On Parabolic Boundary Problems treated in Mixed-Norm Lizorkin-Triebel Spaces

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Mixed-Norm Lizorkin–Triebel Spaces

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Department of Mathematical Sciences

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Based on the papers

J. Johnsen, S. Munch Hansen and W. Sickel, *Characterisation by local means of anisotropic Lizorkin–Triebel spaces with mixed norms*, Journal of Analysis and its Applications, Vol. 32(3), 2013.

J. Johnsen, S. Munch Hansen and W. Sickel, *Anisotropic, mixed-norm Lizorkin–Triebel spaces and diffeomorphic maps*, submitted to Journal of Function Spaces and Applications, 2013.

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Preface

Three years have passed, since I in September 2010 began my journey as a PhD student at Department of Mathematical Sciences, Aalborg University, Denmark. During this period I have participated in a research project, which had been outlined by J. Johnsen and W. Sickel (Jena).

The project aims at providing a more detailed description of boundary value problems for partial differential equations in e.g. mathematical physics. As part of this work we had to generalise a modern technique in harmonic analysis from around the year 1999, and this turned out to be much more involved than first anticipated upon discovering a serious flaw in the existing proof.

The outcome is the present thesis, which for a large part consists of a study of anisotropic, mixed-norm Lizorkin–Triebel spaces in connection with the trace operators occurring in parabolic boundary problems. The main part of the thesis is the following three articles, written jointly with J. Johnsen and W. Sickel:

Characterisation by local means of anisotropic Lizorkin–Triebel spaces with mixed norms, Journal of Analysis and its Applications, Vol. 32(3), 2013.

Anisotropic, mixed-norm Lizorkin–Triebel spaces and diffeomorphic maps, submitted to Journal of Function Spaces and Applications, 2013.

Anisotropic Lizorkin–Triebel spaces with mixed norms — traces on smooth boundaries, preprint (expected submission Sep. 2013).

The first two articles [31, 32] appear in their submitted form in Chapter 4, resp. 5, except for the correction of a few typos, minor adaptation to the present layout (without any essential changes of the content) and when appropriate, references to our other articles are changed to the corresponding chapters. Also, the bibliographies have been merged to a unified list, which can be found at the end of the thesis. The last article [30] appears in Chapter 6.

Acknowledgements

I have looked forward to writing this section of the thesis for a long time and though it is short, it is of utmost importance, since this gives me the opportunity to express my gratitude to the people that have helped me make this journey.

First and foremost, I want to thank my supervisor Jon Johnsen for taking this roller coaster ride with me, for all the little times you have taken to listen when I stood in the doorway to your office looking like a big question mark, and not least for always believing that we could overcome the obstacles, we were faced with. Also I have benefited greatly from your ability to mobilise all your pedagogical skills in those hours, where I have been so frustrated that I could not see clearly.

I would also like to thank Winfried Sickel for welcoming me into this research project and for inviting me to Jena in April 2013. I could not have asked for a better host during my short stay.

In addition, my thanks go to the administrative and technical staff at the department. You have always greeted my questions with a smile and gone out of your way to help with my inquiries. I am also truely grateful to have met fellow PhD student Anita Abildgaard Sillasen, with whom I have shared much more than office space (and a considerable amount of healthier cake).

In November and December 2011 I visited Copenhagen many times to attend a PhD course given by post-doc Heiko Gimperlein. Despite the never-ending rivalry between Zealand and Jutland, I cannot help but to thank you for your hospitality, all our math talks over a cup of coffee and for accompanying me on my runs around The Lakes.

Also attending the course was Kim Petersen, PhD student at University of Copenhagen, with whom I later traveled to the other side of the world, when we both stayed a semester at University of California, Berkeley. I very much enjoyed your company during all our trips to San Francisco; the early morning visits to Tartine Bakery (which almost deserves its own acknowledgement) and the coffee breaks in the afternoon, during which you suffered your share of defeats in card games.

My family deserves a gold medal for their never-ending support and unwavering believe in me. I owe you everything (and even more). A particular thanks to my mother for being my math guru throughout public school and even in the beginning of high school, where you helped me with my algebra exercises.

Cecilie, no words can express my gratitude. For putting a smile on my face every single day of this journey. For laughing away all my silly ideas. For seeing right through me. For everything...

Aalborg, August 2013

Sabrina Munch Hansen

Summary

This PhD thesis is part of a larger project, where the goal is to establish a higher regularity theory for parabolic boundary value problems, when these are considered for non-zero boundary data in a set-up with different integrability properties in the space and time directions. To develop such a theory, a systematic treatment of trace operators is needed; this takes up much of the thesis.

The objective is to find optimal co-domains for the trace operators when these are applied to anisotropic, mixed-norm Lizorkin–Triebel spaces over cylinders, i.e. $F^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$. This requires a thorough study of these scales of function spaces in order to develop the necessary tools.

Chapter 1 contains an introduction and a short survey of earlier contributions to the study of parabolic problems with inhomogeneous boundary conditions; both written in layman's terms. The chapter also has a section on notational preliminaries.

This is in Chapter 2 followed by an introduction to Besov and Lizorkin–Triebel spaces; with focus on the anisotropic, mixed-norm case. Historical remarks and useful properties of these spaces are collected. Furthermore, we compare our results on characterisation by kernels of local means to earlier results and to ongoing research.

In Chapter 3 earlier contributions by P. Weidemaier [63, 65] as well as J. Johnsen and W. Sickel [29] on trace operators occurring in parabolic boundary value problems are discussed. Their point of views differ and as a result, they contribute to the theory in different ways. It is explained how our trace results are related to these works.

Chapter 4 contains generalisations of inequalities in harmonic analysis by V. S. Rychkov [44] to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, i.e. with anisotropies and mixed norms. Moreover, some flaws in [44] are corrected. The inequalities are used to obtain a characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ that can be specialised to the case of kernels of local means.

This particular characterisation is used in Chapter 5 to prove invariance under certain diffeomorphisms of $F^{s,\vec{a}}_{\vec{p},q}$ over both $\mathbb{R}^n$ and cylinders $\Omega \times I$. Since the $F^{s,\vec{a}}_{\vec{p},q}$-spaces contain $L_{\vec{p}}$, it is clear that restrictions on the parameters $\vec{a}$, $\vec{p}$ and on the diffeomorphism are needed for the map $f \mapsto f \circ \sigma$ to leave the spaces invariant.

Exploiting this, we define in Chapter 6 anisotropic, mixed-norm Lizorkin–Triebel spaces over the manifold $\partial\Omega \times I$. These are then used to obtain optimal co-domains for the trace operators at both the flat boundary Ω and the curved boundary $\partial\Omega \times I$.

In Chapter 7 the trace theory is applied to a concrete boundary value problem and necessary conditions are deduced for the existence of a solution of a certain regularity. Finally, Chapter 8 contains some comments on a recent, somewhat related work by S. Mayboroda.

Danish Summary (dansk resumé)

Denne ph.d.-afhandling er en del af et større projekt, hvis mål er at etablere en højere regularitetsteori for parabolske randværdiproblemer, når disse betragtes med ikke-trivielle randdata i en ramme, der tillader forskellige integrabilitetsegenskaber i rum og tid. For at udvikle en sådan teori en det nødvendigt med en systematisk behandling af sporoperatorer; dette udgør en stor del af afhandlingen.

Målet er at finde optimale rådighedsmængder for sporoperatorerne, når disse anvendes på anisotrope Lizorkin–Triebel-rum med blandede normer på cylindere, dvs. $F^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$. Dette kræver et grundigt studie af disse skalaer af funktionsrum for at udvikle de nødvendige værktøjer.

Kapitel 1 indeholder en introduktion og en kort oversigt over tidligere bidrag til studiet af parabolske problemer med inhomogene randbetingelser; afsnittene er ment som en blød indføring i afhandlingens overordnede emne. Yderligere forefindes også et afsnit om notation.

Det efterfølges i kapitel 2 af en introduktion til Besov- og Lizorkin–Triebel-rum med fokus på anisotropier og blandede normer. Her er også samlet historiske bemærkninger og nyttige egenskaber for disse rum. Derudover sammenligner vi vores resultater om karakterisering ved kerner af lokale midler med tidligere resultater og igangværende forskning.

I kapitel 3 diskuteres tidligere bidrag af P. Weidemaier [63, 65] såvel som J. Johnsen og W. Sickel [29] omhandlende sporoperatorer i forbindelse med parabolske randværdiproblemer. Deres synspunkter er forskellige, og som resultat heraf bidrager de på forskellig vis til teorien. Det forklares, hvorledes vores sporresultater relaterer sig til deres.

Kapitel 4 indeholder generaliseringer af uligheder i harmonisk analyse, der skyldes V. S. Rychkov [44], til $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, dvs. med anisotropier og blandede normer. Ydermere rettes nogle fejl i [44]. Ulighederne benyttes til at opnå en generel karakterisering af skalaerne, og denne kan specialiseres til tilfældet med kerner for lokale middelværdier.

Denne karakterisering bruges i kapitel 5 til at bevise, at $F^{s,\vec{a}}_{\vec{p},q}$, både på $\mathbb{R}^n$ og cylindere, er invariante under visse diffeomorfier. Idet $F^{s,\vec{a}}_{\vec{p},q}$-rummene indeholder $L_{\vec{p}}$, er det klart, at restriktioner på parametrene $\vec{a}$, $\vec{p}$ samt på diffeomorfierne er nødvendige, for at afbildningen $f \mapsto f \circ \sigma$ efterlader rummene invariant.

Ved at udnytte dette definerer vi i kapitel 6 anisotrope Lizorkin–Triebel-rum med blandede normer på mangfoldigheden $\partial\Omega \times I$. Dernæst bruges disse til at angive optimale billedrum af sporoperatorerne på både den flade rand Ω og den krumme rand $\partial\Omega \times I$.

I kapitel 7 anvendes sporteorien på et konkret randværdiproblem til at udlede nødvendige betingelser for eksistensen af en løsning med en bestemt regularitet. Afhandlingen rundes af med kapitel 8, hvori et nyligt, og i en vis udstrækning relateret, arbejde af S. Mayboroda kommenteres.

Contents

CHAPTER 1

Introduction

It is probably only mathematicians who walk around thinking about how mathematics play an underlying role in many everyday situations (possibly with the exception of statistics which most people are familiar with). But even so, it is an inescapable fact that e.g. partial differential equations are widely used in various fields, ranging from economics to physics. The ongoing research in this area helps to model real-life phenomena still more accurately and to analyse the models more profoundly.

This PhD thesis contributes to the work of developing a higher regularity theory for inhomogeneous, parabolic boundary value problems; primarily by providing a systematic trace theory for parabolic boundary value problems.

A very simple parabolic partial differential equation is the heat equation

$$\partial_t u - \Delta u = g \quad \text{in} \quad \Omega\times\,]0,T[\,, \tag{1.1}$$
$$u = \varphi \quad \text{on} \quad \partial\Omega\times\,]0,T[\,, \tag{1.2}$$
$$u = u_0 \quad \text{on} \quad \Omega\times\{0\}, \tag{1.3}$$

where $g(x,t)$, $\varphi(x,t)$, $u_0(x)$ are the given data, while $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the Laplacian, $T > 0$ and $\Omega \subset \mathbb{R}^n$ is bounded with $\partial\Omega$ denoting the boundary.

Despite its simple form, the heat equation is important in itself and it is a model for linearised reaction-diffusion equations. Hence results for this equation can give inspiration to the analysis of more complicated reaction-diffusion equations.

In case a solution $u(x,t)$ to (1.1)–(1.3) is continuous on the closure $\overline{\Omega}$ of Ω, then (1.2) simply states that the restriction of u to $\partial\Omega\times\,]0,T[$ equals φ. However, in most cases the solutions will not be continuous functions, hence (1.2) is interpreted using a *trace operator*; and likewise for (1.3).

In the general study of trace operators one tries to match regularity assumptions on u with the number of differentiations performed by the trace operator to determine the co-domain, that makes the operator surjective. This gives decisive information regarding the spaces, where boundary data must be chosen.

1.1 Background

There exists a classical theory, cf. e.g. [37], describing the solutions to (1.1)–(1.3), when $g \in L_p(\Omega\times\,]0,T[)$ for some $p > 1$. More generally, g can belong to a mixed-norm space $L_{\vec{p}}$, where $\vec{p} = (p_1, p_2) \in [1,\infty]^2$, i.e.

$$\Big(\int_0^T \Big(\int_\Omega |g(x,t)|^{p_1}\,dx\Big)^{p_2/p_1} dt\Big)^{1/p_2} < \infty,$$

hence g has different integrability in space and time. This possibility seems very relevant for applications in physics, where e.g. $p_1 = 2$ and $p_2 = \infty$ (replacing the integral by a supremum) can describe a time-bounded kinetic energy of a fluid.

For the homogeneous boundary condition, i.e. $\varphi \equiv 0$, it follows from maximal regularity theory that $u \in W^{2,1}_{\vec{p}}(\Omega \times (\varepsilon,T)) := \{u \in L_{\vec{p}} \mid \partial_t u, \partial^2_{x_1} u, \dots, \partial^2_{x_n} u \in L_{\vec{p}}\}$ for any $\varepsilon > 0$; cf. [1, Ch. III, 4.10] together with the references there for the development of maximal regularity and e.g. [62] for the application to a certain class of parabolic partial differential equations, including (1.1).

In the inhomogeneous case P. Weidemaier was one of the pioneers with some initial investigations, e.g. [64, 66], in the period 1991–2005. One main conclusion is that a Lizorkin–Triebel space with mixed norms is optimal for the boundary data φ in the sense that the trace operator with domain $W^{2,1}_{\vec{p}}(\Omega\times\,]0,T[)$ is surjective.

In 2007 R. Denk, M. Hieber and J. Prüss [9] studied parabolic boundary value problems with inhomogeneous data under the very general assumption that the coefficients in the differential equation take values in a space of operators (on a fixed Banach space). However, the scope in [9] is restricted to a single value of the smoothness parameter of the Lizorkin–Triebel spaces. This is not suitable for a theory of higher regularity of the solution, i.e. whether increased regularity of the data implies increased regularity of the solution.

The year after, a full framework for traces on hyperplanes was developed by J. Johnsen and W. Sickel in [29]. This included the construction of right-inverses, which are essential for a higher regularity theory. The existence of a right-inverse also implies that the trace operator is surjective; an important property, since otherwise it is a priori known that for some data no solution exists. Furthermore, a bounded right-inverse readily gives that the solution depends continuously on the data, which is a necessity for well-posedness of the boundary value problem.

In our paper [30] both the trace results and the construction of right-inverses in [29] are generalised to cylindrical domains $\Omega \times I$. A key ingredient in doing so is a characterisation of the Lizorkin–Triebel spaces by so-called kernels of local means. This can be found in [32], which is based on the work [44] by V. Rychkov, where Lizorkin–Triebel spaces with unmixed norms, i.e. $p_1 = p_2$, is treated.

1.2 Preliminaries

Since each paper contains a section on notation, cf. Section 4.2.1, 5.2.1 and 6.2.1, we here only comment on the notation used in Chapter 2 and 3.

Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{N}$ the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The closure of a set $U \subset \mathbb{R}^n$ is written $\overline{U}$ and $B(0,r)$ is the open ball centered at 0 with radius $r > 0$; the dimension of the surrounding Euclidean space will be clear from the context.

The restriction of a distribution u to an open subset U of $\mathbb{R}^n$ is denoted r_U. Moreover, $t_+ := \max(0,t)$ for $t \in \mathbb{R}$ and $\mathbb{R}^n_+ := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

Multi-index notation is used both in connection with partial derivatives and exponentiation, i.e. for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$,

$$D^\alpha := (-\mathrm{i}\,\partial_{x_1})^{\alpha_1} \cdots (-\mathrm{i}\,\partial_{x_n})^{\alpha_n}, \qquad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

In the following some basic function spaces are introduced; the definitions of Besov and Lizorkin–Triebel spaces are postponed to Section 2.1.

The space $C_0^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, consists of C^∞-functions with compact support in Ω; these are sometimes referred to as test functions. The dual space $\mathcal{D}'(\Omega)$ consists of continuous functionals on $C_0^\infty(\Omega)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all rapidly decreasing C^∞-functions φ in the sense that $x^\alpha D^\beta \varphi(x)$ is bounded for every $\alpha, \beta \in \mathbb{N}_0^n$. The Fourier transformation is for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ defined by

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} \varphi(x)\,dx, \quad \xi \in \mathbb{R}^n,$$

and extends by duality to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions.

For $0 < \vec{p} \le \infty$, where the inequality is understood componentwise, the space $L_{\vec{p}}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions such that

$$\| \, u \,|L_{\vec{p}}(\mathbb{R}^n)\| := \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} |u(x_1,\dots,x_n)|^{p_1} dx_1 \Big)^{p_2/p_1} \cdots dx_n \Big)^{1/p_n} < \infty; \quad (1.4)$$

with the modification of using the essential supremum over x_j in case $p_j = \infty$. Equipped with this quasi-norm, $L_{\vec{p}}(\mathbb{R}^n)$ is a quasi-Banach space, and it is normed if $\vec{p} \ge 1$.

The space $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$, where $0 < q \le \infty$, consists of sequences $(u_k)_{k\in\mathbb{N}_0}$ of Lebesgue measurable functions $u_k : \mathbb{R}^n \to \mathbb{C}$ such that

$$\| \, (u_k)_{k\in\mathbb{N}_0} \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\| := \Big\| \Big(\sum_{k=0}^{\infty} |u_k|^q \Big)^{1/q} \Big| L_{\vec{p}}(\mathbb{R}^n) \Big\| < \infty;$$

with the supremum over k if $q = \infty$. The quasi-norm is abbreviated $\| \, u_k \,|L_{\vec{p}}(\ell_q)\|$, and when $\vec{p} = (p,\dots,p)$ we simplify $L_{\vec{p}}$ to L_p etc.

CHAPTER 2

Besov and Lizorkin–Triebel Spaces

This chapter contains a brief historical overview of Besov and Lizorkin–Triebel spaces followed by an introduction to the anisotropic, mixed-norm versions.

2.1 Historical Overview

Around 1960 the Russian mathematician O. V. Besov introduced the function spaces $B^s_{p,q}(\mathbb{R}^n)$ for $1 < p < \infty$, $1 \le q < \infty$ and $s > 0$ using ideas (to be explained in the following) from the continuous Hölder–Zygmund scale $\mathcal{C}^s$.

J. Peetre followed up in 1967 with a characterisation of these spaces using Fourier analysis, cf. [58] and the references there. This characterisation, which will be our point of departure, relies on a Littlewood–Paley decomposition; the purpose of which is to split a temperate distribution into a countable sum of C^∞-terms with compact spectrum using a partition of unity $1 = \sum_{j=0}^\infty \Phi_j(\xi)$, $\xi \in \mathbb{R}^n$.

This is (for convenience) based on a fixed $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \psi(\xi) \le 1$ on $\mathbb{R}^n$, $\psi(\xi) = 1$ if $|\xi| \le 1$ and $\psi(\xi) = 0$ if $|\xi| \ge 3/2$. Setting $\Phi = \psi - \psi(2\cdot)$, we let

$$\Phi_0(\xi) = \psi(\xi), \qquad \Phi_j(\xi) = \Phi(2^{-j}\xi), \;\; j = 1, 2, \dots \tag{2.1}$$

By continuity of $\mathcal{F}^{-1}$, we have for every $u \in \mathcal{S}'(\mathbb{R}^n)$ that

$$u = \sum_{j=0}^\infty \mathcal{F}^{-1}(\Phi_j \mathcal{F} u) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n),$$

with each term being C^∞ by Paley–Wiener–Schwartz' theorem, cf. [21, Thm. 7.3.1] (the formulation here is from [28]):

Theorem 2.1 ([21, 28]). *When $K \subset \mathbb{R}^n$ is compact, then $u \in \mathcal{E}'(\mathbb{R}^n)$ fulfils* $\operatorname{supp} \hat{u} \subset K$ *if and only if u extends to an entire function $u(x + \mathrm{i}\, y)$ on $\mathbb{C}^n$, which satisfies*

$$\exists N \geq 0,\, C > 0,\;\; \forall x, y \in \mathbb{R}^n : \;\; |u(x + \mathrm{i}\, y)| \leq C(1 + |x + \mathrm{i}\, y|)^N e^{\sup\{\, -y\cdot\xi \mid \xi \in K\}}.$$

The characterisation of the Besov scales using Fourier analysis measures the smoothness of $u \in \mathcal{S}'(\mathbb{R}^n)$ by measuring $\big(2^{js}\mathcal{F}^{-1}(\Phi_j \mathcal{F}u)\big)_{j\in\mathbb{N}_0}$ in terms of $\ell_q(L_{\vec{p}})$-norms (which is meaningful since each element in the sequence is C^∞). The definition is here stated for the full range of parameters:

Definition 2.2. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which*

$$\| \, u \, | B^s_{p,q}(\mathbb{R}^n) \| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \| \, \mathcal{F}^{-1}(\Phi_j \mathcal{F}u) \, | L_p(\mathbb{R}^n) \|^q \Big)^{1/q} < \infty; \tag{2.2}$$

in case $q = \infty$ the sum is replaced by the supremum over all j.

The definition is justified by

Theorem 2.3 ([57, Sec. 2.3.3]). *For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$ the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space (normed if $p, q \geq 1$) and it is independent of the chosen partition of unity (in the sense of equivalent quasi-norms).*

The original definition due to Besov used so-called differences of functions, which for an arbitrary function u on $\mathbb{R}^n$, any constant $h \in \mathbb{R}^n$ and $M = 2, 3, \dots$ are defined as

$$(\Delta^1_h u)(x) = u(x + h) - u(x), \quad (\Delta^M_h u)(x) = \Delta^1_h(\Delta^{M-1}_h u)(x).$$

Instead of stating this definition, we recall, cf. e.g. [57, Thm. 2.5.12] by H. Triebel, that the norm in (2.2) for $0 < p, q \leq \infty$ and $s > n(1/p - 1)_+$ is equivalent to,

$$\| \, u \, | L_p(\mathbb{R}^n) \| + \Big(\int_{\mathbb{R}^n} |h|^{-n-sq} \| \, (\Delta^M_h u) \, | L_p(\mathbb{R}^n) \|^q dh \Big)^{1/q}, \tag{2.3}$$

where M now is an integer chosen such that $M > s$.

Inspired by Peetre's definition of Besov spaces using Fourier analysis, P. I. Lizorkin and H. Triebel, independently of each other, came up with the idea to interchange the ℓ_q- and the L_p-norms. This led to the introduction of Lizorkin–Triebel spaces around 1970, at first for $1 < p, q < \infty$:

Definition 2.4. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. The Lizorkin–Triebel space $F^s_{p,q}(\mathbb{R}^n)$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\| \, u \, | F^s_{p,q}(\mathbb{R}^n) \| := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\Phi_j \mathcal{F}u)|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\| < \infty; \tag{2.4}$$

in case $q = \infty$ the sum is replaced by the supremum over all j.

Results analogous to Theorem 2.3 and (2.3) hold for $F^s_{p,q}(\mathbb{R}^n)$, cf. [57, Sec. 2.3.3, 2.5.10]. In particular, the norm in (2.4) is for $0 < p < \infty$, $0 < q \le \infty$ and $s > \frac{n}{\min(p,q)}$ equivalent to the following, again with $M > s$,

$$\| u \,|L_p(\mathbb{R}^n)\| + \Big\| \Big(\int_{\mathbb{R}^n} |h|^{-n-sq} |(\Delta_h^M u)(\cdot)|^q dh \Big)^{1/q} \Big| L_p(\mathbb{R}^n) \Big\|. \qquad (2.5)$$

We now recall some special cases of the $F^s_{p,q}$- and $B^s_{p,q}$-scales, cf. [57]. First, the Sobolev spaces with integer exponents

$$W^s_p(\mathbb{R}^n) = \{\, u \in L_p(\mathbb{R}^n) \mid D^\alpha u \in L_p(\mathbb{R}^n) \text{ for } |\alpha| \le s\}, \quad s \in \mathbb{N}_0,$$

are a special case of the Lizorkin–Triebel spaces, i.e.

$$F^s_{p,2} = W^s_p, \quad s \in \mathbb{N}_0, \;\; 1 < p < \infty.$$

Secondly, the Besov scales can be specialised to the Hölder–Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$ and also to the non-integer Sobolev spaces $W^s_p(\mathbb{R}^n)$, since (2.3) is a well-known definition of W^s_p, when $s \notin \mathbb{N}_0$. I.e.

$$\begin{aligned} B^s_{\infty,\infty} &= \mathcal{C}^s, \quad s > 0, \\ B^s_{p,p} &= W^s_p, \quad 0 < s \notin \mathbb{N}, \;\; 1 \le p < \infty. \end{aligned}$$

Furthermore, the Besov and Lizorkin–Triebel scales coincide when $p = q$,

$$F^s_{p,p} = B^s_{p,p}, \quad s \in \mathbb{R}.$$

This is sometimes used to give sense to $F^s_{\infty,\infty}(\mathbb{R}^n)$.

These special cases show that it is meaningful to refer to the parameter s as the smoothness index. The other parameters p, q are referred to as the integral, respectively the sum exponent.

Both scales can be defined on open subsets $U \subset \mathbb{R}^n$ by restriction, using r_U to denote restriction to U in the sense of distributions,

$$\overline{F}^s_{p,q}(U) = \{\, u \in \mathcal{D}'(\mathbb{R}^n) \mid \exists \widetilde{u} \in F^s_{p,q}(\mathbb{R}^n) : r_U \widetilde{u} = u\}. \qquad (2.6)$$

(Here we follow Hörmander [22, App. B.2] by placing the bar over F.) Equipped with the quotient quasi-norm (norm if $p, q \ge 1$), i.e.

$$\| u \,|\overline{F}^s_{p,q}(U)\| = \inf_{r_U \widetilde{u} = u} \| \widetilde{u} \,| F^s_{p,q}(\mathbb{R}^n)\|,$$

it is a quasi-Banach space; similarly for $\overline{B}^s_{p,q}(U)$.

2.2 Anisotropic Spaces with Mixed Norms

The anisotropic Besov and Lizorkin–Triebel spaces arise by modifying ψ in (2.1). This is done with an anisotropic distance function $|\cdot|_{\vec{a}}$, where $\vec{a} = (a_1, \dots, a_n) \geq 1$. The purpose is to weight the coordinates x_j differently. Using a quasi-dilation

$$t^{\vec{a}}x := (t^{a_1}x_1, \dots, t^{a_n}x_n), \quad t \geq 0, \;\; x \in \mathbb{R}^n,$$

together with $t^{s\vec{a}}x := (t^s)^{\vec{a}}x$ for $s \in \mathbb{R}$, the function $|x|_{\vec{a}}$ is for $x \in \mathbb{R}^n \setminus \{0\}$ defined as the unique $t > 0$ such that $t^{-\vec{a}}x \in S^{n-1}$ (with $|0|_{\vec{a}} := 0$), i.e.

$$\frac{x_1^2}{t^{2a_1}} + \dots + \frac{x_n^2}{t^{2a_n}} = 1.$$

Indeed, the existence of such t is clear, since $t \mapsto |t^{-\vec{a}}x|$ is continuous on $\mathbb{R}_+$ by the Implicit Function Theorem and $|t^{-\vec{a}}x| \to 0$ for $t \to \infty$, while $|t^{-\vec{a}}x| \to \infty$ for $t \to 0$. The uniqueness follows straightforwardly by monotonicity. Figure 2.1 shows an example in two dimensions of how level curves for $|\cdot|_{\vec{a}}$ may look.

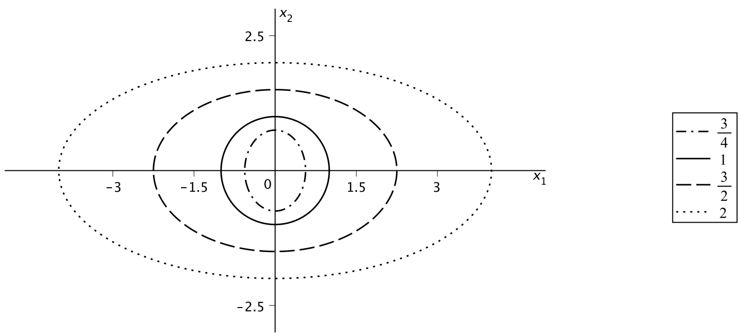

Figure 2.1: Some level curves of $|\cdot|_{(2,1)}$.

Now ψ is modified such that $|\cdot|_{\vec{a}}$ is used instead of $|\cdot|$, i.e. $\psi(\xi) = 1$ for $|\xi|_{\vec{a}} \leq 1$ and $\psi(\xi) = 0$ for $|\xi|_{\vec{a}} \geq 3/2$. Setting $\Phi = \psi - \psi(2^{\vec{a}}\cdot)$, a new partition of unity is obtained by, cf. Figure 2.2,

$$\Phi_0(\xi) := \psi(\xi), \qquad \Phi_j(\xi) := \Phi(2^{-j\vec{a}}\xi), \;\; j = 1, 2, \dots$$

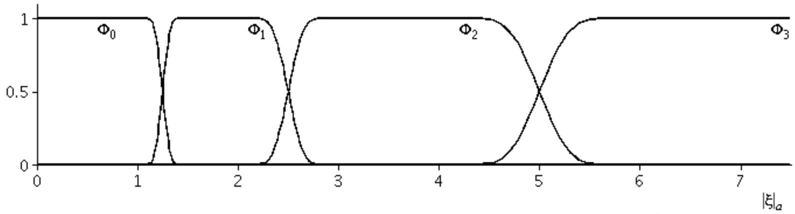

Figure 2.2: The graphs of the first four functions in $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$.

Using this partition of unity in the definition of the Besov and Lizorkin–Triebel spaces gives anisotropic spaces depending on the additional parameter $\vec{a}$. We take it a step further and consider an additional anisotropy, namely on the integrability, by applying the mixed norm $\|\cdot|L_{\vec{p}}\|$ from (1.4). This results in e.g.

Definition 2.5. *Let $0<\vec{p}<\infty$, $0<q\le\infty$ and $s\in\mathbb{R}$. The anisotropic, mixed-norm Lizorkin–Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ consists of the $u\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|\,u\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| := \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|\mathcal{F}^{-1}(\Phi_j\mathcal{F}u)|^q\Big)^{1/q}\Big|L_{\vec{p}}(\mathbb{R}^n)\Big\| < \infty. \tag{2.7}$$

Some properties of $B^{s,\vec{a}}_{\vec{p},q}$ and $F^{s,\vec{a}}_{\vec{p},q}$ can be found in Section 4.2.2 and 5.2. Usually $\vec{a}$ is fixed and therefore not included when stating the requirements for the parameters. In the isotropic case, i.e. $\vec{a}=(1,\dots,1)$, the parameter is even omitted from the spaces, cf. Section 2.1. Moreover, when the results are valid for the full ranges $0<\vec{p}<\infty$, $0<q\le\infty$, $s\in\mathbb{R}$, we often refrain from stating this.

The history of anisotropic, but unmixed Besov spaces dates back as early as the 1970s, where they (with some restrictions on the parameters) were studied e.g. in [48, 49, 56] by H. Triebel and H.-J. Schmeisser and in the monographs [41], [4] by S. M. Nikol'skij, respectively O. V. Besov, V. P. Il'in and S. M. Nikol'skij.

In 1983 Triebel characterised the anisotropic, unmixed Lizorkin–Triebel spaces using Fourier analysis, cf. [57, Sec. 10.1] and the references there. M. Yamazaki proved in [67] many properties of these scales and gave a nice review of $|\cdot|_{\vec{a}}$.

Further historical remarks can be found in Remark 4.4 and [28, Rem. 10].

2.2.1 Characterisation by Local Means

The Lizorkin–Triebel spaces can be characterised in different ways, each with its own strengths and weaknesses. One way is the Fourier-analytical approach in Definition 2.5, another is by kernels of local means in Triebel's sense [58]. Indeed, for $f\in\mathcal{S}'(\mathbb{R}^n)$ and $k\in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} k\subset B(0,1)$, Triebel defined

$$k(t,f)(x)=\int_{\mathbb{R}^n} k(y)f(x+ty)\,dy,\quad x\in\mathbb{R}^n,\ \ t>0,$$

and proved for isotropic spaces (note that it is not a characterisation of $F^s_{p,q}$):

Theorem 2.6 ([58, Sec. 2.4.6])**.** *Let $k_0,k^0\in C^\infty(\mathbb{R}^n)$ with support in $B(0,1)$ and $\int k_0(x)\,dx\neq 0\neq\int k^0(x)\,dx$ together with $k(x):=\Delta^N k^0(x)$ for some $N\in\mathbb{N}$. When $s\in\mathbb{R}$ and $2N>\max\big(s,n(1/p-1)_+\big)$, then for $f\in F^s_{p,q}(\mathbb{R}^n)$,*

$$\|\,k_0(1,f)\,|L_p(\mathbb{R}^n)\|+\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}|k(2^{-j},f)|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^n)\Big\| \tag{2.8}$$

is equivalent to the quasi-norm in (2.7).

We note that for $p \geq 1$, the condition on N reduces to $2N > \max(s, 0)$, hence $n(1/p-1)_+$ is a correction on the number of moment conditions in case $p < 1$.

The original theorem includes several other equivalent quasi-norms, however these are not important for our purpose. The strength of (2.8) is that the calculation of $k(t,f)(x)$ only requires knowledge of f in a ball of radius t around x.

H.-Q. Bui, M. Paluszynski and M. H. Taibleson characterised in [7, 8] the weighted, isotropic Lizorkin–Triebel and Besov scales by kernels of local means, i.e. they proved the equivalence of the quasi-norms for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Later V. S. Rychkov [44] gave a self-contained and more accessible proof in the unweighted case with discrete Littlewood–Paley decompositions. He also exemplified and corrected a mistake in [7], cf. [44, Rem. 2]. To state his result, we let $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ for an $\varepsilon > 0$ fulfil the Tauberian conditions

$$|\mathcal{F}\psi_0(\xi)| > 0 \quad \text{on} \quad \{\,\xi \mid |\xi| < 2\varepsilon\}, \tag{2.9}$$
$$|\mathcal{F}\psi(\xi)| > 0 \quad \text{on} \quad \{\,\xi \mid \varepsilon/2 < |\xi| < 2\varepsilon\} \tag{2.10}$$

and a moment condition of order $M_\psi \in \mathbb{N}_0$ (here $M_\psi = -1$ indicates no moments),

$$D^\alpha(\mathcal{F}\psi)(0) = 0 \quad \text{for all} \quad |\alpha| \leq M_\psi. \tag{2.11}$$

The non-linear Peetre–Fefferman–Stein maximal operator induced by $(\psi_j)_{j\in\mathbb{N}_0}$, where $\psi_j := 2^{jn}\psi(2^j\cdot)$ for $j \geq 1$, is for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $r > 0$ given by

$$\psi^*_{j,r}f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1+2^j|x-y|)^r}, \quad x \in \mathbb{R}^n, \;\; j \in \mathbb{N}_0. \tag{2.12}$$

Theorem 2.7 ([44, Thm. BPT])**.** *When $r > \frac{n}{\min(p,q)}$ and $s < M_\psi + 1$, then there exist $c_1, c_2 > 0$ such that for all $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\begin{aligned} \|\,\psi^*_{0,r}f\,|L_p\| + \|\,2^{js}\psi^*_{j,r}f\,|L_p(\ell_q)\| &\leq c_1\|\,f\,|F^s_{p,q}\| \\ &\leq c_2\|\,\psi_0 * f\,|L_p\| + \|\,2^{js}\psi_j * f\,|L_p(\ell_q)\|. \end{aligned}$$

The theorem is an improvement of the above Theorem 2.6 by Triebel, cf. [44, Sec. 3.1], since k_0, k there can be used as ψ_0, respectively ψ. Indeed, the Tauberian conditions (2.9)–(2.10) are satisfied for ε sufficiently small and $s < M_\psi + 1$ is fulfilled when $2N > s$. Thus the requirement $2N > \max\bigl(s, n(1/p-1)_+\bigr)$ in Theorem 2.6 can be relaxed to $2N > s$ by Theorem 2.7.

Moreover, Theorem 2.7 is a much stronger result in as much as it shows that (2.8) actually characterises $F^s_{p,q}$, since f is not a priori assumed to belong to $F^s_{p,q}$.

Our paper [32] generalises Theorem 2.7 and also [10, Thm. 4.9] by Farkas, where Theorem 2.6 is extended to the anisotropic, but unmixed case. Indeed, using the anisotropic analogue of (2.12), i.e.

$$\psi^*_{j,\vec{a},\vec{r}}f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}}, \quad x \in \mathbb{R}^n, \;\; j \in \mathbb{N}_0, \;\; \vec{r} > 0,$$

where the denominator is adapted to the anisotropic structure of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces introduced in Definition 2.5, the main result in [32] is

Theorem 2.8. *Let $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the Tauberian conditions* (2.9)–(2.10)*, where $|\cdot|$ is replaced by $|\cdot|_{\vec{a}}$, together with the moment condition* (2.11)*. When $s < (M_\psi + 1)\min(a_1, \dots, a_n)$ and $\vec{r} > \min(q, p_1, \dots, p_n)^{-1}$, then the following quasi-norms are equivalent on $\mathcal{S}'(\mathbb{R}^n)$:*

$$\| f \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|, \qquad \| 2^{js}\psi_j * f \,| L_{\vec{p}}(\ell_q)\|, \qquad \| 2^{js}\psi^*_{j,\vec{a},\vec{r}} f \,| L_{\vec{p}}(\ell_q)\|.$$

The proof, which can be found in Section 4.5, relies on generalisations of the arguments in [44] to the anisotropic case with mixed norms. However, at the same time we take the opportunity to correct a fundamental flaw in the article by Rychkov, cf. Remark 4.1 below. The other key ingredient is certain maximal inequalities adapted from [27] to this set-up.

Remark 2.9. *The motivation behind [32] comes from the study of trace operators occurring in parabolic PDE's as will be explained in Chapter 3. However, characterisation by kernels of local means is also useful in approximation theory, e.g. using this Triebel proved in [59, Thm. 3.5] that any element belonging to either $B^s_{p,q}(\mathbb{R}^n)$ or $F^s_{p,q}(\mathbb{R}^n)$ can be expanded uniquely using Daubechies wavelets.*

The broad scope of application makes the fact that the characterisation is not restricted to Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, but is widely applicable, even more important. Indeed, J. Vybiral applied in [61] the method to Besov and Lizorkin–Triebel spaces with dominating mixed smoothness, which generalise the Sobolev spaces

$$S^{\vec{m}}_p W(\mathbb{R}^n) := \Big\{\, f \in L_p(\mathbb{R}^n) \,\Big|\, \sum_{\alpha \le \vec{m}} \| D^\alpha f \,| L_p(\mathbb{R}^n)\| < \infty \Big\}, \quad \vec{m} \in \mathbb{N}^n_0.$$

Later H. Kempka used the method on so-called 2-microlocal Besov and Lizorkin–Triebel spaces with variable integrability, which combine the generalisations of the well-known isotropic spaces to 2-microlocal spaces and to spaces of variable integrability, cf. [34].

In the joint work [35] by H. Kempka and J. Vybiral, the results in [34] are generalised to Besov and Lizorkin–Triebel spaces with both variable integrability and smoothness.

However, neither [61], [34] nor [35] addressed the flaws, cf. Remark 4.1, in [44]. This should strongly indicate that our proof of Theorem 2.8 given in Chapter 4 should be of a wider interest in applied harmonic analysis.

CHAPTER 3

Trace Operators on Lizorkin–Triebel Spaces

The purpose of this chapter is to place our work [30] into a historical context.

3.1 Historical Overview

As mentioned in Section 1.1, there is a classical theory for the solvability of inhomogeneous, parabolic boundary value problems when working with the same integrability in space and time, cf. e.g. [37] by O. A. Ladyženskaja, V. A. Solonnikov and N. N. Uralceva, which was published in Russian in 1967. They discussed both solvability and higher regularity of solutions to second order, parabolic PDE's in the set-up of Sobolev spaces with unmixed norms. Another well-known reference is [38] by G. M. Lieberman.

G. Grubb and V. A. Solonnikov [18] considered solvability and higher regularity for parabolic pseudo-differential boundary problems in the framework of Sobolev spaces with integral exponent $p = 2$. Later Grubb [15] generalised to $p \in \,]1, \infty[\,$.

Weidemaier was one of the first to treat different integrability properties in space and time in connection with parabolic PDE's with inhomogeneous boundary conditions, cf. [63–66]. In [63] he considered PDE's over cylinders

$$\Omega_T := \Omega \times \,]0, T[\,,$$

whereby $T > 0$ and $\Omega \subset \mathbb{R}^n$ is open with compact boundary $\Gamma := \partial\Omega$ and of class $C^{1,1}$; roughly this means that the maps straightening out the boundary have C^1-extensions to the boundary, written $C^1(\overline{\Omega})$, and that the first-order derivatives are Lipschitz continuous, cf. [63], [36, 6.2.2]. When stating results by Weidemaier in the following, these are standing assumptions.

In particular, Weidemaier studied the trace at $\Gamma_T := \partial\Omega\times\,]0,T[$ for elements in the Sobolev space

$$W^{2,1}_{p,q}(\Omega_T) := \bigl\{\, u \in \mathcal{D}'(\Omega_T) \bigm| \partial^\alpha_x u, \partial_t u \in L_q(0,T;L_p(\Omega)) \text{ for } |\alpha| \le 2 \bigr\}, \tag{3.1}$$

where $L_q(0,T;L_p(\Omega))$ consists of the functions $u :]0,T[\to L_p(\Omega)$ that are strongly measurable and for which

$$\|\, u \,|L_q(0,T;L_p(\Omega))\| := \Bigl(\int_0^T \|\, u(t) \,|L_p(\Omega)\|^q \,dt \Bigr)^{1/q} < \infty. \tag{3.2}$$

Weidemaier proved, cf. [63, Thm. 1(i)], that there exist unique, linear and continuous maps $\gamma_{k,m}$, with $k = 1,\dots,n$ and $m = 0,1$, going from $W^{2,1}_{p,q}(\Omega_T)$ into a certain function space over Γ_T such that $\gamma_{k,m} u = (\partial^m_k u)|_{\Gamma_T}$ when

$$u \in W^{2,1}_{p,q}(\Omega_T) \cap \bigl\{\, v \bigm| \forall t \in]0,T[: v(\cdot,t) \in C^1(\overline{\Omega}) \bigr\}. \tag{3.3}$$

For brevity, the maps $\gamma_{k,0}$ are just denoted γ. Even though the $\gamma_{k,m}$ were not shown to be surjective, Weidemaier stated

Our results, which seem to be sharp...

However, in the work [65] from 2002 he returned to this question and proved that surjectivity is obtained by letting a Lizorkin–Triebel space describe the regularity in time. He worked with $F^s_{q,p}(0,T;L_p(\Gamma))$ defined for $1 < p \le q < \infty$ and $0 < s < 1$ to consist of the $u \in L_q(0,T;L_p(\Gamma))$ such that

$$\begin{aligned} &\|\, u \,|F^s_{q,p}(0,T;L_p(\Gamma))\| \\ &:= \Bigl(\int_0^T \Bigl(\int_0^{T-t} h^{-1-sp} \|\, u(t+h) - u(t) \,|L_p(\Gamma)\|^p dh \Bigr)^{q/p} dt \Bigr)^{1/q} < \infty. \end{aligned} \tag{3.4}$$

(The reader is referred to [63] for the definition of $L_p(\Gamma)$, $W^s_p(\Gamma)$.)

The motivation behind this definition comes from (2.5). We now state the sharp result by Weidemaier for γ:

Theorem 3.1 ([65, Thm. 2.3(i), Thm. 2.4])**.** *When $3/2 < p \le q < \infty$, then the map $u \mapsto u|_{\Gamma_T}$, which is well defined for the u in* (3.3)*, has a continuous, surjective extension*

$$\gamma : W^{2,1}_{p,q}(\Omega_T) \to L_q(0,T;W^{2-1/p}_p(\Gamma)) \cap F^{(2-1/p)/2}_{q,p}(0,T;L_p(\Gamma)).$$

The norm on the co-domain is the sum of the two spaces' norms.

The space $W^{2,1}_{p,q}(\Omega_T)$ is, as explained in Section 1.1, a natural framework for studying the parabolic problem in (1.1)–(1.3), since it naturally arises when the inhomogeneous term g belongs to $L_q(0,T;L_p(\Omega))$.

Therefore Theorem 3.1 shows the necessity of Lizorkin–Triebel spaces for the theory of parabolic PDE's. However, the result is based on assumptions on p, q which are not very intuitive. Indeed, it seems very restrictive not to include the case $p > q$, i.e. where elements have the highest order of integrability with respect to x. Also, Theorem 3.1 is not suitable for a study of higher regularity, since the smoothness index is fixed.

J. Johnsen and W. Sickel [29] developed a full framework for trace operators on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$. In stating their results, it will be convenient when considering the trace at e.g. the hyperplane $\{x_k = 0\}$ to use the splitting $\vec{a} = (a', a_k, a'')$, where

$$a' = (a_1, \dots, a_{k-1}), \qquad a'' = (a_{k+1}, \dots, a_n),$$

and likewise for $\vec{p}$. A main result of theirs on the trace $\gamma_{0,1}$ at $\{x_1 = 0\}$ is

Theorem 3.2 ([29, Thm. 2.2, 2.6]). *When*

$$s > \frac{a_1}{p_1} + \sum_{k=2}^{n} a_k \Big(\frac{1}{\min(1, p_2, \dots, p_k, q)} - 1 \Big),$$

then $\gamma_{0,1} : F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-\frac{a_1}{p_1}, a''}_{p'', p_1}(\mathbb{R}^{n-1})$ *is a bounded surjection. Moreover, there exists a bounded right-inverse going the opposite way for all* $s \in \mathbb{R}$.

They also proved a similar theorem for the trace $\gamma_{0,n}$ at $\{x_n = 0\}$:

Theorem 3.3 ([29, Thm. 2.5, 2.6]). *When*

$$s > \frac{a_n}{p_n} + \sum_{k=1}^{n-1} a_k \Big(\frac{1}{\min(1, p_1, \dots, p_k)} - 1 \Big),$$

then $\gamma_{0,n} : F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to B^{s-\frac{a_n}{p_n}, a'}_{p', p_n}(\mathbb{R}^{n-1})$ *is a bounded surjection. Moreover, there exists a bounded right-inverse going the opposite way for all* $s \in \mathbb{R}$.

On the one hand their results extend those by Weidemaier, since they did not work under restrictions on $\vec{p}, q$ and also treated general s. But on the other hand they considered Euclidean spaces and not cylinders.

Another big difference between the two approaches is that Weidemaier defined the trace operators as extensions by continuity, whereas Johnsen and Sickel considered the distributional trace, i.e.

$$\gamma_{0,k} u := u|_{x_k=0} \quad \text{for} \quad u \in C(\mathbb{R}_{x_k}, \mathcal{D}'(\mathbb{R}^{n-1})) \subset \mathcal{D}'(\mathbb{R}^n). \tag{3.5}$$

The inclusion follows, since it can be verified that any $u \in C(\mathbb{R}_{x_k}, \mathcal{D}'(\mathbb{R}^{n-1}))$ identifies uniquely with the distribution $\Lambda_u \in \mathcal{D}'(\mathbb{R}^n)$ given by, cf. [26, Prop. 3.5],

$$\langle \Lambda_u, \varphi \rangle = \int_{\mathbb{R}} \langle u(x_k), \varphi(\cdot, x_k, \cdot) \rangle \, dx_k, \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

In the present work we have chosen the latter method, since this definition seems more natural.

3.2 Cylindrical Domains

As a preparation, we recall the definition of a C^s-diffeomorphism:

Definition 3.4. *A bijective map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a C^s-diffeomorphism, $s \geq 0$, if the components $\sigma_j : \mathbb{R}^n \to \mathbb{R}$ have continuous derivatives up to order $\lfloor s \rfloor$, and they for $s \neq \mathbb{N}_0$ moreover satisfy the Hölder condition*

$$\sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{|D^\alpha \sigma_j(x) - D^\alpha \sigma_j(y)|}{|x-y|^{s-\lfloor s \rfloor}} < \infty.$$

For $s \in \mathbb{N}_0$, σ is called bounded when $\| D^\alpha \sigma_j \,|L_\infty(\mathbb{R}^n)\| < \infty$ for $j = 1, \dots, n$ and $0 < |\alpha| \leq s$, and this also holds for each component of σ^{-1}.

(Hereby $\lfloor s \rfloor$ denotes the smallest integer $k \leq s$.)

The article [30] treats the gap between [65] and [29] as it covers the trace problem for anisotropic, mixed-norm Lizorkin–Triebel spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, where $I :=]0,T[$ for $T > 0$ and $\Omega \subset \mathbb{R}^n$ is C^∞ in the sense of

Definition 3.5. *An open set $\Omega \subset \mathbb{R}^n$ with boundary Γ is C^s for $s \geq 0$, possibly $s = \infty$ in which case Ω is called smooth, when for each boundary point $x \in \Gamma$ there exists a C^s-diffeomorphism λ defined on an open neighbourhood $U_\lambda \subset \mathbb{R}^n$ such that $\lambda : U_\lambda \to B(0,1) \subset \mathbb{R}^n$ is surjective and*

$$\begin{aligned} \lambda(x) &= 0, \\ \lambda(U_\lambda \cap \Omega) &= B(0,1) \cap \mathbb{R}^n_+, \\ \lambda(U_\lambda \cap \Gamma) &= B(0,1) \cap \mathbb{R}^{n-1}, \end{aligned}$$

whereby $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n-1} \times \{0\}$.

Since $\Omega \times I \subset \mathbb{R}^{n+1}$, the parameters $\vec{a}, \vec{p}$ of $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ have $n+1$ entries. When reviewing our results on trace operators in the following, we assume that the first n entries are equal, i.e.

$$\vec{a} = (a_0, \dots, a_0, a_t), \qquad \vec{p} = (p_0, \dots, p_0, p_t). \tag{3.6}$$

Note that we use p_t as the integral exponent in the time direction, whereas Weidemaier uses q, which in our case plays the role of a sum exponent.

3.2.1 The Trace at The Flat Boundary

In the study of this trace, the case $T = \infty$ is included. One of the main results in [30], cf. Section 6.6.1, is here formulated in a simpler version:

Theorem 3.6. *Let $\vec{a}, \vec{p}$ satisfy (3.6) with $a_0 = 1$ and let $\Omega \subset \mathbb{R}^n$ be C^∞. When*

$$s > \frac{a_t}{p_t} + n\Big(\frac{1}{\min(1, p_0)} - 1\Big), \tag{3.7}$$

then the trace operator

$$r_0 : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{B}^{s-\frac{a_t}{p_t}}_{p',p_t}(\Omega) \tag{3.8}$$

is a bounded surjection. Moreover, there exists a bounded right-inverse going in the other direction for all $s \in \mathbb{R}$.

Since $p' = (p_0, \dots, p_0)$, the co-domain in (3.8) is an isotropic Besov space with unmixed norms, cf. Definition 2.2. In the general version stated in Theorem 6.36 below, a_0 can be arbitrary, in which case the co-domain is $\overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)$, when s satisfies (3.7) with a_0 instead of 1.

The construction of a right-inverse relies on the existence of an extension operator $\mathcal{E}_\Omega$ from $\overline{B}^s_{p,q}(\Omega)$ to $B^s_{p,q}(\mathbb{R}^n)$ in the sense that for all $u \in \overline{B}^s_{p,q}(\Omega)$,

$$\mathcal{E}_\Omega u \in B^s_{p,q}(\mathbb{R}^n), \qquad r_\Omega \mathcal{E}_\Omega u = u. \tag{3.9}$$

Such an operator is constructed in the elegant work [45] by Rychkov, where he even constructed a *universal* extension operator, meaning that it works for all admissible parameters of the $\overline{B}^s_{p,q}(\Omega)$-spaces simultaneously. Moreover, the operator is also universal for the $\overline{F}^s_{p,q}(\Omega)$-spaces and works when Ω is either a bounded or a special Lipschitz domain, i.e.

Definition 3.7. *A function $\omega : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is open, is called Lipschitz when there exists a constant $C > 0$ such that for all $x, y \in \Omega$,*

$$|\omega(x) - \omega(y)| \le C|x - y|.$$

A special Lipschitz domain is an open subset $\Omega \subset \mathbb{R}^n$ that lies above the graph of some Lipschitz function ω, that is

$$\Omega = \{ (x', x_n) \in \mathbb{R}^n \mid x_n \ge \omega(x') \}.$$

A bounded Lipschitz domain is a bounded, open subset $\Omega \subset \mathbb{R}^n$, where $\partial\Omega$ can be covered by finitely many open balls $B_j \subset \mathbb{R}^n$ such that for each j, possibly after a rotation, $\partial\Omega \cap B_j$ is part of the graph of a Lipschitz function.

A subset Ω of $\mathbb{R}^n$ is simply called Lipschitz, when it is either a special or a bounded Lipschitz domain.

Rychkov's main result can be found in [45, Thm. 4.1(b)] and it is indeed quite remarkable, since before [45] it was not known whether there existed a universal extension operator covering $0 < p < 1$ or $s < 0$ even for Ω being a half-space.

3.2.2 The Trace at The Curved Boundary

Our work [30] also includes a study of the trace at the curved boundary $\partial\Omega \times I$, cf. Section 6.6.3. In this case we assume that $T < \infty$ and that $\Gamma := \partial\Omega$ is compact. Since the co-domain of this trace is a function space over the manifold $\Gamma \times I$ and it turns out to be a Lizorkin–Triebel space, it is necessary first to define Lizorkin–Triebel spaces over such sets.

In [31] we prepare for this by proving that $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is invariant under the map $u \mapsto u \circ \sigma$ for certain diffeomorphisms σ:

Theorem 3.8. *When $\vec{a}, \vec{p}$ satisfy* (3.6) *and σ is a bounded diffeomorphism on $\mathbb{R}^n$ of the form*

$$\sigma(x) = (\sigma'(x_1, \dots, x_{n-1}), x_n), \quad x \in \mathbb{R}^n,$$

then $u \mapsto u \circ \sigma$ is a linear homeomorphism on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

The proof, which can be found in Section 5.4.1, is based on the characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ by kernels of local means, cf. the more general Theorem 2.8 that may be specialised to such kernels.

Several variants of Theorem 3.8 are included in [31]. Some of them concern cylindrical domains and are used to define Lizorkin–Triebel spaces over $\Gamma \times I$. To state this definition, we first equip $\Gamma \times I$ with e.g. the atlas $\mathcal{F} \times \mathcal{N}$, where $\mathcal{F} = \{\kappa\}$ and $\mathcal{N} = \{\eta\}$ are atlases on Γ, respectively on I. Secondly, we consider any partition of unity $1 = \sum_{j,l\in\mathbb{N}} \psi_j \otimes \varphi_l$, where $1 = \sum \psi_j$, $1 = \sum \varphi_l$ are locally finite (e.g. for every $x \in \Gamma$ only finitely many terms are non-trivial) partitions of unity subordinate to $\mathcal{F}$, respectively to $\mathcal{N}$. (We refer to Section 6.4 for a detailed explanation of the notation used in the following.)

Definition 3.9. *Let $\vec{a}, \vec{p}$ satisfy* (3.6) *and $\Omega \subset \mathbb{R}^n$ be a C^∞-domain. The space $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ consists of the $u \in \mathcal{D}'(\Gamma \times I)$ such that*

$$(\psi_j \otimes \varphi_l) \circ (\kappa(j)^{-1} \times \eta(l)^{-1}) u_{\kappa(j)\times\eta(l)} \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_{\kappa(j)} \times \widetilde{I}_{\eta(l)}), \quad j, l \in \mathbb{N}.$$

Giving meaning to the trace γ at $\Gamma \times I$ requires some work, but the idea is to restrict x_n to 0 in local coordinates; we refer to Section 5.4.1 for the details. To ease notation in the following result for γ, we can due to (3.6) think of x_1 as the variable being restricted to 0 instead of x_n:

Theorem 3.10. *Under the same conditions as in Definition 3.9 and when*

$$s > \frac{a_0}{p_0} + \sum_{k=2}^{n+1} a_k \Big(\frac{1}{\min(1, p_k, q)} - 1 \Big),$$

then

$$\gamma : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I) \tag{3.10}$$

is a bounded surjection. Moreover, there exists a bounded right-inverse going the opposite way for all $s \in \mathbb{R}$.

The bar over the co-domain in (3.10) means that the distributions are restrictions of distributions over the infinitely long cylinder $\Gamma \times \mathbb{R}$, cf. (6.76) below.

The construction of a right-inverse relies also in this case on Rychkov's extension operator, but since the co-domain of γ is anisotropic and with mixed norms, it is necessary to modify his construction. Adapting Rychkov's arguments and using results from [32], e.g. the characterisation in Theorem 2.8, we construct in Theorem 6.34 a universal extension operator

$$\mathcal{E}_u : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n). \tag{3.11}$$

However, it is not sufficient only to have an extension operator for $\mathbb{R}^n_+$, since the right-inverse to γ needs to act on distributions over $\Gamma \times I$. Fortunately, using $\mathcal{E}_u$ it is straightforward to construct an operator which extends distributions over $\mathbb{R}^{n-1} \times \,]-\infty, C[\,$, $C \in \mathbb{R}$, to the whole Euclidean space, cf. Corollary 6.35. Applying cut-off functions, these two extension operators are perfectly sufficient.

3.2.3 Further Remarks

The results outlined in Section 3.2.1 and 3.2.2 as well as further results from [30] are applied to a parabolic boundary problem in Chapter 7. This leads to a crystallization of the necessary compatibility properties in order to have a solution in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$. Moreover, it is shown how a fully inhomogeneous boundary value problem can be reduced to one that is only inhomogeneous in the PDE itself.

CHAPTER 4

Characterisation by Local Means of Anisotropic Lizorkin–Triebel Spaces with Mixed Norms

Publication details

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Abstract:
This is a contribution to the theory of Lizorkin–Triebel spaces having mixed Lebesgue norms and quasi-homogeneous smoothness. We discuss their characterisation in terms of general quasi-norms based on convolutions. In particular, this covers the case of local means, in Triebel's terminology. The main step is an extension of some crucial inequalities due to Rychkov to the case with mixed norms.

4.1 Introduction

This paper is devoted to a study of anisotropic Lizorkin-Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with mixed norms, which has grown out of work of the first and third author, cf. [28, 29].

First Sobolev embeddings and completeness of the scale $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ were covered in [28]. As the foundation for this, the Nikol'skiĭ–Plancherel–Polya inequality for sequences of functions in the mixed-norm space $L_{\vec{p}}(\mathbb{R}^n)$ was established in [28] with fairly elementary proofs. Then a detailed trace theory for hyperplanes in $\mathbb{R}^n$ was worked out in [29], e.g. with the novelty that the well-known borderline $s = \frac{1}{p}$ has to be shifted upwards in some cases, because of the mixed norms.

In the present paper we obtain some general characterisations of the space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, that may be specialised to kernels of local means. We have at least two motivations for this. One is that local means have emerged in the last decade as the natural foundation for a discussion of wavelet bases for Sobolev spaces and their generalisations to the Besov and Lizorkin–Triebel scales; cf. works of Triebel [60, Thm. 1.20] and e.g. Vybiral [61, Thm. 2.12], Hansen [19, Thm. 4.3.1].

Secondly, local means will be crucial for the entire strategy in our forthcoming paper [31], in which we establish invariance of $F^{s,\vec{a}}_{\vec{p},q}$ under diffeomorphisms in order to carry over trace results from [29] to spaces over smooth domains. More precisely, because of the anisotropic structure of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces, we consider them over smooth cylindrical sets in Euclidean space in [31] and develop results for traces on the flat and curved parts of the boundary of the cylinder in [30].

To elucidate the importance of the results here and in [30, 31], we recall that $F^{s,\vec{a}}_{\vec{p},q}$-spaces have applications to parabolic differential equations with initial and boundary value conditions: when solutions are sought in a *mixed-norm* Lebesgue space $L_{\vec{p}}$ (e.g. to allow for different properties in the space and time directions), then $F^{s,\vec{a}}_{\vec{p},q}$-spaces are in general *inevitable* for a correct description of non-trivial data on the *curved* boundary. This conclusion was obtained in works of Weidemaier [64–66], who treated several special cases; the reader may consult the introduction of [29] for details.

To give a brief review of the present results, we recall that the norm $\|\cdot|F^{s,\vec{a}}_{\vec{p},q}\|$ of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is defined in a well-known Fourier-analytic way by splitting the frequency space by means of a Littlewood–Paley partition of unity. But to have "complete" freedom, it is natural first of all to work with convolutions $\psi_j * f$ defined from more arbitrary sequences $(\psi_j)_{j\in\mathbb{N}_0}$ of Schwartz functions with dilations $\psi_j = 2^{j|\vec{a}|}\psi(2^{j\vec{a}}\cdot)$ for $j \geq 1$. This requires both the Tauberian conditions that $\widehat{\psi}_0(\xi)$, $\widehat{\psi}(\xi)$ have no zeroes for $|\xi|_{\vec{a}} < 2\varepsilon$ and $\frac{\varepsilon}{2} < |\xi|_{\vec{a}} < \varepsilon$, respectively; and the moment condition that $D^\alpha\widehat{\psi}(0) = 0$ for $|\alpha| \leq M_\psi$.

Secondly, one may work with anisotropic Peetre–Fefferman–Stein maximal functions $\psi^*_{j,\vec{a}}f$, and with these our main result can be formulated as follows:

Theorem. *For $s < (M_\psi + 1)\min(a_1,\dots,a_n)$ and $0 < p_j < \infty$, $0 < q \leq \infty$, the following quasi-norms are equivalent on the space of temperate distributions:*

$$\|f|F^{s,\vec{a}}_{\vec{p},q}\|,\quad \|\{2^{sj}\psi_j * f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\|,\quad \|\{2^{sj}\psi^*_{j,\vec{a}}f\}_{j=0}^\infty|L_{\vec{p}}(\ell_q)\|.$$

Thus $f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if one (hence all) of these expressions are finite.

In the isotropic case, i.e. when $\vec{a} = (1,\dots,1)$ and unmixed L_p-norms are used, the theorem has been known since the important work of Rychkov [44], albeit in another formulation. In our generalisation we follow Rychkov's proof strategy closely, but with some corrections; cf. Remark 4.1 below.

Another particular case is when the functions ψ_0 and ψ have compact support, in which case the convolutions may be interpreted as local means, as observed by Triebel [58]. Thus we develop the mentioned characterisations by local means for the anisotropic $F^{s,\vec{a}}_{\vec{p},q}$-spaces in Theorem 5.2 below, and as far as we know, already this part of their theory is a novelty. As indicated above, it will enter directly into the proofs of our paper [31].

However, it deserves to be mentioned that the arguments in [31] also rely on a stronger estimate than the inequalities underlying the above theorem. In fact we need to consider parameter dependent functions ψ_θ, $\theta \in \Theta$ (an index set), that satisfy the moment conditions in a uniform way. Theorem 4.18 below gives the precise details and our estimate of

$$\|\,\{2^{sj}\sup_{\theta\in\Theta}\psi^*_{\theta,j,\vec{a}}f\}_{j=0}^\infty\,|L_{\vec{p}}(\ell_q)\|. \tag{4.1}$$

Similar quasi-norms were introduced by Triebel in the proof of [58, Prop. 4.3.2] for the purpose of showing diffeomorphism invariance of the isotropic scale $F^s_{p,q}$. However, he only claimed the equivalence of the quasi-norms for f belonging a priori to $F^s_{p,q}$ and details of proof were not given. Since our estimate of (4.1) is valid for arbitrary distributions $f \in \mathcal{S}'$, it should be well motivated that we develop this important tool with a full explanation here.

Remark 4.1. *The fact that the arguments in [44] are incomplete was observed in the Ph.D. thesis of M. Hansen [19, Rem. 3.2.4], where it was exemplified that in general a certain O-condition is unfulfilled; cf. Remark 4.21 below. Another flaw is pointed out here in Remark 4.10. However, to obtain the full generality with arbitrary temperate distributions in Proposition 4.20 below, we have preferred to reinforce the original proofs of Rychkov. Hence we have found it best to aim at a self-contained exposition in this paper.*

Contents. The paper is organized as follows. Section 4.2 reviews our notation and gives a discussion of the anisotropic spaces of Lizorkin–Triebel type with a mixed norm. Section 4.3 presents some maximal inequalities for mixed Lebesgue norms. Quasi-norms defined from general systems of Schwartz functions subjected to moment and Tauberian conditions are estimated in Section 4.4, following works of Rychkov. In Section 4.5 these spaces are characterised by such general norms, and by local means.

4.2 Preliminaries

4.2.1 Notation

Vectors $\vec{p}=(p_1,\dots,p_n)$ with every $p_i\in\,]0,\infty]$ are written $0<\vec{p}\le\infty$, as throughout inequalities for vectors are understood componentwise; likewise for functions, e.g. $\vec{p}!=p_1!\cdots p_n!$.

By $L_{\vec{p}}(\mathbb{R}^n)$ we denote the set of all functions $u:\mathbb{R}^n\to\mathbb{C}$ that are Lebesgue measurable and such that

$$\|\,u\,|\,L_{\vec{p}}(\mathbb{R}^n)\| := \Big(\int_{\mathbb{R}}\Big(\dots\Big(\int_{\mathbb{R}}|u(x_1,\dots,x_n)|^{p_1}\,dx_1\Big)^{\frac{p_2}{p_1}}\dots\Big)^{\frac{p_n}{p_{n-1}}}\,dx_n\Big)^{\frac{1}{p_n}}<\infty,$$

with the modification of using the essential supremum over x_j in case $p_j=\infty$. Equipped with this quasi-norm, $L_{\vec{p}}(\mathbb{R}^n)$ is a quasi-Banach space; it is normed if $\min(p_1,\dots,p_n)\ge 1$.

Furthermore, for $0<q\le\infty$ we shall use the notation $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ for the space of sequences $(u_k)_{k\in\mathbb{N}_0}=\{u_k\}_{k=0}^{\infty}$ of Lebesgue measurable functions fulfilling

$$\|\,\{u_k\}_{k=0}^{\infty}\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\| := \Big\|\Big(\sum_{k=0}^{\infty}|u_k|^q\Big)^{\frac{1}{q}}\Big|L_{\vec{p}}(\mathbb{R}^n)\Big\|<\infty,$$

with supremum over k in case $q=\infty$. For brevity, we write $\|\,u_k\,|L_{\vec{p}}(\ell_q)\|$ instead of $\|\,\{u_k\}_{k=0}^{\infty}\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|$; as customary for $\vec{p}=(p,\dots,p)$, we simplify $L_{\vec{p}}$ to L_p etc. If $\max(p_1,\dots,p_n,q)<\infty$, sequences of C_0^∞-functions are dense in $L_{\vec{p}}(\ell_q)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth, rapidly decreasing functions; it is equipped with the family of seminorms, using $\langle x\rangle^2:=1+|x|^2$,

$$p_M(\varphi):=\sup\big\{\langle x\rangle^M|D^\alpha\varphi(x)|\,\big|\,x\in\mathbb{R}^n,|\alpha|\le M\big\},\quad M\in\mathbb{N}_0, \tag{4.2}$$

whereby $D^\alpha:=(-\mathrm{i}\,\partial_{x_1})^{\alpha_1}\cdots(-\mathrm{i}\,\partial_{x_n})^{\alpha_n}$ for each multi-index $\alpha\in\mathbb{N}_0^n$; or with

$$p_{\alpha,\beta}(\varphi):=\sup_{x\in\mathbb{R}^n}|x^\alpha D^\beta\varphi(x)|,\quad \alpha,\beta\in\mathbb{N}_0^n. \tag{4.3}$$

The Fourier transformation $\mathcal{F}\varphi(\xi)=\widehat{\varphi}(\xi)=\int_{\mathbb{R}^n}e^{-\mathrm{i}\,x\cdot\xi}\varphi(x)\,dx$ for $\varphi\in\mathcal{S}(\mathbb{R}^n)$ extends by duality to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions.

Throughout, generic constants will mainly be denoted by c or C, and in case their dependence on certain parameters is relevant this will be explicitly stated.

4.2.2 Lizorkin-Triebel Spaces with a Mixed Norm

As a motivation for the general mixed-norm Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, we first mention that for $1<\vec{p}<\infty$ a temperate distribution u belongs to a class $F^{s,\vec{a}}_{\vec{p},2}(\mathbb{R}^n)$ having natural numbers $m_j := \frac{s}{a_j}$ for each $j=1,\dots,n$ if and only if u belongs to the mixed-norm Sobolev space $W^{\vec{m},\vec{a}}_{\vec{p}}(\mathbb{R}^n)$, $\vec{m}=(m_1,\dots,m_n)$, defined by

$$\| u \,|L_{\vec{p}}(\mathbb{R}^n)\| + \sum_{i=1}^{n} \Big\| \frac{\partial^{m_i} u}{\partial x_i^{m_i}} \Big| L_{\vec{p}}(\mathbb{R}^n)\Big\| < \infty. \tag{4.4}$$

This expression defines the norm on $W^{\vec{m},\vec{a}}_{\vec{p}}$, which is equivalent to that on $F^{s,\vec{a}}_{\vec{p},2}$.

More generally, mixed-norm Lizorkin–Triebel spaces generalise the fractional Sobolev (Bessel potential) spaces $H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n)$, since for $1<\vec{p}<\infty$, $s\in\mathbb{R}$,

$$u \in H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n) \iff u \in F^{s,\vec{a}}_{\vec{p},2}(\mathbb{R}^n).$$

Here the norms are also equivalent; the former is given by $\| \mathcal{F}^{-1}(\langle\xi\rangle_{\vec{a}}^{-s}\widehat{u}(\xi))\,|L_{\vec{p}}\|$, whereby $\langle\xi\rangle_{\vec{a}}$ is an anisotropic version of $\langle\xi\rangle$ compatible with $\vec{a}$, cf. the following.

To account for the Fourier-analytic definition of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, we first recall the anisotropic structure used for derivatives. Each coordinate x_j in $\mathbb{R}^n$ is given a weight $a_j \geq 1$, collected in $\vec{a}=(a_1,\dots,a_n)$. Based on the quasi-homogeneous dilation $t^{\vec{a}}x := (t^{a_1}x_1,\dots,t^{a_n}x_n)$ for $t\geq 0$, and $t^{s\vec{a}}x := (t^s)^{\vec{a}}x$ for $s\in\mathbb{R}$, in particular $t^{-\vec{a}}x=(t^{-1})^{\vec{a}}x$, the anisotropic distance function $|x|_{\vec{a}}$ is introduced for $x\neq 0$ as the unique $t>0$ such that $t^{-\vec{a}}x\in S^{n-1}$ (with $|0|_{\vec{a}}=0$); i.e.

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1.$$

For the reader's convenience we recall that $|\cdot|_{\vec{a}}$ is C^∞ on $\mathbb{R}^n\setminus\{0\}$ by the Implicit Function Theorem. The formula $|t^{\vec{a}}x|_{\vec{a}} = t|x|_{\vec{a}}$ is seen directly, and this implies the triangle inequality,

$$|x+y|_{\vec{a}} \leq |x|_{\vec{a}} + |y|_{\vec{a}}. \tag{4.5}$$

The relation to e.g. the Euclidean norm $|x|$ can be deduced from

$$\max\Big(|x_1|^{\frac{1}{a_1}},\dots,|x_n|^{\frac{1}{a_n}}\Big) \leq |x|_{\vec{a}} \leq |x_1|^{\frac{1}{a_1}} + \cdots + |x_n|^{\frac{1}{a_n}}. \tag{4.6}$$

For the above-mentioned weight function, one can e.g. let $\langle\xi\rangle_{\vec{a}} = |(\xi,1)|_{(\vec{a},1)}$, using the anisotropic distance given by $(\vec{a},1)$ on $\mathbb{R}^{n+1}$; analogously to $\langle\xi\rangle$ in the isotropic case.

We pick (for convenience) a fixed Littlewood–Paley decomposition, written $1=\sum_{j=0}^\infty \Phi_j(\xi)$, in the anisotropic setting as follows: Let $\psi\in C_0^\infty$ be a function such that $0\leq\psi(\xi)\leq 1$ for all ξ, $\psi(\xi)=1$ if $|\xi|_a\leq 1$, and $\psi(\xi)=0$ if $|\xi|_a\geq\frac{3}{2}$. Then we set $\Phi=\psi-\psi(2^{\vec{a}}\cdot)$ and define

$$\Phi_0(\xi)=\psi(\xi), \quad \Phi_j(\xi)=\Phi(2^{-j\vec{a}}\xi),\ j=1,2,\dots \tag{4.7}$$

Definition 4.2. *The Lizorkin–Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, where $0 < \vec{p} < \infty$ is a vector of integral exponents, $s \in \mathbb{R}$ a smoothness index, and $0 < q \le \infty$ a sum exponent, is the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\| u \,|\, F^{s,\vec{a}}_{\vec{p},q} \| := \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \left| \mathcal{F}^{-1} \left(\Phi_j(\xi) \mathcal{F}u(\xi) \right)(\cdot) \right|^q \Big)^{\frac{1}{q}} \Big| L_{\vec{p}}(\mathbb{R}^n) \Big\| < \infty.$$

For simplicity, we omit $\vec{a}$ when $\vec{a} = (1, \dots, 1)$ and shall often set

$$u_j(x) = \mathcal{F}^{-1} \left(\Phi_j(\xi) \, \mathcal{F}u(\xi) \right)(x), \quad x \in \mathbb{R}^n, \ j \in \mathbb{N}_0.$$

Occasionally, we need to consider Besov spaces, which are defined similarly:

Definition 4.3. *For $0 < \vec{p} \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ the Besov space $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\| u \,|\, B^{s,\vec{a}}_{\vec{p},q} \| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \| u_j \,| L_{\vec{p}}(\mathbb{R}^n) \|^q \Big)^{\frac{1}{q}} < \infty.$$

Remark 4.4. *The Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}$ have a long history, as they give back e.g. the mixed-norm Sobolev spaces $W^{\vec{m}}_{\vec{p}}$, cf.* (4.4)*. Anisotropic Sobolev (Bessel potential) spaces $H^{s,\vec{a}}_{p}$ with $1 < p < \infty$ (partly for $s > 0$) have been investigated in the monographs of Nikol'skiĭ [41] and Besov, Il'in and Nikol'skiĭ [4]; here the point of departure was a definition based on derivatives and differences. In the second edition [5] also Lizorkin–Triebel spaces with mixed norms were treated in Ch. 6.29–30. For characterisation of $F^{s,\vec{a}}_{p,q}$ by differences we refer also to Yamazaki [68, Thm. 4.1] and Seeger [51].*

The $F^{s,\vec{a}}_{\vec{p},q}$-spaces were considered for $n = 2$ by Schmeisser and Triebel [50], who used the Fourier-analytic characterisation, which we prefer for its efficacy what concerns application of powerful tools from Fourier analysis and distribution theory. (The definition of the anisotropy in terms of $|\cdot|_{\vec{a}}$ is a well-known procedure going back to the 1960s; historical remarks and some basic properties of $|\cdot|_{\vec{a}}$ can be found in e.g. [67].)

For later use we recall some properties of these classes.

Lemma 4.5 ([28, 29])**.** *Each $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is a quasi-Banach space, which is normed if $\vec{p}, q \ge 1$. More precisely, for $u, v \in F^{s,\vec{a}}_{\vec{p},q}$ and $d := \min(1, p_1, \dots, p_n, q)$,*

$$\| u + v \,|F^{s,\vec{a}}_{\vec{p},q} \|^d \le \| u \,|F^{s,\vec{a}}_{\vec{p},q} \|^d + \| v \,|F^{s,\vec{a}}_{\vec{p},q} \|^d.$$

Furthermore, there are continuous embeddings

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

where $\mathcal{S}$ is dense in $F^{s,\vec{a}}_{\vec{p},q}$ for $q < \infty$. Also, the classes $F^{s,\vec{a}}_{\vec{p},q}$ do not depend on the chosen anisotropic decomposition of unity (up to equivalent quasi-norms).

Lemma 4.6 ([29])**.** *For $\lambda > 0$ so large that $\lambda\vec{a} \geq 1$, the space $F^{s,\vec{a}}_{\vec{p},q}$ coincides with $F^{\lambda s,\lambda\vec{a}}_{\vec{p},q}$ and the corresponding quasi-norms are equivalent.*

The lemma suggests to introduce a normalisation for the vector $\vec{a}$, and often one has fixed the value of $|\vec{a}|$ in the literature. In this paper we just adopt the flexible framework with $\vec{a} \geq 1$, though.

Remark 4.7. *In Lemma 4.6 the inequalities $\vec{a} \geq 1$ and $\lambda\vec{a} \geq 1$ are redundant. In fact one can define $F^{s,\vec{a}}_{\vec{p},q}$ for arbitrary $\vec{a} > 0$, as in [29]. This gives another set-up on $\mathbb{R}^n$, where* (4.5)*, and hence* (4.6)*, has to be changed, for then*

$$|x + y|^d_{\vec{a}} \leq |x|^d_{\vec{a}} + |y|^d_{\vec{a}}, \quad d := \min(1, a_1, \dots, a_n). \tag{4.8}$$

The basic results on the $F^{s,\vec{a}}_{\vec{p},q}$-scale can then be derived similarly for $\vec{a} > 0$; only a few constants need to be slightly changed because of (4.8)*. Thus one finds e.g. Lemma 4.6 for all $\lambda > 0$, cf. the end of Section 3 in [29] (the details in [29, Sec. 3] only cover $\vec{a} \geq 1$, but are extended to all $\vec{a} > 0$ as just indicated; in fact $\rho(x, y) = |x - y|_{\vec{a}}$ is then a quasi-distance, a framework widely used by e.g. Stein [54]). However, in view of this lemma, it is simplest henceforth just to assume that $F^{s,\vec{a}}_{\vec{p},q}$ is defined in terms of an anisotropy $\vec{a} \geq 1$; which has been done throughout in the present paper.*

4.2.3 Summation Lemmas

For later reference we give two minor results.

Lemma 4.8. *When $(g_j)_{j\in\mathbb{N}_0}$ is a sequence of nonnegative measurable functions on $\mathbb{R}^n$ and $\delta > 0$, then $G_j(x) := \sum_{k=0}^{\infty} 2^{-\delta|j-k|} g_k(x)$ fulfils for $0 < \vec{p} < \infty$, $0 < q \leq \infty$ that*

$$\| G_j \,|L_{\vec{p}}(\ell_q)\| \leq C_{\delta,q} \| g_j \,|L_{\vec{p}}(\ell_q)\|,$$

whereby the constant is $C_{\delta,q} = (\sum_{k\in\mathbb{Z}} 2^{-\delta|k|\widetilde{q}})^{1/\widetilde{q}}$ for $\widetilde{q} = \min(1, q)$.

Like for the unmixed case in [44, Lem. 2], the above lemma is obtained by pointwise application of Minkowski's inequality to a convolution in $\ell_q(\mathbb{Z})$.

Lemma 4.9. *Let $(b_j)_{j\in\mathbb{N}_0}$ and $(d_j)_{j\in\mathbb{N}_0}$ be two sequences in $[0, \infty]$ and $0 < r \leq 1$. If for some $j_0 \geq 0$ there exist real numbers $C, N_0 > 0$ such that*

$$d_j \leq C 2^{jN_0} \quad \text{for } j \geq j_0, \tag{4.9}$$

and if for every $N > 0$ there exists a real number C_N such that

$$d_j \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r} \quad \text{for } j \geq j_0, \tag{4.10}$$

then the same constants C_N, $N > 0$, fulfil that

$$d_j^r \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k \quad \text{for } j \geq j_0. \tag{4.11}$$

Proof. With $D_{j,N} := \sup_{k\ge j} 2^{(j-k)N} d_k$, (4.10) gives for $j \ge j_0$, $N > 0$,

$$D_{j,N} \le \sup_{k\ge j} C_N \sum_{l\ge k} 2^{(j-l)N} b_l d_l^{1-r} \le C_N \Big(\sum_{l\ge j} 2^{(j-l)Nr} b_l \Big) D_{j,N}^{1-r}. \tag{4.12}$$

Clearly $D_{j_1,N} = 0$ implies $d_j = 0$ for $j \ge j_1$, so (4.11) is trivial for such j. We thus only need to consider the $D_{j,N} > 0$. Now (4.9) yields that $D_{j,N} < \infty$ for all $j \ge j_0$ when $N \ge N_0$, so then (4.11) follows from (4.12) by division by $D_{j,N}^{1-r}$.

Given any $N \in]0, N_0[$, we may in the just proved cases of (4.11) decrease N_0 to N, which gives a version of (4.11) with N in the exponent and the constant C_{N_0}. Analogously to (4.12), one therefore finds from the definition of $D_{j,N}$ that $D_{j,N} \le C_{N_0}^{\frac{1}{r}} (\sum_{l\ge j} 2^{(j-l)Nr} b_l)^{\frac{1}{r}}$ for $j \ge j_0$. Here the right-hand side may be assumed finite (as else (4.11) is trivial for this N), whence we may proceed as before by division in (4.12). □

Remark 4.10. *Lemma 4.9 was essentially crystallised by Rychkov [44, Lem. 3], albeit with three unnecessary assumptions: $d_j < \infty$ (a consequence of* (4.9)*), that $b_j, d_j > 0$ and that $j_0 = 0$. For our proof of Proposition 4.20 below, it is essential to consider $j_0 > 0$, and it would be cumbersome there to reduce to strict positivity of b_j, d_j. In [44] no justification was given for this strictness in the application of [44, Lem. 3], but this is remedied by Lemma 4.9 above.*

4.3 Some Maximal Inequalities

In this section we obtain some maximal inequalities in the mixed-norm set-up. This part of the theory of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces is interesting in its own right, and also important for the authors' work [31]. Moreover, the methods are similar to those adopted in the set-up in Section 4.4 below, but are rather cleaner here.

For distributions u that for some $R > 0$ and $j \in \mathbb{N}$ satisfy

$$\operatorname{supp} \widehat{u} \subset \big\{ \xi \in \mathbb{R}^n \bigm| |\xi_k| \le R\, 2^{ja_k},\ \ k = 1, \dots, n \big\} \tag{4.13}$$

the Peetre–Fefferman–Stein maximal function $u^*(x)$ is given by

$$u^*(x) = \sup_{y\in\mathbb{R}^n} \frac{|u(y)|}{\prod_{l=1}^n (1 + R\, 2^{ja_l}|x_l - y_l|)^{r_l}}, \quad \vec{r} > 0. \tag{4.14}$$

It obviously fulfils

$$|u(x)| \le u^*(x) \le \| u \,| L_\infty \|, \quad x \in \mathbb{R}^n.$$

When u in addition is in $L_{\vec{p}}$, the Nikol'skiĭ–Plancherel–Polya inequality for mixed norms, cf. [28, Prop. 4], gives the finiteness of the right-hand side, hence u^* is finite everywhere. Thus, analogously to [27, Sec. 2], the maximal function is continuous.

To prepare for the theorem below, we first show the following pointwise estimate of $u^*(x)$ by combining the proof ingredients from [27, Prop. 2.2], which the reader may consult for more details. Now their order is crucial:

Proposition 4.11. *When* $0 < \vec{q}, \vec{r} \le \infty$ *then there is a constant* $c_{\vec{q},\vec{r}}$ *such that every* $u \in \mathcal{S}'$ *fulfilling* (4.13) *also satisfies*

$$u^*(x) \le c_{\vec{q},\vec{r}} \left\| \frac{u(x - R^{-1}\,2^{-j\vec{a}}z)}{\prod_{l=1}^n (1+|z_l|)^{r_l}} \,\middle|\, L_{\vec{q}}\,(\mathbb{R}^n_z) \right\| \quad \textit{for } x \in \mathbb{R}^n. \tag{4.15}$$

Proof. Taking $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\psi} \equiv 1$ on $[-1,1] \times \cdots \times [-1,1]$ and such that $\operatorname{supp}\widehat{\psi} \subset [-2,2] \times \cdots \times [-2,2]$, we have $u = \mathcal{F}^{-1}(\widehat{\psi}(R^{-1}2^{-j\vec{a}}\cdot)) * u$, which may be written with an integral since u is C^∞ with polynomial growth,

$$u(y) = \int \cdots \int R^n\, 2^{j|\vec{a}|}\, \psi(R\,2^{j\vec{a}}(y-z))\, u(z)\, dz_1 \cdots dz_n. \tag{4.16}$$

Now $\vec{q} = (q_<, q_\ge)$ is split into two groups $q_<$ and $q_\ge$ according to whether $q_k < 1$ or $q_k \ge 1$ holds. The groups may be interlaced, but for simplicity this is ignored in the notation; the important thing is to treat the two groups separately.

First (4.16) is estimated by the norm of $L_1(\mathbb{R}^n)$, which then is controlled in terms of the norm of $L_{(q_<,1_\ge)}$, whereby interlacing of the groups $q_<$ and $1_\ge$ is unimportant: for fixed y, the spectrum of the integrand in (4.16) is contained in $[-3R\,2^{ja_1}, 3R\,2^{ja_1}] \times \cdots \times [-3R\,2^{ja_n}, 3R\,2^{ja_n}]$, so the Nikol'skiĭ–Plancherel–Polya inequality for mixed norms applies, cf. [28, Prop. 4], which for $q_k < 1$ gives an estimate by the norms of L_{q_k} with respect to z_k; that is,

$$|u(y)| \le c \prod_{q_k<1} (3R\,2^{ja_k})^{\frac{1}{q_k}-1} \left\| R^n 2^{j|\vec{a}|} \psi(R2^{j\vec{a}}(y-\cdot))\, u \,\middle|\, L_{(q_<,1_\ge)} \right\|.$$

(The integration order in this norm is as stated in (4.16).)

Secondly, using Hölder's inequality in the variables where $q_k \ge 1$, and gathering their dual exponents q_k^* in $(q_\ge)^*$, gives for $x \in \mathbb{R}^n$,

$$\begin{aligned} \frac{|u(y)|}{\prod_l (1+R2^{ja_l}|x_l-y_l|)^{r_l}} \le c \prod_{q_k<1} (3R2^{ja_k})^{\frac{1}{q_k}-1} &\left\| \frac{R^n 2^{j|\vec{a}|} u(z)}{\prod_l (1+R2^{ja_l}|x_l-z_l|)^{r_l}} \,\middle|\, L_{\vec{q}} \right\| \\ \times &\left\| \prod_l (1+R2^{ja_l}|y_l-z_l|)^{r_l} \psi(R2^{j\vec{a}}(y-z)) \,\middle|\, L_{(\infty_<,(q_\ge)^*)} \right\|. \end{aligned}$$

Since $\psi \in \mathcal{S}$, a change of coordinates $z_k \mapsto R^{-1}\,2^{-ja_k} z_k$ yields (4.15) with the constant $c_{\vec{q},\vec{r}} = c \prod_{q_k<1} 3^{\frac{1}{q_k}-1} \| \prod_{l=1}^n (1+|z_l|)^{r_l} \psi \,|\, L_{(\infty_<,(q_\ge)^*)} \| < \infty$. □

We now obtain an elementary proof of the *mixed-norm* boundedness of u^*, by adapting the proof of the isotropic L_p-result in [27, Thm. 2.1]:

Theorem 4.12. *Let* $0 < \vec{p} \le \infty$ *and suppose*

$$r_l > \frac{1}{\min(p_1, \dots, p_l)}, \quad l = 1, \dots, n. \tag{4.17}$$

Then there exists a constant c *such that*

$$\| u^* \,|L_{\vec{p}}\| \le c \,\| u \,|L_{\vec{p}}\|$$

holds for all $u \in L_{\vec{p}} \cap \mathcal{S}'$ *satisfying the spectral condition* (4.13).

Proof. We use (4.15) with $q_k = \min(p_1, \dots, p_k)$ for $k = 1, \dots, n$ and calculate the L_{p_j}-norms successively on both sides. Since $p_j \geq q_k$ for all $k \geq j$, we may apply the generalised Minkowski inequality $n - (j-1)$ times, as well as the translation invariance of $dx_1, \dots, dx_n$, which gives

$$\| u^* \,|L_{\vec{p}}\| \leq c_{\vec{q},\vec{r}} \Big(\prod_{l=1}^{n} \|(1+|z_l|)^{-r_l} \,|L_{q_l}\| \Big) \| u \,|L_{\vec{p}}\|.$$

Here (4.17) yields the finiteness of the L_{q_l}-norms. □

The following result is convenient for certain convolution estimates. Since the embedding $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow C^0(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ holds for $s > \vec{a} \cdot \frac{1}{\vec{p}}$, or for $s = \vec{a} \cdot \frac{1}{\vec{p}}$ if $q \leq 1$, it is a result pertaining to continuous functions.

Corollary 4.13. *If $C > 0$ and $\vec{r}$ fulfils (4.17), $d = \min(1, p_1, \dots, p_n)$ yields*

$$\Big\| \sup_{|x-y|<C} |u(y)| \,\Big|L_{\vec{p}}(\mathbb{R}^n_x)\Big\| \leq c \,\| u \,|B^{s,\vec{a}}_{\vec{p},d}\| \quad \textit{for } s = \vec{a}\cdot\vec{r}.$$

Proof. Since $\| \cdot \,|L_{\vec{p}}\|^d$ is subadditive, simple arguments yield

$$\begin{aligned}\Big\| \sup_{|x-y|<C} |u(y)| \,\Big|L_{\vec{p}}(\mathbb{R}^n_x)\Big\|^d &\leq \Big\| \sup_{|x-y|<C} \sum_{j=0}^{\infty} |u_j(y)| \,\Big|\, L_{\vec{p}}(\mathbb{R}^n_x)\Big\|^d \\ &\leq \sum_{j=0}^{\infty} \prod_{\ell=1}^{n} (1 + C\,2^{ja_\ell})^{d\,r_\ell} \| u_j^* \,|L_{\vec{p}}\|^d.\end{aligned}$$

Since $\prod_{\ell=1}^n (1 + C\,2^{ja_\ell})^{d\,r_\ell} \leq (1+C)^{d\,|\vec{r}|}\, 2^{jd\,\vec{a}\cdot\vec{r}}$, the right-hand side is seen to be less than $c \,\| u \,|\, B^{s,\vec{a}}_{\vec{p},d}\|^d$ for $s = \vec{a}\cdot\vec{r}$ by application of Theorem 4.12. □

Remark 4.14. *In [31] Corollary 4.13 enters our estimates for certain $u \in F^{s,\vec{a}}_{\vec{p},q}$ with $\sum_{\ell=1}^n \frac{a_\ell}{\min(p_1,\dots,p_\ell)} < s$. Then one can pick $\vec{r}$ satisfying (4.17) and such that $\vec{a}\cdot\vec{r} < s$, hence elementary embeddings yield*

$$\Big\| \sup_{|x-y|<C} |u(y)| \,\big|L_{\vec{p}}(\mathbb{R}^n_x)\Big\| \leq c \,\| u \,|F^{s,\vec{a}}_{\vec{p},q}\|.$$

4.4 Rychkov's Inequalities

In the systematic theory of the $F^{s,\vec{a}}_{\vec{p},q}$-spaces, it is of course important to dispense from the requirement in Definition 4.2 that the Schwartz functions Φ_j have compact support. In so doing, we shall largely follow Rychkov's treatment of the isotropic case [44].

In the following $\vec{a} = (a_1, \dots, a_n)$ is a fixed anisotropy with $\vec{a} \geq 1$; we set

$$\underline{a} = \min(a_1, \dots, a_n).$$

Throughout this section we consider $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ that fulfil Tauberian conditions in terms of some $\varepsilon > 0$ and/or a moment condition of order M_ψ,

$$|\mathcal{F}\psi_0(\xi)| > 0 \quad \text{on } \{\xi \mid |\xi|_{\vec{a}} < 2\varepsilon\}, \tag{4.18}$$

$$|\mathcal{F}\psi(\xi)| > 0 \quad \text{on } \Big\{\xi \,\Big|\, \frac{\varepsilon}{2} < |\xi|_{\vec{a}} < 2\varepsilon\Big\}, \tag{4.19}$$

$$D^\alpha(\mathcal{F}\psi)(0) = 0 \quad \text{for } |\alpha| \le M_\psi. \tag{4.20}$$

Hereby $M_\psi \in \mathbb{N}_0$, or we take $M_\psi = -1$ when the condition (4.20) is void. Note that if (4.18) is verified for the Euclidean distance, it holds true also in the anisotropic case, perhaps with a different ε; cf. (4.6).

In this section we also change notation by setting

$$\varphi_j(x) = 2^{j|\vec{a}|}\varphi(2^{j\vec{a}}x), \quad \varphi \in \mathcal{S},\ j \in \mathbb{N}. \tag{4.21}$$

For ψ_0 this gives rise to the sequence $\psi_{0,j}(x) := 2^{j|\vec{a}|}\psi_0(2^{j\vec{a}}x)$, but we shall mainly deal with $(\psi_j)_{j\in\mathbb{N}_0}$ that mixes ψ_0 and ψ. Note that $\psi_0 = \psi_{0,0}$.

To elucidate the Tauberian conditions, we recall in the lemma below a well-known fact on Calderón's reproducing formula:

$$u = \sum_{j=0}^\infty \lambda_j * \psi_j * u \quad \text{for } u \in \mathcal{S}'(\mathbb{R}^n). \tag{4.22}$$

Lemma 4.15. *When $\psi_0, \psi \in \mathcal{S}$ fulfil the Tauberian conditions* (4.18), (4.19) *there exist $\lambda_0, \lambda \in \mathcal{S}$ fulfilling* (4.22) *for every $u \in \mathcal{S}'$. Moreover, it can be arranged that $\widehat{\lambda}_0$ and $\widehat{\lambda}$ are supported by the sets in* (4.18), *respectively* (4.19).

Proof. By Fourier transformation (4.22) is carried over to

$$\mathcal{F}\lambda_0(\xi)\,\mathcal{F}\psi_0(\xi) + \sum_{j=1}^\infty \mathcal{F}\lambda(2^{-j\vec{a}}\xi)\,\mathcal{F}\psi(2^{-j\vec{a}}\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{4.23}$$

Finding λ_0, λ reduces to a Littlewood–Paley construction: taking $h \in C_0^\infty$ such that $0 \le h \le 1$ on $\mathbb{R}^n$, $\operatorname{supp} h \subset \{\xi \mid |\xi|_{\vec{a}} < 2\varepsilon\}$ and $h(\xi) = 1$ if $|\xi|_{\vec{a}} \le \frac{3}{2}\varepsilon$, then $\widehat{\lambda_0} := h\,\widehat{\psi_0}^{\,-1}$ and $\widehat{\lambda} := \big(h - h(2^{\vec{a}}\cdot)\big)\widehat{\psi}^{\,-1}$ fulfil (4.23) and the support inclusions.□

A general reference to Calderon's formula could be [14, Ch. 6]. More refined versions have been introduced by Rychkov [46].

To comment on the moment condition, we use for $M \ge -1$ the subspace

$$\mathcal{S}_M := \Big\{\mu \in \mathcal{S}(\mathbb{R}^n) \,\Big|\, D^\alpha(\mathcal{F}\mu)(0) = 0 \quad \text{for all } |\alpha| \le M\Big\}.$$

It is recalled that in addition to the $p_{\alpha,\beta}$ in (4.3) also the following family of seminorms induces the topology on $\mathcal{S}$:

$$q_{N,\alpha}(\psi) := \int_{\mathbb{R}^n} \langle x\rangle^N |D^\alpha\psi(x)|\,dx, \quad N \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n.$$

This is convenient for the fact that moment conditions, also in case of the anisotropic dilation $t^{\vec{a}}$, induce a rate of convergence to 0 in $\mathcal{S}$:

Lemma 4.16. *For $\alpha, \beta \in \mathbb{N}_0^n$ there is an estimate for $0 < t \le 1$, $\nu \in \mathcal{S}$, $\mu \in \mathcal{S}_M$,*

$$p_{\alpha,\beta}\big(t^{-|\vec{a}|}\mu(t^{-\vec{a}}\,\cdot) * \nu\big) \le C_\alpha\, t^{(M+1)\underline{a}} \max p_{0,\zeta}(\widehat{\mu}) \cdot q_{M+1,\gamma}(\widehat{D^\beta \nu}),$$

where the maximum is over all ζ with $|\zeta| \le M+1$ or $\zeta \le \alpha$; and over $\gamma \le \alpha$.

Proof. The continuity of $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}} : L_1 \to L_\infty$ and Leibniz' rule give that

$$\begin{aligned} p_{\alpha,\beta}\big(t^{-|\vec{a}|}\mu(t^{-\vec{a}}\,\cdot) * \nu\big) &= \sup_{z\in\mathbb{R}^n} \left| \mathcal{F}^{-1}\left(D_\xi^\alpha (t^{-|\vec{a}|}\widehat{\mu(t^{-\vec{a}}\,\cdot)}\,\widehat{D^\beta \nu})\right)(z)\right| \\ &\le \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \int t^{a\cdot(\alpha-\gamma)} |D^{\alpha-\gamma}\widehat{\mu}(t^{\vec{a}}\xi)|\, |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi. \end{aligned} \tag{4.24}$$

For $|\alpha-\gamma| \le M$ the integral is estimated using a Taylor expansion of order $N := M - |\alpha-\gamma|$. All terms except the remainder vanish, because μ has vanishing moments up to order M. The integral is therefore bounded by

$$\begin{aligned} &\int t^{\vec{a}\cdot(\alpha-\gamma)} \Big| \sum_{|\zeta|=N+1} \frac{N+1}{\zeta!} (t^{\vec{a}}\xi)^\zeta \int_0^1 (1-\theta)^N \partial_\xi^\zeta D_\xi^{\alpha-\gamma}\widehat{\mu}(\theta t^{\vec{a}}\xi) d\theta \Big|\, |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi \\ &\le t^{(M+1)\underline{a}} \max_{|\zeta|\le M+1} \|D^\zeta \widehat{\mu} \mid L_\infty\| \int |\xi|^{N+1} |D^\gamma \widehat{D^\beta \nu}(\xi)|\, d\xi \\ &\le t^{(M+1)\underline{a}} \max_{|\zeta|\le M+1} p_{0,\zeta}(\widehat{\mu})\, q_{M+1,\gamma}(\widehat{D^\beta \nu}). \end{aligned}$$

For $|\alpha-\gamma| \ge M+1$ the integral in (4.24) is easily seen to be estimated by

$$t^{(M+1)\underline{a}} \max_{\zeta\le\alpha} p_{0,\zeta}(\widehat{\mu})\, q_{0,\gamma}(\widehat{D^\beta \nu}).$$

The claim is obtained by taking the largest of the bounds. □

4.4.1 Comparison of Norms

For any $\vec{r} = (r_1, \dots, r_n) > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we deal in this section with the non-linear maximal operators of Peetre–Fefferman–Stein type induced by $(\psi_j)_{j\in\mathbb{N}_0}$,

$$\psi_j^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{\prod_{\ell=1}^n (1 + 2^{ja_\ell}|x_\ell - y_\ell|)^{r_\ell}}, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0. \tag{4.25}$$

For simplicity their dependence on $\vec{a}$, $\vec{r}$ is omitted. (Compared to (4.14), no R is in the denominator here, as $\psi_j * f$ need not have compact spectrum.)

To give the background, we recall an important technical result of Rychkov:

Proposition 4.17 ([44, (8')]). *Let $\psi_0, \psi \in \mathcal{S}$ be given such that* (4.20) *holds, while $\varphi_0, \varphi \in \mathcal{S}$ fulfil the Tauberian conditions* (4.18), (4.19) *in terms of some $\varepsilon' > 0$. When $0 < p < \infty$, $0 < q \le \infty$ and $s < (M_\psi + 1)\,\underline{a}$ there exists a constant $c > 0$ such that for $f \in \mathcal{S}'$,*

$$\|2^{sj}\psi_j^* f \,|L_p(\ell_q)\| \le c\, \|2^{sj}\varphi_j^* f \,|L_p(\ell_q)\|.$$

We shall extend this to a mixed-norm version, which even covers parameter-dependent families of the spectral cut-off functions; this will be crucial for our results in [31]. So if Θ denotes an index set and $\psi_{\theta,0},\psi_\theta\in\mathcal{S}(\mathbb{R}^n)$, $\theta\in\Theta$, we set $\psi_{\theta,j}(x)=2^{j|\vec{a}|}\psi_\theta(2^{j\vec{a}}x)$ for $j\in\mathbb{N}$. Not surprisingly we need to assume that the ψ_θ fulfil the same moment condition, i.e. uniformly with respect to θ:

Theorem 4.18. *Let $\psi_{\theta,0},\psi_\theta\in\mathcal{S}(\mathbb{R}^n)$ be given such that* (4.20) *holds for some M_{ψ_θ} independent of $\theta\in\Theta$, while $\varphi_0,\varphi\in\mathcal{S}(\mathbb{R}^n)$ fulfil* (4.18), (4.19) *in terms of an $\varepsilon'>0$. Also let $0<\vec{p}<\infty$, $0<q\le\infty$ and $s<(M_{\psi_\theta}+1)\underline{a}$. For a given $\vec{r}$ in* (4.25) *and an integer $M\ge-1$ chosen so large that $(M+1)\underline{a}+s>2\vec{a}\cdot\vec{r}$, we assume that*

$$\begin{aligned}
A &:= \sup_{\theta\in\Theta}\max\|\,D^\alpha\mathcal{F}\psi_\theta\,|L_\infty\| &&<\infty,\\
B &:= \sup_{\theta\in\Theta}\max\|\,(1+|\xi|)^{M+1}\,D^\gamma\mathcal{F}\psi_\theta(\xi)\,|L_1\| &&<\infty,\\
C &:= \sup_{\theta\in\Theta}\max\|\,D^\alpha\mathcal{F}\psi_{\theta,0}\,|L_\infty\| &&<\infty,\\
D &:= \sup_{\theta\in\Theta}\max\|\,(1+|\xi|)^{M+1}\,D^\gamma\mathcal{F}\psi_{\theta,0}(\xi)\,|L_1\| &&<\infty,
\end{aligned}$$

where the maxima are over all α with $|\alpha|\le M_{\psi_\theta}+1$ or $\alpha\le\lceil\vec{r}+2\rceil$, respectively $\gamma\le\lceil\vec{r}+2\rceil$. Then there exists a constant $c>0$ such that for $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$\|2^{sj}\sup_{\theta\in\Theta}\psi^*_{\theta,j}f\,|L_{\vec{p}}(\ell_q)\|\le c(A+B+C+D)\,\|2^{sj}\varphi^*_jf\,|L_{\vec{p}}(\ell_q)\|. \tag{4.26}$$

Hereby $\lceil t\rceil$ denotes the smallest integer $k\ge t$, and $\lceil\vec{r}\rceil:=(\lceil r_1\rceil,\dots,\lceil r_n\rceil)$.

In the proof of the estimate (4.26) we choose $\lambda_0,\lambda\in\mathcal{S}(\mathbb{R}^n)$ by applying Lemma 4.15 to the given $\varphi_0,\varphi\in\mathcal{S}(\mathbb{R}^n)$. Following [44], we then consider the auxiliary integrals

$$I_{j,k}:=\int|\psi_{\theta,j}*\lambda_k(z)|\prod_{l=1}^n(1+2^{ka_l}\,|z_l|)^{r_l}\,dz\,,\quad j,k\in\mathbb{N}_0. \tag{4.27}$$

The integrand may be estimated using that $\psi_{\theta,j}*\lambda_k(z)=2^{k|\vec{a}|}\psi_{\theta,j-k}*\lambda(2^{k\vec{a}}z)$, so the Binomial Theorem and Lemma 4.16 with $\beta=0$, $t^{-1}=2^{j-k}\ge1$ yield

$$\begin{aligned}
|\psi_{\theta,j}*\lambda_k(z)|&\prod_{l=1}^n(1+|2^{ka_l}z_l|)^{r_l}\\
&\le 2^{k|\vec{a}|}\sum_{\alpha\le\lceil\vec{r}\rceil}\binom{\lceil\vec{r}\rceil}{\alpha}p_{\alpha,0}(\psi_{\theta,j-k}*\lambda)\\
&\le C_{\lceil\vec{r}\rceil}2^{(k-j)(M_{\psi_\theta}+1)\underline{a}+k|\vec{a}|}\max{}'\,p_{0,\zeta}(\widehat{\psi_\theta})\cdot q_{M_{\psi_\theta}+1,\gamma}(\widehat{\lambda}),
\end{aligned} \tag{4.28}$$

where $\max'$ denotes a maximum over finitely many multi-indices, in this case over ζ fulfilling $|\zeta|\le M_{\psi_\theta}+1$ or $\zeta\le\lceil\vec{r}\rceil$, respectively $\gamma\le\vec{r}$.

Lemma 4.19. *For any integer $M \geq -1$ there exists a constant $c = c_{M,M_\psi,\vec{r},\lambda_0,\lambda}$ such that for $k, j \in \mathbb{N}_0$,*

$$I_{j,k} \leq c\,(A+B+C+D) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)\underline{a}} & \text{for } k \leq j, \\ 2^{-(k-j)((M+1)\underline{a}-\vec{a}\cdot\vec{r})} & \text{for } j \leq k, \end{cases}$$

when $\psi_{\theta,0}, \psi_\theta \in \mathcal{S}$ and the ψ_θ fulfil (4.20) *for some M_{ψ_θ} independent of $\theta \in \Theta$.*

Proof. First we consider the case $j \geq k \geq 1$, where (4.28) yields

$$\begin{aligned} I_{j,k} &\leq \sup_{z\in\mathbb{R}^n} |\psi_{\theta,j} * \lambda_k(z)| \prod_{l=1}^n (1+2^{ka_l}|z_l|)^{r_l+2} \int \prod_{l=1}^n 2^{-ka_l}(1+|x_l|)^{-2}\,dx \\ &\leq C_{\vec{r}}\, 2^{(k-j)(M_{\psi_\theta}+1)\underline{a}} \max{}' \,\| D^\zeta \widehat{\psi_\theta} \mid L_\infty \| \cdot q_{M_{\psi_\theta}+1,\gamma}(\widehat{\lambda}) \\ &\leq C_{\vec{r},M_{\psi_\theta},\lambda}\, 2^{(k-j)(M_{\psi_\theta}+1)\underline{a}} A. \end{aligned}$$

For $k \geq j \geq 1$ one can replace 2^{ka_l} in (4.27) by 2^{ja_l} at the cost of the factor $2^{(k-j)\vec{a}\cdot\vec{r}}$ in front of the integral. Then the roles of ψ_θ and λ can be interchanged, since the support information on $\widehat{\lambda}$ yields $\lambda \in \bigcap_M \mathcal{S}_M$. This gives, with $\rho = \lceil \vec{r}+2 \rceil$,

$$I_{j,k} \leq c 2^{(k-j)\vec{a}\cdot\vec{r}} \sum_{\alpha\leq\rho} \binom{\rho}{\alpha} p_{\alpha,0}(\psi_\theta * \lambda_{k-j}) \leq C_{M,\vec{r},\lambda} 2^{-(k-j)((M+1)\underline{a}-\vec{a}\cdot\vec{r})} B.$$

Similar estimates are obtained for $I_{j,0}$, $I_{0,k}$ and $I_{0,0}$ with C, D as factors. □

Using Lemma 4.19, the proof given in [44] is now extended to a

Proof of Theorem 4.18. The identity (4.22) gives for $f \in \mathcal{S}'$ and $j \in \mathbb{N}$ that

$$\psi_{\theta,j} * f = \sum_{k=0}^\infty \psi_{\theta,j} * \lambda_k * \varphi_k * f. \tag{4.29}$$

By Lemma 4.19 with M chosen so large that $(M+1)\underline{a} + s > 2\vec{a}\cdot\vec{r}$, there exists a θ-independent constant $c > 0$ such that the summands can be crudely estimated,

$$\begin{aligned} &|\psi_{\theta,j} * \lambda_k * \varphi_k * f(y)| \\ &\qquad \leq \varphi_k^* f(y) \int |\psi_{\theta,j} * \lambda_k(z)| \prod_{l=1}^n (1+2^{ka_l}|z_l|)^{r_l}\,dz \\ &\qquad \leq c\,(A+B+C+D)\,\varphi_k^* f(y) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)\underline{a}} & \text{for } k \leq j, \\ 2^{-(k-j)((M+1)\underline{a}-\vec{a}\cdot\vec{r})} & \text{for } j \leq k. \end{cases} \end{aligned}$$

Here $\varphi_k^* f(y) \leq \varphi_k^* f(x) \max\left(1, 2^{(k-j)\vec{a}\cdot\vec{r}}\right) \prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}$ is easily verified for $x, y \in \mathbb{R}^n$ and $j, k \in \mathbb{N}_0$ by elementary calculations, so therefore

$$\begin{aligned} &\sup_{y\in\mathbb{R}^n} \frac{|\psi_{\theta,j} * \lambda_k * \varphi_k * f(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}} \\ &\qquad \leq c(A+B+C+D)\,\varphi_k^* f(x) \times \begin{cases} 2^{(k-j)(M_{\psi_\theta}+1)\underline{a}} & \text{for } k \leq j, \\ 2^{-(k-j)((M+1)\underline{a}-2\vec{a}\cdot\vec{r})} & \text{for } j \leq k. \end{cases} \end{aligned}$$

Inserting into (4.29) and using that $\delta := \min\big((M_{\psi_\theta}+1)\underline{a}-s, (M+1)\underline{a}-2\vec{a}\cdot\vec{r}+s\big) > 0$ by the assumptions, the above implies for $j \geq 0$,

$$2^{js}\sup_{\theta\in\Theta}\psi^*_{\theta,j}f(x) \leq c(A+B+C+D)\sum_{k=0}^{\infty}2^{ks}\varphi^*_k f(x)\,2^{-|j-k|\delta}.$$

Now Lemma 4.8 yields (4.26). □

4.4.2 Control by Convolutions

Since $\widehat{\psi}$ need not have compact support, Proposition 4.11 is replaced by a pointwise estimate with a sum representing the higher frequencies:

Proposition 4.20. *Let $\psi_0, \psi \in \mathcal{S}$ satisfy the Tauberian conditions* (4.18), (4.19). *For $N, \vec{r}, \tau > 0$ there exists a constant $C_{N,\vec{r},\tau}$ such that for $f \in \mathcal{S}'$ and $j \in \mathbb{N}_0$,*

$$\big(\psi^*_j f(x)\big)^\tau \leq C_{N,\vec{r},\tau}\sum_{k\geq j}2^{(j-k)N\tau}\int\frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^n(1+2^{ka_l}|x_l-z_l|)^{r_l\tau}}\,dz. \tag{4.30}$$

As a proof ingredient we use the $\mathcal{S}'$-order of $f \in \mathcal{S}'(\mathbb{R}^n)$, written $\operatorname{ord}_{\mathcal{S}'}(f)$, that is the smallest $N \in \mathbb{N}_0$ for which there exists $c > 0$ such that, cf. (4.2),

$$|\langle f, \psi\rangle| \leq c\,p_N(\psi) \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n). \tag{4.31}$$

Remark 4.21. *Our proof of Proposition 4.20 follows that of Rychkov [44], although his exposition leaves a heavy burden with the reader, since the application of Lemma 3 there is only justified when* $\operatorname{ord}_{\mathcal{S}'}(f)$ *is sufficiently small; cf. the O-condition* (4.35) *below.*

In a somewhat different context, Rychkov gave a verbal explanation after (2.17) in [46] (with similar reasoning in [19, 55]) that perhaps could be carried over to the present situation. But we have found it simplest to reinforce [44] by showing that the central O-condition is *indeed fulfilled whenever f is such that the right-hand side of* (4.30) *is finite. In so doing, we give the full argument for the sake of completeness.*

Proof. Step 1. First we choose two functions $\lambda_0, \lambda \in \mathcal{S}$ with $\widehat{\lambda} = 0$ around $\xi = 0$ by applying Lemma 4.15 to the given $\psi_0, \psi \in \mathcal{S}$. Using Calderón's reproducing formula, cf. (4.22), on $f(2^{-j\vec{a}}\cdot)$, dilating and convolving with ψ_j, we obtain

$$\psi_j * f = (\lambda_{0,j} * \psi_{0,j}) * (\psi_j * f) + \sum_{k=j+1}^{\infty}(\psi_j * \lambda_k) * (\psi_k * f). \tag{4.32}$$

To estimate $\psi_j * \lambda_k$ we use (4.28) for an arbitrary integer $M_\lambda \geq -1$ to get

$$|\psi_j * \lambda_k(z)| \leq C_{\vec{r}}\frac{2^{j|\vec{a}|}\,2^{(j-k)(M_\lambda+1)\underline{a}}}{\prod_{l=1}^n(1+2^{ja_l}|z_l|)^{r_l}}\max{}'\,p_{0,\zeta}(\widehat{\lambda})\cdot q_{M_\lambda+1,\gamma}(\widehat{\psi}).$$

An analogous estimate is obtained for $\lambda_{0,j} * \psi_{0,j}$, when (4.28) is applied with $t = 1$, $M_{\lambda_0} = -1$. Inserting these bounds into (4.32) yields for $C_{M_\lambda,\vec{r}} = C_{M_\lambda,\vec{r},\lambda_0,\lambda,\psi_0,\psi}$,

$$|\psi_j * f(y)| \le C_{M_\lambda,\vec{r}} \sum_{k=j}^{\infty} 2^{(j-k)(M_\lambda+1)\underline{a}} \int \frac{2^{j|\vec{a}|}|\psi_k * f(y-z)|}{\prod_{l=1}^{n}(1+2^{ja_l}|z_l|)^{r_l}}\, dz. \tag{4.33}$$

Since $j \mapsto 2^{j\vec{a}\cdot\vec{r}}\prod_{l=1}^{n}(1+2^{ja_l}|x_l - z_l|)^{-r_l}$ is monotone increasing, (4.33) entails that for $N = (M_\lambda + 1)\underline{a} - \vec{a}\cdot\vec{r}$,

$$\begin{aligned} \psi_j^* f(x) &\le C_{M_\lambda,\vec{r}} \sum_{k\ge j} 2^{(j-k)(M_\lambda+1)\underline{a}} \int \frac{2^{j|\vec{a}|}|\psi_k * f(z)|}{\prod_{l=1}^{n}(1+2^{ja_l}|x_l-z_l|)^{r_l}}\, dz \\ &\le C_N \sum_{k\ge j} 2^{(j-k)N} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^{n}(1+2^{ka_l}|x_l-z_l|)^{r_l\tau}}\, dz \left(\psi_k^* f(x)\right)^{1-\tau}. \end{aligned} \tag{4.34}$$

Here N can be lowered in the exponent, so (4.34) holds for all $N \ge -\vec{a}\cdot\vec{r}$, with $N \mapsto C_{N,\vec{r}}$ piecewise constant; i.e. constant on intervals having the form $](k-1)\underline{a}, k\underline{a}] - \vec{a}\cdot\vec{r}$ for $k \in \mathbb{N}_0$. Obviously this yields (4.30) in case $\tau = 1$.

Step 2. To cover a given $\tau \in]0,1[$ we apply Lemma 4.9 with b_j as the last integral in (4.34): because of the inequality (4.34), the estimate (4.30) with $C_{N,\vec{r},\tau} = C_N$ follows for all $N > 0$ by the lemma, if we can only verify the last assumption that, for some $N_0 > 0$,

$$d_j := \psi_j^* f(x) = O\left(2^{jN_0}\right). \tag{4.35}$$

In case $\omega \le \vec{r}$ for $\omega = \operatorname{ord}_{\mathcal{S}'} f$, this estimate follows for all $j \ge 0$ from standard calculations by applying (4.31) to the numerator in $\psi_j^* f(x)$.

In the remaining cases, where $\omega > r_l$ for some $l \in \{1,\dots,n\}$, we shall show a similar estimate unless (4.30) is trivial. First we choose $\vec{q}$ such that $\vec{q} \ge \max(r_1,\dots,r_n,\omega)$. Then (4.30) holds true for $\vec{q}$ and the right-hand side gets larger by replacing each q_l with r_l in the denominator. Hence we have for $N > 0$,

$$|\psi_j * f(y)|^\tau \le C_{N,\vec{q},\tau} \sum_{k\ge j} 2^{(j-k)N\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^{n}(1+2^{ka_l}|y_l-z_l|)^{r_l\tau}}\, dz.$$

Using monotonicity as in Step 1, the above is seen to imply, say for $N > \vec{a}\cdot\vec{r}$, $j \in \mathbb{N}_0$ that

$$\left(\psi_j^* f(x)\right)^\tau \le C_{N,\vec{q},\tau} \sum_{k\ge j} 2^{(j-k)(N-\vec{a}\cdot\vec{r})\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_{l=1}^{n}(1+2^{ka_l}|x_l-z_l|)^{r_l\tau}}\, dz.$$

(The constant depends on $\vec{q}$, i.e. on f.) We can assume the sum on the right-hand side is finite for some $j_1 \ge 0$, $N_1 > \vec{a}\cdot\vec{r}$, for else (4.30) is trivial. Then

$$\begin{aligned} &\sup_{m\ge j_1} 2^{(j_1-m)(N_1-\vec{a}\cdot\vec{r})}\psi_m^* f(x) \\ &\qquad \le C_{N_1,\vec{q},\tau}^{1/\tau} \Big(\sum_{k\ge j_1} 2^{(j_1-k)(N_1-\vec{a}\cdot\vec{r})\tau} \int \frac{2^{k|\vec{a}|}|\psi_k * f(z)|^\tau}{\prod_l (1+2^{ka_l}|x_l-z_l|)^{r_l\tau}}\, dz \Big)^{1/\tau} < \infty. \end{aligned}$$

This implies (4.35) at once for $j \geq j_1$ and $N_0 := N_1 - \vec{a} \cdot \vec{r}$, so now Lemma 4.9 yields (4.30) for $j \geq j_1$. When considering the smallest such j_1, the right-hand side of (4.30) is infinite for every $j < j_1$ (any N) so that (4.30) is trivial.

Step 3. For $\tau > 1$ we deduce (4.33) with $r_l + 1$ for all l and afterwards apply Hölder's inequality with dual exponents $\tau, \tau' > 1$ with respect to the Lebesgue measure and the counting measure. Simple calculations then yield (4.30). □

Now we can briefly modify the arguments in [44] to obtain the next result.

Theorem 4.22. *Let $\psi_0, \psi \in \mathcal{S}$ satisfy the Tauberian conditions* (4.18), (4.19). *When $s \in \mathbb{R}$, $0 < \vec{p} < \infty$, $0 < q \leq \infty$ and the $\psi_j^* f$ are given in terms of an $\vec{r}$ satisfying*

$$r_l \min(q, p_1, \dots, p_n) > 1, \quad l = 1, \dots, n,$$

then there exists a constant $c > 0$ such that for $f \in \mathcal{S}'$,

$$\| 2^{sj} \psi_j^* f \,|L_{\vec{p}}(\ell_q)\| \leq c \, \| 2^{sj} \psi_j * f \,|L_{\vec{p}}(\ell_q)\|. \tag{4.36}$$

Proof. The proof relies on the Hardy–Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\operatorname{meas}(B(0,r))} \int_{B(0,r)} |f(x+y)| \, dy.$$

When applied only in one variable x_l, we denote it by M_l; i.e. using the splitting $x = (x', x_l, x'')$ we have $M_l u(x_1, \dots, x_n) := (Mu(x', \cdot, x''))(x_l)$. By assumption on $\vec{r}$, we may pick τ such that $\max_{1 \leq l \leq n} \frac{1}{r_l} < \tau < \min(q, p_1, \dots, p_n)$. This implies that $(1 + |z_l|)^{-r_l \tau} \in L_1(\mathbb{R})$, and since it is also radially decreasing, iterated application of the majorant property of the Hardy–Littlewood maximal function, described in e.g. [54, p. 57], yields a bound of the convolution on the right-hand side of (4.30), hence

$$\psi_j^* f(x) \leq C_{N,\vec{r}}^{1/\tau} \Big(\sum_{k \geq j} 2^{(j-k)N\tau} M_n(\dots M_2(M_1 |\psi_k * f|^\tau) \dots)(x) \Big)^{1/\tau}.$$

Here application of Lemma 4.8 gives

$$\| 2^{js} \psi_j^* f \,|L_{\vec{p}}(\ell_q)\| \leq C_{N,\vec{r}} \, \| 2^{js\tau} M_n(\dots (M_1 |\psi_j * f|^\tau) \dots) | L_{\vec{p}/\tau}(\ell_{q/\tau})\|^{1/\tau},$$

hence (4.36) follows by n-fold application of the maximal inequality of Bagby [2] on the space $L_{\vec{p}/\tau}(\ell_{q/\tau})$, since $\tau < \min(q, p_1, \dots, p_n)$; cf. also [29, Sec. 3.4]. □

4.5 General Quasi-Norms and Local Means

First of all, Theorems 4.18 and 4.22 give very general characterisations of $F^{s,\vec{a}}_{\vec{p},q}$. In fact the next result shows that in Definition 4.2 the Littlewood–Paley partition of unity is not essential: the quasi-norm can be replaced by a more general one in which the summation to 1 or the compact supports, or both, are lost:

Theorem 4.23. *Let $s \in \mathbb{R}$, $0 < \vec{p} < \infty$, $0 < q \le \infty$ and let ψ_0, ψ in $\mathcal{S}(\mathbb{R}^n)$ be given such that the Tauberian conditions* (4.18), (4.19) *are fulfilled together with a moment condition of order M_ψ so that $s < (M_\psi + 1)\min(a_1, \dots, a_n)$, cf.* (4.20)*. When $\psi^*_{j,\vec{a}} f$ is given in terms of an $\vec{r} > \min(q, p_1, \dots, p_n)^{-1}$, cf.* (4.25)*, then the following properties of $f \in \mathcal{S}'(\mathbb{R}^n)$ are equivalent:*

$$\begin{aligned} &\text{(i)} && f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n), \\ &\text{(ii)} && \| \{2^{sj} \psi_j * f\}_{j=0}^\infty | L_{\vec{p}}(\ell_q) \| < \infty, \\ &\text{(iii)} && \| \{2^{sj} \psi^*_{j,\vec{a}} f\}_{j=0}^\infty | L_{\vec{p}}(\ell_q) \| < \infty. \end{aligned}$$

Moreover, the quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}$ is equivalent to those in (ii) *and* (iii)*.*

Proof. Since $\psi_j * f(x) \le \psi^*_{j,\vec{a}} f(x)$ is trivial, clearly (iii) $\implies$ (ii); the converse holds by Theorem 4.22. To obtain (iii) $\implies$ (i), one may in the Lizorkin–Triebel norm estimate the convolutions by $(\mathcal{F}^{-1}\Phi)^*_{j,\vec{a}} f$, and the resulting norm is estimated by the one in (iii) by means of Theorem 4.18 (with a trivial index set like $\Theta = \{1\}$).

That (i) $\implies$ (iii) follows by using Theorem 4.18 to estimate from above by the quasi-norm defined from $(\mathcal{F}^{-1}\Phi)^*_{j,\vec{a}} f$, with all r_l so large that Theorem 4.22 gives control by the $\mathcal{F}^{-1}\Phi_j * f$. □

From the above it is e.g. obvious that the space $F^{s,\vec{a}}_{\vec{p},q}$ does not depend on the Littlewood–Paley partition of unity in (4.7), and that different choices yield equivalent quasi-norms.

As an immediate corollary of Theorem 4.23, there is the following characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ in terms of integration kernels. It has been well known in the isotropic case:

Theorem 4.24. *Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\int k_0(x)\,dx \ne 0 \ne \int k^0(x)\,dx$ and set $k(x) = \Delta^N k^0(x)$ for some $N \in \mathbb{N}$. When $0 < \vec{p} < \infty$, $0 < q \le \infty$ and $s < 2N \min(a_1, \dots, a_n)$, then a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if*

$$\| f \,| F^{s,\vec{a}}_{\vec{p},q} \|^* := \| k_0 * f \,| L_{\vec{p}} \| + \| \{2^{sj} k_j * f\}_{j=1}^\infty \,| L_{\vec{p}}(\ell_q) \| < \infty. \tag{4.37}$$

Furthermore, $\| f \,| F^{s,\vec{a}}_{\vec{p},q} \|^*$ is an equivalent quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

In (4.37), the functions k_j, $j \ge 1$, are given by $k_j(x) = 2^{j|\vec{a}|} k(2^{j\vec{a}} x)$; cf. (4.21).

Remark 4.25. *Obviously, we may choose k_0, k^0 such that both functions have compact support. In this case Triebel termed k_0 and k kernels of* local means, *and in [58, 2.4.6] he proved that* (4.37) *is an equivalent quasi-norm on the f belonging a priori to the isotropic space $F^s_{p,q}$. This was carried over to anisotropic, but unmixed spaces by Farkas [10]. Extension to function spaces with generalised smoothness has been done by Farkas and Leopold [12]; and to spaces of dominating mixed smoothness by Vybiral [61] and Hansen [19].*

Remark 4.26. *Bui, Paluszinki and Taibleson [7] obtained a characterisation, i.e. equivalence for all* $f \in \mathcal{S}'$*, in the isotropic (but weighted) case, which Rychkov [44] simplified to the present discrete Littlewood–Paley decompositions. Our Theorem 4.24 generalises this in two ways, i.e. we prove a characterisation of* $F^{s,\vec{a}}_{\vec{p},q}$ *that has anisotropies both in terms of* $\vec{a}$ *and mixed norms.*

CHAPTER 5

Anisotropic, Mixed-Norm Lizorkin–Triebel Spaces and Diffeomorphic Maps

Publication details

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Abstract:
This article gives general results on invariance of anisotropic Lizorkin–Triebel spaces with mixed norms under coordinate transformations on Euclidean space, open sets and cylindrical domains.

5.1 Introduction

This paper continues a study of anisotropic Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with mixed norms, which was begun in [28, 29] and followed up in our joint work [32].

First Sobolev embeddings and completeness of the scale $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ were established in [28], using the Nikol'skiĭ–Plancherel–Polya inequality for sequences of functions in the mixed-norm space $L_{\vec{p}}(\mathbb{R}^n)$, which was obtained straightforwardly in [28]. Then a detailed trace theory for hyperplanes in $\mathbb{R}^n$ was worked out in [29], e.g. with the novelty that the well-known borderline $s = 1/p$ has to be shifted upwards in some cases, because of the mixed norms.

Secondly, our joint paper [32] presented some general characterisations of $F^{s,\vec{a}}_{\vec{p},q}$, which may be specialised to kernels of local means, in Triebel's sense [58]. One interest of this is that local means have recently been useful for obtaining wavelet bases of Sobolev spaces and especially of their generalisations to the Besov and Lizorkin–Triebel scales. Cf. e.g. works of Vybiral [61, Thm. 2.12], Triebel [60, Thm. 1.20], Hansen [19, Thm. 4.3.1].

In the present paper, we treat the invariance of $F^{s,\vec{a}}_{\vec{p},q}$ under coordinate changes. During the discussions below, the results in [32] are crucial for the entire strategy.

Indeed, we address the main technical challenge to obtain invariance of $F^{s,\vec{a}}_{\vec{p},q}$ under the map

$$f \mapsto f \circ \sigma,$$

when σ is a bounded diffeomorphism on $\mathbb{R}^n$. (Cf. Theorems 5.19 and 5.20 below.) Not surprisingly, this will require the condition on σ that it only affects blocks of variables x_j in which the corresponding integral exponents p_j are equal, and similarly for the anisotropic weights a_j. Moreover, when estimating the operator norm of $f \mapsto f \circ \sigma$, i.e. obtaining the inequality

$$\| f \circ \sigma \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| \le c \| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|,$$

the Fourier-analytic definition of the spaces seems difficult to manage directly, so as done by Triebel [58] we have chosen to characterise $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ in terms of local means, as developed in [32].

However, the diffeomorphism invariance relies not just on the local means, but first of all also on techniques underlying them. In particular, we use the following inequality for the maximal function $\psi^*_j f(x)$ of Peetre–Fefferman–Stein type, which was established in Theorem 4.18 for mixed norms and with uniformity with respect to a general parameter θ:

$$\Big\| \{2^{sj} \sup_{\theta\in\Theta} \psi^*_{\theta,j} f\}_{j=0}^\infty \,\Big| L_{\vec{p}}(\ell_q)\Big\| \le c \,\Big\| \{2^{sj} \varphi^*_j f\}_{j=0}^\infty \,\Big| L_{\vec{p}}(\ell_q)\Big\|.$$

Hereby the 'cut-off' functions ψ_j, φ_j should fulfil a set of Tauberian and moment conditions; cf. Theorem 5.13 below. In the isotropic case this inequality originated in a well-known article of Rychkov [44], which contains a serious flaw (as pointed out in [19]); this and other inaccuracies were corrected in [32].

A second adaptation of Triebel's approach is caused by the anisotropy $\vec{a}$ we treat here. In fact, our proof only extends to e.g. $s < 0$ by means of the unconventional lift operator

$$\Lambda_r = \mathrm{OP}(\lambda_r), \qquad \lambda_r(\xi) = \sum_{j=1}^{n} (1 + \xi_j^2)^{\frac{r}{2a_j}}.$$

Moreover, to cover all $\vec{a} = (a_1, \dots, a_n)$, especially to allow irrational ratios a_j/a_k, we found it useful to invoke the corresponding pseudo-differential operators

$$(1 - \partial_j^2)^\mu = \mathrm{OP}\left((1 + \xi_j^2)^\mu\right)$$

that for $\mu \in \mathbb{R}$ are shown here to be bounded $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-2a_j\mu,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ for all s.

Local versions of our result, in which σ is only defined on subsets of $\mathbb{R}^n$, are also treated below. In short form we have e.g. the following result (cf. Theorem 5.21):

Theorem. *Let $U, V \subset \mathbb{R}^n$ be open and let $\sigma : U \to V$ be a C^∞-bijection on the form $\sigma(x) = (\sigma'(x_1, \dots, x_{n-1}), x_n)$. When $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(V)$ has compact support and all p_j are equal for $j < n$, and similarly for the a_j, then $f \circ \sigma \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ and*

$$\| f \circ \sigma \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\| \le c(\operatorname{supp} f, \sigma) \| f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(V)\|.$$

This is useful for introduction of Lizorkin–Triebel spaces on cylindrical manifolds. However, this subject is postponed to our forthcoming paper [30]. (Already this part of the mixed-norm theory has seemingly not been elucidated before). Moreover, in [30] we also carry over trace results from [29] to spaces over a smooth cylindrical domain in Euclidean space e.g. by analysing boundedness and ranges for traces on the flat and curved parts of its boundary.

To elucidate the importance of the results here and in [30], we recall that the $F^{s,\vec{a}}_{\vec{p},q}$ are relevant for parabolic differential equations with boundary value conditions: when solutions are sought in a *mixed-norm* Lebesgue space $L_{\vec{p}}$ (in order to allow different properties in the space and time directions), then $F^{s,\vec{a}}_{\vec{p},q}$-spaces are in general *inevitable* for a correct description of non-trivial data on the *curved* boundary.

This conclusion was obtained in works of P. Weidemaier [64–66], who treated several special cases; one may also consult the introduction of [29] for details.

Contents. Section 5.2 contains a review of our notation, and the definition of anisotropic Lizorkin–Triebel spaces with mixed norms is recalled, together with some needed properties, a discussion of different lift operators and a pointwise multiplier assertion.

In Section 5.3 results from [32] on characterisation of $F^{s,\vec{a}}_{\vec{p},q}$-spaces by local means are recalled and used to prove an important lemma for compactly supported elements in $F^{s,\vec{a}}_{\vec{p},q}$. Sufficient conditions for $f \mapsto f \circ \sigma$ to leave the spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ invariant for all $s \in \mathbb{R}$ are deduced in Section 5.4, when σ is a bounded diffeomorphism. Local versions for spaces on domains are derived in Section 5.5 together with isotropic results.

5.2 Preliminaries

5.2.1 Notation

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ contains all rapidly decreasing C^∞-functions. It is equipped with the family of seminorms, using $D^\alpha := (-\mathrm{i}\,\partial_{x_1})^{\alpha_1} \cdots (-\mathrm{i}\,\partial_{x_n})^{\alpha_n}$ for each multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\langle x \rangle^2 := 1 + |x|^2$,

$$p_M(\varphi) := \sup \left\{ \langle x \rangle^M |D^\alpha \varphi(x)| \,\middle|\, x \in \mathbb{R}^n, |\alpha| \le M \right\}, \quad M \in \mathbb{N}_0;$$

or with

$$q_{N,\alpha}(\varphi) := \int_{\mathbb{R}^n} \langle x \rangle^N |D^\alpha \varphi(x)|\, dx, \quad N \in \mathbb{N}_0, \ \alpha \in \mathbb{N}_0^n. \tag{5.1}$$

The Fourier transformation $\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} \varphi(x)\, dx$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ extends by duality to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions.

Inequalities for vectors $\vec{p} = (p_1, \dots, p_n)$ are understood componentwise; as are functions, e.g. $\vec{p}! = p_1! \cdots p_n!$. Moreover, $t_+ := \max(0, t)$ for $t \in \mathbb{R}$.

For $0 < \vec{p} \le \infty$ the space $L_{\vec{p}}(\mathbb{R}^n)$ consists of all Lebesgue measurable functions such that

$$\| u \,|\, L_{\vec{p}}(\mathbb{R}^n) \| := \Big(\int_{\mathbb{R}} \Big(\dots \Big(\int_{\mathbb{R}} |u(x_1, \dots, x_n)|^{p_1} dx_1 \Big)^{\frac{p_2}{p_1}} \dots \Big)^{\frac{p_n}{p_{n-1}}} dx_n \Big)^{\frac{1}{p_n}} < \infty,$$

with the modification of using the essential supremum over x_j in case $p_j = \infty$. Equipped with this quasi-norm, $L_{\vec{p}}(\mathbb{R}^n)$ is a quasi-Banach space (normed if $\vec{p} \ge 1$).

Furthermore, for $0 < q \le \infty$ we shall use the notation $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ for the space of all sequences $\{u_k\}_{k=0}^\infty$ of Lebesgue measurable functions $u_k : \mathbb{R}^n \to \mathbb{C}$ such that

$$\| \{u_k\}_{k=0}^\infty \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \| := \Big\| \Big(\sum_{k=0}^\infty |u_k(\cdot)|^q \Big)^{1/q} \Big| L_{\vec{p}}(\mathbb{R}^n) \Big\| < \infty,$$

with supremum over k in case $q = \infty$. This quasi-norm is often abbreviated to $\| u_k \,|L_{\vec{p}}(\ell_q)\|$; when $\vec{p} = (p, \dots, p)$ we simplify $L_{\vec{p}}$ to L_p. If $\max(p_1, \dots, p_n, q) < \infty$, then sequences of C_0^∞-functions are dense in $L_{\vec{p}}(\ell_q)$.

Generic constants will primarily be denoted by c or C and when relevant, their dependence on certain parameters will be explicitly stated. The notation $B(0, r)$ stands for the ball in $\mathbb{R}^n$ centered at 0 with radius $r > 0$, while $\overline{U}$ denotes the closure of a set $U \subset \mathbb{R}^n$.

5.2.2 Anisotropic, Mixed-Norm Lizorkin–Triebel Spaces

The scales of mixed-norm Lizorkin–Triebel spaces refines the scales of mixed-norm Sobolev spaces, cf. [29, Prop. 2.10], hence the history of these spaces goes far back in time; the reader is referred to Remark 4.4 and [28, Rem. 10] for a brief historical overview, which also list some of the ways to define Lizorkin–Triebel spaces.

Our exposition uses the Fourier-analytic definition, but first we recall the anisotropic distance function $|\cdot|_{\vec{a}}$, where $\vec{a}=(a_1,\dots,a_n)\in[1,\infty[^n$, on $\mathbb{R}^n$ and some of its properties. Using the quasi-homogeneous dilation $t^{\vec{a}}x:=(t^{a_1}x_1,\dots,t^{a_n}x_n)$ for $t\geq 0$, the function $|x|_{\vec{a}}$ is for $x\in\mathbb{R}^n\setminus\{0\}$ defined as the unique $t>0$ such that $t^{-\vec{a}}x\in S^{n-1}$ ($|0|_{\vec{a}}:=0$), i.e.

$$\frac{x_1^2}{t^{2a_1}}+\dots+\frac{x_n^2}{t^{2a_n}}=1.$$

By the Implicit Function Theorem, $|\cdot|_{\vec{a}}$ is C^∞ on $\mathbb{R}^n\setminus\{0\}$. We also recall the quasi-homogeneity $|t^{\vec{a}}x|_{\vec{a}}=t|x|_{\vec{a}}$ together with (cf. [28, Sec. 3])

$$\begin{gathered} |x+y|_{\vec{a}}\leq|x|_{\vec{a}}+|y|_{\vec{a}}, \\ \max(|x_1|^{1/a_1},\dots,|x_n|^{1/a_n})\leq|x|_{\vec{a}}\leq|x_1|^{1/a_1}+\dots+|x_n|^{1/a_n}. \end{gathered}\tag{5.2}$$

The definition of $F^{s,\vec{a}}_{\vec{p},q}$ uses a Littlewood–Paley decomposition, $1=\sum_{j=0}^\infty\Phi_j(\xi)$, which (for convenience) is based on a fixed $\psi\in C_0^\infty$ such that $0\leq\psi(\xi)\leq 1$ for all ξ, $\psi(\xi)=1$ if $|\xi|_{\vec{a}}\leq 1$ and $\psi(\xi)=0$ if $|\xi|_{\vec{a}}\geq 3/2$; setting $\Phi=\psi-\psi(2^{\vec{a}}\cdot)$, we define

$$\Phi_0(\xi)=\psi(\xi),\quad \Phi_j(\xi)=\Phi(2^{-j\vec{a}}\xi),\quad j=1,2,\dots\tag{5.3}$$

Definition 5.1. *The Lizorkin–Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with $s\in\mathbb{R}$, $0<\vec{p}<\infty$ and $0<q\leq\infty$ consists of all $u\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|\,u\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|:=\Big\|\Big(\sum_{j=0}^\infty 2^{jsq}\left|\mathcal{F}^{-1}\left(\Phi_j(\xi)\mathcal{F}u(\xi)\right)(\cdot)\right|^q\Big)^{1/q}\Big|L_{\vec{p}}(\mathbb{R}^n)\Big\|<\infty.$$

The number q is called the sum exponent and the entries in $\vec{p}$ are integral exponents, while s is a smoothness index. Usually the statements are valid for the full ranges $0<\vec{p}<\infty$, $0<q\leq\infty$, so we refrain from repeating these. Instead we focus on whether $s\in\mathbb{R}$ is allowed or not. In the isotropic case, i.e. $\vec{a}=(1,\dots,1)$, the parameter $\vec{a}$ is omitted.

We shall also consider the closely related Besov spaces, recalled using the abbreviation

$$u_j(x):=\mathcal{F}^{-1}\left(\Phi_j(\xi)\mathcal{F}u(\xi)\right)(x),\quad x\in\mathbb{R}^n,\ j\in\mathbb{N}_0.\tag{5.4}$$

Definition 5.2. *The Besov space $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ consists of all $u\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|\,u\,|B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)(\mathbb{R}^n)\|:=\Big(\sum_{j=0}^\infty 2^{jsq}\|\,u_j\,|L_{\vec{p}}(\mathbb{R}^n)\|^q\Big)^{1/q}<\infty.$$

In [28, 29] many results on these classes are elaborated, hence we just recall a few facts. They are quasi-Banach spaces (normed if $\min(p_1,\dots,p_n,q)\geq 1$) and the quasi-norm is subadditive, when raised to the power $d:=\min(1,p_1,\dots,p_n,q)$,

$$\| u+v\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|^d \leq \| u\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|^d + \| v\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|^d, \quad u,v\in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n).$$

Also the spaces do not depend on the chosen anisotropic decomposition of unity (up to equivalent quasi-norms) and there are continuous embeddings

$$\mathcal{S}(\mathbb{R}^n)\hookrightarrow F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\hookrightarrow \mathcal{S}'(\mathbb{R}^n), \tag{5.5}$$

where $\mathcal{S}$ is dense in $F^{s,\vec{a}}_{\vec{p},q}$ for $q<\infty$.

Since for $\lambda>0$, the space $F^{s,\vec{a}}_{\vec{p},q}$ coincides with $F^{\lambda s,\lambda\vec{a}}_{\vec{p},q}$, cf. [29, Lem. 3.24], most results obtained for the scales when $\vec{a}\geq 1$ can be extended to the case $0<\vec{a}<1$ (for details we refer to Remark 4.7).

The subspace $L_{1,\mathrm{loc}}(\mathbb{R}^n)\subset\mathcal{D}'(\mathbb{R}^n)$ of locally integrable functions is equipped with the Fréchet space topology defined from the seminorms $u\mapsto\int_{|x|\leq j}|u(x)|\,dx$, where $j\in\mathbb{N}$. By $C_{\mathrm{b}}(\mathbb{R}^n)$ we denote the Banach space of bounded, continuous functions, endowed with the sup-norm.

Lemma 5.3. *Let $s\in\mathbb{R}$ and $\alpha\in\mathbb{N}_0^n$ be arbitrary.*

(i) *The differential operator D^α is bounded $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\to F^{s-\vec{a}\cdot\alpha,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.*

(ii) *For $s>\sum_{\ell=1}^n\left(\frac{a_\ell}{p_\ell}-a_\ell\right)_+$ there is an embedding $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^n)$.*

(iii) *The embedding $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\hookrightarrow C_{\mathrm{b}}(\mathbb{R}^n)$ holds true for $s>\frac{a_1}{p_1}+\cdots+\frac{a_n}{p_n}$.*

Proof. For part (i) the reader is referred to [29, Lem. 3.22], where a proof using standard techniques for $F^{s,\vec{a}}_{\vec{p},q}$ is indicated (though the reference should have been to Proposition 3.13 instead of 3.14 there).

Part (ii) is obtained from the Nikol'skij inequality, cf. [28, Cor. 3.8], which allows a reduction to the case in which $p_j\geq 1$ for $j=1,\dots,n$, while $s>0$; then the claim follows from the embedding $F^{0,\vec{a}}_{\vec{p},1}\hookrightarrow L_{1,\mathrm{loc}}$.

Part (iii) follows at once from [29, (3.20)]. □

A local maximisation over a ball can be estimated in $L_{\vec{p}}$, at least for functions in certain subspaces of $C_{\mathrm{b}}(\mathbb{R}^n)$; cf. Lemma 5.3(iii):

Lemma 5.4 ([32]). *When $C>0$ and $s>\sum_{l=1}^n\frac{a_l}{\min(p_1,\dots,p_l)}$, then*

$$\Big\|\sup_{|x-y|<C}|u(y)|\,\Big|L_{\vec{p}}(\mathbb{R}^n_x)\Big\|\leq c\,\|u\,|F^{s,\vec{a}}_{\vec{p},q}\|.$$

Next we extend a well-known embedding to the mixed-norm setting. Let $C^\rho_*(\mathbb{R}^n)$ denote the Hölder class of order $\rho>0$, which by definition consists of all $u\in C^k(\mathbb{R}^n)$, where $k\in\mathbb{N}_0$ and $k<\rho\leq k+1$, satisfying

$$\|u\|_\rho:=\sum_{|\alpha|\leq k}\sup_{x\in\mathbb{R}^n}|D^\alpha u(x)|+\sum_{|\alpha|=k}\sup_{x-y\in\mathbb{R}^n\setminus\{0\}}|D^\alpha u(x)-D^\alpha u(y)|\,|x-y|^{k-\rho}<\infty.$$

Lemma 5.5. *For $\rho > 0$, $s \in \mathbb{R}$ with $s \le \rho$ the embedding $C^\rho_*(\mathbb{R}^n) \hookrightarrow B^{s,\vec{a}}_{\infty,\infty}(\mathbb{R}^n)$ holds true.*

Proof. This follows by modifying [20, Prop. 8.6.1] to the anisotropic case, i.e.

$$\| u \,|B^{s,\vec{a}}_{\infty,\infty}\| = \sup_{j\in\mathbb{N}_0} 2^{sj} \sup_{x\in\mathbb{R}^n} |\mathcal{F}^{-1}(\Phi_j \mathcal{F}u)(x)| \le c_\rho \|u\|_\rho. \tag{5.6}$$

The expressions in the Besov norm are for $j \ge 1$ estimated using that $\mathcal{F}^{-1}\Phi$ has vanishing moments of arbitrary order,

$$\mathcal{F}^{-1}(\Phi_j\mathcal{F}u)(x) = \int \mathcal{F}^{-1}\Phi(y)\Big(u(x-2^{-j\vec{a}}y) - \sum_{|\alpha|\le k} \frac{\partial^\alpha u(x)}{\alpha!}(-2^{-j\vec{a}}y)^\alpha\Big)\,dy.$$

A Taylor expansion of order $k-1$, where $k \in \mathbb{N}$ is chosen such that $k < \rho \le k+1$, yields an estimate of the parenthesis by

$$\begin{aligned}\Big|\sum_{|\alpha|=k} \frac{k}{\alpha!}\big(2^{-j\vec{a}}y\big)^\alpha \int_0^1 (1-\theta)^{k-1}\,\big(\partial^\alpha u(x-2^{-j\vec{a}}\theta y) - \partial^\alpha u(x)\big)\,d\theta\Big| \\ \le \sum_{|\alpha|=k} \frac{k}{\alpha!}\,\big|2^{-j\vec{a}}y\big|^k\,\|u\|_\rho\,\big|2^{-j\vec{a}}y\big|^{\rho-k}\int_0^1 (1-\theta)^{k-1}d\theta \le c'_\rho\,\big|2^{-j\vec{a}}y\big|^\rho\,\|u\|_\rho.\end{aligned}$$

Now we obtain, since $\vec{a} \ge 1$,

$$\sup_{x\in\mathbb{R}^n} |\mathcal{F}^{-1}(\Phi_j\mathcal{F}u)(x)| \le c'_\rho\, 2^{-j\rho}\|u\|_\rho \int |\mathcal{F}^{-1}\Phi(y)||y|^\rho\,dy \le c_\rho\, 2^{-j\rho}\|u\|_\rho.$$

This bound also works for $j = 0$, if c_ρ is large enough, so (5.6) holds for $\rho \ge s$. □

As a tool we also need to know the mapping properties of certain Fourier multipliers $\lambda(D)u := \mathcal{F}^{-1}(\lambda(\xi)\hat{u}(\xi))$. For generality's sake, we give

Proposition 5.6. *When $\lambda \in C^\infty(\mathbb{R}^n)$ for some $r \in \mathbb{R}$ has finite seminorms of the form*

$$C_\alpha(\lambda) := \sup\big\{\, 2^{-j(r-\vec{a}\cdot\alpha)}|D^\alpha\lambda(2^{j\vec{a}}\xi)| \,\big|\, j \in \mathbb{N}_0,\ 1/4 \le |\xi|_{\vec{a}} \le 4 \,\big\}, \quad \alpha \in \mathbb{N}_0^n,$$

then $\lambda(D)$ is continuous on $\mathcal{S}'(\mathbb{R}^n)$ and bounded $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-r,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$, with operator norm $\|\lambda(D)\| \le c_{\vec{p},q}\sum_{|\alpha|\le N_{\vec{p},q}} C_\alpha(\lambda)$.

Proof. The quasi-homogeneity of $|\cdot|_{\vec{a}}$ yields that $|D^\alpha\lambda(\xi)| \le cC_\alpha(\lambda)(1+|\xi|_{\vec{a}})^{r-\vec{a}\cdot\alpha}$, hence every derivative is of polynomial growth, cf. (5.2), so $\lambda(D)$ is a well-defined continuous map on $\mathcal{S}'$.

Boundedness follows as in the proof of [29, Prop. 3.15], mutatis mutandis. In fact, only the last step there needs an adaptation to the symbol $\lambda(\xi)$, but this is trivial because finitely many of the constants $C_\alpha(\lambda)$ can enter the estimates. □

5.2.3 Lift Operators

The invariance under coordinate transformations will be established below using a somewhat unconventional lift operator Λ_r, $r \in \mathbb{R}$,

$$\Lambda_r u = \operatorname{OP}(\lambda_r(\xi))u = \mathcal{F}^{-1}\big(\lambda_r(\xi)\widehat{u}(\xi)\big), \qquad \lambda_r(\xi) = \sum_{k=1}^{n} (1+\xi_k^2)^{r/(2a_k)}. \tag{5.7}$$

To apply Proposition 5.6, we derive an estimate uniformly in $j \in \mathbb{N}_0$ and over the set $\frac{1}{4} \le |\xi|_{\vec{a}} \le 4$: while the mixed derivatives vanish, the explicit higher order chain rule in Appendix A on page 65 yields

$$\begin{aligned} &|D^{\alpha_l}_{\xi_l}(2^{-jr}\lambda_r(2^{j\vec{a}}\xi))| \\ &\le \sum_{k=1}^{\alpha_l} c_k (2^{-2ja_l} + \xi_l^2)^{\frac{r}{2a_l}-k} 2^{j(\alpha_l a_l - 2ka_l)} \sum_{\substack{k=n_1+n_2 \\ \alpha_l = n_1+2n_2}} (2(2^{ja_l}\xi_l))^{n_1} 2^{n_2} < \infty. \end{aligned} \tag{5.8}$$

Indeed, the precise summation range gives $\alpha_l = n_1 + 2(k-n_1)$, so the harmless power $2^{n_1+n_2}$ results. (Note that this means that $|D^\alpha \lambda_r(2^{j\vec{a}}\xi)| \le C_\alpha 2^{j(r-\vec{a}\cdot\alpha)}$.)

The symbol λ_r has no zeros, and for $\lambda_r(\xi)^{-1}$ it is analogous to obtain such estimates of $D^\alpha(2^{jr}\lambda_r(2^{j\vec{a}}\xi)^{-1})$ uniformly with respect to j, using Appendix A and the above. So Proposition 5.6 gives both that Λ_r is a homeomorphism on $\mathcal{S}'$ (although $\Lambda_r^{-1} \ne \Lambda_{-r}$) and the proof of

Lemma 5.7. *The map Λ_r is a linear homeomorphism $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-r,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ for $s \in \mathbb{R}$.*

In a similar way one also finds the next auxiliary result.

Lemma 5.8. *For any $\mu \in \mathbb{R}$ the map $(1-\partial^2_{x_k})^\mu u = \operatorname{OP}((1+\xi_k^2)^\mu)u$, whereby $k \in \{1,\dots,n\}$, is a linear homeomorphism $F^{s,\vec{a}}_{\vec{p},q} \to F^{s-2\mu a_k,\vec{a}}_{\vec{p},q}$ for all $s \in \mathbb{R}$.*

A standard choice of an anisotropic lift operator is obtained by associating each $\xi \in \mathbb{R}^n$ with $(1,\xi) \in \mathbb{R}^{1+n}$, which is given the weights $(1,\vec{a})$, and by setting

$$\langle \xi \rangle_{\vec{a}} = |(1,\xi)|_{(1,\vec{a})}.$$

This is in C^∞, as $|\cdot|_{(1,\vec{a})}$ is so outside the origin. (Note the analogy to $\langle \xi \rangle = \sqrt{1+|\xi|^2}$.) Moreover, $\partial^\alpha \langle \xi \rangle^t_{\vec{a}}$ is for each $t \in \mathbb{R}$ estimated by powers of $|\xi|$, cf. [67, Lem. 1.4]. Therefore there is a linear homeomorphism $\Xi^t_{\vec{a}} \colon \mathcal{S}' \to \mathcal{S}'$ given by

$$\Xi^t_{\vec{a}} u := \operatorname{OP}(\langle \xi \rangle^t_{\vec{a}})u = \mathcal{F}^{-1}\left(\langle \xi \rangle^t_{\vec{a}}\, \widehat{u}(\xi)\right), \quad t \in \mathbb{R}.$$

In our mixed-norm set-up it is a small exercise to show that it restricts to a homeomorphism

$$\Xi^t_{\vec{a}} \colon F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-t,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \quad \text{for all } s \in \mathbb{R}. \tag{5.9}$$

Indeed, invoking Proposition 5.6, the task is as in (5.8) to show a uniform bound, and by using the elementary properties of $\langle\xi\rangle_{\vec{a}}$ (cf. [67, Lem. 1.4]) one finds for $t-\vec{a}\cdot\alpha\ge 0$,

$$\big|D^\alpha(2^{-jt}\langle 2^{j\vec{a}}\xi\rangle^t_{\vec{a}})\big| = 2^{j(\vec{a}\cdot\alpha-t)}\Big|D^\alpha_\eta\langle\eta\rangle^t_{\vec{a}}\big|_{\eta=2^{j\vec{a}}\xi}\Big| \le c2^{j(\vec{a}\cdot\alpha-t)}\langle 2^{j\vec{a}}\xi\rangle^{t-\vec{a}\cdot\alpha}_{\vec{a}} \le c\langle\xi\rangle^{t-\vec{a}\cdot\alpha}.$$

When $t-\vec{a}\cdot\alpha\le 0$, then $|\xi|^{t-\vec{a}\cdot\alpha}_{\vec{a}}$ is the outcome on the right-hand side. But the uniformity results in both cases, since the estimates pertain to $\frac{1}{4}\le|\xi|_{\vec{a}}\le 4$.

We digress to recall that the classical fractional Sobolev space $H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n)$, for $s\in\mathbb{R}$ and $1<\vec{p}<\infty$, consists of the $u\in\mathcal{S}'$ for which $\Xi^s_{\vec{a}}u\in L_{\vec{p}}(\mathbb{R}^n)$; with $\|\,u\,|H^{s,\vec{a}}_{\vec{p}}\| := \|\,\Xi^s_{\vec{a}}u\,|L_{\vec{p}}\|$. If $m_k := s/a_k\in\mathbb{N}_0$ for all k, then $H^{s,\vec{a}}_{\vec{p}}$ coincides (as shown by Lizorkin [39]) with the space $W^{(m_1,\dots,m_n)}_{\vec{p}}(\mathbb{R}^n)$ of $u\in L_{\vec{p}}$ having $\partial^{m_k}_{x_k}u$ in $L_{\vec{p}}$ for all k.

This characterisation is valid for $F^{s,\vec{a}}_{\vec{p},2}$, $1<\vec{p}<\infty$, in view of the identification

$$u\in H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n) \iff u\in F^{s,\vec{a}}_{\vec{p},2}(\mathbb{R}^n), \tag{5.10}$$

which by use of Ξ^s reduces to the case $L_{\vec{p}}=F^{0,\vec{a}}_{\vec{p},2}$. The latter is a Littlewood–Paley inequality that may be proved with general methods of harmonic analysis, cf. [29, Rem. 3.16].

A general reference on mixed-norm Sobolev spaces is the classical book of Besov, Il'in and Nikol'skij [4, 5]. Schmeisser and Triebel [50] treated $F^{s,\vec{a}}_{\vec{p},q}$ for $n=2$.

Remark 5.9. *Traces on hyperplanes were considered for $H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n)$ by Lizorkin in [39] and for $W^{\vec{m}}_{\vec{p}}(\mathbb{R}^n)$ by Bugrov [6], who raised the problem of traces at $\{x_j=0\}$ for $j<n$. This was solved by Berkolaiko, who treated traces in the $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$-scales for $1<\vec{p}<\infty$ in e.g. [3]. The range $0<\vec{p}<\infty$ was covered on $\mathbb{R}^n$ for $j=1$ and $j=n$ in [29], and in our forthcoming paper [30] we carry over the trace results to $F^{s,\vec{a}}_{\vec{p},q}$-spaces over a smooth cylindrical domain $\Omega\times]0,T[$.*

Remark 5.10. *We take the opportunity to correct a minor inaccuracy in [29], where a lift operator (also) called Λ_r unfortunately was defined to have symbol $(1+|\xi|^2_{\vec{a}})^{r/2}$. However, it is not in $C^\infty(\mathbb{R}^n)$ for $\vec{a}\ne(1,\dots,1)$; this can be seen from the example for $n=2$ with $\vec{a}=(2,1)$, where [67, Ex. 1.1] gives the explicit formula*

$$|\xi|_{\vec{a}} = 2^{-1/2}\big(\xi_2^2+(\xi_2^4+4\xi_1^2)^{1/2}\big)^{1/2}.$$

Here an easy calculation shows that $D_{\xi_1}|\xi|^2_{\vec{a}}$ is discontinuous along the line $(\xi_1,0)$, which is inherited by the symbol e.g. for $r=2$. The resulting operator is therefore not defined on all of $\mathcal{S}'$. However, this is straightforward to avoid by replacing the lift operator in [29] by the better choice Ξ^r given in (5.9). This gives the space $H^{s,\vec{a}}_{\vec{p}}(\mathbb{R}^n)$ in (5.10).

5.2.4 Paramultiplication

This section contains a pointwise multiplier assertion for the $F^{s,\vec{a}}_{\vec{p},q}$-scales. We consider the densely defined product on $\mathcal{S}' \times \mathcal{S}'$, introduced in [24, Def. 3.1] and in an isotropic set-up in [43, Ch. 4],

$$u \cdot v := \lim_{j \to \infty} \mathcal{F}^{-1}\left(\psi(2^{-j\vec{a}}\xi)\mathcal{F}u(\xi)\right) \cdot \mathcal{F}^{-1}\left(\psi(2^{-j\vec{a}}\xi)\mathcal{F}v(\xi)\right), \tag{5.11}$$

which is considered for those pairs (u, v) in $\mathcal{S}' \times \mathcal{S}'$ for which the limit on the right-hand side exists in $\mathcal{D}'$ and is independent of ψ. Here $\psi \in C_0^\infty$ is the function used in the construction of the Littlewood–Paley decomposition (in principle the independence should be verified for all $\psi \in C_0^\infty$ equalling 1 near the origin; but this is not a problem here).

To illustrate how this product extends the usual one, and to prepare for an application below, the following is recalled:

Lemma 5.11 ([24]). *When $f \in C^\infty(\mathbb{R}^n)$ has derivatives of any order of polynomial growth, and when $g \in \mathcal{S}'(\mathbb{R}^n)$ is arbitrary, then the limit in* (5.11) *exists and equals the usual product $f \cdot g$, as defined on $C^\infty \times \mathcal{D}'$.*

Using this extended product, we introduce the usual space of multipliers

$$M(F^{s,\vec{a}}_{\vec{p},q}) := \left\{ u \in \mathcal{S}' \;\middle|\; u \cdot v \in F^{s,\vec{a}}_{\vec{p},q} \text{ for all } v \in F^{s,\vec{a}}_{\vec{p},q} \right\}$$

equipped with the induced operator quasi-norm

$$\| \, u \,|M(F^{s,\vec{a}}_{\vec{p},q})\| := \sup\left\{ \| \, u \cdot v \,|F^{s,\vec{a}}_{\vec{p},q}\| \;\middle|\; \| \, v \,|F^{s,\vec{a}}_{\vec{p},q}\| \le 1 \right\}.$$

As Lemma 5.5 at once yields $C^\infty_{L_\infty} \subset \bigcap_{s>0} B^{s,\vec{a}}_{\infty,\infty}$ (a well-known result in the isotropic case) for $C^\infty_{L_\infty} := \{ g \in C^\infty \,|\, \forall\alpha\colon D^\alpha g \in L_\infty \}$, the next result is in particular valid for $u \in C^\infty_{L_\infty}$:

Lemma 5.12. *Let $s \in \mathbb{R}$ and take $s_1 > s$ such that also*

$$s_1 > \sum_{\ell=1}^{n} \left(\frac{a_\ell}{\min(1, q, p_1, \ldots, p_\ell)} - a_\ell \right) - s. \tag{5.12}$$

Then each $u \in B^{s_1,\vec{a}}_{\infty,\infty}$ defines a multiplier of $F^{s,\vec{a}}_{\vec{p},q}$ and

$$\| \, u \,|M(F^{s,\vec{a}}_{\vec{p},q})\| \le c \, \| \, u \,|B^{s_1,\vec{a}}_{\infty,\infty}\|.$$

Proof. The proof will be brief as it is based on standard arguments from paramultiplication, cf. [24], [43, Ch. 4] for details. In particular we shall use the decomposition

$$u \cdot v = \Pi_1(u, v) + \Pi_2(u, v) + \Pi_3(u, v).$$

The exact form of this can also be recalled from the below formulae.

In terms of the Littlewood–Paley partition $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$ from Definition 5.1, we set $\Psi_j = \Phi_0 + \cdots + \Phi_j$ for $j \geq 1$ and $\Psi_0 = \Phi_0$. These are used in Fourier multipliers, now written with upper indices as $u^j = \mathcal{F}^{-1}(\Psi_j \widehat{u})$.

Note first that $s_1 > 0$, whence $B^{s_1,\vec{a}}_{\infty,\infty} \hookrightarrow L_\infty$, which is useful since the dyadic corona criterion for $F^{s,\vec{a}}_{\vec{p},q}$, cf. [29, Lem. 3.20], implies the well-known simple estimate

$$\| \Pi_1(u,v) \,|F^{s,\vec{a}}_{\vec{p},q}\| \leq c \| u \,|L_\infty\| \, \| v \,|F^{s,\vec{a}}_{\vec{p},q}\|.$$

Furthermore, since

$$s_2 := s_1 + s > \sum_{\ell=1}^{n} \frac{a_\ell}{\min(1, q, p_1, \ldots, p_\ell)} - |\vec{a}|,$$

using the dyadic ball criterion for $F^{s,\vec{a}}_{\vec{p},q}$, cf. [29, Lem. 3.19], we find that

$$\begin{aligned} \| \Pi_2(u,v) \,|F^{s_2,\vec{a}}_{\vec{p},q}\| &\leq c \| 2^{js_2} u_j v_j \,|L_{\vec{p}}(\ell_q)\| \\ &\leq c \sup_{k\in\mathbb{N}_0} 2^{ks_1} \| u_k \,|L_\infty\| \, \|2^{js}|v_j| \,|L_{\vec{p}}(\ell_q)\| \\ &\leq c \| u \,|B^{s_1,\vec{a}}_{\infty,\infty}\| \, \| v \,|F^{s,\vec{a}}_{\vec{p},q}\|. \end{aligned}$$

To estimate $\Pi_3(u,v)$ we first consider the case $s > 0$ and pick $t \in]s, s_1[$. The dyadic corona criterion together with the formula $v^j = v_0 + \cdots + v_j$ and a summation lemma, which exploits that $t - s_1 < 0$ (cf. [67, Lem. 3.8]), give

$$\begin{aligned} \| \Pi_3(u,v) \,|F^{t,\vec{a}}_{\vec{p},q}\| &\leq c \sup_{k\in\mathbb{N}_0} 2^{ks_1} \| u_k \,|L_\infty\| \, \|2^{(t-s_1)j} v^{j-2} \,|L_{\vec{p}}(\ell_q)\| \\ &\leq c \| u \,|B^{s_1,\vec{a}}_{\infty,\infty}\| \left\| 2^{(t-s_1)j} \sum_{k=0}^{j} |v_k| \,\middle|L_{\vec{p}}(\ell_q) \right\| \\ &\leq c \| u \,|B^{s_1,\vec{a}}_{\infty,\infty}\| \, \| v \,|F^{t-s_1,\vec{a}}_{\vec{p},q}\|. \end{aligned} \tag{5.13}$$

Since $t - s_1 < 0 < s$ implies $F^{s,\vec{a}}_{\vec{p},q} \hookrightarrow F^{t-s_1,\vec{a}}_{\vec{p},q}$, and also $F^{t,\vec{a}}_{\vec{p},q} \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}$ holds, the above yields

$$\| \Pi_3(u,v) \,|F^{s,\vec{a}}_{\vec{p},q}\| \leq c \| u \,|B^{s_1,\vec{a}}_{\infty,\infty}\| \, \| v \,|F^{s,\vec{a}}_{\vec{p},q}\|. \tag{5.14}$$

For $s \leq 0$ the procedure is analogous, except that (5.13) is derived for $t \in]0, s_1+s[$, which is non-empty by assumption (5.12) on s; then standard embeddings again give (5.14).

In closing, we remark that as required, the product $u \cdot v$ is independent of the test function ψ appearing in the definition. Indeed, for $q < \infty$ this follows from Lemma 5.11, which gives the coincidence between this product on $\mathcal{S}' \times \mathcal{S}$ and the usual one, hence by density of $\mathcal{S}$, cf. (5.5), and the above estimates, the map $v \mapsto u \cdot v$ extends uniquely by continuity to all $g \in F^{s,\vec{a}}_{\vec{p},q}$. When $q = \infty$, then the embedding $F^{s,\vec{a}}_{\vec{p},\infty} \hookrightarrow F^{s-\varepsilon,\vec{a}}_{\vec{p},1}$ for $\varepsilon > 0$ yields the independence using the previous case. □

5.3 Characterisation by Local Means

Characterisation of Lizorkin–Triebel spaces $F^s_{p,q}$ by local means is due to Triebel, cf. [58, 2.4.6], and it was from the outset an important tool in proving invariance of the scale under diffeomorphisms.

An extensive treatment of characterisations of mixed-norm spaces $F^{s,\vec{a}}_{\vec{p},q}$ in terms of quasi-norms based on convolutions, in particular the case of kernels of local means, was given in [32], which to a large extent is based on extensions to mixed norms of inequalities in [44]. For the reader's convenience we recall the needed results.

Throughout this section we consider a fixed $\vec{a} \geq 1$ with $\underline{a} := \min(a_1, \dots, a_n)$ and functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ that fulfil Tauberian conditions in terms of some $\varepsilon > 0$ and/or a moment condition of order $M_\psi \geq -1$ ($M_\psi = -1$ means that the condition is void),

$$
\begin{aligned}
|\mathcal{F}\psi_0(\xi)| > 0 \quad &\text{on} \quad \big\{\, \xi \,\big|\, |\xi|_{\vec{a}} < 2\varepsilon \big\}, && (5.15)\\
|\mathcal{F}\psi(\xi)| > 0 \quad &\text{on} \quad \{\, \xi \mid \varepsilon/2 < |\xi|_{\vec{a}} < 2\varepsilon \}, && (5.16)\\
D^\alpha(\mathcal{F}\psi)(0) = 0 \quad &\text{for} \quad |\alpha| \leq M_\psi. && (5.17)
\end{aligned}
$$

Note by (5.2) that in case (5.15) is fulfilled for the Euclidean distance, it holds true also in the anisotropic case, perhaps with a different ε.

We henceforth change notation, from (5.4) to

$$
\varphi_j(x) = 2^{j|\vec{a}|}\varphi(2^{j\vec{a}}x), \quad \varphi \in \mathcal{S}, \;\; j \in \mathbb{N}, \tag{5.18}
$$

which gives rise to the sequence $(\psi_j)_{j\in\mathbb{N}_0}$. The non-linear Peetre–Fefferman–Stein maximal operators induced by $(\psi_j)_{j\in\mathbb{N}_0}$ are for an arbitrary $\vec{r} = (r_1, \dots, r_n) > 0$ and any $f \in \mathcal{S}'(\mathbb{R}^n)$ given by (dependence on $\vec{a}$ and $\vec{r}$ is omitted)

$$
\psi_j^* f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\psi_j * f(y)|}{\prod_{\ell=1}^n (1 + 2^{ja_\ell}|x_\ell - y_\ell|)^{r_\ell}}, \quad x \in \mathbb{R}^n, \;\; j \in \mathbb{N}_0. \tag{5.19}
$$

Later we shall also refer to the trivial estimate

$$
|\psi_j * f(x)| \leq \psi_j^* f(x). \tag{5.20}
$$

Finally, for an index set Θ, we consider $\psi_{\theta,0}, \psi_\theta \in \mathcal{S}(\mathbb{R}^n)$, $\theta \in \Theta$, where the ψ_θ satisfy (5.17) for some M_{ψ_θ} independent of $\theta \in \Theta$, and also $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ that fulfil (5.15)–(5.16) in terms of an $\varepsilon' > 0$. Setting $\psi_{\theta,j}(x) = 2^{j|\vec{a}|}\psi_\theta(2^{j\vec{a}}x)$ for $j \in \mathbb{N}$, we can state the first result relating different quasi-norms.

Theorem 5.13 ([32]). *Let $0 < \vec{p} < \infty$, $0 < q \leq \infty$ and $-\infty < s < (M_{\psi_\theta} + 1)\underline{a}$. For a given $\vec{r}$ in (5.19) and an integer $M \geq -1$ chosen so large that $(M+1)\underline{a} - 2\vec{a}\cdot\vec{r} + s > 0$, we assume that*

$$
\begin{aligned}
A &:= \sup_{\theta\in\Theta} \max \| D^\alpha \mathcal{F}\psi_\theta \,|L_\infty\| &&< \infty,\\
B &:= \sup_{\theta\in\Theta} \max \| (1+|\xi|)^{M+1}\, D^\gamma \mathcal{F}\psi_\theta(\xi) \,|L_1\| &&< \infty,\\
C &:= \sup_{\theta\in\Theta} \max \| D^\alpha \mathcal{F}\psi_{\theta,0} \,|L_\infty\| &&< \infty,\\
D &:= \sup_{\theta\in\Theta} \max \| (1+|\xi|)^{M+1}\, D^\gamma \mathcal{F}\psi_{\theta,0}(\xi) \,|L_1\| &&< \infty,
\end{aligned}
$$

where the maxima are over α such that $|\alpha| \le M_{\psi_\theta} + 1$ or $\alpha \le \lceil \vec{r} + 2 \rceil$, respectively over γ with $\gamma_j \le r_j + 2$. Then there exists $c > 0$ such that for $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$
\big\| \{2^{sj} \sup_{\theta\in\Theta} \psi^*_{\theta,j} f\}_{j=0}^\infty \,\big|L_{\vec{p}}(\ell_q)\big\| \le c(A+B+C+D) \,\big\| \{2^{sj} \varphi^*_j f\}_{j=0}^\infty \,\big|L_{\vec{p}}(\ell_q)\big\|.
$$

It is also possible to estimate the maximal function in terms of the convolution appearing in its numerator:

Theorem 5.14 ([32])**.** *Let $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy the Tauberian conditions in* (5.15)–(5.16)*. When $s \in \mathbb{R}$, $0 < \vec{p} < \infty$, $0 < q \le \infty$ and*

$$
\frac{1}{r_l} < \min(q, p_1, \dots, p_n), \quad l = 1, \dots, n,
$$

then there exists a constant $c > 0$ such that for $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$
\big\| \{2^{sj} \psi^*_j f\}_{j=0}^\infty \,\big|L_{\vec{p}}(\ell_q)\big\| \le c \,\big\| \{2^{sj} \psi_j * f\}_{j=0}^\infty \,\big|L_{\vec{p}}(\ell_q)\big\|.
$$

As a consequence of Theorems 5.13 and 5.14 (the first applied for a trivial index set like $\Theta = \{1\}$), we obtain the characterisation of $F^{s,\vec{a}}_{\vec{p},q}$ by local means:

Theorem 5.15 ([32])**.** *Let $k_0, k^0 \in \mathcal{S}$ such that $\int k_0(x)\,dx \ne 0 \ne \int k^0(x)\,dx$ and set $k(x) = \Delta^N k^0(x)$ for some $N \in \mathbb{N}$. When $0 < \vec{p} < \infty$, $0 < q \le \infty$ and $-\infty < s < 2N\underline{a}$, then a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if (cf.* (5.18) *for the k_j)*

$$
\| f \,|F^{s,\vec{a}}_{\vec{p},q}\|^* := \| k_0 * f \,|L_{\vec{p}}\| + \|\{2^{sj} k_j * f\}_{j=1}^\infty \,|L_{\vec{p}}(\ell_q)\| < \infty. \tag{5.21}
$$

Furthermore, $\| f \,|F^{s,\vec{a}}_{\vec{p},q}\|^*$ is an equivalent quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

Application of Theorem 5.15 yields a useful result regarding Lizorkin–Triebel spaces on open subsets, when these are defined by restriction, i.e.

Definition 5.16. *Let $U \subset \mathbb{R}^n$ be open. The space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ is defined as the set of all $u \in \mathcal{D}'(U)$ such that there exists a distribution $f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ satisfying*

$$
f(\varphi) = u(\varphi) \quad \text{for all} \quad \varphi \in C_0^\infty(U). \tag{5.22}
$$

We equip $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ with the quotient quasi-norm (norm if $\vec{p}, q \ge 1$)

$$
\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\| := \inf_{r_U f = u} \| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|.
$$

In (5.22) it is tacitly understood that on the left-hand side φ is extended by 0 outside U. For this we henceforth use the operator notation $e_U\varphi$. Likewise r_U denotes restriction to U, whereby $u = r_U f$ in (5.22).

The Besov spaces $\overline{B}^{s,\vec{a}}_{\vec{p},q}(U)$ can be defined analogously. The quotient norms have the well-known advantage that embeddings and completeness can be transferred directly from the spaces on $\mathbb{R}^n$. However, the spaces are probably of little interest, if ∂U does not satisfy some regularity conditions, because we then expect (as in the isotropic case) that they do not coincide with those defined intrinsically.

Lemma 5.17. *Let $U \subset \mathbb{R}^n$ be open. When $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ has the infimum quasi-norm derived from the local means in Theorem 5.15 fulfilling* $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$ *for an $r > 0$, and*

$$\operatorname{dist}(\operatorname{supp} f, \mathbb{R}^n \setminus U) > 2r$$

holds for some $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ with compact support, then

$$\| f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\| = \| e_U f \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|. \tag{5.23}$$

In other words, the infimum is attained at $e_U f$ for such f.

Proof. For any other extension $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$, the difference $g := \widetilde{f} - e_U f$ is non-zero in $\mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} e_U f \cap \operatorname{supp} g = \emptyset$. So by the properties of r,

$$\operatorname{supp}(k_j * e_U f) \cap \operatorname{supp}(k_j * g) = \emptyset, \quad j \in \mathbb{N}_0.$$

Since $g \neq 0$ there is some j such that $\operatorname{supp}(k_j * g) \neq \emptyset$, hence $k_j * g(x) \neq 0$ on an open set disjoint from $\operatorname{supp}(k_j * e_U f)$. This term therefore effectively contributes to the $L_{\vec{p}}$-norm in (5.21) and thus $\| \widetilde{f} \,| F^{s,\vec{a}}_{\vec{p},q}\| = \| e_U f + g \,| F^{s,\vec{a}}_{\vec{p},q}\| > \| e_U f \,| F^{s,\vec{a}}_{\vec{p},q}\|$, which shows (5.23). $\square$

5.4 Invariance under Diffeomorphisms

The aim of this section is to show that $F^{s,\vec{a}}_{\vec{p},q}$ is invariant under suitable diffeomorphisms $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ and from this deduce similar results in a variety of set-ups.

5.4.1 Bounded Diffeomorphisms

A one-to-one mapping $y = \sigma(x)$ of $\mathbb{R}^n$ onto $\mathbb{R}^n$ is here called a diffeomorphism if the components $\sigma_j : \mathbb{R}^n \to \mathbb{R}$ have classical derivatives $D^\alpha \sigma_j$ for all $\alpha \in \mathbb{N}^n$. We set $\tau(y) = \sigma^{-1}(y)$.

For convenience σ is called a *bounded diffeomorphism* when σ and τ also satisfy

$$C_{\alpha,\sigma} := \max_{j\in\{1,\dots,n\}} \| D^\alpha \sigma_j \,| L_\infty\| < \infty, \tag{5.24}$$

$$C_{\alpha,\tau} := \max_{j\in\{1,\dots,n\}} \| D^\alpha \tau_j \,| L_\infty\| < \infty. \tag{5.25}$$

In this case there are obviously positive constants (when $J\sigma$ denotes the Jacobian matrix)

$$c_\sigma := \inf_{x\in\mathbb{R}^n} |\det J\sigma(x)| > 0, \qquad c_\tau := \inf_{y\in\mathbb{R}^n} |\det J\tau(y)| > 0. \tag{5.26}$$

E.g., by the Leibniz formula for determinants, $c_\sigma \geq 1/(n!\prod_{|\alpha|=1} C_{\alpha,\sigma}) > 0$.

Conversely, whenever a C^∞-map $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ fulfils (5.24) and $c_\sigma > 0$, then τ is C^∞ (since $J\tau(y) = \frac{1}{\det J\sigma(\tau(y))} \operatorname{Adj} J\sigma(\tau(y))$, where Adj denotes the adjugate, each $\partial_j\tau_k$ is in C^m if τ is so) and using e.g. Appendix A it is seen by induction over $|\alpha|$ that τ fulfils (5.25). Hence σ is a bounded diffeomorphism.

Recall that for a bounded diffeomorphism σ and a temperate distribution f, the composition $f\circ\sigma$ denotes the temperate distribution given by

$$\langle f\circ\sigma, \psi\rangle = \langle f, \psi\circ\tau|\det J\tau|\rangle \quad \text{for} \quad \psi\in\mathcal{S}. \tag{5.27}$$

It is continuous $\mathcal{S}'\to\mathcal{S}'$ as the adjoint of the continuous map $\psi\mapsto\psi\circ\tau|\det J\tau|$ on $\mathcal{S}$: since $|\det J\tau|$ is in $C^\infty_{L_\infty}$, continuity on $\mathcal{S}$ can be shown using the higher-order chain rule to estimate each seminorm $q_{N,\alpha}(\psi\circ\tau)$, cf. (5.1), by $\sum_{|\beta|\leq|\alpha|} q_{N,\beta}(\psi)$ (changing variables, $\langle\sigma(\cdot)\rangle$ can be estimated by using the Mean Value Theorem on each σ_j).

We need a few further conditions, due to the anisotropic situation: one can neither expect $f\circ\sigma$ to have the same regularity as f, e.g. if σ is a rotation; nor that $f\circ\sigma\in L_{\vec p}$ when $f\in L_{\vec p}$. On these grounds we first restrict to the situation in which

$$a_0 := a_1 = a_2 = \ldots = a_{n-1}, \qquad p_0 := p_1 = \ldots = p_{n-1} \tag{5.28}$$

and

$$\sigma(x) = (\sigma'(x_1,\ldots,x_{n-1}), x_n) \quad \text{for all} \quad x\in\mathbb{R}^n. \tag{5.29}$$

To prepare for Theorem 5.19 below, which gives sufficient conditions for the invariance of $F^{s,\vec a}_{\vec p,q}$ under bounded diffeomorphisms of the type (5.29), we first show that it suffices to have invariance for sufficiently large s:

Proposition 5.18. *Let σ be a bounded diffeomorphism on $\mathbb{R}^n$ on the form (5.29). When (5.28) holds and there exists $s_1\in\mathbb{R}$ with the property that $f\mapsto f\circ\sigma$ is a linear homeomorphism of $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ onto itself for every $s > s_1$, then this holds true for all $s\in\mathbb{R}$.*

Proof. It suffices to prove for $s\leq s_1$ that

$$\| f\circ\sigma\,|F^{s,\vec a}_{\vec p,q}\| \leq c\,\| f\,|F^{s,\vec a}_{\vec p,q}\| \tag{5.30}$$

with some constant c independent of f, as the reverse inequality then follows from the fact that the inverse of σ is also a bounded diffeomorphism with the structure in (5.29).

First $r > s_1 - s + 2a_n$ is chosen such that $d_0 := \frac{r}{2a_0}$ is a natural number. Setting $d_n = \frac{r}{2a_n}$ and taking $\mu\in[0,1[$ such that $d_n - \mu\in\mathbb{N}$, we have that $r_\mu := r - 2\mu a_n > s_1 - s$.

Now Lemma 5.7 yields the existence of $h \in F^{s+r,\vec{a}}_{\vec{p},q}$ such that $f = \Lambda_r h$, i.e.

$$f = (1-\partial^2_{x_n})^{d_n-\mu}(1-\partial^2_{x_n})^{\mu} h + \sum_{k=1}^{n-1}(1-\partial^2_{x_k})^{d_0} h. \tag{5.31}$$

Setting $g_1 = \big((1-\partial^2_{x_n})^{\mu} h\big)\circ\sigma$ and $g_0 = h\circ\sigma$, we may apply the higher-order chain rule to e.g. $h = g_0 \circ \tau$ (using denseness of $\mathcal{S}$ in $\mathcal{S}'$ and the $\mathcal{S}'$-continuity of composition in (5.27), Appendix A extends to $\mathcal{S}'$). Taking into account that $\tau(x) = (\tau'(x'), x_n)$, and letting prime indicate summation over multi-indices with $\beta_n = 0$, we obtain

$$f = \sum_{l=0}^{d_n-\mu} \eta_{n,l}\, \partial^{2l}_{x_n} g_1 \circ \tau + \sum_{k=1}^{n-1} \sum_{|\beta|\le 2d_0}{}' \eta_{k,\beta}\, \partial^{\beta} g_0 \circ \tau, \tag{5.32}$$

where $\eta_{n,l} := (-1)^l \binom{d_n-\mu}{l}$ and the $\eta_{k,\beta}$ are functions containing derivatives at least of order 1 of τ, and these can be estimated, say by $c\prod_{1\le m\le 2d_0}\langle \partial^m_{x_k}\tau\rangle^{2d_0}$. Composing with σ and applying Lemma 5.3(i) gives for $d := \min(1, q, p_0, p_n)$, when $\|\cdot\|$ denotes the $F^{s,\vec{a}}_{\vec{p},q}$-norm,

$$\begin{aligned}
&\| f\circ\sigma \|^d \\
&\le \sum_{l=0}^{d_n-\mu} |\eta_{n,l}|^d \,\| \partial^{2l}_{x_n} g_1 \|^d + \sum_{k=1}^{n-1}\sum_{|\beta|\le 2d_0}{}' \| \eta_{k,\beta}\circ\sigma \,|M(F^{s,\vec{a}}_{\vec{p},q})\|^d \,\| \partial^{\beta} g_0 \|^d \qquad (5.33)\\
&\le c\,\| g_1 \,|F^{s+r_\mu,\vec{a}}_{\vec{p},q}\|^d + \| g_0\,|F^{s+r,\vec{a}}_{\vec{p},q}\|^d \sum_{k=1}^{n-1}\sum_{|\beta|\le 2d_0}{}' \| \eta_{k,\beta}\circ\sigma\,|M(F^{s,\vec{a}}_{\vec{p},q})\|^d.
\end{aligned}$$

According to the remark preceding Lemma 5.12, the last sum is finite because $\eta_{k,\beta} \in C^{\infty}_{L_\infty}$. Finally, since $s + r_\mu > s_1$ and $s + r > s_1$, the stated assumption means that $h \mapsto g_1$ and $h \mapsto g_0$ are bounded, which in view of $r_\mu + 2\mu a_n = r$ and Lemmas 5.7 and 5.8 yields

$$\| f\circ\sigma \|^d \le c\| h\,|F^{s+r_\mu+2\mu a_n,\vec{a}}_{\vec{p},q}\|^d + \| h\,|F^{s+r,\vec{a}}_{\vec{p},q}\|^d \le c\| f\,|F^{s,\vec{a}}_{\vec{p},q}\|^d;$$

proving the boundedness of $f \mapsto f\circ\sigma$ in $F^{s,\vec{a}}_{\vec{p},q}$ for all $s \in \mathbb{R}$. □

In addition to the reduction in Proposition 5.18, we adopt in Theorem 5.19 below the strategy for the isotropic, unmixed case developed by Triebel [58, 4.3.2], who used Taylor expansions for the inner and outer functions for large s.

While his explanation was rather sketchy, our task is to account for the fact that the strategy extends to anisotropies and to mixed norms. Hence we give full details. This will also allow us to give brief proofs of additional results in Section 5.4.2 and 5.5 below.

To control the Taylor expansions, it will be crucial for us to exploit both the local means recalled in Theorem 5.15 and the parameter-dependent set-up in Theorem 5.13. This is prepared for with the following discussion.

The functions k_0 and k in Theorem 5.15 are for the proof of Theorem 5.19 chosen (as we may) so that N in the definition of k fulfils $s < 2N\underline{a}$ and so that both are *even* functions and

$$\operatorname{supp} k_0,\ \operatorname{supp} k \subset \bigl\{\, x \in \mathbb{R}^n \bigm| |x| \le 1 \bigr\}. \tag{5.34}$$

The set Θ in Theorem 5.13 is chosen to be the set of $(n-1)\times(n-1)$ matrices $\mathcal{B} = (b_{i,k})$ that, in terms of the constants $c_\sigma, C_{\alpha,\sigma}$ in (5.26) and (5.24), respectively, satisfy

$$|\det \mathcal{B}| \ge c_\sigma, \tag{5.35}$$

$$\max_{i,k} |b_{i,k}| \le \max_{|\alpha|=1} C_{\alpha,\sigma} =: C_\sigma. \tag{5.36}$$

Splitting $z = (z', z_n)$, we set $g(z) = z'^{\gamma'} k(z)$ for some $\gamma' \in \mathbb{N}_0^{n-1}$ (chosen later) and define

$$\psi_\theta(y) = g(\mathcal{A}y', y_n) \tag{5.37}$$

where θ is identified with $\mathcal{A}^{-1} := J\sigma'(x')$, which clearly belongs to Θ (for each x').

To verify that the above functions ψ_θ, $\theta \in \Theta$, satisfy the moment condition (5.17) for an M_{ψ_θ} such that the assumption $s < (M_{\psi_\theta} + 1)\underline{a}$ in Theorem 5.13 is fulfilled, note that

$$\widehat{\psi_\theta}(\xi) = |\det \mathcal{A}|^{-1} \mathcal{F}g\bigl({}^t\mathcal{A}^{-1}\xi', \xi_n\bigr).$$

Hence $D^\alpha \widehat{\psi_\theta}$ vanishes at $\xi = 0$ when $D^\alpha \widehat{g} = D^\alpha(-D_{\xi'})^{\gamma'}\widehat{k}(\xi)$ does so. Since $\widehat{k}(\xi) = -|\xi|^{2N}\widehat{k^0}(\xi)$ and $\widehat{k^0}(0) \ne 0$, we have that $D^\alpha \widehat{g}(0) = 0$ for α satisfying $|\alpha| + |\gamma'| \le 2N - 1$. In the course of the proof below, cf. Step 3, we obtain a θ-independent estimate of $|\gamma'|$, hence of M_{ψ_θ}.

Moreover, the constant A in Theorem 5.13 is finite: basic properties of the Fourier transform give the following estimate, where the constant is independent of $\mathcal{A}^{-1} \in \Theta$:

$$\begin{aligned}
\| D^\alpha \mathcal{F}\psi_\theta \,|L_\infty\| &\le \int |y^\alpha g(\mathcal{A}y', y_n)|\,dy \\
&= |\det \mathcal{A}^{-1}| \int |z_n^{\alpha_n}|\,|(\mathcal{A}^{-1}z')^{\alpha'}|\,|g(z)|\,dz \\
&\le c(\alpha, C_\sigma) \int_{|z|\le 1} |k(z)|\,dz.
\end{aligned}$$

To estimate B we exploit that $\mathcal{F}\colon B^{n/2}_{2,1}(\mathbb{R}^n) \to L_1(\mathbb{R}^n)$ is bounded according to Szasz's inequality (cf. [50, Prop. 1.7.5]) and obtain

$$\| (1+|\cdot|)^{M+1} D^\gamma \mathcal{F}\psi_\theta \,|L_1\| \le c\, \| y^\gamma g(\mathcal{A}y', y_n) \,|B^{M+1+\frac{n}{2}}_{2,1}\| \le c(\gamma, C_\sigma, C_\tau)\, \| k \,|C_0^m\|,$$

when $m \in \mathbb{N}$ is chosen so large that $m > M + 1 + n/2$. In fact, the last inequality is obtained using the embeddings $C_0^m \hookrightarrow H^m \hookrightarrow B^{M+1+n/2}_{2,1}$ and the estimate

$$\| y^\gamma \psi_\theta \,|C_0^m\| = \sup \bigl| \partial^\alpha \bigl(y^\gamma (\mathcal{A}y')^{\gamma'} k(\mathcal{A}y', y_n)\bigr)\bigr| \le c(\gamma, C_\sigma, C_\tau)\, \| k \,|C_0^m\|.$$

This relies on the higher-order chain rule, cf. Appendix A, and the support of k: it suffices to use the supremum over $|\alpha| \leq m$ and $\{y \in \mathbb{R}^n \mid |\mathcal{A}y'|^2 + y_n^2 \leq 1\}$, and for a point in this set $|y'| \leq \|\mathcal{A}^{-1}\||\mathcal{A}y'| \leq c(C_\sigma)$, so we need only estimate on an $\mathcal{A}$-independent cylinder.

Replacing k by k_0 in the definition of g and setting $\psi_{\theta,0}(y) = g(\mathcal{A}y', y_n)$, the finiteness of C and D follows analogously. The Tauberian properties follow from $\int k_0 \, dx \neq 0 \neq \int k^0 \, dx$.

Hence all assumptions in Theorem 5.13 are satisfied, and we are thus ready to prove our main result:

Theorem 5.19. *If σ is a bounded diffeomorphism on $\mathbb{R}^n$ on the form in* (5.29)*, then $f \mapsto f \circ \sigma$ is a linear homeomorphism $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ when* (5.28) *holds.*

Proof. According to Proposition 5.18, it suffices to consider $s > s_1$, say for

$$s_1 := K_0 a_0 + (n-1)\frac{a_0}{p_0} + \frac{a_n}{\min(p_0, p_n)}, \tag{5.38}$$

whereby K_0 is the smallest integer satisfying

$$K_0 a_0 > (n-1)\frac{a_0}{p_0} + \frac{a_n}{\min(p_0, p_n)}. \tag{5.39}$$

We now let $s \in]s_1, \infty[$ be given and take some $K \geq K_0$, i.e. K solving (5.39), such that

$$K a_0 + (n-1)\frac{a_0}{p_0} + \frac{a_n}{\min(p_0, p_n)} < s < 2K a_0. \tag{5.40}$$

(The interval thus defined is non-empty by (5.39), and the left end point is at least s_1.)

Note that (5.40) yields that every $f \in F^{s,\vec{a}}_{\vec{p},q}$ is continuous, cf. Lemma 5.3(iii); so are even the derivatives $D^\beta f$ for every $\beta = (\beta_1, \dots, \beta_{n-1}, 0)$ with $|\beta| \leq K$, since $s - \beta \cdot \vec{a} = s - |\beta| a_0 > \vec{a} \cdot 1/\vec{p}$.

Step 1. For the norms $\| f \circ \sigma \,|F^{s,\vec{a}}_{\vec{p},q}\|$ and $\| f \,|F^{s,\vec{a}}_{\vec{p},q}\|$ in inequality (5.30), which also here suffices, we use Theorem 5.15 with $2N > (K-1)(2K-1) + s/\underline{a}$.

By the symmetry of k_0 and k in (5.34), we shall estimate

$$k_j * (f \circ \sigma)(x) = \int_{|z| \leq 1} k(z) \, f(\sigma(x + 2^{-j\vec{a}} z)) \, dz, \quad j \in \mathbb{N}, \tag{5.41}$$

together with the corresponding expression for k_0, where k is replaced by k_0.

First we Taylor expand the entries in $\sigma'(x') := (\sigma_1(x'), \dots, \sigma_{n-1}(x'))$ to the order $2K - 1$. So for $\ell = 1, \dots, n-1$ there exists $\omega_\ell \in]0, 1[$ such that

$$\sigma_\ell(x' + z') = \sum_{|\alpha'| < 2K} \frac{\partial^{\alpha'} \sigma_\ell(x')}{\alpha'!} z'^{\alpha'} + \sum_{|\alpha'| = 2K} \frac{\partial^{\alpha'} \sigma_\ell(x' + \omega_\ell z')}{\alpha'!} z'^{\alpha'}. \tag{5.42}$$

For convenience, we let $\sum'_\alpha$ denote summation over multi-indices $\alpha \in \mathbb{N}_0^n$ having $\alpha_n = 0$ and define the vector of Taylor polynomials, respectively entries of a remainder R,

$$P_{2K-1}(z') = \sum_{|\alpha|\le 2K-1}{}' \frac{\partial^\alpha \sigma'(x')}{\alpha!} z^\alpha, \qquad R_\ell(z') = \sum_{|\alpha|=2K}{}' \frac{\partial^\alpha \sigma_\ell(x'+\omega_\ell z')}{\alpha!}\, z^\alpha.$$

Applying the Mean Value Theorem to f, cf. (5.40), now yields an $\widetilde{\omega} \in]0,1[$ so that

$$\begin{aligned}|k_j * (f\circ\sigma)(x)| \le \Big| \int_{|z|\le 1} k(z)\, f(P_{2K-1}(2^{-ja'}z'), x_n + 2^{-ja_n} z_n)\, dz\Big| \\ + \sum_{d=1}^{n-1} \int_{|z|\le 1} \big| k(z)\, \partial_{x_d} f(y', x_n + 2^{-ja_n} z_n)\, R_d(2^{-ja'} z')\big|\, dz,\end{aligned} \tag{5.43}$$

when $y' := P_{2K-1}(2^{-ja'}z') + \widetilde{\omega}(R_1(2^{-ja'}z'), \dots, R_{n-1}(2^{-ja'}z'))$. Using (5.24) and (5.42), it is obvious that this y' fulfils

$$|\sigma(x) - (y', x_n + 2^{-ja_n} z_n)| \le |\sigma'(x') - y'| + |2^{-ja_n} z_n| < C \tag{5.44}$$

for each $z \in \operatorname{supp} k$ and a constant C depending only on n and $C_{\alpha,\sigma}$ with $|\alpha| \le 2K$.

Step 2. Concerning the remainder terms in (5.43), we exploit (5.44) to get

$$\begin{aligned}&\int_{|z|\le 1} \big| k(z)\, \partial_{x_d} f(y', x_n + 2^{-ja_n} z_n)\, R_d(2^{-ja'} z')\big|\, dz \\ &\qquad \le 2^{-2jKa_0} \Big(\sum_{|\alpha'|=2K} \frac{\|\partial^{\alpha'} \sigma_d \,|L_\infty\|}{\alpha'!}\Big) \int_{|z|\le 1} |k(z)|\, dz \sup_{|\sigma(x)-y|<C} |\partial_{x_d} f(y)|.\end{aligned}$$

The exponent in 2^{-2jKa_0} is a result of (5.28) and the chosen Taylor expansion of $\sigma(x + 2^{-j\vec{a}} z)$, and since $s - 2Ka_0 < 0$ the norm of ℓ_q is trivial to calculate, whence

$$\begin{aligned}\Big\| 2^{js} \int_{|z|\le 1} \big| k(z)\, \partial_{x_d} f(y', x_n + 2^{-ja_n} z_n)\, R_d(2^{-ja'} z')\big|\, dz \,\Big| L_{\vec{p}}(\ell_q)\Big\| \\ \le c \Big\| \sup_{|\sigma(x)-y|<C} |\partial_{x_d} f(y)| \,\Big| L_{\vec{p}}(\mathbb{R}^n_x)\Big\|.\end{aligned} \tag{5.45}$$

Now we use that $p_1 = \ldots = p_{n-1}$ to change variables in the resulting integral over $\mathbb{R}^{n-1}$, with τ' denoting $(\sigma')^{-1}$. Since Lemma 5.4 in view of (5.40) applies to $\partial_{x_d} f$, $d = 1, \dots, n-1$, the right-hand side of the last inequality can be estimated, using also Lemma 5.3(i), by

$$c\Big(\sup_{y\in\mathbb{R}^{n-1}} |\det J\tau'(y)|\Big)^{1/p_0} \| \partial_{x_d} f \,| F^{s-a_d,\vec{a}}_{\vec{p},q}\| \le c \| f \,| F^{s,\vec{a}}_{\vec{p},q}\|. \tag{5.46}$$

Step 3. To treat the first term in (5.43), we Taylor expand $f(\cdot, x_n)$, which is in $C^K(\mathbb{R}^{n-1})$. Setting $P(z') = P_{2K-1}(z') - P_1(z')$, expansion at the vector $P_1(2^{-ja'}z')$ gives

$$
\begin{aligned}
f(P_{2K-1}&(2^{-ja'}z'), x_n + 2^{-ja_n}z_n) \\
&= \sum_{0\le|\beta|\le K-1}{}' \frac{D^\beta f(P_1(2^{-ja'}z'), x_n + 2^{-ja_n}z_n)}{\beta!} P(2^{-ja'}z')^\beta \\
&\quad + \sum_{|\beta|=K}{}' \frac{D^\beta f(y', x_n + 2^{-ja_n}z_n)}{\beta!} P(2^{-ja'}z')^\beta,
\end{aligned} \tag{5.47}
$$

where y' is a vector analogous to that in (5.43) and satisfies (5.44), perhaps with another C.

To deal with the remainder in (5.47), note that the order was chosen to ensure that, in the powers $P(2^{-ja'}z')^\beta$, the l'th factor is the β_l'th power of a sum of terms each containing a factor $2^{-ja_0|\alpha'|}$ with $|\alpha'| \ge 2$. Hence each $|\beta| = K$ in total contributes by $O(2^{-2jKa_0})$. More precisely, as in Step 2 we obtain

$$
\begin{aligned}
\int_{|z|\le1} &\Big| k(z) \sum_{|\beta|=K}{}' \frac{D^\beta f(y', x_n + 2^{-ja_n}z_n)}{\beta!} P(2^{-ja'}z')^\beta \Big| \, dz \\
&\le 2^{-j2Ka_0} \int_{|z|\le1} |k(z)| \, dz \Big(\sum_{2\le|\alpha|\le 2K-1} C_{\alpha,\sigma} \Big)^K \sum_{|\beta|=K}{}' \sup_{|\sigma(x)-y|<C} |D^\beta f(y)|.
\end{aligned}
$$

In view of (5.40), Lemma 5.4 barely also applies to $D^\beta f$ for $|\beta| = K$, so the above gives

$$
\begin{aligned}
\Big\| 2^{sj} \int_{|z|\le1} & k(z) \sum_{|\beta|=K}{}' \frac{D^\beta f(y', x_n + 2^{-ja_n}z_n)}{\beta!} P(2^{-ja'}z')^\beta \, dz \Big| L_{\vec p}(\ell_q) \Big\| \\
&\le c \Big(\sup_{y\in\mathbb{R}^{n-1}} |\det J\tau'(y)| \Big)^{\frac{1}{p_0}} \sum_{|\beta|=K}{}' \| D^\beta f \,|F^{s-\beta\cdot\vec a,\vec a}_{\vec p,q}\| \le c \| f \,|F^{s,\vec a}_{\vec p,q}\|.
\end{aligned}
$$

Now it remains to estimate the other terms resulting from (5.47), i.e.

$$
\sum_{0\le|\beta|\le K-1}{}' \int_{|z|\le1} k(z) \frac{D^\beta f(P_1(2^{-ja'}z'), x_n + 2^{-ja_n}\, z_n)}{\beta!} P(2^{-ja'}z')^\beta \, dz.
$$

Using the multinomial formula on $P(z') = \sum'_{2\le|\gamma|\le2K-1} z^\gamma \partial^\gamma \sigma'(x')/\gamma!$ and g, ψ_θ discussed in (5.37), the above task is finally reduced to controlling terms like

$$
\begin{aligned}
I_{j,\beta,\gamma}&(\sigma'(x'), x_n) \\
&:= 2^{-2j|\beta|a_0} \int_{|z|\le1} g(z) D^\beta f(\sigma'(x') + 2^{-ja_0} J\sigma'(x')z', x_n + 2^{-ja_n}z_n) \, dz \qquad (5.48) \\
&= 2^{-2j|\beta|a_0} |\det \mathcal{A}| \int \psi_\theta(y) D^\beta f(\sigma'(x') + 2^{-ja_0}y', x_n + 2^{-ja_n}y_n) \, dy.
\end{aligned}
$$

Note that in g, ψ_θ we have $2 \le |\gamma| \le |\beta|(2K-1)$ and $|\beta| \le K-1$, $\beta_n = 0 = \gamma_n$.

Step 4. Before we estimate (5.48), it is first observed that all previous steps apply in a similar way to the convolution $k_0 * (f \circ \sigma)$; except in this case there is no dilation, thus the ℓ_q-norm is omitted and the function ψ_θ is replaced by $\psi_{\theta,0}$.

So, when collecting the terms of the form (5.48) with finitely many β, γ in both cases (omitting remainders from Steps 2–3), we obtain with two changes of variables and (5.20),

$$\begin{aligned}
&\Big\| \sum_{\beta,\gamma}{}' I_{0,\beta,\gamma}(\sigma'(x'),x_n) \,\Big|\, L_{\vec p}\Big\| + \Big\| 2^{js} \sum_{\beta,\gamma}{}' I_{j,\beta,\gamma}(\sigma'(x'),x_n) \,\Big|\, L_{\vec p}(\ell_q)\Big\| \\
&\le c \sum_{\beta,\gamma}{}' \big(\sup_{y\in\mathbb{R}^{n-1}} |\det J\tau'(y)|\big)^{\frac{1}{p_0}} \Big(\Big\| \int \psi_{\theta,0}(y) D^\beta f(x-y)\,dy \,\Big|\, L_{\vec p}\Big\| \\
&\qquad\qquad + \Big\| 2^{j(s-2|\beta|a_0)} \int \psi_\theta(y) D^\beta f(x-2^{-j\vec a}y)\,dy \,\Big|\, L_{\vec p}(\ell_q)\Big\| \Big) \\
&\le c \sum_{\beta,\gamma}{}' \Big\| \Big\{ 2^{j(s-2|\beta|a_0)} \sup_{\theta\in\Theta} \psi^*_{\theta,j} D^\beta f \Big\}_{j=0}^\infty \,\Big|\, L_{\vec p}(\ell_q)\Big\|.
\end{aligned} \tag{5.49}$$

Here we apply Theorem 5.13 to the family of functions $\psi_{\theta,0}, \psi_\theta$ with the φ_j chosen as the Fourier transformed of the system in the Littlewood–Paley decomposition, cf. (5.3). Estimating $|\gamma|$, the ψ_θ satisfy the moment condition (5.17) with $M_{\psi_\theta} := 2N - 1 - (K-1)(2K-1)$, which fulfils $s < (M_{\psi_\theta}+1)\underline{a}$, because of the choice of N in Step 1. So, by applying Theorem 5.14 and Lemma 5.3(i), using $s - 2|\beta|a_0 \le s - \beta\cdot\vec a$, the above is estimated thus:

$$\begin{aligned}
&\Big\| \Big\{ 2^{js} \sum_{\beta,\gamma}{}' I_{j,\beta,\gamma}(\sigma'(x'),x_n) \Big\}_{j=0}^\infty \,\Big|\, L_{\vec p}(\ell_q)\Big\| \\
&\qquad \le c(A+B+C+D) \sum_{\beta,\gamma}{}' \Big\| \Big\{ 2^{j(s-2|\beta|a_0)} (\mathcal{F}^{-1}\Phi_j)^* D^\beta f \Big\}_{j=0}^\infty \,\Big|\, L_{\vec p}(\ell_q)\Big\| \\
&\qquad \le c \sum_{\beta,\gamma}{}' \| D^\beta f \,|\, F^{s-2|\beta|a_0,\vec a}_{\vec p,q} \| \le c \| f \,|\, F^{s,\vec a}_{\vec p,q} \|.
\end{aligned}$$

This proves the necessary estimate for the given $s > s_1$. □

5.4.2 Groups of bounded diffeomorphisms

It is not difficult to see that the proofs in Section 5.4.1 did not really use that x_n is a single variable. It could just as well have been replaced by a whole group of variables x'', corresponding to a splitting $x = (x', x'')$, provided σ acts as the identity on x''.

Moreover, x' could equally well have been 'embedded' into x'', that is x'' could contain variables x_k both with $k < j_0$ and with $k > j_1$ when $x' = (x_{j_0}, \dots, x_{j_1})$ (but no interlacing); in particular the changes of variables yielding (5.46) would carry over to this situation when $p_{j_0} = \ldots = p_{j_1}$. It is also not difficult to see that Proposition 5.18 extends to this situation when $a_{j_0} = \ldots = a_{j_1}$ (perhaps with several g_1-terms, each having a value of μ).

Thus we may generalise Theorem 5.19 to situations with a splitting into $m \geq 2$ groups, i.e. $\mathbb{R}^n = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_m}$ where $N_1 + \ldots + N_m = n$, namely when

$$\vec{p} = (\underbrace{p_1, \ldots, p_1}_{N_1}, \underbrace{p_2, \ldots, p_2}_{N_2}, \;\ldots\;, \underbrace{p_m, \ldots, p_m}_{N_m}), \tag{5.50}$$

$$\vec{a} = (a_1, \ldots, a_1, a_2, \ldots, a_2, \;\ldots\;, a_m, \ldots, a_m), \tag{5.51}$$

$$\sigma(x) = (\sigma_1'(x_{(1)}), \sigma_2'(x_{(2)}), \;\ldots\;, \sigma_m'(x_{(m)})) \tag{5.52}$$

with arbitrary bounded diffeomorphisms σ_j' on $\mathbb{R}^{N_j}$ and $x_{(j)} \in \mathbb{R}^{N_j}$.

Indeed, viewing σ as a composition of $\sigma_1 := \sigma_1' \otimes \mathrm{id}_{\mathbb{R}^{n-N_1}}$ etc. on $\mathbb{R}^n$, the above gives

$$\| f \circ \sigma \,| F^{s,\vec{a}}_{\vec{p},q} \| \leq c \| f \circ \sigma_m \circ \ldots \circ \sigma_2 \,| F^{s,\vec{a}}_{\vec{p},q} \| \leq \ldots \leq c \| f \,| F^{s,\vec{a}}_{\vec{p},q} \|.$$

Theorem 5.20. *The map* $f \mapsto f \circ \sigma$ *is a linear homeomorphism on* $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ *when* (5.50)–(5.52) *hold.*

5.5 Derived Results

5.5.1 Diffeomorphisms on Domains

The strategies of Proposition 5.18 and Theorem 5.19 also give the following local version. E.g., for the paraboloid $U = \{ x \mid x_n > x_1^2 + \ldots + x_{n-1}^2 \}$ we may take σ to consist in a rotation around the x_n-axis; cf. (5.29).

Theorem 5.21. *Let* $U, V \subset \mathbb{R}^n$ *be open and* $\sigma : U \to V$ *a* C^∞*-bijection as in* (5.29). *If* (5.28) *is fulfilled and* $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(V)$ *has compact support, then* $f \circ \sigma \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ *and*

$$\| f \circ \sigma \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(U) \| \leq c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(V) \| \tag{5.53}$$

holds for a constant c depending only on σ *and the set* $\operatorname{supp} f$.

Proof. Step 1. Let us consider $s > s_1$, cf. (5.38), and adapt the proof of Theorem 5.19 to the local set-up. We shall prove the statement for the $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(V)$ satisfying $\operatorname{supp} f \subset K \subset V$ for some arbitrary compact set K. First we fix $r \in]0,1[$ so small that

$$6r < \min\big(\operatorname{dist}(K, \mathbb{R}^n \setminus V), \operatorname{dist}(\sigma^{-1}(K), \mathbb{R}^n \setminus U) \big). \tag{5.54}$$

Then, by Lemma 5.17, we have $\| f \circ \sigma \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(U) \| = \| e_U (f \circ \sigma) \,| F^{s,\vec{a}}_{\vec{p},q} \|$ when Theorem 5.15 is utilised for $k_0, k \in \mathcal{S}$, say so that $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$; cf. also (5.34). Extension by 0 outside U of $f \circ \sigma$ is redundant, for it suffices to integrate over $x \in W := \operatorname{supp}(f \circ \sigma) + \overline{B}(0,r)$. However, to apply the Mean Value Theorem, cf. (5.43), we extend f by 0 instead, i.e. we consider (5.41) with integration over $|z| \leq r$ and with f replaced by $e_V f$.

Since $e_V f$ inherits the regularity of f (cf. Lemma 5.17) and $\partial^\alpha\sigma$ can be estimated on the compact set W, the proof of Theorem 5.19 carries over straightforwardly. E.g. one obtains a variant of (5.46), where $|\det J\tau'(x')|^{1/p_0}$ is estimated over $\{x' \mid \exists x_n\colon (x',x_n)\in\sigma(W)\}$, and the integration is then extended to $\mathbb{R}^n$, which by Lemma 5.17 yields

$$\Big\| \sup_{|x-y|<C} |\partial_{x_d} e_V f(y)| \,\big|L_{\vec p}(\mathbb{R}^n_x)\Big\| \le c\,\| e_V f\,|F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)\| = c\,\| f\,|\overline{F}^{s,\vec a}_{\vec p,q}(V)\|.$$

To estimate the first term in (5.43) in this local version, the argumentation there is modified as above and the index set Θ is chosen to be the set of all $(n-1)\times(n-1)$ matrices satisfying (5.35) with infimum over $x\in W$ and (5.36) with $C_\sigma := \max_{\substack{1\le j\le n,\\ |\alpha|=1}} \sup_{x\in W} |D^\alpha\sigma_j(x)|$.

Before applying Theorem 5.13 to the new estimate (5.49), the integration is extended to $\mathbb{R}^n$ (using $e_V f$). Then application of Theorems 5.13 and 5.14 together with Lemma 5.17 finish the proof for $s>s_1$.

Step 2. For $s\le s_1$ we use Lemma 5.7 to obtain an $h\in F^{s+r,\vec a}_{\vec p,q}(\mathbb{R}^n)$ such that $e_V f=\Lambda_r h$; thus the identity (5.31) holds in $\mathcal{D}'(\mathbb{R}^n)$ for $e_V f$ and h. Applying r_V to both sides and using that it commutes with differentiation on $C_0^\infty(\mathbb{R}^n)$, hence on $\mathcal{D}'(\mathbb{R}^n)$, we obtain (5.32) as an identity in $\mathcal{D}'(V)$ for the new $g_0 := (r_V h)\circ\sigma$ and $g_1 := \big(r_V(1-\partial^2_{x_n})^\mu h\big)\circ\sigma$.

Composing with σ yields an identity in $\mathcal{D}'(U)$, when $\eta_{k,\beta}\circ\sigma$ is treated using cut-off functions. E.g. we can take $\chi,\chi_1\in C_0^\infty(U)$ such that $\chi\equiv 1$ on the set $\operatorname{supp}(f\circ\sigma)+\overline{B}(0,r) =: W_r$ and such that $\operatorname{supp}\chi\subset W_{2r}$, while $\chi_1\equiv1$ on W_{3r} and $\operatorname{supp}\chi_1\subset W_{4r}$. This entails

$$\chi\cdot f\circ\sigma = \sum_{l=0}^{d_n-\mu} \eta_{n,l}\chi\,\partial^{2l}_{x_n}(\chi_1 g_1) + \sum_{k=1}^{n-1}\sideset{}{'}\sum_{|\beta|\le 2d_0} \eta_{k,\beta}\circ\sigma\cdot\chi\,\partial^\beta(\chi_1 g_0). \tag{5.55}$$

Using e_U on both sides, Lemmas 5.17 and 5.12 imply (with $\mathbb{R}^n$ omitted),

$$\begin{aligned}\| f\circ\sigma\,|\overline{F}^{s,\vec a}_{\vec p,q}(U)\|^d &\le c\sum_{l=0}^{d_n-\mu} \big\| e_U(\partial^{2l}_{x_n}(\chi_1 g_1))\,\big|F^{s,\vec a}_{\vec p,q}\big\|^d\\ &\quad + c\sideset{}{'}\sum_{|\beta|\le 2d_0} \big\| e_U(\partial^\beta(\chi_1 g_0))\,\big|F^{s,\vec a}_{\vec p,q}\big\|^d.\end{aligned}$$

As e_U and differentiation commute on $\mathcal{E}'(U)\ni\chi_1 g_j$, Lemma 5.3(i) leads to an estimate from above. But Lemma 5.17 applies since the supports are in W_{4r}, so with $\widetilde{\chi}_1 := \chi_1\circ\tau$ we find that the above is less than or equal to

$$\begin{aligned}&c\big\| e_U(\chi_1 g_1)\,\big|F^{s+r_\mu,\vec a}_{\vec p,q}\big\|^d + c\| e_U(\chi_1 g_0)\,|F^{s+r,\vec a}_{\vec p,q}\|^d\\ &\quad= c\big\|\big(\widetilde{\chi}_1\cdot r_V(1-\partial^2_{x_n})^\mu h\big)\circ\sigma\,\big|\overline{F}^{s+r_\mu,\vec a}_{\vec p,q}(U)\big\|^d + c\big\|(\widetilde{\chi}_1\cdot r_V h)\circ\sigma\,\big|\overline{F}^{s+r,\vec a}_{\vec p,q}(U)\big\|^d.\end{aligned}$$

Using Step 1 and Lemmas 5.12, 5.8, 5.7 and 5.17, this entails

$$\begin{aligned}\| f\circ\sigma \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\|^d &\le c\| (1-\partial^2_{x_n})^\mu h\,|F^{s+r_\mu,\vec{a}}_{\vec{p},q}\|^d + c\| h\,|F^{s+r,\vec{a}}_{\vec{p},q}\|^d\\ &\le c\| \Lambda_r^{-1} e_V f\,|F^{s+r,\vec{a}}_{\vec{p},q}\|^d \le c\| f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(V)\|^d,\end{aligned}$$

which shows the local theorem for $s\le s_1$. □

There is also a local version of Theorem 5.20, with similar proof, namely

Theorem 5.22. *Let $\sigma_j : U_j \to V_j$ for $j=1,\dots,m$ be C^∞-bijections, where $U_j, V_j \subset \mathbb{R}^{N_j}$ are open. When $\vec{a},\vec{p}$ fulfil* (5.50)–(5.51) *and $f\in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U_1\times\dots\times U_m)$ has compact support, then* (5.53) *holds true for $U = U_1\times\dots\times U_m$ and $V = V_1\times\dots\times V_m$.*

As a preparation for our upcoming work [30], we include a natural extension to the case of an infinite cylinder, where $\operatorname{supp} f$ is only required to be compact on cross sections:

Theorem 5.23. *Let $\sigma : U\times\mathbb{R}\to V\times\mathbb{R}$, where $U,V\subset\mathbb{R}^{n-1}$ are open, be a C^∞-bijection on the form* (5.29)*. If* (5.28) *holds and $f\in\overline{F}^{s,\vec{a}}_{\vec{p},q}(V\times\mathbb{R})$ has $\operatorname{supp} f\subset K\times\mathbb{R}$, whereby $K\subset V$ is compact, then $f\circ\sigma\in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})$ and*

$$\| f\circ\sigma\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})\| \le c(\operatorname{supp} f,\sigma)\| f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|.$$

Proof. We adapt the proof of Theorem 5.21: in Step 1 we take $r\in]0,1[$ so small that $6r$ is less than both $\operatorname{dist}(K,\mathbb{R}^{n-1}\setminus V)$ and $\operatorname{dist}(\sigma'^{-1}(K),\mathbb{R}^{n-1}\setminus U)$. Since the extension by 0 of f, i.e. $e_{V\times\mathbb{R}}f$, is well defined, as $K\subset V$ is compact, it is an immediate corollary to the proof of Lemma 5.17 that

$$\| f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(V\times\mathbb{R})\| = \| e_{V\times\mathbb{R}} f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|. \tag{5.56}$$

Then the rest of the proof for $s>s_1$ follows that of Theorem 5.21, now with $W := (\sigma'^{-1}(K)+\overline{B}(0,r))\times\mathbb{R}$.

For $s\le s_1$ we have $e_{V\times\mathbb{R}}f = \Lambda_r h$ for some $h\in F^{s+r,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$; cf. Lemma 5.7. Hence (5.32) holds as an identity in $\mathcal{D}'(V\times\mathbb{R})$ for $g_1 := \big(r_{V\times\mathbb{R}}(1-\partial^2_{x_{n+1}})^\mu h\big)\circ\sigma$ and $g_0 := (r_{V\times\mathbb{R}}h)\circ\sigma$.

The $\eta_{k,\beta}\circ\sigma$ are controlled using cut-off functions $\chi,\chi_1\in C^\infty_{L_\infty}(U)$ with similar properties in terms of the sets $W_r = (\sigma'^{-1}(K)+\overline{B}(0,r))\times\mathbb{R}$. Thus we obtain (5.55) in $\mathcal{D}'(U\times\mathbb{R})$.

Now, as in (5.56) it is seen that $f\circ\sigma$ and $e_{U\times\mathbb{R}}(\chi\cdot f\circ\sigma)$ have identical norms, so the estimates in Step 2 of the proof of Theorem 5.21 finish the proof, mutatis mutandis. □

5.5.2 Isotropic Spaces

Going to the other extreme, that is when also $a_n = a_0$ and $p_n = p_0$, then the Lizorkin–Triebel spaces are invariant under any bounded diffeomorphism (i.e. without (5.29)), since in this case we can just change variables in all coordinates, in particular in (5.45)–(5.46).

Moreover, we can adapt Proposition 5.18 by taking $d_n = d_0$ and $\mu = 0$ in the proof; and the set-up prior to Theorem 5.19 is also easily modified to the isotropic situation. Hence we obtain

Corollary 5.24. *When $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is any bounded diffeomorphism, then the map $f \mapsto f \circ \sigma$ is a linear homeomorphism of $F^s_{p,q}(\mathbb{R}^n)$ onto itself for all $s \in \mathbb{R}$.*

This is known from work of Triebel [58, Thm. 4.3.2], which also contains a corresponding result for $B^s_{p,q}$. (It is this proof we extended to mixed norms in the previous section.) The result has also been obtained recently by Scharf [47], who covered all $s \in \mathbb{R}$ by means of an extended notion of atomic decompositions.

In an analogous way, we also obtain an isotropic counterpart to Theorem 5.21:

Corollary 5.25. *When $\sigma : U \to V$ is a C^∞-bijection between open subsets U, V of $\mathbb{R}^n$, then $f \circ \sigma \in \overline{F}^s_{p,q}(U)$ for every $f \in \overline{F}^s_{p,q}(V)$ having compact support and*

$$\| f \circ \sigma \,|\overline{F}^s_{p,q}(U)\| \le c(\operatorname{supp} f, \sigma)\| f \,|\overline{F}^s_{p,q}(V)\|.$$

A The Higher-Order Chain Rule

For convenience we give a formula for higher order derivatives of a composite map

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{C}.$$

Namely, when f, g are C^k and $x_0 \in \mathbb{R}^n$ then for any multi-index γ with $1 \le |\gamma| \le k$,

$$\partial^\gamma (g \circ f)(x_0) = \sum_{1\le|\alpha|\le|\gamma|} \partial^\alpha g(f(x_0)) \sum_{\substack{\forall j:\ \alpha_j = \sum n_{\beta^j} \\ \gamma = \sum_{j,\beta^j} n_{\beta^j}\beta^j}} \gamma! \prod_{\substack{j=1,\dots,m \\ 1\le|\beta^j|\le|\gamma|}} \frac{1}{n_{\beta^j}!}\Big(\frac{\partial^{\beta^j} f_j(x_0)}{\beta^j!}\Big)^{n_{\beta^j}}. \tag{5.57}$$

The first sum is over multi-indices $\alpha = (\alpha_1, \dots, \alpha_m)$, which in the second are split

$$\alpha_1 = \sum_{1\le|\beta^1|\le|\gamma|} n_{\beta^1}, \quad \dots, \quad \alpha_m = \sum_{1\le|\beta^m|\le|\gamma|} n_{\beta^m}$$

into integers $n_{\beta^j} \ge 0$ (parametrised by $\beta^j = (\beta^j_1, \dots, \beta^j_n) \in \mathbb{N}^n_0$, with upper index j) that fulfil the constraint

$$\gamma = \sum_{j=1}^m \sum_{1\le|\beta^j|\le|\gamma|} n_{\beta^j}\beta^j. \tag{5.58}$$

Formula (5.57) and (5.58) result from Taylor's limit formula:

$$g(y+y_0)=\sum_{|\alpha|\le k} c_\alpha y^\alpha + o(|y|^k).$$

It holds for $y\to 0$ if *and only if* $c_\alpha=\frac{1}{\alpha!}\partial^\alpha g(y_0)$ for all $|\alpha|\le k$. (Necessity is seen recursively for $y\to 0$ along suitable lines; sufficiency from the integral remainder.)

Indeed, $k=|\gamma|$ suffices, and with $y=f(x+x_0)-f(x_0)$ Taylor's formula applies both to g and to each entry f_j (by summing over an auxiliary $\beta^j\in\mathbb{N}_0^n$),

$$g(f(x+x_0))=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\partial^\alpha g(f(x_0))y_1^{\alpha_1}\dots y_m^{\alpha_m}+o(|y|^k) \tag{5.59}$$
$$=\sum_{|\alpha|\le k}\partial^\alpha g(f(x_0))\prod_{j=1}^m\frac{1}{\alpha_j!}\Big(\sum_{1\le|\beta^j|\le k}\frac{x^{\beta^j}}{(\beta^j)!}\partial^{\beta^j}f_j(x_0)+o(|x|^k)\Big)^{\alpha_j}+o(|y|^k).$$

The first remainder is $o(|x|^k)$ as $o(|y|^k)/|x|^k=o(1)(|f(x+x_0)-f(x_0)|/|x|)^k\to 0$. Using the binomial formula and expanding $\prod_{j=1}^m$, the other remainders are also seen to contribute by terms that are $o(|x|^k)$ or better. Thus a single $o(|x|^k)$ suffices.

Hence we shall expand $(\dots)^{\alpha_j}$ using the multinomial formula. So we split $\alpha_j=\sum n_{\beta^j}$, with integers $n_{\beta^j}\ge 0$, in the sum over all multi-indices $\beta^j\in\mathbb{N}_0^n$ with $1\le|\beta^j|\le k$. The corresponding multinomial coefficient is $\alpha_j!/\prod_{\beta^j}(n_{\beta^j})!$, so (5.59) yields

$$g(f(x+x_0)) \tag{5.60}$$
$$=\sum_{|\alpha|\le k}\partial^\alpha g(f(x_0))\prod_{j=1}^m\sum_{\alpha_j=\sum n_{\beta^j}}\prod_{1\le|\beta^j|\le k}\frac{1}{n_{\beta^j}!}\Big(\frac{x^{\beta^j}}{\beta^j!}\partial^{\beta^j}f_j(x_0)\Big)^{n_{\beta^j}}+o(|x|^k).$$

Calculating these products, of factors having a choice of $\alpha_j=\sum n_{\beta^j}$ for every $j=1,\dots,m$, one obtains polynomials x^ω associated to the multi-indices $\omega=\sum_{j=1}^m\sum_{1\le|\beta^j|\le k}n_{\beta^j}\beta^j$.

For $|\omega|>k$ these are $o(|x|^k)$, hence contribute to the remainder. Thus modified, (5.60) is Taylor's formula of order k for $g\circ f$, so that $\partial^\gamma(g\circ f)(x_0)/\gamma!$ is given by the coefficient of x^ω for $\omega=\gamma$, which yields (5.57)–(5.58).

This concise proof has seemingly not been worked out before, so it should be interesting in its own right. E.g. the Taylor expansions make the presence of the β^j obvious, and the condition $\gamma=\sum_{j,\beta^j}n_{\beta^j}\beta^j$ is natural. Also the constants $\gamma!/\prod n_{\beta^j}!$ and $(\beta^j)!^{-n_{\beta^j}}$ lead to easy applications. Clearly $\partial^\alpha g(f(x_0))$ is multiplied by a polynomial in the derivatives of $f_1,\dots,f_m$, which has degree $\sum_{j=1}^m\sum_{\beta^j}n_{\beta^j}=\sum_j\alpha_j=|\alpha|$.

The formula (5.57) itself is well known for $n=1=m$ as the Faa di Bruno formula; cf. [33] for its history. For higher dimensions, the formulas seem to have been less explicit.

The other contributions we know have been rather less straightforward, because of reductions, say to f, g being polynomials (or to finite Taylor series), and/or by use of lengthy combinatorial arguments with recursively given polynomials, which replace the sum over the β^j in (5.57); such as the Bell polynomials that are used in e.g. [42, Thm. 4.2.4].

Closest to the present approach, we have found the contributions [53] and [13] in case of one and several variables, respectively.

CHAPTER 6

Anisotropic Lizorkin–Triebel Spaces with Mixed Norms — Traces on Smooth Boundaries

Publication details

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Abstract:
This article deals with trace operators on anisotropic Lizorkin–Triebel spaces with mixed norms over cylindrical domains, where the boundary is sufficiently smooth. As a preparation we include a rather self-contained exposition of Lizorkin–Triebel spaces on manifolds and extend these results to mixed-norm Lizorkin–Triebel spaces on cylinders in Euclidean space.

6.1 Introduction

The present paper departs from the work [29] of the first and third author dealing with traces on hyperplanes of anisotropic Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with mixed norms.

The application of such spaces to parabolic differential equations is to some extent known. It was outlined in the introduction to [29] how they apply to fully inhomogeneous boundary value problems: for such problems the $F^{s,\vec{a}}_{\vec{p},q}$-spaces are in general *inevitable* for a correct description of the boundary data.

Previously, a somewhat similar conclusion had been obtained in works of Weidemaier [64–66] (and also by Denk, Hieber and Prüss [9]). He discovered the necessity of isotropic Lizorkin–Triebel spaces (for vector-valued functions) for an optimal description of the time regularity of the boundary data. However, with integral exponents p_x and p_t in the space and time directions, respectively, Weidemaier worked under the technical restriction that $p_x \leq p_t$.

For the reader's sake, it is recalled that the main purpose of [29] was to extend the classical theory of trace operators to the $F^{s,\vec{a}}_{\vec{p},q}$-scales. However, because the mixed norms do not allow a change of integration order, this meant that the techniques had to be worked out both for the 'inner' and 'outer' traces given on, say smooth functions as

$$u(x_1, x'') \mapsto u(0, x''), \quad \text{resp.} \quad u(x', x_n) \mapsto u(x', 0).$$

When $u \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, then in the first case the trace was proved to be surjective on the mixed-norm Lizorkin–Triebel space $F^{s-a_1/p_1,a''}_{p'',p_1}(\mathbb{R}^{n-1})$ having the specific sum exponent $q = p_1$, while in the second case the trace space is (as usual) a Besov space, namely $B^{s-a_n/p_n,a'}_{p',p_n}(\mathbb{R}^{n-1})$.

As indicated, only traces on hyperplanes were covered in [29]; but the study included (almost) necessary and sufficient conditions on s in relation to $\vec{a}$, $\vec{p}$ and q, also in combination with normal derivatives (Cauchy traces), and existence and continuity of right-inverses. Furthermore, Weidemaier's restriction on the integral exponents was never encountered with the framework and methods adopted in [29].

These investigations in [29] are in this work followed up with a general study of trace operators and their right-inverses in the scales $F^{s,\vec{a}}_{\vec{p},q}$ of anisotropic Lizorkin–Triebel spaces with mixed norms defined on smooth cylinders $\Omega \times I$ and their curved boundaries $\partial\Omega \times I$.

In doing so, it is a main technical question to obtain invariance of the spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ under the map $f \mapsto f \circ \sigma$, when $U \subset \mathbb{R}^n$ is open and σ is a C^∞-bijection. We addressed this question in our joint paper [31], where we proved invariance e.g. under the restriction that σ only affects groups of coordinates x_j for which the corresponding p_j are equal in the vector of integral exponents $\vec{p} = (p_1, \dots, p_n)$; and similarly for the a_j.

This was done by generalising Triebel's method in [58, 4.3.2]. Indeed, having reduced to large s using a lift operator, it relies on Taylor expansion of the inner and outer functions, whereby most terms are manageable when the $F^{s,\vec{a}}_{\vec{p},q}$-spaces are normed via kernels of local means developed in [32]; an underlying parameter-dependent estimate obtained in [32] finally gives control over the effects of the Jacobian matrices.

In this paper, we develop the consequences for trace operators. E.g. the trace r_0 at $\{t = 0\}$ of $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, where $I :=]0, T[$, is given a meaning in a pedestrian way using an arbitrary extension of u to $\mathbb{R}^{n+1}$ and applying the trace at $\{t = 0\}$ from [29]. We obtain, using the splitting $\vec{p} = (p', p_t)$ with all entries in p' being equal and likewise for $\vec{a}$, the following, cf. Theorem 6.36 below:

Theorem. *For s sufficiently large (cf.* (6.57) *below) the operator r_0 is a bounded* surjection*,*

$$r_0 : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega).$$

Furthermore, r_0 has a right-inverse K_0 and it is bounded for every $s \in \mathbb{R}$,

$$K_0 : \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I).$$

The process of giving meaning to the curved trace γ of u is more involved, since it requires to first work locally and then observing that the local pieces define a global trace. After this has been done, we obtain using the splitting $\vec{p} = (p_1, p'')$, where $p_1 = \ldots = p_n$ and likewise for $\vec{a}$, cf. Theorem 6.44 below:

Theorem. *When $\partial\Omega$ is compact and s is sufficiently large (cf.* (6.72)*), the operator γ is a bounded* surjection*,*

$$\gamma : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I).$$

Furthermore, γ has a right-inverse K_γ and it is bounded for every $s \in \mathbb{R}$,

$$K_\gamma : \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I).$$

The right-inverse K_γ is constructed using the right-inverse in [29, Thm. 2.6] to the trace at $\{x_1 = 0\}$ and Rychkov's universal extension operator in [45], which is modified such that it applies to anisotropic, mixed-norm Lizorkin–Triebel spaces over half-spaces, cf. Section 6.5.

We also give in Theorem 6.43 an *explicit* construction of a right-inverse Q_Ω (of r_0 on $\mathbb{R}^{n+1}$) having the support preserving property

$$Q_\Omega : \mathring{B}^{s,a'}_{p',p_t}(\overline{\Omega}) \to \mathring{F}^{s,\vec{a}}_{\vec{p},q}(\overline{\Omega} \times \mathbb{R}) \quad \text{for all } s \in \mathbb{R}.$$

Finally, we analyse in Section 6.6.4 and 6.6.5 traces at the curved corner $\Gamma\times\{0\}$ associated to $\Omega\times I$ and follow up by giving the resulting compatibility properties for solutions of the heat equation.

Contents. Section 6.2 contains a review of our notation and the definition of anisotropic Lizorkin–Triebel spaces with mixed norms is recalled, together with some needed properties and a pointwise multiplier assertion. Moreover, a basic lemma for elements in $F^{s,\vec{a}}_{\vec{p},q}$ with compact support on cross sections of the cylindrical domain is proved.

In Section 6.3 sufficient conditions for $f\mapsto f\circ\sigma$ to leave the spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ invariant for a certain range of the parameters, including negative values of s, are recalled.

Section 6.4 contains first a preparatory treatment of unmixed Lizorkin–Triebel spaces on general C^∞-manifolds and these results are then extended to $F^{s,\vec{a}}_{\vec{p},q}$-spaces on the curved boundary of a cylinder.

Rychkov's universal extension operator in [45] is modified to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)$ in Section 6.5. Moreover, its properties on temperate distributions are analysed in addition.

Finally, Section 6.6 contains a discussion of the trace at the flat as well as at the curved boundary of a cylindrical domain, including some applications to e.g. the Dirichlet boundary problem for the heat equation.

6.2 Preliminaries

6.2.1 Notation

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of the rapidly decreasing C^∞-functions and it is equipped with the family of seminorms, using $D^\alpha := (-\mathrm{i}\partial_{x_1})^{\alpha_1}\cdots(-\mathrm{i}\partial_{x_n})^{\alpha_n}$ for each multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathrm{i}^2=-1$ and $\langle x\rangle^2 := 1+|x|^2$,

$$p_M(\varphi) := \sup\big\{\,\langle x\rangle^M|D^\alpha\varphi(x)|\,\big|\,x\in\mathbb{R}^n, |\alpha|\le M\big\},\quad M\in\mathbb{N}_0.$$

By duality, the Fourier transformation $\mathcal{F}\varphi(\xi)=\widehat{\varphi}(\xi)=\int_{\mathbb{R}^n}e^{-\mathrm{i}x\cdot\xi}\varphi(x)\,dx$ for $\varphi\in\mathcal{S}$ extends to the dual space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions.

Throughout, inequalities for vectors $\vec{p}=(p_1,\dots,p_n)$ are understood componentwise; as are functions, e.g. $\vec{p}!=p_1!\cdots p_n!$, while $t_+:=\max(0,t)$ for $t\in\mathbb{R}$.

For $0<\vec{p}\le\infty$ the space $L_{\vec{p}}(\mathbb{R}^n)$ consists of the Lebesgue measurable functions such that

$$\|\,u\,|\,L_{\vec{p}}(\mathbb{R}^n)\| := \Big(\int_{\mathbb{R}}\Big(\cdots\Big(\int_{\mathbb{R}}|u(x_1,\dots,x_n)|^{p_1}dx_1\Big)^{\frac{p_2}{p_1}}\cdots\Big)^{\frac{p_n}{p_{n-1}}}dx_n\Big)^{\frac{1}{p_n}}<\infty;$$

in case $p_j=\infty$, the essential supremum over x_j is used. When equipped with this quasi-norm, $L_{\vec{p}}(\mathbb{R}^n)$ is a quasi-Banach space (normed if $\vec{p}\ge 1$).

In addition, we shall for $0 < q \leq \infty$ denote by $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ the space of sequences $(u_k)_{k\in\mathbb{N}_0}$ of Lebesgue measurable functions $u_k : \mathbb{R}^n \to \mathbb{C}$ such that

$$\| (u_k)_{k\in\mathbb{N}_0} \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\| := \Big\| \Big(\sum_{k=0}^{\infty} |u_k|^q \Big)^{1/q} \Big| L_{\vec{p}}(\mathbb{R}^n) \Big\| < \infty;$$

with the supremum over k in case $q = \infty$. For brevity, $\| (u_k)_{k\in\mathbb{N}_0} \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|$ is written $\| u_k \,|L_{\vec{p}}(\ell_q)\|$ and when $\vec{p} = (p, \dots, p)$, $L_{\vec{p}}$ is simplified to L_p etc. We recall that sequences of C_0^∞-functions are dense in $L_{\vec{p}}(\ell_q)$ if $\max(p_1, \dots, p_n, q) < \infty$.

Generic constants will be denoted by c or C, with their dependence on certain parameters explicitly stated when relevant.

Lastly, the closure of an open set $U \subset \mathbb{R}^n$ is denoted $\overline{U}$ and $B(0, r)$ is the ball centered at 0 with radius $r > 0$; the dimension of the surrounding Euclidean space will be clear from the context or otherwise stated explicitly.

6.2.2 Anisotropic, Mixed-Norm Lizorkin–Triebel Spaces

This section only contains the Fourier-analytic definition of the mixed-norm Lizorkin–Triebel spaces and a few essential properties used in this paper; for an actual introduction to these spaces we refer the reader to [28] and [29, Sec. 3].

First we recall the definition of the anisotropic distance function $|\cdot|_{\vec{a}}$, where $\vec{a} = (a_1, \dots, a_n) \in [1, \infty[^n$, on $\mathbb{R}^n$ and some of its properties. Using the quasi-homogeneous dilation $t^{\vec{a}}x := (t^{a_1}x_1, \dots, t^{a_n}x_n)$ for $t \geq 0$, the function $|x|_{\vec{a}}$ is for $x \in \mathbb{R}^n \setminus \{0\}$ defined as the unique $t > 0$ such that $t^{-\vec{a}}x \in S^{n-1}$ ($|0|_{\vec{a}} := 0$), i.e.

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1.$$

For basic properties of $|\cdot|_{\vec{a}}$ we refer to [28, Sec. 3].

The Fourier-analytic definition also relies on a Littlewood–Paley decomposition, i.e. $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$, which is based on a (for convenience fixed) $\psi \in C_0^\infty$ such that $0 \leq \psi(\xi) \leq 1$ for all ξ, $\psi(\xi) = 1$ if $|\xi|_{\vec{a}} \leq 1$ and $\psi(\xi) = 0$ if $|\xi|_{\vec{a}} \geq 3/2$. Setting $\Phi = \psi - \psi(2^{\vec{a}}\cdot)$, we define

$$\Phi_0(\xi) = \psi(\xi), \qquad \Phi_j(\xi) = \Phi(2^{-j\vec{a}}\xi), \quad j = 1, 2, \dots \tag{6.1}$$

Definition 6.1. *The Lizorkin–Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < \vec{p} < \infty$ and $0 < q \leq \infty$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\| u \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| := \Bigg\| \Bigg(\sum_{j=0}^{\infty} 2^{jsq} \left|\mathcal{F}^{-1}\left(\Phi_j(\xi)\mathcal{F}u(\xi)\right)(\cdot)\right|^q \Bigg)^{1/q} \Bigg| L_{\vec{p}}(\mathbb{R}^n) \Bigg\| < \infty.$$

The number q is called a sum exponent and the entries in $\vec{p}$ integral exponents, while s is a smoothness index. In case $\vec{a} = (1, \dots, 1)$, the parameter $\vec{a}$ is omitted.

When studying traces on the flat boundary of a cylinder, Besov spaces are inevitable:

Definition 6.2. *The Besov space $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < \vec{p}, q \le \infty$ consists of the $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\| u \,|B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| := \Big(\sum_{j=0}^{\infty} 2^{jsq} \| \mathcal{F}^{-1} (\Phi_j \mathcal{F} u) \; |L_{\vec{p}}(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty.$$

Both $F^{s,\vec{a}}_{\vec{p},q}$ and $B^{s,\vec{a}}_{\vec{p},q}$ are quasi-Banach spaces (normed if $\min(p_1, \dots, p_n, q) \ge 1$) and the quasi-norm is subadditive when raised to the power $d := \min(1, p_1, \dots, p_n, q)$,

$$\| u + v \,|F^{s,\vec{a}}_{\vec{p},q}\|^d \le \| u \,|F^{s,\vec{a}}_{\vec{p},q}\|^d + \| v \,|F^{s,\vec{a}}_{\vec{p},q}\|^d, \quad u, v \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n). \tag{6.2}$$

Different choices of anisotropic decomposition of unity give the same space (with equivalent quasi-norms) and there are continuous embeddings

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

where $\mathcal{S}$ is dense in $F^{s,\vec{a}}_{\vec{p},q}$ for $q < \infty$.

Lemma 6.3. *For $\lambda > 0$ so large that $\lambda \vec{a} \ge 1$, the spaces $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ coincide with $B^{\lambda s,\lambda\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, respectively $F^{\lambda s,\lambda\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ and the corresponding quasi-norms are equivalent.*

The proof of this lemma for Besov spaces follows that of Lizorkin–Triebel spaces, which can be found in [29, Lem. 3.24]. Indeed, the only exception is that [29, Lem. 3.23] needs to be adapted to Besov spaces, but this is easily done using the modifications indicated just above Lemma 3.21 there.

In view of Lemma 6.3, most results obtained for the scales when $\vec{a} \ge 1$ can be extended to the range $0 < \vec{a} < \infty$ (for details we refer to Remark 4.7).

The Banach space $C_{\mathrm{b}}(\mathbb{R}^n)$ of continuous, bounded functions is equipped with the sup-norm, while the subspace $L_{1,\mathrm{loc}}(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ of locally integrable functions is endowed with the Fréchet space topology defined from the seminorms $u \mapsto \int_{|x| \le j} |u(x)| \, dx$, where $j \in \mathbb{N}$.

Lemma 6.4 ([31])**.** *Let $s \in \mathbb{R}$ and $\alpha \in \mathbb{N}_0^n$ be arbitrary.*

(i) *The differential operator D^α is bounded $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \to F^{s-\alpha\cdot\vec{a},\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.*

(ii) *For $s > \sum_{\ell=1}^n \big(\frac{a_\ell}{p_\ell} - a_\ell\big)_+$ there is an embedding $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^n)$.*

(iii) *The embedding $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}^n)$ holds for $s > \frac{a_1}{p_1} + \cdots + \frac{a_n}{p_n}$.*

Next, we recall a paramultiplication result and refer to Section 5.2.4 for details,

Lemma 6.5. *Let $s \in \mathbb{R}$ and take $s_1 > s$ such that also*

$$s_1 > \sum_{\ell=1}^{n} \Big(\frac{a_\ell}{\min(1, q, p_1, \dots, p_\ell)} - a_\ell \Big) - s. \tag{6.3}$$

Then each $u \in B^{s_1, \vec{a}}_{\infty,\infty}(\mathbb{R}^n)$ defines a multiplier of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ and

$$\| \, u \cdot v \, | F^{s,\vec{a}}_{\vec{p},q} \| \le c \, \| \, u \, | B^{s_1,\vec{a}}_{\infty,\infty} \| \cdot \| \, v \, | F^{s,\vec{a}}_{\vec{p},q} \|, \quad v \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n).$$

In particular, it holds for $u \in C^\infty_{L_\infty}(\mathbb{R}^n) := \{ \, g \in C^\infty \, | \, \forall \alpha \in \mathbb{N}^n_0 : \, D^\alpha g \in L_\infty \}$.

The characterisation of $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ by kernels of local means as developed in Theorem 4.24 is utilised below, hence it is included here for convenience, using the notation

$$\varphi_j(x) = 2^{j|\vec{a}|} \varphi(2^{j\vec{a}} x), \quad \varphi \in \mathcal{S}, \;\; j \in \mathbb{N}. \tag{6.4}$$

Theorem 6.6. *Let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ such that $\int k_0(x) \, dx \neq 0 \neq \int k^0(x) \, dx$ and set $k(x) = \Delta^N k^0(x)$ for some $N \in \mathbb{N}$. When $0 < \vec{p} < \infty$, $0 < q \le \infty$, and $s < 2N \min(a_1, \dots, a_n)$, then a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ if and only if*

$$\| \, f \, | F^{s,\vec{a}}_{\vec{p},q} \|^* := \| \, k_0 * f \, | L_{\vec{p}} \| + \| \{ 2^{sj} k_j * f \}_{j=1}^\infty \, | L_{\vec{p}}(\ell_q) \| < \infty.$$

Furthermore, $\| \, f \, | F^{s,\vec{a}}_{\vec{p},q} \|^*$ is an equivalent quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

We also recall the definition of $F^{s,\vec{a}}_{\vec{p},q}$-spaces over open sets. Here we use the notation introduced by Hörmander [22, App. B.2] and place a bar over F etc., to indicate that it is a space of restricted distributions.

Definition 6.7. *Let $U \subset \mathbb{R}^n$ be open. The space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ is defined as the set of $u \in \mathcal{D}'(U)$ such that there exists a distribution $f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ satisfying*

$$f(\varphi) = u(\varphi) \quad \textit{for all} \quad \varphi \in C^\infty_0(U). \tag{6.5}$$

We equip $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ with the quotient quasi-norm

$$\| \, u \, | \overline{F}^{s,\vec{a}}_{\vec{p},q}(U) \| := \inf_{r_U f = u} \| \, f \, | F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \|,$$

which is a norm if $\vec{p}, q \ge 1$. (Besov spaces over open sets are defined analogously.)

The space $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\overline{U})$ consists of the distributions in $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, which are supported in the closed set $\overline{U}$.

Recall that since $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is a quasi-Banach space, $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ is so too by the usual arguments for quotient spaces modified to exploit the subadditivity in (6.2).

In (6.5) it is tacitly understood that on the left-hand side φ is extended by 0 outside U. For this we henceforth use the operator notation $e_U\varphi$. Likewise r_U denotes restriction to U, whereby $u = r_U f$ in (6.5). We shall refer to such f as an extension of u.

Remark 6.8. *Theorem 6.6 induces an equivalent quasi-norm $\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\|^*$ on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ by taking the infimum of $\| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|^*$ for $r_U f = u$.*

As a preparation we include a slightly modified version of Lemma 5.17:

Lemma 6.9. *Let $U \subset \mathbb{R}^n$ be open. When $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})$ is normed as in Remark 6.8 using kernels of local means with* $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$ *for an $r > 0$, and when $K \subset U$ is a compact set fulfilling*

$$\operatorname{dist}(K, \mathbb{R}^n \setminus U) > 2r, \tag{6.6}$$

then it holds for every $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})$ with $\operatorname{supp} f \subset K \times \mathbb{R}$ *that*

$$\| f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})\|^* = \| e_{U\times\mathbb{R}} f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^*.$$

That is, the infimum is for such f attained at $e_{U\times\mathbb{R}}f$.

Proof. For an arbitrary extension $\widetilde{f}$ of f, it holds for $g := \widetilde{f} - e_{U\times\mathbb{R}}f$ that $\operatorname{supp} e_{U\times\mathbb{R}}f \cap \operatorname{supp} g = \emptyset$, hence by (6.6),

$$\operatorname{supp}(k_j * e_{U\times\mathbb{R}}f) \cap \operatorname{supp}(k_j * g) = \emptyset, \quad j \in \mathbb{N}_0.$$

When $g \neq 0$, there exists $j \in \mathbb{N}_0$ such that $\operatorname{supp}(k_j * g) \neq \emptyset$, thus $k_j * g(x) \neq 0$ on an open set disjoint from $\operatorname{supp}(k_j * e_{U\times\mathbb{R}}f)$. This term therefore effectively contributes to the $L_{\vec{p}}$-norm in the local means characterisation, yielding

$$\| \widetilde{f} \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^* > \| e_{U\times\mathbb{R}} f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^*. \qquad \square$$

For temperate distributions vanishing in the time direction, we let $e_{I\to I'}$ denote extension by 0 from $\mathbb{R}^{n-1} \times I$ to $\mathbb{R}^{n-1} \times I'$ for open intervals $I \subset I'$. Then we similarly get

Lemma 6.10. *Let $I =]b,c[$ and $I' =]a,c[$ where $-\infty \le a < b < c \le \infty$. When the space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times I)$ is normed as in Remark 6.8 using kernels of local means with* $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$ *for an $r > 0$, then it holds for all $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1}\times I)$ satisfying $f(\cdot,t) = 0$ for $t \in]b, b+2r[$ that*

$$\| f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times I)\|^* = \| e_{I\to I'} f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times I')\|^*. \tag{6.7}$$

A similar equality holds for extension from $I =]a,b[$ to I', when $f(\cdot,t) = 0$ for $t \in]b-2r, c[$.

Proof. The inequality $\leq$ follows immediately, since the distributions considered in the infimum on the right-hand side in (6.7) also are considered on the left-hand side.

To prove equality we assume that $<$ holds. Then there exists an extension $\widetilde{f}$ of f which is not among the distributions considered in the infimum on the right-hand side, and which, with an infimum over $r_{\mathbb{R}^{n-1}\times I'}h = e_{I\to I'}f$, moreover fulfils

$$\| \widetilde{f} \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| < \inf \| h \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|. \tag{6.8}$$

Actually it suffices to consider those h for which $h \equiv 0$ on $\mathbb{R}^{n-1}\times\,]-\infty, b+2r[\,$. Indeed, for any other h the distribution $(1-\chi(t))h(\cdot,t)$, where $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $t \in\,]-\infty, a[$ and $\chi(t) = 0$ for $t \in]b,\infty[\,$, has a smaller quasi-norm than h. This can be verified similarly to the proof of Lemma 6.9, using that the distance between $\operatorname{supp} h \cap (\mathbb{R}^{n-1}\times\,]-\infty, a[\,)$ and $\operatorname{supp}(1-\chi)h$ is at least $2r$.

Now for such h we have $\operatorname{supp} h \subset \mathbb{R}^{n-1}\times [b+2r,\infty[\,$, and since $\widetilde{f}(t) \not\equiv 0$ for $a < t < b$ it is easily seen by the proof strategy of Lemma 6.9 that

$$\| \widetilde{f} \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| > \| h \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|,$$

which contradicts (6.8). □

For simplicity of notation the * on the quasi-norm is omitted in the following.

6.3 Invariance under Diffeomorphisms

To introduce Lizorkin–Triebel spaces on manifolds, it is essential that the spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ for certain open subsets $U \subset \mathbb{R}^n$ are invariant under suitable C^∞-bijections σ. An extensive treatment of this subject can be found in [31], but for convenience we recall the needed results. These hold for $0 < \vec{p} < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$ unless additional requirements are specified. First a result pertaining to isotropic spaces,

Theorem 6.11. *When $\sigma : U \to V$ is a C^∞-bijection between open sets $U, V \subset \mathbb{R}^n$ and $f \in \overline{F}^{s}_{p,q}(V)$ has compact support, then $f\circ\sigma \in \overline{F}^{s}_{p,q}(U)$ and*

$$\| f\circ\sigma \,|\overline{F}^{s}_{p,q}(U)\| \leq c \| f \,|\overline{F}^{s}_{p,q}(V)\| \tag{6.9}$$

holds for a constant c depending only on σ and the set $\operatorname{supp} f$.

In the anisotropic situation it cannot be expected, e.g. if σ is a rotation, that $f\circ\sigma$ has the same regularity as f, nor that $f\circ\sigma \in L_{\vec{p}}$ when $f \in L_{\vec{p}}$. We therefore restrict to

$$\vec{p} = (\underbrace{p_1,\dots,p_1}_{N_1},\ \dots\ ,\underbrace{p_m,\dots,p_m}_{N_m}), \quad N_1+\dots+N_m = n,\ \ m \geq 2, \tag{6.10}$$

and $\vec{a}$ having the same structure.

Theorem 6.12. *Let $\sigma_j : U_j \to V_j$ for $j = 1, \dots, m$ be C^∞-bijections, where $U_j, V_j \subset \mathbb{R}^{N_j}$ are open. When $\vec{a}$, $\vec{p}$ fulfil* (6.10) *and $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U_1 \times \cdots \times U_m)$ has compact support, then* (6.9) *holds for $U = U_1 \times \cdots \times U_m$ and $V = V_1 \times \cdots \times V_m$.*

For traces at the curved boundary of cylinders, the next special case is useful:

Theorem 6.13. *Let $U, V \subset \mathbb{R}^{n-1}$ be open and let $\sigma : U \times \mathbb{R} \to V \times \mathbb{R}$ be a C^∞-bijection on the form*

$$\sigma(x) = (\sigma'(x_1, \dots, x_{n-1}), x_n) \quad \textit{for all} \quad x \in U \times \mathbb{R}.$$

When $\vec{a}, \vec{p}$ satisfy (6.10) *with $m = 2$, $N_1 = n - 1$, $N_2 = 1$ and $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(V \times \mathbb{R})$ has* $\operatorname{supp} f \subset K \times \mathbb{R}$, *whereby $K \subset V$ is compact, then $f \circ \sigma \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})$ and*

$$\| f \circ \sigma \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})\| \le c(\operatorname{supp} f, \sigma) \| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|.$$

The above three theorems can be found with proofs as Theorems 5.21, 5.22 and 5.23, respectively. As needed, we shall tacitly apply these results in situations with $n+1$ variables, the last of which is interpreted as time. Then we let $t = x_{n+1}$.

6.4 Function Spaces on Manifolds

To develop Lizorkin–Triebel spaces over cylinders and to settle the necessary notation, we first review distributions on manifolds.

6.4.1 Distributions on Manifolds

To allow comparison with existing literature on partial differential equations, we follow [17, Sec. 8.2] and [21, Sec. 6.3]. E.g. a diffeomorphism is in the following a bijective C^∞-map between open sets, and we recall

Definition 6.14. *An n-dimensional manifold X is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$. The manifold X is C^∞ (or smooth), if it is equipped with a C^∞-structure, i.e. a family $\mathcal{F}$ of homeomorphisms κ mapping open sets $X_\kappa \subset X$ onto open sets $\widetilde{X}_\kappa \subset \mathbb{R}^n$, with $X = \bigcup_{\kappa \in \mathcal{F}} X_\kappa$, such that the maps*

$$\kappa \circ \kappa_1^{-1} : \kappa_1(X_\kappa \cap X_{\kappa_1}) \to \kappa(X_\kappa \cap X_{\kappa_1}), \quad \kappa, \kappa_1 \in \mathcal{F}, \tag{6.11}$$

are diffeomorphisms, and $\mathcal{F}$ contains every homeomorphism $\kappa_0 : X_{\kappa_0} \to \widetilde{X}_{\kappa_0}$, for which the compositions in (6.11) *with $\kappa = \kappa_0$ are diffeomorphisms.*

A subfamily of $\mathcal{F}$ where the X_κ cover X is called a (compatible) atlas, and $\mathcal{F}_1 \subset \mathcal{F}_2$ means that every chart κ in $\mathcal{F}_1$ is also a member of $\mathcal{F}_2$. (The definition of a C^∞-manifold X means that a maximal atlas has been chosen on the set X.)

Unless otherwise stated, X denotes an n-dimensional C^∞-manifold and $\mathcal{F}$ is the maximal atlas. A partition of unity $1 = \sum_{j\in\mathbb{N}} \psi_j(x)$ with $\psi_j \in C_0^\infty(X)$ and $\psi_j(x) \geq 0$ for $x \in X$ is said to be subordinate to $\mathcal{F}$ (instead of to the covering $X = \bigcup_{\kappa\in\mathcal{F}} X_\kappa$), when for each $j \in \mathbb{N}$ there exists a chart $\kappa(j) \in \mathcal{F}$ such that $\operatorname{supp}\psi_j \subset X_{\kappa(j)}$. It is locally finite, when $1 = \sum \psi_j(x)$ for every $x \in X$ has only finitely many non-trivial terms in some neighbourhood of x. Note that for each compact set $K \subset X$, this finiteness extends to an open set $U \supset K$.

We recall the definition of a distribution on a C^∞-manifold, using the notation $\varphi^* u$ for the pullback of a distribution u by a function φ; when u is a function then $\varphi^* u = u \circ \varphi$.

Definition 6.15. *The space $\mathcal{D}'(X)$ consists of all the families $\{u_\kappa\}_{\kappa\in\mathcal{F}}$, where $u_\kappa \in \mathcal{D}'(\widetilde{X}_\kappa)$ and which for all $\kappa, \kappa_1 \in \mathcal{F}$ fulfil*

$$u_{\kappa_1} = (\kappa \circ \kappa_1^{-1})^* u_\kappa \quad \text{on} \quad \kappa_1(X_\kappa \cap X_{\kappa_1}). \tag{6.12}$$

($\mathcal{D}'(X)$ only identifies with the dual of $C_0^\infty(X)$ in case there is a positive density on X; cf. [21, Ch. 6].)

Each $u \in C^k(X)$, $k \in \mathbb{N}_0$, can be identified with the family $u_\kappa := u \circ \kappa^{-1}$ of functions in $C^k(\widetilde{X}_\kappa)$, which evidently transform as in (6.12). Thus $C^k(X) \subset \mathcal{D}'(X)$ is obvious. For any $u \in \mathcal{D}'(X)$, the notation $u \circ \kappa^{-1}$ is also used to denote u_κ.

In (6.12) restriction of e.g. u_κ to $\kappa(X_\kappa \cap X_{\kappa_1})$ is tacitly understood. To ease notation we will in the rest of the paper, when composing with a chart, suppress such restriction to the chart's co-domain.

Lemma 6.16 ([21, Thm. 6.3.4])**.** *For any atlas $\mathcal{F}_1 \subset \mathcal{F}$, each family $\{u_\kappa\}_{\kappa\in\mathcal{F}_1}$ of elements $u_\kappa \in \mathcal{D}'(\widetilde{X}_\kappa)$ fulfilling (6.12) for $\kappa, \kappa_1 \in \mathcal{F}_1$ is obtained from a unique $v \in \mathcal{D}'(X)$ by "restriction" to $\mathcal{F}_1$, i.e. $v \circ \kappa^{-1} = u_\kappa$ for every $\kappa \in \mathcal{F}_1$.*

So if an open set $U \subset \mathbb{R}^n$ is seen as a manifold X, then $\mathcal{F}_1 = \{\mathrm{id}_U\}$ at once gives $\mathcal{D}'(U) \hookrightarrow \mathcal{D}'(X)$; the surjectivity of this map follows by gluing together, cf. [21, Thm. 2.2.4].

For $Y \subset X$ open, the restriction of $u \in \mathcal{D}'(X)$ to Y is $r_Y u := \{r_{\kappa(Y\cap X_\kappa)} u_\kappa\}$, where κ runs through the charts in $\mathcal{F}$ for which $X_\kappa \cap Y \neq \emptyset$. If instead of $\mathcal{F}$ we only consider an atlas $\mathcal{F}_1 \subset \mathcal{F}$, then the corresponding subfamily identifies with a distribution $u_Y \in \mathcal{D}'(Y)$, cf. Lemma 6.16, and since this is unique, $r_Y u = u_Y$, i.e. it is sufficient to consider an arbitrary atlas when determining the restriction of a distribution.

A distribution $u \in \mathcal{D}'(X)$ is said to be 0 on an open set $Y \subset X$ if $r_Y u = 0$. Using this,

$$\operatorname{supp} u := X \setminus \bigcup \{\, Y \subset X \text{ open} \mid u = 0 \text{ on } Y \}, \tag{6.13}$$

and it is easily seen that for any atlas $\mathcal{F}_1 \subset \mathcal{F}$,

$$\operatorname{supp} u = \bigcup_{\kappa_1\in\mathcal{F}_1} \kappa_1^{-1}(\operatorname{supp} u_{\kappa_1}). \tag{6.14}$$

The space $\mathcal{E}'(X)$ consists of the distributions $u \in \mathcal{D}'(X)$ having compact support, while $\mathcal{E}'(K)$ for an arbitrary $K \subset X$ consists of the $u \in \mathcal{E}'(X)$ with $\operatorname{supp} u \subset K$. Any $u \in \mathcal{E}'(Y)$, where $Y \subset X$ is open, has an "extension by 0"; even locally:

Corollary 6.17. *When $Y \subset X$ is open and $u \in \mathcal{E}'(Y)$, then there exists $v \in \mathcal{E}'(X)$ such that $r_Y v = u$ and $\operatorname{supp} v = \operatorname{supp} u$. Moreover, when given $u_\kappa \in \mathcal{E}'(\widetilde{X}_\kappa)$ for a single $\kappa \in \mathcal{F}$, then there exists a distribution $v \in \mathcal{E}'(X)$ such that $v_\kappa = u_\kappa$ and $\operatorname{supp} v = \kappa^{-1}(\operatorname{supp} u_\kappa)$.*

Proof. In the case that $\operatorname{supp} u \subset X_\kappa \subset Y$ for some $\kappa \in \mathcal{F}$, then there exists an open set $U \subset X$ such that $\operatorname{supp} u \subset U \subset \overline{U} \subset X_\kappa$ (X is normal). The family $\mathcal{F}_1 := \{\kappa\} \cup \{\kappa_1 \in \mathcal{F} \,|\, \overline{U} \cap X_{\kappa_1} = \emptyset\}$ is an atlas, since its domains covers X. Setting $v_\kappa = u_\kappa$ and $v_{\kappa_1} = 0$ for the other $\kappa_1 \in \mathcal{F}_1$, the family $\{v_{\kappa_1}\}_{\kappa_1 \in \mathcal{F}_1}$ clearly transforms as in (6.12), hence defines a $v \in \mathcal{D}'(X)$, cf. Lemma 6.16. From (6.14) it is clear that $\operatorname{supp} v = \operatorname{supp} u$; and $r_Y v = u$ is evident in the atlas $\mathcal{F}_1$.

In the general case, we use that any $u \in \mathcal{E}'(Y)$ can be written as a finite sum $u = \sum \psi_j u$, where $1 = \sum \psi_j$ is a locally finite partition of unity subordinate to the atlas $\{\,\kappa|_{Y \cap X_\kappa} \,|\, \kappa \in \mathcal{F} : Y \cap X_\kappa \neq \emptyset\}$ on Y. Since for each summand, $\operatorname{supp} \psi_j u \subset Y \cap X_{\kappa(j)}$ is compact, the above gives the existence of a $v_j \in \mathcal{D}'(X)$ such that $r_Y v_j = \psi_j u$ and $\operatorname{supp} v_j = \operatorname{supp} \psi_j u$. Because the restriction operator is linear, taking $v = \sum v_j$ proves the statement.

For the last part, consider X_κ as a manifold with the atlas containing only the chart κ. Lemma 6.16 gives a $w \in \mathcal{D}'(X_\kappa)$ such that $w_\kappa = u_\kappa$, hence the special case above applied to w and $Y = X_\kappa$ gives, that there exists a $v \in \mathcal{E}'(X)$ such that $v_\kappa = w_\kappa$, and $\operatorname{supp} v = \operatorname{supp} w$. $\square$

6.4.2 Isotropic Lizorkin–Triebel Spaces on Manifolds

Since we later need a few isotropic results, and since the proofs are much cleaner for isotropic spaces, we shall fix ideas in this section by working with arbitrary $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let us add that most references on isotropic spaces over manifolds just describe the outcome without referring directly to the general definitions in [21, Ch. 6], thus being inadequate for our generalisations here.

Manifolds in General

We first recall that when $U \subset \mathbb{R}^n$ is open, $u \in \mathcal{D}'(U)$ is said to belong to the Lizorkin–Triebel space $\overline{F}^s_{p,q}(U)$ locally, if $\varphi u \in \overline{F}^s_{p,q}(U)$ for all $\varphi \in C_0^\infty(U)$; the set of such elements is denoted $F^s_{p,q;\mathrm{loc}}(U)$. Here we use the notation without bar, since φ has compact support in U. The space can be generalised to

Definition 6.18. *The local Lizorkin–Triebel space $F^s_{p,q;\mathrm{loc}}(X)$ consists of all the $u \in \mathcal{D}'(X)$ such that $u_\kappa \in F^s_{p,q;\mathrm{loc}}(\widetilde{X}_\kappa)$ for every $\kappa \in \mathcal{F}$.*

For $u \in \mathcal{D}'(X)$ to belong to $F^s_{p,q;\mathrm{loc}}(X)$, it suffices that u_{κ_1} is in $F^s_{p,q;\mathrm{loc}}(\widetilde{X}_{\kappa_1})$ for each κ_1 in an atlas $\mathcal{F}_1 \subset \mathcal{F}$. Indeed given $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$, a partition of unity yields a reduction to the case where $\operatorname{supp}(\varphi \circ \kappa) \subset X_\kappa \cap X_{\kappa_1}$, and the transition rule in (6.12) gives

$$(\kappa \circ \kappa_1^{-1})^*(\varphi u_\kappa) = \varphi \circ (\kappa \circ \kappa_1^{-1}) u_{\kappa_1}.$$

Since $\varphi \circ (\kappa \circ \kappa_1^{-1})$ is in $C_0^\infty(\kappa_1(X_\kappa \cap X_{\kappa_1}))$, the product is in $\overline{F}^s_{p,q}(\kappa_1(X_\kappa \cap X_{\kappa_1}))$ by assumption on $\mathcal{F}_1$; so by Theorem 6.11 one has $\varphi u_\kappa \in \overline{F}^s_{p,q}(\widetilde{X}_\kappa)$.

For example, when X is an open set $U \subset \mathbb{R}^n$, the identification $\mathcal{D}'(X) \simeq \mathcal{D}'(U)$ implies that $F^s_{p,q;\mathrm{loc}}(X) \simeq F^s_{p,q;\mathrm{loc}}(U)$ as it according to the above suffices to consider the atlas $\{\mathrm{id}_U\}$.

For a partition of unity $1 = \sum_{j=1}^\infty \psi_j$ subordinate to $\mathcal{F}$, we use for brevity

$$\widetilde{\psi}_j := \psi_j \circ \kappa(j)^{-1}.$$

The partition is of course already subordinate to $\mathcal{F}_1 := \{\, \kappa(j) \mid j \in \mathbb{N}\}$, which by the above suffices for determining $F^s_{p,q;\mathrm{loc}}(X)$. This is moreover true, when the cut-off functions $\widetilde{\psi}_j$ of a locally finite partition of unity are invoked:

Lemma 6.19. *A distribution $u \in \mathcal{D}'(X)$ belongs to $F^s_{p,q;\mathrm{loc}}(X)$ if and only if*

$$\widetilde{\psi}_j u_{\kappa(j)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)}), \quad j \in \mathbb{N}. \tag{6.15}$$

Proof. Since $\widetilde{\psi}_j \in C_0^\infty(\widetilde{X}_{\kappa(j)})$, this condition is necessary for u to be in $F^s_{p,q;\mathrm{loc}}(X)$.

Conversely, for an arbitrary $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$ we obtain $\varphi u_\kappa = \sum_{j \in I} \psi_j \circ \kappa^{-1} \varphi u_\kappa$ with summation over a finite index set $I \subset \mathbb{N}$, because $(\psi_j)_{j \in \mathbb{N}}$ is locally finite. As $\operatorname{supp}(\psi_j \circ \kappa^{-1}\varphi) \subset \kappa(X_\kappa \cap X_{\kappa(j)})$, Lemma 5.17 and then Theorem 6.11 applied to $\kappa(j) \circ \kappa^{-1}$ yields, cf. (6.12),

$$\begin{aligned} &\| \, \varphi u_\kappa \, | \overline{F}^s_{p,q}(\widetilde{X}_\kappa) \| \\ &\qquad \le c_\kappa \sum_{j \in I} \big\| \, \widetilde{\psi}_j \cdot (\varphi \circ \kappa \circ \kappa(j)^{-1}) \cdot u_{\kappa(j)} \, \big| \overline{F}^s_{p,q}(\kappa(j)(X_\kappa \cap X_{\kappa(j)})) \big\|. \end{aligned} \tag{6.16}$$

After multiplication with $\chi_j \in C_0^\infty(\widetilde{X}_{\kappa(j)})$ chosen such that $\chi_j \equiv 1$ on $\operatorname{supp} \widetilde{\psi}_j$, we obtain by applying Lemma 6.5 with some $s_1 > s$ satisfying (6.3) and suppressing extension by 0 to $\mathbb{R}^n$ that

$$\begin{aligned} &\| \, \varphi u_\kappa \, | \overline{F}^s_{p,q}(\widetilde{X}_\kappa) \| \\ &\qquad \le c_\kappa \sum_{j \in I} \| \, \varphi \circ \kappa \circ \kappa(j)^{-1} \chi_j \, | B^{s_1}_{\infty,\infty}(\mathbb{R}^n) \| \| \, \widetilde{\psi}_j u_{\kappa(j)} \, | \overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)}) \|. \end{aligned} \tag{6.17}$$

The right-hand side is by (6.15) finite, hence $u_\kappa \in F^s_{p,q;\mathrm{loc}}(\widetilde{X}_\kappa)$ for each $\kappa \in \mathcal{F}$. □

The space $F^s_{p,q;\mathrm{loc}}(X)$ can be topologised through a separating family of quasi-seminorms,

$$\mu_j(u) := \| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\|, \quad j \in \mathbb{N}. \tag{6.18}$$

Indeed, if u is non-zero in $F^s_{p,q;\mathrm{loc}}(X)$, then there exists $\kappa \in \mathcal{F}$ and $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$ such that $\varphi u_\kappa \neq 0$, i.e. $\| \varphi u_\kappa \,|\overline{F}^s_{p,q}(\widetilde{X}_\kappa)\| > 0$. So (6.17) gives that $\mu_j(u) > 0$ for at least one $j \in \mathbb{N}$.

Going a step further, one obtains an equivalent family of quasi-seminorms even for a "restricted" family $\{v_{\kappa_1}\}_{\kappa_1 \in \mathcal{F}_1}$:

Lemma 6.20. *Let $1 = \sum \varphi_k$ be a locally finite partition of unity subordinate to some atlas $\mathcal{F}_1 \subset \mathcal{F}$ and let $\widetilde{\varphi}_k = \varphi_k \circ \kappa_1(k)^{-1}$. When a family of distributions $v_{\kappa_1} \in \mathcal{D}'(\widetilde{X}_{\kappa_1})$, $\kappa_1 \in \mathcal{F}_1$, transforms as in* (6.12) *and $\widetilde{\varphi}_k v_{\kappa_1(k)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa_1(k)})$ for every $k \in \mathbb{N}$, then there exists a unique $u \in F^s_{p,q;\mathrm{loc}}(X)$ such that $u_{\kappa_1} = v_{\kappa_1}$ for all $\kappa_1 \in \mathcal{F}_1$ and*

$$\| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\| \le c_j \max \| \widetilde{\varphi}_k u_{\kappa_1(k)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa_1(k)})\|, \quad j \in \mathbb{N}, \tag{6.19}$$

with maximum over $k \in \mathbb{N}$ for which $\operatorname{supp}\psi_j \cap \operatorname{supp}\varphi_k \neq \emptyset$, *cf.* (6.13).

Proof. There exists a unique distribution $u \in \mathcal{D}'(X)$ such that $u_{\kappa_1} = v_{\kappa_1}$ for all $\kappa_1 \in \mathcal{F}_1$, cf. Lemma 6.16, and using (6.17) with $\varphi = \widetilde{\psi}_j$ and $1 = \sum \varphi_k$ as the partition of unity readily shows (6.19). Consequently $u \in F^s_{p,q;\mathrm{loc}}(X)$. ☐

The opposite inequality of (6.19) can be shown similarly from (6.16)–(6.17), hence we obtain

Corollary 6.21. *The space $F^s_{p,q;\mathrm{loc}}(X)$ can be equivalently defined from any atlas $\mathcal{F}_1 \subset \mathcal{F}$. Lemma 6.19 holds for any locally finite partition of unity subordinate to $\mathcal{F}_1$, and the resulting system of quasi-seminorms is equivalent to* (6.18).

As a preparation we include an obvious consequence of the proof of Corollary 6.17:

Corollary 6.22. *When given $u_\kappa \in \mathcal{E}'(\widetilde{X}_\kappa) \cap \overline{F}^s_{p,q}(\widetilde{X}_\kappa)$ for a single $\kappa \in \mathcal{F}$, then there exists $v \in \mathcal{E}'(X) \cap F^s_{p,q;\mathrm{loc}}(X)$ such that $v_\kappa = u_\kappa$ and* $\operatorname{supp} v = \kappa^{-1}(\operatorname{supp} u_\kappa)$.

When an open set $U \subset \mathbb{R}^n$ is seen as a manifold X, then $F^s_{p,q;\mathrm{loc}}(U)$ obviously coincides with $F^s_{p,q;\mathrm{loc}}(X)$, since it by Corollary 6.21 suffices to consider $\mathcal{F}_1 = \{\mathrm{id}_U\}$ and any partition of unity $1 = \sum_{j=1}^\infty \psi_j$ on U. On $F^s_{p,q;\mathrm{loc}}(U)$, the family in (6.18) gives the usual structure of a Fréchet space if $p, q \ge 1$, and in general we have:

Theorem 6.23. *The space $F^s_{p,q;\mathrm{loc}}(X)$ is a complete topological vector space with a translation invariant metric; for $p, q \ge 1$ it is locally convex, hence a Fréchet space.*

Proof. It follows straightforwardly from [17, Thm. B.5], which is based on a separating family of seminorms, that the separating family $(\mu_j^d)_{j \in \mathbb{N}}$, whereby $d := \min(1, p, q)$, of subadditive functionals can be used to construct a topology, which turns $F^s_{p,q;\mathrm{loc}}(X)$ into a topological vector space. Indeed, only a minor modification in the proof of continuity of scalar multiplication is needed, since the μ^d_j are not positive homogeneous — unless $p, q \geq 1$, and in this case the positive homogeneity implies that $F^s_{p,q;\mathrm{loc}}(X)$ is locally convex.

A translation invariant metric can be defined as in [17, Thm. B.9], i.e.

$$d'(u,v) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mu_j(u-v)^d}{1+\mu_j(u-v)^d}, \tag{6.20}$$

and the arguments there yield that d' defines the same topology as $(\mu^d_j)_{j\in\mathbb{N}}$.

For an arbitrary Cauchy sequence (u_m) in $F^s_{p,q;\mathrm{loc}}(X)$, the sequence $(\widetilde{\psi}_j u_{m,\kappa(j)})$, where $u_{m,\kappa(j)} := u_m \circ \kappa(j)^{-1}$, is Cauchy in $\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$ for each $j \in \mathbb{N}$. Since this space is complete, there exists $\widetilde{v}_{\kappa(j)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$ such that

$$\| \widetilde{\psi}_j u_{m,\kappa(j)} - \widetilde{v}_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\| \to 0 \quad \text{for} \quad m \to \infty. \tag{6.21}$$

Clearly $\widetilde{v}_{\kappa(j)} \in \mathcal{E}'(\widetilde{X}_{\kappa(j)})$, hence it follows from Corollary 6.22 that there exists a $v^{(\kappa(j))} \in \mathcal{E}'(X) \cap F^s_{p,q;\mathrm{loc}}(X)$ so that $\operatorname{supp} v^{(\kappa(j))} = \kappa(j)^{-1}(\operatorname{supp} \widetilde{v}_{\kappa(j)}) \subset \operatorname{supp} \psi_j$ and $v^{(\kappa(j))}_{\kappa(j)} = \widetilde{v}_{\kappa(j)}$.

To find a limit for (u_m), we note that $\widetilde{u}_{\kappa(j)} := \sum_{l\in\mathbb{N}} v^{(\kappa(l))}_{\kappa(j)}$ is well defined in $\mathcal{D}'(\widetilde{X}_{\kappa(j)})$, since on every compact set $K \subset \widetilde{X}_{\kappa(j)}$ there are only finitely many non-trivial terms. This family transforms as in (6.12), for in $\mathcal{D}'(\kappa(j)(X_{\kappa(j)} \cap X_{\kappa(k)}))$,

$$\widetilde{u}_{\kappa(k)} \circ \kappa(k) \circ \kappa(j)^{-1} = \sum_l v^{(\kappa(l))}_{\kappa(k)} \circ \kappa(k) \circ \kappa(j)^{-1} = \sum_l v^{(\kappa(l))}_{\kappa(j)} = \widetilde{u}_{\kappa(j)}.$$

Since $\widetilde{\psi}_j \widetilde{u}_{\kappa(j)} = \sum_l \widetilde{\psi}_j v^{(\kappa(l))}_{\kappa(j)}$ has finitely many terms, hence belongs to $\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$, existence of $u \in F^s_{p,q;\mathrm{loc}}(X)$ with $u_{\kappa(j)} = \widetilde{u}_{\kappa(j)}$ for all j follows from Lemma 6.20.

To show the convergence of u_m to u in $F^s_{p,q;\mathrm{loc}}(X)$, we rely on extra copies of the locally finite partition of unity to estimate by finitely many terms,

$$\mu_j(u_m-u)^d \leq \sum_{\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k \neq \emptyset} \big\| \widetilde{\psi}_j \big(\psi_k \circ \kappa(j)^{-1} u_{m,\kappa(j)} - v^{(\kappa(k))}_{\kappa(j)}\big) \,\big|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\big\|^d.$$

For $k \neq j$ the domains can clearly be changed to $\kappa(j)(X_{\kappa(j)} \cap X_{\kappa(k)})$, since $v^{(\kappa(k))}$ and the ψ_k have compact support in $X_{\kappa(k)}$. Using Theorem 6.11, each term can then be estimated by,

$$c\big\| \psi_j \circ \kappa(k)^{-1}(\widetilde{\psi}_k u_{m,\kappa(k)} - \widetilde{v}_{\kappa(k)}) \,\big|\overline{F}^s_{p,q}\big(\kappa(k)(X_{\kappa(j)} \cap X_{\kappa(k)})\big)\big\|^d.$$

By means of a cut-off function equal to 1 on the compact supports, one can extend by 0 to $\mathbb{R}^n$ and apply Lemmas 6.5 and 5.17, which yields

$$\mu_j(u_m-u)^d \leq c \sum_{\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k \neq \emptyset} \| \widetilde{\psi}_k u_{m,\kappa(k)} - \widetilde{v}_{\kappa(k)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(k)})\|^d.$$

Each term converges to 0, cf. (6.21), hence $F^s_{p,q;\mathrm{loc}}(X)$ is complete. □

Compact Manifolds

For trace operators on cylinders, compact manifolds are of special interest, since the intersection of the curved and the flat boundary is often of such nature.

When X is compact there exists a finite atlas $\mathcal{F}_0$ and a partition of unity $1 = \sum_{\kappa\in\mathcal{F}_0} \psi_\kappa$ such that $\operatorname{supp}\psi_\kappa \subset X_\kappa$ is compact for each $\kappa \in \mathcal{F}_0$. The space $F^s_{p,q;\mathrm{loc}}(X)$ is in this case just denoted $F^s_{p,q}(X)$, since the elements satisfy a global condition according to

Theorem 6.24. *When X is a compact C^∞-manifold, then $F^s_{p,q}(X)$ is a quasi-Banach space (normed if $p, q \geq 1$) when equipped with*

$$\| u \,| F^s_{p,q}(X)\| := \Big(\sum_{\kappa\in\mathcal{F}_0} \| \widetilde{\psi}_\kappa u_\kappa \,| \overline{F}^s_{p,q}(\widetilde{X}_\kappa)\|^d \Big)^{1/d}, \quad d := \min(1, p, q), \tag{6.22}$$

and $\| \cdot \,| F^s_{p,q}(X)\|^d$ is subadditive.

Proof. Positive homogeneity and subadditivity are inherited from the quasi-norms on the $\overline{F}^s_{p,q}(\widetilde{X}_\kappa)$ and then the quasi-triangle inequality follows for $d < 1$ by using dual exponents $\frac{1}{d}, \frac{1}{1-d}$,

$$\| u + v \,| F^s_{p,q}(X)\| \leq 2^{\frac{1-d}{d}} \big(\| u \,| F^s_{p,q}(X)\| + \| v \,| F^s_{p,q}(X)\|\big), \quad u, v \in F^s_{p,q}(X).$$

For any $u \in F^s_{p,q}(X)$ with $\| u \,| F^s_{p,q}(X)\| = 0$, clearly $\widetilde{\psi}_\kappa u_\kappa = 0$ on $\widetilde{X}_\kappa$ for every $\kappa \in \mathcal{F}_0$. Moreover, $\psi_\kappa \circ \kappa_1^{-1} u_{\kappa_1} = 0$ for $\kappa, \kappa_1 \in \mathcal{F}_0$ with $X_\kappa \cap X_{\kappa_1} \neq \emptyset$, as (6.12) applies on $\kappa_1(X_\kappa \cap X_{\kappa_1})$. Therefore $u_{\kappa_1} = \sum_{\kappa\in\mathcal{F}_0} (\psi_\kappa \circ \kappa_1^{-1}) u_{\kappa_1} = 0$ for all $\kappa_1 \in \mathcal{F}_0$, hence $u = 0$.

Completeness follows from Theorem 6.23, since we for X compact have a partition of unity with only finitely many non-zero elements, hence the topology there is equal to the one defined from (6.22). □

6.4.3 Isotropic Besov Spaces on Manifolds

For later reference, it is briefly mentioned that all the definitions and results in Section 6.4.2 can be adapted to Besov spaces $B^s_{p,q;\mathrm{loc}}(X)$. E.g. they are complete, when endowed with the quasi-seminorms

$$\mu_j(u) := \| \widetilde{\psi}_j u_{\kappa(j)} \,| \overline{B}^s_{p,q}(\widetilde{X}_{\kappa(j)})\|, \quad j \in \mathbb{N},$$

and for $p, q \geq 1$ even Fréchet spaces. Moreover, when X is compact, $B^s_{p,q}(X)$ is a quasi-Banach space under the norm

$$\| u \,| B^s_{p,q}(X)\| := \Big(\sum_{\kappa\in\mathcal{F}_0} \| \widetilde{\psi}_\kappa u_\kappa \,| \overline{B}^s_{p,q}(\widetilde{X}_\kappa)\|^d \Big)^{1/d}, \quad d := \min(1, p, q). \tag{6.23}$$

Indeed, this results as the arguments in Section 6.4.2 merely rely on Lemma 6.5 and Theorem 6.11. For one thing, the paramultiplication result in the lemma is simply replaced by a Besov version, cf. [24], [43] or [58, 4.2.2], while we now indicate the needed modifications of the invariance result in Theorem 6.11:

The proof of Theorem 5.21, i.e. Theorem 6.11, was divided into two steps. For large s, the arguments carry over to $B^s_{p,q}$ using [57, Sec. 2.7.1, Rem. 2] instead of Lemma 5.3(iii) and also using the characterisation of isotropic Besov spaces by kernels of local means, cf. [44, Thm. BPT] or [59, Thm. 1.10]. This characterisation also readily gives a variant of Lemma 5.17 for $B^s_{p,q}$.

Then Lemma 5.4 is replaced by Corollary 4.13 and it is noted that Theorem 5.13, which can be found with proof on page 33, carries over to the quasi-norm $\|\cdot|\ell_q(L_p)\|$. Indeed, the only modification is to apply the inequality in [44, (21)] instead of Lemma 4.8 in the last line of the proof.

Finally, the reference to Theorem 5.14 is changed to [44, (23)]. However, Rychkov's starting point [44, (34)] was flawed, as mentioned in Remark 4.1, but it can be derived from our anisotropic version in Proposition 4.20, as the elementary inequality $\prod(1+|2^{ja_l}z_l|)^{r_0} \geq (1+|2^{j\vec{a}}z|)^{r_0}$ brings us back at once to the isotropic maximal functions. Our anisotropic dilations by $2^{j\vec{a}}$ disappear when invoking the majorant property of the maximal function (cf. its proof in [54, p. 57]).

For small s, the lift operator

$$I_r u = \mathcal{F}^{-1}(\langle\xi\rangle^r \mathcal{F}u) \tag{6.24}$$

is used instead of (5.7), because application of [57, 2.3.8] then readily gives an $h \in B^{s+r}_{p,q}(\mathbb{R}^n)$ for some even integer $r > s_1 - s$, such that $e_V f = I_r h$. Since

$$I_r h = (1-\Delta)^{\frac{r}{2}} h,$$

the rest of the proof is easily carried over to a full proof of the fact that a C^∞-bijection $\sigma : U \to V$ sends $\overline{B}^s_{p,q}(V)$ boundedly into $\overline{B}^s_{p,q}(U)$.

6.4.4 Mixed-Norm Lizorkin–Triebel Spaces on Curved Boundaries

As a motivation, we first note that in case of evolution equations, the function $u(x,t)$, depending on the location x in space and the time t, describes to each t in an open interval $I \subset \mathbb{R}$ the state of a system (as a function of x in an open subset $\Omega \subset \mathbb{R}^n$). Thus solutions are sought in $C_{\mathrm{b}}(\mathbb{R}, L_{\vec{r}}(\Omega))$, say for some $\vec{r} \geq 1$, equipped with the norm

$$\sup_{t\in I} \| u(x,t)\,|L_{\vec{r}}(\Omega)\|.$$

Thus it should be natural to work in the scale of mixed-norm Lizorkin–Triebel spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, in which t is taken as the outer integration variable in the norm of $L_{\vec{p}}$; i.e. we take $t = x_{n+1}$ with associated weight a_t and integral exponent p_t (when it eases notation, they will be written with $n+1$ as index).

The results in Section 6.4.2 can be carried over to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ under the assumptions that

$$a_0 := a_1 = \ldots = a_n, \qquad p_0 := p_1 = \ldots = p_n, \tag{6.25}$$

and that Ω is C^∞ in the sense adopted e.g. by [17]:

Definition 6.25. *An open set $\Omega \subset \mathbb{R}^n$ with boundary Γ is C^∞ (or smooth), when for each boundary point $x \in \Gamma$ there exists a diffeomorphism λ defined on an open neighbourhood $U_\lambda \subset \mathbb{R}^n$ such that $\lambda : U_\lambda \to B(0,1) \subset \mathbb{R}^n$ is surjective and*

$$\begin{aligned}\lambda(x) &= 0,\\ \lambda(U_\lambda \cap \Omega) &= B(0,1) \cap \mathbb{R}^n_+,\\ \lambda(U_\lambda \cap \Gamma) &= B(0,1) \cap \mathbb{R}^{n-1},\end{aligned}$$

whereby $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n-1} \times \{0\}$.

The unit ball in $\mathbb{R}^n$ will below be denoted by B and in $\mathbb{R}^{n-1}$ by B'.

Curved Boundaries in General

Let $I \subset \mathbb{R}$ be an open interval. As $\Gamma \times I$ is a C^∞-manifold, $\mathcal{D}'(\Gamma \times I)$ is a special case of Definition 6.15 and therefore the results regarding distributions on manifolds in Section 6.4.1 are applicable. The manifold can be equipped with e.g. the atlas $\mathcal{F} \times \mathcal{N}$, where $\mathcal{F} = \{\kappa\}$ and $\mathcal{N} = \{\eta\}$ are maximal atlases on Γ, respectively on I.

Locally finite partitions of unity $1 = \sum \psi_j(x)$ and $1 = \sum \varphi_l(t)$ subordinate to $\mathcal{F}$, respectively to $\mathcal{N}$ give a locally finite partition of unity $1 = \sum \psi_j \otimes \varphi_l$ on $\Gamma \times I$. Note that we formally should sum with respect to a fixed enumeration of the pairs (j, l) in $\mathbb{N} \times \mathbb{N}$, but for simplicity's sake we avoid this. (The sums are locally finite anyway.) As above, we use the notation $\widetilde{\psi_j \otimes \varphi_l} = (\psi_j \otimes \varphi_l) \circ (\kappa(j)^{-1} \times \eta(l)^{-1})$.

Since the maximal atlas on $\Gamma \times I$ contains charts that do not respect the splitting into t and the x-variables, it is not obviously useful for the anisotropic spaces. We have therefore chosen to adopt Lemma 6.19 as our point of departure for the $F^{s,\vec{a}}_{\vec{p},q}$-spaces on the curved boundary. Because Γ is of dimension $n-1$, it is noted that the parameters $\vec{a}$, $\vec{p}$ for these spaces only contain n entries.

Definition 6.26. *The space $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ consists of the $u \in \mathcal{D}'(\Gamma \times I)$ for which*

$$\widetilde{\psi_j \otimes \varphi_l u}_{\kappa(j)\times\eta(l)} \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_{\kappa(j)} \times \widetilde{I}_{\eta(l)}), \quad j, l \in \mathbb{N}.$$

The family in (6.18) and Corollary 6.21 adapted to this set-up, cf. Theorem 6.12, give that

$$\mu_{j,l}(u) := \big\| \widetilde{\psi_j \otimes \varphi_l u}_{\kappa(j)\times\eta(l)} \,\big|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_{\kappa(j)} \times \widetilde{I}_{\eta(l)})\big\|, \quad j, l \in \mathbb{N}, \tag{6.26}$$

is a separating family of quasi-seminorms and that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ can be equivalently defined from any atlas $\mathcal{F}_1 \times \mathcal{N}_1$, where $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{N}_1 \subset \mathcal{N}$; with the same topology.

Theorem 6.27. *The space* $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ *is a complete topological vector space with a translation invariant metric; for* $p_0, p_t, q \geq 1$ *it is locally convex, hence a Fréchet space.*

Proof. For $d := \min(1, p_0, p_t, q)$ the separating family $(\mu^d_{j,l})_{j,l\in\mathbb{N}}$, cf. (6.26), is used to construct a topology as in Theorem 6.23. This immediately gives that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ is a topological vector space and even locally convex, when $d \geq 1$.

The metric is in this case obtained by letting the $\mu_{j,l}$ enter the summation formula for $d'(u,v)$, cf. (6.20), as any enumeration of the (j,l) gives the same sum; adapting the arguments in the proof of [17, Thm. B.9] to two summation indices is straightforward.

Completeness follows as in the isotropic case, but with application of Theorem 6.12 instead of Theorem 6.11 when showing the convergence. □

Curved Boundaries in the Compact Case

When Γ is compact, a finite atlas on the boundary can e.g. be obtained from the composite maps $\kappa = \widetilde{\gamma}_{0,n} \circ \lambda$, where $\widetilde{\gamma}_{0,n} \colon (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, 0)$ in local coordinates. Indeed, according to Definition 6.25 and the compactness of Γ there exists on Γ a finite open cover $\{U_\lambda\}$, where λ runs in an index set Λ, which together with Ω gives an open cover of $\overline{\Omega}$. Each $\lambda \in \Lambda$ induces a diffeomorphism $\kappa : \Gamma_\kappa \to B'$ on $\Gamma_\kappa := U_\lambda \cap \Gamma$ by $\kappa = \widetilde{\gamma}_{0,n} \circ \lambda$. These maps form an atlas $\mathcal{F}_0$ on Γ and thereby an atlas $\{\kappa \times \mathrm{id}_{\mathbb{R}}\}_{\kappa\in\mathcal{F}_0}$ on $\Gamma \times \mathbb{R}$.

A partition of unity is obtained by using a function $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ on $\Omega \setminus \bigcup_\lambda U_\lambda$ to slightly generalise [17, Thm. 2.16]. This yields a family of functions $\{\psi_\lambda\} \cup \{\psi\}$ with $\psi_\lambda \in C^\infty_0(U_\lambda)$ and $\psi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset \Omega$ such that $\sum_\lambda \psi_\lambda(x) + \psi(x) = 1$ for $x \in \overline{\Omega}$. (Existence of such χ is similar to [17, Cor. 2.14], where K need not be compact.)

In addition, the functions $\psi_\kappa := \psi_\lambda|_\Gamma \in C^\infty_0(\Gamma_\kappa)$ constitute a finite partition of unity of Γ subordinate to $\mathcal{F}_0$. Hence $1 = \sum_{\kappa\in\mathcal{F}_0} \psi_\kappa \otimes \mathbb{1}_{\mathbb{R}}$, with $\mathbb{1}_{\mathbb{R}}$ denoting the characteristic function of $\mathbb{R}$, is a partition of unity on $\Gamma \times \mathbb{R}$.

Recalling that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ is equivalently defined from any atlas $\mathcal{F}_1 \times \mathcal{N}_1$, where $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{N}_1 \subset \mathcal{N}$, we obtain

Theorem 6.28. *Let* Γ *be compact and* $J \subset \mathbb{R}$ *be a compact interval. The space*

$$\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J) := \{\, u \in F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times \mathbb{R}) \mid \operatorname{supp} u \subset \Gamma \times J\} \tag{6.27}$$

is closed and a quasi-Banach space (normed if $\vec{p}, q \geq 1$*), when equipped with the quasi-norm*

$$\| u \,|\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J)\| := \Big(\sum_{\kappa\in\mathcal{F}_0} \| \widetilde{\psi_\kappa \otimes \mathbb{1}_{\mathbb{R}} u_{\kappa\times\mathrm{id}_{\mathbb{R}}}} \,|\overline{F}{}^{s,\vec{a}}_{\vec{p},q}(B' \times \mathbb{R})\|^d \Big)^{1/d}, \tag{6.28}$$

where $d := \min(1, p_0, p_t, q)$*. Furthermore,* $\| \cdot \,|\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J)\|^d$ *is subadditive.*

The support condition in (6.27) means $\bigcup_{\kappa\in\mathcal{F}_0}(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})(\operatorname{supp} u_{\kappa\times\mathrm{id}_{\mathbb{R}}})\subset\Gamma\times J$, cf. (6.14), hence

$$\operatorname{supp} u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\subset B'\times J. \tag{6.29}$$

This implies that each summand in (6.28) is finite, since the factor $\mathbb{1}_{\mathbb{R}}$ can be replaced by some $\chi\in C_0^\infty(\mathbb{R})$ where $\chi=1$ on J; and this χ can be chosen as a finite sum of the φ_l from Definition 6.26.

Proof. By the same arguments as in Theorem 6.24, the expression in (6.28) is a quasi-norm. It gives the same topology on $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ as the family $(\mu_{j,l}^d)_{j,l\in\mathbb{N}}$, since there exist $c_1,c_2>0$ such that for each $u\in\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$, cf. (6.26),

$$c_1\mu_{j,l}(u)^d\le \|\,u\,|\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)\|^d\le c_2\sum_{j',l'\in\mathbb{N}}{}' \mu_{j',l'}(u)^d, \tag{6.30}$$

where the prime indicates that the summation is over finitely many integers.

Indeed, Theorem 6.12 yields that $\mu_{j,l}(u)^d$ is bounded from above by

$$\begin{aligned}&\sum_{\kappa\in\mathcal{F}_0}\big\|\,(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\kappa(j)^{-1}\times\eta(l)^{-1})\widetilde{\psi_j\otimes\varphi_l u}_{\kappa(j)\times\eta(l)}\,\big\|^d\\ &\qquad\le c\sum_{\kappa\in\mathcal{F}_0}\big\|\,\widetilde{\psi_\kappa\otimes\mathbb{1}}_{\mathbb{R}}(\psi_j\otimes\varphi_l)\circ(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\,|\overline{F}{}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_\kappa\times\mathbb{R})\big\|^d,\end{aligned}$$

where $\|\cdot\|$ is the norm on $\overline{F}{}^{s,\vec{a}}_{\vec{p},q}\big(\kappa(j)\times\eta(l)(\Gamma_{\kappa(j)}\cap\Gamma_\kappa\times\mathbb{R}_{\eta(l)})\big)$. Using for each $\kappa\in\mathcal{F}_0$ some function $\chi_\kappa\in C^\infty_{L_\infty}(\mathbb{R}^n)$ chosen such that $\chi_\kappa=1$ on $\operatorname{supp}\widetilde{\psi}_\kappa\cap\operatorname{supp}(\psi_j\circ\kappa^{-1})$ and $\operatorname{supp}\chi_\kappa\subset\kappa(\Gamma_{\kappa(j)}\cap\Gamma_\kappa)$, we extend by 0 to $\mathbb{R}^{n+1}$ and apply Lemma 6.5 to obtain the left-hand inequality in (6.30).

The right-hand inequality can be shown similarly by first replacing $\mathbb{1}_{\mathbb{R}}$ in (6.28) with some $\chi\in C_0^\infty(\mathbb{R})$ where $\chi=1$ on J, as discussed above.

To prove that $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ is closed, we consider an arbitrary sequence $(u_m)_{m\in\mathbb{N}}$, which belongs to $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ and converges in $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R})$ to some u. Since $u_{m,\kappa\times\mathrm{id}_{\mathbb{R}}}$ converges to $u_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ in $\mathcal{D}'(B'\times\mathbb{R})$ and (6.29) holds for each $u_{m,\kappa\times\mathrm{id}_{\mathbb{R}}}$, it follows that $\operatorname{supp} u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\subset B'\times J$, whence $\operatorname{supp} u\subset\Gamma\times J$.

Completeness follows immediately, since each Cauchy sequence in $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ converges to some u in $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R})$ and closedness of $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ then gives that $\operatorname{supp} u\subset\Gamma\times J$. □

6.5 Rychkov's Universal Extension Operator

A key ingredient in the construction of right-inverses to the trace operators is a modification of Rychkov's extension operator, introduced in [45] for bounded or special Lipschitz domains $\Omega \subset \mathbb{R}^n$, cf. Definition 3.7,

$$\mathcal{E}_{u,\Omega} : \overline{F}^{s}_{p,q}(\Omega) \to F^{s}_{p,q}(\mathbb{R}^n). \tag{6.31}$$

The linear and bounded operator $\mathcal{E}_{u,\Omega}$ works for all $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, cf. [45, Thm 4.1]; and it also applies to Besov spaces ($p = \infty$ included). Thus it was termed a *universal* extension operator.

In Section 6.6.3 below it will be clear that we for $\Omega = \mathbb{R}^n_+$ also need an extension operator for anisotropic spaces with mixed norms. We therefore modify $\mathcal{E}_{u,\Omega}$ accordingly, relying on the proof strategy in [45], yet we present significant simplifications in the proof of Proposition 6.30 and add e.g. Proposition 6.31. The reader may choose to skip the proofs in a first reading.

We take another approach than Rychkov when defining $\overline{\mathcal{S}}'(\mathbb{R}^n_+)$; this can be justified by [45, Prop. 3.1] and the remark prior to it. Similarly to [16, App. A.4] we use the following distribution spaces:

Definition 6.29. *For any open set $U \subset \mathbb{R}^n$, the space $\overline{\mathcal{S}}'(U)$ is defined as the set of $f \in \mathcal{D}'(U)$ for which there exists $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ such that $r_U \widetilde{f} = f$.*

The spaces $\mathring{\mathcal{S}}(\overline{U})$ and $\mathring{\mathcal{S}}'(\overline{U})$ consist of the functions in $\mathcal{S}(\mathbb{R}^n)$, respectively the distributions in $\mathcal{S}'(\mathbb{R}^n)$ supported in $\overline{U}$.

We define the convolution $\varphi * f(x)$ for $x \in \mathbb{R}^n_+$, when $f \in \overline{\mathcal{S}}'(\mathbb{R}^n_+)$, cf. Definition 6.29, and when $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has its support in the opposite half-space $\overline{\mathbb{R}}^n_-$, that is $\varphi \in \mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_-)$. This is done by using an arbitrary extension $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ of f, i.e.

$$\varphi * f(x) := \langle \widetilde{f}, \varphi(x - \cdot) \rangle, \quad x \in \mathbb{R}^n_+, \tag{6.32}$$

which is well defined, since it as a function on $\mathbb{R}^n_+$ clearly does not depend on the choice of extension $\widetilde{f}$.

This is used in a variant of Calderón's reproducing formula (cf. Proposition 6.30 below),

$$f = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n_+), \tag{6.33}$$

to give meaning to each $\psi_j * (\varphi_j * f)$; cf. (6.4) for the subscript notation. Indeed, $\varphi_j * \widetilde{f} \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ is an extension of $\varphi_j * f$ by (6.32), so (6.32) also yields

$$\psi_j * (\varphi_j * f)(x) := \psi_j * (\varphi_j * \widetilde{f})(x), \quad x \in \mathbb{R}^n_+. \tag{6.34}$$

The idea in Rychkov's extension operator $\mathcal{E}_u$ is to use *another* extension of $\varphi_j * f$, namely

$$e_+(\varphi_j * f)(x) := \begin{cases} 0 & \text{for } x \in \overline{\mathbb{R}}^n_-, \\ \varphi_j * \widetilde{f}(x) & \text{for } x \in \mathbb{R}^n_+; \end{cases}$$

for brevity, we use $e_+ = e_{\mathbb{R}^n_+}$ and $r_+ = r_{\mathbb{R}^n_+}$. Indeed, $e_+(\varphi_j * f)$ is C^∞ for $x_n \neq 0$, hence measurable, and in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Moreover $e_+(\varphi_j * f)$ is in $\mathcal{S}'(\mathbb{R}^n)$, because it is $O((1+|x|^2)^N)$ for a large N. Using (6.32), we obtain the alternative formula

$$\psi_j * (\varphi_j * f)(x) = \psi_j * e_+(\varphi_j * f)(x), \quad x \in \mathbb{R}^n_+. \tag{6.35}$$

Here we can exploit that $\psi_j * e_+(\varphi_j * f)$ is defined on all of $\mathbb{R}^n$, hence by substituting this into the right-hand side of (6.33), $\mathcal{E}_u$ is obtained simply by letting x run through not just $\mathbb{R}^n_+$, but $\mathbb{R}^n$, i.e.

$$\mathcal{E}_u(f) := \sum_{j=0}^{\infty} \psi_j * e_+(\varphi_j * f) \quad \text{for } f \in \overline{\mathcal{S}}'(\mathbb{R}^n_+). \tag{6.36}$$

To make this description more precise, we first justify (6.33). So we recall that a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ fulfils moment conditions of order L_φ, when

$$D^\alpha(\mathcal{F}\varphi)(0) = 0 \quad \text{for} \quad |\alpha| \le L_\varphi.$$

Proposition 6.30. *There exist 4 functions $\varphi_0, \varphi, \psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ supported in $\mathbb{R}^n_-$ and with $L_\varphi, L_\psi = \infty$ such that* (6.33) *holds for all $f \in \overline{\mathcal{S}}'(\mathbb{R}^n_+)$.*

Proof. We shall exploit the existence of a real-valued function $g \in \mathcal{S}(\mathbb{R})$ with

$$\int g(t)\,dt \neq 0, \qquad \int t^k g(t)\,dt = 0 \quad \text{for all} \quad k \in \mathbb{N},$$

and $\operatorname{supp} g \subset [1,\infty[$. (This may be obtained as in [45, Thm. 4.1(a)].)

With $\varphi_0(x) := g(-x_1)\cdots g(-x_n)/c^n$ for $c = \int g\,dt$, the properties of g immediately give

$$\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n \mid x_k < 0, \ \ k = 1,\dots,n\},$$
$$\int \varphi_0\,dx = 1, \qquad \int x^\alpha \varphi_0(x)\,dx = 0 \quad \text{for} \quad |\alpha| > 0.$$

Thus the support of $\varphi := \varphi_0 - 2^{-|\vec{a}|}\varphi_0(2^{-\vec{a}}\cdot)$ lies in $\mathbb{R}^n_-$, and $L_\varphi = \infty$ since

$$\int x^\alpha \varphi(x)\,dx = \int x^\alpha \varphi_0(x)\,dx - 2^{\vec{a}\cdot\alpha} \int x^\alpha \varphi_0(x)\,dx = 0 \quad \text{for all } |\alpha| \ge 0.$$

The functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ are conveniently defined via $\mathcal{F}$,

$$\begin{aligned} \widehat{\psi}_0(\xi) &= \widehat{\varphi}_0(\xi)(2 - \widehat{\varphi}_0(\xi)^2), \\ \widehat{\psi}(\xi) &= \big(\widehat{\varphi}_0(\xi) + \widehat{\varphi}_0(2^{\vec{a}}\xi)\big)\big(2 - \widehat{\varphi}_0(\xi)^2 - \widehat{\varphi}_0(2^{\vec{a}}\xi)^2\big). \end{aligned} \tag{6.37}$$

Since $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j\vec{a}}\xi) = \widehat{\varphi}_0(2^{-j\vec{a}}\xi) - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)$ for $j \ge 1$, we obtain by basic algebraic rules,

$$\begin{aligned} \widehat{\psi}_j(\xi)\widehat{\varphi}_j(\xi) &= \big(2 - \widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big)\big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big) \\ &= 2\big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big) - \big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^4 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^4\big). \end{aligned}$$

This gives a telescopic sum:

$$\sum_{j=0}^{\infty} \widehat{\psi}_j(\xi)\widehat{\varphi}_j(\xi) = 2 \lim_{N\to\infty} \widehat{\varphi}_0(2^{-N\vec{a}}\xi)^2 - \lim_{N\to\infty} \widehat{\varphi}_0(2^{-N\vec{a}}\xi)^4 = 1, \tag{6.38}$$

using that $\widehat{\varphi}_0(0) = 1$. As the convergence is in $\mathcal{S}'(\mathbb{R}^n)$, the inverse Fourier transformation yields

$$\sum_{j=0}^{\infty} \psi_j * \varphi_j = \delta. \tag{6.39}$$

The fact that $L_\psi = \infty$ is obvious from (6.37), since $D^\alpha \widehat{\varphi}_0(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$. The inclusion $\operatorname{supp} \psi_0 \subset \mathbb{R}^n_-$ is clear, because $\psi_0 = \varphi_0 * (2\delta - \varphi_0 * \varphi_0)$. Similarly $\operatorname{supp} \psi \subset \mathbb{R}^n_-$, since ψ is a sum of convolutions of functions with such support.

To show (6.33), we note that when $\widetilde{f} \in \mathcal{S}'(\mathbb{R}^n)$ fulfils $r_+\widetilde{f} = f$, then by (6.39),

$$\widetilde{f} = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * \widetilde{f}) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n). \tag{6.40}$$

More precisely, to circumvent that the summands in (6.39) need not have compact supports, one can show that $\sum_{j<N} \widehat{\psi}_j \widehat{\varphi}_j \mathcal{F}\widetilde{f}$ converges to $\mathcal{F}\widetilde{f}$ in $\mathcal{S}'(\mathbb{R}^n)$ by using (6.38) and a test function in $\mathcal{S}(\mathbb{R}^n)$. Then (6.34) gives,

$$f = r_+\widetilde{f} = \sum_{j=0}^{\infty} r_+\big(\psi_j * (\varphi_j * \widetilde{f})\big) = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n_+),$$

in view of the continuity of $r_+ : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n_+)$. □

As a novelty, one can now show directly that $\mathcal{E}_u$ has nice properties on the space $\overline{\mathcal{S}}'(\mathbb{R}^n_+)$ of restricted temperate distributions:

Proposition 6.31. *The series for $\mathcal{E}_u(f)$ in (6.36) converges in $\mathcal{S}'(\mathbb{R}^n)$ whenever $f \in \overline{\mathcal{S}}'(\mathbb{R}^n_+)$, and the induced map $\mathcal{E}_u : \overline{\mathcal{S}}'(\mathbb{R}^n_+) \to \mathcal{S}'(\mathbb{R}^n)$ is w^*-continuous.*

Remark 6.32. *The space $\overline{\mathcal{S}}'(\mathbb{R}^n_+)$ is endowed with the seminorms $f \mapsto |\langle \widetilde{f}, \varphi\rangle|$ for $\varphi \in \mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_+)$ and $r_+\widetilde{f} = f$, using the well-known fact that it is the dual of $\mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_+)$. (I.e. $f_\nu \to 0$ means that for some (hence every) net $\widetilde{f}_\nu$ of extensions, one has $\langle \widetilde{f}_\nu, \varphi\rangle \to 0$ for all $\varphi \in \mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_+)$.)*

Proof. It suffices according to the limit theorem for $\mathcal{S}'$ to obtain convergence of

$$\sum_{j=0}^{\infty} \langle e_+(\varphi_j * f), \check{\psi}_j * \eta\rangle \quad \text{for } \eta \in \mathcal{S}(\mathbb{R}^n), \tag{6.41}$$

where $\check{\psi}(x) = \psi(-x)$ as usual.

Since $L_\psi = \infty$, it follows at once from Lemma 4.16 that the second entry tends rapidly to zero, i.e. for any seminorm p_M one has

$$p_M(\check{\psi}_j * \eta) = O(2^{-jN}) \quad \text{for every } N > 0. \tag{6.42}$$

For the first entries, a test against an arbitrary $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives, for some M,

$$\begin{aligned} |\langle e_+(\varphi_j * f), \phi\rangle| &= \Big| \int \langle \tilde{f}(y), \varphi_j(x-y)\rangle 1_{\mathbb{R}^n_+}(x)\phi(x)\,dx \Big| \\ &= |\langle 1_{\mathbb{R}^n_+} \otimes \tilde{f}(x, x-y), \phi \otimes \varphi_j)\rangle_{\mathbb{R}^n\times\mathbb{R}^n}| \\ &\le c p_M(\phi \otimes \varphi_j) \le c' p_M(\phi) p_M(\varphi_j). \end{aligned} \tag{6.43}$$

Here $p_M(\varphi_j) = p_M(2^{j|\vec{a}|}\varphi(2^{j\vec{a}}\cdot)) = O(2^{j(|\vec{a}|+Ma^0)})$ grows at a fixed rate. Therefore the choice $\phi = \check{\psi}_j * \eta$ shows via (6.42) that the series has rapidly decaying terms, hence converges.

To obtain continuity of $\mathcal{E}_u$, it suffices to show that $T\eta := \sum_{j=0}^\infty \check{\varphi}_j * \big(\mathbb{1}_{\mathbb{R}^n_+}(\check{\psi}_j * \eta)\big)$ defines a transformation $T : \mathcal{S}(\mathbb{R}^n) \to \mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_+)$ satisfying

$$\langle \mathcal{E}_u(f), \eta\rangle = \langle \widetilde{f}, T\eta\rangle \quad \text{for all } \eta \in \mathcal{S}(\mathbb{R}^n). \tag{6.44}$$

To this end we may let $\mathbb{1}_{\mathbb{R}^n_+}$ act first in (6.43), which via (6.41) gives

$$\langle \mathcal{E}_u(f), \eta\rangle = \sum_{j=0}^\infty \big\langle \widetilde{f}, \int \check{\psi}_j * \eta(x) \mathbb{1}_{\mathbb{R}^n_+}(x)\varphi_j(x-y)\,dx\big\rangle. \tag{6.45}$$

The integral is in $\mathcal{S}(\mathbb{R}^n)$ as a function of y (cf. the theory of tensor products), and since $\operatorname{supp}\varphi_j \subset \overline{\mathbb{R}}^n_-$ it is only non-zero for $y_n \ge x_n > 0$. Hence the summands in $T\eta$ belong to $\mathring{\mathcal{S}}(\overline{\mathbb{R}}^n_+)$, so T has range in this subspace, if its series converges in $\mathcal{S}(\mathbb{R}^n)$. But by the completeness, this follows since any seminorm p_M applied to $\int \check{\psi}_j * \eta(x)\mathbb{1}_{\mathbb{R}^n_+}(x)\varphi_j(x-y)\,dx$ is estimated by $cp_M(\check{\varphi}_j)p_{M+n+1}(\check{\psi}_j * \eta)$, which tends rapidly to 0 as above.

Finally, (6.45) now yields (6.44) by summation in the second entry. □

In the next convergence result, the familiar dyadic corona condition, cf. e.g. [29, Lem. 3.20], has been weakened to one involving convolution with a function ψ satisfying a moment condition of infinite order. It appeared implicitly in [45].

Lemma 6.33. *Let $(g^j)_{j\in\mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$ such that*

$$\|\,(g^j)\,\| := \|\,2^{js}G^j\,|L_{\vec{p}}(l_q)\| < \infty,$$

where for some $\vec{r} > 0$,

$$G^j(x) = \sup_{y\in\mathbb{R}^n} \frac{|g^j(y)|}{\prod_{l=1}^n (1 + 2^{ja_l}|x_l - y_l|)^{r_l}}, \quad x \in \mathbb{R}^n.$$

When $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ with $L_\psi = \infty$, then $\sum_{j=0}^{\infty} \psi_j * g^j$ converges in $\mathcal{S}'(\mathbb{R}^n)$ for any such $(g^j)_{j\in\mathbb{N}_0}$ and

$$\Big\| \sum_{j=0}^{\infty} \psi_j * g^j \,\Big|F^{s,\vec{a}}_{\vec{p},q}\Big\| \le c_{q,s} \| \,(g^j)\, \| \tag{6.46}$$

with a constant $c_{q,s}$ independent of $(g^j)_{j\in\mathbb{N}_0}$.

Proof. By assumption $\| \,(g^j)\, \| < \infty$, hence $G^j(\widetilde{x}) < \infty$ for an $\widetilde{x} \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, implying $|g^j(x)| \le G^j(\widetilde{x}) \prod_{l=1}^{n}(1 + 2^{ja_l}|\widetilde{x}_l - x_l|)^{r_l}$. Thereby, g^j belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and grows at most polynomially, thus g^j and therefore also $\psi_j * g^j$ are in $\mathcal{S}'(\mathbb{R}^n)$.

Using Φ_l from (6.1), the following estimate holds for $l \in \mathbb{N}_0$, $x \in \mathbb{R}^n$,

$$|\mathcal{F}^{-1}\Phi_l * \psi_j * g^j(x)| \le \int |\mathcal{F}^{-1}\Phi_l * \psi_j(z)||g^j(x-z)|\,dz \le I_{j,l} \cdot G^j(x), \tag{6.47}$$

where

$$I_{j,l} = \int |\mathcal{F}^{-1}\Phi_l * \psi_j(z)| \prod_{l=1}^{n}(1 + 2^{ja_l}|z_l|)^{r_l}\,dz.$$

Since $L_\psi = \infty = L_{\mathcal{F}^{-1}\Phi}$, a straightforward application of Lemma 4.19 yields the following estimate of the anisotropic dilations in $I_{j,l}$: for every $M > 0$ there is some $C_M > 0$ such that

$$I_{j,l} \le C_M 2^{-|l-j|M} \quad \text{for all } j, l \in \mathbb{N}_0.$$

For $M = \varepsilon + |s|$, where $\varepsilon > 0$ is arbitrary, we obtain from (6.47),

$$2^{ls}|\mathcal{F}^{-1}\Phi_l * \psi_j * g^j(x)| \le c_s 2^{js} 2^{-|l-j|\varepsilon} G^j(x), \tag{6.48}$$

which implies, using $|j-l| \ge j-l$,

$$\| \,\psi_j * g^j \,|F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}\| \le c_s\Big(\sum_{l=0}^{\infty} 2^{(-|j-l|-2l)\varepsilon}\Big)\| \,2^{js}G^j \,|L_{\vec{p}}\| \le c_s 2^{-j\varepsilon}\| \,(g^j)\, \|.$$

This yields for $d := \min(1, p_1, \dots, p_n)$,

$$\sum_{j=0}^{\infty} \| \,\psi_j * g^j \,|F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}\|^d \le c_s^d \| \,(g^k)\, \|^d \sum_{j=0}^{\infty} 2^{-j\varepsilon d} < \infty,$$

hence $\sum_{j=0}^{\infty} \psi_j * g^j$ converges in the quasi-Banach space $F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}$ and thus in $\mathcal{S}'$.

Finally, by (6.48) and Lemma 4.8 applied to $(2^{js}G^j)_{j\in\mathbb{N}_0}$,

$$\begin{aligned}\Big\| \sum_{j=0}^{\infty} \psi_j * g^j \,\Big|F^{s,\vec{a}}_{\vec{p},q}\Big\| &\le c_{q,s}\Big\| \Big(\sum_{j=0}^{\infty} 2^{-|l-j|\varepsilon} 2^{js} G^j\Big)_{l\in\mathbb{N}_0} \Big|L_{\vec{p}}(\ell_q)\Big\| \\ &\le c_{q,s}\| \,2^{js}G^j \,|L_{\vec{p}}(\ell_q)\|,\end{aligned}$$

which shows (6.46). □

We recall a variant φ_j^+ of the Peetre–Fefferman–Stein maximal operators induced by $(\varphi_j)_{j\in\mathbb{N}_0}$, with $\varphi_0,\varphi\in\mathcal{S}$ supported in $\mathbb{R}^n_-$; i.e. for $f\in\overline{\mathcal{S}}'(\mathbb{R}^n_+)$ and $\vec{r}>0$,

$$\varphi_j^+ f(x)=\sup_{y\in\mathbb{R}^n_+}\frac{|\varphi_j*f(y)|}{\prod_{l=1}^n(1+2^{ja_l}|x_l-y_l|)^{r_l}},\quad x\in\mathbb{R}^n_+,\ j\in\mathbb{N}_0. \tag{6.49}$$

Now we are ready to state the main theorem of this section:

Theorem 6.34. *When $\varphi_0,\varphi,\psi_0,\psi\in\mathcal{S}(\mathbb{R}^n)$ are as in Proposition 6.30, then*

$$\mathcal{E}_u(f):=\sum_{j=0}^{\infty}\psi_j*e_+(\varphi_j*f) \tag{6.50}$$

is a linear extension operator from $\overline{\mathcal{S}}'(\mathbb{R}^n_+)$ to $\mathcal{S}'(\mathbb{R}^n)$, i.e. $r_+\mathcal{E}_uf=f$ in $\mathbb{R}^n_+$ for every $f\in\overline{\mathcal{S}}'(\mathbb{R}^n_+)$. Moreover, $\mathcal{E}_u:\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)\to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is bounded for all $s\in\mathbb{R}$, $0<\vec{p}<\infty$ and $0<q\le\infty$.

Proof. First it is shown using (6.49) that for an arbitrary $f\in\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)$ and $\vec{r}>\min(q,p_1,\dots,p_n)^{-1}$,

$$\|\,2^{js}\varphi_j^+f\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n_+)\|\le c\|\,f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)\|. \tag{6.51}$$

Besides φ_j^+f, we shall use the well-known maximal operator φ_j^*f, where the supremum in (6.49) is replaced by supremum over $\mathbb{R}^n$. Hence for every $g\in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ such that $r_+g=f$, we get from (6.32) that

$$\varphi_j^+f(x)=\sup_{y\in\mathbb{R}^n_+}\frac{|\varphi_j*g(y)|}{\prod_{l=1}^n(1+2^{ja_l}|x_l-y_l|)^{r_l}}\le\varphi_j^*g(x),\quad x\in\mathbb{R}^n_+.$$

This yields (6.51) when combined with the following, obtained from techniques behind Theorem 4.23:

$$\inf_{r_+g=f}\|\,2^{js}\varphi_j^*g\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|\le c\inf_{r_+g=f}\|\,g\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|=c\|\,f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)\|. \tag{6.52}$$

More precisely, since we only have $L_\varphi=\infty$ available, it is perhaps simplest to exploit that the Tauberian conditions are fulfilled by the functions $\mathcal{F}^{-1}\Phi_0$, $\mathcal{F}^{-1}\Phi$ appearing in the definition of $F^{s,\vec{a}}_{\vec{p},q}$, cf. (6.1). Then Theorem 4.18 yields that the quasi-norm on the left-hand side in (6.52) is estimated by $\|\,2^{js}(\mathcal{F}^{-1}\Phi_j)^*g\,|L_{\vec{p}}(\ell_q)\|$, which in turn is estimated by $\|\,g\,|F^{s,\vec{a}}_{\vec{p},q}\|$ using Theorem 4.22.

To apply Lemma 6.33, we estimate $\|\,(e_+(\varphi_j*f))\,\|$ using the extension of (6.49) to $\mathbb{R}^n$, that is

$$\widetilde{\varphi}_j^+f(x):=\sup_{y\in\mathbb{R}^n_+}\frac{|\varphi_j*f(y)|}{\prod_{l=1}^n(1+2^{ja_l}|x_l-y_l|)^{r_l}},\quad x\in\mathbb{R}^n,\ j\in\mathbb{N}_0,$$

with which it is immediate to see that

$$\| (e_+(\varphi_j * f)) \| = \| 2^{js} \widetilde{\varphi}_j^+ f \,| L_{\vec{p}}(\ell_q) \|.$$

A splitting of the integral on the right-hand side in one over $\mathbb{R}^n_+$, respectively one over $\mathbb{R}^n_-$ yields, using the obvious inequality $\widetilde{\varphi}_j^+ f(x', x_n) \le \varphi_j^+ f(x', -x_n)$ for $x \in \mathbb{R}^n_-$ and (6.51), cf. Lemma 6.33,

$$\| \mathcal{E}_u f \,| F^{s,\vec{a}}_{\vec{p},q} \| \le c \| (e_+(\varphi_j * f)) \| \le 2c \| 2^{js} \varphi_j^+ f \,| L_{\vec{p}}(\ell_q)(\mathbb{R}^n_+) \| \le 2c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \|.$$

Finally, continuity of $r_+ : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n_+)$ together with (6.35) and Proposition 6.30 give

$$r_+(\mathcal{E}_u f) = \sum_{j=0}^{\infty} r_+\big(\psi_j * e_+(\varphi_j * f)\big) = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) = f,$$

hence $\mathcal{E}_u f$ is an extension of f. □

In the study of trace operators, it will be necessary to extend from more general domains. Indeed, using the splitting $x = (x', x_n)$ on $\mathbb{R}^n$ and writing $f(x', C - x_n)$ as $f(\cdot, C - \cdot)$, the fact that $x \mapsto (x', C - x_n)$ is an involution easily gives a universal extension from the half-line $]-\infty, C[$:

Corollary 6.35. *For any $C \in \mathbb{R}$, the operator*

$$\mathcal{E}_{u,C} f(x) := \mathcal{E}_u\big(f(\cdot, C - \cdot)\big)(x', C - x_n)$$

is a linear and bounded extension from $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[)$ to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

Proof. The quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is invariant under translations $\tau_h u = u(\cdot - h)$, cf. [29, Prop. 3.3], and under the reflection $\mathcal{R}u = u(\cdot, -\cdot)$, when Φ_0, Φ are invariant under $\mathcal{R}$, as we may assume up to equivalence. So, clearly $u(x', C - x_n)$ belongs to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ and has the same quasi-norm as u.

By Definition 6.7, this readily implies that the change of coordinates is also continuous from $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[)$ to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]0, \infty[)$. Thus

$$\begin{aligned} \| \mathcal{E}_{u,C} f \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \| &\le c \| \mathcal{E}_u(f(\cdot, C - \cdot)) \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \| \\ &\le c \| f(\cdot, C - \cdot) \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]0, \infty[) \| \\ &\le c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[) \|, \end{aligned}$$

and the linearity of $\mathcal{E}_{u,C}$ follows directly from the linearity of $\mathcal{E}_u$. □

In comparison with the well-known half-space extension by Seeley [52], we note that the above construction is applicable for all $s \in \mathbb{R}$, even in the mixed-norm case. Also it has the advantage that several results from [32] can be utilised, making the argumentation less cumbersome.

6.6 Trace Operators

Under the assumption in (6.25), we study the trace at the flat boundary of a cylinder $\Omega \times I$, where $\Omega \subset \mathbb{R}^n$ is C^∞ and $I :=]0,T[$, possibly $T = \infty$. The trace at the curved boundary is studied only for $T < \infty$ and under the additional assumption that $\partial\Omega$ is compact. The associated operators are

$$\begin{aligned} r_0 &: f(x_1,\dots,x_n,t) \mapsto f(x_1,\dots,x_n,0), \\ \gamma &: f(x_1,\dots,x_n,t) \mapsto f(x_1,\dots,x_n,t)|_\Gamma. \end{aligned}$$

As a preparation (for a discussion of compatibility conditions), the chapter ends with a discussion of traces on both the flat and the curved boundary at the corner $\partial\Omega \times \{0\}$ of the cylinder.

For the reader's sake, we recall some notation from [29], namely that the trace at the hyperplane where $x_k = 0$ is denoted by $\gamma_{0,k}$:

$$\gamma_{0,k} : f(x_1,\dots,x_n,t) \mapsto f(x_1,\dots,0,\dots,x_n,t). \tag{6.53}$$

It will be convenient for us to use $p' := (p_1,\dots,p_{k-1})$, $p'' := (p_{k+1},\dots,p_n,p_t)$, analogously for $\vec{a}$, and $r_l := \max(1,p_l)$. Furthermore, we recall that $x_{n+1} = t$, $a_{n+1} = a_t$, $p_{n+1} = p_t$, hence we shall work with $\vec{a},\vec{p}$ of the form, cf. (6.25),

$$\vec{a} = (a_0,\dots,a_0,a_t), \qquad \vec{p} = (p_0,\dots,p_0,p_t) < \infty, \tag{6.54}$$

where the finiteness of $\vec{p}$ is assumed in order to apply the results in [29].

6.6.1 The Trace at the Flat Boundary

The trace r_s, defined by evaluation at $t = s$, is for each $s \in I$ well defined on the subspace,

$$C(I,\mathcal{D}'(\Omega)) \subset \mathcal{D}'(\Omega \times I), \tag{6.55}$$

where the embedding can be seen by modifying the proof of [26, Prop. 3.5]. On the smaller subspace $C(\overline{I},\mathcal{D}'(\Omega))$ consisting of the elements having a continuous extension in t to $\mathbb{R}$, even the trace r_0 is well defined (and it induces a similar operator also denoted r_0). Indeed, for $u \in C(\overline{I},\mathcal{D}'(\Omega))$ all extensions f are equal in $\Omega \times I$ and by continuity therefore also at $t = 0$, hence

$$r_0 u := f(\cdot,0). \tag{6.56}$$

Now, it was shown in [29, Thm. 2.4] that

$$F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}) \hookrightarrow C_{\mathrm{b}}(\mathbb{R},L_{r'}(\mathbb{R}^n)) \quad \text{when} \quad s > \frac{a_t}{p_t} + n\Big(\frac{a_0}{\min(1,p_0)} - a_0\Big), \tag{6.57}$$

and this induces an embedding $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \hookrightarrow C(\overline{I},L_{r'}(\Omega))$, so the trace r_0 can be applied to $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, i.e. for an arbitrary extension f in $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$,

$$r_0 u = r_\Omega f(\cdot,0). \tag{6.58}$$

To define a right-inverse of r_0 when applied to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, we recall that a bounded right-inverse K_{n+1} of the analogous trace $\gamma_{0,n+1}$ on Euclidean space, cf. [29, Thm. 2.6],

$$K_{n+1} : B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n) \to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}), \quad s \in \mathbb{R}, \tag{6.59}$$

is given by the following, where $\psi \in C^\infty(\mathbb{R})$ so that $\psi(0) = 1$ and $\operatorname{supp}\widehat{\psi} \subset [1, 2]$,

$$K_{n+1}v(x) := \sum_{j=0}^{\infty} \psi(2^{ja_{n+1}}x_{n+1})\mathcal{F}^{-1}(\Phi_j(\xi', 0)\mathcal{F}v(\xi'))(x'). \tag{6.60}$$

Theorem 6.36. *When $\vec{a}, \vec{p}$ fulfil* (6.54) *and s satisfies the inequality in* (6.57)*, then*

$$r_0 : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)$$

is a bounded surjection *and it has a right-inverse K_0. More precisely, the operator K_0 can be chosen so that $K_0 : \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ is bounded for all $s \in \mathbb{R}$.*

Proof. The analogue of this theorem on Euclidean spaces, cf. [29, Thm. 2.5], yields for any $f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$ the existence of a constant c (only depending on $s, \vec{p}, q, \vec{a}$) such that

$$\big\| \gamma_{0,n+1}f \,\big|B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n)\big\| \le c\| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|.$$

Choosing f in (6.58) so the right-hand side is bounded by $2c\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)\|$, we obtain boundedness of r_0, since $r_\Omega(\gamma_{0,n+1}f) = r_0u$, cf. (6.53).

A right-inverse K_0 is constructed using K_{n+1} in (6.59) and Rychkov's extension operator in (6.31):

$$K_0 := r_{\Omega\times I} \circ K_{n+1} \circ \mathcal{E}_{u,\Omega} : \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I). \tag{6.61}$$

(Since (6.31) applies only to isotropic spaces over $\Omega \subset \mathbb{R}^n$, one can exploit (6.54) to make rescalings $(s, a') \leftrightarrow s/a_0$, cf. Lemma 6.3.)

It is bounded for all $s \in \mathbb{R}$, because K_{n+1} and $\mathcal{E}_{u,\Omega}$ are so. Finally, (6.58) yields for any $v \in \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)$,

$$r_0 \circ K_0v = r_\Omega(K_{n+1} \circ \mathcal{E}_{u,\Omega}v)(x_1, \dots, x_n, 0) = r_\Omega \circ \gamma_{0,n+1} \circ K_{n+1} \circ \mathcal{E}_{u,\Omega}v = v,$$

hence K_0 is a right-inverse of r_0. □

6.6.2 A Support Preserving Right-Inverse

As a further preparation for a discussion of parabolic boundary problems, we now present a support preserving right-inverse to the trace at $\{t = 0\}$. It is useful in reduction to problems with homogeneous boundary conditions. At no extra cost, general $\vec{a}$ and $\vec{p}$ are treated in most of this section.

It is known from [29] that whenever $s > \frac{a_t}{p_t} + \sum_{k\le n}\big(\frac{a_k}{\min(1,p_1,\dots,p_k)} - a_k\big)$, then r_0 is bounded,

$$r_0 : F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n\times\mathbb{R}) \to B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n).$$

The particular right-inverse in (6.60) shall now be replaced by a finer construction of a right-inverse Q having the useful property that

$$\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \implies \operatorname{supp} Qu \subset \overline{\mathbb{R}}^n_+ \times \mathbb{R}. \tag{6.62}$$

Roughly speaking the idea is to replace the use of Littlewood–Paley decompositions by kernels of local means $(k_j)_{j\in\mathbb{N}_0}$. That is, we tentatively take Q of the form

$$Qu(x,t) = \sum_{j=0}^{\infty} \eta(2^{ja_t}t) k_j * u(x). \tag{6.63}$$

Hereby the auxiliary function $\eta \in \mathcal{S}(\mathbb{R})$ is again chosen with $\eta(0) = 1$ and such that $\operatorname{supp}\widehat{\eta} \subset [1,2]$.

The main reason for this choice of Qu is that the property (6.62) will eventually result when the kernels k_j are so chosen that

$$\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \implies \operatorname{supp} k_j * u \subset \overline{\mathbb{R}}^n_+. \tag{6.64}$$

By the support rule for convolutions, this follows if $\operatorname{supp} k_j \subset \overline{\mathbb{R}}^n_+$. However, in order to choose the k_j, we shall first take functions $\varphi_0, \varphi, \psi_0, \psi$ in $\mathcal{S}(\mathbb{R}^n)$ with support in $\overline{\mathbb{R}}^n_+$ and satisfying

$$\int \varphi_0\,dx = 1 = \int \psi_0\,dx, \qquad L_\varphi = \infty = L_\psi, \tag{6.65}$$

in such a way that by setting e.g. $\psi_j(x) = 2^{j|\vec{a}|}\psi(2^{j\vec{a}}x)$, one has Calderon's reproducing formula

$$u = \sum_{j=0}^{\infty} \psi_j * \varphi_j * u \quad \text{ for } u \in \mathcal{S}'(\mathbb{R}^n). \tag{6.66}$$

Existence of these functions may be obtained as in the proof of Proposition 6.30, simply by omitting the reflection in the definition of φ_0 and proceeding with the argument for formula (6.40) in the proof there.

Now we can simply obtain $\operatorname{supp} k_j \subset \overline{\mathbb{R}}^n_+$ by choosing

$$k_0 = \psi_0 * \varphi_0, \quad k = \psi * \varphi.$$

Then (6.66) states that $u = \sum_{j\ge 0} k_j * u$, which together with the condition $\eta(0) = 1$ will imply that Q is a right-inverse of r_0.

Since the supports of the k_j are only confined to be in the half-space $\overline{\mathbb{R}}^n_+$, we refer to the k_j as kernels of *localised* means. (Triebel termed them local in case the supports are compact.)

In addition, we need to recall an $\mathcal{S}'$-version of [26, Prop. 3.5]:

Lemma 6.37. *There is an (algebraic) embedding* $C_{\mathrm{b}}(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{R}^n \times \mathbb{R})$ *given by*

$$\langle \Lambda_f, \psi \rangle = \int_{\mathbb{R}} \langle f(t), \psi(\cdot, t) \rangle_{\mathbb{R}^n} \, dt$$

for each continuous, bounded map $f : \mathbb{R} \to \mathcal{S}'(\mathbb{R}^n)$ *and* $\psi \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$.

Proof. By the boundedness, the family $\{f(t)\}_{t \in \mathbb{R}}$ is equicontinuous, so for some $M > 0$ we have $|\langle f(t), \phi \rangle| \le c p_M(\phi)$ for all $t \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Hence the integrand is continuous and estimated crudely by $c p_{M+2}(\psi)/(1+t^2)$, so Λ_f makes sense and $|\langle \Lambda_f, \psi \rangle| \le c\pi p_{M+2}(\psi)$. □

Using this lemma, we can now improve on (6.63) by giving Qu a more precise meaning as an element of $C_{\mathrm{b}}(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^n_x))$. Namely, $Qu(\cdot, t)$ is the distribution given on $\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\langle Qu(\cdot, t), \phi \rangle = \sum_{j=0}^{\infty} \eta(2^{j a_t} t) \langle k_j * u, \phi \rangle. \tag{6.67}$$

This will be clear from the proof of

Proposition 6.38. *The operator Q is a well-defined w^*-continuous linear map* $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n \times \mathbb{R})$ *having range in* $C_{\mathrm{b}}(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^n_x))$. *It is a right-inverse of r_0 preserving supports in $\overline{\mathbb{R}}^n_+$ in the strong form*

$$\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \implies \forall t \in \mathbb{R} : \operatorname{supp} Qu(\cdot, t) \subset \overline{\mathbb{R}}^n_+. \tag{6.68}$$

In particular, $Q : \mathring{\mathcal{S}}'(\overline{\mathbb{R}}^n_+) \to \mathring{\mathcal{S}}'(\overline{\mathbb{R}}^n_+ \times \mathbb{R})$, *cf. Definition 6.29.*

Remark 6.39. *We can of course add that* (6.68) $\implies$ (6.62)*, for we may apply Lemma 6.37 to $f = Qu$ and consider the $\psi(x, t)$ that vanish for $x_n \ge 0$: when* (6.68) *holds, the integrand is identically* 0. *(Unlike* (6.68)*, property* (6.62) *is meaningful also without continuity of Qu with respect to t.)*

Proof. It is first noted that $\sum \langle k_j * u, \phi \rangle$ converges absolutely for each test function $\phi \in \mathcal{S}'(\mathbb{R}^n)$. In fact, using the notation $\check{k}_j(x) = k_j(-x)$, the estimate

$$|\langle u, \check{k}_j * \phi \rangle| \le c p_M(k_j * \phi) \le c 2^{-jN}$$

holds for any $N > 0$; this follows from the infinitely many vanishing moments, i.e. $L_k = \infty$, cf. Lemma 4.16.

Hence $\sum \langle k_j * u, \phi \rangle \eta(2^{j a_t} t)$ is a Cauchy series for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ as $\eta(2^{j a_t} t)$ is a bounded sequence for fixed t. Since it converges, Qu is defined in $\mathcal{S}'(\mathbb{R}^n)$ for each t.

The convergence is absolute and uniform in t, so $t \mapsto \langle Qu(t), \phi \rangle$ is continuous; and bounded by $c \sum |\langle k_j * u, \phi \rangle|$. Therefore Qu is in the subspace $C_{\mathrm{b}}(\mathbb{R}_t, \mathcal{S}'(\mathbb{R}^n_x))$, cf. Lemma 6.37.

Consequently $r_0 Qu$ is defined by evaluation at $t = 0$ and therefore equals $\sum \eta(0) k_j * u(x)$, hence gives back u because of (6.66). Using the convergence in $\mathcal{S}'(\mathbb{R}^n)$, the support preservation in (6.68) is immediate from (6.64) by test against any $\phi \in C_0^\infty(\mathbb{R}^n)$ vanishing for $x_n \geq 0$.

Finally, continuity of Q follows at once if $\langle Qu, \psi\rangle = \langle u, T\psi\rangle$ for $\psi \in \mathcal{S}(\mathbb{R}^{n+1})$, i.e. if Q is the transpose of $T : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^n)$ given by

$$(T\psi)(x) = \int_{\mathbb{R}} \sum_{j=0}^{\infty} \check{k}_j * \psi(x,t) \eta(2^{ja_t} t)\, dt.$$

This series is Cauchy in $\mathcal{S}(\mathbb{R}^{n+1})$, for a seminorm p_M applied to the general term is less than $p_M(\eta(2^{ja_t}t)) = O(2^{ja_t M})$ times $p_M(\check{k}_j * \psi)$, which decays rapidly as $L_k = \infty$. Denoting the sum by $S(x,t)$, also $x \mapsto \int S(x,t)\, dt$ is a Schwartz function, so $T\psi$ is well defined and by the definition of tensor products we get, using (6.67) and Lemma 6.37,

$$\langle u, T\psi\rangle = \langle u \otimes 1, S\rangle = \int \langle u, S(\cdot,t)\rangle\, dt = \int \langle Qu(\cdot,t), \psi(\cdot,t)\rangle\, dt = \langle Qu, \psi\rangle. \quad \square$$

Before we go deeper into the boundedness of Q in the scales of Lizorkin–Triebel spaces, we first sum up the fundamental estimate in the next result. In the isotropic case it goes back at least to the trace investigations of Triebel [57, p. 136].

Proposition 6.40. *For $\vec{p} = (p_1, \dots, p_n, r)$ in $]0,\infty[^{\,n+1}$, a real number $a > 0$ and $0 < q \leq \infty$ there is a constant c with the property that*

$$\big\| \big\{ v_j \otimes 2^{j\frac{a}{r}} f(2^{ja}\cdot) \big\}_{j=0}^{\infty} \,\big|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1}) \big\| \leq c \Big(\sum_{j=0}^{\infty} \| v_j \,|L_{p'}(\mathbb{R}^n)\|^r \Big)^{1/r},$$

whenever (v_j) is a sequence of measurable functions on $\mathbb{R}^n$ and $f \in C(\mathbb{R})$ is such that $t^N f(t)$ is bounded for some $N > 0$ satisfying $Nr > 1$.

Proof. To save a page of repetition from [29, Sec. 4.2.3], we leave it to the reader to carry over the proof given there with a few notational changes. (Note that f itself is bounded, so the arguments there extend to our case without any Schwartz class assumptions on f.) $\square$

Theorem 6.41. *The operator Q is for $0 < \vec{p} < \infty$, $0 < q \leq \infty$ a bounded map*

$$Q : B^{s,a'}_{p',p_t}(\mathbb{R}^n) \to F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^n \times \mathbb{R}) \quad \text{for all } s \in \mathbb{R}.$$

Proof. By means of an auxiliary function $\mathcal{F}\widetilde{\eta} \in C_0^\infty(\mathbb{R})$ fixed such that $\mathcal{F}\widetilde{\eta} = 1$ on $[1,2] \supset \operatorname{supp}\widehat{\eta}$ and $\operatorname{supp}\mathcal{F}\widetilde{\eta} \subset]0,\infty[$, we may rewrite Qu in terms of convolutions on $\mathbb{R}^{n+1}$ as follows, using that $k_j = \psi_j * \varphi_j$ and with the understanding that for $j = 0$ the first factor is $\psi_0 \otimes \widetilde{\eta}$,

$$Qu = \sum_{j=0}^{\infty} \widetilde{\eta}_j * \eta(2^{ja_t}\cdot)(t) \cdot k_j * u(x) = \sum_{j=0}^{\infty} (\psi \otimes \widetilde{\eta})_j * \big(\varphi_j * u \otimes \eta(2^{ja_t}\cdot)\big).$$

Now we may invoke Lemma 6.33 as the function $\psi \otimes \widetilde{\eta}$ has all its moments equal to 0, because its Fourier transformed function is supported in a half-plane disjoint from the origin in $\mathbb{R}^{n+1}$. This gives an estimate of the Lizorkin–Triebel norm as follows,

$$\| Qu \,|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \leq c\| \{2^{(s+\frac{a_t}{p_t})j}(\varphi_j * u \otimes \eta(2^{ja_t}\cdot))^*_j\}_{j=0}^{\infty} \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1})\|.$$

Here the maximal function $(\cdot)^*_j$ considered in the lemma allow us to estimate the j^{th} term by

$$\sup_{y,y_t} |2^{sj}\varphi_j * u(y) 2^{j\frac{a_t}{p_t}}\eta(2^{ja_t}y_t)| \prod_{l=1}^{n+1}(1+2^{ja_l}|x_l-y_l|)^{-r_l} \leq v_j(x)2^{j\frac{a_t}{p_t}}f(2^{ja_t}t),$$

if we set

$$\begin{aligned} v_j &= \sup_{y} |2^{sj}\varphi_j * u(y)| \prod_{l=1}^{n}(1+2^{ja_l}|x_l-y_l|)^{-r_l}, \\ f(t) &= \sup_{y_t} |\eta(y_t)|(1+|t-y_t|)^{-r_t}. \end{aligned}$$

To invoke Proposition 6.40, we note that v_j, f are continuous (by an argument similar to e.g. [27, (6)–(7)]) and, moreover, $\sup |t^N f(t)| < \infty$ for $0 < N \leq r_t$. We therefore apply the proposition for $r = p_t$, $a = a_t$ and note that if we fix the above parameter r_t such that $r_t p_t > 1$, then $Np_t > 1$ is fulfilled at least for $N = r_t$. This gives

$$\begin{aligned} \| Qu \,|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| &\leq c\| \{v_j \otimes 2^{j\frac{a_t}{p_t}}f(2^{ja_t}\cdot)\}_{j=0}^{\infty} \,|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1})\| \\ &\leq c\Big(\sum_{j=0}^{\infty} \| v_j \,|L_{p'}(\mathbb{R}^n)\|^{p_t}\Big)^{1/p_t}. \end{aligned}$$

So by writing v_j in terms of the Peetre–Fefferman–Stein maximal function $\varphi^*_j u(x)$,

$$\| Qu \,|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \leq c\Big(\sum_{j=0}^{\infty} \|2^{sj}\varphi^*_j u \,|L_{p'}(\mathbb{R}^n)\|^{p_t}\Big)^{1/p_t} \leq c\| u \,|B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|.$$

The last inequality is essentially known from [44, (4)], but to account for effects of the flaws pointed out in Remark 4.1, let us briefly note the following: if we apply [44, (21)] to the very last formula in the proof of Theorem 4.18, then we get an estimate of the above sum by $\| 2^{sj}(\mathcal{F}^{-1}\Phi)^*_j u \,|\ell_{p_t}(L_{p'})\|$. This can be controlled by the $\ell_{p_t}(L_{p_0})$-norm of the convolutions $2^{sj}\mathcal{F}^{-1}\Phi_j * u$ (i.e. by the stated $\| u \,|B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|$) by following the argument for [44, (23)], after the remedy discussed in Section 6.4.3, say for simplicity with $r_0 := r_1 = \ldots = r_n$ and $r_0 p_0 > n$.$\square$

Remark 6.42. *By combining Proposition 6.38 and Theorem 6.41, one directly obtains*

$$Q : \overset{\circ}{B}{}^{s,a'}_{p',q}(\overline{\mathbb{R}}^n_+) \to \overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}) \quad \textit{for all } s \in \mathbb{R}.$$

The operator Q is now used to replace the particular right-inverse to r_0 in (6.61) by an operator Q_Ω that preserves support in $\overline{\Omega}$.

The construction uses the partition of unity $1 = \sum_\lambda \psi_\lambda + \psi$ on $\overline{\Omega}$ constructed in Section 6.4.4 as well as cut-off functions $\eta_\lambda \in C_0^\infty(\mathbb{R}^n)$, $\lambda \in \Lambda$, chosen such that $\operatorname{supp} \eta_\lambda \subset B$ and $\eta_\lambda = 1$ on $\operatorname{supp} \widetilde{\psi}_\lambda$. Moreover, $\eta_\Omega \in C^\infty_{L_\infty}(\mathbb{R}^n)$, cf. Lemma 6.5 for the definition of $C^\infty_{L_\infty}$, and $\operatorname{supp} \eta_\Omega \subset \Omega$ with $\eta_\Omega = 1$ on $\operatorname{supp} \psi$.

Theorem 6.43. *When $\vec{a}, \vec{p}$ satisfy* (6.54), *$0 < q \le \infty$ and $s \in \mathbb{R}$, then the operator Q_Ω defined by*

$$Q_\Omega u := \sum_\lambda e_{U_\lambda \times \mathbb{R}}\big((\eta_\lambda Q u_\lambda) \circ (\lambda \times \mathrm{id}_\mathbb{R})\big) + \eta_\Omega Q(\psi u), \quad u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n), \tag{6.69}$$

where $u_\lambda := e_B\big((\psi_\lambda u) \circ \lambda^{-1}\big)$, is bounded,

$$Q_\Omega : B^{s,a'}_{p',p_t}(\mathbb{R}^n) \to F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}),$$

and $r_0 Q_\Omega u = u$ whenever $u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n)$ fulfils $\operatorname{supp} u \subset \overline{\Omega}$.

Moreover, Q_Ω has range in $C(\mathbb{R}_t, \mathcal{D}'(\mathbb{R}^n_x))$ and preserves supports in $\overline{\Omega}$ in the strong form

$$\operatorname{supp} u \subset \overline{\Omega} \implies \forall t \in \mathbb{R} : \operatorname{supp} Q_\Omega u(\cdot, t) \subset \overline{\Omega}. \tag{6.70}$$

Proof. For the terms in the sum over λ in (6.69), we note that the multiplication result in [58, 4.2.2] together with the Besov version of Theorem 6.11, cf. Section 6.4.3, imply

$$u_\lambda = e_B\big((\psi_\lambda u) \circ \lambda^{-1}\big) \in B^{s,a'}_{p',p_t}(\mathbb{R}^n). \tag{6.71}$$

(These results apply to isotropic Besov spaces, so we use Lemma 6.3 to make rescalings $(s, a') \leftrightarrow s/a_0$, cf. (6.54).)

Theorem 6.41 and the paramultiplication result in Lemma 5.12 now gives that $\eta_\lambda Q u_\lambda \in F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$, hence according to Theorem 5.23,

$$(\eta_\lambda Q u_\lambda) \circ (\lambda \times \mathrm{id}_\mathbb{R}) \in \overline{F}^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(U_\lambda \times \mathbb{R}).$$

As $\operatorname{supp} \eta_\lambda \subset B$, Lemma 6.9 gives that extension of this composition by 0 belongs to $F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$.

For the last term in (6.69), it is an immediate consequence of [58, 4.2.2] that $\psi u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n)$, since $\psi \in C^\infty_{L_\infty}$ (as $\psi = 1 - \sum_\lambda \psi_\lambda$ on $\overline{\Omega}$ and $\partial\Omega$ is compact).

This shows that $Q_\Omega u \in F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$ and by applying the quasi-norm estimates in the theorems and lemmas referred to above, we obtain

$$\| Q_\Omega u \,|F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le c \| u \,| B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|.$$

Furthermore, it follows from Proposition 6.38 that $Q_\Omega u \in C(\mathbb{R}_t, \mathcal{D}'(\mathbb{R}^n_x))$ and therefore the effect of r_0 on $Q_\Omega u$ is simply restriction to $t = 0$, cf. (6.56). Hence for $u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n)$,

$$r_0 Q_\Omega u = \sum_\lambda e_{U_\lambda}\big((\eta_\lambda Q u_\lambda)(\lambda(\cdot), 0)\big) + \eta_\Omega Q(\psi u)(\cdot, 0).$$

Since Q according to Proposition 6.38 is a right-inverse of r_0, this sum equals the following, by using (6.71) as well as the properties of $\eta_\lambda, \eta_\Omega$, and in the final step that $\operatorname{supp} u \subset \overline{\Omega}$,

$$\sum_\lambda e_{U_\lambda}\big((\eta_\lambda u_\lambda) \circ \lambda\big) + \psi u = \sum_\lambda e_{U_\lambda}\big(\eta_\lambda \circ \lambda \cdot \psi_\lambda u\big) + \psi u = \sum_\lambda \psi_\lambda u + \psi u = u.$$

Finally, the support preserving property in (6.70) follows from (6.68). Indeed, when $\operatorname{supp} u \subset \overline{\Omega}$, then the support of each u_λ is contained in $\overline{\mathbb{R}}^n_+$ and therefore $\operatorname{supp}(\eta_\lambda Q u_\lambda)(\lambda(\cdot), t) \subset \overline{\Omega}$ for all $t \in \mathbb{R}$, which immediately gives that $\operatorname{supp} Q_\Omega u(\cdot, t) \subset \overline{\Omega}$. $\square$

6.6.3 The Trace at the Curved Boundary

We now address the trace γ of distributions in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, where for simplicity $I =]0, T[$, $T < \infty$, and Ω is smooth as in Definition 6.25 with compact boundary Γ.

Preliminaries

The trace is first worked out locally and then it is observed that the local pieces define a global trace. In this process we use that the trace $\gamma_{0,1}$ is a bounded surjection, cf. [29, Thm. 2.2],

$$\begin{aligned} &\gamma_{0,1} : F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}) \to F^{s-\frac{a_0}{p_0},a''}_{p'',p_0}(\mathbb{R}^n) \\ &\quad \text{for } \ s > \frac{a_0}{p_0} + (n-1)\Big(\frac{a_0}{\min(1,p_0,q)} - a_0\Big) + \Big(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Big). \end{aligned} \tag{6.72}$$

This is also valid for $\gamma_{0,n}$ in view of (6.54) and we prefer to work with this, for locally the boundary Γ is defined by the equation $x_n = 0$, as usual. For the s in (6.72), we have by [29, Thm. 2.1], since $r_k := \max(1, p_k)$,

$$F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}, L_{r''}(\mathbb{R}^n)) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{6.73}$$

So when $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ for such s, an extension f in the corresponding space on $\mathbb{R}^n$ is a function and for this we right away get

$$f \circ (\lambda^{-1} \times \mathrm{id}_{\mathbb{R}}) \in L_{1,\mathrm{loc}}(B \times \mathbb{R}). \tag{6.74}$$

Moreover, if we work locally with cut-off functions $\psi \in C_0^\infty(U_\lambda)$, $\varphi \in C_0^\infty(\mathbb{R})$, then Lemma 6.5 yields $\psi\otimes\varphi f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$. Changing coordinates, Theorem 6.13 implies that $(\psi \otimes \varphi f) \circ (\lambda^{-1} \times \mathrm{id}_{\mathbb{R}})$ is in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(B \times \mathbb{R})$, hence it extends by 0 to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$. By (6.72),

$$\gamma_{0,n}\big((\psi \otimes \varphi f) \circ (\lambda^{-1} \times \mathrm{id}_{\mathbb{R}})\big) \in F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n).$$

Strictly speaking, we should have inserted the extension by 0, namely $e_{B\times\mathbb{R}}$, before applying $\gamma_{0,n}$, but we have chosen not to burden notation with this. Now restriction to $B' \times \mathbb{R}$ gives an element in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B' \times \mathbb{R})$, and since it is easily seen using (6.73) that restriction to $\{x_n = 0\}$ and $e_{B\times\mathbb{R}}$ can be interchanged, we obtain

$$(\psi \otimes \varphi f) \circ (\lambda^{-1}(\cdot, 0) \times \mathrm{id}_{\mathbb{R}}) \in \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B' \times \mathbb{R}). \tag{6.75}$$

Furthermore, to describe the range of γ, we introduce for an open interval $I' \supset I$ the restriction (with notation as in Section 6.4.4)

$$r_I : F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I') \to F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I),$$

which for any $v \in F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times I')$ is defined as the distribution arising from the family $\{r_{B'\times I} v_{\kappa\times\mathrm{id}_{I'}}\}_{\kappa\in\mathcal{F}_0}$ of distributions on $B' \times I$, cf. the paragraph on restriction just below Lemma 6.16.

Using r_I, we also introduce a space of restricted distributions (in the time variable only),

$$\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times I) := r_I F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times \mathbb{R}) = r_I \mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J) \tag{6.76}$$

valid for any compact interval $J \supset I$. Since $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J)$ is a quasi-Banach space, cf. Theorem 6.28, the space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times I)$ is so too when equipped with

$$\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times I)\| := \inf_{r_I v=u} \| v \,|\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma \times J)\|. \tag{6.77}$$

The Definition

To give sense to γu in $\mathcal{D}'(\Gamma \times I)$, it is first observed that (6.74) induces invariantly defined functions. Indeed, in view of the identity $\kappa^{-1}(\cdot) = \lambda^{-1}(\cdot, 0)$, we set

$$f_\kappa = f \circ (\lambda^{-1}(\cdot, 0) \times \mathrm{id}_{\mathbb{R}}) \in L_{1,\mathrm{loc}}(B' \times \mathbb{R})$$

and as distributions they transform as in (6.12), since

$$f_\kappa \circ (\kappa \circ \kappa_1^{-1} \times \mathrm{id}_{\mathbb{R}}) = f_{\kappa_1} \quad \text{on} \quad \kappa_1(\Gamma_{\kappa_1} \cap \Gamma_\kappa) \times \mathbb{R}. \tag{6.78}$$

Hence by Lemma 6.16 there exists a unique $v \in \mathcal{D}'(\Gamma \times \mathbb{R})$ with

$$v_{\kappa\times\mathrm{id}_{\mathbb{R}}} = f_\kappa. \tag{6.79}$$

That v is in $F^{s-a_0/p_0,a''}_{p'',p_0;\mathrm{loc}}(\Gamma\times\mathbb{R})$ is a special case of (6.75), cf. Definition 6.26.

Note that the distribution v does not depend on the atlas $\mathcal{F}_0$, for when another atlas $\mathcal{F}_1$ in the same way induces a distribution v_1, then formula (6.78) read with κ running through $\mathcal{F}_0$ and κ_1 running through $\mathcal{F}_1$ implies that both v and v_1 result by "restriction" from the distribution w induced by $\mathcal{F}_0\cup\mathcal{F}_1$.

Now we define the trace γu in $\mathcal{D}'(\Gamma\times I)$ by

$$\gamma u = r_I v. \tag{6.80}$$

Indeed, to verify that γu is independent of the chosen f, it suffices to derive that for any two extensions $f_1,f_2\in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$, the following identity holds for each $\lambda\in\Lambda$ and $(x',t)\in B'\times I$:

$$f_1\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t) = f_2\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t). \tag{6.81}$$

To do so, we choose $\psi\in C_0^\infty(U_\lambda)$, $\varphi\in C_0^\infty(\mathbb{R})$ such that $\psi(\lambda^{-1}(x',0))\neq 0$ and $\varphi(t)\neq 0$. Since f_1,f_2 coincide in $\Omega\times I$, the functions

$$e_{B\times\mathbb{R}}\big((\psi\otimes\varphi f_j)\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}})\big)(x,t),\quad j=1,2,$$

are identical for $(x,t)\in B\times I$ with $x_n>0$. Letting $x_n\to 0^+$ therefore gives the same limits in $L_{r''}(\mathbb{R}^{n-1}\times I)$, cf. (6.73), in particular they coincide in $L_{r''}(B'\times I)$. As $(\psi\otimes\varphi)\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t)\neq 0$, this yields (6.81).

Furthermore, (6.81) can be used to show that γ does not depend on the Lizorkin–Triebel space satisfying (6.72). For when u belongs to two different Lizorkin–Triebel spaces, we can take f_1 above to be an extension in one of the spaces and f_2 to be an extension in the other. The identity in (6.81) then gives that γu belongs to the intersection of the corresponding Lizorkin–Triebel spaces over the curved boundary.

We also note that the trace γ has the natural property that $r_I\circ\gamma=\gamma\circ r_I$ on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I')$ for any open interval $I'\supset I$.

Finally, γ applied to any $u\in r_{\Omega\times I}C(\mathbb{R}^{n+1})$ gives the expected, namely $r_{\Gamma\times I}\widetilde{u}$ for any extension $\widetilde{u}\in C(\mathbb{R}^{n+1})$ of u. Indeed using (6.80),

$$\begin{aligned}(\gamma u)_{\kappa\times\mathrm{id}_I} &= r_{B'\times I}\big(\widetilde{u}\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})\big)\\ &= (r_{\Gamma\times I}\widetilde{u})\circ(\kappa^{-1}\times\mathrm{id}_I) = (r_{\Gamma\times I}\widetilde{u})_{\kappa\times\mathrm{id}_I},\end{aligned}$$

which shows that γu equals a restriction, $r_{\Gamma\times I}\widetilde{u}$, of the continuous function $\widetilde{u}$.

The Theorem

To construct a right-inverse of γ, we use a bounded right-inverse K_n of $\gamma_{0,n}$, where because of (6.54) we may refer to [29, Thm. 2.6] for a right-inverse of the similar trace $\gamma_{0,1}$ in (6.72),

$$K_n : F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n)\to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}),\quad s\in\mathbb{R}, \tag{6.82}$$

given by, cf. just above (6.60) for the ψ,

$$K_n v(x) := \sum_{j=0}^{\infty} \psi(2^{ja_n} x_n) \mathcal{F}^{-1}\big(\Phi_j(\xi', 0, \xi_{n+1}) \mathcal{F} v(\xi', \xi_{n+1})\big)(x', x_{n+1}).$$

Hereby we have set $p'' = (p_0, \dots, p_0, p_t) \in]0, \infty[^{\,n}$, which results when $p_n = p_0$ is left out; cf. (6.54).

Theorem 6.44. *When Γ is compact, $\vec{a}, \vec{p}$ satisfy* (6.54) *and (s,q) fulfils the inequality in* (6.72)*, then*

$$\gamma : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{F}^{s-a_0/p_0, a''}_{p'',p_0}(\Gamma \times I)$$

is a bounded surjection, *which has a right-inverse K_γ. More precisely, the operator K_γ can be chosen such that $K_\gamma : \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ is bounded for every $s \in \mathbb{R}$.*

Proof. Since the space $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I)$, cf. (6.76), does not depend on how the compact interval $J \supset I$ is chosen, it is fixed in the following. Moreover, γu does not depend on the extension f of u, thus we take f such that $\operatorname{supp} f \subset \mathbb{R}^n \times J$. By (6.79) and (6.76), $\gamma u = r_I v$ is in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I)$.

To prove boundedness, note that v according to (6.14) belongs to $\mathring{F}^{s,a''}_{p'',q}(\Gamma \times J)$, since

$$\operatorname{supp} v \subset \bigcup_{\lambda \in \Lambda} \big(\lambda^{-1}(\cdot, 0) \times \mathrm{id}_{\mathbb{R}}\big)(B' \times J) = \Gamma \times J. \tag{6.83}$$

Hence it can be inferred from Theorem 6.28 that

$$\begin{aligned} &\| \gamma u \,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I)\|^d \\ &\quad\le \inf_{\substack{r_{\Omega\times I} f = u \\ \operatorname{supp} f \subset \mathbb{R}^n \times J}} \sum_{\lambda \in \Lambda} \big\| (\psi_\lambda \otimes \mathbb{1}_{\mathbb{R}} f) \circ (\lambda^{-1}(\cdot, 0) \times \mathrm{id}_{\mathbb{R}}) \,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B' \times \mathbb{R})\big\|^d. \end{aligned} \tag{6.84}$$

By choosing first a cut-off function on $\mathbb{R}$, we can use the infimum norm to fix f such that $\| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le 2\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)\|$. Using the arguments leading up to (6.75) and the boundedness of $\gamma_{0,n}$, cf. (6.72), each summand in (6.84) can be estimated by

$$c\| (\psi_\lambda \otimes \mathbb{1}_{\mathbb{R}} f) \circ (\lambda^{-1} \times \mathrm{id}_{\mathbb{R}}) \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(B \times \mathbb{R})\|^d.$$

Finally, applying Theorem 6.13 and Lemma 6.5, since $\psi_\lambda \otimes \mathbb{1}_{\mathbb{R}} \in C^\infty_{L_\infty}(\mathbb{R}^{n+1})$, we obtain

$$\| \gamma u \,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I)\| \le c\| f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le 2c\| u \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)\|.$$

The construction of a right-inverse K_γ uses that for any $w \in \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I)$ there exists $v \in \mathring{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times J)$ such that $r_I v = w$. It is easily verified that

$$w^\kappa := r_{\mathbb{R}^{n-1}\times I}\big(e_{B'\times\mathbb{R}}(\widetilde{\psi}_\kappa \otimes \mathbb{1}_{\mathbb{R}} v_{\kappa \times \mathrm{id}_{\mathbb{R}}})\big) \tag{6.85}$$

is independent of the extension v; and clearly w^κ is in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^{n-1}\times I)$ with support in $B'\times I$. For $\chi_1,\chi_2\in C_0^\infty(\mathbb{R})$ such that $\chi_1+\chi_2\equiv 1$ on a neighbourhood of I and such that χ_1, χ_2 vanish before the right, respective the left end point of I, we let, cf. Theorem 6.34 and Corollary 6.35,

$$w^\kappa_{\text{ext}} = \mathcal{E}_u(\chi_1 w^\kappa) + \mathcal{E}_{u,T}(\chi_2 w^\kappa),$$

where extension by 0 to $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}\times]-\infty, T[$ before application of $\mathcal{E}_u$, respectively $\mathcal{E}_{u,T}$ is understood. Lemma 6.10 gives that this extension does not change the regularity of the distributions, hence w^κ_{ext} belongs to $F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n)$; and moreover $r_{\mathbb{R}^{n-1}\times I} w^\kappa_{\text{ext}} = w^\kappa$.

Now using K_n in (6.82) and functions $\eta_\lambda\in C_0^\infty(\mathbb{R}^n)$, $\lambda\in\Lambda$, such that $\operatorname{supp}\eta_\lambda\subset B$ and $\eta_\lambda = 1$ on $\operatorname{supp}\widetilde{\psi}_\lambda$, we define (using the v-independence of w^κ_{ext})

$$K_\gamma w = r_{\Omega\times I}\sum_{\lambda\in\Lambda} e_{U_\lambda\times\mathbb{R}}(\eta_\lambda K_n w^\kappa_{\text{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}}).$$

Boundedness of K_γ is a consequence of first using Lemma 6.9 together with Theorem 6.13, $d := \min(1,p_0,p_t,q)$,

$$\begin{aligned}\| K_\gamma w\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|^d &\le \sum_{\lambda\in\Lambda}\|(\eta_\lambda K_n w^\kappa_{\text{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U_\lambda\times\mathbb{R})\|^d\\ &\le c\sum_{\lambda\in\Lambda}\|\eta_\lambda K_n w^\kappa_{\text{ext}}\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^d,\end{aligned}$$

and then Lemmas 6.5 and 6.10 as well as the mapping properties of K_n, $\mathcal{E}_u$, $\mathcal{E}_{u,T}$,

$$\begin{aligned}\| K_\gamma w\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|^d &\le c\sum_{\substack{\kappa\in\mathcal{F}_0\\ j=1,2}}\|\chi_j w^\kappa\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^{n-1}\times I)\|^d\\ &\le c\sum_{\kappa\in\mathcal{F}_0}\|(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}}v)\circ(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B'\times\mathbb{R})\|^d.\end{aligned}$$

The extension v is chosen arbitrarily among those in $\overset{\circ}{F}{}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times J)$ satisfying $r_I v = w$, thus taking the infimum over all such v yields the boundedness of K_γ, cf. (6.77) and (6.28).

To verify that K_γ is indeed a right-inverse, we use that an extension of $K_\gamma w$ is

$$f = \sum_{\lambda\in\Lambda} e_{U_\lambda\times\mathbb{R}}(\eta_\lambda K_n w^\kappa_{\text{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}}).$$

Hence the definition of γ, cf. (6.80), gives that $\gamma(K_\gamma w) = r_I h$, where $h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}} = f\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})$. We shall now prove that $r_{B'\times I}h_{\kappa_1\times\mathrm{id}_R} = w_{\kappa_1\times\mathrm{id}_I}$ for each $\kappa_1\in\mathcal{F}_0$. Indeed,

$$r_{B'\times I}h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}} = r_{B'\times I}\sum_{\lambda\in\Lambda}(\eta_\lambda K_n w^\kappa_{\text{ext}})\circ(\lambda\circ\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}}), \tag{6.86}$$

where extension by 0 from $\kappa_1(\Gamma_{\kappa_1}\cap\Gamma_\kappa)\times\mathbb{R}$ to $B'\times\mathbb{R}$ in each term is understood. Using that K_n is a right-inverse of $\gamma_{0,n}$ and that $w^\kappa_{\mathrm{ext}}=w^\kappa$ on $\kappa(\Gamma_{\kappa_1}\cap\Gamma_\kappa)\times I$, each summand in (6.86) equals, cf. also (6.85), (6.12),

$$(\eta_\lambda w^\kappa)\circ(\lambda\circ\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})=(\eta_\lambda\circ\lambda\cdot\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}.$$

As $\eta_\lambda\circ\lambda\equiv 1$ on $\operatorname{supp}\psi_\kappa$ and $\sum\psi_\kappa\equiv 1$ on Γ, we finally obtain, using that $r_I v=w$,

$$r_{B'\times I}h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}=r_{B'\times I}\Big(v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}\sum_{\lambda\in\Lambda}(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})\Big)=w_{\kappa_1\times\mathrm{id}_I},$$

hence K_γ is a right-inverse of γ. □

6.6.4 The Traces at the Corner

The trace from either the flat or the curved boundary to the corner $\Gamma\times\{0\}\simeq\Gamma$ cannot simply be obtained by applying r_0 and then γ, or vice versa, since these operators are defined on spaces over the whole cylinder.

In the following, under the assumptions that $I=\,]0,T[$ is finite and Γ compact, the trace operators $r_{0,\Gamma}$, γ_Γ will therefore be introduced (the subscript Γ indicates that we end up at the manifold $\Gamma\times\{0\}\simeq\Gamma$). We note that focus will not be on optimality regarding the co-domains, since the purpose of this section merely is to prepare for a discussion of compatibility conditions in connection with PDEs; and from this point of view the interesting question is whether the following identity holds in $\mathcal{D}'(\Gamma)$,

$$r_{0,\Gamma}\circ\gamma u=\gamma_\Gamma\circ r_0 u. \tag{6.87}$$

Recall that when working with spaces on the boundary, the anisotropy and the vector of integral exponents only have n entries. Since it is different entries that need to be left out, depending on whether we are studying $\Gamma\times I$ or Ω, it will in the following be convenient to use $a''=(a_1,\dots,a_{n-1},a_t)$ as well as $a'=(a_1,\dots,a_n)$; and likewise for p', p''. Moreover, (6.54) is a standing assumption on $\vec{a}$, $\vec{p}$.

We assume that s satisfies the inequality in (6.57) adapted to vectors of n entries, i.e. for the trace from the curved boundary $\Gamma\times I$,

$$s>\frac{a_t}{p_t}+(n-1)\Big(\frac{a_0}{\min(1,p_0)}-a_0\Big), \tag{6.88}$$

and for the trace from the flat boundary Ω,

$$s>\frac{a_0}{p_0}+(n-1)\Big(\frac{a_0}{\min(1,p_0)}-a_0\Big). \tag{6.89}$$

Remark 6.45. *When $v\in\overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$ for a compact interval J and s fulfils* (6.88)*, then for each $\kappa\in\mathcal{F}_0$,*

$$v_{\kappa\times\mathrm{id}_{\mathbb{R}}}\in C_{\mathrm{b}}(\mathbb{R}_t,L_{1,\mathrm{loc}}(B')).$$

This follows if for every compact set $K \subset B'$, the map $t \mapsto v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\cdot,t)$ is continuous with values in $L_1(K)$. In Theorem 6.28 we may, if necessary, change the partition of unity (using some $\varphi \in C_0^\infty(B')$ equalling 1 on K) such that $\psi_\kappa \equiv 1$ on $\kappa^{-1}(K)$. Then $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is in $\overline{F}^{s,a''}_{p'',q}(B'\times\mathbb{R})$, which because of (6.57) *in view of* (6.88) *is contained in $C_{\mathrm{b}}(\mathbb{R}_t, L_1(B'))$. Hence $v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is in $L_1(K)$, continuously in time.*

The Curved Boundary

For $w \in \overline{F}^{s,a''}_{p'',q}(\Gamma\times I)$ there exists a $v \in \overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$, where $J \supset I$ is any compact interval, such that $r_I v = w$, cf. (6.76). By exploiting that $v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is continuous with respect to t, cf. Remark 6.45, we define for $x\in\Gamma$,

$$r_{0,\Gamma}w(x) = \sum_{\kappa\in\mathcal{F}_0} \psi_\kappa(x) v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\kappa(x),0) \tag{6.90}$$

with the understanding that the product $\psi_\kappa(x)v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\kappa(x),0)$ is defined to be 0 outside Γ_κ. On Γ_κ the product is meaningful, since $v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is in $C_{\mathrm{b}}(\mathbb{R}, L_{1,\mathrm{loc}}(B'))$.

The trace $r_{0,\Gamma}$ in (6.90) is independent of the chosen $v \in \overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$, since for any two extensions v_1, v_2 in this space, $\widetilde{\psi}_\kappa \cdot r_{B'\times I} v_{j,\kappa\times\mathrm{id}_{\mathbb{R}}}$, $j=1,2$, coincide on $B'\times I$, hence by continuity also on $B'\times\{0\}$.

Moreover, the trace depends neither on the atlas nor on the subordinate partition of unity. Indeed, considering another atlas $\mathcal{F}_1$ with a subordinate partition of unity $1 = \sum_{\kappa_1\in\mathcal{F}_1}\varphi_{\kappa_1}$, we have on Γ, cf. (6.12) for the atlas $\mathcal{F}_0\cup\mathcal{F}_1$,

$$\sum_\kappa \psi_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\kappa(\cdot),0) = \sum_{\kappa,\kappa_1}\psi_\kappa\varphi_{\kappa_1} v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\kappa_1(\cdot),0) = \sum_{\kappa_1}\varphi_{\kappa_1} v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\kappa_1(\cdot),0).$$

In the following theorem the co-domain of the trace is $B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)$; the definition and properties of this space follow from Section 6.4.3, since it coincides with an isotropic space in view of (6.54) and Lemma 6.3. Note that we have abbreviated the $(n-1)$-vector $(a_0,\dots,a_0)$ to a_0, and similarly for p_0.

Theorem 6.46. *When a'', p'' are as above with $0<p''<\infty$ and s satisfies* (6.88)*, then $r_{0,\Gamma}$ is a bounded operator,*

$$r_{0,\Gamma}\colon \overline{F}^{s,a''}_{p'',q}(\Gamma\times I) \to B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma).$$

Proof. From Remark 6.45 we have that $v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is in $C_{\mathrm{b}}(\mathbb{R}, L_{1,\mathrm{loc}}(B'))$, hence using the bounded trace operator, cf. [29, Thm. 2.5] and (6.88),

$$\gamma_{0,n}\colon F^{s,a''}_{p'',q}(\mathbb{R}^n) \to B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\mathbb{R}^{n-1}), \tag{6.91}$$

it is readily seen that

$$\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\cdot,0) = r_{B'}\gamma_{0,n}e_{B'\times\mathbb{R}}(\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}).$$

Since $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}} \in F^{s,a''}_{p'',q}(B'\times\mathbb{R})$, we therefore have by (6.91) that $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\cdot,0)$ belongs to $\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(B')$. Now Corollary 6.22 adapted to Besov spaces, cf. Section 6.4.3, implies that $r_{0,\Gamma}w \in B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)$.

To prove that $r_{0,\Gamma}$ is bounded, we use (6.23) to estimate $\| r_{0,\Gamma}w \,|B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)\|^d$, where $d := \min(1,p_0,p_t)$, by

$$\sum_{\kappa,\kappa_1\in\mathcal{F}_0} \Big\| \psi_{\kappa_1}\circ\kappa^{-1}\cdot\widetilde{\psi}_\kappa v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\kappa_1\circ\kappa^{-1}(\cdot),0)\,\Big|\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\kappa(\Gamma_\kappa\cap\Gamma_{\kappa_1}))\Big\|^d.$$

After a change of coordinates $x\mapsto\kappa\circ\kappa_1^{-1}(x)$ and a slight restriction of the domain to a suitable open subset W such that $\overline{W}\subset\kappa_1(\Gamma_{\kappa_1}\cap\Gamma_\kappa)$, and finally multiplication by a $\chi_{\kappa_1}\in C_0^\infty(B')$ where $\chi_{\kappa_1}\equiv 1$ on $\operatorname{supp}\widetilde{\psi}_{\kappa_1}$, this can be estimated by, cf. [58, 4.2.2] for an s_1 large enough,

$$c\sum_{\kappa,\kappa_1\in\mathcal{F}_0}\Big(\sum_{|\alpha|\le s_1}\| e_{B'}(\psi_\kappa\circ\kappa_1^{-1}\chi_{\kappa_1})\,|L_\infty\|\Big)^d\,\big\| e_{B'}(\widetilde{\psi}_{\kappa_1}v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\cdot,0))\,\big\|^d.$$

Here $\|\cdot\|$ is the quasi-norm on $B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\mathbb{R}^{n-1})$ and the constant c contains $\sup_{\overline{W}}|\det J(\kappa\circ\kappa_1^{-1})|^d$ as a finite factor (J denotes the Jacobian matrix). Now boundedness of $\gamma_{0,n}$ in (6.91) gives

$$\Big\| r_{0,\Gamma}w\,\Big|\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)\Big\|^d \le c\sum_{\kappa_1\in\mathcal{F}_0}\|\,\widetilde{\psi}_{\kappa_1}v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}\,|\overline{F}^{s,a''}_{p'',q}(B'\times\mathbb{R})\|^d,$$

hence taking the infimum over all admissible v (as we may since $r_{0,\Gamma}$ is independent of the extension) proves that $r_{0,\Gamma}$ is bounded. □

The Flat Boundary

In this section we consider the trace operator γ_Γ, which simply is the trace at Γ of distributions defined on Ω. In view of (6.87) and Theorem 6.36, the domain of interest for γ_Γ is the unmixed Besov space $\overline{B}^{s,a'}_{p',q}(\Omega)$, which according to Lemma 6.3 even equals an isotropic space, cf. (6.54).

The operator is defined by carrying over the definition and results for γ in Section 6.6.3. Indeed, we remove the time dependence and use the Besov space result in [11, Thm. 1] for $\gamma_{0,n}$. An embedding similar to (6.73) also holds in the case of Besov spaces, cf. [11, Prop. 1] and (6.89), and Theorems 6.28 and 6.13 are replaced by the Besov versions, cf. Section 6.4.3, of Theorems 6.24 and 6.11 respectively. Recalling that the $(n-1)$-vector $(a_0,\dots,a_0)$ is abbreviated a_0, and likewise for p_0, this yields

Theorem 6.47. *When $a'=(a_0,\dots,a_0)\in[1,\infty[^n$, $p'=(p_0,\dots,p_0)\in]0,\infty[^n$ and s satisfies (6.89), then γ_Γ is a bounded operator,*

$$\gamma_\Gamma : \overline{B}^{s,a'}_{p',q}(\Omega)\to B^{s-a_0/p_0,a_0}_{p_0,q}(\Gamma).$$

We note that, as usual for Besov spaces, the sum exponent is not changed and, moreover, a formula similar to the one in (6.90) for $r_{0,\Gamma}$ holds for γ_Γ. I.e. for any extension f of $w \in \overline{B}^{s,a'}_{p',q}(\Omega)$, with (6.79)–(6.80) adapted to γ_Γ for the f_κ, we have when extension by 0 outside Γ_κ is suppressed,

$$\gamma_\Gamma w = \sum_{\kappa\in\mathcal{F}_0} \psi_\kappa \cdot f_\kappa \circ \kappa. \tag{6.92}$$

Indeed, $\bigl(\sum_{\kappa\in\mathcal{F}_0} \psi_\kappa \cdot f_\kappa \circ \kappa\bigr)_{\kappa_1} = \sum_{\kappa\in\mathcal{F}_0} \psi_\kappa \circ \kappa_1^{-1} \cdot f_{\kappa_1} = f_{\kappa_1} = (\gamma_\Gamma w)_{\kappa_1}$ for each $\kappa_1 \in \mathcal{F}_0$. This formula is convenient in a discussion of compatibility conditions, cf. the next section.

6.6.5 Applications

Without proof, we now indicate, by merely adapting [18, Ch. 6] to the present set-up, what the above considerations yield in a study of e.g. the heat equation. That is, for $\Delta = \partial^2_{x_1} + \ldots + \partial^2_{x_n}$ we consider

$$\partial_t u - \Delta u = g \quad \text{in} \quad \Omega\times I, \tag{6.93}$$

$$\gamma u = \varphi \quad \text{on} \quad \Gamma\times I, \tag{6.94}$$

$$r_0 u = u_0 \quad \text{on} \quad \Omega\times\{0\}. \tag{6.95}$$

Under the assumption that $\vec{a} = (1,\ldots,1,2)$ and $\vec{p} = (p_0,\ldots,p_0,p_t) < \infty$, we give in the theorem below necessary conditions for the existence of a solution u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, when γ and r_0 in (6.94), (6.95) make sense, i.e. when s fulfils the two conditions

$$\begin{aligned} s &> \frac{1}{p_0} + (n-1)\Bigl(\frac{a_0}{\min(1,p_0,q)} - a_0\Bigr) + \Bigl(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Bigr) \quad \text{and} \\ s &> \frac{2}{p_t} + n\Bigl(\frac{1}{\min(1,p_0)} - 1\Bigr). \end{aligned} \tag{6.96}$$

Theorem 6.48. *Let $\vec{a}$, $\vec{p}$ and s fulfil the requirements above. When the boundary value problem in* (6.93)–(6.95) *has a solution $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, then the data (g,φ,u_0) necessarily satisfy*

$$g \in \overline{F}^{s-2,\vec{a}}_{\vec{p},q}(\Omega\times I), \quad \varphi \in \overline{F}^{s-\frac{1}{p_0},a''}_{p'',p_0}(\Gamma\times I), \quad u_0 \in \overline{B}^{s-\frac{2}{p_t}}_{p_0,p_t}(\Omega).$$

Moreover, for all $l \in \mathbb{N}_0$ fulfilling both

$$2l < s - \frac{1}{p_0} - \frac{2}{p_t} - (n-1)\Bigl(\frac{1}{\min(1,p_0)} - 1\Bigr) \tag{6.97}$$

and

$$2l < s - \frac{1}{p_0} - (n-1)\Bigl(\frac{a_0}{\min(1,p_0,q)} - a_0\Bigr) - \Bigl(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Bigr), \tag{6.98}$$

the data are compatible in the sense that

$$r_{0,\Gamma}\partial_t^l\varphi = \gamma_\Gamma\Big(\Delta^l u_0 + \sum_{j=0}^{l-1}\Delta^j r_0(\partial_t^{l-1-j} g)\Big), \tag{6.99}$$

which reduces to $r_{0,\Gamma}\varphi = \gamma_\Gamma u_0$ *for* $l = 0$ *(the sum is void).*

We note that the corrections containing the minima in (6.96) and (6.97)–(6.98) amount to 0 in the classical case in which $\vec{p}, q \geq 1$.

Remark 6.49. *In the construction of solutions to e.g.* (6.93)–(6.95)*, it is well known from [18, Thm. 6.3] that the problem for* $p_0 = 2 = p_t$ *is solvable, when the data* (g, φ, u_0) *are subjected to the compatibility* conditions *in* (6.99)*. For general* p_0, p_t *a first step could be to reduce to the case in which* $\varphi \equiv 0$, $u_0 \equiv 0$*. This can be achieved by combining the surjectivity of* γ *in Theorem 6.44 with the support preserving right-inverse* Q_Ω *(of* r_0*) analysed in Theorem 6.43.*

CHAPTER 7

Applications to the Heat Equation

In this chapter we apply the trace results from Chapter 6 to the boundary value problem,

$$\partial_t u - \Delta u = g \quad \text{in} \quad \Omega \times I, \tag{7.1}$$

$$\gamma u = \varphi \quad \text{on} \quad \Gamma \times I, \tag{7.2}$$

$$r_0 u = u_0 \ \text{ on } \ \Omega \times \{0\}, \tag{7.3}$$

where Ω is C^∞ with compact boundary Γ and $I :=]0, T[\,,\ T < \infty$.

More precisely, we deduce necessary conditions for the existence of a solution u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, where

$$\vec{a} = (1, \ldots, 1, 2), \qquad \vec{p} = (p_0, \ldots, p_0, p_t) < \infty, \tag{7.4}$$

and conclude with a few observations regarding sufficient conditions.

Grubb and Solonnikov [18] studied, as mentioned in Section 3.1, necessary and sufficient conditions for the solvability of parabolic pseudo-differential boundary value problems in the framework of Sobolev spaces with different smoothness in space and time, but with L_2-integrability in all directions. Later Grubb [15] generalised this to L_p-integrability for $1 < p < \infty$.

Both articles include a thorough discussion of compatibility conditions and treat the question of higher regularity of solutions. It will, however, be too far-reaching to review the results in [15, 18] in details due to the heavy machinery involved.

As briefly mentioned in Section 1.1, Weidemaier was one of the first to study inhomogeneous, parabolic boundary value problems in the set-up of different integrability properties in space and time.

For second order parabolic equations of the form,

$$\partial_t u - \sum_{i,j=1}^{n} a_{ij} \partial_{x_i} \partial_{x_j} u = g \text{ in } \Omega \times I, \tag{7.5}$$

where $(a_{ij})_{i,j}$ is positive definite, he proved the following result:

Theorem 7.1 ([64, Thm. 1.3]). *Let $\Omega \subset \mathbb{R}^n$ be bounded and $C^{2+\varepsilon}$ for some $\varepsilon > 0$. Assume that*

(i) $1 < p \le q < \infty, \quad \frac{1}{p} + \frac{2}{q} \neq 2,$

(ii) $g \in L_q(0,T; L_p(\Omega))$,

(iii) $\varphi \in L_q(0,T; W_p^{2-1/p}(\Gamma)) \cap F_{q,p}^{(2-1/p)/2}(0,T; L_p(\Gamma))$,

(iv) $u_0 \in B_{p,q}^{2(1-1/q)}(\Omega)$,

(v) $u_0(\cdot) = \varphi(\cdot, 0)$ *on* Γ *when* $\frac{1}{p} + \frac{2}{q} < 2$.

Then there exists a unique solution $u \in W_{p,q}^{2,1}(\Omega_T)$ to (7.5) *with boundary conditions* (7.2)–(7.3). *Moreover,*

$$\| u \,| W_{p,q}^{2,1}(\Omega \times I) \| \le c(p,q,T) \big(\| u_0 \| + \| g \| + T^{-\frac{1}{2p}} \| \varphi \| \big),$$

where the norms are those from the spaces in (ii)–(iv).

We refer to (3.1)–(3.2) and (3.4) for the function spaces used in the theorem. Moreover, $C^{2+\varepsilon}$-domains are recalled in Definition 3.5, and the norm in (iii) can be found in Theorem 3.1. In (iv) notation is chosen without bars as in [64], because Weidemaier seemingly did not recall the definition of these spaces.

Denk, Hieber and Prüss [9] also worked on parabolic boundary problems with inhomogeneous data in the mixed-norm framework. However, since their set-up has many features, only the essence of their main results will be given here (using our notation). Indeed, they studied equations of the type

$$\partial_t u + \mathcal{A}(t,x,D) u = g(t,x),$$

where $\mathcal{A}(t,x,D)$ is a differential operator of order $2m$ with coefficients taking values in the space of bounded operators on a Banach space E. The equation is considered together with similarly general boundary conditions, where the coefficients also take values in this space.

Under the assumption that Ω is a connected C^{2m}-domain with compact boundary, they treated necessary and sufficient conditions for the existence and uniqueness of a solution in $W_q^1(I; L_p(\Omega; E)) \cap L_q(I; W_p^{2m}(\Omega; E))$. Among the conditions are that $u_0 \in \overline{B}_{p,q}^{2m(1-1/p)}(\Omega; E)$ and $g \in L_q(I; L_p(\Omega; E))$ as well as compatibility conditions on these data.

7.1 Necessary Compatibility Properties

This section is based on [18, Ch. 6]; we merely concretise and adapt the general arguments there to the case of integral exponent $\vec{p}$ instead of 2.

In (7.2)–(7.3) the traces γ, r_0 are according to Theorems 6.44 and 6.36 defined on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ when

$$\begin{aligned} s &> \frac{1}{p_0} + (n-1)\Big(\frac{1}{\min(1,p_0,q)} - 1\Big) + \Big(\frac{2}{\min(1,p_0,p_t,q)} - 2\Big) \quad \text{and} \\ s &> \frac{2}{p_t} + n\Big(\frac{1}{\min(1,p_0)} - 1\Big). \end{aligned} \tag{7.6}$$

From Lemma 5.3(i) and the above-mentioned theorems, it is clear that the data must satisfy

$$g \in \overline{F}^{s-2,\vec{a}}_{\vec{p},q}(\Omega \times I), \quad \varphi \in \overline{F}^{s-\frac{1}{p_0},a''}_{p'',p_0}(\Gamma \times I), \quad u_0 \in \overline{B}^{s-\frac{2}{p_t}}_{p_0,p_t}(\Omega). \tag{7.7}$$

After application of r_0 to u, one can apply the trace γ_Γ in Theorem 6.47, when

$$s > \frac{1}{p_0} + \frac{2}{p_t} + (n-1)\Big(\frac{1}{\min(1,p_0)} - 1\Big). \tag{7.8}$$

Using (6.58), (6.92) for an arbitrary extension f of u, we obtain

$$\gamma_\Gamma(r_0 u) = \gamma_\Gamma\big(r_\Omega f(x,0)\big) = \sum_{\kappa \in \mathcal{F}_0} \psi_\kappa f(\cdot,0) = f(\cdot,0). \tag{7.9}$$

According to Theorem 6.46, the assumption on s in (7.8) also makes it well defined to apply the trace $r_{0,\Gamma}$ to γu. This gives, cf. (6.80), (6.90),

$$r_{0,\Gamma}(\gamma u) = \sum_{\kappa \in \mathcal{F}_0} \psi_\kappa f(\cdot,0) = f(\cdot,0), \tag{7.10}$$

thus

$$\gamma_\Gamma(r_0 u) = r_{0,\Gamma}(\gamma u). \tag{7.11}$$

Analysing the compositions $\gamma_\Gamma \circ r_0$, $r_{0,\Gamma} \circ \gamma$, it is obvious that

$$\gamma_\Gamma(r_0 u), \;\; r_{0,\Gamma}(\gamma u) \in B^{s-\frac{1}{p_0}-\frac{2}{p_t}}_{p_0,p_t}(\Gamma).$$

However, when s is sufficiently large, then (7.11) may be supplemented by higher order compatibility properties involving e.g. time derivatives ∂_t^l, $l \in \mathbb{N}$. To prepare for a discussion of these, we first need to define ∂_t^l on distributions v in $\mathcal{D}'(\Gamma \times I)$. Indeed, since it by Lemma 6.16 suffices to consider the atlas $\{\kappa \times \mathrm{id}_I\}_{\kappa \in \mathcal{F}_0}$, where $\mathcal{F}_0$ is defined just above Theorem 6.28, it is easily seen that $\{\partial_t^l v_{\kappa \times \mathrm{id}_I}\}_{\kappa \in \mathcal{F}_0}$ defines a distribution in $\mathcal{D}'(\Gamma \times I)$, which we denote $\partial_t^l v$. Utilising that the charts in $\mathcal{F}_0$ are just the identity in the time direction, it is straightforward to verify that ∂_t^l is bounded,

$$\partial_t^l : \overline{F}^{s,a''}_{p'',q}(\Gamma \times I) \to \overline{F}^{s-a_t l,a''}_{p'',q}(\Gamma \times I).$$

As a further preparation, we also include

Lemma 7.2. *The space $r_{\Omega\times I}\mathcal{S}(\mathbb{R}^{n+1})$ is dense in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ for $q<\infty$.*

This follows straightforwardly from the denseness of $\mathcal{S}(\mathbb{R}^{n+1})$ in $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$, cf. Lemma 4.5, and the continuity of $r_{\Omega\times I}: F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\to\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$.

We now apply the distributional derivative ∂_t^l followed by $r_{0,\Gamma}$ to φ, i.e.

$$r_{0,\Gamma}\partial_t^l\varphi = r_{0,\Gamma}\partial_t^l(\gamma u), \tag{7.12}$$

for the $l\in\mathbb{N}$ satisfying

$$2l < s-\frac{1}{p_0}-\frac{2}{p_t}-(n-1)\Big(\frac{1}{\min(1,p_0)}-1\Big), \tag{7.13}$$

where we recall that ∂_t has weight 2, cf. (7.4).

In the following, we rewrite the expression on the right-hand side in (7.12) to obtain a property involving only the data (g,φ,u_0). The first step is to interchange ∂_t^l and γ, which requires that $\gamma\partial_t^l u$ is well defined, i.e.

$$2l < s-\frac{1}{p_0}-(n-1)\Big(\frac{1}{\min(1,p_0,q)}-1\Big)-\Big(\frac{2}{\min(1,p_0,p_t,q)}-2\Big). \tag{7.14}$$

It is easily seen that the operators can be interchanged on $r_{\Omega\times I}\mathcal{S}(\mathbb{R}^{n+1})$, by exploiting that the effect of γ on this space is simply restriction to the boundary, cf. Section 6.6.3. Due to Lemma 7.2 as well as the continuity of ∂_t^l and γ, it therefore also holds true on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$; in case $q=\infty$, we rely on the denseness of $r_{\Omega\times I}\mathcal{S}(\mathbb{R}^{n+1})$ in $\overline{F}^{s-\varepsilon,\vec{a}}_{\vec{p},1}(\Omega\times I)$, where $\varepsilon>0$ is chosen so small that γ can still be applied (possible since all inequalities are strict). Hence

$$\partial_t^l\gamma u = \gamma\partial_t^l u.$$

Now the arguments in (7.9)–(7.10) are applied to $\partial_t^l u$ instead of u, as we may in view of (7.13)–(7.14) since $\partial_t^l u\in\overline{F}^{s-2l,\vec{a}}_{\vec{p},q}(\Omega\times I)$. This gives

$$r_{0,\Gamma}(\gamma\partial_t^l u) = \gamma_\Gamma(r_0\partial_t^l u).$$

By induction, using that u solves the heat equation,

$$\partial_t^l u = \Delta^l u + \sum_{j=0}^{l-1}\Delta^j\partial_t^{l-1-j}g,\quad l\in\mathbb{N},$$

and since a denseness argument as above gives that r_0 and Δ can be interchanged, we obtain from (7.12) and the rewritings thereof,

$$r_{0,\Gamma}\partial_t^l\varphi = \gamma_\Gamma\Big(\Delta^l u_0 + \sum_{j=0}^{l-1}\Delta^j r_0(\partial_t^{l-1-j}g)\Big).$$

These are properties involving only the data (g, φ, u_0). Using matrix notation, where the number of rows is the maximal l satisfying (7.13)–(7.14), we have

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & -r_{0,\Gamma} & \gamma_\Gamma \\ \gamma_\Gamma r_0 & -r_{0,\Gamma}\partial_t & \gamma_\Gamma \Delta \\ \gamma_\Gamma r_0(\partial_t + \Delta) & -r_{0,\Gamma}\partial_t^2 & \gamma_\Gamma \Delta^2 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} g \\ \varphi \\ u_0 \end{bmatrix}. \tag{7.15}$$

From (7.15) it is immediately clear that the triples (g, φ, u_0) having the compatibility properties form a closed, linear subspace. Moreover, e.g. u_0 only enters via $\gamma_\Gamma \Delta^k u_0$, while φ only appears with its traces $r_{0,\Gamma}\partial_t^k \varphi$ at $\Gamma \times \{0\}$.

The above proves

Theorem 7.3. *Let $\vec{a}, \vec{p}$ fulfil* (7.4)*. When the boundary value problem* (7.1)*–*(7.3) *has a solution u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ with (s, q) satisfying* (7.6)*, then the data must be specified as in* (7.7)*.*

Moreover, for all $l \in \mathbb{N}_0$ satisfying both (7.13) *and* (7.14)*, the data (g, φ, u_0) are compatible in the sense that*

$$r_{0,\Gamma}\partial_t^l \varphi = \gamma_\Gamma \Big(\Delta^l u_0 + \sum_{j=0}^{l-1} \Delta^j r_0(\partial_t^{l-1-j} g) \Big),$$

which for $l = 0$ reduces to (the sum is void)

$$r_{0,\Gamma}\varphi = \gamma_\Gamma u_0.$$

Comparing Theorem 7.1 and [9, Thm. 2.3] on one hand with Theorem 7.3 on the other, it is clear that the latter offers more flexibilty when it comes to the function spaces. Weidemaier's and Denk, Hieber and Prüss' results on the other hand work for less regular domains; and in some respects give more complete answers to the existence and uniqueness of solutions to the boundary value problems they studied.

7.2 On Sufficient Conditions

The purpose of this section is to reduce the problem of finding a solution u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ to (7.1)–(7.3) to a problem concerning a PDE with homogeneous boundary conditions. In the following we shall for brevity denote the unit ball in $\mathbb{R}^n$ by B and $\chi := \mathbb{1}_{\mathbb{R}^n_+}$. Moreover, we shall use the partition of unity $1 = \sum_\lambda \psi_\lambda + \psi$ on $\overline{\Omega}$ constructed in Section 6.4.4.

We let s satisfy (7.6), (7.8) and assume for simplicity that $s - \vec{a} \cdot \alpha$ for any $\alpha \neq 0$ fails to fulfil at least one of the conditions. Furthermore, we simplify to

$$p_0, p_t, q \geq 1, \tag{7.16}$$

hence, since $\vec{a} \cdot \alpha \geq 1$, the requirements on s reduce to

$$\frac{1}{p_0} + \frac{2}{p_t} < s < \frac{1}{p_0} + \frac{2}{p_t} + 1. \tag{7.17}$$

Furthermore, it is assumed that the data belong to the spaces given in (7.7) and that φ, u_0 are compatible, i.e. they satisfy (7.11).

In case we can prove the existence of a $w \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ such that $\gamma w = \varphi$ and $r_0 w = u_0$, then solving (7.1)–(7.3) reduces to finding a solution $v \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ to

$$\begin{aligned} \partial_t v - \Delta v &= g - (\partial_t - \Delta) w && \text{in } \ \Omega \times I, & (7.18)\\ \gamma v &= 0 && \text{on } \ \Gamma \times I, & (7.19)\\ r_0 v &= 0 && \text{on } \ \Omega \times \{0\}, & (7.20) \end{aligned}$$

since $v + w$ then solves (7.1)–(7.3). This homogeneous problem can probably be studied using semigroup methods as in e.g. [64, Thm. 1.3] and [9]. However, due to time constraints such a study has not been possible in this PhD project.

The surjectivity of γ, cf. Theorem 6.44, gives the existence of a $w_1 \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$ such that $\gamma w_1 = \varphi$. In the following we construct

$$\widetilde{u} \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \ \text{ with } \ \gamma \widetilde{u} = 0 \ \text{ and } \ r_0 \widetilde{u} = u_0 - r_0 w_1 =: u_1. \tag{7.21}$$

Then w can be taken as $\widetilde{u} + w_1$.

For such a $\widetilde{u}$ to even exist, we first note that u_1 and 0 are compatible. Indeed, by repeating the calculations in (7.9)–(7.10) for w_1 instead of u and using (7.11), we obtain

$$\gamma_\Gamma u_1 = \gamma_\Gamma u_0 - r_{0,\Gamma} \gamma w_1 = 0. \tag{7.22}$$

To construct $\widetilde{u}$ we first need an operator, which maps u_1 to an element of the same Besov space, but over $\mathbb{R}^n$, in such a way that the support is contained in $\overline{\Omega}$:

Proposition 7.4. *When $1 \leq p < \infty$, $0 < q < \infty$ and*

$$\frac{1}{p} < s < \frac{1}{p} + 1, \tag{7.23}$$

then for any $u \in \overline{B}^{s}_{p,q}(\Omega)$ with $\gamma_\Gamma u = 0$, the operator

$$K_\Omega u := \sum_\lambda e_{U_\lambda}(\chi \circ \lambda \cdot \psi_\lambda f) + \psi f, \tag{7.24}$$

is well defined. Moreover, K_Ω is bounded,

$$K_\Omega : \{\, u \in \overline{B}^{s}_{p,q}(\Omega) \mid \gamma_\Gamma u = 0 \,\} \to B^s_{p,q}(\mathbb{R}^n),$$

and has the properties that $r_\Omega K_\Omega u = u$ and $\operatorname{supp} K_\Omega u \subset \overline{\Omega}$ for such u.

Before proving the proposition, we first recall a paramultiplication result:

Lemma 7.5 ([25, (2.55)]). *When $0 < p, q \le \infty$ and*

$$\max\Big(\frac{1}{p} - 1, \frac{n}{p} - n\Big) < s < \frac{1}{p}, \tag{7.25}$$

then χ is a multiplier for $B^s_{p,q}(\mathbb{R}^n)$.

The proof of Proposition 7.4 also utilises a characterisation of Besov spaces involving all derivatives up to a certain order:

Lemma 7.6. *Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. For any $m \in \mathbb{N}_0$,*

$$u \in B^s_{p,q}(\mathbb{R}^n) \iff \sum_{|\alpha| \le m} \| D^\alpha u \,|B^{s-m}_{p,q}(\mathbb{R}^n)\| < \infty; \tag{7.26}$$

and the sum is an equivalent quasi-norm on $B^s_{p,q}(\mathbb{R}^n)$.

Proof. For the implication from right to left we recall, cf. e.g. [17, (5.2)], that $1 = \langle\xi\rangle^{-2m} \sum_{|\alpha|\le m} C_{m,\alpha}\xi^{2\alpha}$, hence using the lift operator in (6.24) gives

$$u = I_{-2m} \sum_{|\alpha| \le m} C_{m,\alpha} D^{2\alpha} u.$$

Now the lifting property in [57, 2.3.8] yields

$$\| u \,|B^s_{p,q}\| \le c \sum_{|\alpha| \le m} C_{m,\alpha} \| D^{2\alpha} u \,|B^{s-2m}_{p,q}\| \le c \sum_{|\alpha| \le m} C_{m,\alpha} \| D^{\alpha} u \,|B^{s-m}_{p,q}\| < \infty.$$

The other implication is trivial. □

We shall apply this lemma to χf, where $f \in B^s_{p,q}(\mathbb{R}^n)$ with $\gamma_{0,n} f = 0$. Hence we must verify that χf and $D_{x_j}(\chi f)$ belong to $B^{s-1}_{p,q}(\mathbb{R}^n)$ for $j = 1, \dots, n$. Since χ has a singularity at $x_n = 0$, the derivative for $j = n$ is considered separately:

Lemma 7.7. *Let $1 \le p < \infty$ and $0 < q < \infty$. When*

$$\frac{1}{p} < s < \frac{1}{p} + 1,$$

then $D_{x_n}(\chi f) = \chi D_{x_n} f$ in $B^{s-1}_{p,q}(\mathbb{R}^n)$ for every $f \in B^s_{p,q}(\mathbb{R}^n)$ with $\gamma_{0,n} f = 0$.

Proof. The space $\mathcal{S}$ is dense in $B^s_{p,q}$ by [29, Lem. 3.5] because $p, q < \infty$, hence there exists a sequence $(f_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}$ converging to f in $B^s_{p,q}$. Since multiplication with χ is readily seen to be continuous on L_p and $B^s_{p,q} \hookrightarrow L_p$, we have that $\lim_{k\to\infty}(\chi f_k) = \chi f$ in L_p. Thus by Leibniz' rule we have in $\mathcal{D}'$,

$$\begin{aligned} D_{x_n}(\chi f) &= \lim_{k\to\infty} \big((D_{x_n}\chi) f_k + \chi D_{x_n} f_k\big) \\ &= \lim_{k\to\infty} \big(-\mathbb{1}_{\mathbb{R}^{n-1}}(x') \otimes \mathrm{i}\,\delta_0(x_n) f_k(x', 0) + \chi D_{x_n} f_k\big). \end{aligned} \tag{7.27}$$

The embedding $B^s_{p,q}(\mathbb{R}^n) \subset C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{n-1}))$ and the continuity of the trace $\gamma_{0,n} : B^s_{p,q}(\mathbb{R}^n) \to B^{s-1/p}_{p,q}(\mathbb{R}^{n-1})$, cf. [26, Thm. 1.4, resp. Thm. 1.1], yield,

$$\lim_{k\to\infty} f_k(\cdot, 0) = \lim_{k\to\infty} \gamma_{0,n} f_k = \gamma_{0,n} f.$$

By the assumption that $\gamma_{0,n} f = 0$ and the continuity of the tensor product in the second entry (which without difficulty is seen from its definition, cf. e.g. [21, Thm. 5.1.1]), the first term on the right-hand side in (7.27) therefore equals 0.

For the second term, Lemma 7.5 can be applied as $D_{x_n} f_k \in B^{s-1}_{p,q}(\mathbb{R}^n)$. Hence we obtain $D_{x_n}(\chi f) = \chi D_{x_n} f$ in $B^{s-1}_{p,q}(\mathbb{R}^n)$. □

With the above three lemmas, we now give a

Proof of Proposition 7.4. Each term in (7.24) is treated separately. That the last term ψf is in $B^s_{p,q}$ follows from the same arguments as for ψu in Theorem 6.43.

For the other terms, repeating the arguments for (6.71) gives

$$f_\lambda := e_B\big((\psi_\lambda f) \circ \lambda^{-1}\big) \in B^s_{p,q}(\mathbb{R}^n). \tag{7.28}$$

Now we use Lemma 7.6 with $m = 1$ to verify that $\chi f_\lambda \in B^s_{p,q}(\mathbb{R}^n)$. First the derivative $D_{x_n}(\chi f_\lambda)$ is treated by exploiting that $\gamma_\Gamma u = 0$ implies $(\psi_\lambda f) \circ \lambda^{-1}(\cdot, 0) = 0$ on $\{\, x \in B \,|\, x_n = 0\}$, cf. Section 6.6.4 and (6.79)–(6.80). Since $\operatorname{supp} \psi_\lambda \circ \lambda^{-1} \subset B$, we therefore have that $\gamma_{0,n} f_\lambda = 0$, hence $D_{x_n}(\chi f_\lambda)$ belongs to $B^{s-1}_{p,q}$ by Lemma 7.7.

The other derivatives $D_{x_j}(\chi f)$, $j = 1, \dots, n-1$, are straightforward, because χ for fixed x_n is either identically 0 or 1 and therefore $D_{x_j}(\chi f_\lambda)$ either equals 0 or $D_{x_j} f_\lambda$. Thus Lemma 7.5 yields that $D_{x_j}(\chi f_\lambda) = \chi D_{x_j} f_\lambda \in B^{s-1}_{p,q}$, and it also readily gives that χf_λ itself belongs to this space, hence $\chi f_\lambda \in B^s_{p,q}$ follows from Lemma 7.6.

By the isotropic Besov versions of Theorem 6.11 and Lemma 5.17, cf. Section 6.4.3, this implies that the terms $e_{U_\lambda}(\chi \circ \lambda \cdot \psi_\lambda f)$ in (7.24) are in $B^s_{p,q}$. So $K_\Omega u \in B^s_{p,q}(\mathbb{R}^n)$, and since λ maps $U_\lambda \cap \Omega$ to $B \cap \mathbb{R}^n_+$ we have

$$r_\Omega K_\Omega u = r_\Omega\Big(\Big(\sum_\lambda \psi_\lambda + \psi\Big) f\Big) = u,$$

whereas outside Ω, $K_\Omega u$ is 0 due to the supports of χ, ψ_λ, ψ. This shows that $\operatorname{supp} K_\Omega u \subset \overline{\Omega}$ and that $K_\Omega u$ is independent of the chosen extension f.

Boundedness of K_Ω follows using once again the Besov versions of Lemma 5.17 and Theorem 6.11, which for $d := \min(1, q, p)$ yields,

$$\| K_\Omega u \,|B^s_{p,q}(\mathbb{R}^n)\|^d \le c \sum_\lambda \| \chi \cdot (\psi_\lambda f \circ \lambda^{-1}) \,|\overline{B}^s_{p,q}(B)\|^d + \| \psi f \,|B^s_{p,q}(\mathbb{R}^n)\|^d.$$

Since $\overline{B}^s_{p,q}(B)$ is equipped with the quotient quasi-norm and $\chi f_\lambda \in B^s_{p,q}(\mathbb{R}^n)$, each term in the sum over λ can be estimated by, cf. also Lemmas 7.6 and 7.5,

$$\| \chi f_\lambda \,|B^s_{p,q}\|^d \le c \sum_{|\alpha|\le 1} \| \chi D^\alpha f_\lambda \,|B^{s-1}_{p,q}\|^d \le c \| f_\lambda \,|B^s_{p,q}\|^d.$$

Therefore the multiplication result [58, 4.2.2] for Besov spaces gives

$$\| K_\Omega u \,|B^s_{p,q}\|^d \le c \sum_\lambda \| \psi_\lambda f \,|B^s_{p,q}\|^d + \| \psi f \,|B^s_{p,q}\|^d \le c\| f \,|B^s_{p,q}\|^d,$$

hence taking the infimum over all admissible f (as we may, since K_Ω is independent of the extension) proves the boundedness of K_Ω. □

Finally, returning to (7.21), we verify that $\widetilde{u} := r_{\Omega\times I}(Q_\Omega K_\Omega u_1)$ first of all belongs to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ and secondly fulfils the needed boundary conditions, namely $r_0\widetilde{u} = u_1$ and $\gamma\widetilde{u} = 0$. In our treatment of Q_Ω it will be crucial that we have Theorem 6.43.

Since u_1 is in $\overline{B}^{s-2/p_t}_{p_0,p_t}(\Omega)$, it follows straightforwardly from Proposition 7.4 and Theorem 6.43 that $\widetilde{u} \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$. Moreover, the definition of r_0 in (6.58) immediately implies, when choosing $Q_\Omega K_\Omega u_1$ as the extension of $\widetilde{u}$,

$$r_0\widetilde{u} = r_\Omega(K_\Omega u_1) = u_1.$$

Also in the definition of γ, cf. (6.79)–(6.80), the extension $Q_\Omega K_\Omega u_1$ of $\widetilde{u}$ is considered. Since $\gamma_\Gamma u_1 = 0$ by (7.22), Proposition 7.4 and (6.70) yield the decisive property that

$$\operatorname{supp} Q_\Omega K_\Omega u_1(\cdot,t) \subset \overline{\Omega} \quad \text{for all } t\in\mathbb{R}.$$

Hence $Q_\Omega K_\Omega u_1(\lambda^{-1}(x),t) = 0$ for $x\in B$ with $x_n<0$, and by (6.73) it is therefore also 0 on $B' \simeq B\cap\{x_n=0\}$. This shows that $\gamma\widetilde{u}=0$.

To sum up, we have by the construction of $\widetilde{u}$ reduced the fully inhomogeneous problem in (7.1)–(7.3) to the one in (7.18)–(7.20), where both boundary conditions are homogeneous.

CHAPTER 8

Final Remarks

In this short chapter we draw attention to the PhD thesis [40] by S. Mayboroda, since this is a recent, and to some extend related, work.

S. Mayboroda extended in her PhD thesis, cf. [40], results from [23], where D. Jerison and C. Kenig studied the inhomogeneous Poisson equation on Lipschitz domains with homogeneous Dirichlet boundary conditions. The focus in [40] was well-posedness of this problem for inhomogeneous Dirichlet or Neumann boundary conditions. To treat this question, she developed a trace theory in Lipschitz domains, which will be briefly outlined below.

We refer to [40, Sec. 3.4] for a definition of Besov and Lizorkin–Triebel spaces over the boundary of a Lipschitz domain and to (2.6) here for the spaces over a Lipschitz domain. Using these definitions, Mayboroda proved

Theorem 8.1 ([40, Thm. 1.1.3]). *Let $\Omega \subset \mathbb{R}^n$ be Lipschitz. When*

$$0 < q \leq \infty, \quad \frac{n-1}{n} < p \leq \infty \quad \textit{and} \quad (n-1)\max\big((1/p-1),0\big) < s < 1,$$

then the restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : \overline{B}^{s+1/p}_{p,q}(\Omega) \to B^s_{p,q}(\partial\Omega),$$

which furthermore has a linear, bounded right-inverse.

The theorem contains a similar statement for Lizorkin–Triebel spaces ($q = \infty$ is understood when $p = \infty$), but in this case existence of a linear, bounded right-inverse relies on an additional assumption on the parameters, cf. [40, (1.1.11)].

Boundedness of Tr and that it is at all well defined was established there using non-smooth atomic decompositions of $B^s_{p,p}(\mathbb{R}^n)$ and afterwards interpolation to cover the whole range of parameters. The construction of a right-inverse was based on Rychkov's extension operator in (3.9) and so-called layer potentials associated with $\partial\Omega$, cf. [40, Sec. 2.3] for a short introduction to these.

In comparision, our proof of boundedness of γ in Theorem 6.44, where Ω is C^∞, relies on the characterisation by kernels of local means in Theorem 4.24, while the right-inverse is constructed using our mixed-norm version of Rychkov's extension operator and a right-inverse to the trace $\gamma_{0,n}$, cf. Theorem 6.34, respectively (6.82).

The work of Mayboroda is not directly related to our work, since she treated Lipschitz domains, whereas we focus on C^∞-domains. However, it indicates that the development of a systematic trace theory in different contexts is of a wider interest.

Bibliography

[1] H. Amann. *Linear and quasilinear parabolic problems. Vol. 1: Abstract linear theory*, volume 89 of *Monographs in Mathematics*. Birkhäuser, Basel, 1995.

[2] R. J. Bagby. An extended inequality for the maximal function. *Proc. Amer. Math. Soc*, 48(2):419–422, 1975.

[3] M. Z. Berkolaĭko. Theorems on traces on coordinate subspaces for some spaces of differentiable functions with anisotropic mixed norm. *Dokl. Akad. Nauk SSSR*, 282(5):1042–1046, 1985. Engl. transl. Soviet Math. Dokl. 31 (1985), No. 3, 540–544.

[4] O. V. Besov, V. P. Il'in, and S. M. Nikol'skij. *Integral representations of functions and imbedding theorems.* V. H. Winston, 1979.

[5] O. V. Besov, V. P. Il'in, and S. M. Nikol'skij. *Integral representations of functions and imbedding theorems.* Nauka, second edition, 1996. (In Russian).

[6] J. S. Bugrov. Functional spaces with mixed norm. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:1137–1158, 1971. Engl. transl. Math USSR **5** (1971), 1145–1167.

[7] H.-Q. Bui, M. Paluszynski, and M. H. Taibleson. A maximal function characterization of weighted Besov–Lipschitz and Triebel–Lizorkin spaces. *Studia Math.*, 119:219–246, 1996.

[8] H.-Q. Bui, M. Paluszynski, and M. H. Taibleson. Characterization of the Besov–Lipschitz and Triebel–Lizorkin spaces. The case $q < 1$. In *Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996)*, pages 837–846, 1997. Special Issue.

[9] R. Denk, M. Hieber, and J. Prüss. Optimal L^p–L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.*, 257(1):193–224, 2007.

[10] W. Farkas. Atomic and subatomic decompositions in anisotropic function spaces. *Math. Nachr.*, 209:83–113, 2000.

[11] W. Farkas, J. Johnsen, and W. Sickel. Traces of anisotropic Besov-Lizorkin-Triebel spaces—a complete treatment of the borderline cases. *Math. Bohem.*, 125(1):1–37, 2000.

[12] W. Farkas and H.-G. Leopold. Characterisations of function spaces of generalised smoothness. *Ann. Mat. Pura Appl. (4)*, 185(1):1–62, 2006.

[13] L. E. Fraenkel. Formulae for high derivatives of composite functions. *Math. Proc. Cambridge Philos. Soc.*, 83(2):159–165, 1978.

[14] M. Frazier, B. Jawerth, and G. Weiss. *Littlewood–Paley theory and the study of function spaces*, volume 79 of *CBMS Regional Conference Series in Mathematics.* 1991.

[15] G. Grubb. Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces. *Math. Z.*, 218(1):43–90, 1995.

[16] G. Grubb. *Functional calculus of pseudodifferential boundary problems*, volume 65 of *Progress in Mathematics.* Birkhäuser, Boston, second edition, 1996.

[17] G. Grubb. *Distributions and Operators.* Springer, 2009.

[18] G. Grubb and V. A. Solonnikov. Solution of parabolic pseudo-differential initial-boundary value problems. *J. Differential Equations*, 87(2):256–304, 1990.

[19] M. Hansen. *Nonlinear approximation and function spaces of dominating mixed smoothness.* PhD thesis. Friedrich-Schiller-University Jena, 2010.

[20] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*, volume 26 of *Mathématiques & Applications (Berlin) [Mathematics & Applications].* Springer, Berlin, 1997.

[21] L. Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition.

[22] L. Hörmander. *The analysis of linear partial differential operators. III.* Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.

[23] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130:161–219, 1995.

[24] J. Johnsen. Pointwise multiplication of Besov and Triebel–Lizorkin spaces. *Math. Nachr.*, 175:85–133, 1995.

[25] J. Johnsen. Elliptic boundary problems and the Boutet de Monvel calculus in Besov and Triebel-Lizorkin spaces. *Math. Scand.*, 79(1):25–85, 1996.

[26] J. Johnsen. Traces of Besov spaces revisited. *Z. Anal. Anwendungen*, 19(3):763–779, 2000.

[27] J. Johnsen. Pointwise estimates of pseudo-differential operators. *J. Pseudo-Differ. Oper. Appl.*, 2(3):377–398, 2011.

[28] J. Johnsen and W. Sickel. A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin–Triebel spaces with mixed norms. *J. Funct. Spaces Appl.*, 5(2):183–198, 2007.

[29] J. Johnsen and W. Sickel. On the trace problem for Lizorkin–Triebel spaces with mixed norms. *Math. Nachr.*, 281(5):669–696, 2008.

[30] J. Johnsen, W. Sickel, and S. M. Hansen. Anisotropic Lizorkin–Triebel spaces with mixed norms — traces on smooth boundaries. In preparation.

[31] J. Johnsen, W. Sickel, and S. M. Hansen. Anisotropic, mixed-norm Lizorkin–Triebel spaces and diffeomorphic maps. *J. Funct. Spaces Appl.*, 2013. Submitted.

[32] J. Johnsen, W. Sickel, and S. M. Hansen. Characterisation by local means of anisotropic Lizorkin–Triebel spaces with mixed norms. *Journal of Analysis and its Applications*, 32(3):257–277, 2013.

[33] W. P. Johnson. The curious history of Faà di Bruno's formula. *Amer. Math. Monthly*, 109(3):217–234, 2002.

[34] H. Kempka. 2-microlocal Besov and Triebel–Lizorkin spaces of variable integrability. *Rev. Mat. Complut.*, 22(1):227–251, 2009.

[35] H. Kempka and J. Vibyral. Spaces of variable smoothness and integrability: Characterizations by local means and ball means of differences. *J. Fourier Anal. Appl.*, 18(4):852–891, 2012.

[36] A. Kufner, O. John, and S. Fučík. *Function spaces.* Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis. Noordhoff International Publishing, Leyden, 1977.

[37] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasi-linear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[38] G. M. Lieberman. *Second order parabolic differential equations.* World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[39] P. I. Lizorkin. Multipliers of Fourier integrals and estimates of convolutions in spaces with mixed norm. Applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 34:218–247, 1970. Engl. transl. Math. USSR Izv. **4** (1970).

[40] S. Mayboroda. *The Poisson problem on Lipschitz domains.* PhD thesis. University of Missouri–Columbia, 2005.

[41] S. M. Nikol'skij. *Approximation of functions of several variables and imbedding theorems.* Springer, 1975.

[42] L. Rodino. *Linear partial differential operators in Gevrey spaces.* World Scientific Publishing Co. Inc., River Edge, NJ, 1993.

[43] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications.* Berlin, 1996.

[44] V. S. Rychkov. On a theorem of Bui, Paluszynski, and Taibleson. *Tr. Mat. Inst. Steklova*, 227(Issled. po Teor. Differ. Funkts. Mnogikh Perem. i ee Prilozh. 18):286–298, 1999.

[45] V. S. Rychkov. On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains. *J. London Math. Soc. (2)*, 60:237–257, 1999.

[46] V. S. Rychkov. Littlewood–Paley theory and function spaces with A_p^{loc} weights. *Math. Nachr.*, 224:145–180, 2001.

[47] B. Scharf. Atomic representations in function spaces and applications to pointwise multipliers and diffeomorphisms, a new approach. *Math. Nachr.*, 286(2-3):283–305, 2013.

[48] H.-J. Schmeisser. Anisotropic spaces. II. Equivalent norms for abstract spaces, function spaces with weights of Sobolev-Besov type. *Math. Nachr.*, 79:55–73, 1977.

[49] H.-J. Schmeisser and H. Triebel. Anisotropic spaces. I. Interpolation of abstract spaces and function spaces. *Math. Nachr.*, 73:107–123, 1976.

[50] H.-J. Schmeisser and H. Triebel. *Topics in Fourier analysis and function spaces*, volume 42 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology].* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987.

[51] A. Seeger. A note on Triebel–Lizorkin spaces. In *Approximation and function spaces (Warsaw, 1986)*, volume 22 of *Banach Center Publ.*, pages 391–400. PWN Polish Sci. Publ. Warsaw, 1989.

[52] R. T. Seeley. Extension of C^∞ functions defined in a half space. *Proc. Amer. Math. Soc.*, 15:625–626, 1964.

[53] K. Spindler. A short proof of the formula of Faà di Bruno. *Elem. Math.*, 60(1):33–35, 2005.

[54] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series.* Princeton University Press, Princeton, NJ, 1993. Monographs in Harmonic Analysis III.

[55] J.-O. Strömberg and A. Torchinsky. *Weighted Hardy spaces*, volume 1381 of *Lecture Notes in Mathematics*. Springer, Berlin, 1989.

[56] H. Triebel. *Interpolation theory, function spaces, differential operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[57] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser, Basel, 1983.

[58] H. Triebel. *Theory of function spaces II*, volume 84 of *Monographs in Mathematics*. Birkhäuser, Basel, 1992.

[59] H. Triebel. *Theory of function spaces III*, volume 100 of *Monographs in Mathematics*. Birkhäuser, Basel, 2006.

[60] H. Triebel. *Function spaces and wavelets on domains*, volume 7 of *EMS Tracts in Mathematics*. European Mathematical Society, Zürich, 2008.

[61] J. Vybiral. Function spaces with dominating mixed smoothness. *Dissertationes Math. (Rozprawy Mat.)*, 436, 2006.

[62] W. V. Wahl. The equation $u' + A(t)u = f$ in a Hilbert space and L^p-estimates for parabolic equations. *J. London Math. Soc. (2)*, 25:483–497, 1982.

[63] P. Weidemaier. On the trace theory for functions in Sobolev spaces with mixed L_p-norm. *Czechoslovak Mathematical Journal*, 44(1):7–20, 1994.

[64] P. Weidemaier. Existence results in L_p-L_q spaces for second order parabolic equations with inhomogeneous Dirichlet boundary conditions. In *Progress in partial differential equations, Vol. 2 (Pont-à-Mousson, 1997)*, volume 384 of *Pitman Res. Notes Math. Ser.*, pages 189–200. Longman, Harlow, 1998.

[65] P. Weidemaier. Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm. *Electron. Res. Announc. Amer. Math. Soc.*, 8:47–51, 2002.

[66] P. Weidemaier. Lizorkin–Triebel spaces of vector-valued functions and sharp trace theory for functions in Sobolev spaces with a mixed L_p-norm in parabolic problems. *Mat. Sb.*, 196(6):3–16, 2005.

[67] M. Yamazaki. A quasihomogeneous version of paradifferential operators. I. Boundedness on spaces of Besov type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(1):131–174, 1986.

[68] M. Yamazaki. A quasihomogeneous version of paradifferential operators. II. A symbol calculus. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 33(2):311–345, 1986.