoning and understanding when working with fractions. An example could be that $\frac{1}{3}$ is interpreted as bigger than $\frac{1}{2}$ because 3 is bigger than 2. This study explores how natural numbers can detract from the understanding of fractions in contrast to Study 2, which investigated how whole number arithmetic operations were positively related to fraction comparisons.

Study 5: How does students' fraction proficiency develop and how do other mathematical topics support this development?
Study 5 explored how high- and low-performing students developed their fraction proficiency during fourth grade. The students followed the same curriculum during the school year, and I had developed instructional material in fractions that was used in an intervention period around Christmas in the school year 2018/19. The developed instructional material considered the fraction instructional material in particular, which exhibited a greater focus on fraction equivalence compared with the content in the most common mathematics books used in Denmark (see Chapters 5.2 and 6.3).

1.5 Overview of the dissertation

After this brief introduction to the project (see Fig. 3), I present its overall methodological and philosophical foundations in Chapter 2. In Chapter 3, I introduce the terminology related to fractions and give a short historical overview of the development of fractional notation. In Chapter 4, the relevant literature is reviewed to clarify what is known about how students learn to understand fractions. Thus, this chapter includes four reviews: *4.2 Review (1): Mathematical knowledge and fraction proficiency* aims at elaborating on what it means to understand mathematics and fraction proficiency. *4.3 Review (2): Natural number bias* and *4.4 Review (3): Number knowledge development* sum up how fractions can be viewed as a component of a student's overall development in number knowledge, and I develop and unite the theoretical framework. Lastly, *4.5 Review (4): Fraction interventions* provides an overview over an analysis of how previous intervention studies have been carried out on fraction interventions targeting students with mathematical difficulties/struggling learners.

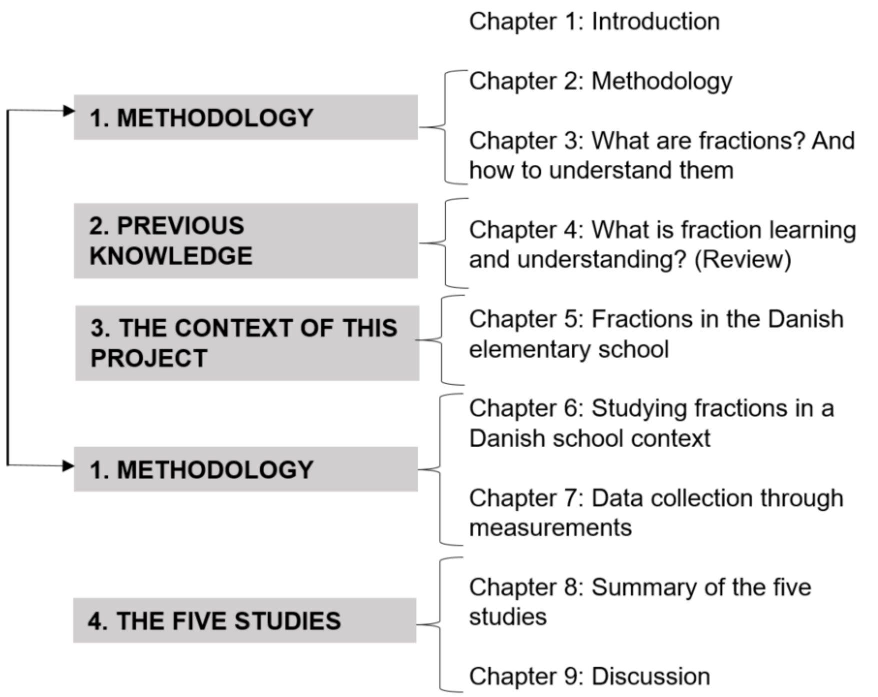

Fig. 3 Overview of the chapters

Chapter 5 introduces and analyses how fractions are presented in the official Danish curriculum, and a simple content analysis of three commonly used books of mathematics is conducted. Chapter 6 outlines and discusses the projects and how I studied fourth grade students' fraction proficiency. In Chapter 7, I present the methodological considerations connected to data collection through measurements. Chapter 8 is a summary of the five studies described in the five papers. Finally, in Chapter 9, I discuss the results, methodological choices for the PhD project, the implications for instruction, and the contribution to the field, including suggestions for further research. The five papers are placed at the end of the dissertation; however, they will be removed from the final publication. One paper is still under revision for a journal (Study 5), and one paper is still a manuscript (Study 1), so they cannot be published elsewhere beforehand. (All five papers will be published with open access.)

The structure of the dissertation can be seen as follows: The overall aim for the introduction to the five studies can be seen as containing four main elements: First, a methodology element. Second, a previous knowledge element. Third, an element that describes and analyses the Danish context of the dissertation. Fourth, the actual studies and their results. The methodology element is divided into two parts that I placed before the descriptions of the five studies. The aim of the introduction to the five studies is to elaborate the methodology behind the studies, elaborate the context in which the studies were conducted, and create coherence and transparency of the current research project's development.

Chapter 2: Methodology

As mentioned in the introduction, this PhD project draws on pragmatism as a research methodology. The theoretical framework of this approach primarily originates from Dewey's theoretical work, and it provides the basis for the enquiry-based research methodology of this PhD project. The consequences of this research strategy are explained and discussed in order to improve the transparency of the project. The purpose of this chapter is to explain the methodological approach to the choices made in the process and how to interpret knowledge generated from this PhD project. First, the nature of pragmatism is introduced. Next, central concepts in Dewey's theories are explained and reflected (experience and enquiry), and finally, the enquiry phases are explained in the context of the current PhD project.

2.1 The nature of pragmatism

A common oversimplification of pragmatism has been merely asking, 'What works?' and this oversimplification has been a persistent problem throughout the last century. Fortunately, there have been ongoing discussions about the nature of pragmatism, which have also created a more varied understanding of its nature (Goldkuhl, 2012; Morgan 2014; Silva et al., 2018)

The common simplified question 'What works?' is not in itself an accurate conceptualisation of pragmatism; other questions are needed to capture its multifaceted framework. We must instead raise questions in our research such as 'why' or 'how' questions, for example: Why do we define this as a problem in itself? Why do we do our research this way and not another? (Dewey, [1938]1986; Morgan, 2014). In this context, the questions become: 'Why do we define fraction difficulties as a problem?' 'How and why do we investigate fraction proficiency in school?' When we ask these types of questions, we focus on our different choices in the research process. For example, why do we choose to say having difficulties in learning fractions is a problem, or why do we choose to use a Pearson correlation coefficient analysis in looking for answers? It is a simplification of pragmatism only to ask, 'What works?' because in reducing the method to that question, we ignore our choices about both the problems we will investigate and the essence of those problems.

Pragmatism has been seen as a paradigm (e.g., Goldkuhl, 2012) or as a methodological approach (e.g., Parvaiz et al., 2016) in which it is essential for the researchers to ask the 'right questions'. Determining the 'right questions' must involve the values of the researcher, and the researchers must therefore also question these values or beliefs. Therefore, pragmatism is not based purely on either a quantitative or a qualitative approach. What method the researcher chooses is determined by the question or enquiry (Fendt et al., 2008; Morgan, 2014; Onwuegbuzie & Leech, 2005). My choice of method in this research has been driven by the problem observed and by exploring this problem by questioning and enquiring further into the topic. Therefore, pragmatism has been my methodological foundation because it asserts that the problem determines how we investigate and thereby capture the multifaceted field of mathematical educational research. There is no theory or method that determines how to explore and investigate the field; it all depends on the research process and its transparency. In the current PhD project, four of the five studies are based on quantitative research. Primarily, I chose to collect my data through the measurement tool developed in Study 1. Different methods and statistical analyses are used in each study. The choice of data, statistical models, and analysis is driven by the overarching research question in the PhD project: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?*

To summarise, the essence of pragmatism is not connected to a particular method, but the choice of method is based on the investigation of the problem.

2.2 Experience as the bridge

As previously mentioned, this PhD project is primarily based on Dewey's theories of pragmatism; a framework in which experience is a central concept. For most of his life, Dewey developed and conceptualised pragmatism by orientating it towards human experience. The central theme of Dewey's theory is the attempt to overcome the epistemological barriers between the observer and the observed (Dewey, [1920]1986, [1933]1986). As he states:

> Experience includes what men do and suffer, what they strive for, love, believe, and endure, and how men act and are acted upon, the ways in which they do and suffer, desire, and enjoy, see, believe, imagine—in short, processes in *experiencing.* Experience denotes the planted field, the sowed seeds, the reaped harvests, the changes of night and day, spring and autumn, wet and dry, heat and cold, that are observed, feared, longed for; it also denotes the one who plants and reaps, who works and rejoices, hopes, fears, plans, invokes magic or chemistry to aid him, who is downcast or triumphant. It is 'double-barrelled' in that it recognizes in its primary integrity no division between act and material, subject and object, but contains them both in an unanalysed totality. 'Thing' and 'thought'…are singlebarreled; they refer to products discriminated by reflection out of primary experience (Dewey, 1925, p. 8).

According to Dewey, experience must be seen as both the subject's being and acting in the world, not as the subject's being outside and looking into the world (Elkjaer, 2000). Moreover, 'experience' often implies that a subject passively senses and observes an object external to the subject itself, but this is not how Dewey defines experience – there are no divisions of act and object, or of subject and object. Overall, Dewey's theories and ideas can be seen as founded on the idea of an organic unity. There has been a critique of the idea of organic unity where the principle of continuity of experience defines the concept of experience that transcends the boundaries, which can be seen as a simplification. For example, Rorty (1998) states that this can be seen as an attempt to 'marry Hegel with Darwin' (p. 291). The broader discussion of the implications exceeds the scope of this dissertation. However, it is important to raise the critique because unification can be seen as a simplification; yet, I argue that Dewey's theoretical framework of organism unity makes it possible to capture not only both sides of subject and object, but the overall complexity of acting and being in the world.

Dewey's concept of experience was defined in his later work as transactional. Transaction refers to an interpretation of reality that is not static or isolated but that exists in the relationships or exchanges with other events. Transactions means that the elements, humans, and surroundings in reality influence one another and are therefore changed by this influence. In contrast to the term 'interaction', according to which the elements are not changed, the focus of the term 'transaction' is on the relation between the elements (Brinkmann, 2011; Dewey & Bentley, [1949]1973). Dewey's theory tries to overcome the gap between the observer and the observed through human experience, meaning experience is not to be seen merely as subjective, but is both subjective and objective because it is transactional in nature (Brinkmann, 2011). In this way, Dewey argues that there are several ways to interpret the world; there is no single point of view that can reveal the entire picture because the nature of the world is based on experience. Therefore, knowledge is not seen as final or true but instead continues to develop and change, as Dewey ([1938]1986) argued: 'The history of science also shows that when hypotheses have been taken to be finally true and unquestionable, they have obstructed enquiry and kept science committed to doctrines that later turned out to be invalid' (p. 145). Hence, Dewey rejects the existence of direct, exact knowledge and emphasises that all knowledge has mediational and inferential aspects (Dewey, [1938]1986).

This does not mean that there is no true reality; however, it means that reality is constantly changing because of our actions. Any attempt to find a stable, enduring, external reality outside ourselves is not possible because of our constant action in the same reality (Dewey, [1920]1986, 1925, [1933]1986, [1938]1986). As a result, the findings of this project cannot be considered enduring reality but must be seen as a matrix of enquiry into why fractions can be difficult to learn – I only experience the mediated reality, and I mediate my reality by the methods chosen for this current project. I chose to primarily collect data about students' fraction knowledge by my developed fraction measurement tool (described in Study 1), and data generated from this measurement tool mediates the reality as well as me as a researcher mediating what I observe as a problem. That students showing more difficulty comparing equal fractions might not have occurred if the measurement had a different design or if I, as a researcher, did not observe the problem.

In connection with this interpretation/understanding of reality, it must be emphasised that Dewey underlines the importance of actions. Actions create the essential gap between pragmatism and most versions of interpretivism (e.g., relativism) because, according to interpretivism, we are free to interpret our experiences in whatever way we want to. Hence, actions have outcomes that are often quite predictable, and we build our lives around experiences that link actions and their outcomes. We are not free to interpret our experience in any direction we choose, because we must consider the outcomes of the actions. That students show difficulties in understanding fractions is an experience shared by both the teachers and the students themselves; however, saying that this difficulty is a result of a poor number sense can be seen as a hypothesis that needs to be explored by, for example, making an intervention working with number knowledge that leads to students being better at fractions. Even though Dewey denies that there is an unchangeable, real knowledge, experiences create predictable knowledge or, as Dewey calls it, 'warranted assertibility' (Dewey, [1938]1986).

For this reason, I do not consider the knowledge generated by the different studies in the PhD project as 'true knowledge'; instead, knowledge developed during the project must be interpreted as warranted assertibility.

2.3 Enquiry: the basis for the project

Enquiry is always embedded in the framework of biological and cultural operations. Dewey's emphasis on cultural factors specifies that every act of enquiry is based on a background of culture and therefore takes effect in the modification of the conditions out of which it grows (Dewey, [1938]1986). Experience and enquiry are not limited to the private subject; they are centred on a context or culture. My cultural background as a teacher and the Danish school system will influence the conditions out of which the enquiry grows, and so will the research culture of which I am a part in my study of fourth-grade classes in a municipality and in the research group in Aarhus Teacher Education and Aalborg University. Enquiry must be seen as organic; that is, it will be shaped by the conditions of the surroundings.

Dewey argues that enquiry and questioning must be closely connected and related in the term 'meaning'. He explains the relationship between enquiry and questioning by arguing that when we enquire into a phenomenon or a problem, we must also be in the process of questioning it. Problems grow out of actual situations, and the nature of a problem must be defined according to the elements in a given situation that are experienced and settled in observations (Dewey, [1938]1986). In this project, my problem is founded on the observations that students have difficulties solving mathematical problems that involve fractions and that I must continue to be in a process of questioning this problem. During the research process, I tried to question how the students' problems in learning the concept of fractions developed through the different studies in each paper (see Chapter 8 for a summary of the studies). The given situation is described and analysed based on the different curricula in Chapter 5.

Dewey further defines the situations that motivate enquiry as indeterminate situations, meaning that the situation of an organism must be interpreted in the environmental context of objects and events as well as placed in the timeframe of past, present, and future (Dewey, [1938]1986). Here it is clear that Dewey's framework also had a biological, organic approach. An indeterminate situation is further described as an ongoing automatically habitual activity that does not satisfy a need in a situation. The term 'indeterminate' is central to Dewey's theory and emphasises the significance of environmental objects and events in a given situation. The unique experience is connected to indeterminateness in any given circumstance, and it controls the enquiry until the enquiry has transformed the situation into a determined one (if the enquiry is successful). Therefore, even though knowledge is warranted assertibility, it is still possible to predict or determine what a result will be.

The current research project's starting point – students' difficulties in learning fractions – can be seen as an indeterminate situation in which I continue enquiring and questioning: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* In the enquiry process, I try to find new knowledge about the answer to the question; however, the knowledge is still seen as warranted assertibility. The enquiry must be based on and determined by judgement connected to the question of 'why'. This means that each study choice, such as data collection methods and statistical analysis, is connected to recognised problems in the study. For example, in Study 1, how can we measure and study fraction proficiency? Can it be done by a curriculum-based measurement, or is it better to interview students? Do my test items measure fraction proficiency? Can using confirmatory factor analysis (CFA) explore whether the items are related by a latent factor? If not, I should use an exploratory factor analysis (EFA) instead.

Four of the five studies are based on quantitative data collection and must be seen as having some advantages in moving from the indeterminate to the determinate; the many observations make it possible to find determinate patterns. However, the quantitative data collection will contain the issue of whether the complexity of the intermediate situation is reduced too much, or whether an important variable is not captured. In the cultural complexity context, the students are unique individuals, and the teachers have various backgrounds, and the situation is connected to the measurement situation (the student might be given the right opportunity to show their fraction proficiency in a test situation that differs from the regular classroom instructions). This complexity cannot be fully captured in my quantitative data collection, and it will not ever be possible to capture the complexity in any given situation. Qualitative data will have the same problem. The complexity will also change constantly, so we constantly act in the situation and thereby change it. I am changing the situation by conducting a measurement and trying to capture and investigate students' difficulties with fractions, and in doing this, I also change the reality by my action. As Dewey would say, reality is constantly changing because of our actions. (Dewey, [1920]1986, 1925, [1933]1986, [1938]1986). To compensate for the reduced complexity of the intermediate situation in the measurement data collection, I also made observations and interviewed students in the intervention phase (phase 4). These data will be analysed in future studies, although it will still be a reduced picture of fraction understanding – it will be a smaller picture with more details.

To summarise, the importance of questioning in the enquiry process must be emphasised. Dewey argues that enquiry and questioning must be closely connected and related in the term *meaning*. He explains the relationship between enquiry and questioning by arguing that when we enquire into a phenomenon or a problem, we must also be in the process of questioning it. Problems grow out of actual situations, and the nature of a problem must be defined according to the elements in a given situation that are experienced and settled in observations. In this current PhD project, the problem is founded on my experience that some students seem to have difficulties when working with mathematical problems and tasks that involve fractions and that I must continue to be in the process of questioning this problem. In other words, during the research process, I tried to question whether an understanding of the students' difficulties in learning the concept of fractions was developed through the different studies in each paper.

2.4 The phases of enquiry

'Enquiry is the controlled or directed transformation of an indeterminate situation into one that is thus determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole' (Dewey, [1938]1986). The creation of this unified whole is an ongoing process, and this dissertation should be interpreted as a picture of this process. In other words, knowing comes about when enquiry leads to an understanding that goes beyond ordinary apprehension.

Dewey defines the six phases of enquiry in *How We Think* ([1933]1986) and in *Logic* ([1938]1986) (see Fig. 4). In the following, I will present the terms for the phases used by Dewey in *Logic* (pp. 109–122) and the corresponding but slightly different terms used in *How We Think* (pp. 200–210) in parentheses. Although only five phases are described in *How We Think*, I found that the sixth phase described in *Logic* is especially important in this context because it describes the difference between scientific knowledge and common knowledge. The section is structured with an introduction to the phase defined by Dewey, followed by examples of how this phase has influenced questionings connected to this PhD project.

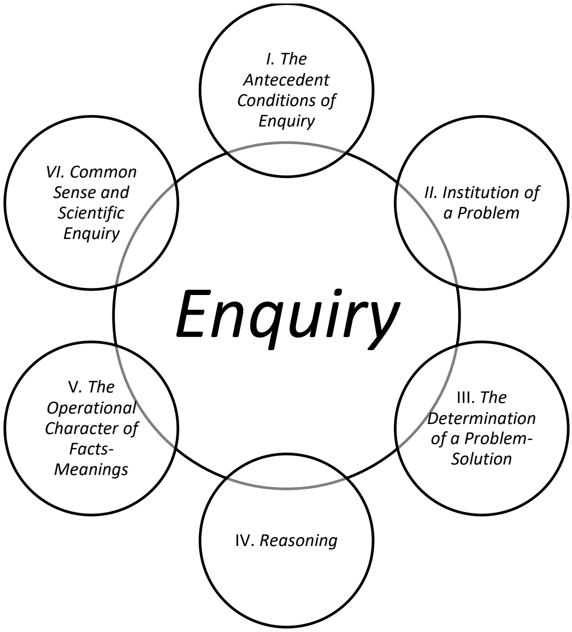

Fig. 4 The phases of the enquiry process

I) *The Antecedent Conditions of Enquiry: The Indeterminate Situation (Suggestion)*. A perplexing or problematic situation arrests or grows out of an observed or direct activity. Dewey used a broad variety of terms to define indeterminate situations (disturbed, troubled, ambiguous, confused, full of conflicting tendencies, etc.). 'It is the situation that has these traits. We are doubtful because the situation is inherently doubtful' (Dewey [1938]1986, p. 109). In *How we think* ([1933]1986) the term 'suggestion' is used for the first phase, and it is a slightly different definition. In *How we think* ([1933]1986), this first phase looks at different kinds of suggestions to go further into the enquiry. Therefore, Dewey here emphasises that when we find ourselves 'in a hole', we need to come up with different suggestions on how to solve the problem. In *Logic* ([1938]1986), the focus is on defining the condition for the problem, whereas in *How we Think* ([1933]1986), the focus is on how the condition makes us come up with the need for enquiry. This might seem like two distinctions, but I will argue that they are related and alike. Hence, the suggestions are creating the condition, and the condition is creating the suggestion. We act in the world and create the world.

In this PhD project, the antecedent condition of enquiry can be seen as a process in which we must ask ‘Why?’ or question the situations: ‘Why do we need fractions (the fraction notation)?’ ‘Why is the concept of fraction complex?’ ‘What is the complexity?’ ‘Do we need fractions? (If we do not, we do not have a problem)’ ‘What is the concept of the fraction?’ ‘Alternatively, why am I experiencing this situation as a problem?’

II) Institution of a Problem (Intellectualisation). The unspecified situation becomes an issue for the enquiry process because it is subjected to enquiry (Dewey, [1933]1986, [1938]1986). In other words, we need to determine whether the situation requires enquiry. Is it a problem in general?

Here I need to question the situation: ‘Is it a general problem?’ ‘Where is the problem of fractions situated?’ Here, this phase is connected to the reviews in Chapter 4, which explore the international investigation of the problem. In Chapter 5, the introduction of fractions in the Danish school system is analysed. The reviews are placed in the same section, and the reviews also influence and play an important role in the other phases, so as outlined, it is a dynamic process.

III) *The Determination of a Problem-Solution (The guiding idea, hypothesis).* Dewey defines this phase as the phase in which a possible solution that is founded on factual conditions is suggested. These conditions are secured by observations. Dewey does not define what kind of observations these are; it must be determined by the situation or the enquiry. This means ideas and expected consequences (forecasts) of what will happen when planned operations are put into practice (Dewey, [1938]1986).

In this phase of the project, I examined the data from the first collection, looking at conditions such as the students’ difficulties with comparing fractions (Study 2). An implicit hypothesis was made that there was a relation between the students’ ability to solve multiplicative whole-number tasks (e.g., 12 × 74 and 78 ÷ 3) and their ability to compare fractions, for example ($\frac{1}{4} > \frac{1}{5}$). Another example is in Study 4, where I made the following explicit hypothesis: *Students who use this intuitive reasoning from natural numbers have a tendency to do so across different kinds of tasks.* In Study 5, the *Determination of a Problem-solution (Phase III*) can be seen in the developed fraction instruction material, which is a suggested solution to the problem. This instruction development was based on the findings from Review 4 (see Chapter 4.5).

IV) *Reasoning (Reasoning)*. This can be seen as a process where reasoning about the developed meaning-contents of ideas is connected to their relation to other ideas (Dewey, [1938]1986).

In every study, this Phase IV is primarily seen in the discussion section, for example, in Study 1's discussion of fraction proficiency measurement or Study 5's high- and low-performing differences in development. Another example of reasoning could be in Study 1 where we had an ongoing discussion about whether the Rash analysis would contribute new information or not. In Study 3, which is theoretical, the focus is reasoning about the importance of fraction equivalence, and this study mainly connects to theoretical reasoning.

V) *The Operational Character of Facts-Meanings (Testing the Hypothesis by Action).* According to Dewey, both observed facts and ideas are operational and must cooperate to come together as a whole despite their differences; the facts must serve as evidence for the hypothesis (Dewey, [1938]1986). In this phase, the meaning of the act in action must be examined. There are slightly different terms used in *Logic* ([1938]1986) and in *How We Think* ([1933]1986), where testing the hypothesis phase in *How We Think* ([1933]1986) might be seen as being split into two phases in *Logic* ([1938]1986) and is therefore also part of Phase VI. However, I choose to put it next to *the Operational Character Phase*, and I see it primarily as evaluating the facts from testing the hypothesis. Yet, this choice is open to discussion.

In this PhD project, intervention material was tested on high- and low-performing student groups in Study 5, meaning that the hypothesis that the material created an opportunity for both high-and low-performing students was tested. The hypothesis in Study 2 was tested by analysing the relation between students' answers to whole number arithmetic and fraction comparison tasks. In Study 4, the relation between all natural number bias aspects was investigated by analysing the collected data.

VI) *Common sense and scientific enquiry* (this phase is not described in *How We Think*). The difference between common sense and scientific enquiry is connected to differences in (a) subject or topic matters, which of course are connected to what kind of problems are in focus, (b) distance from the immediate subject, and (c) differences in the degree of precision, control, and systematicity. Solutions to common-sense problems are based on the habitual culture of a group and therefore reflect the group's culture, whereas science-based enquiry is a more disinterested enquiry that is not connected to one group. Alternatively, it could be said that the focus of common sense is on qualities (e.g., good, bad), and the focus of science is on relations (e.g., position, motion). Dewey defines the 'world' of common sense as 'the environment in which human beings are *directly* involved' (Dewey, [1938]1986, p. 66). In contrast, scientific enquiry must lack 'direct involvement of human beings in the *immediate* environment' (Dewey, [1938]1986, p. 67), and therefore it is somewhat distant from present

> needs and wants. Hence, scientific knowledge is ‘attainment of knowledge…for its own sake,’ and it is ‘attaining confirmed facts, “laws” and theories’ (Dewey, [1938]1986, pp. 66–67). This means that scientific knowledge is judged not on its presentation, but on systematic relations of coherence and consistency (Dewey, [1938]1986).

That this PhD project is a scientific enquiry, and not based in common sense enquiry, can be seen in three outlined differences mentioned above: (a) The topic of fraction understanding and learning is not one that is normally connected to common sense; however, as a mathematics teacher, the topic could also be enquired about in an everyday setting in a classroom. (b) Where my research project differs is the distance between me as an external researcher and the immediate subject: the students learning fractions in fourth grade. The intermediate situation was conducted in the municipality of Syddjurs. The use of quantitative methods to enable the collection of a greater amount of data across classrooms can be seen as a way to ensure independence from the particular situation in each classroom. However, the data were collected in the same municipality, so they could be particular to Syddjurs. The differences between schools (large or small, urban or rural, private or public, see Chapter 6.2.1) can be seen as a way to ensure that the observations are not founded on a specific type of classroom. (c) There are differences in the degree of precision, control, and systematicity. The precision can be seen in the development securing the accuracy of the measurement tool (Study 1), and control can be seen in the continued evaluation and discussion of my statistical script written by my counsellor and research partner. The systematics can be found in the search strategy behind the reviews in Appendices A–D.

The question of whether we can confirm or replicate our findings is also central to scientific enquiry. In Study 5, for example, we can confirm the discovered pattern in the first data in the delayed data set, which confirms our findings in the first data set. Other questions need to be asked, such as in Study 2: ‘Do we find the same relation between whole number arithmetic and fraction comparison tasks in grades other than fourth grade?’ The overall question of whether we can replicate our findings in other data sets and classes must be further investigated.

Overall, Dewey’s enquiry process can be charaterised as dynamic and nonlinear, and in the context of the PhD project, it is important to emphasise that the five studies influence each other in continued cycles; for example, Study 1 continued to influence the discussion on how we observe or collect data for the enquiry. Each study explores different parts of the overarching problem. Dewey also emphasises that the sequence is not fixed, and some phases can be expanded; in other words, the process is dynamic (Dewey, [1933]1986).

Chapter 3: What are fractions? And how to understand them

During this PhD project, I became more and more aware of the need for understanding historical and cultural influence to answer the question of why we need fraction notation today. There is also a need for a semantic framework to capture the multifaceted complex concept of fraction.

In this chapter, I try to capture the development of the written fraction notation and its multifaceted structure. The first part of the chapter contains a definition of the terminology connected to fractions in an educational context. This is followed by a brief historical overview of the development of fractional notation, and at the end of the chapter there is a theoretical presentation of the multifaceted construct of the fraction concept. The purpose of the section is to describe and explain the terminology connected to fractions as it is used in the five studies of this dissertation to make a foundation for how to understand fractions. When describing the historical development of fraction notation, it is the written notation that is described, whereas when I later describe the concept of fractions, it is an elaboration of the semantic meaning of the concept. By doing so, the aim is to create a foundation for studying the overarching research question: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* I need to describe, capture, and define the need for fraction notation and the nature of multifaceted concept fractions. It can be seen as a further elaboration of the *Antecedent Conditions of Enquiry: The Indeterminate Situation Phase* I (Dewey, [1938]1986), where I try to answer, 'Why do we need fractions (the fraction notation)?' and 'Why is the concept of fractions complex?'

3.1 Terminology of fractions

The term 'fraction' comes from the Latin term frangere (fractus) which means 'to break'. The traditional representation is 'a part of a whole' or a 'number of equal parts'. A fraction's notation consists of three parts: a numerator, a denominator, and a line that separates the two numbers (World Encyclopedia, n.d.).

As previously mentioned, fractions are connected to a symbolic notation of rational numbers. Fractional notation is defined as $\frac{a}{b}$, where the denominator b can be any non-zero quantity, which means that any rational number can be written as a fraction; however, not every fraction is necessarily a rational number. For example, $\frac{2}{3}$ is a rational number, but $\frac{1}{\sqrt{2}}$ is not (Lamon, 2012).

The use of terminology associated with fractions in an educational context is not consistent. Payne (1976) describes how there is a great variety of terms used in connection to fractions in the literature of mathematics education, such as fractional numbers and fraction symbols. He claims that the choice of term is primarily based on the personal preferences of the writers. Kieren (1995), whose theoretical framework will be described later in the chapter, gives his perspective on terminology use:

> I am taking the liberty of using the terms 'fractional numbers', 'rational numbers' and 'rational numbers of arithmetic' loosely and interchangeably; I am thinking about children perhaps aged 7 to 12, as they come to learn to deal with the non-negative rational numbers and their operations through using standard and nonstandard fractional language. (p. 35)

To make it even more complex, the term 'rational numbers' has been used interchangeably with 'fraction' in the elementary school setting (Lamon, 2012). It has both referred to the mathematical definition (elements of a quotient field) and as a topic in elementary school (Olive, 1999). The relationship between the term 'fraction' and rational numbers has been described by Behr et al. (1992) as follows: 'Rational numbers are elements of an infinite quotient field consisting of infinite equivalence classes, and the elements of the equivalence classes are fractions' (p. 296). Therefore, equivalence classes play a central role in the mathematical interpretation of fractions. In building the connection between rational numbers and fractions, x is a rational number, if integers a and b exist, such that $bx = a$ (Kieren, 1993).

Other researchers have stressed the need for a clearer definition of the term 'fraction' (Lamon, 2012; Thompson & Saldanha, 2003). Lamon (2012) argues that careless use of the term can cause difficulties in communicating. She therefore makes a distinction in the use of the term 'fraction', which coexists in mathematics education: a numeral and an abstract sense of a number. First, the numeral refers to a fraction's bipartite symbol, where fractions refer to a notational system – a particular form for writing numbers: $\frac{a}{b}$, a particular notation form where a and b are written with a bar/line between them. The second interpretation involves fractions that are synonymous with positive rational numbers in a school setting (Lamon, 2007, 2012).

I am taking the freedom of using the term 'fraction' based on Lamon's (2012) second interpretation. Here, in this context, the term 'fraction' is defined as a notation where both the numerator and the denominator consist of natural numbers, and therefore the fraction is also a positive rational number. When using the term 'fraction notation', it refers to the written notation in the form of two integers, one above and one below a horizontal bar, for example $\frac{1}{3}$. When I use the term 'fraction concept', it refers to the broader multifaceted concept connected to the understanding of fractions.

3.2 The historical development of the fractions notation

The concept of fractions has historically been connected to breaking up or dividing in the setting of food or trade, for example, in a market place (Streetland, 1991). The first known descriptions were made by the Babylonians. They made a fraction system that was funded in the base of sixty: half = 30, one-third = 20, quarter = 15 (Aldosray, 2016; Cajori, 1928; Miller, 2017). The Egyptians used a form of notation that dates back to 4000 BCE. They used a hieroglyphic inscription system with a special notation system for unit fractions where the reciprocal of any integer was notated by placing an oval sign, which meant 'mouth', indicating 'part', above the number above the integer. For example, the fraction $\frac{1}{8}$ would appear as  in the Ahmes Papyrus. The oval eventually developed into a dot, and $\frac{1}{8}$ would later appear as  where the numerator is two horizontal lines, each representing the quantity of four. Occasionally, they used a special sign for fractions in the form of $\frac{n}{(n+1)}$. However, the commonly used notation was the unit fractions (having a numerator of 1). Overall, a vertical form of a fraction notation was used and developed in ancient Egypt (Cajori, 1928; Merzbach & Boyer, 2011). It is important to emphasise that Egypt was not the only place where vertical notation was used, and it had most likely been used in many earlier civilisations.

It is worth mentioning that the tradition of vertical notation was not recognised worldwide. For example, the Greeks also needed a notation for fractions, and they developed their own systems. However, this notation was rather unclear, and often the context around the fraction was essential when reading the fraction correctly (Cajori, 1928). The following example is alphabetic numerals from this system. They used diacritical as a mark that was placed after the denominator of the unit fraction. (The Greek number system $\beta = 2$ and $\delta = 4$). This means, that $\beta' = \frac{1}{2}$ and that $\delta\beta' = \frac{1}{42}$.However, the last notation can also be $40\,\frac{1}{2}$ (Allen, 1997; Miller, 2017).

Many similar notations were used in Greek ancient civilisation, with increasing sophistication of the notation. The late Diophantus (AD 200/214 to 284/298) has been recognised as being the first found Greek mathematician that used a vertical notation form identical to our modern fraction notation, although with the denominator and numerator in reversed positions (Allen, 1997), and like the Egyptians, he usually only used the unit fraction. Although later the Greeks used the vertical notation form, they had a sophisticated understanding of numbers (Cajori, 1928; Eves, 1976). Overall it seems that the Greeks lacked a common notation of fractions, which meant that fractions were excluded from common use in the number system. The reason for this is an ongoing discussion of whether this was based on an imperfect notation or whether this missing notation was genuine 'conceptual divergence in numbers'. The Greeks' mathematical understanding of magnitude less than one differed from our modern mathematical understanding of numbers (Høyrup, 2004).

The ancient Greek mathematicians were rather late in their use of the vertical notation. Approximately 300 years before, in 150AD, the Indian Jain mathematicians wrote ‘Sthananga Sutra’, which contained their collected work on numbers theory, arithmetic, and fraction operations. Our modern notation of fractions, known as bhinnarasi, also appears to have been developed in India by three mathematicians: Aryabhatta (476–550), Brahmagupta (598–668), and Bhaskara (1114–1185). Their work with numbers resulted in forming the fraction bipartite notation system where they placed the numerators above the denominators without a bar between them (Cajori, 1928; Miller, 2017; Plofker, 2016) in contrast to Diophantus, who had them around the other way.

The Moroccan mathematician Al-Hassar is famous for using the notation with the horizontal bar between the numerator and denominator for the first time (Aldosray, 2016; Saidan, 1996). However, this notation was used more than a thousand years before, and Al-Hassars’ fame in the Western world could be due to his work being translated into Latin (Saidan, 1996). The first European mathematician to use the bar notation was Fibonacci (1175–1250), who described the horizontal bar using the Latin term ‘virga’ (Cajori, 1928; Plofker, 2016).

In the Middle Ages, fraction notations with a bar were generally found in Latin manuscripts, but when printing was invented as a way to duplicate, the bar was often left out, probably because of typographical problems (Miller, 2017). In 1585, Simon Sten (1548–1620) wrote ‘D Thiende’; in this work, he describes how natural numbers can be extended by using decimal fractions (Streetland, 1991).

This fraction notation, which places the horizontal bar between the numerator and the denominator, was developed over centuries (Edwards, 1979; Miller, 2017; Thompson & Saldanha, 2003). In the last 300 years, it has been consistent as a notation, even though the diagonal fraction bar (solidus or virgule) was found in a handwritten document from 1718 by Thomas Twining’s Ledger, it was properly used before this (Miller, 2017). The diagonal bar was probably invented because the horizontal bar was typographically problematic. Nowadays, the use of this diagonal notation is increasing because it is directly available on any computer’s keyboard. Therefore, the horizontal bar (–) might in the future be replaced with the slash symbol (/) as it is easier to write on a keyboard.

To look at the fraction notation separated from the general development of rational number notation is a simplification, and the development of the notation is of course influenced by the general development of the mathematical field. For example, in connection to the development of *differential calculus*, the multifaceted concept of fraction was articulated by the French Mathematician Jean-Baptiste le Rond d’Alembert (1717-1783), who questioned how to understand $\frac{dy}{dx}$. He argued that the concept of limit should not be thought of as a derivative that merely symbolised one quotient (a result of a calculation), but should be interpreted as symbols representing one magnitude (Edwards, 1979; Thompson & Saldanha, 2003). The importance of this question is easily missed because it was not placed in the context of fractions but in the context of *differential calculus*, but it is important in later discussions of the multifaceted structure of fractions. Seeing fractions as a rate of change as two independent numbers was an incoherent conception; hence, the common interpretation of a *ratio* as two numbers present, for example, in the fraction notation, was really one number. This means that when looking at the fraction notation, it is a number and not a calculation (Thompson & Saldanha, 2003).

However, this unified conception of fraction notation as a number was not adopted as consensus among mathematicians, and this might be lucky. As Vogel demonstrated in 1936, the whole terminology for *ratios* claiming not to be seen as numbers but a *relation between a pair of numbers* is fortunate because it supported the need for fractions and the connection to the terminology for fractions. Keeping the *ratio* understanding of fractions can be why fractions are saved in a theoretically acceptable way. We need this notation to illustrate the relation between two pairs of natural numbers (Høyrup, 2004); even though we have another notation form for rational numbers (decimals and percentages), they do not capture this relation. The concept of *ratios* needing a language that is connected to the practice of fractions is a fortunate accident that means that we have the need for the fraction notation.

These mathematical questions are connected to the need for fraction notation, and how to understand this notation further leads to other mathematical concepts such as rates-as-numbers and continuity of functions. These concepts are the foundation that leads to the development of a formal construction of rational number systems – and of course, also real number systems (Thompson & Saldanha, 2003).

It would be out of the context of this dissertation to summarise this development. Nevertheless, I want to emphasise that the mathematical development of rational and real number systems consists of many concepts that are typically first introduced in graduate mathematics courses. The content of the Danish textbook will further be introduced and analysed in Chapter 5.2.

To summarise, the fractional notation system has been developed over thousands of years and will continue to evolve in the future. Overall, fractional notation was invented and developed because of the need for a way to depict where reciprocals of integers were present, where the notation, for example, represented ‘one part of three’ or ‘one divided by three’. The need for this notational form in mathematics is still relevant. The phrase, ‘$\frac{1}{4}$ pizza’ gives a different interpretation to ‘0.25 pizza’ or ‘25% pizza’. In addition, the fraction notation form with the horizontal bar has several advantages in common calculation and algebra, such as the tasks $24 \div 4 \times 6$; if the calculation is read from left to right, the answer is 36, and if multiplication sign is used first, then it is 1. There is no consensus about what is right (Cajori, 1928), and to determine which order to do this, the calculation brackets must be used: $(24 \div 4) \times 6$

or 24 ÷ (4 × 6). If the fraction notation is used, this problem is avoided: $\frac{24}{4} \times 6$ or $\frac{24}{4\times6}$. The advanges of this notation are further found when reducing an expression in algebra, for example, when reducing the expression $(24\text{a} + 12\text{ab}) \div (9a - 6ac) = 3\text{a}(8 + 4\text{b}) \div 3\text{a}(3 - 2\text{c}) = (8\text{a} + 4\text{b}) \div (3 - 2\text{c})$.

In the fraction notation form, the same process would be $\frac{24\text{a}+12\text{ab}}{9\text{a}-6\text{ac}} = \frac{3\text{a}(8+4\text{b})}{3\text{a}(3-2\text{c})} = \frac{8\text{a}+4\text{b}}{3-2\text{c}}$. It seems easier to find that 3a divided by 3a is equal to 1 because 3a is just above and below the bar, and that $\frac{8\text{a}+4\text{b}}{3-2\text{c}}$ is a new expression, rather than a calculation as $(8\text{a} + 4\text{b}) \div (3 - 2\text{c})$. My understanding of that is the result of years of schooling in a Danish school system, however.

The fraction notation makes it possible to describe the world mathematically in a way we need. We need a multi-faceted notation, but at the same time, this complexity makes it difficult to fully understand and learn.

3.3 The multi-faceted construct of the fraction concept

A way to describe the multifaceted structure of fractions is the five subconstructs: *part-whole*, *measure*, *operator*, *quotient*, and *ratio* which have been added to the semantic concept of fractions. Kieren (1976) originally developed this framework of subconstructs. Both Vergnaud (1983) and Freudenthal (1983), as well as Kieren (1976), independently identified subconstructs, aspects, or objects of fractions in the mid-1970s and early 1980s. All three researchers came up with frameworks that broadly consist of the same *objects*, *constructs*, or *subconstructs*, using slightly different terms and definitions. Vergnaud (1983) defines the concept of fractions founded in the broader context of the multiplicative conceptual field, and Freudenthal (1983) defines the concept of fractions based more on the development of different aspects of partitioning. Kieren introduced the theoretical framework of rational numbers as a set of interrelated, but distinct subconstructs: *part-whole, measure, operator*, *quotient,* and *ratio* (Kieren, 1980, 1976).

These have generally been accepted and applied by researchers as the five subconstructs that comprise the fraction concept (Behr et al., 1993; Charalambous & Pitta-Pantazi, 2007; Kieren, 1980; Tsai & Li, 2017). However, it is important to emphasise that there have been other ways to describe this complexity and that the five subconstructs are not unchangeable or universal (Hecht et al., 2003; Ohlsson, 1987, 1988; Rapp et al., 2015). For example, Ohlsson made semantic interpretations of fractions and suggested three basic senses: comparison between quantities, division of quantities, and counteracting changes (Ohlsson, 1987). Later, Ohlsson developed his definition and defined fractions as containing the concept of rational numbers, binary vectors, and composite fractions (Ohlsson, 1988). However, fraction has also been seen as having structural-operational duality where fractions (rational numbers) are seen as structural (pair of integers that are members of a defined set of pairs) and operational (the result of a division of the two integers) (Sfard, 1991). Another definition of fractions is how much is present in a rational quantity (*part-whole* and *measure*) (Hecht et al., 2003). A way to hypothesise the relation between decimals and fractions can be seen in Fig. 5. The figure was developed by Rapp et al. (2015). Here fractions are illustrated as connected to countable discrete data, and here a fraction represents the *ratio* between numerator and denominator, meaning that it is a bipartite format ($\frac{a}{b}$) that is defined as the value of the part (the numerator) and the whole (the denominator). In contrast, the decimal is connected to continuous data; here, a decimal represents the one-dimensional magnitude of a fraction ($\frac{a}{b} = c$) set in the standard base-10 metric system. In this framework, the fraction notation represents a two-dimensional relation, whereas the responding decimal is a one-dimensional magnitude (English & Halford, 1995; Rapp et al., 2015). This means that the one-dimensional magnitude .04 responds to the bipartite format $\frac{4}{100}$.

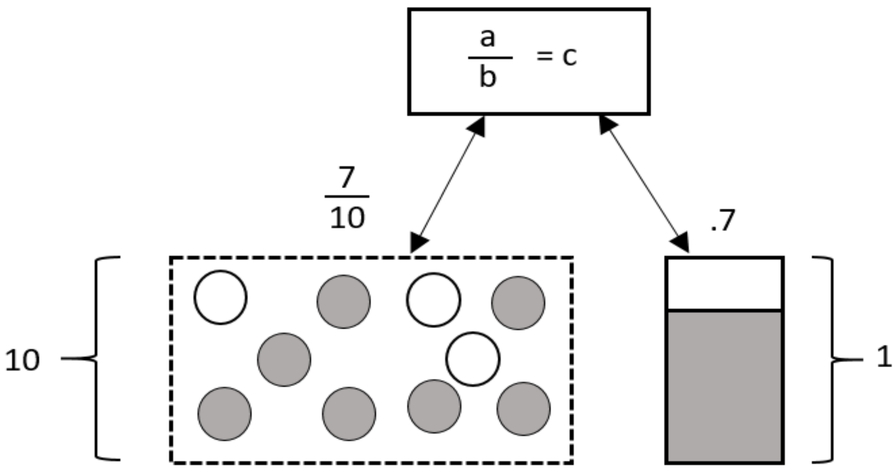

Fig. 5 Hypothesised alignment of decimals and fractions with discrete and continuous entities. Figure developed by Rapp et al. (2015)

The above-mentioned studies show both multiple concepts and frameworks of fractions, and there is no universal consensus of the interpretation of fractions. In this dissertation, the five subconstructs – *part-whole*, *measure*, *operator*, *quotient*, and *ratio* (five-part model) developed from Kieren's (1976) theoretical framework–are used as a reference point since this frame captures the multifaceted complex concept of fractions. It is one exemplary clear model that includes or overlaps with most of the other above suggested frameworks. In his early work, Kieren (1976) recognised four subconstructs: *measure*, *ratio*, *quotient*, and *operator*. The concepts of this *part-whole* were implied as being embedded in each of the four constructs as a base, and the *part-whole* construct was not included as a separate construct.

Later, Behr et al. (1983) in the Rational Number Project further developed Kieren's (1976) work by suggesting that the *part-whole* or partitioning subconstruct should be considered a distinct subconstruct of fractions (see Fig. 6) and added rate and decimals, but they broadly recognised and used the five-part model. In the same text where Behr et al. adds the new subconstructs, they introduce the five-part model, and in Kieren's article from 1980, the model of the five subconstructs was likewise defined and explained.

Later, Kieren developed his five-part model framework for rational numbers (1988, 1993, and 1995). In his later work, he re-established the model with three underlying concepts (partitioning, equivalence, and unit forming). This development can broadly be explained as focusing on *quotient* as the foundation for rational numbers. This later model has not had the same explanatory power as the previous simpler five-part model, but it is important to emphasise that his model has developed over time, and in this context, the model underlines that knowledge shall not be seen as universal, but as warranted assertibility.

The recognition of the original five-part model during the last decade might be explained by its simplicity and therefore functionality in the research field. A body of research has used his five subconstruct model (1980), consisting of *part-whole*, *measure*, *quotient*, *operator*, and *ratio* which, in this dissertation, will be referred to as his five-part model (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007, 2012; Tsai & Li, 2017). The framework based on Kieren's work has been criticised for not exhausting or not including other possible interpretations (Ohlsson, 1987), and that the subconstruct model can be interpreted as a semantic top-down analysis of rational numbers, which can be seen as an adult understanding of fractions. It is uncertain whether it describes students' constructs of fractional knowledge (Olive & Lobato, 2008).

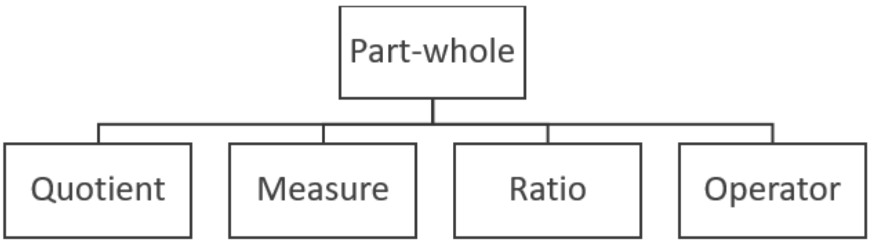

Fig. 6 The Five-Part Model Note. The original model with the five subconstructs of fractions connected to the different operations of fractions and to problem solving (Behr et al., 1983).

In the context of this dissertation, Kieren's earlier five-part model of subconstructs is used as a framework to understand the multifaceted nature of fractions, particularly in Study 1 in the development of the fraction proficiency measurement tool. In Study 3, the theoretical framework is used as the foundation for the analysis of the different equivalence conceptions. Therefore, the measurement tool is the foundation for data collection. Kieren's framework is implicitly present in Study 5, where I follow high- and low-performing students' fraction proficiency development during fourth grade.

I will describe the content of the five subconstructs in the next sections. They are further unfolded in Study 3, where they are used as a framework for analysing the two conceptions of equivalence.

3.3.1 The *part-whole* subconstruct

The *part-whole* subconstructs are based on the student's ability to partition either a continuous quantity or several discrete objects into equal-sized parts or sets (Behr et al., 1983). This subconstruct is often the first subconstruct introduced for children in school. It describes the number of equal-sized partitioned parts denoted by denominator *b*, and the numerator defines the number of parts (Barnette-Clarke et al., 2010; Charalambous & Pitta-Pantazi, 2007; Kieren, 1980; Marshall, 1993; Tsai & Li, 2017).

The representation of *part-whole* differs when the constructs are connected to a continuous quantity, for example, the area of a pizza or length of a road, where the *part-whole* is taken from a group of discrete quantities, such as a box of candies (Beckmann, 2011). The two types, continuous quantity versus discrete quantity, also demand different types of cognitive structures. The student performs significantly better when the task's content is in the form of discrete examples compared to continuous (Hiebert & Tonnesse, 1978). The *part-whole* construct is often the first representation introduced, and it is the most frequent model used in the classroom setting where teachers use it to introduce and explain the concept of fractions (Fuchs et al., 2013; Hiebert & Tonnesse, 1978).

3.3.2 The *quotient* subconstruct

Another one of the five subconstructs is *quotient*, where the notation $\frac{a}{b}$ can also refer to the mathematical operation of division. Hence, $\frac{a}{b}$ can be seen as a ÷ b, representing a *quotient*. The interpretation of representation can be explained as follows: the denominator stands for the number of recipients, and the numerator is the quantity that has to be shared (Behr et al., 1993; Kieren, 1993, 1976; Marshall, 1993; Middleton et al., 2001). This process involves a minimum of two stages of interpretation: The first stage includes an interpretation of the fraction as an operation, and this means a pathway to understanding the equivalence or transfer between $\frac{8}{4}$ and 2 or see $\frac{1}{4}$ as the result of 1÷ 4 (Behr et al., 1983; George, 2017; Marshall, 1993).

Furthermore, division is the only whole operation that includes whole numbers, where a rational number can be the outcome, and it is a way to connect and develop students' whole number understanding so they have a concept of numbers that includes rational numbers (Bright et al., 1988; Middleton et al., 2001). Toluk and Middleton (2001) conducted a case study where they observed students' progress in fraction schemes and the concept of the operation division in their development from the *part-whole* subconstruct concept towards a conceptualisation of the *quotient* subconstruct. Studies have also shown that proficiency in long division supports students' development of fractions (Siegler & Pyke, 2013; Ye et al., 2016). In other words, the *quotient* subconstruct involves a process where a two-entity versus a one-entity phenomenon is present. The process starts by first looking at the fractions as two quantities (the numerator and the denominator). Second, the numerator is seen as a divisor, and the denominator is viewed as the dividend. Third, the process of partitive or quantitative division of a single quantity is the result. This method further leads to two different forms of division: partitive or measurement (Behr et al., 1993). This description can be traced back to d'Alembert's question of whether a fraction should be thought of as merely a symbol that represents one number instead of as a pair of symbols representing a *ratio* of two numbers. Here, it can be said that the answer lies in the phases of a process.

3.3.3 The *measure* subconstruct

The *measure* subconstruct contains two interpretations: the first is that the fraction can be understood as a numerical value and the second that the fraction can be seen as a *measure*, for example, a distance or a size (Charalambous & Pitta-Pantazi, 2007) when the subconstruct *measure* is defined as a fraction used to determine a distance. Here, the distance is connected to an interpretation unit fraction, which is used repeatedly to measure a distance. It is therefore often connected to a number line (Charalambous & Pitta-Pantazi, 2007; Marshall, 1993). The *measure* is defined as a distance to a certain point from the starting point in a unit fraction distance (Behr et al., 1993; Kieren, 1976; Marshall, 1993). The term 'certain point' is used because not all points on a number line can be defined as a fraction (these are irrational numbers). The subconstruct therefore includes a determination of the unit and the starting and ending points. When introduced to fractions, the starting point would often be zero; however, the starting point may also be points other than zero, such as the distance between 1 and 2 $\frac{7}{11}$ (Marshall, 1993). Hence, the subconstruct can also contain other representations such as stripes, chips, areas, etc. (Kieren, 1976; Lamon, 2012). In Kieren's (1976) original definition, both partitive and measurement division are used, and Lamon (2012) followed this definition by including and emphasising measuring units as an important part of the understanding of measure (Lamon, 2007, 2012).

The *measure* subconstruct can overall be seen as based on four approaches: (a) to recognise a fraction as a unique number, (b) to understand the density property (infinite number of fractions between two given fractions), (c) equal partitioning, and (d) to describe a unit fraction as a unit of measure that can be used repeatedly to measure distances (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007, 2012; Marshall, 1993; Pantziara & Philippou, 2012). This is further described in Study 3.

3.3.4 The *ratio* subconstruct

A *ratio* can be defined as the relationship between two relative magnitudes: a numerator and denominator. Thus, it can be seen as a comparative index or proportionality rather than as a number (Behr et al., 1993; Kieren, 1993; Lamon, 2012). This is the only subconstruct; no partitioning of an object is required (Marshall, 1993). To illustrate the difference between *quotient* and *ratio*, *ratio* can be seen as four apples for every five students. This concept differs from *quotient* that can be seen as five students sharing four apples, meaning every student gets $\frac{4}{5}$ of an apple. A *ratio* can likewise be seen as a part-part relationship, where two similar units are compared. For instance, there are four boys for every five girls (George, 2017). The term *ratio* is not well defined, and there is no consensus about the terminology (like fractions). As a result, it also has several understandings and definitions; for example, there is no consensus of '*ratio*' and 'rate' and how these differ (Beckmann et al., 2015). Some researchers have distinguished the difference of the terms to whether the compared quantities are the same or not (e.g., Ohlsson, 1988). When having an understanding of fractions based on the *ratio*, it is central to look at the relations between the two different whole numbers represented in the numerator and the denominator (e.g., in $\frac{1}{2}$ the denominator is twice the size of the numerator). This subconstruct can therefore be seen as strongly connected to the concept of equivalence and explains why $\frac{2}{3}$ is equal to $\frac{4}{6}$ (Kieren, 1976). The proportional relation between *a* and *b* means that if there is a change in *a*, it will lead to predictable change in *b*. The constancy in the notation indicates that the *ratio* is constant (Behr et al., 1993; Charalambous & Pitta-Pantazi, 2007; Marshall, 1993; Wong & Evans, 2007).

3.3.5 The *operator* subconstruct

The *operator* subconstruct can be defined as a given value or area that needs to be operated on so we can find a second size of value or region. The fraction works as an *operator* that operates on another value and can be illustrated as a function machine (Behr et al., 1993; Kieren, 1976; Tsai & Li, 2017). One example of a task that includes an *operator* subconstruct could be where a student is asked to transform a figure into a new figure that is $\frac{3}{4}$ of the original size (Marshall, 1993). In the context of continuous quantities, the *operator* first shrinks and then stretches the original object.

For example, for $\frac{3}{4}$ of 12, first you stretch by multiplying 12 by 3 equal to 36, then you divide by 4, which gives the result of 9. It is also possible to do it the other way around: first shrink then stretch. In the context of discrete entities, the fraction $\frac{\mathrm{a}}{\mathrm{b}}$ operates on a set of objects to find a new set with $\frac{a}{b}$ times as many objects (Behr et al., 1983).

Behr et al. (1993) later developed these subconstructs by defining five different interpretations. They specifically focused on the two interpretations of stretcher-shrinker and duplicator/partition-reducer in their analysis. Here, the stretcher-shrinker interpretation is defined as the result of the operation is the same number of units of different size, whereas the result of a duplicator/partition-reducer operation is a different number of units of the same size (see Study 3 for further elaboration). The different interpretations of the *operator* have been seen as a way to understand the fraction multiplier (Lamon, 2012; Marshall, 1993) and overall, this subconstruct *operator* requires an understanding of composition, reversibility, and proportionality (Kieren, 1976).

3.4 Summary

As mentioned in the beginning of this chapter, it was clear during my enquiry process that I needed to explore the historical development of the fraction notation to fully understand the need for this representation. The notation of fractions is a product of a long historical development and might continue to be developed, especially because of the extended use of symbols that are included on a keyboard. Overall, the development tells us that we still need this notation to describe the world mathematically. Therefore, the fundamental question is whether we need to understand fractions as outlined in the previous chapter as part of the first phase in the enquiry process (*The Antecedent Conditions of Enquiry Phase I*). My answer is yes, we need the notation to describe the world mathematically.

The last part of the chapter aimed to explain the concept of fractions and illustrate the complexity of the concept. Kieren's five-part model is chosen as the theoretical framework for the definition of fractions since it illustrates and captures fractions' complex multifaceted nature and this can illustrate and elaborate why the topic of fractions often leads to difficulties in students' learning processes. It explains the semantic complexity of fractions. It is important to emphasise that the subconstructs are seen as parts, and they work flexibly together. In my work on this dissertation, especially Study 3, I developed a new illustration of the interaction between subconstructs from Behr et al.'s original figure, and I wanted to emphasise the overlapping subconstructs and that they are working together (see Fig. 7). The difference between Figs. 6 and 7 is that the newly developed Fig. 7 emphasises that all subconstructs overlap.

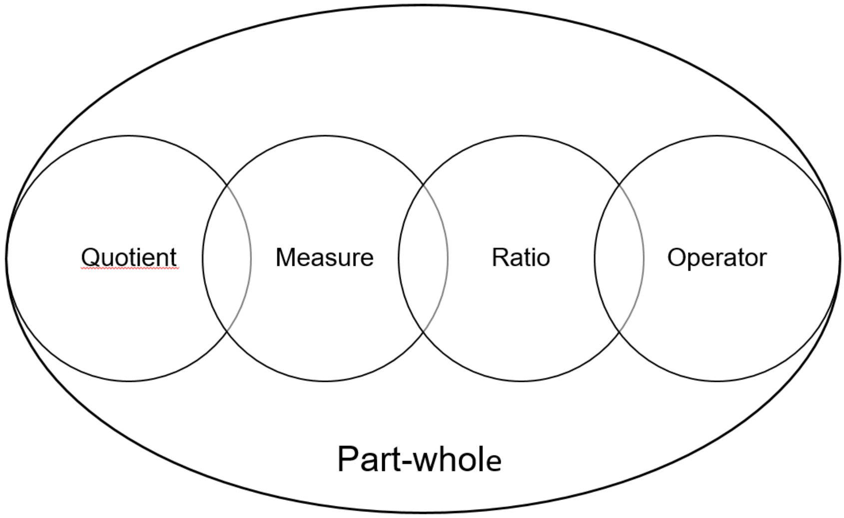

Fig. 7 The theoretical model linking the five subconstructs of fractions, developed from Behr et al.'s model (1983) and used in Study 3

Because I continued to ask questions in my enquiry, I began to question whether *quotient* should play the same role as *part-whole* because it can be a bridge between natural and rational numbers. This bridge is based on the fact that *quotient* is seen as the result of the whole number division. The importance of division is supported by the findings in Study 2, where the relation between whole number arithmetic and fraction comparison tasks is investigated. This study found that both the correct answers in division and multiplication had a stronger relation to the correct answers in fraction comparison tasks than addition and subtraction. The later studies of Kieren (1993) also emphasised the importance of the *quotient*; therefore, the latest semantic figure I developed can be seen in Fig. 8. Future empirical research needs to be done to further explore the theoretical figure.

The measurement tool developed for this PhD project is developed from this subconstruct framework together with the official Danish curriculum. This measurement tool is described, analysed, and validated in Study 1. The study is based primarily on the *quotient* subconstruct.

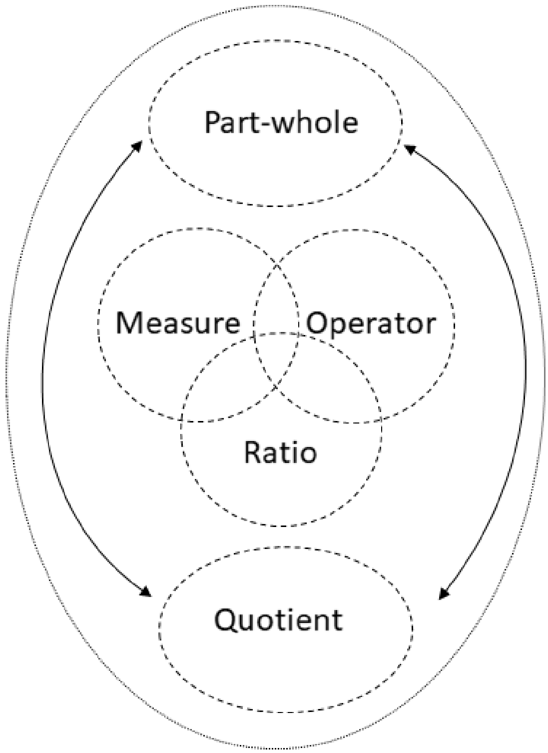

Fig. 8 New developed semantic model emphasising *quotient* equal to *part-whole*

Chapter 4: What is fraction learning and understanding?

As the terms ‘understanding/knowledge/fractions proficiency’, ‘natural number bias’, ‘number knowledge development’, and ‘fraction intervention’ are central for this project, the aim of this chapter is to elaborate and describe these terms through four different literature reviews.

As described in Chapter 2, the project is an ongoing process of enquiry. I did four literature reviews connected to different stages of enquiries, mainly the three phases at the beginning of the process: *Institution of a Problem (Phase II), Determination of a Solution (Phase III)*, and *Reasoning (Phase IV)*, and at the end of the process: *Common Sense and Scientific Enquiry (Phase VI),* where the scientific enquiry is supported by finding similar or contradictory conclusions in other studies. It is important to emphasise that reviews of prior knowledge of the subject will be part of the other phases of the enquiry as well, always influencing the choices made in the research process.

The reviews were a dynamic process. During the writing and review process, I continued going back and making new searches or redefining and adapting the search string, and therefore the reviews were present in the remaining enquiry Phases I) *The Antecedent Conditions of Enquiry* and V) *The Operational Character of Facts-Meanings*. The reviews were not made according to a linear process in which one review was finished before beginning the next. Rather, the reasoning from a previous review often influenced the next, which in turn would lead me back to the previous review. This process and method will be explained further in the next section. The four reviews will be reported in the following order: 1) mathematical knowledge and fraction proficiency, 2) natural number bias, 3) number knowledge development, and 4) fraction intervention.

4.1 Method used for making the reviews

The objective of the systematic reviews was to survey prior research on mathematical understanding and fraction proficiency, fraction interventions, natural number bias, and number knowledge development. These reviews followed an established protocol (Petticrew & Roberts, 2006; Zins, 2000). Following this protocol, I have created a systematic search strategy in which search terms, databases, and search hits are documented and evaluated (see Appendices B, C, D, and E). This is done to secure the demands for systematicity in the scientific enquiry process, as previously outlined in Chapter 2.4 (Dewey, [1938]1986).

Overall, the literature search comprised four steps. First I selected the databases and specific search locations for each review (see Appendix A). To select the search words, I conducted a pilot search using single words and phrases in the selected databases within a period covering a suitable time limitation for each search. For example, natural number knowledge included studies from the year 2005 to 2020; the year 2005 was the year Ni and Zhou (2005) published their study about the bias. The experience of these pilot searches was the foundation for developing models that were finally used in the database search. In other locations (e.g., Danish or Scandinavian journals), I looked for single words, as done in the first step.

In the database search, my chosen language was English. This meant that only Danish literature that had been published internationally would be found. In addition, I had selected research that involved alternative sources of the Scandinavian National Mathematical Centre, such as the Nationellt Centrum för Matematikutbildning (NCM) Matematikksenteret, Nasjonalt senter for matematikk i opplæring or the specific Danish journals *MoNa* and the Scandinavian journal *Tangenten, Nomad, and Nämnaren*. When looking into these Scandinavian journals and centres, I chose Scandinavian words for the search, for example, brøk*, bråk* or fraction* in addition to the English terms, so these journals could be published in Scandivian languages as well as English. Upon completion of the process, the fourth step included an evaluation of the protocol with a research librarian. In some reviews, I decided that the search protocol had to be adapted in order to deliver more precise results, so it was repeated with new models that reflected an overall consistency throughout the search (see Appendices B, C, D, and E). Later, feedback from reviews meant that the search protocol was repeated or new words were added.

4.2 Review (1): Mathematical knowledge and fraction proficiency

This first review's aim is to enquire further into mathematical knowledge and fraction proficiency and thereby inform the overall enquiry process of the problem: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?*

The enquiry process needed an elaboration of how mathematical understanding could be described before I could look further into how students developed the concept of fractions. I chose to see and define mathematical understanding as containing both procedural and conceptual knowledge (National Research Council, 2001; Rittle-Johnson et al., 2015; Star, 2005; Thompson & Saldanha, 2003). This definition has been used in several theories of learning and development in the mathematical educational field (e.g., Hiebert & LeFevre, 1986; Jordan et al., 2013; Rittle-Johnson et al., 2001, 2015; Star, 2005). This definition of understanding as containing both procedural/conceptual duality has a long history. The duality is first seen in modern times in Skemp's (1976) distinction between instrumental and relational understanding, which has been commonly used over the last four decades. Other researchers have defined the dual nature of knowledge not as different types of mathematical understandings, but as complementarity in knowledge (Maciejewski & Star, 2016). This approach to complementarity knowledge can be found in Gray and Tall's (1994) two terms *process* and *concept*, and in Sfard's (1991) process and object duality. In the official Danish curriculum, there is a duality between two terms *færdigheder* (skills/ability) and *viden* (knowledge). This duality will be analysed further in Chapter 5.1.

The two types of knowledge, procedural and conceptual, dominate the discourse in mathematics education studies. In the context of this dissertation, I take the position that the conceptual and procedural distinction of knowledge type is a productive framework to describe knowledge[1]. See Appendix B for search protocols.

4.2.1 Conceptual knowledge

Conceptual knowledge is often discussed in the literature based on the definition from Hiebert and Lefevre (1986) as 'knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information' (pp. 3–4). Byrnes (1992) further described conceptual knowledge simplified as 'knowing that' and in

[1] I need to emphasise that I am aware of Dewey's own theoretical framework for 'knowing' or knowledge which simplified can be seen mostly as connected, active, and problem-oriented. However this PhD project is based in methodology of pragmatism, and in this enquiry process a central question is: 'How can fraction knowledge be defined?', and to answer this question I found that the theoretical framework of conceptual and procedural knowledge contributed with central points and definitions.

detail as ‘relational representations’ which ‘consist of two or more represented entities, that are mentally linked through a relation of some sort’ (p. 236). Kieren (1993) defined conceptual knowledge as ‘the interweaving of the intuitive and formal knowledge on a personal basis’ (p. 49). Overall, conceptual knowledge is defined as the ability to see interconnections between things rather than seeing knowledge as discrete bits of information – ‘knowing that’ (Byrnes & Wasik, 1991).

Conceptual knowledge in the context of fractions could involve understanding the magnitude of fractions; for example, $\frac{1}{2}$ can refer to either a pizza where half is eaten or half of the students in a class (Cramer et al., 2002; Hecht & Vagi, 2012). Fraction magnitude can also be ordering fractions from smallest to largest (e.g., Hecht et al., 2003; Smith et al., 2005). Conceptual understanding could involve solving $\frac{1}{2} + \frac{1}{4}$ by shading corresponding pieces of a circle or using a number line (Hecht & Vagi, 2012). However, these operations could also be seen as procedural, depending on the students’ solving process. Conceptual knowledge can be seen as including understanding of the previously learned subconstructs of fractions: *measure*, *ratio*, *operation*, *quotient*, and *part-whole* (Lenz et al., 2020).

When looking more deeply into the literature, it can be seen that definitions of conceptual knowledge differ in their level of detail. The definition of conceptualisation by Hallett et al. (2010) remains implicit to some extent because they define conceptual knowledge as ‘the ability to see interconnections’ (p. 396) without further specification of what this involves in a subject-specific way. With regard to mathematics, on the other hand, Lin et al. (2013) defined conceptual knowledge as ‘the relationships and interconnections of ideas which explain and give meaning to mathematical procedures’ (p. 42). When looking at conceptual knowledge in the context of fractions specifically, both Hecht et al. (2003) and Jordan et al. (2013) specified conceptual knowledge as mainly based on the *part-whole* and *measure* aspects of fractions. (This leads back to previous definitions of subconstruct fractions in Chapter 3.)

4.2.2 Procedural knowledge

Procedural knowledge has been defined in terms of knowledge of procedures in the solving process and as sequential – knowing what to do next (Hiebert & Lefevre, 1986) – or as the ‘knowing how’ to do something (Hallett et al., 2010). Byrnes (1992) further defined it as ‘goal-directed action sequences’ (p. 236), meaning students’ ability to put together an action sequence to solve a problem. This can also be described as knowledge of what actions to take next in the mathematical solving process (Rittle-Johnson et al., 2001; Rittle-Johnson & Rittle-Johnson, 2017). This kind of knowledge has been connected to algorithms (Hiebert & LeFevre, 1986). As Hiebert and Lefevre (1986) put it:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols. (pp. 7–8)

By these definitions, procedural knowledge can be seen as the ability to execute action sequences to solve problems. This type of knowledge is tied to specific problem types and has been interpreted as not widely generalisable (Rittle-Johnson et al., 2001). In the context of fractions, procedural knowledge can refer to the ability to solve fraction-based tasks or problems correctly. For example, this process could include fraction arithmetic such as finding the common denominator by multiplying the two denominators when adding fractions (Hallett et al., 2010). In other words, procedural knowledge in fractions refers to the ability to carry out tasks or solve problems in fractions accurately – 'knowing how' (Byrnes & Wasik, 1991; Durkin & Rittle-Johnson, 2015; Hecht & Vagi, 2012). This does not mean that procedural knowledge is superficial as a mechanical procedure might suggest. It also has a deeper meaning of students using procedures flexibly and innovatively in their problem-solving process (Maciejewski & Star, 2016; Star, 2005).

4.2.3 The interaction between the two types of knowledge

The importance of conceptual knowledge is important in learning fraction procedures and emphasises the importance of conceptual knowledge in the learning process (Hallett et al., 2010; Hecht & Vagi, 2010; Siegler et al., 2011). Historically, many studies of the development of conceptual and procedural knowledge have been based on detecting which of these two kinds of knowledge needs to be developed first for mastery of a given mathematical subject or topic. The relations between the two types of knowledge might be unidirectional, meaning that students begin by developing conceptual knowledge and then procedural knowledge (Byrnes, 1992; Geary, 1994; Halford, 1993; Siegler & Crowley, 1994) or going from some kind of procedural knowledge to conceptual knowledge (Karmiloff-Smith, 1996; Siegler & Stern, 1998), or bidirectional, according to an iterative model in which each of the two types of knowledge is developed in mutual support of the other (Hecht & Vagi, 2010, 2012) in an iterative process (Rittle-Johnson et al., 2001; Rittle-Johnson & Rittle-Johnson, 2017).[2]

According to Rittle-Johnson et al. (2015), conceptual knowledge in the literature shows a tendency to play a greater role in newer research, but procedural knowledge has not been given the same focus historically. Procedural knowledge has been linked

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[2] More than one type of development has been defined where the knowledge types have not been directly causally related (inactivation view) (Schneider & Stern, 2010).

to traditional instructional methods in classrooms in terms of mastery of algorithms, whereas conceptual knowledge has been associated with the reform approaches in which mathematics is seen as a sense-making activity. Star (2005) argues that the two different definitions of knowledge types (procedural and conceptual) have been interpreted as connected to different quality of knowledge (superficial and deep). Procedural knowledge has been seen as more superficial, whereas conceptual knowledge has been considered deeper. This conflation has stifled or misled research on procedural knowledge acquisition and performance; procedural knowledge has come to be viewed inaccurately as more superficial than conceptual knowledge and less supportive of overall mathematical understanding. The conceptualisations of procedural knowledge seem to be more homogeneous, describing procedural knowledge as knowledge referring to algorithms for solving mathematical tasks, although this is a simplification of the concept (Baroody et al., 2007; Rittle-Johnson et al., 2015; Rittle-Johnson & Rittle-Johnson, 2017; Star, 2005; Star & Stylianides, 2013). The same tendency is found in fraction research. Here, fraction concepts are often defined as including understanding *part-whole*, fraction notation ($\frac{1}{2}$), and fraction magnitude (e.g., $\frac{2}{5}$, $\frac{2}{4}$ and $\frac{2}{3}$ can be ranked from smallest to largest.

Researchers have suggested that concepts and procedures are interdependent and learned through mutual interaction in an interactive process (Maciejewski & Star, 2016; Rittle-Johnson et al., 2001, 2015; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Rittle-Johnson, 2017; Schneider & Stern, 2010). Rittle-Johnson et al. (2001) showed that there is an underlying relationship between conceptual and procedural knowledge in which development of 'the relations between conceptual and procedural knowledge are bidirectional, and that improved procedural knowledge can lead to improved conceptual knowledge, as well as the reverse' (p. 360). They argue that conceptual and procedural knowledge is an iterative process, meaning that one type of knowledge supports increases in the other type, which in turn supports increases in the first. This bidirectional relationship means that procedural and conceptual knowledge are equal in importance, depending on each other to produce a deeper understanding of mathematical concepts (see Fig. 9). When learning fraction procedures (e.g., the fraction addition algorithm), it also supports the conceptual knowledge of fractions (e.g., fraction magnitude or relation to decimals). When developing conceptual knowledge, developing the understanding of fraction magnitudes also supports the development of procedural knowledge. Both approaches to knowledge are fundamental concepts and support each other. Connected or joined conceptual and procedural knowledge can be described as an indication of deeper understanding for students in mathematics.

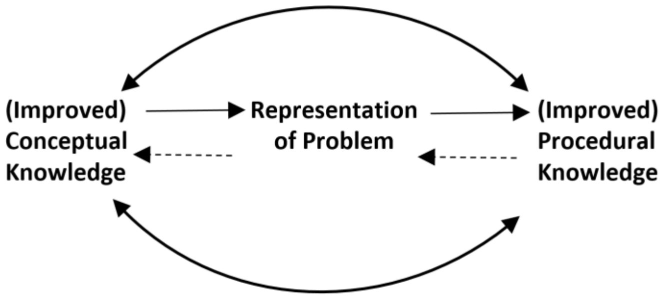

Fig. 9 Iterative model of the development of conceptual and procedural knowledge (Rittle-Johnson et al., 2001, p. 347)

That the two types of knowledge cannot be seen as separate and unconnected is supported by several studies that show a positive high correlation between conceptual and procedural fraction knowledge (e.g., Hallett et al., 2010; Schneider & Stern, 2010). Hallett et al. (2010) reported the individual differences in the conceptual and procedural fraction knowledge of fourth and fifth graders in the United Kingdom (N = 318). The researchers formed two different scales (conceptual and procedural). The correlation between the two scales was high and significant ($r = .68$, $p < .001$). Almost the same correlation was found by Jordan et al. (2013) between the two kinds of knowledge ($r = .62$, $p < .001$) in their study of sixth-grade students' knowledge of fraction in the US. Both studies together give no support to separating conceptual and procedural knowledge into two different scales in a measurement. Only a few empirical studies of knowledge of fractions separate them (exploring two latent variables). In only one study, by Schneider and Stern (2010), an analysis was made splitting the conceptual and procedural knowledge into two latent variables in a confirmatory factor analysis. The study was conducted in fifth and sixth grades in Germany (N = 230). These latent factors showed high correlations ($r > .93$, $p < .001$), and factor analysis showed that the two-factor model distinguished between conceptual and procedural knowledge, and the one-dimensional model was of equally adequate fit.

A new study by Lenz et al. (2020) compared a two-dimensional factor model including a 'conceptual-procedural' model with a one-dimensional model including just one underlying factor. The difference between the goodness-of-fit of the two models was found to be significant ($\Delta\chi^2$ = one, N = 235, DF = 29,162, $p < .01$) and the best model fit was the two-dimensional 'conceptual-procedural' model. Although the two-dimensional model showed a better fit, the one-dimensional model showed an acceptable fit as well (TLI = .89, CFI = .93, RMSEA = .10). Correlations between the conceptual indicator variables ranged from .48 to .57 ($p < .001$) and the four procedural indicators ranged from .32 – .56 ($p < .001$) but the correlations between conceptual and procedural indicators ranged from .32 to .53 ($p < .001$). There was no clear difference shown between the correlations among the same types of indicators (conceptual/conceptual and procedural/procedural) and between the two types of indicators (conceptual/procedural).

The contradictory results may be explained by the previously described multidisciplinary nature of the field of mathematics education, which influences the definition of the two kinds of knowledge. Star and Stylianides (2013) found that psychologists and mathematics educators use the same terms – procedural and conceptual knowledge – to refer to different types of mathematical knowledge, but the two terms are used differently between disciplines. Whereas mathematics education research tends to view the two kinds of knowledge as focusing on qualities within a mathematics setting, in psychology research, the terms tend to be based on the nature of knowledge per se and not as connected to discipline-specific knowledge. Because of the multidisciplinary nature of the field, it is important to be aware of how the terms are used. I take the position that the distinctions between the conceptual and procedural definitions of knowledge types are a productive framework. However, as the above-mentioned studies illustrate, the two types of knowledge are interrelated as part of an iterative process, so I will treat them as contributing equally to students' development of fraction proficiency. In the analysis of the data from Study 1, where the developed measurement tool will be validated, there will be no distinction between the two types of knowledge, but rather an overall measurement of fraction proficiency. The intervention material is designed to develop both the student's conceptual and procedural knowledge.

4.2.4 Fraction proficiency

In the mathematics education community, the term 'mathematical proficiency' has been broadly accepted. The term was first introduced in 2001 by the National Research Council in the report 'Adding it Up: Helping Children Learn Mathematics'. The Council reviewed the best available research on mathematics learning and then defined five strands of mathematics proficiency, which included both conceptual and procedural knowledge. These strands are summarised in Table 1.

The five strands are not isolated concepts that need to be developed separately from one another, but are linked and interdependent. Developing just one or two strands will not support students' efforts to become mathematically proficient (National Research Council, 2001). The term 'proficiency' attempts to capture the essence of what it means to learn mathematics. This report, where the term was first introduced, was edited by Jeremy Kilpatrick, Jane Swafford, and Bradford Findell, who argued that the terms *expertise*, *competence*, *knowledge*, and *facility* do not fully cover all aspects of what it means to be 'competent in mathematics'. Instead, the interaction between them must be included in the concept, and the term *proficient* is meant to capture the intended complexity.

Table 1 Proficiency (National Research Council, 2001, p. 5)

Strands	Definition	Explanation
Conceptual understanding	The comprehension of mathematical concepts, operations and relations.	Using and understanding number magnitude when making estimations.
Procedural fluency	A skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.	Being capable of moving between different representations.
Strategic competence	The ability to formulate, represent, and solve mathematical problems.	Having multiple strategies for calculations, flexible use of numbers while doing mental computations.
Adaptive reasoning	The capacity for logical thought, reflection, explanation, and justification.	Judging whether answers are reasonable.
Productive disposition	The habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.	To see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off and to see oneself as an effective learner and 'doer' of mathematics.

The term *proficiency* was developed in an American school tradition and culture, which are different in many ways from the Scandinavian tradition. In the Danish school system, a central term is *competencies*, which is often used to describe the content of mathematical education (Niss & Højgaard, 2002, 2019). However, in this dissertation, I chose the concept *proficiency* to describe mathematical learning as it fits the present understanding of conceptual and procedural knowledge better than a framework of different mathematical competencies.

The term *proficiency* has been broadly adopted into the education field of mathematics, but few researchers have used terms like *fraction proficiency* or rational *number proficiency*. During the literature review, only 24 peer-reviewed articles appeared in the search (see the second part of the search protocol in Appendix B).

When evaluating the studies, I only found two that attempted to define *fraction proficiency* as a concept (Brown & Quinn, 2007; Tsai & Li, 2017). In the context of their study on the relationships between fraction proficiency and algebra, Brown and Quinn (2007) defined it as a state in which 'not only a student is able to understand fraction concepts, but also that the student is able to manipulate fractions for accurate computation without the aid of a calculator' (p. 9). Their definition suggests that proficiency is not only connected to the concept of fractions, but also to computations. When looking at the original definition of mathematical proficiency, you can see the first four strands in their definition; the fifth, productive disposition, is not explicit in it. However, their definition includes both conceptual and procedural knowledge.

In contrast to Brown and Quinn's definition, fraction proficiency in Tsa and Li (2017) is defined as 'conceptual comprehension, procedural skills and the ability to approach daily situations involving fractions' (p. 246). This is close to the overall definition of proficiency but does not contain explicit reference to strategic competence or adaptive reasoning. However, because all five strands are described as being interconnected, they might be implicit in their definition. Furthermore, Tsa and Li (2017) proposed a framework for developing fraction proficiency that contained the following five dimensions:

- The five constructs of fractions (based on Kieren's subconstruct as described in Chapter 3.3).
- The concept of equivalent fractions.
- The procedural fluency for and conceptual understanding of fraction operations.
- The relationship between fractions, decimals, and percentages and the transition between different forms of representations involving fractions (Tsai & Li, 2017).

4.2.5 Summary of fraction proficiency

Although their framework is interesting, the taxonomy between the five dimensions and the definition of overall mathematical proficiency is not clear. As will be outlined in Study 3, I see fraction equivalence as an important concept in the five subconstructs and not as a separate dimension. Moreover, the concept of equivalence is closely connected to a conceptual understanding of fraction procedures when adding fractions (e.g., finding the common denominator). Equivalence plays an important role when describing the relationship between fractions, decimals, and percentages; for example, $\frac{1}{4}$ is equal to $\frac{25}{100}$. The five subconstructs are also a foundation for describing this relationship. These two studies offered a definition of the term *fraction proficiency*, while others in the review used the term without further definition (e.g., Ennis & Losinski, 2019; Vitoria et al., 2017).

Overall, the multifaceted construct of fractions makes it difficult to produce a description of fraction development within the framework of fraction proficiency. After reviewing and synthesising the theoretical and empirical studies, I suggest the theoretical framework for defining fraction proficiency presented in Table 2. As Rittle-Johnson et al. (2001) also argued, I see conceptual and procedural knowledge as an iterative process, and both knowledge types are combined in my definition of proficiency.

Table 2 Fraction proficiency

Strands	Definition
Conceptual understanding of fractions	Comprehension of fractional notation, magnitude, operations, and relations (e.g., equivalence, density, and other rational number representations); a flexible understanding of five subconstructs (described in Chapter 4.2.1).
Fraction procedural fluency	Skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (e.g., fraction arithmetic, algebra, or conversion between fractions, decimals, and percentages).
Fraction strategic competence	The ability to formulate, represent, and solve mathematical problems involving fractions.
Fraction adaptive reasoning	The capacity for logical thought, reflection, explanation, and justification, particularly as connected to estimation of fraction size.
Fraction productive disposition	The habitual inclination to see fractions as sensible, useful, and worthwhile, understanding the necessity of the notation form, coupled with a belief in diligence and one’s own efficacy when using fractions.

As the National Research Council (2001) also emphasises, the five strands in their original framework of ‘proficiency’ are interrelated and cannot be separated from each other. The measurement tool developed in Study 1 and the students’ development of fraction proficiency in Study 5 are based on this definition. The time limitation of a computerised test made it difficult to evaluate all the strands, especially ‘fraction productive disposition’, ‘fraction strategic competence’, and ‘fraction adaptive reasoning’. Kieren’s subconstructs were, as previously mentioned, central elements in the development of the test (Chapter 3; Behr et al., 1984; Charalambous & Pitta-Pantazi, 2007; Kieren, 1976; Marshall, 1993).

4.3 Review (2): Natural number bias

During the research process exploring *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* it was clear that the natural number bias was a central aspect of answering parts of this overarching research question. As previously mentioned, this second review was closely connected to *Institution of a Problem Phase II*, Study 4. The search protocol can be seen in Appendix C.

4.3.1 Natural number bias

One of the major difficulties many researchers agree on is associated with problems in students' learning process of fractions is a tendency to let whole number knowledge interfere with their concept of fractions and rational numbers (e.g., English & Halford, 1995; Meert et al., 2010; Ni & Zhou, 2005; Van Hoof et al., 2018).

This tendency was first called 'whole number bias' by Ni and Zhou (2005); later, the term 'natural number bias' was used by other researchers, such as Vamvakoussi, Van Dooren, and Verschaffel (2012). The tendency to let whole number knowledge interfere with rational numbers has been described by many researchers in both the educational and psychological fields. They agreed that there was a distinct difficulty associated with the students' whole number knowledge interfering with their concept of fractions (or rational numbers) (e.g., Behr et al., 1993; Streetland, 1991); however, Ni and Zhou were the first to formulate and describe it as a bias. In the present chapter, a literature review about natural number bias was conducted. This included studies from 2005 until the present, indicating that impetus was taken to begin with the first year that a whole number bias was defined.

Overall, the term 'whole number bias' has been defined as a major tendency to inappropriately apply natural number properties to the concept of rational numbers (Ni & Zhou, 2005). The later developed term 'natural number bias' has the same intended meaning (Christou & Vosniadou, 2012; Gómez et al., 2014; Obersteiner et al., 2013; Van Hoof et al., 2013, 2018), yet, the term must be seen as more specific. There is a mathematical distinction between whole numbers and natural numbers. Whereas natural numbers can be described in everyday language as those used for 'counting' (cardinal numbers or positive integers that include 1, 2, 3, …), whole numbers are defined as integers, which consist of natural numbers as well as integers and zero (or exact positive and negative numbers plus zero).

Current studies only focus on fractions that contain only positive whole numbers in the numerator and the denominator, so the term *natural number bias* is more appropriate (Christou, 2015; Gómez et al., 2014; Van Hoof, Verschaffel, et al., 2015). Hence, the term *natural number bias* is used in this current PhD project in order to make a more precise description of the bias based on the procedures and properties of natural numbers that influence different concepts of fractions.

The overall tendency for studies connected to the research group in Belgium (Leuwen) or Greece is to use the term *natural number bias,* whereas studies from the US use the term *whole number bias*.

4.3.2 Who is affected by the natural number bias?

Research has shown that everyone seems to have a tendency for natural number bias in some form. It has been detected in elementary school students' answers (McMullen et al., 2018; Meert et al., 2010; Reinhold et al., 2020; Resnick et al., 2019) in high school students (DeWolf & Vosniadou, 2015; Obersteiner et al., 2016; Van Hoof, Vandewalle, et al., 2015), in adults (Fu et al., 2020; Vamvakoussi et al., 2012), and in expert mathematicians (e.g., Obersteiner et al., 2013). The bias can be detected and is present among children who have just learned fractions and also in adults with a lot of math experience, particularly longer experience with fractions. Obersteiner et al. (2016) found evidence for a natural number bias in eighth-grade students, but they did not find the same traces of this bias in expert mathematicians. These researchers argued that while students found their answers using their intuition about natural numbers, experts would rely on their intuition about algebra, meaning they were unaffected by this bias as they do not use natural number strategies. This finding indicates that instruction or experience with rational numbers can be used by students to develop strategies to overcome a natural number bias. This is further supported by Rinne et al. (2017) who found that bias decreases with increasing experience with fractions.

They found that students who showed a partial understanding of fractions by choosing fractions with smaller numbers were more likely to adopt normative comparison strategies earlier than those with larger number bias. Exploratory factor analysis showed that over time, children appeared to increasingly represent fractions as discrete magnitudes when simpler strategies were unavailable. These results support the integrated theory of numerical development which posits that an understanding of numbers as magnitudes unifies the process of learning whole numbers and fractions. The findings contrast with conceptual change theories, which propose that children must move from a view of numbers as counting units to a new view that accommodates fractions to overcome the natural number bias.

This outcome is further supported by Kainulainen et al. (2017) who found that third- to fifth-grade students' development from natural number-based reasoning to a mathematically correct concept of fractions was almost non-existent over a one-year period. Instead, students appeared to develop intermediary concepts before acquiring a mathematically correct understanding of rational numbers. Second, a cross-cultural study by McMullen et al. (2018) made a comparative study where they found that

Finnish and Flemish students were affected in similar ways by the natural number bias.

The overall finding was that older students and adults have not completely overcome the natural number bias and that its development seems to take place gradually over time.

4.3.3 Different aspects of the natural number bias

According to my search, the first to propose categorisation of different aspects of natural number bias was Vamvakoussi et al. (2012). They detected three different aspects: comparison, arithmetical operations, and density property. Later Van Hoof, Vandewalle, et al. (2015) detected four aspects in their review: *density*, *operations*, *size*, and *representations* (see Fig. 10). The definition of *size* overlaps with Vamvakoussi et al.’s (2012) definition of the term comparison, whereas representation is a new aspect that Van Hoof, Vandewalle, et al. (2015) defined. These different aspects of natural number bias are used in the coding process in Study 4. It is important to emphasise that in other studies only three aspects have been defined as *density*, *size*, and *operations* (e.g., Van Hoof, Verschaffel, et al., 2015), indicating that the definition of the aspect is not universal and unchangeable.

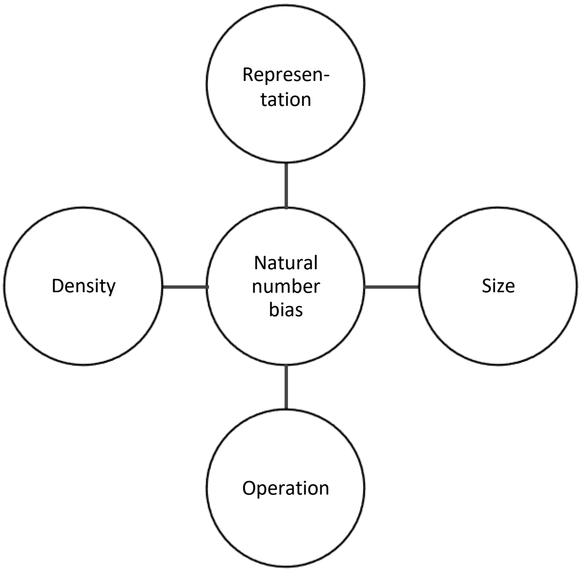

Fig. 10 The model of four aspects of natural number bias used in Study 4

Density

The first aspect of *density* is described as the contrast between natural numbers and rational numbers and is based on the distinction that natural numbers are discrete (you can always count which number comes next on the number line), whereas rational numbers are dense (you do not know which number comes next, so there are infinite numbers between two rational numbers). This difference can lead to the natural number bias that there is a finite number of numbers between two pseudo-successive numbers; for example, between $\frac{1}{2}$ and $\frac{1}{4}$, there is only $\frac{1}{3}$ (McMullen & Van Hoof, 2020; Vamvakoussi et al., 2012; Van Hoof, Vandewalle, et al., 2015

I found nine studies that examined *density* as a natural number bias aspect (Iuculano & Butterworth, 2011; McMullen et al., 2018; Vamvakoussi et al., 2012; Vamvakoussi & Vosniadou, 2010; Van Hoof et al., 2016, 2018; Van Hoof, Janssen, et al., 2015; Van Hoof, Vandewalle, et al., 2015; Van Hoof, Verschaffel, et al., 2015).

Vamvakoussi and Vosniadou's (2010) study concluded that all age groups from grades 7 to 11 showed natural number bias, meaning that they had an idea of discreteness that a fraction had a unique 'successor' like natural numbers. This bias appeared to be unconnected to a unique cultural school system, but affects students across countries (McMullen et al., 2018). In Van Hoof, Verschaffel, et al.'s (2015) study, the aspect of *density* was shown to be the strongest natural number bias compared to *size* and *operation*.

Representation

Van Hoof, Vandewalle, et al. (2015) described the representation aspect of natural number bias as another difference between natural numbers and rational numbers. While a natural number can be seen as having only one symbolic representation, each rational number has an infinite number of symbolic representations (e.g., $\frac{1}{2}$ is equal to 0.5 or $\frac{1}{4}$ is equal to $\frac{2}{8}$). This trend could indicate that students have been unable to accept or understand that a fraction and a decimal can represent the same number (Vamvakoussi et al., 2012) or that a fraction can be interpreted as two separate natural numbers instead of representing a single number (Stafylidou & Vosniadou, 2004; Van Hoof, Vandewalle, et al., 2015).

Size

The fast majority of studies investigated the numerical *size* of rational numbers where students make their reasoning about the *size* of rational numbers based on their concept of natural numbers (e.g., that $\frac{1}{3}$ is bigger than $\frac{1}{2}$ because 3 is bigger than 2, or $\frac{2}{3}$ is smaller than $\frac{3}{7}$ because both 2 and 3 are smaller than $\frac{3}{7}$) (Rinne et al., 2017; Van

Hoof, Vandewalle, et al., 2015; Van Hoof, Verschaffel, et al., 2015). When a comparison answer can be based on prior natural number knowledge, it is called a congruent task, for example, $\frac{7}{8} > \frac{1}{2}$, whereas when it is not true, the comparison is defined as an incongruent item, for example $\frac{2}{3} > \frac{3}{7}$ (Van Hoof, Janssen, et al., 2015). This tendency has explained the student difficulty of seeing a fraction as one number instead of two separate numbers (Van Hoof , Vandewalle, et al., 2015), so the *size* and *representation* aspects must therefore be closely connected, and this explains why the *representation* aspect was dropped in later studies (e.g., Van Hoof et al., 2018).

Meert et al. (2010) found that response times were slower for fractions with common numerators compared to fractions with common denominators, which they proposed indicated an interference of the *size* of the denominators when choosing the larger fraction (e.g., the problem of $\frac{2}{3}$ compared to $\frac{2}{4}$ takes longer to solve than $\frac{3}{5}$ compared $\frac{4}{5}$).

Obersteiner et al. (2013) further looked into response time and found that mathematician experts process fraction comparisons differently depending on whether there is a common component present. When the fraction had the same denominator or numerator, the experts were affected by the natural number bias (showing a longer response time when solving problems with different denominators). When the fractions did not contain any common components, there was no natural number bias, which they argued indicated the use of a more holistic strategy in the solving process. Obersteiner and Tumpek (2016) supported this conclusion via an eye-tracking study where they found that participants preferred componential strategies when there were common numerators or common denominators, and they preferred holistic over componential strategies for fraction pairs without common components.

Furthermore, Gómez et al. (2015) showed that students with a natural number bias were significantly faster at solving fraction comparison tasks than proficient students. They suggested that this reflects a difference in the way these two groups reasoned about fractions; where natural number-biased students based their reasoning on the natural numbers concept, proficient students used more complex strategies.

Vamvakoussi et al. (2012) found a natural number bias in their experiment that measured university students' speed and accuracy when solving various fraction tasks where half of the task targeted congruent, and the other half contained incongruent fraction pairs. With congruent pairs (e.g., $\frac{6}{9}$ and $\frac{2}{3}$), natural number reasoning will lead to the correct answers while whole number reasoning produces wrong answers with incongruent pairs ($\frac{2}{3}$ and $\frac{3}{5}$). Their results found no significant difference in correct versus incorrect answers, but it took significantly longer for respondents to compare incongruent pairs versus congruent pairs. This trend was further investigated by Van Hoof et al. (2013) in their study of whether fraction comparisons were controlled by distance stimuli in first- and fifth-year secondary school students. Their results also found longer reaction times for incongruent fraction pairs. A new study by Obersteiner et al. (2020) found a reverse bias defined as the misconception that a smaller component is always the larger fraction, such as $\frac{2}{3}$ is bigger than $\frac{4}{5}$. This reverse bias was greater among participants with lower mathematics experience. In addition, these researchers found that when the fractions were close to 0 or 1, there was a decrease in the detected natural number bias.

Overall, the many studies about *size* where participants were asked to compare or order fractions reveal the complexity of the concept of natural number bias. Several factors could affect the natural number bias observed in fraction comparison tasks, such as the strength and preciseness of rational number representations. The strength of the rational number representations was likely dependent on the students' experience with fractions, and their answers could depend on the length of time that had passed since they last received instruction on the topic (Alibali & Sidney, 2015).

Operation

The last aspect, *operation*, can be defined as students using their assumptions or rules connected to whole number arithmetic when they are solving fraction operation tasks. This aspect appears when students assume that addition and multiplication will always produce larger results, and subtraction and division will always lead to smaller results (Vamvakoussi et al., 2012; Van Hoof, Vandewalle, et al., 2015). The can also be connected to the common mistake made when adding or subtracting to change both the numerator and the denominator, for example $\frac{1}{4} + \frac{3}{4} = \frac{4}{8}$ (Tian & Siegler, 2017). In the review of literature, eight studies targeted this (Sidney & Alibali, 2017; Vamvakoussi et al., 2012; Van Hoof, Verschaffel, et al., 2015).

In Obersteiner et al.'s (2016) study, the students were given a task in which the congruent problem answers based only on natural numbers led the students to the correct answer; for example, can $4 < x \times 4$ be true? (The correct answer is yes.) However, the incongruent problem answers given by respondents who relied on natural numbers led to an incorrect answer; for example, can $4 > x \times 4$ be true? While the correct answer is still 'yes', the natural number bias answer would be 'no'. Younger students based their answers on their intuition about natural numbers and therefore showed a tendency for bias. In contrast, expert mathematicians relied on an algebraic understanding and were therefore unaffected. Their study focused only on multiplication and division. Vamvakoussi and colleagues investigated intuitive reasoning about all four *operation*s as algebraic hypotheses in connection to their response times. Their studies on *operation*s showed that students made more mistakes with incongruent items, and that it took more time to arrive at the correct answers. Overall, they found that the natural number bias was deeply rooted in people's reasoning about arithmetic (Vamvakoussi et al., 2012).

Schumacher and Malone (2017) investigated natural number bias in subtraction and addition tasks. Their research showed that students were more likely to have a tendency to use a natural bias on tasks with unlike denominators. They found that below-average students who received small-group tutoring had a lower tendency for a natural number bias compared to students who only participated in the regular instruction.

4.3.4 Summary of natural number bias

The four aspects defined by Van Hoof, Vandewalle et al. (2015) are multifaceted in their structure, and their theoretical framework should be discussed. In contrast to the study by Van Hoof, Verschaffel, et al. (2015), I did not find any indication of the fact that the aspects can be seen as aspect of an overall tendency of natural number bias.

As mentioned in Study 4, it should also be discussed whether we should understand these four dimensions as an aspect of the natural number bias or as four different natural number bias constructs. Another question is whether the four aspects can be seen as adequate, or whether there are others. For example, it is clear that the concept of equal fractions differs from the other answers that students gave when comparing fractions in Study 2.

The definition of *representation* includes the concept of a fraction as a 'single magnitude' or *quotient* and thereby the student sees the notation as two separate natural numbers that do not interact. This can be seen as one process of understanding the fraction notation (e.g., interpreting $\frac{a}{b}$ as one magnitude), whereas understanding the different *equivalence* classes as the same magnitude is another (e.g., $\frac{a}{b} = \frac{na}{nb}$).

4.4 Review (3): Number knowledge development

The third review is central for exploring the overarching research question: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* where we need to enquire into what 'development of fractions' means. The development of number knowledge is therefore in focus in this third review. In this chapter, the two central theories of number knowledge development are explained: first, the *conceptual change theory* and then the *integrated theory*. The integrated theory is further described in Study 5, and the review is primarily connected to an overall reasoning (*Reasoning Phase IV*) in the research process. (The search protocol can be seen in Appendix D.)

4.4.1 Conceptual change theory

In connection to the natural number bias, the theoretical framework that has often been used to explain and describe why the bias appears is the conceptual change theory (Vosniadou, 1994).This approach to learning can be seen as based in both science education and cognitive-developmental research and has, of course, also been applied to mathematics education (Brown, 2015; Van Dooren & Inglis, 2015; Vosniadou & Verschaffel, 2004). Conceptual change theory emphasises that the fraction learning process must include a conceptual change in the students' concept of numbers. This conceptual change theory proposes that students encounter natural numbers more frequently than rational numbers in their first years of schooling and before beginning in school. As a result, they have already developed a concept of what numbers are, and their actions are based on these first experiences with natural numbers. One experience could be that numbers always get bigger with multiplication, while they get smaller with division. Thus, students require a conceptual change of these initial natural numbers concepts when rational numbers are introduced (e.g., McMullen et al., 2015; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004, 2010; Vosniadou & Verschaffel, 2004). The different nature of rational numbers, which students need to change their conception of, can be seen in Table 3. Overall conceptual change theory is described in the following quote by Stafylidou & Vosniadou (2004, p. 504):

a) The knowledge acquisition process is not always a process of enriching existing conceptual structures. Sometimes the acquisition of new information requires the radical reorganisation of what is already known.
b) Learning that requires the reorganisation of existing knowledge structures is more difficult and time consuming than learning that can be accomplished through enrichment. Moreover, it is likely that in the process of reorganisation, students will create misconceptions.
c) Many misconceptions are synthetic models that reveal students' attempts to assimilate the new information to their existing knowledge base.

Table 3 Conceptual change between natural numbers and fractions/rational numbers (Stafylidou & Vosniadou, 2004)

Numerical value	**Natural numbers**	**Fractions**
Symbolic representation	One number (presupposition of discreteness).	Two numbers and a line (presupposition of density).
Ordering	Supported by the natural numbers' sequence (counting on). Existence of a successor or a preceding number. No number between two different numbers.	Not supported by the natural numbers' sequence. There is no unique successor or a unique preceding number. Infinity.
Relationship to the unit	The unit is the smallest number.	No unique smallest number.
Operations		
Addition–subtraction	Supported by the natural numbers' sequence.	Not supported by the natural numbers' sequence.
Multiplication	Multiplication makes the number bigger.	Multiplication makes the number either bigger or smaller.
Division	Division makes the number smaller.	Division makes the number either smaller or bigger.

In the conceptual change theory, researchers argue that the explanation for natural number bias is caused by how the 'bias' interferes with fraction learning as a temporary 'misconception' of rational numbers (Vamvakoussi & Vosniadou, 2010). In their study, Vamakoussi and Vosniadou (2010) argue that before students are introduced to rational numbers that include fractions, they have already formed a 'coherent explanatory framework of numbers' (p 186). This framework can be defined as a domain-specific understanding of numbers, such as counting. Students' concept of numbers is connected to the domain, which is the basis for their understanding. Vamakoussi and Vosniadou (2010) further explain this as follows:

> Within the framework theory approach to conceptual change, the phenomenon of students' misconceptions due to faulty natural number reasoning (Moss, 2005; Ni & Zhou, 2005) can be explained not as occasional intrusions of students' prior knowledge, but as an indication that students draw heavily on their initial understandings of numbers to make sense of rational numbers. (p. 187)

This means that initial understanding is the key to understanding new types of numbers. However, when students encounter this new information or experience with numbers, it is more demanding and requires more time to understand because the new information is likely not compatible with their initial concept of numbers (Van Hoof, Janssen, et al., 2015).

According to Van Hoof et al. (2018), there is no consensus in the conceptual change theory regarding whether a student's preliminary concepts of numbers can be characterised as relatively independent fragments (e.g., DiSessa, 2013) or if these early concepts of numbers are a more coherent idea (e.g., Vosniadou, 2013). However, in both interpretations, conceptual change is not an all-or-nothing matter, but a gradual process with many intermediate states (Van Hoof et al., 2018; Vosniadou, 2013). This change should not be seen as a specific point in time where it will take place, according to Vamvakoussi and Vosniadou's (2010) theoretical framework; it is transition or development from a natural number to a rational number perception, and transition is a gradual and time-consuming process.

This transition process from natural numbers to rational numbers generates synthetic conceptions; an intermediate state that can be explained as a bridge between the student's previous concept of whole numbers and the new concept of rational numbers. In their framework, Vamvakoussi and Vosniadou (2010) further emphasise that students rely primarily on additive mechanisms to add new information to prior knowledge in an all-or-nothing way. Natural number bias can be seen when students develop a synthetic concept of numbers, which includes rational numbers, but the synthetic conception includes a misconception. The integrated theoretical approach therefore argues that the development of rational numbers from natural numbers moves through intermediary concepts, which support the theory of a slow and gradually developing conceptual change in rational numbers (Kainulainen et al., 2017). It is important to emphasise that conceptual change with rational numbers is seen as a complex phenomenon that consists of different subconstructs (McMullen et al., 2015).

Overall, the conceptual change theories differ, but they are all founded on an underlying assumption that there is a conceptual difference between an early understanding of whole numbers and a later-developing understanding of fractions (Brown, 2015; Fu et al., 2020; McMullen et al., 2018; Vamvakoussi & Vosniadou, 2010). The earlier concept of whole numbers is explained as interfering with the later-developing concept of rational numbers (Vamvakoussi et al., 2012, 2018). To sum up, the conceptual change theories and the understanding of fractions and other rational numbers requires a substantial change in the basic concept of numbers (DeWolf & Vosniadou, 2015; McMullen et al., 2015; Van Hoof et al., 2018).

4.4.2 Integrative theory

In contrast to the conceptual change theory, Siegler et al. (2011) suggested an integrated framework named integrative theory of numerical development. This theory proposes that whole number knowledge can be seen as the basis for developing an understanding of fractions. In integrative theory, the number line plays an important role because both the magnitude of fractions and whole numbers (natural numbers) can be represented on a mental number line. Schneider and Siegler (2010) do not view the development of fractions as a conceptual change but instead argue that over time, students become gradually able to understand the concept of fractions as a holistic magnitude instead of composites of a set of whole numbers with one in the numerator and the other in the denominator (Schneider & Siegler, 2010).

The integrative theory is further supported by Torbeyns et al.'s (2015) study, where they argued that there are two different ways in which integrative theory varies from conceptual change theories. First, they propose that conceptual change theories do not take into account that whole number knowledge has been proven to have a significant positive impact on fraction learning. Second, they argue that overgeneralisation of whole number knowledge (which natural number bias is founded on) is not the only difficulty students have; there are many other difficulties with fractions that cannot be explained by a natural number bias or the need for a conceptual change.

Siegler et al. (2012) viewed the integrative theory of numerical development of fractions learning as a process of expanding on whole number magnitude representations. In this theory, conceptual changes still take place, but they are seen as isolated and fine-grained changes when it comes to fraction learning (Rinne et al., 2017). The natural number bias holds that this theoretical framework is not a conceptual mistake but is instead viewed as a phase where students try to emphasise rational numbers in their conceptual knowledge of numbers, and the bias will decrease during experiences with rational numbers.

These gradual numerical development expansion support the integrative theory of numerical development, which posit that an understanding of numbers as magnitudes unifies the understanding of numbers. Meaning that understanding of fractions is integrated in the knowledge of magnitude. Fractions must be taught in the context of natural numbers, meaning that throughout the teaching process, the student must be given the opportunity to learn whole number properties and fraction properties and to contrast and link the two together because natural number magnitude knowledge supports the fraction learning process (Rinne et al., 2017) and therefore also supports students in overcoming their tendency to hold a natural number bias.

The integrative theory argues that a central idea of numerical development is the expansion process of the numbers where magnitude plays an important part. This process contains four overlapping steps described here by Siegler (2016, p. 341):

1. Representing the magnitudes of non-symbolic numbers with increasing precision.
2. Connecting small symbolic numbers to their non-symbolic referents.
3. Extending understanding from smaller to larger whole numbers.
4. Accurately representing the magnitudes of rational numbers.

The representation of non-symbolic numbers could estimate the sum of dot arrays, whereas processes connecting symbolic numbers to the symbolic referent include a written number and should be connected to the correct number of objects. The process of extending the small numbers to larger whole numbers is logical when understanding larger numbers and also how the decimal system is connected to whole numbers (e.g., 30 is bigger than 3). Accuracy in representing the magnitude of fractions is, for example, estimating placing $\frac{3}{4}$ on the number line.

The integrative theories of development of mathematical concepts can be seen as a more holistic understanding of learning mathematics and not as the development of one topic after another (Siegler, 2016). Proponents of the integrative theory of numerical development therefore propose that fraction learning is based on a process of expanding on the whole number magnitude concept and not a fundamental conceptual change (Rinne et al., 2017; Siegler et al., 2012). Natural number bias can be seen as a phase that some students require in this expanding process. A longitudinal study by Rinne et al. (2017) supported the integrative theory where the concept of numbers is founded in the understanding of different magnitudes that unify the process of learning whole numbers and fractions. The overall development of the number concept from natural to rational numbers does not require an overall conceptual change (Dyson et al., 2020; Fu et al., 2020; Rinne et al., 2017; Schneider & Siegler, 2010; Siegler et al., 2011). The integrative theory argues that students have a tendency for natural number bias at the beginning of learning fractions, but this will gradually decrease with more experience with fractions, and in the process, students will create new strategies to solve fraction tasks (Fu et al., 2020).

The students' development of rational numbers from natural numbers appears to move through an intermediary concept, which supports the theory of a slow and gradual development of a conceptual change in rational numbers (Kainulainen et al., 2017). These gradual numerical developments support the integrative theory of numerical development, which posit that an understanding of numbers as magnitudes unifies where fractions are integrated into the knowledge of whole numbers. Fractions must be taught in the context of natural numbers, which means that throughout the teaching process, the student must be given the opportunity to learn whole number properties and fraction properties. Both should be contrasted and linked together because whole number magnitude knowledge supports the fraction learning process (Rinne et al., 2017) and therefore also helps students to overcome their tendency for natural number bias. These new studies are in contrast with conceptual change theories, which suggest that students should develop their concept of natural numbers first as counting units and then the more complex concept of rational numbers (e.g., fractions to overcome the natural number bias).

In the Danish context, this integrated approach can be seen as related to the framework of mathematical landscape theory (Lindenskov, 2006, 2010). This is a metaphor which sees mathematics as a landscape that students expand and move through, meaning that mathematical development is considered more as an integrative process, where mathematics is seen as a whole which is more in alignment with an integrative learning process.

4.4.3 The dissertation's theoretical perspective on number knowledge development

This dissertation proposes that the two theories of conceptual change and integrative theory should not be seen as opposite to each other but instead as an *integrated conceptual change framework* (Fig. 11).

I do not see the two theories[3] in opposition to each other. Looking at the development or process of going from natural numbers to rational numbers can be understood as an integrated development where natural numbers play an important role in the development of the concept of rational numbers in the expansion process of the overall number concept. When the starting point and ending point are viewed as separate from the process of looking at the difference between the two points, or between the two concepts of numbers, this can be seen as a change in concept when the starting point is held up against the ending point. For the sample, the student realises that $1\frac{1}{2}$ is a number between 1 and 2, next finding also that $1\frac{1}{4}$ and $1\frac{2}{3}$ are between 1 and 2, and thereafter finding new fractions between the two numbers, then realising that there is a density between the two whole numbers, finishing by realising there are infinite numbers of fractions (rational numbers) between every whole number.

[3] It is important to emphasise that the two theories must not be interpreted as fully-fledged theories using Niss' (2007) terms. In Niss' framework, they must be seen as sub-theories both based in an individual notion.

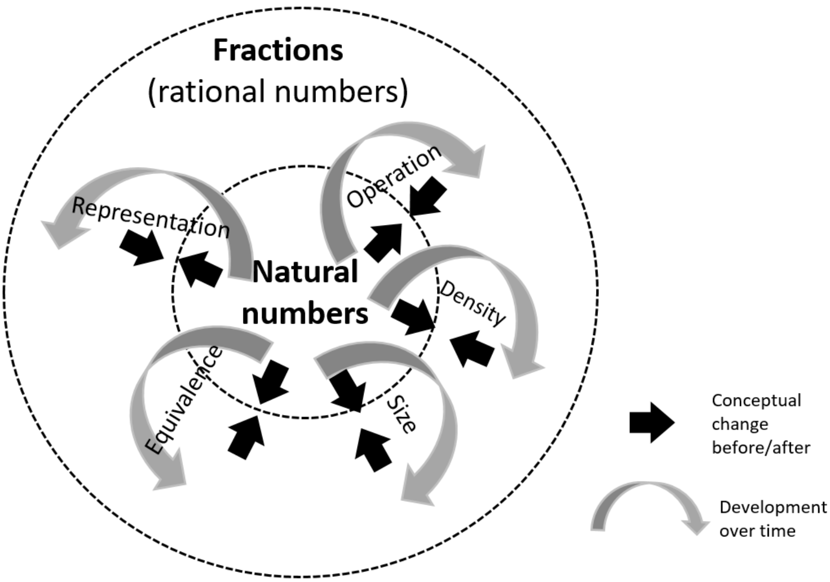

Fig. 11 The integrated conceptual change framework

The two theoretical frameworks already have similarities where conceptual change describes the process of development as a slow change (DeWolf & Vosniadou, 2015; Van Dooren et al., 2015; Van Hoof, Janssen, et al., 2015) while the integrative theories also acknowledge the need for new concepts or small changes in concepts (Siegler, 2016; Torbeyns et al., 2015). Hence, I argue that it makes sense that the two frameworks are not seen as opposite to each other but as two sides of the same coin focusing on different aspects in the number knowledge development process. I argue that the two theories can be combined in one theoretical perspective, *integrated conceptual change framework or model,* where the change is defined by the objects (natural numbers or rational numbers), and the integration defines the process. From this perspective, the development cannot be viewed as an overall conceptual change; it is more correct to look at the number knowledge as an expansion of the knowledge of natural numbers. As another explanation, a conceptual change must be considered as a point in time, whereas integrated development is the movement through time. From a mathematical standpoint, rational numbers are not separate concepts from natural numbers. Therefore, rational numbers include the interpretation of whole numbers.

Several subconstructs are required for supporting the expansion of the number concept, for example *part-whole*, *quotient,* etc. This expansion leads the student to overcome their natural number bias if they have a tendency to this. To overcome the natural number bias *size*, the *measure* subconstruct could be supported by comparing the fraction size on the number line, or the understanding of density could be supported by the subconstruct *quotient.* If four children share eight buns, they get two buns each. $\frac{8}{4}$ = 2, and if four children share 9 buns they get $2\frac{1}{4}$ each (this will further be described in the developed intervention instruction material in Chapter 6.2.2).

When looking at how this framework was generated, it was first based on Study 2, where I explored the connection between whole number arithmetic and fraction comparison: in other words, how whole number operations and fraction comparisons are related or integrated. It is an investigation of how a whole number operation relates to (supports) fractions – the integration. The same integration is seen in Study 5, where I explore how instructions from other mathematical topics support the development of fraction proficiency (multiplication/division and equations). Study 4 explores how natural numbers distract the concept of fractions. Future research should look further into which instructions support the development of this *integrated conceptual change* and how this differs between subgroups. During the last few decades, several fraction intervention studies have been conducted, and the generated knowledge from these studies is described in the next review (Review 4).

4.5 Review (4): Fraction interventions

As previously described, this fourth review was made in connection with the development of the instructional material used in Study 5. The aim was to collect previous knowledge about fraction intervention. This was used to inform the enquiry process, primarily *Determination of a Problem-Solution Phase* (*Phase III*) and *Common Sense and Scientific Enquiry Phase* (*Phase VI*). I used the review to develop an instruction material that could help me explore and try to solve the observed problem with students' difficulties with developing the multifaceted concept of fractions. The review's conclusions supported and secured the quality of the material. In addition these previous findings from this review were also used to discuss and *Reasoning* (*Phase IV*) with my findings in Study 5 (. Lastly the reviews findings also supported the *Scientific Enquiry* (*Phase VI*). The search protocol can be seen in Appendix E.

The first review was made by Misquitta (2011), who examined studies published between 1990 and 2008 that focused on fraction instruction and targeted struggling learners. This review resulted in the inclusion or identification of 10 empirical studies. The overall results show that four different kinds of interventions were found to be effective for improving the development of mathematics for struggling learners and for teaching fractions: 1) graduated sequences, 2) anchored instruction, 3) strategy instruction, and 4) direct instruction. Ad 1) The graduated sequences can be defined as a graduated progress in the instruction. This could be graduated sequence, concrete-representational-abstract (CRA). The use of CRA-sequence in the fraction instruction showed positive results (Butler et al., 2003; Jordan et al., 1999). Ad 2) The second intervention components anchored instruction was primarily based on instruction using videodiscs of real-world problems targeting fraction, and the aim was to improve students' problem-solving. (Bottge, 1999; Bottge et al., 2002). Ad 3) The third intervention components that were found to have a positive effect on students' fraction leaning was strategy instruction including fraction strategy steps (Joseph & Hunter, 2001; Test & Ellis, 2005). Ad 4) Lastly, the fourth component, direct instruction, emphasises the need for a direct and explicit approach when teaching fractions to students with difficulties (Bottge & Hasselbring, 1993; Flores & Kaylor, 2007; Gersten & Kelly, 1992; Kelly et al., 1990). The review further emphasises the overall importance of both procedural and conceptual knowledge in fraction instruction.

The second review was conducted by Shin and Bryant (2015), who further extended Misquitta's (2011) review by expanding the inclusion years to cover studies from 1975–2014. This review included 17 publications and targeted students in third to twelfth grade. The review examined studies where there were related topics aligned with the Common Core State Standards for Mathematics (CCSSM), so the majority of studies focused on fraction learning that was aligned with the CCSSM. The review contained specific instructional components embedded in the interventions. This review also found that explicit and systematic instruction in connection with visual representations of fractions showed significant and highly positive outcomes for developing fraction concepts and skills. Furthermore, the review found that heuristic approaches combined with explicit instruction led to improvising struggling students' concept of fractions, which in the previous review by Misquitta (2011) was defined as strategy instruction (Joseph & Hunter, 2001; Test & Ellis, 2005). They further identified three studies that revealed a positive effect of using a concrete-representational-abstract (CRA) approach in the instruction of fractions by Hughes (2011), Reneau (2012), and Watt (2013).

The third review was made by Roesslein and Codding[4] (2019). In contrast with Shin and Bryant's review, they included studies that focused more on elementary-level students (K-6) and did not include studies from seventh to twelfth grade. They included a total of 12 studies published from 2012 to 2017, most of which focused primarily on conceptual learning or both procedural and conceptual knowledge (of fractions). The most common conceptual approach was supporting students in developing fraction magnitude, word problems, equivalence, or number lines (e.g., Fuchs et al., 2013; Fuchs, Schumacher, et al., 2016; Sharp & Shih, 2017), and fewer studies have the shared focus of both conceptual and procedural learning (e.g. Watt & Therrien, 2016). Only one study targeted isolated procedural knowledge (Everett et al., 2014). This can be seen as a support for Star's (2005) previous mentioned point that conceptual knowledge has been more in focus than procedural knowledge.

[4] I read the review after the intervention, so it did not have a great influence on the intervention instruction material, and the finding of the reviews was used as background knowledge for the discussion in Study 5– and in the overall inquiry process of this PhD project.

It is noteworthy that two studies focused on equivalency understanding using *ratio*-based problem solving (Hunt, 2014; Westenskow & Moyer-Packenham, 2016). Overall, the majority of studies (11 out of the 12) utilised concrete and/or visual representations of fractions. They found a large effect size generated across studies when the intervention content consisted of explicit, systematic instruction, visual representation – specially the use of the number line. Medium to large effect sizes were found for the equivalency or magnitude understanding of fractions. In addition, the results indicated promising effects when the intervention consisted of a multicomponent of representation targeting a variety of fraction skills, especially the number line.

These three previously conducted reviews (Misquitta, 2011; Roesslein & Codding, 2019; Shin & Bryant, 2015) were included my review, but I only included studies targeting second to sixth grade. The choice of only including these grades is founded in the fact that this PhD project targeted fourth grade students, so studies relevant for this group could be considered two grades before and after. I made a new search including the word: fraction* and interventions* OR instruction* and included newer studies from the years 2018–2020. Six newer studies were found (Barbieri et al., 2020; Flores et al., 2018, 2020; Fuchs et al., 2020; Hacker et al., 2019; Soni & Okamoto, 2020). These studies were not taken into account when designing the intervention in 2018, but they were used later in confirming or disconfirming my findings (*Scientific Enquiry Phase VI*). I made local searches in Scandinavian Journal, but none of the included studies were conducted in a Danish context; the only included study from the Scandinavian context was in Finland (Kiili et al., 2018).

The summarised results from the fourth review about different fraction interventions show the importance of multiple representations, which particularly include the use of the number line. The use of a concrete-representational-abstract (CRA) sequence[5] in the instruction material can be a way to obtain explicit and systematic instruction. The educational use of the CRA was further supported when looking at the large effect sizes in studies using this approach (Bouck et al., 2017; Butler et al., 2003; Jordan et al., 1999). However, none of these CRA fraction studies from 2017 and before explored the CRA-integrated approach (CRA-I) where the phases of the CRA are integrated into the instruction so that students use manipulatives, representations, and abstract symbolic notation in each lesson and almost every task if possible. Later

[5] The CRA approach has also been implemented in the Singaporean curriculum where it has been a prominent component of the official curriculum (Ministry of Education Singapore, 2012). Here it is named concrete-pictorial-abstract (CPA). The main difference between CRA and CPA is that the representational approach is simplified and called pictorial. The overall research project works with different elements of the Singaporian curriculum, but my project focuses on the use of CRA and creating an inclusive learning environment for both high and low performing students.

studies explored the integrated approach (Flores et al., 2018, 2020; Morano et al., 2020). Other effective fraction intervention contents were strategy instruction (Joseph & Hunter, 2001; Test & Ellis, 2005; Zhang et al., 2016) and explicit or direct instruction[6] (Fuchs et al., 2013, 2014, 2017; Fuchs, Malone, et al., 2016; Fuchs, Schumacher, et al., 2016; Hunt, 2014; Kelly et al., 1990).

There could be a publication bias within these reviews, where studies showing statistically significant effects are more likely to be published than those not showing such effects (Rosenthal, 1979).

The instruction material developed in the current PhD project is founded on the results from these reviews. The material is used in Study 5. However, as previously mentioned, this only includes studies published before June 2018. Later published works are used to reconfirm or discuss my findings in the last phase (*Scientific Enquiry Phase VI*) of the enquiry process. This will further be elaborated in connection to the intervention design in Chapter 6.3.

[6] Direct instruction defined when the children are taught for example fractions as the main subject as opposite to when the subject is for example equations, but fractions are used. The latter is explicit instruction in equations and indirectly in fractions. The instruction material is both characterized by being structured, systematic, and scaffolded and is often characterised as an important part of explicit instructions (e.g., Hughes et al., 2017).

Chapter 5: Fractions in the Danish elementary school

In terms of further exploration into the inquiry phase, *Institution of a Problem* (*Phase II*), this chapters is set in a Danish context to explore the context of where the students' problems with fractions are observed. The aim for this chapter is therefore to describe and analyse the educational context in which the PhD project is settled. I will therefore in this chapter outline how the topic of fractions is settled in the context of the Danish Curriculum. In Denmark students have to have ten years of fundamental schooling – this means that there is ten years of compulsory school from grade 0 to grade 9. It may be confusing, but the reason is that the Danish school system starts with grade 0. This means that the students begin in grade 0 and finish after grade 9, which then makes ten years of schooling. It is not mandatory to go to school. Parents are allowed to home school their children, but the vast majority of children are attending either public schools (about 80%) or private schools (about 20%). All children shall by law be instructed or follow the mandatory parts of the official curriculum. However, the ten years of schooling is mandatory for all Danish children, and the final examines are based on the objectives from the curriculum. Therefore, the instruction material and the measurement tool developed for this project must logically also be founded in the official curriculum. Therefore, I will first introduce how fractions are described and embedded in the official curriculum published by the Danish Ministry of Education (Danish Ministry of Education, 2015, 2019)[7].

In addition to the intended official Danish curriculum, there is the enacted curriculum in the classroom. In Denmark, we have a high degree of local independent on each school, and teachers have a high degree of autonomy (of course many work in teams). For example, they are allowed to choose what materials are used in the classroom. This means that they can freely choose a mathematic book system or combine systems from several private publishers or choose not to use a published book and make instruction materials by themselves. However, because of the schools' financial situation, each school usually has one published book system they use through all classes[8]. Every publisher is also free to publish, and no official has to approve the quality of the content in the books. The consequences of the high degree of local independence and the teachers' autonomy make it questionable to generalise about the enacted curriculum in the Danish mathematics classroom based on the content of

[7] The current Danish Curriculum 'Forenklede Fælles Mål' was developed from 2013 to 2015. The curriculum was resolved in 2014 and implemented in the school year of 2015/16. In 2019, the curriculum was revised, and the primary adjustment was that some objectives were no longer mandatory.

[8] Often there is a web portal connected to the book system.

the Danish mathematic books. However, their content can be seen as a part of the intended curriculum, which of course influences the enacted curriculum.

It is difficult to get an overall view if the enacted curriculum in the classrooms because of the local independence in the Danish school system – Every teacher is free to choose their material or create their own. However, to show the tendency, I make a small content analysis of the published mathematic books' fraction approaches. The aim of this simple content analysis is to illustrate how my developed instruction material in Study 5 differs from the commonly used instruction material. To sum up, the structure of this chapter is first an introduction to how fractions are embedded in the official curriculum. Hereafter, there is a small fraction content analysis of four commonly used mathematic books. The analysis consists of coding the fraction tasks into the different subconstructs (Charalambous & Pitta-Pantazi, 2007; Kieren, 1980, 1988) introduced in the tasks, and next, an analysis of the content of the first two pages introducing fractions in fourth grade.

5.1 Official curriculum

The latest official curriculum was published by the Danish Ministry of Education in 2015 (Danish Ministry of Education, 2015, 2019). It is structured into four main areas: *Mathematical competencies* (Matematiske competenceområder), *Numbers and algebra* (Tal og algebra), *Geometry and measurement* (Geometri og måling), and *Statistics and probability* (Statistik og sandsynlighed). For each of the four areas, there is a different competency objective that targets a different grade level: grades 3, 6, and 9. In each Competency objective, there are different *skills* and *knowledge* objectives (Færdigheds- og vidensområder og -mål), and these are formed in different phases. When I started on this project, every objective in January 2018 was mandatory; however, later in March 2018, the phases described for each *skills* and *knowledge* objective were changed to be guidelines and not mandatory for teachers and school leaders to follow. This means that when this current instruction material was used in the school year 18/19, the content of the two columns, Numbers and Calculation strategies, were not mandatory in the tables below[9]. However, the areas of competency and the Competency objectives were mandatory as well as the Special attention notes (Særlige opmærksomhedspunkter).

[9] It was a brief period between 2015–2018 where the content of the phases were mandatory.

Table 4 The official Danish Curriculum after third grade (Danish Ministry of Education, 2014, 2019)

AFTER THIRD GRADE						
			SKILLS AND KNOWLEDGE OBJECTIVES			
			NUMBERS		**CALCULATION STRATEGIES**	
AREA OF COMPETENCY	**COMPETENCY OBJECTIVE**	**PHASES**	Skills	Know-ledge	Skills	Know-ledge
NUMBERS AND ALGEBRA	**The student can develop methods for calculating with natural numbers**	1	The student can utilise natural numbers for the description of amount and sequence.	The student has knowledge of simple natural numbers.	The student can do simple calculations with natural numbers.	The student has knowledge about strategies for simple calculations with natural numbers
		2	The student can utilise multi-digit natural numbers for the description of amount and sequence.	The student has knowledge about the role of natural numbers in the constructio n of the decimal system.	The student can develop methods for addition and subtraction with natural numbers.	The student has knowledge of strategies for mental arithmetic, estimation, and arithmetic with written notes and digital tools.
		3	**The student can recognise simple decimal numbers and fractions in everyday situations**	**The student has knowledge of simple decimal numbers and fractions.**	The student can develop methods for multiplication and division with natural numbers.	The student has knowledge of strategies for multiplication and division.
Special attention notes after third grade: *The student can use three-digit numbers to describe magnitude and order (area of competency: Numbers and algebra/Numbers). *The student can add and subtract simple natural numbers by using mental calculations or using a calculator (area of competency: Numbers and algebra/Calculation strategies).						

Note. There is some difficulties translating from Danish to English; therefore, in Danish and English, words do not necessarily have the exact same meaning.

5.1.1 After third grade

For a better overview, see Table 4. Under each of the objectives for *skills* and *knowledge*, there are different topics, such as *numbers* and *calculation strategies*. These topics are divided into two columns. The Ministry does not explicitly describe the first column to the left as the *skills* object and the right column as the knowledge objective, but this can be seen as an implicit structure in the present curriculum (Danish Ministry of Education, 2019). In the following chapter, these two columns are defined as the *skills string* and *knowledge string*.

The first-time fraction is described in Phase 3 in Numbers in both the *skills and knowledge string*. Students must be able to recognise decimals and fractions in everyday life in the *skills string*. Here you might see some parallels to the first part in the productive disposition strand in the definition of proficiency (National Research Council, 2001). Here the students need to realise the habitual inclination meaning to see mathematics as useful and worthwhile. As well as realising and recognising that fractions are used in their every day and therefore are useful. Whether the two strings *skills* and *knowledge* can be seen as the duality procedural and conceptual knowledge will be further elaborated in next section. However, it is clear that there is a duality between the two strings where *skills* is connected to 'doing something' (here it is recognising) or 'knowing something' (here it is knowing simple fractions).

There are no description of fraction in the calculating strategies column (light grey in the table). This mean that students after third grade primarily have to recognise fractions and have knowledge about fractions.

The mandatory *Competency objective* (the underlined text in Table 4) after third grade only describes how a student can develop methods for calculating natural numbers, and it does not consist of rational numbers. In addition, rational numbers are not mentioned in any of the special attention notes attaches to third grade (see last row in the table).

5.1.2 After sixth grade

The second time fractions is described is in Phase 1 for grades 4–6 under *skills and knowledge objectives*: numbers. (See Table 5) This is when students should be able to use decimals and fractions in everyday situations, and the students have a knowledge of the fractions' and decimals' structures in the decimal system. The taxonomy in the *skills string* is going from recognizing a fraction (in the previous phase after third grade) to using that fraction in the next phase; that is, from a more passive approach to an active approach. Still, it is connected to everyday situations and has parallels to the productive disposition string in conceptualisation of proficiency made by the National Council in 2001. Again there is a duality between doing and knowing between the two strings *skills* and *knowledge*.

Table 5 The official Danish Curriculum after sixth grade (Danish Ministry of Education, 2014, 2019)

AFTER SIXTH GRADE						
			SKILLS AND KNOWLEDGE OBJECTIVES			
AREA OF COMPETENCY	**COMPETENCY OBJECTIVE**	**PHASES**	**NUMBERS**		**CALCULATION STRATEGIES**	
			Skills	Know-ledge	Skills	Know-ledge
NUMBERS AND ALGEBRA	**The student can utilise rational numbers and variables for descriptions and calculations**	1	**The student can utilise/use decimal numbers and *fractions* in everyday situations.**	**The student has knowledge of *fractions* and decimals structures in the decimal system.**	The student can do calculations with the four operations using natural numbers, including calculations about the everyday economy.	The student has knowledge about the four operations with natural numbers, including using a spreadsheet.
		2	The student can utilise negative whole numbers.	The student has knowledge about negative whole numbers.	**The student can develop methods for operations with decimals, simple *fractions*, and negative whole numbers.**	**The student has knowledge of strategies for calculations with decimals, simple *fractions*, and negative whole numbers.**
		3	The student can utilise percent, simple potencies and pi.	The student has knowledge about the concept of percent, simple potencies, and pi.	The student can utilise calculations with percent, including using digital tools.	The student has knowledge of strategies for calculations with percent.
Special attention marks/goals after sixth grade: *The student can choose an appropriate arithmetic operation/calculation when solving simple everyday problems and set up a simple expression of an arithmetic operation (area of competency: Numbers and algebra/Calculation strategies). * The student can complete calculation processes within all four operations, including estimations and the use of a calculator as well as simple natural numbers by using mental calculations or using a calculator (area of competency: Numbers and algebra/Calculation strategies).						

Note. There is some difficulties translating from Danish to English; therefore, in Danish and English, words do not necessarily have the exact same meaning.

The next time fraction is mentioned is in Phase 2 in Proficiency and knowledge objectives after sixth grade, but now it is no longer under *numbers*; now it is placed under *calculation* strategies (the light grey column to the right in Table 5). In the *skills string*, it is mentioned that the student can develop methods for operations with simple fractions. Whether these operations include all four operations or only two (e.g., addition and subtraction) is not clear. However, it can be seen as a parallel to the procedural fluency string (National Research Council, 2001) or to procedural knowledge (e.g., Star, 2005; Star & Stylianides, 2013).

Under the *knowledge string*, it is stated that the student has knowledge of strategies for calculations with simple fractions. This means that, under the *skills* string, the student must develop a method. But in the knowledge string, they must have the knowledge of strategies. This means that the two strings differ in their approach, and the easy solution would be to use the two different approaches (conceptual and procedural) and connect *skills* to *procedural knowledge* and knowledge to *conceptual knowledge*. However, *conceptual knowledge is not just* knowledge of strategies for calculations; it could also be defined as procedural knowledge; that is, knowing the strategies or procedures (e.g., Star, 2005; Star & Stylianides, 2013). Conceptual understanding refers to rich relationships between different concepts as a connected web of knowledge (Hiebert & LeFevre, 1986). It would be described as focusing on the relation or differences between operation with decimals, simple fractions, and negative whole numbers; for example, realising the conceptual differences between multiplying whole number and fractions. Therefore, it will not be correct to say that knowledge is similar to conceptual knowledge; it could be knowledge to understanding the procedure, and as Star (2005) argued, there is also a deeper understanding connected to procedure knowledge, and conceptual knowledge consists of a quality and can be deep knowledge of why the procedure works. And as previous mentioned, the relation between conceptual and procedural knowledge is an iterative process (Rittle-Johnson et al., 2001). With this in mind, the two strings, *skills* and knowledge, must be seen as working together and be interconnected to develop methods (*skills*) and strategies (*knowledge*).

When looking at the two different terms *methods* and *strategies*, it is important to question whether this is the same or not? In the Danish tradition, *method* has often been defined as ways to do something, and strategies has been defined as ways to think (Pind, 2018). This again indicates that the *skills* column is more connected to ‘doing’ or procedural knowledge, whereas strategies can be seen more connected to knowledge. Whether this is a more conceptual approach is not clear. Therefore, the connection between knowledge or the web of knowledge is not described.

5.1.3 After ninth grade

The fraction is mentioned under the topic ‘Numbers’ in Phase 1 in the learning trajectories for after ninth grade. See Table 6.

Here, it again describes how the student can use fractions, but this time it is not connected to everyday situations. This can be seen as a development from fractions in a context setting and without a context setting – or you can see it as a more concrete setting to a more abstract setting. It is almost the same as described in the last section however percent is added. This in an indication of the order of how the different representations of rational numbers are introduced: first fraction decimals and last percent. There is an ongoing discussion described in the literature review of Tian and Siegler (2018) about when representation should first be introduced. Previously, researchers had argued that decimals are easier to master because of their shared base-10 structures, which cover both whole numbers and decimal notations, and the various fraction forms had led to the stance that decimals are easier to master (e.g., DeWolf et al., 2015a; Hurst & Cordes, 2016).

In the Danish curriculum, there is not a clear order for how the two rational numbers, fractions and decimals, should be introduced, but it is clear that percentages are introduced later. The connection between the three representations is first described in the phase between grades seven to nine.

Under the knowledge string, the student has knowledge of the connection between decimals, fractions and percentages. It is notable that the connections between representation have not been described earlier; it is clear that there is a clear parallel to the conceptual knowledge characterised as a connected web of knowledge (Hiebert & LeFevre, 1986). Under calculation strategies, fractions are not explicitly mentioned, but it is described that the student can do complex calculations with rational numbers –this must be interpreted as though the student can do complex calculations with fractions. Hereafter, neither fractions nor rationale numbers are mentioned, but in Phase 3 under numbers, it describes how a student can utilise real numbers where rational numbers are included.

Table 6 The official Danish Curriculum after ninth grade (Danish Ministry of Education, 2014, 2019)

AFTER NINTH GRADE						
			SKILLS AND KNOWLEDGE OBJECTIVES			
			NUMBERS		CALCULATION STRATEGIES	
AREA OF COMPETENCY	COMPETENCY OBJECTIVE	PHASES	Skills	Know-ledge	Skills	Know-ledge
NUMBERS AND ALGEBRA	**The student can utilise real numbers and algebraic expressions in mathematical inquiries**	1	**The student can utilise/use decimals, *fractions,* and percent.**	**The student has knowledge of the connection between decimals, *fractions,* and percent.**	The student can do complex calculations with **rational numbers.**	The student has knowledge of the order of operations.
		2	The student can utilise potencies and roots.	The student has knowledge of potencies and roots.	The student can create calculations about growth, including the rate of growth.	The student has knowledge of the percent growth for growth calculations in a spreadsheet, including knowledge of rates, loans and savings.
		3	The student can utilise/ **real numbers.**	The student has knowledge of **irrational numbers.**	The student can perform calculation potencies and roots.	The student has knowledge of calculation rules, potencies, and roots.

Special attention notes after ninth grade:
*The student can complete simple percent calculations with the use estimation and calculator (area of competency: Numbers and algebra/Numbers).
*The student can set in numbers instead of variables in a simple formula (area of competency: Numbers and algebra/Formulas and algebraic expression).

Note. There is some difficulties translating from Danish to English; therefore, in Danish and English, words do not necessarily have the exact same meaning.

To summarise, in *Numbers* there is a development in *proficiency* from 'recognizing fractions in everyday situations' to 'utilising fractions in everyday situations' to 'utilising a fraction is not specific in everyday situations'. Whereas when we are looking at the *knowledge* string for *Numbers*, it changes from 'knowledge of simple fractions' and 'knowledge of fractions in the decimal system' to 'knowledge of the connection between fractions and the two other representations of rational numbers: decimals and percentages'. In calculation strategies, the *skills string* is described as a set of developing methods for fraction operations and the connection to strategies described in the knowledge.

5.2 Analysing the content of the instructional materials

The following three textbooks are the most commonly used in the Danish school system in fourth grade: *Matematrix 4* (Gregersen et al., 2006), *Multi 4* (Mogensen et al., 2011), *Kontext+ 4* (Lindhardt et al., 2014)[10]. The textbook series typically consists of one book for each school year. It is important to note that mathematical textbooks in Denmark do not have to be licensed or approved by the Ministry of Education. Therefore, private publishers are free to publish and sell textbooks to the Danish school system. Authors of these books are usually skilled mathematics teachers who have received higher education. Each of the books' chapters cover an individual mathematical topic such as division, areas, equations, decimal numbers.

Through a simple analysis of the fraction problems in the three fourth grade textbooks, I coded each of the tasks in the fraction chapter into one of the five subconstructs: *part-whole*, *quotient*, *measure*, *ratio*, and *operator* (Behr et al., 1983; Charalambous & Pitta-Pantazi, 2007; Kieren, 1976; Tsai & Li, 2017) and specified whether the task was contextualised in an everyday situation or not. I only conducted a simple analysis of each main basic textbook, which did not require extra materials, such as photocopied materials, worksheets or online materials into account. I only analysed the fourth grade text books and not the overall progress throughout the book systems. If it had been possible, I would prefer another researcher to carry out the coding separately to secure the reliability and validity of the analysis, thus qualifying the analysis. In other words, it was not possible for me to calculate the inter-rater reliability between two different codings. The sole individual coding and analysis is why I call it a simple.

I found that between 28% and 52% of the tasks described in the textbooks were contextualised through everyday situations; for example, using pieces of cake or

[10] There is a fourth commonly used system named *Format* (Madsen et al., 2009), but the students' book did not contain a separate chapter about the topic of fractions, and I only found four tasks about fractions in a chapter about numbers; therefore I did not include this system. I did not include *MatLab* (Kaas et al., 2020) either because the system was published after I made the instruction material. *MatLab* was published in the Summer of 2020.

pizza. As previously reported by Ni and Zhou (2005), the *part-whole* representations are the main representation in the early instruction in the US. The same tendency was found in the Danish book systems where the *part-whole* subconstruct dominated all the instruction material, except for that of *Multi 4* (Mogensen et al., 2011) where measurement tasks in the form of number lines were predominant. The use of the number line is aligned by previous fraction intervention reviews (Chapter 4.4; see results from reviews by Misquitta, 2011; Roesslein & Codding, 2019; Shin & Bryant, 2015) where the use of the number line in the intervention showed significant good results. Overall, emphasis on *ratios* did not dominate the books, nor did the idea of a fraction as a *quotient* (see Table 7).

Table 7 Simple content analysis of the three commonly used mathematic books

Book	Con text	No context	Part-Whol e	Measure	Ratio	Quotient	Operator	Equivalence
Matematrix 4 (2006)	.28	.72	.26	.17	.02	.21	.10	.01
Multi 4 (2011)	.33	.67	.22	.28	.07	.09	.31	-
Kontext+ 4 (2014)	.52	.48	.55	.09	-	.11	.18	.05

Note. Matematrix 4 (Gregersen et al., 2006), *Multi 4* (Mogensen et al., 2011), *Kontext+ 4* (Lindhardt et al., 2014). Not every task could be coded as primarily containinig one of the five subconstructs *part-whole, meausure, ration, quotient or operator.*

The percentage of tasks involving equivalence was also low: between 1% and 5% in the books, and one book (*Multi 4*) consisted of no tasks involving equivalence (Mogensen et al., 2011).

Matematrix 4 (Gregersen et al., 2006) has the highest percentage of the *quotient* subconstruct (21%). There were no explicit explanations or examples for the conceptual differences between natural numbers and fractions (rational numbers). The same result was reported by Debou and Verschetze's thesis in 2012, who investigated the three most commonly used textbooks for elementary school mathematics in Flanders. Their analysis showed that textbooks paid no explicit attention to the conceptual differences between natural and rational numbers (refered in Van Hoof, Verschaffel, et al., 2015).

Overall, my content simple analysis[11] shows a great variation in content between each book as *Multi 4* (Mogensen et al., 2011) focuses more on measurement than the others

[11] I define the analysis as simple; therefore another researcher did not participate and confirm the coding to secure the reliability.

(whereas *Matematrix 4* (Gregersen et al., 2006) focuses more on *quotient*). However, in all the books, above 20% of the tasks targeted the subconstruct of *part-whole*, and very few tasks targeted *ratio*.

Each book's layout and approach to the first tasks (in the introduction of the fraction chapter) differ. One thing the books have in common is that the students are not meant to write or draw directly in them; they have to solve the tasks on a separate piece of paper. I analysed the first two pages about fractions from each book.

5.2.1 *Matematrix 4* (2006)

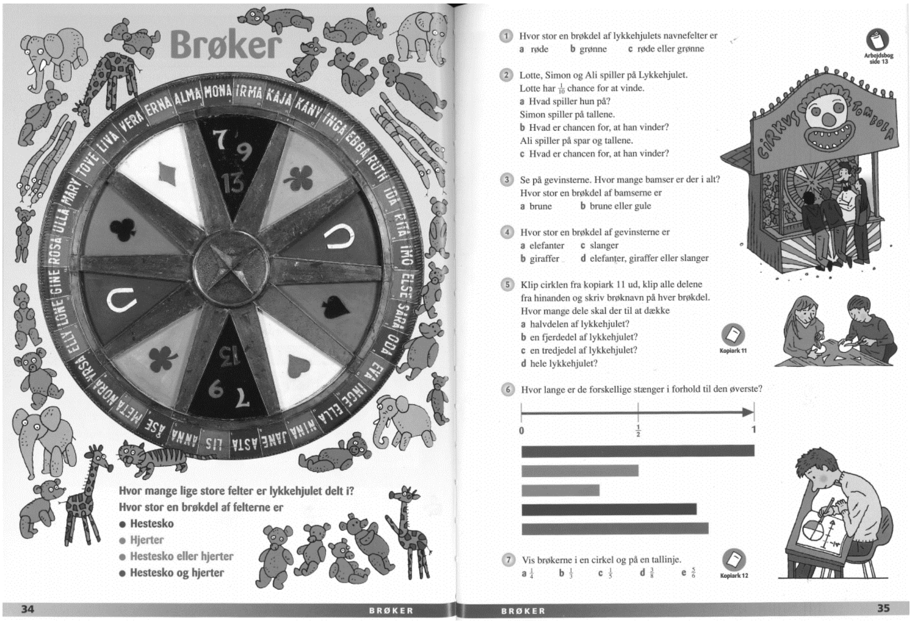

Fig. 12 Introduction pages to fractions in the mathematic book *Matematrix 4* (Gregersen et al., 2006). Reprinted by permission from *Alinea*.

The book introduces fractions with a wheel of fortune and asks how many equal sized parts the wheel has been divided into. See Fig. 12. The task numbers 1 to 5 on page 35 are connected to the wheel on page 34. The word problem on page 35 can be translated as follows:

1. How big a fraction of the name plates on the wheel of fortune is
   a. red b. green c. red or green

p. 35 in Matematrix 4 (Gregersen et al., 2006)

The name plates are placed around the circle on page 34 and can be a little difficult to find. The tasks can be seen as embedded in the *part-whole* subconstruct – not the 'normal pizza slides representation' – where the representation of the fraction must be found in name plates around the circle. Each name represents a part, and the whole is all the names; therefore, the names must be seen as discrete quantities.

Task number two is a word problem about three people playing on the wheel of fortune, perhaps the characters drawn on the right side of the following page (35). The text can be translated as follows:

> 2. Lotte, Simon, and Ali play on the wheel of fortune. Lotte has a $\frac{1}{10}$ chance of winning.
>   a. What did Lotte bet on?
>   Simon bets on the numbers.
>   b. What is his chance of winning?
>   Ali bets on spades and the numbers,
>   c. What is his chance of winning?
>
> p. 35 in *Matematrix 4* (Gregersen et al., 2006)

The first question is connected to finding and recognising $\frac{1}{10}$ on the wheel of fortune. The most obvious way to solve this involves counting whether there are ten slices in the middle and finding a unique one. Again, this is based primarily on the *part-whole* subconstruct. The next word problem is number 3:

> 3. Look at the prizes. How many teddies are there in total?
>   How big is the fraction of teddies that are
>   a. brown   b. brown or yellow
>
> p. 35 in *Matematrix 4* (Gregersen et al., 2006)

To answer this question, the students need to find the prizes on pages 34 and have the background knowledge that teddies are often the prize from playing the wheel of fortune (as pictured in the drawings around the wheel of fortune). The students need to have a great overview of the two pages thanks to their multi-modality nature (i.e., pictures, texts, drawings, and diagrams). The content addresses *part-whole* with discrete data (how big is the part out of the whole). In a), the students are asked to find the whole in order to solve the next task. The part is the brown teddies in a) and both the brown and yellow teddies in b).

In word problem number 4, the students continue working with the prizes in the wheel of fortune:

> 4. Which fraction of the prizes are
>   a. elephants?   c. snakes?
>   b. giraffes?   d. elephants, giraffes, or snakes?
>
> p. 35 in *Matematrix 4* (Gregersen et al., 2006)

The content is again *part-whole* with discrete data therein. This task could have easily been a sub-question of word problem number 3. The last question about the wheel of fortune is word problem 5:

5. Cut out the circles on the copies. Separate all parts from each other and write the name of the fraction on each part. How many parts are necessary to cover… a. half of the wheel of fortune? b. a quarter of the wheel of fortune? c. a third of the wheel of fortune? d. the whole wheel of fortune? p. 35 in *Matematrix 4* (Gregersen et al., 2006)

The content is still primarily based on the subconstruct of *part-whole*. However, the subconstruct *measure* can be present as well; the students have to cover the wheel of fortune by putting its parts next to each other. There are 12 parts cut out of the circle; therefore, this word problem's content also involves realising the equivalence between $\frac{1}{2}$ and $\frac{3}{6}$ and so on ($\frac{3}{12} = \frac{1}{4}$, $\frac{4}{12} = \frac{1}{3}$, and $\frac{12}{12} = 1$). This task is the first enactive task (Bruner, 1966) in this introduction to fractions, seeing as the cut-outs are physical manipulatives for the students. The context of the word problem changes in problem number 6:

6. How long are the different rods in comparison to the one on the top? p. 35 in *Matematrix 4* (Gregersen et al., 2006)

The content is not based on an everyday context, but rather the rod figures of different length. The subconstruct that is primarily present in this task is *measure*; the length of the rods is important when measuring, not the area. To support this, the number line introduced at the beginning of the task indicates that the blue rod is equal to 1. It has been previously shown that the number line is an important tool for or representation in the students' development of fraction knowledge (Barbieri et al., 2020; Dyson et al., 2020; Hamdan & Gunderson, 2017; Soni & Okamoto, 2020).
The final task on the two introduction pages is number 7:

7. Show the fraction in a circle and a number line. a) $\frac{1}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{5}$ d) $\frac{3}{8}$ e) $\frac{5}{6}$ p. 35 in *Matematrix 4* (Gregersen et al., 2006)

The task only consists of the fraction notation system and is not based on any context unless one considers it is based on the mathematical symbol language. The task progresses from unit fractions to non-unit fractions. The subconstruct in this task is predominantly *measure* when the students show the fraction on the number line, whereas it is *part-whole* when the fraction is shown in a circle. In sum, the two introduction pages on fractions in *Matematrix 4* (pp. 34–35) shift from tasks contextualised in everyday life to tasks free of any particular context. Primarily, the subconstruct used is *part-whole* in the beginning, but in the last two tasks include *measure*.

The representations of fractions used on the two pages include the wheel of fortune, the circle, and the number line (introduced in the last two tasks); this aligns with previous studies emphasising the utility of multiple representations (Flores et al., 2018; Westenskow & Moyer-Packenham, 2016). The illustrations (e.g., referring to the prizes) are not necessarily placed beside the tasks and can be difficult to find. Overall, the two pages are an example of the multimodality often present in many mathematics books. The use of the wheel of fortune relates to the topic of probability introduced in a later chapter, so if the teacher is following the progression of the book, this topic will be introduced later in the school year. The aim of the wheel of fortune is to contextualise the task for the students; however, it is debatable how many Danish students have played on a wheel of fortune by the age of 10, and it is rather difficult to see how the wheel works from the illustration.

Task 5 includes the use of concert materials or manipulatives as the students are asked to cut out a circle from the worksheet. The progression of the tasks is not explicit. However, there seems to be shift from enactive tasks to pictural representations and end with abstract symbolic language, as often seen in CRA studies (e.g., Flores et al., 2018; Hughes, 2011; Morano, 2017).

5.2.*2 Multi 4* (2011)

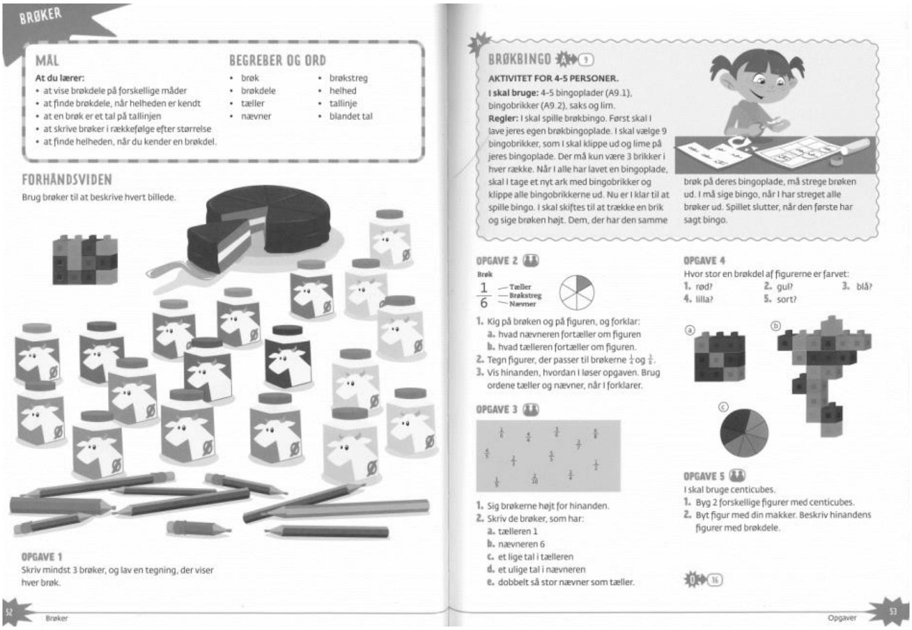

Fig. 13 Introduction pages to fractions in the mathematic book *Multi 4* (Mogensen et al., 2011). Reprinted by permission from *Gyldendal*.

On top of the first fraction introduction page in the *Multi 4* book (Mogensen et al., 2011) is a box page where the aims and goals (Mål) for this chapter are stated in the left column, and the right column includes an overview of the mathematic concepts and terms connected to the chapter (Fig. 13).

Hereafter, the first page (p. 52) appears to be based on background knowledge (Forhåndsviden) stated in the page headline. On this page, the students are asked to use a fraction to describe each picture, which implies there is more than one picture. However, there is only one big picture on the page (p. 52), or it appears to be one picture as the pencils overlap the milk bottles. As a result, I assume that 'each' refers to the different elements on the pages: a) the cake, b) the milk bottles, c) centicubes, and d) pencils. They include both discrete and 'semi-discrete' entities (both the cake and the coloured centicube figure indicate that they consist of parts; however, they can also be seen as continuous entities). The most obvious subconstruct in this presentation is the *part-whole.*

The first problem on this page can be translated into the following task:

> Task 1
>
> 1. Write at least three fractions down and make a drawing that shows each fraction.
>
> p. 52 in *Multi 4* (Mogensen et al., 2011)

This task refers to the picture above. It is not explicit which subconstruct the students would use in their solving process for this task. Most likely it would be *part-whole*. Therefore, the task above is based on this subconstruct, but it is not necessarily the method that students would choose to solve the task.

The process goes from the symbolic notation of a fraction to a corresponding drawing representing this fraction; or one can say that it moves from the abstract to a drawing representation, which is in direct contrast to the normal CRA-sequence approach, where students go from concrete representation to abstract form (Butler, 2003; Flores et al., 2018; Hughes, 2011; Morano, 2017; Morano et al., 2020). There is no given context in this task. On page 53, the first activity is a game (e.g., bingo), where the students are meant to play against each other in small groups. Hereafter, the next word problem is introduced next to a circle diagram showing a green $\frac{1}{6}$ slice:

> Task 2
>
> 1. Look at the fraction and on the figure and explain:
>   a. What does the denominator say about the figure?
>   b. What does the numerator say about the figure?
>
> 2. Draw figures that illustrate the fractions $\frac{1}{4}$ and $\frac{3}{8}$.
> 3. Show each other how you solved the task. Use the words ‘numerator’ and ‘denominator’ when you explain.
>
> p. 53 in *Multi 4* (Mogensen et al., 2011)

There is no context in this task, and it can be seen more as an instruction-text genre rather than a word-problem genre. In this task, the primary subconstruct is the *part-whole* connected to the first part of the task. In the last part of the task, it is not explicit how the students illustrate the fractions when they are following the instruction. Therefore, it is explicit which subconstructs are present in the task. The oral explanation of fractions is emphasised by the content.

As a result, students are asked to discuss their explanations to each other. The process in the first task includes looking at both the fraction notation and the figure and explaining how they correspond to each other. We can see this as an RA-process, where both the abstract notation and the representation are present. It is not an explicit process that moves from representation to abstract form but should instead be recognised as a two-way process.

The next task, Task 3, is a new instruction. These fractions are placed in a grey box above the task.

> Task 3
> 1.Say the fractions out loud to each other.
> 2. Write the fractions down that have
> a) 1 as a numerator
> b) 6 as a denominator
> c) an equal number in the numerator
> d) an unequal number in the denominator
> e) a denominator that is twice as big as the numerator.
> p. 53 in *Multi 4* (Mogensen et al., 2011)

The task includes an instruction where the students need to be familiar with the following mathematical terms: *numerator*, *denominator*, *equal*, and *unequal*. The subconstruct that this task is based on is not explicit; it depends on the students' own interpretation of the symbol that they are writing. The task is based on the abstract symbolic notation system and other representations are not used in this task. The next task on the page is Task 4:

> Task 4
> How large is the coloured fractions of the figures?
> 1. Red? 2. Yellow? 3. Blue? 4. Purple? 5. Black?
> p. 53 in *Multi 4* (Mogensen et al., 2011)

The figures that the text refers to are placed under the text (two centicubes figures and one circle representation). Here, again, the subconstruct is mainly *part-whole*, and the task consists of three different representations: a square, a complex figure, and a circle. The first two are illustrated as they are made of centicubes.

The figure can be seen as a semi-discreet entity, meaning that the entities are not separated but placed close together. There is clear indication of where the parts begin and stop, and they can easily be counted. The final task on the page is Task 5:

> Task 5
> You shall use centicubes to
> 1. build two different figures of centicubes
> 2. switch figures with your peers and describe each other's figures with fractions.
> p. 53 in *Multi 4* (Mogensen et al., 2011)

This is the first time that the students are asked to use concrete materials enactively. The subconstruct connected to this task is *part-whole*.

Overall, the progression in the difficulty level is clear. The aim is to understand the fraction notation and be familiar with the mathematical terms used in connection to fractions – both in oral and written language. The progression in the use of representation on the page is not explicit for the reader. On the first page, the students are asked to use fractions to describe the picture but are first introduced to the fraction notation on the next page.

The use of concrete material is first introduced in the last task on page 53. This is in contrast to the progression in the CRA-approach, which goes from the concrete to the abstract (Jordan et al., 1999; Kim et al., 2015; Morano et al., 2020). However, many different representations are used to illustrate fractions as previous intervention studies have also emphasised (Flores et al., 2018; Westenskow & Moyer-Packenham, 2016).

5.2.3 *Kontext+ 4* (2014)

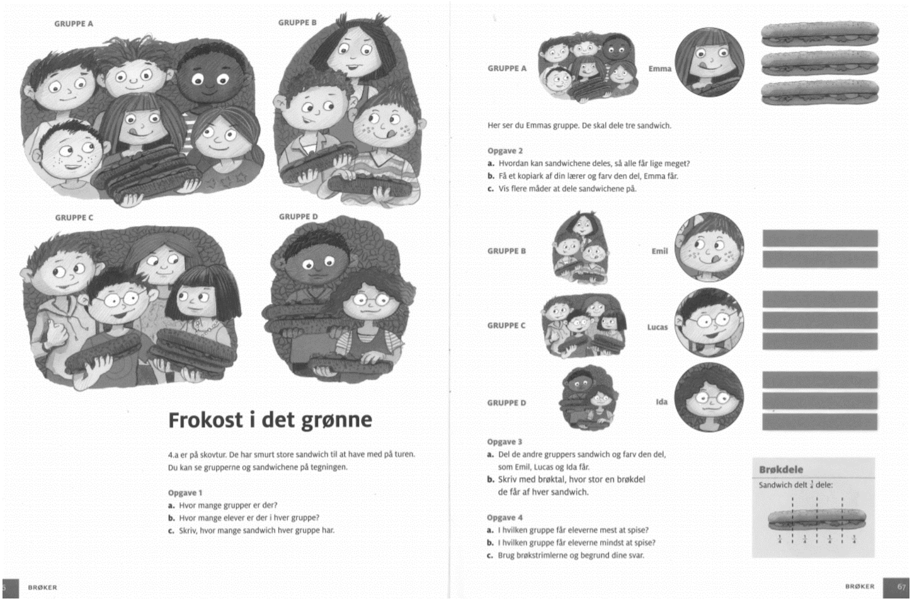

Fig. 14 Introduction pages[12] to fractions in the mathematics book *Kontext+ 4.* Reprinted by permission from *Alinea*.

The next book is *Kontext+ 4* (Lindhardt et al., 2014). See Fig. 14. Here, the headline on the first page about fractions is 'Picnic in the Forest'. The topic of the chapter is in

[12] There are two pages (64–65) before these page (66–57) about fractions which contain classroom tasks about fractions and the aims for this chapter. However, there were no word problems, and therefore I chose the next two pages.

the bottom corner of the page with small green letters: Fractions (brøker). The first picture on page 66 shows students placed in four different groups, and the members of the groups vary. The first word problem is:

> 4.a is on picnic. They have made large sandwiches to bring to the picnic.
> You can see the groups and the sandwiches on the drawing.
> Task 1
> a. How many groups are there?
> b. How many students are there in each group?
> c. How many sandwiches are there in each group?
> p. 66 in *Kontext+ 4* (Lindhardt et al., 2014)

The task is set in the everyday context of a picnic – most Danish students have experienced being on a trip to the forest. The three questions do not require any knowledge of fractions – they can be seen as a way to make the students aware and read the information in the drawing.

On the next page (p. 67), one group is downscaled from the previous page, showing group A, and a portrait of a girl from the group is placed on the right of the group picture. Her name, Emma, is written next to this portrait. On the right of the picture of Emma, there are three equal sized sandwiches placed on top of each other.

Under these pictures, the word problem continues:

> Here you can see Emma's group. They have to share the three sandwiches.
> Task 2
> a. How can the sandwiches be shared so they all get an equal amount of sandwich?
> b. Get a copy from your teacher and colour the part Emma gets.
> c. Show other ways to divide the sandwiches.
>
> p. 67 in *Kontext+ 4* (Lindhardt et al., 2014)

The content of task 2 is about equal sharing and equal parts and can be seen as a foundation to later work with equivalence. The subconstruct is primarily *quotient* based on partition division. The text in point 'b' and 'c' can be characterised as an instruction and not as a question. The solving process can be seen as based in an concrete representation.

Next on the page (p. 67), the other three groups are cut out of the picture shown on the previous page and, as above, one student from each group is taken out and portrayed and named to the right of the group picture. However, now the sandwiches are translated into light grey (brown) rods.

Task 3
a. Divide the other groups' sandwiches and colour the parts Emil, Lucas, and Ida get.
b. Write with a fraction how big a piece they get of each sandwich.
p. 67 in *Kontext+ 4* (Lindhardt et al., 2014)

This task is continuing the word problem above, comprised of the other three groups put into the same problem – therefore, the task is still based on the *quotient* subconstruct approach. Here, it is explicitly stated that students have to interpret the fraction as a number, which leads back to the previous discussion of the interpretation of the fraction notation: Is it a number or a relationship between two numbers? (e.g., Kieren, 1995; Lamon, 2012; Thompson & Saldanha, 2003). In this context, it is emphasised that it must be interpreted as a number. This is the first time the abstract fraction notation is required. Looking further into the progression, the students are instructed to make the abstraction looking at the rods/stripes as representations of sandwiches – this abstraction is supported by colouring them the same colour as the sandwich bread. In the task to colour the stripes on the handed copy sheet, this can be seen as a representational-abstract (RA) approach to the solving process (Butler et al., 2003). This means that there is no enactive concrete materials present – but it is a representational approach in the form of colouring a picture; there are no physical materials that show the fraction or part.

The next and last word problem on the pages is:

Task 4
a. In which group do the students get the most to eat?
b. In which group do the students get the least to eat?
c. Use the stripes/rods to reason for your answer.
p. 67 in *Kontext+ 4* (Lindhardt et al., 2014)

The word problem setting continues in the context of the picnic. It is again a sharing situation where the approach to fractions must be seen as based on the subconstruct *quotient*. The students are asked to use their coloured stripes in their reasoning process – again, this demonstrates the RA-approach as in the previous tasks.
Overall, in contrast to the other two books, this third book, *Kontext+ 4* (Lindhardt et al., 2014) introduces fractions primarily using the subconstruct *quotient* approach, where the students have to share or divide sandwiches. The *quotient* subconstruct was emphasised in Kieren's later work (Kieren, 1988, 1993), and using parting or division as the base for understanding fractions can be seen as a way to support how natural numbers and fractions are connected (see Fig. 8). Therefore, the students' understanding of whole numbers and division naturally creates the need for fractions (rational numbers)s. Natural numbers and rational numbers are connected by the division operation, which is the only operation that creates rational numbers when working with two natural numbers (Hannula, 2003; Middleton et al., 2001). This could also be seen as an attempt to create a web of connected knowledge, previously

defined as conceptual knowledge (Hiebert & LeFevre, 1986; Star & Stylianides, 2013).

It is an implicit assumption that the sandwiches are equal in size; otherwise, the task makes very little sense. Another assumption is that the students share equally, meaning that they share the sandwich fairly among themselves, and either it is the same type of sandwich, or the type of sandwich does not matter. This means that whether the students get egg or ham is not important (in contrast to the everyday experience where the type of sandwich plays an important role in the sharing process).

To summarise, this analysis has looked at three examples of introductions to fractions in fourth grade from three commonly used fourth grade books. The first two books primarily based their introductions on the *whole* subconstruct, whereas the last, *Kontext+ 4*, was based on the *quotient* subconstruct. When looking at the overall content analysis of the different uses of subconstructs, *Kontext+ 4* was not the book system with the highest percentage of use of the *quotient* subconstruct (about 10% of the tasks were containing this subconstruct). *Matematrix 4*, on the other hand, showed the highest percentage of overall use of the *quotient* subconstruct (see Table 7).

In particular, the first *Matematrix 4* (2006) and the last *Kontext+ 4* (2014) try to use the same everyday setting during the first to fifth word problem (wheel of fortune and picnic), whereas *Multi 4* (2011) starts by using the everyday setting in the first setting to activate the students' background knowledge. Later in the book, the majority of the tasks are not set in the everyday context; instead they are mostly based on the aim to understand and recognise the fraction notation. Overall, there is no explicit progression or an overall concrete material, representational-abstract (CRA) approach. There are three very different system layouts. However, they all fulfil the trajectories in the official national curriculum. Recognising and understanding fractions in an everyday setting is central. As previously mentioned, which books are used in each of the Danish mathematics classrooms is a local choice made by each school, normally by the teachers or the mathematics teacher group. Of course, it depends on the school's economic resources – a new book system or books are not affordable every school year.

5.3 Summary

To summarise this dissertation's chapter about the Danish curriculum: There was very little focus on fraction equivalence either in the official or instructed curriculum when I started my project in 2018. *Kontext+ 4* (Lindhardt et al., 2014) contained a little by introducing equal parts. And as outlined in Study 3, I found equivalence important as it supports an understanding by creating flexible knowledge using this understanding in different mathematical contexts, for example, transforming fractions to decimals or percentages as in $\frac{1}{4} = \frac{25}{100} = 25\%$. Therefore, I find equivalence to support a flexible understanding of fractions and create a web of knowledge as I will describe in Study 3 (Chapter 8.3).

The progression in the use of enactive manipulatives was not clear in the above; however, many tasks in the book used manipulatives, such as centicubes or cutting out circles. However, an explicit progression between concrete, picture, and abstract (CRA) representation was not present in any of the books. In particular, the use of the number line representation varied across books).

This finding, in addition to the knowledge generated from the literature reviews, led me to develop different and new instruction material that could support students with mathematics difficulties as well as high-performing students in an inclusive classroom environment. This meant that the aim for content instruction material was to include a clear explicit progression in the use of different representations, aiming for using a Concrete, Representational and Abstract approach (CRA) and using many different representations, including the number line. In addition, the content is a CRA-integrated approach (CRA-I; Strickland & Maccini, 2013) where the phases of the CRA are integrated into the instruction, meaning students use manipulatives, representations, and abstract symbolic notation in each lesson and almost every task if possible. Therefore, I hypothesise that a more integrated use develops a more flexible understanding, where the different uses of the different approaches side by side support the development of a flexible procedural and conceptual knowledge of fractions (Hiebert & LeFevre, 1986; Rittle-Johnson et al., 2015; Star & Stylianides, 2013).

In addition to fulfilling the requirements of the official curriculum's trajectories (Danish Ministry of Education, 2014, 2019), I made sure the introduction of fractions was set in an everyday setting. This instruction material was used in Study 5 and will be further described in the next chapter. It is seen as a part of the *Determination of a Problem-Solution Phase III* in the enquiry process.

Chapter 6: Studying fractions in a Danish school context

As described in Chapter 1, there is one timeline for research project planning and data collection, and one inquiry process where five studies are overlapping, integrating and in some level inform each other in a nonlinear process. Although the research process was not linear, it did provide more in-depth insight into the project. The tracks in the timeline included 1) literature review, 2) measurement/assessment, and 3) intervention. The scope and schedule of the current PhD project is the focus of this chapter.

Next, I specifically chose to focus on the intervention designed used in Study 5. Therefore, the complexity of Study 5 requires a broader and more detailed description of the instruction material (Chapter 6.21) and implementation of the instruction (Chapter 6.2.2) than elaborated in Study 5. Finally, the ethical considerations I confronted during the research for this PhD project are described in Chapter 6.4.

6.1 Project scope and timeline

As mentioned in the introduction, the project consists of five phases:

- Phase 1: First experience as a teacher.
- Phase 2: Initial project start-up.
- Phase 3: First data collection and measurement.
- Phase 4: Intervention instruction and measures.
- Phase 5: Completion.

Each phases from phase 2 through phase 5 was implemented on three tracks. See Fig. 15:

- Track 1) Knowledge Collection: collect and review previous studies and data.
- Track 2) Measurement: analyse collected studies and data; conduct measurement
- Track 3) Intervention: develop instructional materials (see Chapter 6.2.2).

Data were included from two independent collections in phase 3 (first data collection) and phase 4 (the intervention phase). To avoid confusing, the different phases connected to the timeline with the different phases connected to the overall enquiry process based on Dewey's theory, I used Roman numerals when the described phases are connected to the overall enquiry process. I use Arabic numeral when describing phases connected to the projects time line.

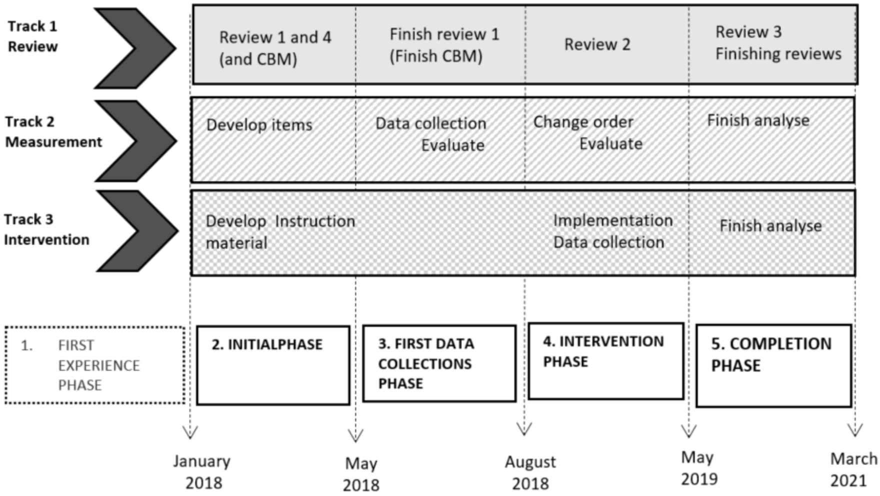

Fig. 15 The three different tracks during each phase

As mentioned in the introduction, the dissertation is based on five studies; all of which generate different answers to the overarching research question: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?*The five studies were conducted during distinct phases of the timeline (see Fig. 16).

The five studies will be further elaborated on in Chapter 8.

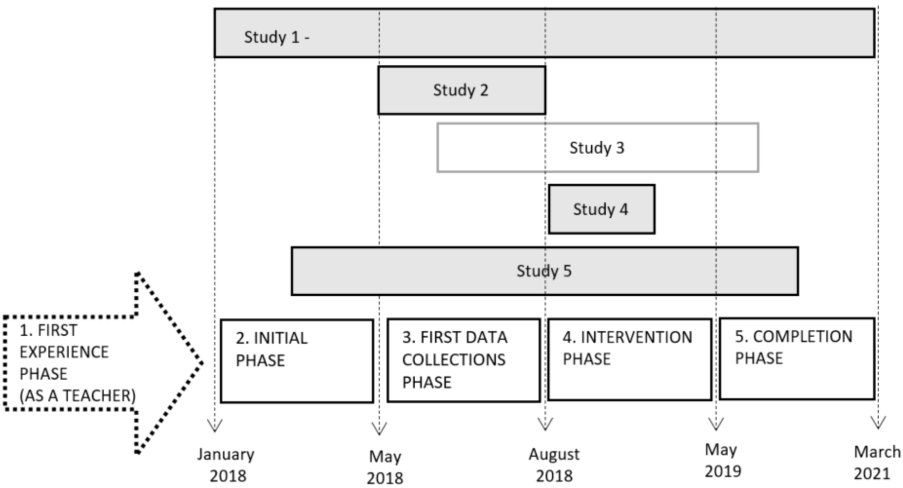

Fig. 16 The five studies connected to the project's phases and timeline

6.2 Elaboration of the intervention phase

The following sections further elaborate on the intervention described in phase 4 in the time line of the project. This phase was an intricate puzzle with many different components that needed to be explained in Study 5. In addition, the participating schools, the instructional materials, and the implementation of the instruction intervention will also be described.

6.2.1 Participants

The data from phase 4 were selected from 11 schools in the same municipality. The participating school characteristics can be seen in Table 8. The largest school had four fourth grade classes (96 students), and the smallest school had one fourth grade class (13 students). In total, 446 students were enrolled. The average age was 10 years and four months (SD = 0.028) at the beginning of the 2018–2019 school year. The participating schools' ethnicity was 92.7% Danish origin, 6.8% non-Western immigrants, and 0.5% Western immigrants. The teachers who participated in the project were 22 Danish schoolteachers and eight local mathematics consultants. The municipal school director decided that all schools in the region should participate. Because all schools in the municipality participated in the project, the teachers were assigned to participate in the research. This confirmed that the teachers participating in the project are considered representatives, and they do not have a tendency to be first movers. In other words, the schools, students, and teachers enrolled in the current PhD project can be seen as representatives for the Danish schools in general.

Table 8 Participating school characteristics retrieved from the Danish Ministry of Education (2020)

School	School size (students)	Type	Stu-dent per class	Well-being	Average absence	Average graduation mark	Teacher comp	Secon dary edc.
A	516	SCS -P	22.9	.89	.07	6.3	.88	.67
B	578	SCS -P	22.9	.89	.07	6.7	.87	.85
C	802	BCS -P	23.5	.93	.04	6.8	.88	.89
D	370	CS-P	18,7	.92	.07	6.2	.88	.86
E	394	CS-P	20.3	.93	.06	6.2	.90	.65
F	303	CS-P	22.0	.93	.06	7.6	.66	.62
G	236	CS-P	21.0	.90	.06	7.4	.80	-
H	330	CS-P	19.5	.90	.07	6.5	.92	.82
I	291	CS-Pr	20.2	-	-	6.4	-	.74
J	150	CS-P	21.7	.95	.07	-	.97	-
K	301	CS-P	20.3	.93	.05	7.1	.85	.47

Note. SCS-P: Small City School – Public; BCS-P: Big City School – Public; CS-P: Country School – Public; CS-Pr: Country School – Privat; Teacher comp: Percent of educated teachers; Secondary edc.: Continuing to secondary education

The 22 teachers had quite different experience. One had just finished her education, and another had 30 years of experience as a teacher. Twenty were educated as mathematics teachers and two had teaching degrees, but not in mathematics. There were five data collection or measuring points during the school year described in Study 5.

6.2.2 Developed instructional materials

The material developed was named *T-MAT Brøker* (*T-MAT Fractions*) where T stands for the T in Track (Teacher Routine and Content Knowledge). Therefore this material was a part of the longitudinal research project conducted by the research group at VIA University College. The content of the instruction materials used in Study 5 was based on Kieren's five subconstructs (see Table 9). The material was designed to take proximately seven weeks in total.

Table 9 Topics in the developed instruction material during the seven weeks

	Topic	**Hvad er en brøk?**	**What is a fraction?**	**Subconstructs**
Four weeks	Module 1	Lige store dele	Equal parts	Part-whole/*quotient* in introduction
	Module 2	Hvad er en brøk?	What is a fraction?	*Part-whole*
	Module 3	At sætte en brøk på en tallinje	To place a fraction on a number line	*Measurement*
	Module 4	At forlænge en brøk	To expand a fraction	*Ratio/part-whole*
	Module 5	At forkorte en brøk	To simplify a fraction	*Ratio/part-whole*
	Topic	**Blandede tal**	**Mixed numbers**	**Part-whole**
Three weeks	Module 1	Hvad betyder blandede tal?	What do mixed numbers means?	*Measurement/part-whole*
	Module 2	Uægte brøker	Improper fraction	*Measurement/part-whole*
	Module 3	Fra blandede tal til uægte brøker	From mixed numbers to improper fractions	*Measurement/part-whole*
	Topic	**Del af en mængde**	**Part of a group**	
	Module 1	Find del af en mængde	Find a part of a group	*Operator/part-whole*
	Module 2	Hvor meget er?	How much is?	*Operator/part-whole*
	Module 3	Hvad er det hele?	What is the whole (group)?	*Operator/part-whole*

The progression of the developed material is based on Bruner's three phases: enactive, symbolic and abstract (Bruner, 1966) previously described as the CRA-sequence and shown to be beneficial in the review of Chapter 4.4 (e.g., Butler et al., 2003; Flores et al., 2018; Morano, 2017), and the approach had also been developed into the Singaporean official curriculum as CPA-sequence (Ministry of Education Singapore, 2012). The layout and design of the Singaporean mathematics books were therefore an inspiration for the layout in my materials. In the instruction materials, the three phases were used as models for students' structural support in their math problem-solving process, meaning that the CRA-sequence was used to make an explicit support (Shin & Bryant, 2015).

All students were given physical manipulatives in the form of fraction bricks to use when solving problems in the enactive phase (e.g., plastic circles, paper, thread or plastic blocks). When solving the problem in the representable phase, the book was illustrated with different drawn representations (e.g., pie charts or block models) to support the students in their problem-solving process when drawing. Often, all three pictures were presented simultaneously on the page. This structure was symbolised by three different icons: 1) an apple in the enactive concrete process, 2) a pencil when it is representational pictorial process, and 3) an equal sign when it is symbolic or abstract (see Fig. 17).

On the first course day, every class received physical materials to ensure that all classes had the same materials available to support the problem-solving process. According to the findings in Misquitta's (2011) review, graduated instruction sequences are essential for fraction learning, and the use of CRA-sequences had a positive effect on the students fraction learning process. Westenskow & Moyer-Packenham (2016) further broaden this so having multiple representations of fractions support students with difficulties in mathematics. As a structure, CRA seems to be a way to secure a structure when using multiple representations. The aim for the developed material was to create an inclusive environment where all subgroups of students could be supported in their problem-solving process, from low- to high-performing students, using multiple representations. This was based a CRA-integrated approach (CRA-I; Strickland & Maccini, 2013). The CRA-phases are integrated into the instruction to use manipulatives, pictures, and abstract symbolic notation in each lesson and as many tasks as possible.

This approach is different from traditional CRA, in which the phases are separated and students must master each phase before transitioning to the next (e.g., Butler et al., 2003). In other words, CRA-I instruction integration of the phases is essential. Therefore, students work with physical manipulatives, pictorial representations, and abstract symbolic notation in the same lesson (e.g., Flores et al., 2018; Strickland & Maccini, 2013). This approach supports the teachers' ability to make explicit connections across representations supporting a web of knowledge and thereby supporting a conceptual understanding. Therefore, I hypothesise that a more inclusive learning environment is being created in which the students are supported by different representations depending on which representation makes sense in their unique solving process. For example, low-performing students can transition to more concrete representations, whereas high-performing students are allowed to use abstract symbols when solving a task. Compared to the traditional CRA-instructional sequence, CRA-I provides additional support as students transition across phases by enabling teachers and students to cross-reference specific, representational, and abstract models during instruction.

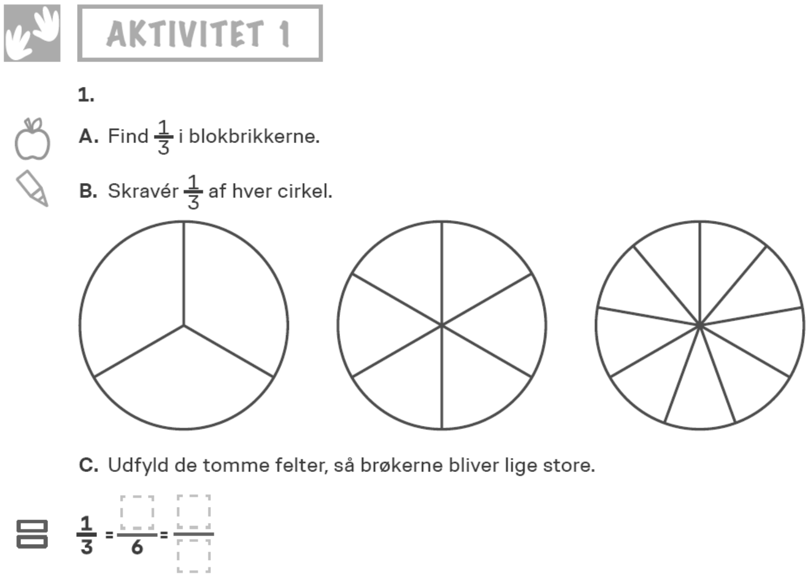

Fig. 17 A task from the instruction material T-MAT fractions where the three icons illustrate the three different representation in the CRA-sequence (Apple, Pencil and Equal Sign)

The tasks at the end of the chapter often used the R-A approach, but we emphasised that any students who still require the physical materials were welcome to continue using them to support them in their working process. CRA-I is especially helpful in supporting the transition from the representational to the abstract phase. This phase is often challenging for students having difficulty with mathematics because they often struggle to conceptualise abstract concepts (Hudson, Miller, & Butler, 2006). Based on the intervention review in Chapter 4.5, it was essential to use several representations in the instruction (e.g., Butler et al., 2003; Flores et al., 2018; Westenskow & Moyer-Packenham, 2016) and that there should be focus on strategy instruction (Joseph & Hunter, 2001; Test & Ellis, 2005). Therefore, pie charts, number lines, and block models were used in the instructional materials. It is noteworthy that two studies which focused on equivalency understanding using *ratio*-based problem solving (Hunt, 2014; Westenskow & Moyer-Packenham, 2016) showed a positive effect.

The progression further developed into how the students and class collaborate during each module of the instructional process. At the beginning of each new topic, there is class enquiry discussion on the new topic facilitated by the teacher. Examples of questions are ‘What does it mean when some parts are equal?’ or ‘How can we describe wholes and parts?’ The students work together first when solving a problem or a task and are eventually asked to work alone. When the students were meant to work alone, I made two levels of tasks; a more advanced level 2 and an easier level 1. The students were free to choose the level they found challenging by supervision from their teacher. The structure includes talks including the entire class, groups of four, pairs, and solo work. This was illustrated in the material with symbols for different numbers of hands. Another difference is that my instruction material was intended to be used only once; therefore, students could write and draw in the books. In particular, drawing was an essential part of the problem-solving process in many tasks.

I created a similar content analysis of the developed fraction instruction material as described in Chapter 5.2. The content analysis of the commonly-used mathematics books in Denmark from the chapter is embedded in Table 10. I included only those books published at the time for the development of my instruction material T-MAT fraction. One significant difference in the instruction material is the focus on equivalence.

Table 10 Simple content analysis of existing and new developed instruction materials

Book	Context	No context	*Part-Whole*	*Measure*	*Ratio*	*Quotient*	*Operator*	Equivalence
Matematrix 4 (2006)	.28	.72	.26	.17	.02	.21	.10	.01
Multi 4 (2011)	.33	.67	.22	.28	.07	.09	.31	-
Kontext+ 4 (2014)	.52	.48	.55	.09	-	.11	.18	.05
T-MAT fraction	**.47**	**.53**	**.34**	**.13**	**.04**	**.07**	**.22**	**.26**

Note. Matematrix 4 (Gregersen et al., 2006), *Multi 4* (Mogensen et al., 2011), *Kontext+ 4* (Lindhardt et al., 2014). Not every task could be coded as primarily containing one of the five subconstructs *part-whole*, *measure*, *ratio*, *quotient*, or *operator*.

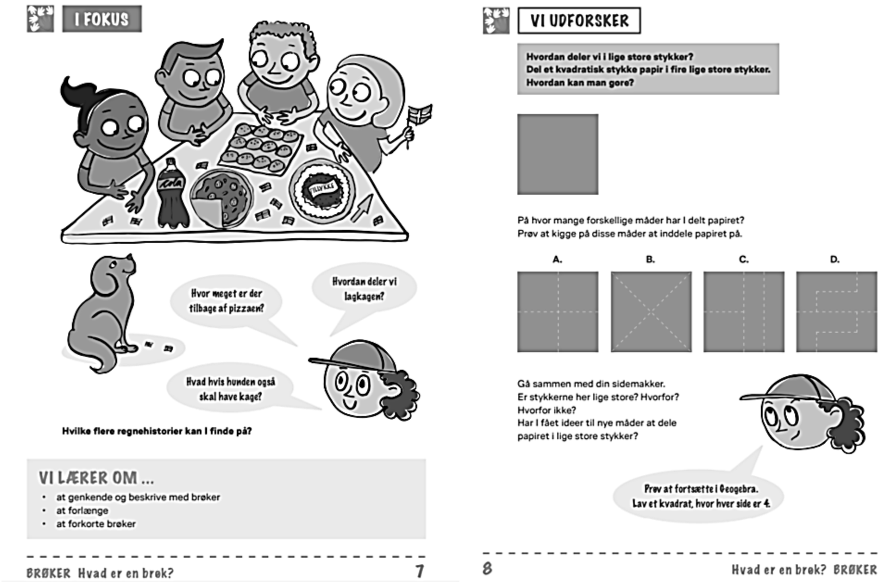

Fig. 18 Introduction pages to fractions in the developed material.

When looking at the introduction pages of my developed instruction material, the first page is set in the context of a birthday party where four children need to share. See Fig. 18. Therefore, the subconstruct *quotient* is present. However, there is also the part from the whole in the pizza; therefore, the subconstruct *part-whole* is also present, so both subconstructs are present. The birthday setting is consistent with the official Danish curriculum, in which first-time fractions are introduced. They must recognise fractions in everyday situations. This is also how all three Danish mathematics books started introducing fractions; they were all presented in an everyday setting. Therefore, this project's proposed instructional materials are consistent with Danish tradition and established the introduction in a context familiar to the student. Under the picture of the birthday party, a girl asks different questions:

> How much pizza is left?
> How do we share the birthday cake?
> What if the dog is also given cake?
> In the blue box below the picture, it tells what 'we are learning' in this chapter:
> Recognising and describing fractions.
> Expanding and reducing fractions.
>
> p. 7 in *T-MAT fraction* (developed material)

On top of the next page (page 8), there is placed a grey box. The following questions are asked:

> How can we divide into equal parts?
> Divide a square piece of paper into four equal parts.
> How can you do this?
>
> p. 8 in *T-MAT fraction* (developed material)

Under the box, different examples are showing how the paper can be divided into equal parts. Between the examples, there is one question and an instruction:

> How many different ways did you find to divide the paper?
> Look at these ways to divide the paper.
>
> p. 8 in *T-MAT fraction* (developed material)

Last on the page are four new questions:

> Talk to your peer.
> Are the parts equal? Why?
> Why not?
> Have you found new ways to divide the paper into equal parts?
>
> p. 8 in *T-MAT fraction* (developed material)

Again, the girl in the right corner makes a request:

> Try to continue in GeoGebra.
> Make a square where each side is 4.
>
> p. 8 in *T-MAT fraction* (developed material)

The instructional material introduced fractions differently compared to the introduction in the three books. The material began with the assumption that when working with fractions, the parts need to be equal, and parts can be equal even though they do not have the same shape; *Kontext+ 4* (Lindhardt et al., 2014) was the only Danish mathematics books with content about this (see Chapter 5.3). Therefore, the starting point must be an understanding of the equal part. For example, every quarter in $\frac{3}{4}$ needs to be the same size.

The primary difference between the developed instructional materials and the other commonly-used mathematics books is that the first task is inactive. The students are asked to divide a piece of paper into equal parts; the abstract notion of the fraction is not introduced until later. In all of the standard books, the fraction notation was introduced in the first two pages. Later, my observations in the classroom showed that this was an important step. Therefore, about eight to ten students in each class experienced difficulty understanding that equal parts do not have the same shape; they can have different shapes and still be equal. This is the first step in developing equivalence in this material, and later in the chapter this is further developed to expand or reduce fractions.

Another difference is that our instructional material's layout differed from the other three books, especially *Matematrix 4* (Gregersen et al., 2006) and *Multi 4* (Mogensen et al., 2011). Therefore, there were fewer tasks on each side. The focus on equivalence differs from the commonly used mathematics book published when the intervention took place.

6.2.3 Implementation and fidelity

In March 2018, before the 2018–2019 school year, all school leaders and fourth-grade teachers were informed by the mathematics coordinator of the municipality about the current PhD project in connection with an informational meeting. In May, the first informational meeting was held with all fourth-grade teachers and consultants. According to Century and Cassata (2016), this first meeting could be called organisational and environmental, both the characteristics of the specific setting (e.g., the classroom at school) and its broader ecology (e.g., the municipality). Another factor that Century and Cassata (2016) discussed is implementation over time. Time will always be a factor in the implementation and diffusion of practices. All these factors will, of course, influence the implementation. However, the time factor was considered using the implementation model in the project, which was the Q-model (QUEST-model) for teachers' professional development.

The Q-model is based on the principles of PLCs and action teaching (Mogensen et al., 2015; Nielsen et al., 2013). The Q-model was developed during a research and development project called QUEST (qualifying the in-service education of science teachers). See Fig. 19. This implementation was possible because of the help and support from the research group connected to TRACK at VIA University College.

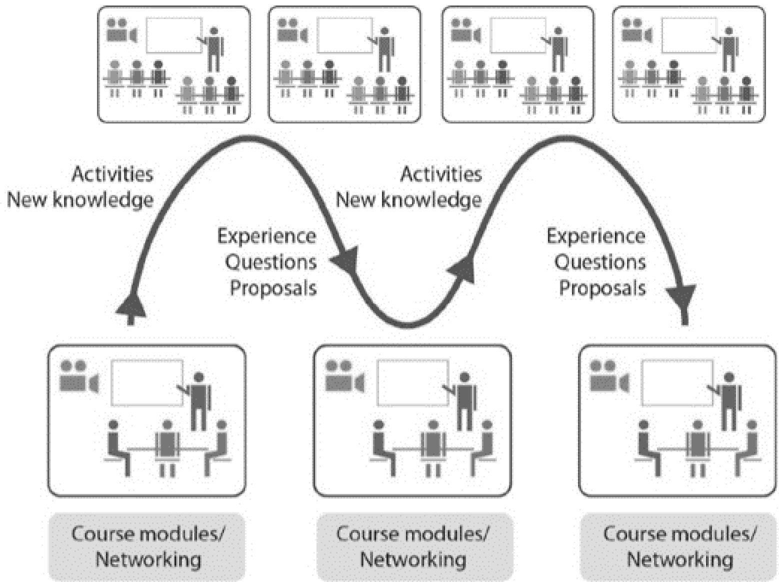

Fig. 19 Teacher courses according to the Q-model

The implementation follows a rhythm of full-day seminars during which participating teachers are introduced to elements of the instruction material by periods of individual engagement in their own practice and collaborative inquiries planned and organised at a local school-based level. Previous studies have shown that the Q-model created good results regarding developing and changing the practice of science teacher learning communities (Mogensen et al., 2015; Nielsen et al., 2013). For example, after two years, 90% of the participating teachers reported that the project had changed and improved their teaching, and 88% reported that the project had led to more collaborations among the science teachers in the participating schools. The Q-rhythm was chosen as the implementation model so participating teachers could develop and adapt the key elements of the fraction instruction material into their own teaching practices in a sustainable way. This implementation was supported by the TRACK group; therefore, fraction instruction was already inserted into the ongoing Q-rhythm of this research project.

Before fraction instruction, the teachers participated in a course module which consisted of three courses within periods where they could develop and try materials and activities. The module activities followed a five-step process, according to the Q-rhythm (Nielsen et al., 2013). The first step consisted of one course day for participating mathematics teachers and mathematics consultants where the theoretical fraction framework and instruction materials were introduced. They were also given concrete student materials, for example, books, plastic circles, and blocks for each student to support the concrete phase in the CRA-sequence (Butler et al., 2003; Flores et al., 2018; Morano, 2017). Research has shown that confidence and training in new routines are especially important if practitioners are to change their practice Wahlgren, 2009). Hence, teachers were given time to understand and practice elements. They had time to plan their implementations and active learning in their own practice. Observation and field notes from the course provided data on expectations with the instructions in different teaching practices for collective reflections. The first school group started the intervention here, and the delayed school group started after the second course. See further description in Study 5.

In step two, teachers taught fraction materials and activities on their own and informed and collaborated with their mathematics team about their active learning process. The teachers documented their practice situations using video to support analyses and collective discussions. This collective discussion could be based on the teacher's observations in the classroom or in the video films made by the students while working with the topic of fractions (e.g., how different classroom discussions can enhance students' fraction proficiency).

Step three comprised a one-day seminar in which teachers from the first group of schools shared their experiences with their own teaching practices with the delayed group. Small videos from students' work with fraction tasks were shared and discussed between the teachers. Time was again allocated for new input and for planning the second action learning steps in their own teaching practice. Data were collected by field notes from the course and the small film teachers had recorded.

Step four was like step two in that teachers would develop and experiment with mathematical activities in their own practice. Now, all teachers taught fractions to both the first group and the delayed group. Unlike step two, however, step four would further require teachers to observe their students' difficulties or progress.

Step five was the final course day. Teachers would present their experiences from the classrooms and evaluate how they experienced the fraction instruction period.

In addition, I followed two classes during the fraction instruction time, made observations and interviewed students during my visit. This was done to obtain a first-hand observation of how the instructional material was used by the teachers and how the students worked with the material. These qualitative observations were important for me as a researcher because they provided unique insights in the classrooms regarding students' reasoning about fractions during their regular school day. However, I did not use this qualitative data explicitly in any studies although they were important implicit observations from daily school life which informed the research process.

6.3 Ethical considerations

Ethical considerations were a central concern during the project, both in connection with the participants and the quality of research. The studies were conducted under the Danish Code of Conduct for Research Integrity (Ministry of Higher Education and Science, 2014). I have striven to be transparent and honest in the description of the analytical approach so it may be replicated, and the data verified. All data are securely stored on the computer and server. All students' written assessments, such as the reading test, were securely stored in a locked room in a locked closet. All school principals gave permission for the research project, and consent letters were sent to parents under the Danish Code of Conduct and General Data Protection Regulation GDPR (Regulation (EU) 2016/679) . In the classes where I observed and interviewed students, all parents gave their active consent for their children to participate (see Appendix F). All regulations and forms were handled and supervised by the legal department of VIA University College. All students' data were anonymised as soon as possible during the data cleaning process. Research ethics involve more than simply following rules and regulations. During the project, I dealt with different issues and problems that required reflection on how to act ethically. In the following section, I provide examples of how I dealt with them.

6.3.1 Teacher level

The director of the municipality agreed to participate in the project. The teachers were assigned to be a part of the research, which ensured that the teachers participating in the project could be considered representatives of teachers, but it also produced dilemmas. The teachers were not necessarily positive about the project, the instructional material, or adjusting their practice. In addition, due to the requirement of assessing their students during the school year, it was important that the curriculum based measurement (CBM) be as short as possible.

The fact remained that the project was forced on the teachers, and they did not volunteer to participate. We addressed this issue at the informational meeting where we emphasised that their participation was really appreciated. We also voiced our concerns and informed the teachers that we hoped they would appreciate the opportunity to be a part of the project once they received more information.

Consideration for the students' daily lives and school and teacher autonomy was central. Consequently, I created a delayed group in Study 5 that could not start simultaneously because of local differences in school year activities. During the project, I also attempted to ensure that the teachers' need for support and freedom was acknowledged and emphasised that they were active participants in the project, not passive pieces in a chess game. This was a collaborative process in which I needed their help. It was important that the teachers did not feel that they were being evaluated through their students’ results. My interest was in the students’ progress, and in a way, I was evaluating my instructional materials.

6.3.2 Student level

An introductory letter was sent to all parents or guardians who had children participating in the project. All students’ parents were asked to give consent for their child to participate in the project. Because of the size of the project, we chose passive consent when collecting large-scale data. In other words, the parents were supposed to write if they did not want to let their child participate. This was discussed in the research group and with the legal department. Danish Data Protection Agency rules, which are based on EU regulations, and Danish Data Protection Regulations do not require active consent according to the legal department. However, I would still have preferred to find ways to obtain active consent without overwhelming the teachers.

Because these data have statistical purposes, they were transformed into anonymous data as soon as possible so measures would no longer be connected to any student by name or UniC. This process had limitations; therefore, the age, school identification (ID), and class ID variables remained to give some information about the students and the school size. All data were stored on a research server with the highest security level, meaning that secured research computers would have access. I was very concerned about the measurement situation and about whether students found the test situation intimidating. On the teacher course I emphasised that the students should by all means not have a bad feeling about the measurement situation and that the teachers needed to talk about it was for helping getting knowledge about how students over all develop their knowledge and not whether how well each student did. It was likewise important whether the students developed or not. The students were not given the measurement result directly after the measurement, which meant that the results did not generate a competition between the students by comparing results. It was emphasised that the teachers could make a judgement whether some students should not participate because they found that the students were not comfortable in the situation. If they excluded students they just needed to make a notification to me. No teacher excluded students. The importance of the short measurement and the time limitation was also emphasised as an important aspect when the teachers evaluated the students’ well-being when taking the measurement, therefore it did not take too much time, and students could overview the task. This made the student feel fine while conducting the measurement.

As mentioned, during the intervention phase 4 connected to the time line in the project, I made observations and conducted interviews with students so I could follow the interventions in the classroom. For this part of the study, I chose to actively collect parents’ consent; the children of parents who did not respond were not filmed or interviewed. The problem with passive consent is that not all parents have the resources or energy to read a message about the study and their child’s role and to give permission. Therefore, some children do not have the same protection as those whose parents have the energy to read this type of information. When consent must be given actively, all children receive the same protection. Previous studies have shown that students who return parental consent are less likely to come from ethnic minority backgrounds and more likely to be female, live with both parents, and have more highly educated parents (e.g., Esbensen et al., 2008). Accordingly, on the one hand, there is a risk of bias.

When I was interviewing students, I always began by emphasising that they were free to refuse to participate, and if they got tired during the interview, they were always welcome to say they wanted to stop the interview at any time. I furthermore emphasised that their faces would not be seen on the recording and explained what it meant to be anonymous with words fourth graders could understand. Power structure (Brinkmann, 2014; Kvale & Brinkmann, 2015) is a consideration when minors are interviewed, and I, an adult, could be found frightening. Therefore, I did not interview students during the first weeks of instruction; I wanted students to feel safe around me and know my face before they sat down with me.

An important part of my research was to distribute the knowledge generated through the project. I was asked by the Mathematics Teachers Association and the Ministry of Education to share information about the project with teachers. As a part of these talks, I wanted to show five minutes of a recording where a student explained how to compare two fractions. The students were anonymised in the film, but I chose from films of both gifted students and students who showed difficulties with fractions. I selected a recording of high-performing students' explanations because I thought that they would not have the same vulnerability as low-performing students who reveal great difficulties. Even though I had already received consent from both the parents and the students, I anticipated that the students might regret this consent later and become self-conscious about revealing their difficulties to others even though they were fully anonymous. Thus, I had to collect their active consent once more because the parents and the students had only given their consent for the recordings to be shown in research, not as part of a broader scale of communication about the project.

My research project should not be a burden on teachers or their students, so the courses about fractions had to be planned so they could be placed at appropriate times in the school year without interfering too much with the teachers' instructional obligations.

Chapter 7: Quantitative data collection

As previously mentioned, the theoretical framework of pragmatism is typically connected to qualitative research studies, not quantitative, because measurements are often a central data collection method in the quantitative research process. Measurements were not often central to the methodology of pragmatism when used in the Danish educational research field. However, Dewey emphasised that the problem determined what method should be used in the inquiry process ([1933]1986, [1938]1986). Using a measurement tool generating quantitative data, I try to explore the overarching problem: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* Using quantitative methods does not conflict with the methodology of pragmatism. The relative lack of quantitative research studies based on pragmatism might be explained by the fact that knowledge is seen as warranted assertibility within this framework, and knowledge generated from statistical analyses is traditionally viewed as unchangeable fact. However, this does not mean that the quantitative method by nature cannot be used in the framework of pragmatism. Whether to use these methods depends on the question we want to investigate. Nonetheless, when using the method, the knowledge derived from the analyses must be seen as warranted assertibility. The quantitative method in the form of a measurement provides new possibilities to link actions and their outcomes. For example, it is interesting examining the outcome from using the fraction instruction material in the classroom. It might result in an improvement of the students' answers and measurement. Whether there is any improvement or not, it is still an outcome. Even though knowledge is warranted assertibility, the measurement tool provides an opportunity to determine a result. In other words, the measurement tool makes it possible for the inquirer to transform the indeterminacy in any given circumstance in each classroom into a determined situation.

When collecting data through measurement, it is central to secure the accuracy of the measurement tool. Even though the first study validated the measurement tool developed for this project, I chose to write a separate chapter about how to secure both the reliability and validity of the measurement tool used. Hence, it is central for the findings in the inquiry process of empirical studies in this PhD project, especially Study 5, where the students' fraction proficiency was followed. Therefore, this chapter will be a brief introduction to the central terms and considerations required when developing a measurement.

7.1 What is a measurement?

This research project is based on a quantitative approach in which curriculum-based tests are used to collect data. For this study, I developed a curriculum based measurement (CBM) for measuring fractions, which I called *CBM-fractions.* The CBMs are characterised as short measurements targeting the curriculum or part of the curriculum (Anselmo et al. 2017; Deno 2003; Fuchs et al. 1999). However, within the test terminology field, there are many different terms, and clarification may be in order. A mathematical test is supposed to measure one's learning outcome – each student's ability within a specific topic, differentiating between what students have already mastered and have yet to master. A test can be defined as a measuring method of a person's ability, knowledge or performance in a particular domain. A test can also be explained as a systematic procedure for describing or observing one or more characteristics of a person using a numerical category system (Nitko, 1983).

A *test* is defined as: 'An evaluative device or procedure in which a systematic sample of a test takers' behaviour in a specified domain is obtained and scored using a standardised process' (American Educational Research Association et al., 2014, p. 224).

An *assessment* in mathematics has been defined as a means of judging the student's mathematical capability, performance, and achievements. An assessment addresses the outcome of mathematics teaching at the student level (Niss, 1993).

These two terms have been used synonymously, but the two definitions reveal that *test* is more connected to test takers, whereas an *assessment* is broader and also connected to programmes that need to be evaluated. In different definitions, measurement properties are constantly being used within various scientific and cultural contexts. The COSMIN Project Initiative began in 2006 to develop a taxonomy of measurement properties and a consensus of definitions for evaluating Patient-reported Outcome Measures (Mokkink et al., 2018; Prinsen, et al., 2018). It is an international initiative that consists of multidisciplinary researchers from areas such as psychometrics, epidemiology, and qualitative research. These researchers developed a taxonomy of measurement properties that is relevant for evaluating a measurement's instrument. This framework creates great insight into what kind of validity and reliability analysis is necessary for evaluating the accuracy of a measurement. However, even though it is a multidisciplinary research, it is founded in a psychological or medical approach for measuring. Therefore, I needed to develop this taxonomy to fit better into the framework of educational research. For obtaining better educational terminology, I included the *Standard for Educational and Psychological Testing Developed* (SEPTD) by the American Educational Research Association (AERA), the American Psychological Association (APA) and the National Council on Measurement in Education (NCME) published in 2014. An example of this integration of the psychological (COSMIN) and educational (SEPTD) approaches to measurement is, for example, the COSMIN model uses the term *patient*, whereas SEPTD, of course, uses the term *student*. Another term, for example, *predicted validity*, is not present in the COSMIN model. However, the term is present in SEPTD, and other terms from COSMIN are not present in SEPTD. The sections below aim to create an overview of what kind of analysis is needed for evaluating the measurement developed for this project.

Overall, there are two key elements in the evaluation process of measurements and assessments: validity and reliability. See Fig. 20.

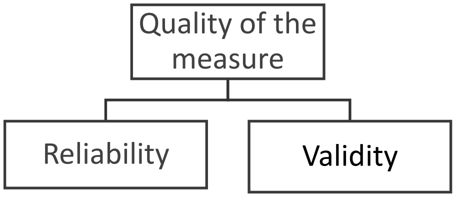

Fig. 20 Two key elements in evaluating the quality of a measurement

7.2 Reliability

Reliability is a key issue in the theoretical framework of measurement studies. It can be defined as how reliable a measurement instrument is in a consistent and predictable way. For the scale to be reliable, the score must represent some true state of the variable being assessed, meaning that the score should not change unless there is an actual change in the variable (DeVellis, 2017). *Reliability* has been defined by two approaches. First, *reliability* is the correlation between two summarised scores on two equivalent tests – presuming that conducting one test does not influence the second time the test is conducted. The second term, associated with reliability, has been used more generally to describe the consistency of scores across replications of procedures (e.g., terms of standard errors; (American Educational Research Association et al., 2014). In the framework of COSMIN, reliability contains three properties: internal consistency, retest reliability, and measurement error (Mokkink et al., 2018; Prinsen et al., 2018). See Fig. 21.

Internal consistency is defined as the internal relations of each item and the total variance in the score(American Educational Research Association et al., 2014). Furthermore, it is important to highlight that it is the degree of interrelation among items on a scale, and it is often reported as Cronbach's alpha. The importance of a unidimensional scale is high when Cronbach's alpha is reported; therefore, the alpha should be calculated for each of these scales. It is closely connected to structural validity because the construction and validation of the scale affects its internal consistency (Mokkink et al., 2018; Prinsen et al., 2018). In this project, Cronbach's alpha was used to estimate the intermediate consistency reported in Study 1 as $\alpha = .90$ ($N = 663$), which is considered good (Cortina, 1993). As explored in Study 1, it could be discussed whether the developed measurement tool contained two subscales: *Meaning* and *Symbol*. If I had chosen this division, Cronbach's alpha needs to be reported for each of the subscales.

Retest reliability is connected to stability as well as the test-retest study and can be defined as the reliability coefficient obtained by conducting the same test a second time with the same practitioners after a time interval and the correlation between the two test scores (American Educational Research Association et al., 2014). Often, the Pearson's correlations (PPC) have been used to estimate this coefficient. However, an infraclass correlation, two-way mixed model may be preferred because the model takes both the variance within the portion and between multiple time points into account (Mokkink et al., 2018; Prinsen et al., 2018). The test-retest reliability was reported to be PCC (147) = .90, $p < 0.0001$. Correspondingly, the individual test-retest value is considered acceptable (American Educational Research Association et al., 2014).

A measurement error, or an error of measurement, can be defined as the disparity between an observed score and the true score. It is also called the *standard error measurement*, *systematic error*, *random error*, or true score(American Educational Research Association et al., 2014). A *measurement error* can be defined as a systematic and random error of a student's test score that is not connected to the true changes in the topic or area to be measured (Mokkink et al., 2018; Prinsen et al., 2018). This process led to the following analysis. The *internal consistency* was estimated using Cronbach's alpha while the *retest reliability* was estimated by the Pearson correlation, which is reported in study one.

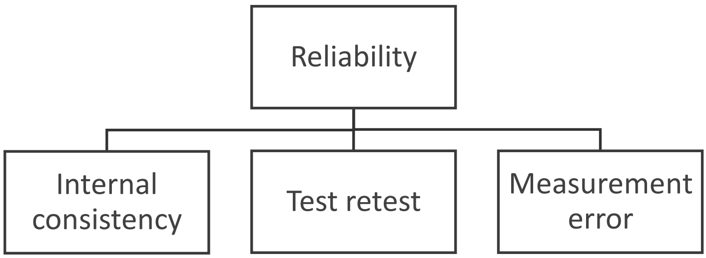

Fig. 21 Three kinds of reliability

7.3 Validity

A test is considered valid if it measures what it claims to measure (Kelley, 1927). Validity is connected to a test's development in that it evaluates whether the test can accumulate evidence to support a specific interpretation of a score. Overall, there are three primary types of validity connected to tests: criterion, constructed and content validity. See Fig. 22.

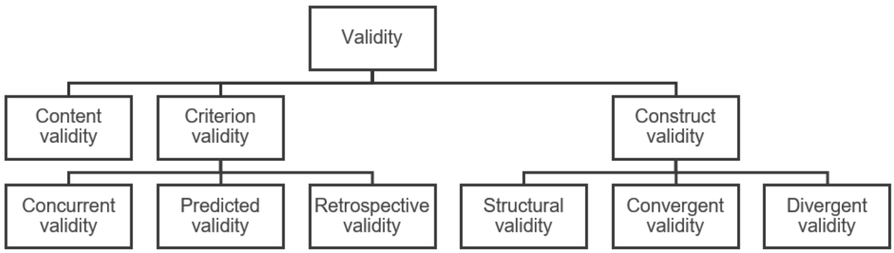

Fig. 22 A diagram of the types of validity

7.3.1 Content validity

Content validity is defined as the extent to which a test measures a proper sample of the topic that we want to study. For example, if the measurement is developed to study fraction proficiency in fourth grade, *content validity* can be defined as how well the item represents the topic fractions at that level. In the context of this study, content validity is based on experienced teachers' evaluations of the test items after a pilot test in their classes as described in study one. In my future work, I will be the teacher and can rank each item with (+, 0, –) besides a more qualitative evaluation. This was done in a study by Al-Shehhi et al. (2019).

7.3.2 Criterion validity

Criterion validity or *criterion-related validity* is defined as the degree to which the scores of a test or measurement are an adequate reflection of a 'gold standard' measurement (Mokkink et al., 2018; Prinsen et al., 2018). This concept leads to consideration of what can be defined as a 'gold standard'. As Christ et al. (2005, 2008) explored in the context of CBM, what is a suitable 'gold standard'? There is currently no other standardised fraction test in a Danish or Scandinavian context. Internationally, Rodrigues et al. (2019) has three fraction measures under development; however, their studies were published late in the process of the current dissertation. This means no tests can be considered perfect for use as a gold standard for the new test. We used the National Tests with Grade 3 validations. In another study connected to the previously mentioned large longitudinal research project Teacher

Routine and Content Knowledge on Teacher Education at VIA University College, the teacher's rating of the student's level in mathematics was collected (rating from 1 to 5) and had been used to validate the overall CBM at the beginning of fourth grade, where one of the subscores was the current CBM-fractions. The analysis from this study showed CBM-fractions, and a teacher rating showed r (179) = .63 ($p < 0.001$).

Criterion validity is often categorised into three types: *concurrent validity*, *predictive validity*, and *retrospective validity*. *Concurrent validity* can be defined as the degree to which two measurement scores are related. The two measurements must be conducted about the same time (American Educational Research Association et al., 2014). *Predicted validity* is how well a measurement score correlates with another measurement conducted at some point after the first (American Educational Research Association et al., 2014). Many CBM-studies have been conducted to make a predicted validation (e.g., Kettler & Albers, 2013; Shapiro & Gebhardt, 2012). The last type of criterion validity is *retrospective*, defined as the extent to which a present measurement can show a correlation to a previously obtained measurement. Because the national test was conducted one year before the measurement tool was developed for this study, we may consider this validation for retrospective validation. However because I consider the validation form primarily to be based on *convergent validity* (will be described below) I do not go further into the *retrospective* criterion validity.

7.3.3 Construct validity

Construct validity can be defined as the degree to which the measurement scores match the hypothesis of what the measurements evaluate; this could be the relationship or distinctions between the measurement score and other measurement scores. Unlike criterion validity, there is no demand for a 'gold standard' for the measurements used for validation (Mokkink et al., 2018; Prinsen et al., 2018). There is no consensus on what sub element this type of validation consists of. In the context of this study, the focus is structural validity and convergent and divergent validity.

Structural validity can be defined as the degree to which measurement scores are sufficient for an evaluation of the dimensions of the constructed scale or subscale. Therefore, this validation is closely connected to the reliability of the internal consistency analyses (Mokkink et al., 2018; Prinsen et al., 2018). Often an item to a response/Rasch or confirmatory factor analysis is a way to evaluate structural validity in the project.

Convergent validity is an evaluation of which levels of measurement scores have a strong relationship with scores from measurements that are conceptually similar to the measurement (American Educational Research Association et al., 2014). This validity is closely connected to criterion validity, but the criterion validity demands of the golden standard are not present in the convergent validity (Mokkink et al., 2018; Prinsen et al., 2018). In this context, it can be discussed whether the National Test should be considered a gold standard for fraction proficiency so it can be considered criterion validity, or if it is more appropriate to compare the validation against the National Test as convergent validation. I would argue that the validity evaluation in Study 1 must be seen as a convergent validation.

Divergent or *discriminant validity* is the opposite of convergent validity: an evaluation of the degree to which a measure diverges from another measure. Therefore, the other measure can be considered conceptually unrelated to the first measure. In this context, it could be related to the reading test score to evaluate whether the test measures the students' reading skills rather than their fraction proficiency.

7.4 Summary

The quality of the measurement consists of several levels and considerations. In the context of this study, the assessment needs to be short and effective and not take up more teaching time than necessary, but it must still have a high degree of accuracy. To secure quality, several analyses were conducted and described in Study 1. Developing a measurement is a long process, and the measurement may still need to be further developed. It is important to emphasise that because it is a computerised test, it provides limited data about the students' problem-solving process. This will be further discussed in Chapter 9.

The methodological choice of using a measurement tool to collect quantitative data must be a central question in the enquiry process. Indeed, which kind of data the developed knowledge is based on will always be central in a scientific enquiry. I could have chosen another method to inquire into my observed and experienced problem: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* This choice would have contributed to other insights, but it is the premises when making a methodical choice, and as outlined in Chapter 2, the choices influences the insights that can be developed.

Chapter 8: Summary of the five studies

In this chapter, each study and its corresponding article is summarised with a focus on the purpose, analysis, and results. The full texts of the five articles are included at the end of the dissertation and sent to the assessment committee. The articles are not included in the published dissertation – therefore two of the articles have not yet been accepted or published. As described in the Methodology chapter, the knowledge generated in each study is considered part of the enquiry process. However, as shown in Chapter 6.4, the studies overlap, and the knowledge of each does not equally inform the others (see Fig. 23). However all studies inform the overarching research question.

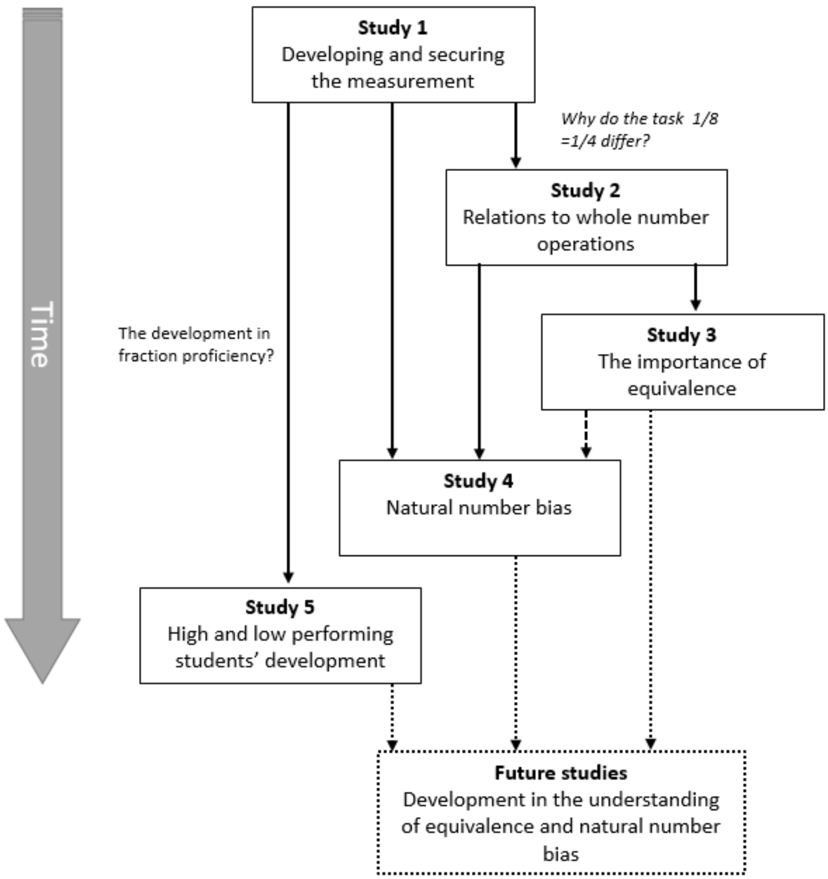

Fig. 23 Information flow between the five studies

A more complex flow diagram of the five studies' interaction is shown in Fig. 23. As previously described in the introduction, Study 1 contains the development and evaluation of a measurement tool targeting fractions in fourth grade. This developed measuring tool can be seen as the basis for the data collection in this PhD project, and therefore the other four studies are connected directly or indirectly to this. Consequently, it is of methodological importance to utilise an accurate measurement tool when collecting quantitative data. In the process of analysing the students' answers, I found a difference in their responses when comparing fractions. When they compared the equal fractions $\frac{1}{4}$ with $\frac{2}{4}$, they showed a different pattern than when they compared $\frac{5}{11}$ with $\frac{3}{5}$. Comparing $\frac{1}{4}$ with $\frac{2}{4}$ was found to be significantly more difficult. This finding motivated me to question and enquire further into this observed difference. This led to Study 2, in which the students' answers to fraction comparison tasks were related to their answers to whole number arithmetic tasks (addition, subtraction, multiplication, and division). My curiosity about why equivalence was particularly difficult led me to explore the conceptions of equivalence further in Study 3, which examines two conceptions of fraction equivalence theoretically.

The findings from Study 2, showing the students' difficulties comparing fractions, led me to explore how I could explain the students' different types of wrong answers and how some of these answers could be explained by the influence of natural numbers in Study 4 (Natural-Number Bias Patterns in Answers to Different Fraction Tasks). As shown in Fig. 23, Study 5, (regarding the differences in high- and low-performing students' fraction proficiency development) was primarily connected to Study 1. Therefore the measurement tool used in Study 5 was developed in Study 1.

8.1 Study 1

Development and evaluation of a curriculum-based measurement targeting fractions in fourth grade

I co-wrote this paper with Professor Rasmus Waagepetersen (Aalborg University).

Manuscript will be submitted to *Assessment for Effective Intervention*.

The first paper described a study aimed at explaining and investigating a measurement tool that would evaluate the curriculum-based measurement that was designed for the project. Consequently, the aim was to investigate the validity and reliability of this newly developed, robust indicator. Investigating how different kinds of reliability and validity are defined or related has previously been described in Chapter 7. The theoretical framework used for developing this measurement was Kieren's five subconstructs, which are described in Chapter 3.3.

Resume of study 1

The measurement tool developed for this PhD project was named CBM-fractions. The measuring tool was founded in the curriculum-based measurement (CBM) approach. CBM was originally developed during the mid-1970s by Stan Deno at the University of Minnesota's Institute for Research on Learning Disabilities (Stecker et al., 2005). The CBMs are characterised as fluency and short measurements (e.g., Anselmo et al. 2017; Deno 2003; Fuchs et al. 1999). The CBM-fractions measurement was computerised and had a time limitation of 10 minutes. It contained 36 test items, all targeting fractions.

The primary research question for this CBM development study is as follows: The study addressed the following overarching question: *How accurate is the CBM-fractions instrument for measuring fourth grade students at the end of the school year?* To elaborate this, we addressed the following research questions in Study 1:

1. *What is the validity, of the CBM-fractions instrument including structural, convergent, and divergent validity?*
2. *What is the reliability of the CBM-fractions instrument including internal consistency and test-retest?*

Methods used in Study 1

First, the structural validity of the CBM-fraction was evaluated using a Generalised subdimension (GSM) model. A one-factor model and a two-factor model were compared using the Akaike information criterion (AIC) and Bayesian information criterion (BIC). We then constructed a one-parameter logistic IRT (Rasch) model to estimate the difficulty level for each item. Next, we estimated the different kinds of reliability evaluations. The internal consistency was evaluated by Cronbach's alpha. A Pearson's correlation coefficient (PCC) was estimated to evaluate the test-retest reliability. Finally, convergent, and divergent validity was calculated using PCC for the CBM-fraction and the different national test scores.

Results from Study 1

The first analysis to evaluate the structural validity GSM-analysis was conducted by examining a one-factor and a two-factor model. The analysis showed that the two-factor model had a marginally better fit (AIC = 15131.67 and BIC = 15459.94) with smaller AIC and BIC compared to the one-factor model (AIC = 15191.66 and BIC = 15515.43). Hence, a high correlation was found between the two latent variables (.85). We argue that the high correlation consequently shows a close relation between the two factors, which supports that the developed measurement tool can be seen as a robust indicator of fraction proficiency. Meaning that it makes little sense to separate the measurement into two subscales (see Table 11).

Table 11 Indicators of the CFA Models

	Obs	11(model)	Df	AIC	BIC	Corr
All students						
One-factor	663	-7519.737	73	15185.47	15513.74	
Two-factor	663	-7489.037	74	15126.07	15458.83	.85

Note. Including common gender effect (part of Table 4, Study 1)

To further evaluate the structural validity, a Rasch scaling analysis was conducted. This showed that the items ranged in difficulty level from -1.86 (item 2) to 4.10 (item 15.2).

Reliability was reported as an internal consistency of Cronbach alpha $\alpha = .90$ (N = 663), which is considered good but is influenced by the number of items (Cortina, 1993). The test-retest reliability was evaluated to be PCC (147) = .90, $p < 0.0001$. As a result, the individual test-retest value is considered acceptable (Koo & Li, 2016). Convergent validity was evaluated by the PCC for the CBM-fraction by the three national tests' subscore, whereas the divergent validity was evaluated by a sentence reading test (Sætningslæseprøve fra Hoegrefe). The estimations showed that all correlations were significant ($p < 0.0001$) (see Table 12).

Table 12 Correlation between the CBM-fractions and validity measures

	1	2	3	4	5	6
1 CBM-fractions	1 (663)					
2 NT total points	.57* (632)	1 (632)				
3 NT: Numbers and algebra	.51* (632)	.82* (632)	1			
4 NT: Geometry	.45* (632)	.83* (632)	.59* (632)	1		
5 NT: Probability and statistics	.50* (632)	.87* (632)	.64* (632)	.61* (632)	1	
6 Reading NB	.31* (575)	.33* (575)	.28* (575)	.25* (575)	.32* (575)	1

Note. *p < 0.0001, NB (_) = N for each correlation. Not all students took the reading tests, and one school was not required to take national tests

Discussion of Study 1

The evaluation of a measurement is important for the project investigation of the overarching problem of *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* Specifically, 'How can we investigate?' is what Study 1 attempts to explore.

Even though the measurements were valid, it does not mean that they could not be developed further (e.g., the test-retest correlation might be improved if the test time was longer, meaning that the time restriction could be set to 15 minutes rather than 10).

When a test is computerised, it always brings limited information about the students' solving process. Therefore, the information contained in each answer must be narrowed down as right or wrong. The order of each item should also be reconsidered to ensure the right progression in the measurement. However, overall, the results confirm that the CBM developed for this project provides a valid test score of students' fraction proficiency.

8.2 STUDY 2

Students' ability to compare fractions related to proficiency in the four operations

I co-wrote this paper with Senior Researcher PhD Peter Sunde (Aarhus University).

Published in *CERME-Eleventh Congress of the European Society for Research in Mathematics Education (2019)*.

As previously mentioned, the second paper was founded on my curiosity to explore and describes a study where I investigate why answers in fraction comparison tasks differ, and why fraction equivalence tasks in particular show a different pattern. The difference was found when looking into the students' answers in the process of developing the measurement tool in Study 1. This data was collected during the first data collection phase in the project (phase 3), where I was in the process of developing the measurement tool and pilot testing the design. Hence, the measurement was a pilot where the items were evaluated by students and teachers. This first test contained more items (110 items), including problems on the four operations, fractions, and algebra as well as word problems. The test was time restricted to 45 minutes.

Resume of Study 2

The second paper reports a study that investigated the relationship between fourth-grade students' ability to solve three fraction comparison tasks $\frac{1}{4} > \frac{1}{5}$, $\frac{1}{4} = \frac{2}{8}$, and $\frac{5}{11} < \frac{3}{5}$ and their ability to solve a whole number arithmetic task for each of the four operations: 68 + 753, 547 – 64, 12 × 74 and 78 ÷ 3.

The overarching research question is as follows: *How do students' abilities to solve arithmetic tasks in the four operations (e.g., division) relate to their abilities to answer items that require them to compare fractions?*

Our hypothesis was that the students' proficiencies in division and multiplication would show a stronger association with their abilities to compare fractions than between their proficiency in addition or subtraction and their ability to compare fractions. For these reasons, we hypothesise that the concept of fraction is more closely connected to multiplicative reasoning.

Methods used in Study 2

Participants were fourth-grade students (ages 10–11), and the test was conducted at the end of the school year 17/18. The test was computerised and had a time limitation of 45 minutes; therefore, only the students who reached the fraction items (items 50 and 52) were included in the study (N = 99). First, we compared the students' responses to the three fraction comparison tasks to each other and examined the different patterns in the correct, wrong, and missing answers.

Hereafter, the analysis focused on how the different answers to one fraction comparison were associated with the other fraction tasks and with the answers to the four arithmetic operation items. The analysis was conducted as a binary logistic regression function. The p-values for the significance of associations were estimated using 2 x 2 contingency X^2-test logistic regression models.

Results from Study 2

The analysis showed that the students had greater difficulties solving the equivalent fraction item $\frac{1}{4} = \frac{2}{8}$. Only about 30% of the students answered $\frac{1}{4} = \frac{2}{8}$ correctly, whereas 49% answered $\frac{5}{11} < \frac{3}{5}$ correctly. The two non-equal fraction items showed 'different' answer patterns; therefore, both items had about 50% of the students answering correctly, while approximately 30% answered incorrectly, and 20% chose not to answer (Fig. 24).

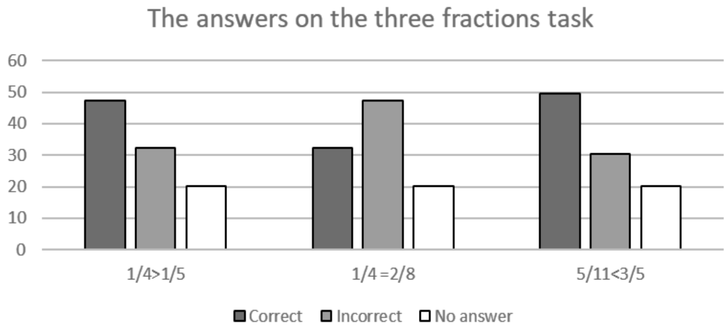

Fig. 24 Percentage of correct, incorrect or no answer to the three fraction items (N = 99). (Fig. 2, Study 2)

The results of the non-equal fraction item $\frac{5}{11} < \frac{3}{5}$ were highly significantly positively associated with the results of both the multiplication items (OR = 4.5, p = 0.0009) and division items (OR = 3.9, p = 0.003). In contrast, the results of equal fraction items $\frac{1}{4} = \frac{2}{8}$ were only associated positively with the division item (OR = 2.6, p = 0.02) (Table 13). The lowest associations were found between the results of two fraction items, the results of addition and subtraction items (all four ORs: 1.7–2.4: non-significant) and between equal fraction items $\frac{1}{4} = \frac{2}{8}$ and multiplication items (OR = 2.1: non-significant).

Table 13 Coefficients of association (log-odds ratios, see text). (Table 1, Study 2)

	$\frac{1}{4}=\frac{2}{8}$	$\frac{5}{11}<\frac{3}{5}$	68 + 775	547 – 64	12 ×74	78 ÷ 3
$\frac{5}{11}<\frac{3}{5}$	2.40 ****					
68+753	0.85	0.53				
547-64	0.75	0.88	1.91 ****			
12×74	0.72	1.51 ***	0.83	1.60 ***		
78 ÷ 3	0.96 *	1.36 **	1.02 *	0.68	1.68 ****	
N	*99*	*99*	*142*	*142*	*142*	*142*

Discussion of Study 2

Overall, the statistical analysis showed that the equal fraction task differs from non-equal fraction tasks. Only about 30% of the students answered the equal fraction item correctly, whereas approximately 50% answered the two non-equal fraction items correctly. This result was unexpected because the equal fraction tasks consist of the commonly known unit fraction $\frac{1}{4}$, which frequently occurs in instructional material of fractions in fourth grade, whereas the fraction $\frac{5}{11}$ in the non-equal fraction task is a rare fraction notation in the school curriculum overall. The same pattern was found in later independent data collections in the intervention phase in the time line of the project (phase 4), confirming this pattern. When looking at the pattern, it needs to be questioned whether the reason could be the design of the item. This means that it tested the students' knowledge about the symbols >, <, and =. However, the same item design existed earlier in the test, but in these the student had to compare decimal numbers, and the answers to these tasks did not show that the item including the equal sign was more difficult than other items.

When examining the differences in the two fraction items associated with the four operation items, the result of the non-equal $\frac{5}{11}<\frac{3}{5}$ fraction was highly associated and significant to the results both for multiplication and division. However, there was no significant association with the results of either the addition or subtraction items. This pattern result agrees with our hypothesis that fractions have a stronger relationship to multiplicative reasoning than additive reasoning.

The same pattern was not found in an association between the non-equal fraction item and the four arithmetic items. There was only a modest association between the fraction task and division. Hence, the equal fraction item differs from the for non-equal fraction item. Overall, these results support the notion that an understanding of fractions is closely connected to multiplicative reasoning; however, it is essential to pursue further investigation since there seems to be a different pattern in the students' concept of fraction comparison when the comparison is based on equivalence compared to non-equivalence. This difference could be based on different kinds of natural number bias.

8.3 Study 3

Two conceptions of fraction equivalence

I co-wrote this paper with PhD Mette Bjerre (VIA University College).

Published in *Educational Studies in Mathematics*.

This third study reported in Paper 3 was founded on my curiosity to explore fraction equivalence. This curiosity was based on my analysis of students' answers in Study 2 to the fraction comparison tasks $\frac{1}{4} = \frac{2}{8}$. As reported in Study 2, almost 50% of students could not answer correctly on this test item by the end of fourth grade. This made me wonder whether and why fraction equivalence was an important concept, and what kinds of conceptions are connected to fraction equivalence. This study is a theoretical study in which equivalence is investigated through a semantic framework of fractions.

Resume of study 3

The fourth paper's study consisted of a mathematical analysis that distinguished two different approaches to equivalence: *proportional equivalence* and *unit equivalence*. These two approaches have distinctly different approaches to concepts and meanings when developing an understanding of fraction equivalence. The first *unit equivalence* is based on unit understanding, while the other *proportional equivalence* is grounded in proportionality understanding. The implicit research question is as follows: *How can we define fraction equivalence, and why does fraction equivalence matter?*

Method used in Study 3

First, we defined the mathematical definition of equivalence and distinguished the two different conceptions of fraction equivalence: *unit equivalence* and *proportional equivalence*. Our theoretical analysis is based on the different fraction subconstructs defined by Kieren (1976). We use his framework to analyse how the two conceptions are present in each of the five subconstructs. Hereafter, we analyse the equivalence influences on fraction arithmetic reasoning, in particular, focusing on fraction addition and subtraction.

Results and discussion of Study 3

When looking at the two fractions $\frac{2}{3}$ and $\frac{4}{6}$ they obviously represent the same magnitude. A way to understand that these two fractions are equal and thereby that their equivalence would be to draw two circles, partitioning one circle into three equal parts and painting two parts, then partitioning the second circle into six equal parts and painting four. In this way, it can be seen that the two fractions are in fact equal because an equal area of the two circles is painted. However, that will only be true if the two circles have the same size to begin with, meaning that the unit is the same (Fig. 25). We define this as unit equivalence. However, fractions can also be interpreted differently as the ratio between the numerator and denominator, so we can also interpret the equivalence of fractions when they describe the same proportionality. This type of equivalence we call proportional equivalence. An example could be: ‘A boy has eaten two pieces of his small kid-sized pizza, which was in six equal pieces. His father eats one piece of a family-sized pizza, which was in three equal pieces. The boy and his father have eaten the same fraction of each of their pizzas. It is central to emphasise that *unit equivalence* automatically contains *proportional equivalence*, but *proportional equivalence* does not contain *unit equivalence*.

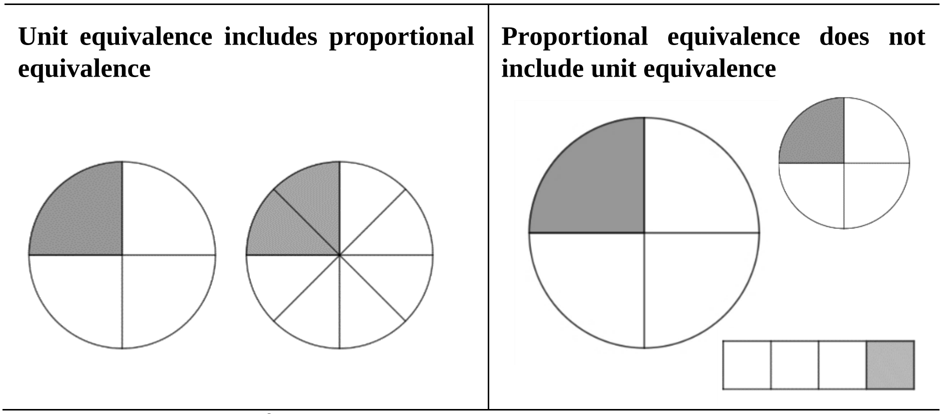

Fig. 25 Two conceptions of equivalence (Fig. 3, study 3)

In the first part of the analysis, we found that the *part-whole*, *quotient*, *measure*, and *operator* subconstructs contained both proportional and unit equivalence. In contrast, we found that the *ratio* subconstruct, which is based on a proportional relation between the numerator and denominator, contained only proportional equivalence. This means that the two equivalence conceptions can develop a parallel interpretation of equivalent fractions.

The analysis revealed new mathematical conceptions and perspectives on equivalence. Furthermore, the analysis detected different areas where the knowledge of fraction equivalence was necessary for developing an understanding of fraction arithmetic within addition and subtraction. With respect to the alterable perspectives of the multifaceted concept of a fraction, our analysis revealed mathematical concepts and potential perspectives on equivalence that had not previously been combined into one framework.

The importance of equivalence was further elaborated in the analysis of different fraction arithmetic reasoning. For example, a whole number subtracted by a fraction first partitioning of the wholes is needed when subtracting a fraction from a natural number. This could be the task $2 - \frac{1}{4}$, where one of the two wholes must be partitioned before performing the subtraction. This operation requires an understanding of equivalence where 2 can be seen as either equal to $1\frac{4}{4}$ or $\frac{8}{4}$ (Fig. 26).

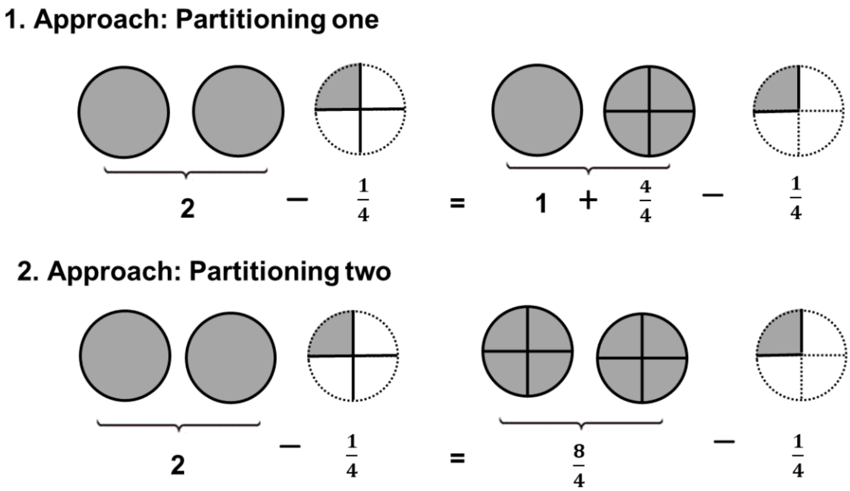

Fig. 26 Different partitioning approaches (Fig. 17, Study 3)

The different perspectives on equivalence could provide new insights into known difficulties with fractions for students. Students' proficiency in both equivalences might lead them to be better prepared to learn algebra, percentages, and linear proportionality; for example, it should be understood that the *ratio* or the proportional relationship stays the same when $\frac{2(3a+5b)}{4} = \frac{3a+5b}{2}$ is reduced.

8.4 Study 4

Natural number bias pattern in answers to different fraction tasks

I co-wrote this paper with Professor Rasmus Waagepetersen (Aalborg University).

Published in *Bringing Nordic mathematics education into the future. Papers from NORMA 20. Preceedings of the Ninth Nordic Conference on Mathematics Education* (In press).

As previously described, this fourth study further explores the different answer patterns in the fraction measurement tool. The data for this study was collected in the intervention phase (Phase 4) where the design of the fraction measurement tool was further developed. This aim of this study was to investigate the different answer patterns. I began in Study 2, where I investigated the different answer patterns between whole number arithmetic and fraction comparison tasks. I wanted to explore how different incorrect answers could be explained by the natural number bias aspects. Review 2 about natural number bias was made primarily while this study was conducted

Resume of study 4

In this study, the role of natural number bias was further investigated. The data from the first data collection came from the developed test administered at the beginning of fourth grade. Here, we focused on two students' wrong answers and coded whether they were based on one of the four aspects of natural number bias: *representation*, *density*, *operation*s, or *size*. The hypothesis was that the four different natural number bias aspects could be closely related; thus, students with a tendency to transfer their concept of natural numbers in one context are more likely to make the same transfer into others. The aim was to investigate if and how these four aspects are related.

The primary research question for this paper is as follows: *How are students' different natural number biases related to each other, and is there a pattern that indicates an overall tendency towards natural number bias?*

Methods used in Study 4

The data used in this study consist of answers from 484 fourth-grade students from the beginning of the 2018/19 school year. In total, 235 girls and 249 boys took part in the study. The test consisted of 36 items and was time-restricted (10 minutes), which meant that not all students finished the test. Therefore, we only included the 484 students who finished all items from 1 to 22 in the test. We selected 14 items where the students could provide answers that were influenced by natural number bias aspects: R (*representation*), S (*size*), D (*density*), and O (*operation*s).

The analysis consisted of three steps: First, the different natural number bias aspects were coded from the students' answers. Next, a descriptive statistics analysis of each of the 14 items was conducted to obtain an overview. The aggregated variables were obtained by counting the number of natural number bias aspect errors. We accounted for non-normality in the distribution of the data by studying the relations between the four variables, using Spearman's rank-order correlation coefficient (rs). To correct for multiple testing, a Bonferroni correction was used to estimate the significance level.

Results from Study 4

In general, most of the wrong answers can be explained by a natural number bias aspect. The highest proportions of correct answers were found in the items connected to *representation* (0.60–0.70), while the lowest proportions of correct answers were found in the items connected to *operations* (0.01–0.00). Table 14 The correlation matrix between the four aspects of natural number bias is shown in Table 14. The proportion of natural number bias mistakes associated to the aspect of *representation* ranged from 0.11 to 0.30, whereas the proportion of natural number bias mistakes associated with the aspect of *size* ranged from 0.47 to 0.54. No strong correlation was found between any of the four aspects, and none of them were statistically significant (see Table 14).

Table 14 The correlation matrix between the four aspects of natural number bias

Aspect of natural number bias	1	2	3	4
1. Representation	1			
2. Size	-.001 (ns)	1		
3. Density	0.117 (ns)	0.035 (ns)	1	
4. Operation	0.051 (ns)	0.050 (ns)	-0.065 (ns)	1

Note. Spearman's rank-order correlation coefficient (r_s), significance levels: $p < 0.05$ (overall) and $p < 0.0083$ (Bonferroni corrected) for individual correlations, (ns): not significant (Table 3, Study 4)

Discussion of Study 4

The analysis followed three perspectives:

First, natural number bias could explain why a majority of the students provided the wrong answers in the fraction tasks. The lowest proportion of wrong answers influenced by a natural number bias aspect was in the four items linked to representation while items connected to operations demonstrated the highest proportion of answers influenced by natural number bias. This finding might be explained by the fact that the tests were conducted at the beginning of fourth grade when students had little experience with fraction addition and based their solving processes on their knowledge of natural numbers (Kainulainen et al., 2017; Ni & Zhou, 2005; Van Hoof, Vandewalle, et al., 2015). This strategy could be viewed as a mediating phase in the integrated conceptual change framework, which I developed and argued for in Chapter 4.4.3, where I combined the two theoretical frameworks integrated theory (e.g., Tian & Siegler, 2017) and conceptual change theory (e.g., Van Dooren et al., 2015). There is movement through their expansion of their number knowledge.

Second, Spearman's correlations (Table 14) indicated that the four aspects of natural number bias are not significantly related to each other. This finding contradicted the hypothesis that the students who have a tendency to apply their knowledge of natural numbers to fractions will show the same tendency across different aspects.

Third, the correlations between the four aspects have led to a discussion on whether we should define four natural number bias aspects as *types* instead of *aspects*. The term 'aspect' indicates a close connection between aspects of the same natural number bias, which does not seem to be the case here. Hence, it would be better to define them as types of natural number biases.

8.5 Study 5

Differences in high- and low-performing students' fraction proficiency development

I co-wrote this paper with Professor Rasmus Waagepetersen (Aalborg University), PhD Pernille Sunde (VIA University College), PhD Mette Bjerre (VIA University College), and Professor Pirjo Aunio (University of Helsinki).

The fifth study reported in Paper 5 investigated the differences in the development of fraction proficiency for high- and low-performing students during their fourth-grade school year. The study's data collection was finished during the intervention phase in the timeline of the project (phase 4). The developed fraction instruction material used in the intervention is elaborated in Chapter 6.6.2. The study aimed to follow the students' development of fraction proficiency over time and see how the development differed (or not) between the high-and low-performing students. Study 5 was a major focus when I started on my PhD project, and it took a lot of planning, information, design, and development. The study is primarily connected to Study 1, and the measurement tool developed in this study was the foundation for the data collection.

Resume of study 5

The data consisted of 21 fourth-grade classes (N = 398) from which two groups were selected: the first group included the highest performing 25% of students (n = 99), and the second group contained the lowest performing 25% students (n = 100) according to national test scores. The fourth-grade students' fraction proficiency was studied for eight months at five distinct measurement time points. The research design allowed the observation during this time span, and all classes followed the same instruction structure for these topics throughout the school year. At the beginning of the school year, multiplication and division were introduced, and then an instruction period for fractions followed. Finally, equation instructions were introduced. This action made it possible to investigate how other mathematical topics (e.g., multiplication, division, and equations) influenced fraction proficiency. This study addressed the following overarching research question: *How do high- and low-performing students differ in their development of fraction proficiency during fourth grade, and how do the different groups benefit from different forms of instruction?*

During this period, the students were instructed in fractions for seven weeks. This instruction used different representations with the aim of creating an inclusive classroom environment.

Methods used in Study 5

Due to local curriculum activities, the fraction instruction periods were delayed by four weeks at six schools. The delayed data set was therefore considered a control or confirmation of the identified pattern in the first data set.

The primary parameters of interest were the changes in the mean test scores for each test for the low- and high-performing students and the differences in these changes. The analysis was founded on the test scores using a multiple linear regression model based on the following parameters: α, β_{1ow}, β_{high}, λ_{12}, λ_{23}, λ_{34}, λ_{45} and δ_{12}, δ_{23}, δ_{34}, δ_{45}. In addition, we added gender as a variable with the associated parameter α to assess the expected differences in scores between the two genders.

Results from Study 5

The analysis showed that the high-performing group developed their fraction proficiency beyond the period where they received instruction in fractions (see Table 15).

Table 15 The results of a mixed-model analysis for the first and delayed data sets (Table 2, Study 5)

	First data				Delayed data			
	Coef.	Std.	t	Pr(>|t|)	Coef	Std.	t	Pr(>|t|)
β_{low}	4.66	0.90	5.15	<0.0001 ***	5.88	0.84	6.99	<0.0001 ***
λ_{12}	-0.03	0.63	-0.05	0.96	-0.27	0.60	-0.45	0.65
λ_{23}	3.59	0.65	5.53	<0.0001 ***	0.88	0.62	1.43	0.16
λ_{34}	0.61	0.69	0.88	0.38	3.29	0.64	5.12	<0.0001 ***
λ_{45}	-0.31	0.69	-0.46	0.65	-0.05	0.63	-0.08	0.94
β_{high}- β_{low}	5.18	1.06	4.88	<0.0001 ***	4.28	1.05	4.08	<0.0001 ***
A	3.97	0.91	4.37	<0.0001 ***	0.745	0.94	0.79	0.43
δ_{12}	2.48	0.87	2.87	0.004 **	3.91	0.86	4.53	<0.0001 ***
δ_{23}	-0.48	0.89	-0.54	0.59	0.19	0.89	0.21	0.83
δ_{34}	1.84	0.94	1.96	0.05	-0.81	0.94	-0.86	0.39
δ_{45}	0.47	0.93	0.50	0.62	1.36	0.93	1.47	0.142
η_{12}	2.45	0.59	4.14	<0.0001 ***	3.64	0.62	5.83	<0.0001 ***
η_{23}	3.11	0.61	5.09	<0.0001 ***	1.06	0.64	1.67	0.10
η_{34}	2.45	0.63	3.90	<0.0001 ***	2.49	0.68	3.64	<0.0001 ***
η_{45}	0.16	0.63	0.25	0.81	1.31	0.68	1.94	0.05

Note The Pr(>t) column contains a p-value for each parameter that is based on a t-test for the hypothesis that the parameter is zero

They also developed their fraction proficiency when they received instruction in multiplication and division. In contrast, the low-performing students only developed their fraction proficiency during the first weeks of the fraction instruction period.

The results indicated that there were significantly disparate changes (parameter δ_{12}) for high- and low-performing students between the first and second test. In other words, a difference was found in the changes between low- and high-performing students during the first period where they received instruction in multiplication and division in both data sets. (see Figs. 27 and 28).

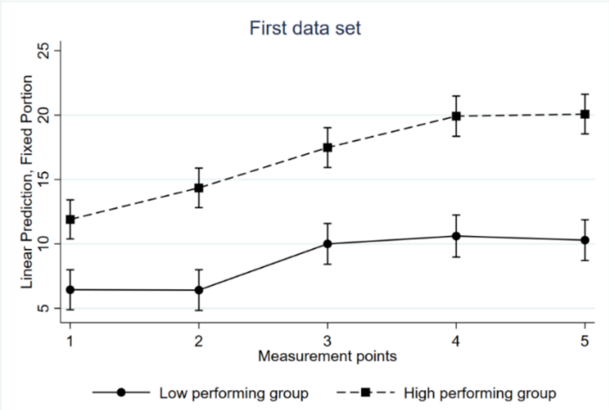

Fig. 27 A multiple linear regression model with 95% confidence intervals shown for the first data set

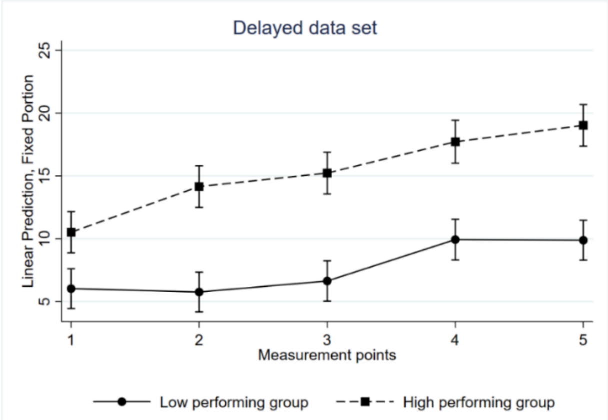

Fig. 28 A multiple linear regression model with 95% confidence intervals shown for the delayed data set

Discussion of Study 5

The two groups (high-performing and low-performing) differed in their development of fraction proficiency during the school year. One important distinction was that low-performing students did not develop their *fraction proficiency* unless they were instructed in fractions, and this pattern was found for both low-performing groups only. In contrast, both of the high-performing student groups showed another pattern; their fraction proficiency also developed outside the fraction instruction period and was supported during instruction in whole-number multiplication and division. However, instruction in equations did not have any influence on their fraction proficiency score, and this pattern was found in both the high- and low-performing groups.

It is noteworthy that the effects of instructions in other mathematical topics were not present for low-performing student groups. In other words, high-performing groups' fraction proficiency seemed to be supported when they had instruction in multiplicative principles in form of division and multiplication, while the same development was not found in low-performing groups. Overall, high-performing students showed no negative growth during the entire period while low-performing students only demonstrated significant positive growth if they received extra fraction instruction on a basic level. The gap between the two groups was widened primarily outside the fraction instruction period. I argue that this finding may indicate that high-performing students have a more integrated concept of mathematics compared to low-performing students. Meaning that they develop a web of knowledge (Hiebert & LeFevre, 1986). However, more research needs to be done to explore this in the future.

Chapter 9: Discussion

Knowledge generated from this PhD project shall not be seen as universal or unchangeable, but as warranted assertibility. Every finding must be seen in light of this so each finding is reported from the current point in the ongoing inquiry process. Consequently, the discussion in this chapter is based on my asserted findings from the five studies and the recommendation for classroom practices indicated by those five studies. This chapter discusses the results from the studies, the project, its methodical choices, implications for education, and contributions to the field.

9.1 Results from the five studies

In this PhD project, I sought to address the following research question: *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* I explored and investigated the question by engaging in the outlined inquiry process. This section addresses the fundamental question above, focusing on results concerning the students' difficulties in developing their concept of fractions. The discussion is structured around three main results that are founded in the five studies: a) aspects of natural number bias, b) fraction equivalence, and c) development of fraction proficiency. Each of the five studies varies in how they contribute to and inform these three main outcomes. Therefore, denotative boxes indicate which studies contributed primarily to which part of the discussion. Study 1 is not mentioned in any of the boxes but must be seen as implicit underlaying all the studies because it was the base of the quantitative data collection).

Study II
Study IV

9.1.1 Types of natural number bias

As previously outlined in Chapter 4.4.3, my suggested framework, *integrated conceptual change,* proposes an understanding of number knowledge development that combines *integrative* and *conceptual change* theories. This theory is compound and consists of both the process and the shift during time and includes natural number bias. *Integrated conceptual change framework* includes and combines different explanations for expanding students' number knowledge – namely, that students need a conceptual change in their number knowledge and this change must be seen as an integrated process over time. This means that in order to develop their number knowledge from natural numbers to also include rational numbers, the students develop conceptual change in their number knowledge. These needed changes can be found in the different types of natural number bias. Whether to use the term *types* or *aspects* can be discussed, and as outlined in Study 4, I argue for the use of the term *types*. However, at the same time it is not a new separate knowledge of numbers – the number knowledge must be seen as one integrated knowledge, including new aspects of numbers.

Previously, natural number bias has been defined with three or four aspects: density, operations, size, and sometimes representation (see the review in Chapter 4.3). My findings in Study 4 showed that the four aspects of natural number bias were not related or intricately connected. This is contrary to the findings of a previous research study by Van Hoof, Verschaffel, et al. (2015) where an overall natural number bias was found. Their conclusion on the overall whole number bias was that three aspects (*density*, *size*, and *operation*) were found across grades. Hence, their study showed the natural number bias was not found to be equally strong for each of the three aspects in their study, indicating that the aspects differ. However, the results from Study 4 in the present PhD project led to the conclusion that it is difficult to define natural number bias as one overall tendency of natural number bias when analysing the correlation between the students' answers coded as containing natural number bias. Instead, the study found that it could be better to use the term 'types' or 'kinds' to emphasise that there seems to be different whole number biases at stake. The discrepancy between my study and the study by Van Hoof, Verschaffel, et al.'s (2015) could be explained by the fact that my Study 4 had another aim, which was not finding the natural number aspect across grades but across each student's answers, thereby determining whether some students have an overall tendency toward natural number bias. This is a methodical explanation for the different conclusion. In addition, other methodical differences between the two studies make it difficult to compare the two conclusions. For example, item design for future research in a Danish context where the same test item is used as in Van Hoof, Verschaffel, et al. (2015) would be a better avenue for discussing results between studies.

Another important perspective to include in the discussion of Study 4's findings that the four aspects were not related to each other is that the data used in this study were obtained at the beginning of fourth grade and that the pattern that was found in this study could change as students move through the school system. However, for the present, the four aspects of natural numbers (density, operation, representation, and size) do not seem to be closely connected. Therefore, it might be wrong to see them as aspects of the same natural number bias. In Study 4, I argue that it might be more appropriate to use the term 'natural number bias types' instead of 'natural number bias aspect'. However, this demands further research on this particular area, and the current PhD project's knowledge is, as stated, warranted assertibility.

Another discussion is whether the three or four types of natural number biases are an adequate or fulfilling framework for understanding the biases. Fraction equivalence might be a fifth type of natural number bias because it appears that the equivalence task differs from the other non-equal comparison tasks. For example, the understanding that $\frac{1}{3}$ is bigger than $\frac{1}{4}$ even though 4 is larger than 3, and that $\frac{1}{4}$ is equal to $\frac{2}{8}$ represent two different mathematical concepts that need to be changed in students' number knowledge. The first, that $\frac{1}{4}$ is smaller than $\frac{1}{3}$, can be seen as a new understanding of how number notation's size or magnitude differs between natural number and fraction notations. Developing the understanding that $\frac{1}{4}$ is equal to $\frac{2}{8}$ can be seen as a conceptual change wherein each natural number represents a unique magnitude or amount, whereas in the understanding of fractions, two different notations can represent the same magnitude or amount. Van Hoof, Vandewalle et al. (2015) argue that the *representation* aspect includes the understanding that fractions and decimals can represent the same number and that $\frac{1}{2}$ is seen as one number ($\frac{1}{2}$ = 0.5). This could include fraction equivalence, meaning that according to students' perception, $\frac{1}{2} = \frac{2}{4} = 0.5$. However, it is noteworthy that fraction equivalence classes must also be connected to the subconstruct *ratio*. Therefore, ratio is connected to understanding fractions as equivalent classes: $\left[\frac{1}{2}\right] = \left\{\frac{2}{4}, \frac{4}{8}, \frac{6}{12}, \ldots\right\}$ (Behr et al. 1983; Kieren 1976, 1980), as also concluded in the theory of fraction equivalence conceptions in Study 3. This means that to fully understand fraction equivalence, the student must also understand the invariance between the numerator and denominator and see the proportional relationship between the two integers.

Equivalence is not conceiving of fraction equivalence as based only on the understanding of a fraction as 'one number' or a decimal. I therefore hypothesise that it would be better to define equivalence as a bias type of its own. This seems to differ in answer patterns compared to non-equal fraction comparison tasks according to Study 2. Therefore, the figure presented in Chapter 4.4.3 might be incomplete and at least one more type of natural number bias, *equivalence bias*, must be added into the figure (see Fig 29).

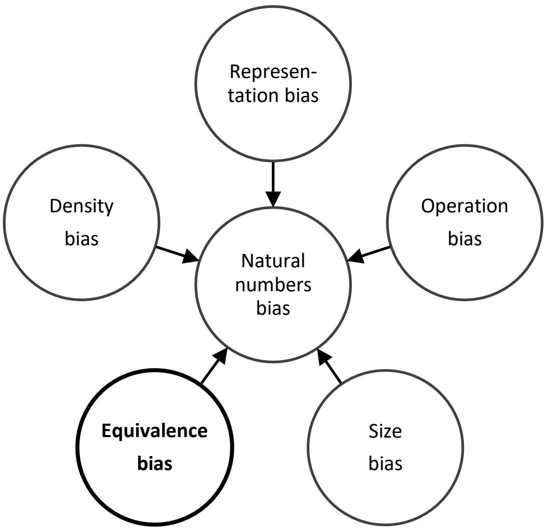

Fig. 29 The suggested five types of natural number biases. Note: Equivalence is separated from size

9.1.2 Fraction equivalence

Study II
Study III

I argue that fraction equivalence is a major concept in understanding fractions because it is a way to support students' development of their fraction knowledge, both procedural and conceptual. For example, finding the common denominator when adding $\frac{1}{4}$ and $\frac{1}{3}$ makes little sense if the students do not see the equivalence between $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{3} = \frac{4}{12}$ (whether this is founded in a procedural or conceptual understanding depends on the individual student's solving process). After reviewing the content of Danish mathematics books (see Chapter 5.2), I found that little attention has been paid to the development of the interpretation of equivalence.

As detected and analysed in Study 3, there are two different conceptions of equivalence: *proportional* and *unit equivalence*. The two different approaches work together and are combined. For example, unit equivalence will always contain proportional equivalence. To further elaborate the two different approaches in overall number knowledge development when a child learns natural numbers, the child learns to connect a set of entities to a unique number symbol (Levine et al., 2010). For example, the number four refers to an exact amount across different representations: four pencils, four chairs, four flowers and so forth. Children learn to see that the natural number represents a specific magnitude across different representations. However, when the child is learning fractions, they must learn to recognise the number as a proportional relation. For example, one dog out of four dogs, one piece of pizza out of four or – even more difficult – two boys out of eight children. All these different representations stand for the fraction $\frac{1}{4}$. All show the proportional equivalence between the different representations. Recognising, for example, *four* in different representations allows for counting or adding the different entities. However, recognising one quarter requires the proportional relation between two integers. Previous studies have shown that different representations create different advantages or challenges for students in their learning processes with fractions (e.g., Hamdan & Gunderson, 2017; Sidney et al., 2019), and therefore different representations also create differences in their understanding of equivalence. Study 3 further revealed how different representations support or confuse the two conceptions of fraction equivalence.

Furthermore, fraction equivalence can be interpreted as a quotient (Behr et al., 1992; Charalambous & Pitta-Pantazi, 2007; Kieren, 1980), meaning the invariance of the multiplicative relationship between the numerators and a denominator (Behr et al., 1992; Ni, 2001). In Study 2, the results revealed the only significant relation between the equal fraction comparison task and one of the four arithmetic operations. This relation was the operation division, but the relation was small. However, the pattern between the division and the fraction equivalence comparison tasks indicates that there is a relation between division and fraction equivalence. This is not surprising; division of two whole numbers is the only operation whose result can be a fraction – a quotient. As mentioned in Chapter 3.4, this led me to further develop the figure of the five subconstructs and to emphasise the subconstruct *quotient* in the figure so that the subconstruct *quotient* shares the same role as the subconstruct *part-whole* in the illustration (see Fig. 8). This is aligned with Kieren's (1993) later-developed framework, which also emphasised the importance of the *quotient*. The relation between division was also found in Study 5 in the high-performing students who developed their fraction proficiency while receiving instruction in both multiplication and division.

As outlined in Study 3, fraction equivalence can be viewed as an important part of developing an understanding of the fraction in connection to other mathematical concepts, for example, in order to support the understanding of algebra ($\frac{a^2+a}{4a^2+8a} = \frac{a+1}{4a+8}$) and percentages ($\frac{1}{4} = \frac{25}{100} = 25\%$). The conclusion from this theoretical study can be supported by future empirical studies.

9.1.3 Development of fraction proficiency over time and in relation to other topics

Study II
Study V

One primary outcome of the research was to explain how to support students' understanding of fractions. Two of the studies found that multiplicative understanding influenced the students' answers (Studies 2 and 5). First, students showed that their ability to solve whole number multiplication and division arithmetic was reflected positively in their responses when comparing fractions. The same pattern was not observed when reviewing whole number operations in addition or subtraction. Second, instruction in topics such as multiplication and division can support students' development of fraction proficiency. This pattern was found in Study 5, where instruction in multiplicative concepts in particular was found to be influential in high-performing students' fraction proficiency. The vital role of multiplicative reasoning in fraction learning is well-known (e.g., Lamon, 2012). However, the question of whether instruction in multiplicative concepts may influence high-performing students' fraction proficiency had not previously been investigated. Low-performing students only learn fractions when they receive instruction.

I argue that the high-performing students in Study 5 seem to demonstrate a more integrated concept of numbers where topics influence on each other, whereas the same integrated mathematical development was not found in the low-performing group. Thereby, the study reveals that the gap between low-performing and high-performing students' fraction proficiency will continue to widen over time because of the different pattern in how they learn. In other words, this can be simplified to the claim that high-performing students learn mathematics, whereas low-performing students learn topics in mathematics. This can be seen as a fundamental difference in their conceptual knowledge; low-performing students do not show the same web of knowledge as high-performing students. To use the landscape metaphor often employed in a Danish cultural context (Lindenskov, 2006, 2010), high-performing students can see the entire landscape. In contrast, low-performing students only see small parts of the maps and cannot connect them; they see one mountain and not the mountain range. The implication of this finding of this difference for classroom practices will be discussed in Chapter 9.3.

When looking at number knowledge development through an *integrated conceptual change framework* perspective (my suggested combined framework in Chapter 4.4.3), the finding of the difference between the high- and low-performing students could be explained by the difference in the level of integrated understanding between the two groups. In other words, high-performing students show an integrated development in their number knowledge, whereas low-performing students have potential for developing an understanding of fractions, decimals or percentages, but these topics are not integrated and are not part of an overall number knowledge that also includes natural numbers.

9.2 Project and methodical choices

Mathematical education research is an interdisciplinary field, and this project reflects this (Ernest et al., 2016; Niss, 2007; Sierpinska et al., 1993; Williams et al., 2016), providing a small picture of the complexity this multifaceted field involves. Hence, the project consists of studies with a more psychological approach founded in empirical educational studies and a more theoretical approach founded in the mathematical concept of fractions. I could argue that there may be a lack of exploration into the sociological characteristics of this field, for example, analysing observation of students' and teachers' interaction within the instruction situations. However, the methodical choices were determined by questions within the inquiry process, meaning questions connected to the overarching research question, *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?*

These served as the foundation for my choices. Therefore, it makes little sense to state that other methods could be used; it is better to inquire whether other questions could have been asked or whether there was another method to use for this investigation into possible answers. As outlined in Chapter 2, questions are central for the methodological framework of pragmatism. My actions, in the form of methodical choices, lead to these findings from the studies. Though enquiry is always embedded in the framework of biological and cultural operations, the knowledge developed from this PhD project must also be seen as embedded in the Danish school system both in classrooms and schools but also in university and teacher education, where my research project is based.

9.2.1 Discussion of the data collection through measurement

As outlined, pragmatism is not connected to one method but is based on observing a problem and investigating it. As outlined in Chapter 2, the current PhD project is based on the problems I have observed regarding students' difficulties with developing an understanding of fractions. When trying to investigate the problem, I developed a measurement tool whose aim was to measure students' fraction proficiency (analysed in Study 1). This raises the question of whether this measurement can capture the multifaceted structure of fractions. The quantitative measurement will have some limitation in capturing the complexity of the topic and, for example, qualitative interviews of students might be a better way to investigate this complexity. However, conducting qualitative interviews does not guarantee capturing the students' understanding and difficulties with the multifaceted concept of fractions; it will only be an external representation of students' understanding (Goldin & Shteingold, 2001; Rittle-Johnson & Alibali, 1999). However, future analyses of the conducted interviews during *intervention phase 5* might reveal new insights into the complexity. The benefits from using the developed measurement tool provide possibilities of finding patterns in students' answers that would otherwise not have been possible. If I had not used a quantitative data collection method, it would not have been possible to investigate an overall pattern of different types of answers (Study 2) or to determine an overall difference between high- and low-performing students' fraction proficiency during time period (Study 5).

Another question that needs to be raised regarding the use of a measurement tool is whether students are being given the opportunity to show their understanding of fractions in a given test. The students' performance on a test must be viewed as their performance in a specific test situation; this is a limitation. In addition, the students' problem-solving process was not captured in the computerised test. The students' fraction proficiency may have been different than the test results shown if it had been investigated through observation during regular instruction in a classroom. It is plausible that the students know or understand more than they show under measurement conditions. Future analyses of the interviews made in a more informal setting in the classroom will undoubtedly reveal new insights as well as new questions. As stated, the knowledge developed through this PhD project is seen as warranted assertibility.

A third question would be whether and how measurements could be further developed and improved. The long development process of these measurements shows that it is an ongoing process. It is important to continually improve and question measurement tools in the inquiry process; this is a central part of the data collections. This means also determining what I investigate and even more importantly, what I do not investigate. The fraction proficiency measurement could be further developed, specifically the two-string productive disposition strategic competence defined by the National Research Council (2001). This aspect of proficiency was not captured due to my choices in measurement design. The time limitation of the test demanded that cuts be made. Future development of these strings could be included, but test time would have had to been extended. It was unfortunate that this test was not completed before the intervention. However, to ensure that the content validity was as high as possible, outside experts (fourth-grade teachers) confirmed that the test measured fractions as taught in the Danish curriculum.

Lastly, I find that it is important to raise a question related to the quality of the measurement: were we measuring what we think we were measuring? How high was the content validity? Was the CBM measuring fraction proficiency? Study 1 was essential for addressing these questions and took a great majority of my time. When evaluating content validity, different test items might have been added to target density differently. An example would be asking a student to write a fraction between $\frac{1}{5}$ and $\frac{1}{4}$. However, my knowledge at the beginning of the project was lacking and evolved during the inquiry process study. In other words, I would now recommend adding items. In addition, I would have had the teacher rate each item to ensure content validity, as Al-Shehhi et al. (2019) did.

The computerised tests have some limitations in their design. However, an attempt was made to overcome some of these limitations by observing and interviewing students. A perfect way to explore students' knowledge does not exist. The methods used will simply be a representation of students' thinking. The computerised test made it possible to have a large dataset that could confirm our findings and thereby support the *Inquiry's Scientific Phase VI*. Therefore, it offered a way to explore and create a picture of the students' difficulties with learning fractions.

9.2.2 Discussion of the intervention's research design

As previously mentioned in Chapter 1.3, when I started on the PhD. project the plan was to use a quasi-experimental design with a control group. However, this design changed during the first year of my project for several reasons.
Primarily, an important reason for not using quasi-experimental design was that the control group would not have a designed intervention. Therefore, students in the control group received regular instruction based on the three mathematics books analysed in Chapter 5.3. Even though I attempted to gather data on the instruction about fractions in the control group, the results were insufficient because many teachers did not respond to the survey. As a result, it was not possible to include accurate variables about the length of fraction instruction in the control group. Consequently, any attempt at comparison would be unreasonable because the length of fraction instruction in the intervention group was seven weeks; I did not know how long the instruction period lasted in the control group. Without this information, I might compare students with seven weeks of fraction instruction in the intervention study to students who only received three weeks of comparable instruction in the control group.

A designated control group would be preferred, though due to study limitations, I was unable to construct another intervention design for the control group. Instead, I changed the design to focus primarily on high- and low-performing students' fraction proficiency development during grade four. Therefore, as described in Study 5, the project was designed as a delayed project component. As described in the chapter on methodology, the project is based on a process of inquiry in which the project continues to be developed and explored. As a result, the theoretical knowledge about quantitative studies grew, prompting the change in the research design. In addition, the difficulty with quasi-experimental designs and their effect size lies in the importance of a longitudinal view because the effect might decrease over time and therefore a delayed post-test would be important for the quality of the quasi-study (Aunio et al., 2005). These reservations influenced my choice to avoid a study based on a quasi-experiment.

A way to improve the research design might be to use a cross-over design (Maclure & Mittleman, 2000), in which the first group would start with multiplication/division and after that, fractions. In contrast, the delayed group would start with fractions, followed by instruction in multiplication/division. This approach would help analyse several concepts, such as whether multiplicative reasoning improves if the instruction takes place after – as opposed to before – the fraction instruction period.

Another improvement could be to consider a mixed-method design in which the interviews conducted during *intervention phase 4* would be used in an integrated investigation of the field. Despite these reconsiderations and drawbacks, I found that my PhD. project contributed to how the quantitative data can be useful when investigating student difficulties in developing their understanding of fractions.

9.2.3 The included students

The composition of the group of included students in the five studies must be taken into consideration and examined.

In Study 1, when analysing the accuracy of the GSM model, I chose to include both students who had answered all items in the measurement and students who had started on the measurement in the evaluations of structural validity of the measurement. This was done to secure that we found the same pattern in the indicators when comparing the results for the two evaluations.

Because of time limitations in the CBM, not all students were able to answer all questions. This inclusion or exclusion of students may have created a bias, especially in Study 2, but as explained there I emphasise that the included students must be considered to be above average and thereby further emphasise the problem with comparing equal fractions. In addition, the implication must be considered that knowledge may be associated with students demonstrating a certain speed when solving a task. Future studies must consider this effect.

In Study 5, I set the cut-off score at 25% before I started on the analysis. This means that the 25% was taken out of the total population, not taken out of each class. The consequent was that some classes had two students in the low group and other classes had eight students in this group. Nevertheless, all classes had both students in the low-and high-performing group, confirming that the variance between schools and classes were not high (also shown in Study 5). Overall, as outlined in Chapter 6.2.1, the schools and classes can be seen as average Danish fourth-grade classes, and therefore the students included in the cut-off score of 25% must be seen as including students characterised as those with mathematic difficulties.

A cut-off score other than 25% could have been discussed or applied. The estimated cut-off score varies across studies (Geary, 1994; Mazzocco, 2007; Swanson et al., 2018). These cut-off points have ranged from the 5th to the 46th percentile (Mazzocco, 2007; Swanson et al., 2018). In the intervention review in Chapter 4.4, the highest cut-off score was the 40th percentile (Westenskow & Moyer-Packenham, 2016). There is no consensus, and the cut-off score is often used as a means to have an operational way to delineate the students with mathematical difficulties. There has been some consensus among researchers when using norm-referenced math scores that scores below the 7th or 11th percentile identify students with math learning disabilities. Further, scores between the 11th and 25th percentile identified students with math difficulties (Mazzocco, 2007; Mazzocco et al., 2013; Mazzocco & Räsänen, 2013; Swanson et al., 2018). That percentile is also the cut-off score used in Study 5, in which the 25 lowest and highest performing students in the national test's sub-score numbers and algebra were used. However, it is important to emphasise that several studies have shown that each subgroup's heterogeneity when using strict cut-off score criteria is changing (Swanson et al., 2018; Mazzocco et al., 2007). Therefore, each group's differences varied. For example, students' cognitive measures varied greatly within the group (Geary, 2011; Mazzocco, 2007; Swanson et al., 2018). Moreover, the students in both the high- and low-performing groups in Study 5 cannot be seen as homogeneous. However, the study aimed to follow students' development during that time, and a pattern appeared to differ significantly between the high- and low-performing students. This pattern was confirmed in the delayed groups.

Another consideration could be whether it was appropriate to only use the national test score in *numbers and algebra*, or if the overall score from the other sub-scores in the national test should have been taken. I found that the *numbers and algebra* variable best explained the difficulties with numbers and was the best indicator of high- or low-performing students. I must emphasise that the Danish national test has been criticised for not being accurate when using it on a student level (VIVE – The Danish Center for Social Science Research, 2020) – it is only accurate on a school or class level. Therefore, I need to emphasise that when I choose to use the National Test score it is because of the high number of participating students (N=199). The high number made it possible to look at an overall tendency rather than on a separate unique students' level. The use of National Test score was an operational choice I made.

9.2.4 Concerns about following the development during time period

Lastly, a central consideration for the inquiry process of the overarching research question *How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in grade four?* is the concerns connected to the limited time (eight months) in which the students were followed. Hence, following the students' development over only eight months in Study 5 did not allow sufficient time to detect the full impact of the instruction on fraction proficiency. Especially for the low-performing group, a longitudinal study following the students during a period of three to five years will provide more insights and knowledge about student development. As previously mentioned, studies have found that low-performing students showed little growth in their fraction knowledge during their time in the school system (Jordan et al., 2017; Siegler & Pyke, 2013). Therefore, following the students over a longer period would be important: Do the high performing students maintain their progress? Alternatively, do the low performing students start their growth in fraction proficiency later?

Time as a factor is also an important perspective in evaluating the fact that we did not find any relation between the different types of natural number biases (in Study 5). This may be because the students were just beginning the process of learning fractions and would later develop different generalisations across the multi-conceptual framework of fractions. Additionally, the process of their rational number knowledge development had just started. Therefore, the concept and difficulties with the concept were not fully developed within the students. The integrated development is still in progress, as I argue in my developed theoretical *framework of integrated conceptual change.* Therefore, following the students over a longer period and evaluating their natural number bias and fraction proficiency would be valuable for the inquiry's *Scientific Phase VI.* The question is, can we rediscover the same patterns in later grades and thereby confirm the findings?

9.3 Recommendations for classroom practice

This section offers three perspectives relevant to the teaching and learning of fractions in the classroom. The first recommendation is that teachers should offer their students sufficient opportunities to acquire fraction equivalence concepts and understand the two constructions of equivalence: proportional or unit equivalence. More attention on equivalence can support the conceptual understanding for developing a flexible concept of fractions. This means that fraction equivalence can support the understanding of fraction arithmetic (e.g., common numerator), the connection between fractions and percentage (e.g., $\frac{4}{25} = \frac{16}{100} = 16\%$) and between fraction and algebra (e.g., $\frac{4(a+3b)}{8} = \frac{a+3b}{2}$) (Study 3).

Second, teachers must have time to identify the concepts that students need in order to understand how rational numbers, including fractions, differ from natural numbers, but also how natural numbers are included in rational numbers. Teachers must pay attention to the *integrated conceptual change* the students need to experience while overcoming their natural number biases. Because the different kinds of natural number biases do not seem to be related, as pointed out in Study 4, overcoming one bias does not mean overcoming all natural biases.

Third, students with mathematical difficulties may benefit from direct instruction, as previously recommended by the review of Shin and Bryant (2015). However, these detailed instructions should also focus on how different mathematical subjects are integrated. They must be supported in an integrated development so that students do not only learn 'division' or 'fractions', but also learn general mathematics. Specifically, their multiplicative understanding is a foundation to support fraction understanding (Study 2 and 5)

9.4 Contributions and future research

As mentioned, I see the knowledge developed during this PhD project as 'warranted assertibility'. Therefore, when answering the overarching research question, these answers must not be seen as universal or unchangeable answers or contributions.

How can we investigate and explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?

I addressed the first part of the question, *How can we investigate students' difficulties with developing the multifaceted concept of fractions in fourth grade?* in Study 1, where I developed a measurement targeting fraction proficiency. As outlined in Chapter 9.2.1, this type of computerised measurement has both limitations and advantages. However, it is a way to investigate the complex field of difficulties with fractions.

The second part of the question, *How can we explain students' difficulties with developing the multifaceted concept of fractions in fourth grade?* was addressed through the following studies in the PhD project. The three primary contributions to answer this part of the question are founded on three main perspectives: Different natural number bias types, equivalence, and development over time. This means that students' difficulties can be explained and founded on their tendency to different kinds of natural number biases, the difficulties with conceptions of equivalence, and their fraction proficiency not being connected or supported by their multiplicative concepts of whole numbers. These three primary contributions were developed during exploration and inquiry into the complex research field of fractions and will be further outlined in the following sections covering different kinds of natural number biases, equivalence, and development over time.

9.4.1 Different kinds of natural number biases

Natural number bias must be seen in the light of the broader development of number knowledge. As I argue in Chapter 4.4.3, it is important that the two theories of *conceptual change* and *integrative theory* are not considered opposites to each other, but combined as an *integrated conceptual change framework* (Fig. 11). In my combined framework, the process of transitioning from whole numbers to rational numbers can be seen as an integrated type of development in which whole numbers play an essential role in the emergence of the concept of rational numbers. Thereby, rational numbers act as an expansion of students' overall number concepts. However, when the starting and ending points of the expanding process are examined separately from the process, the difference between the two number concepts may be interpreted as a change in concept, for example, change in the understanding of *density* or *size*.

Several subconstructs such as the *quotient* are needed in the student's expanding process of number knowledge. Subconstructs must support a change in the students' concepts and help overcome their natural number biases. However, as demonstrated in Studies 2 and 5, natural numbers shall not be seen narrowly as distractors; they also offer support in the learning process of fractions.

In summary, my suggested *integrated conceptual change framework* expresses conceptual change as the comparison between points in time, whereas integrated development is defined by movement through time between the points. Natural number biases can be seen as a concept of numbers that need to be changed when developing an understanding of fractions. However, fraction understanding is still an integrated part of the overall number knowledge. This framework needs to be further investigated as argued in Chapter 4.4.3.

9.4.2 Equivalence

A more theoretical contribution of this PhD project is defining the two equivalence types and how they are essential for developing flexible understanding when performing fraction arithmetic. Study 3 theoretically explores the importance of understanding that proportional equivalence differs from unit equivalence. Future empirical research on the two equivalence conceptions is needed to elaborate on their significance, especially the similarity between learning natural and rational numbers. For example, when learning natural numbers, it is essential that one counts the numbers of different things (e.g., spoons, pencils, etc.) and realises that there can be both three spoons and three pencils present, meaning there are the same amount present in this case. However, this does not mean that three pencils are equal to three spoons. The child recognises the relationship between the number and the amount. When learning fractions, students must develop the ability to recognise the proportional equivalence instead of the counted amount. For example, instead of counting to three, one must focus on the proportional relation when recognising fractions.

9.4.3 Development of fraction understanding

Finally, it appears that high- and low-performing students differ in their development of fraction proficiency. High-performing students' fraction proficiency seems to be supported by instruction in multiplication and division, whereas the same support does not appear for low-performing students.

As shown in Studies 2 and 5, natural number arithmetic, especially multiplication and division, seems to play a role in the understanding of fractions, and thereby the importance of multiplicative understanding within natural numbers is a central aspect for the development of fraction understanding. As shown in Study 5, the gap between the two groups widens outside the fraction instruction period. This finding may indicate that high-performing students have a more integrated concept of mathematics than low-performing students. Future research needs to explore how low-performing students can be supported in order to develop a more integrated concept of mathematics or be able to make a transfer of knowledge between mathematical concepts and thereby develop their web of knowledge as well.

Overall, from a mathematical theoretical standpoint, rational numbers are not separate from natural numbers, but both number concepts are included in the overall rational number knowledge. Therefore, seeing it as an integrated expanding process is important. Though my suggested framework, *integrated, conceptual change* leaves several important questions of how to interpret the *integrated conceptual change* in the framework of mathematical knowledge containing both procedural and conceptual knowledge. To further develop the framework in the future, it could be central to actively add 'procedural' into the framework – making it an *integrated conceptual and procedural change*. Hence, as I argued in Chapter 4.2.3, procedural and conceptual knowledge must be seen as parts of an iterative process (Rittle-Johnson et al., 2001; Star, 2005; Star & Stylianides, 2013). Consequently, a change in concept knowledge will most likely also lead to a change in procedural knowledge, and vice versa. However, future empirical and theoretical research needs to further investigate the framework.

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Appendices

Appendix A. Databases included

Databases

Academic Search Elite

Australian Education Index (AUEI)

British Education Index

Education Research Complete

Education Resources Information Center (ERIC)

FIS Bildung Literaturdatenbank

Idunn

NORA (Norwegian Open Research Archives)

ProQuest Education Journal

PsycCRITIQUES

PsycINFO

ScienceDirect

Teacher Reference Center

The Danish National Research Database

Web of Science Core Collection

Other locations

MoNa (MONA - Matematik- og Naturfagsdidaktik) (Journal peer review)

Tangenten - Caspar Forlag AS – Tidsskrift for matematikkundervisning (Journal)

NOMAD: Nordic Studies in Mathematics Education (Journal peer review)

Nämnaren (Journal)

Nationellt Centrum för Matematikutbildning (NCM)

Matematikksenteret, Nasjonalt senter for matematikk i opplæring

Google Scholar

The Danish Evaluation Institute (EVA)

UC Knowledge

2. Word and phrase search

The searches were conducted in EBSCOhost with chosen databases:

Academic Search Elite

British Education Index

Education Research Complete

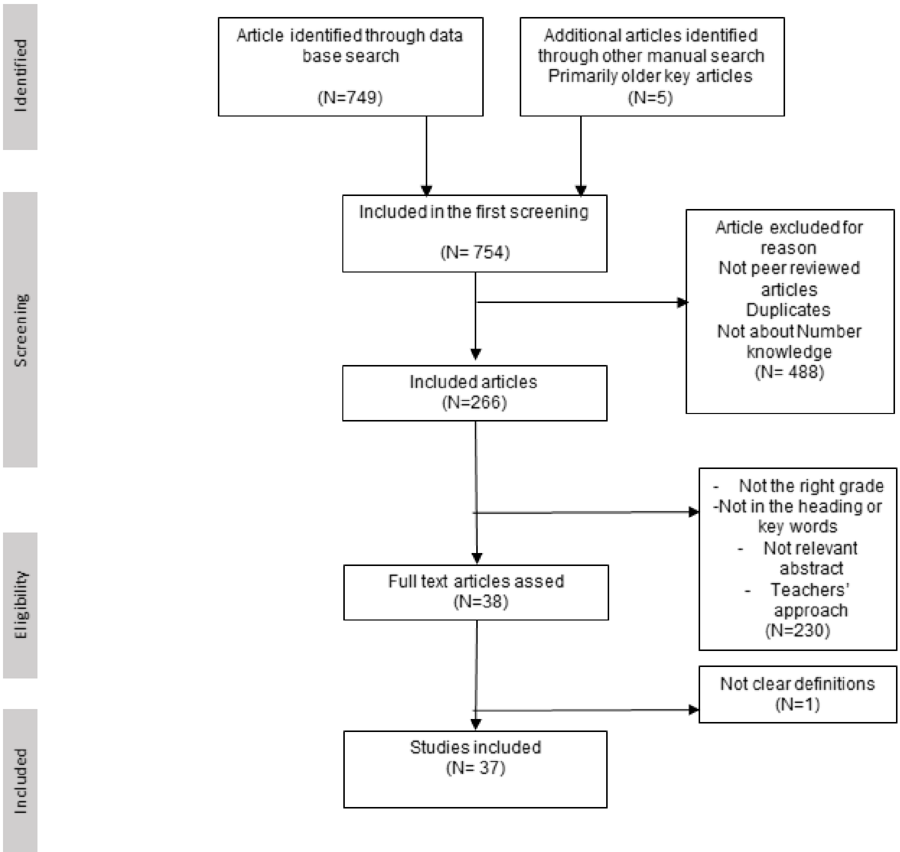
ERIC (Education Resource Information Center)

PsycCRITIQUES

Teacher Reference Center

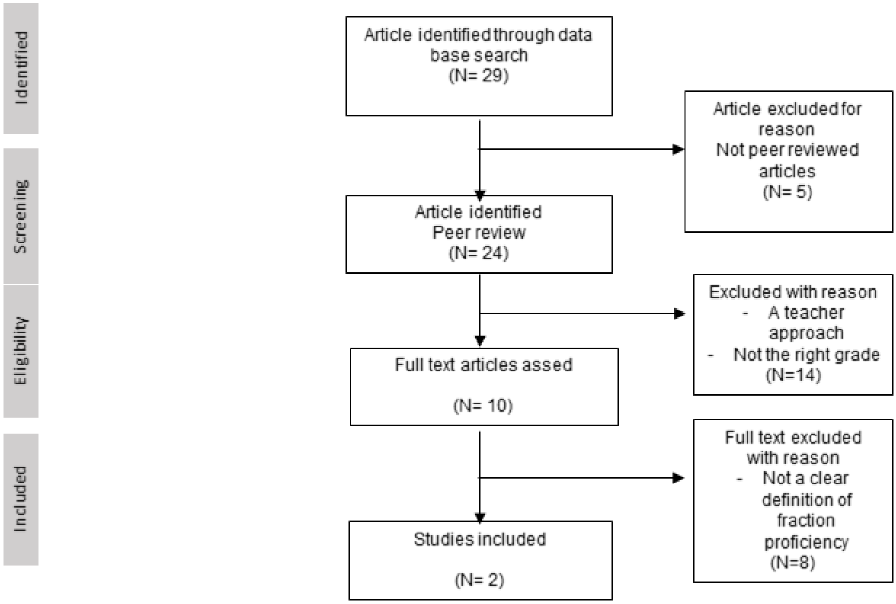
Appendix B. First Review: Mathematical knowledge and fraction proficiency

First Conceptual and procedural knowledge



Search words: ("conceptual knowledge" OR "procedural knowledge") AND "learning" AND "fraction*" AND ("elementary school" OR "primary school")

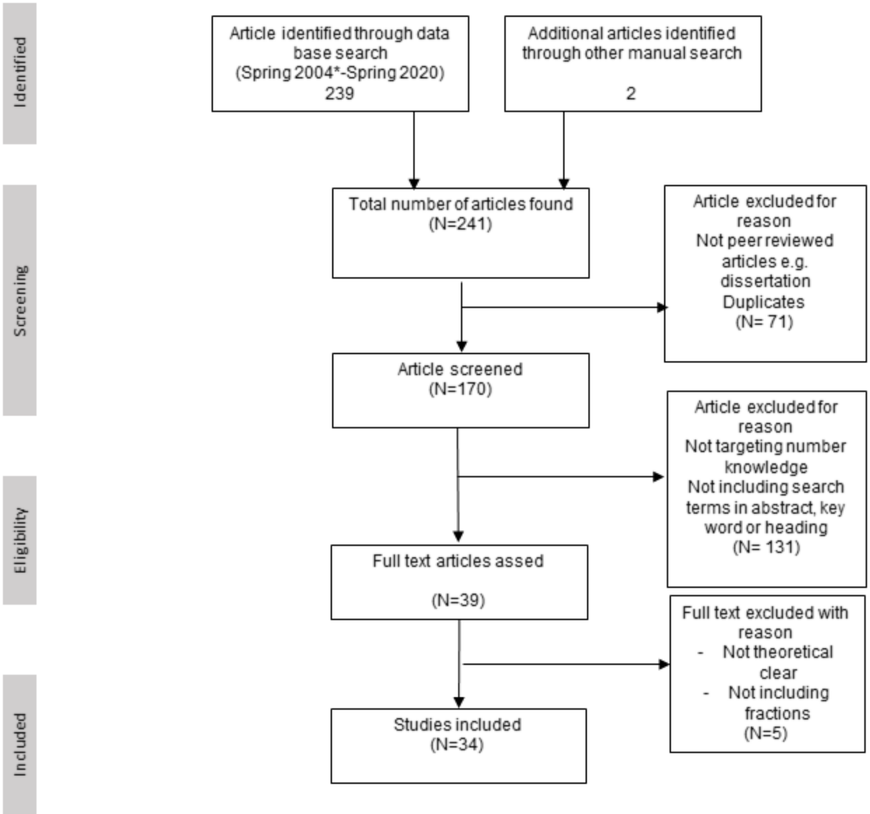
Fraction proficiency



Search 1: "fraction proficiency" OR "rational number proficiency"

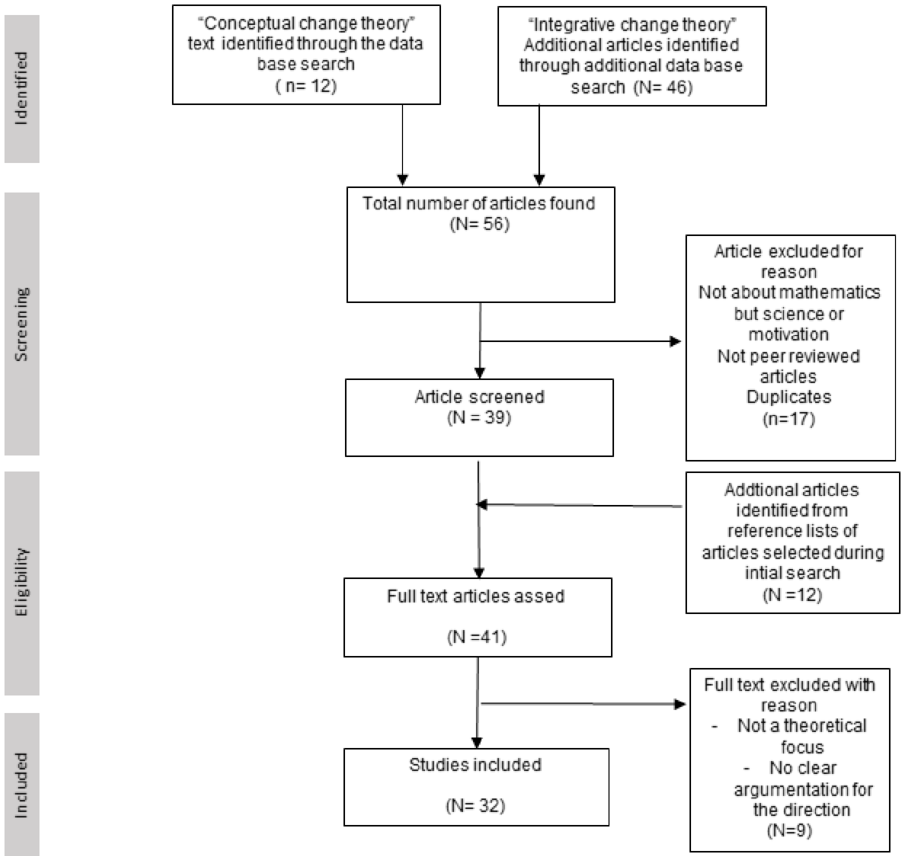
Search 2: "fraction*" OR "proficiency"

Appendix C. Second Natural Number bias



Search words: ("whole number bias" OR "natural number bias") AND "fraction*"

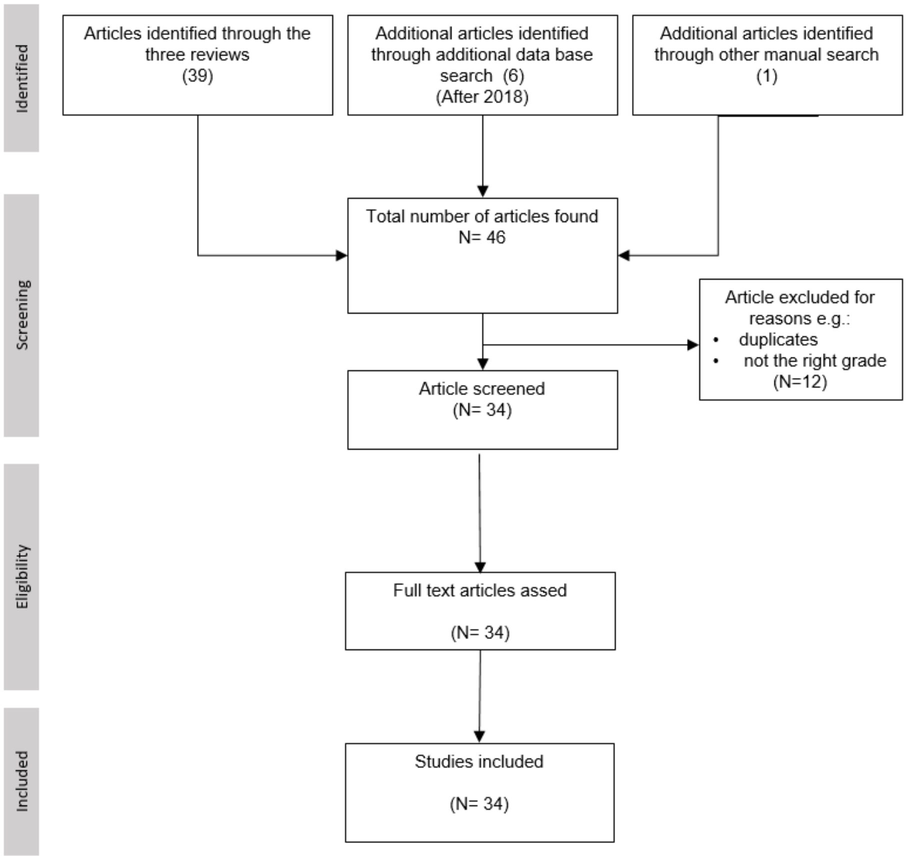
Appendix D. Third Review – Number knowledge development



Search 1 "conceptual change theory" AND "mathematic*" AND "learning*" NOT "motivation*" NOT "neuro science"

Search 2 "integrative theory" OR "integrated theory" OR "numerical development" AND "rational number*"

Appendix E. Fourth review: Fraction interventions



Search words: ("fraction intervention" OR "fraction instruction") AND "struggling learner*" AND "learning difficulties" AND "math*"

Appendix F. Consent letter

Kære forælder på x

Jeres barns matematiklærer har sagt ja til, at klassen må deltage i et forskningsprojekt om "elevers udvikling af brøkbegrebet". Når klassen i deres årsplaner arbejder med brøker, vil jeg derfor gerne komme ud og observere i klassen. Da klassen tidligere er blevet fulgt i første klasse i forbindelse med "tal og regning" er det en unik mulighed at følge klassen igen, idet det giver mulighed for at følge elevernes matematiske udvikling over en længere tidsperiode. Som en del af projektet vil udvalgte matematiktimer blive videooptaget, og elevernes faglige udvikling vil blive fulgt gennem forskellige aktiviteter. Videooptagelserne vil blive brugt i fuldt anonymiseret form til forskning. Det vil ikke på nogen måde være muligt at genkende dit barn i artikler eller generelle forskningsresultater.

Desuden vil cirka seks elever blive tilfældigt udvalgt i samråd med klassens matematiklærere til at deltage i tre små interviews gennem de fire uger, hvor klassen arbejder med brøker. Det vil sige, hver elev deltager i et interview i starten af brøkforløbet, ét i midten og ét i slutningen. Hvert interview tager cirka 20 minutter. Interviewene vil blive videooptaget, så de kan indgå i en analyse af, hvordan brøker bliver forklaret og forstået gennem arbejdet med dem. De små interviews foretages, når det passer ind i planlægningen, og når det kan lade sig gøre gerne i den understøttende undervisning.

Det er vigtigt, at vi får ny viden om elevers udvikling og forståelse af brøkbegrebet, da brøker har vist sig at være et centralt område at forstå for elevernes fortsatte matematiske udvikling. Det er derfor værdsat, at klassens matematiklærere har givet lov til, at jeg må komme ud på x. Jeg skal derfor bede om dit samtykke til at bruge optagelserne af dit barn til forskning. Derfor skal du angive nedenfor, hvad du vil give tilladelse til, underskrive og sende brevet med retur til klassens matematiklærer. Du skal også udfylde og sende sedlen retur, hvis du *ikke* ønsker, at dit barn skal deltage. Har du spørgsmål er du velkommen til at kontakte mig X, mobil: XXXXXXX

De bedste hilsner

Pernille Ladegaard Pedersen, lektor ved læreruddannelsen i Aarhus og ph.d.-studerende ved AAU

Undertegnede giver følgende tilladelser vedr. mit barns deltagelse i forskningsprojektet i skoleåret 2018/2019 (Sæt kryds):

	JA	**NEJ**
1. Mit barn må videofilmes i forbindelse med matematikundervisningen		
2. Mit barn må udtages og filmes i individuelle interviews om dets forståelse af		

Barnets fulde navn   Fødselsdato og år

Dato   Forældreunderskrift

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