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Published in:
I E E E Transactions on Education

DOI (link to publication from Publisher):
10.1109/TE.2023.3285718

Publication date:
2024

Document Version
Accepted author manuscript, peer reviewed version

A Critical Look at the Laplace Transform Method in Engineering Education

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***Abstract*—Contribution: This article presents a new look at teaching the Laplace transform for engineering students by emphasising the obsolescence of the current method of finding the inverse Laplace transform when solving differential equations, and by recognising the important role of a computer-assisted environment in helping the students understand the main idea behind the Laplace transform, instead of asking the students to repeat computational processes by hand. Background: The Laplace transform is a widely used integral transform that has important applications in many areas of engineering, and therefore, has a central place in the curricula for engineering education. However, according to several research articles, many students experience great difficulties understanding the Laplace transform. Research Question: Is the use of partial fractions and Laplace transform tables necessary for a proper *conceptual* understanding of the Laplace transform method? Methodology: Using the Anthropological Theory of the Didactic as an educational platform, the current teaching of the Laplace transform method, is analysed. A parallel discussion of the teaching of logarithms at the upper secondary school level is drawn, where, previously, this also took place using the tables of logarithms, but now the reliance on calculators is overwhelming. The authors suggest a method of teaching the Laplace transform in a computer-assisted environment. Findings: In the light of the shift in computer hardware and software, the authors conclude by calling for innovation in and revision of engineering education through bridging the gap between procedures and understanding, by using computer software, where it is suitable.**

***Index Terms*—Laplace transform, CAS-assisted environment, anthropological theory of the didactic, engineering education, mathematics education.**

I. Introduction

ORDINARY and partial differential equations describe the time evolution of certain physical quantities. Examples of these quantities are the current in an electric circuit, the oscillation of a vibrating membrane and the heat flow through an insulated conductor. The differential equations are usually supplemented with initial conditions that describe the state of the modelled system at time $t = 0$.

A very powerful method for solving such equations is that of the Laplace transform, which is defined as a mapping between the time domain and the domain of the complex variable s: [1]

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} f(t)e^{-st}dt \tag{1}$$

The complex-frequency variable $s = \sigma + j\omega$ is used to transform a large class of time functions, such as constants, sinusoids, and exponentials, which are frequently used in circuit analysis and control systems.

The Laplace transform is named after the French mathematician Pierre Simon Laplace (1749–1827), who did the theoretical background work. In his work on probability theory and his search for solutions of differential equations, Laplace studied integral transforms of the form

$$F(s) = \int_a^b K(s,t)f(t)dt \tag{2}$$

which obviously include the Laplace transform defined in (1). Laplace took a critical step forward by applying the idea of *transformation*, rather than just looking for a solution in the form of an integral. According to [2], the modern use of the Laplace transform is a relatively recent development, and less than three generations ago, other approaches were still in widespread use, such as the *operational calculus*, developed by the English electrical engineer Oliver Heaviside (1850–1925).

The Heaviside calculus was popular among electrical engineers in the 1920s and 1930s, especially after the publication of the book *Electric Circuit Theory and the Operational Calculus* [3] in 1926 by the American electrical engineer John R. Carson (1886–1940). Three years later, a German edition of this book was published, and soon induced a debate between the adherents of the Heaviside method and those of the modern Laplace transform. In the 1950s, mathematicians began to realise the advantages of the modern Laplace transform, which led to its complete acceptance among them. For a more detailed history of the development of the Laplace transform, the reader can consult the two papers [4] and [5].

The Laplace transform literally converts the original differential equation into an algebraic equation. The latter, being much easier to solve than the first, is then transformed back to the time domain, using the inverse Laplace transform. Several textbooks on linear circuit analysis (e.g. [6]) and control systems (e.g. [7]) introduce the Laplace transform both as a method of solving a linear differential equation, and in the definition of a transfer function, which is merely an algebraic representation of the differential equation, describing a dynamic system [1]. The Laplace transform yields an expression $Y(s)$ that seldom appears in Laplace transform tables. The procedure required here is to decompose the function $Y(s)$ into so-called partial fractions to determine the time-response function $y(t)$.

The partial fraction decomposition itself is often a tedious and time-consuming technique, and it should be mastered in order to match an entry in Laplace transform tables. It is interesting to note that the same technique is used to evaluate many indefinite integrals with the help of tables of indefinite integrals in calculus books, as in, e.g., [8]. The use of Laplace transform tables is also analogous to the use of the somewhat old-fashioned tables of logarithms in mathematics at the secondary school: after transforming a product of numbers to a sum of logarithms, one previously uses the tables to find the number (the *antilogarithm*!) corresponding to that sum. However, the logarithm itself is usually defined as an exponent, not by using tables [9].

The authors wish here to raise the question whether the use of partial fractions and Laplace transform tables is necessary for a proper *conceptual* understanding of the Laplace transform method. This is the research question of this *concept* paper.

II. Literature Review

The research literature on the Laplace transform is still scarce. However, the authors did an online search and found four sources that are relevant to this study. All these sources concluded that the students find the Laplace transform difficult.

In her PhD thesis [10], the author argues that, although the Laplace transform is considered a difficult concept for many students to discern, the teaching methods used to introduce the Laplace transform play a crucial role in how the student understand the Laplace transform.

In [11], the authors mentioned some obstacles that could arise in the teaching and learning of Laplace transforms and concluded that it is one of the most difficult topics for electrical engineering students to grasp, when learning electric circuit theory. They pointed out that what makes transient response difficult is that the mathematics used is rather advanced.

The authors of the paper [12] interviewed 22 university teachers regarding the difficulties involved in the learning, and the relevance of, the Laplace transform in engineering education. They concluded that the teachers did not have a unified view on either the difficulties involved in learning, or the importance of, the Laplace transform. While educational research often focuses on the students' conceptions and misconceptions of the Laplace transform, the paper also points to the importance of studying the conceptions of the instructors themselves.

In the article [13], the authors, who all taught similar courses, concluded that students find the Laplace transform difficult, mainly because there is significant confusion in its definition, as presented in many of the standard textbook on the subject. Specifically, most engineering textbooks define the Laplace transform as

$$F(s) = \mathcal{L}(f(t)) = \int_{0^-}^{\infty} f(t)e^{-st}dt \tag{3}$$

where the lower limit of the integral is set to 0^- so that discontinuities and impulses are included at the time $t = 0$, while mathematicians use 0^+ as the lower limit.

III. Using a Computer Environment to Study a Mathematical Topic

Contemporary educational researchers and practitioners seem to agree on the basic principle that teaching should not just *transmit* knowledge to students but should also *promote* the students to work independently, in order to learn. There are many general educational models suggesting more elaborate characteristics of the ways in which students could work autonomously, in terms of taxonomies that describe different degrees of autonomy of the students' work. One such model is a CAS-assisted learning environment (CAS stands for Computer Algebra System). From a didactical perspective, CAS-assisted learning describes an interactive educational environment where computer software is integrated in teaching to help the students visualise a certain concept or discern a specific topic. In this regard, a vast number of research articles and PhD theses (e.g., [14], [15]) indicate that computer-based interventions facilitate the students' conceptual learning and have a significant impact on their learning.

Studying a mathematical subject such as the Laplace transform, simply means doing mathematical work in that subject for educational purposes [16]. In a teaching and learning context, as well as in professional research, working in an area of mathematics is attempting to solve a set of problems, by using techniques and creating new concepts. The new capabilities of modern powerful computational tools can give rise to expanded possibilities, not only in solving tasks within a mathematical field, but also in helping students to become aware of some aspects of a mathematical concept, which may remain implicit in the conventional paper-and-pencil instrumentation [17].

Thus, following [18], it is argued that working in a mathematical field, involves three structural levels:

1) Tasks
2) Techniques
3) Theories

Tasks constitute the first structural level. They do not just include individual problems but rather more general *types* of problems. For instance, within the topic of Laplace transform, the task "solve the given differential equation using the Laplace transform method" refers to a certain type of problems but not to others.

The second level is that of techniques. A technique means a method of doing tasks. Using techniques helps in arranging and differentiating tasks. For instance, the technique to solve the task "find the time function that corresponds to a given function in the Laplace domain" depends on the type of the function, e.g. whether partial fraction decomposition should be used or not.

The third level is that of theories. While the first two levels are related to an action to be executed, this level is related to discourse and validation. Here, the consistency and effectiveness of techniques are validated through mathematical concepts, theorems, structures, and properties.

It should be emphasised that this three-level structure is merely an assumed didactical model, which, hopefully, will help in reflecting on whether the use of CAS can positively contribute to the students' conceptual understanding, since tasks and techniques play an important role in an engineering classroom.

Anthropological and socio-cultural approaches seem to be more appropriate to understand the role played by instruments in didactical situations, and are able to rehabilitate the role of techniques in mathematical activities. Therefore, this simple didactical model leads to the concept of praxeology introduced in a more elaborate approach, namely the Anthropological Theory of the Didactic (ATD), that was proposed by the French mathematician Yves Chevallard in 1991 [19]. A brief account of this approach is presented in the next section.

IV. The Anthropological Theory of the Didactic

The Anthropological Theory of the Didactic (ATD) is a relatively recent approach to the didactics of mathematics. It views mathematics as a product of a human activity in the sense that mathematical concepts and thinking modes depend on the social and cultural contexts where they develop. According to ATD, a body of knowledge is subject to a *transformation* from the moment it is produced, put into use, selected, and designed to be taught, until it is actually taught in a given educational institution. As a consequence, mathematical objects are not absolute, eternal objects, but are entities that are created from the *practices* of given institution. In this theory, an institution has to be understood in a very broad sense: a ministry of education, a department of engineering, and even a book publisher, are examples of institutions. Any social or cultural practice takes place in such an institution. These practices or *praxeologies*, as they are called by Chevallard, consist of two blocks: the practical block, formed by types of problems or tasks and by the techniques used to solve them, and the knowledge (or theory) block that provides the discourse necessary to justify and interpret the practical block. The knowledge block itself is structured in two levels: the technology, which provides a first level of justification and explanation of the technique used, and the theory that constitutes a deeper level of justification. A praxeology is therefore a quadruple (T, τ, θ, Θ) that consists of four components: a *task* T, a *technique* τ, a *technology* θ, and a *theory* Θ.

ATD imparts a wider sense to a technique than its usual meaning in educational discourse, comprising not just recognised routines for standard tasks, but more complex assemblies of reasoning and routine work, whereas mainstream mathematics education research delimits the technique component τ more narrowly in terms of routine manipulations, computational procedures, and algorithmic skills [20].

At the university, a task can, for example, be an exercise taken from a mathematics text, such as the task T of solving a first-order differential equation. The students need a technique τ to solve the task, such as separating the variables in the equation. The technology θ may consist of validating the method used, such as verifying that the differential equation is separable. The technology may itself be explained and justified by the theory component. For example, the existence and uniqueness of a solution of a first-order differential equations is the theory Θ that explains the technology θ.

V. The Didactical Challenge of Teaching the Laplace Transform

From the instructor's perspective, the textbook represents the knowledge to be taught, together with the course syllabus. However, presenting the definition of the Laplace transform as it is given in many standard textbooks, is what Freudenthal [21] would call an *anti-didactical inversion*: it may not make sense to the students, since meaning, motivation, and curiosity are taken away from them, not to mention the reasons why other definitions or attempts failed.

A. How Is the Laplace Transform Method Introduced to the Students

A brief description of a teaching practice in the Laplace transform method will be presented here. The teaching method is currently used by the first author of this article in an introductory course in modelling and simulation of dynamic systems, given to third year students in Sustainable Design at Aalborg University, Copenhagen, Denmark. The teaching is partially based on the didactic paradigm called *questioning the world* [22], where the instructor's role is like that of a supervisor of scientific research, rather than the still dominant paradigm of *visiting monuments*. This *didactic contract* is therefore different from traditional teaching in an engineering or mathematics classroom: the students should be given the opportunity to *experience* the reasoning behind the Laplace transform *before* it is *abstracted*, and *motivate* them to study it.

Two teaching sessions, of approximately three hours each, are allocated to the Laplace transform method in the course: the first one provides a thorough introduction to the Laplace transform method, while the second session is reserved to exercises in transforming differential equations into the Laplace domain *by hand* and *using* MATLAB[1] to obtain the solution in the time domain. The first author starts the first teaching session by writing the following equation on the board

$$y''(t) + 3y'(t) + 2y(t) = 1, \quad y(0) = 0, \quad y'(0) = 0 \tag{4}$$

and ask the students the following questions:

1) What dynamic system could the equation model?
2) What does the time function $y(t)$ represent in the system you chose?
3) What does the number 1 stand for in your system?

The students investigate these questions by any means available, e.g. books, notes, computers and the web. Through these questions, it may be clear that the purpose is to embed the intended teaching situation in the Laplace transform method within an engineering context: the equation describes the same *dynamics* of many kinds of engineering systems, such as a mechanical system that consists of a mass, damper, and a spring, or an RLC electric circuit, shown in Fig. 1, where $y(t)$ represents the charge on the capacitor C and the input voltage $e_{\text{in}}(t)$ is a *unit* voltage pulse for $t \geq 0$.

[1]MATLAB is a programming language and a numeric computing environment, developed by the software company MathWorks.

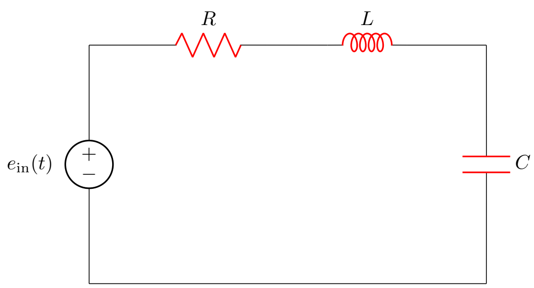

Fig. 1: An RLC circuit.

The answers of these questions give rise to a discussion in the class of the various systems that could be modelled by the equation. The following questions are then added to the previous list:

4) Consider a homogeneous linear second-order differential equation of your choice. Where does its characteristic equation come from?
5) What is the method of undetermined coefficients about?
6) What is the solution of equation (4)?

The rationale behind these questions is twofold:

- To prepare the students for another method of solving the differential equation, different from the one they have studied in their first year, so that they will, hopefully, relate their previous knowledge to the new method they will encounter.
- To give them an opportunity to work as research mathematicians, like, for example Laplace, in his search for solutions of differential equations. For example, seeking a solution of the form $y = e^{rt}$ will lead to the characteristic equation of a homogeneous linear second-order differential equation.

The next step is to prepare the students to the introduction of the Laplace transform method, by giving them the following task:

7) Multiply each side of equation (4) by e^{-st} and integrate from $t = 0$ to $t = \infty$. What kind of function do you get?

It is the authors' experience that many students struggle to evaluate integrals like $\int_0^\infty y'(t)e^{-st}dt$. Due to the time constraint of the teaching session, the authors reason that it would be pedagogically sound to intervene rather than leaving these students frustrated for a long time. The students are reminded of the *method of integration by parts*, and asked to search for worked examples of this method. In this regard, a related question is asked:

8) If you know the result of $\int_0^\infty y'(t)e^{-st}dt$, can you find $\int_0^\infty y''(t)e^{-st}dt$ by inspection?

Here, the aim is to lead the students to an important *discovery*: the second derivative in the time domain *corresponds* to the square of the variable s. This is in fact related to the students' previous knowledge about characteristic equations, where the "square" corresponds to the second derivative.

It is clear that these last two questions are more *closed* than the previous ones, in the sense that they are *directing* the students towards a specific *target* knowledge. The purpose here is to let the students acquire a *specific piece* of knowledge, namely the Laplace transform method, and not just certain modes of work and thought. Moreover, one cannot afford to discard the knowledge accumulated by Laplace, Heaviside and others, and require the students to reconstruct it anew.

Through *interactive* and CAS-assisted lecturing, the students are guided to recover the time function $y(t)$ from the expression they find, using MATLAB.

Finally, the culmination of these questions involves the *institutionalisation* of the method of the Laplace transform to solve linear differential equations: the students are confronted with the formal definition of the Laplace transform, its properties and its use, as given in their textbook. The session ended by asking the students whether they have met an analogous process in their upper secondary mathematics. It is of course the process of taking the logarithms of both sides of an exponential equation, that is meant here.

B. An Anthropological Analysis of the Laplace Transform Method

The study of the life of the theory of Laplace transform at an engineering department requires the identification of the praxeology which brings it into play. At the undergraduate engineering level, the task block T is usually "solve a linear differential equation by the Laplace transform method". Here, an essential role is played by the technique τ: it consists of transforming the equation into the Laplace domain first, using routine sub-tasks, and then transforming the Laplace function found back into the time domain, either by using partial fraction decomposition and a Laplace table, or by letting a CAS tool do the work and get an answer. In either case, techniques may not have only a pragmatic value which permits them to produce results, they may also have an *epistemological* value, since they can be a source of new questions about mathematical knowledge, and thus may contribute to the understanding of the objects they involve.

The technology θ, being a discourse on the *praxis*, consists of the general properties of Laplace transforms that explain the transformed equation, as well as the Laplace transform tables, as it is precisely these tables that justify the technique of partial fraction decomposition and validate the answers found: the aim is to match the Laplace function found with an entry in these tables in order to find the final solution in the time domain.

Regarding the theory component Θ, it constitutes a deeper level of justification of how the Laplace tables themselves are compiled: there is *essentially* a one-to-one correspondence between time-domain functions and their Laplace transforms [23], and the following result provides the *raison d'ˆetre* or the *rationale* of the technology θ:

The time-domain function $f(t)$ can be *recovered* by using the inverse-transform integral

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_\Gamma F(s)e^{st}ds \quad , \quad t \geq 0 \qquad (5)$$

where Γ is a contour within the region of convergence that goes from $s = \sigma - j\infty$ to $s = \sigma + j\infty$. The evaluation of this integral involves a knowledge of complex-variable theory [24] and is *beyond the reach* of undergraduate engineering students.

Thus, the theory component Θ is *missing* from the praxeology quadruple (T, τ, θ, Θ) in the sense that, even if equation (5) appears in a textbook, the students are not required to use it or even understand it: the final justification of the techniques is given by the Laplace tables, not by the theory behind them.

The instructor, being the director of the didactic situation in the classroom, is certainly affected by this mathematical constraint: It significantly determines her/his practice and ultimately the praxeology *actually taught*. The instructor is therefore restricted to design tasks whose techniques are solely justified by the *Laplace transform pairs* she/he gives to the students. This issue is illustrated by reconsidering the task of solving equation (4):

Find the solution of the initial-value problem

$$y''(t) + 3y'(t) + 2y(t) = 1, \quad y(0) = 0, \quad y'(0) = 0 \tag{6}$$

using the Laplace transform.

The Laplace transform of the equation is given by

$$s^2Y(s)) + 3sY(s) + 2Y(s) = \frac{1}{s} \tag{7}$$

Solving for $Y(s)$, one obtains

$$Y(s) = \frac{1}{s(s^2+3s+2)} \tag{8}$$

Showing the Laplace transform in action, this paper-and-pencil technique clearly has an *epistemological* value, since it is linked to knowledge of the linearity property of the Laplace transform, as well as its basic merit, which is, converting a differential equation into an algebraic one. In fact, the conceptual value attained in taking the Laplace transform of a differential equation is very similar to that of taking the logarithms of both sides of an exponential equation. For example, a common task in upper secondary mathematics is solving an exponential equation [9], such as

$$2^x = 5 \tag{9}$$

whose *logarithmic transform*

$$x = \frac{\ln(5)}{\ln(2)} \tag{10}$$

illustrates the concept that exponentiation and taking logarithms are inverse operations. Therefore, the purpose of both the Laplace transform and the *logarithm transform* is to make tedious calculations easier and they are quite analogues procedures: the Laplace transform converts a differential equation into an algebraic equation, while the logarithmic transform converts hard multiplication into easy addition:

$$\ln(ab) = \ln(a) + \ln(b) \tag{11}$$

Returning to the original task, in order to find the inverse Laplace transform $\mathcal{L}^{-1}\{Y(s)\} = y(t)$, the instructor *expects* that the students perform an algebraic manipulation in the form of partial fraction decomposition:

$$\frac{1}{s(s^2+3s+2)} = \frac{1}{s(s+1)(s+2)} \tag{12}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \tag{13}$$

After some manipulation, the constants A, B, and C are determined, yielding the expression

$$Y(s) = \frac{\frac{1}{2}}{s} + \frac{-1}{s+1} + \frac{\frac{1}{2}}{s+2} \tag{14}$$

Using a table of Laplace transforms, one finally get the solution in the time domain:

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \tag{15}$$

It should be noted that, partial fraction decomposition becomes more tedious and painstaking, if the denominator of the Laplace function includes complex or repeated poles. However, the manipulation itself can have an epistemological value in *another* mathematical topic, namely, the knowledge of elementary algebraic properties and manipulations of expressions, but it retains *little* conceptual value for the Laplace transform method itself: what is being done here is engaging the student to apply a repetitive computational process, like a computer does, in order to match an entry in a Laplace table. If the student gets the correct answer, all what the instructors have done, would be a verification that they have trained a well-prepared student to be replaced by a computer.

C. Using a CAS Environment in the Laplace Transform Method

Many modern computer software are available to find the inverse Laplace transform of equation (8). As mentioned before, the first author uses MATLAB in the course. MATLAB is one of the most popular CAS programs that is used in mathematics and engineering classrooms at many universities worldwide [25]. In MATLAB, the inverse Laplace transform can be found using the following code:

```
%% Inverse Laplace transform in MATLAB
% Define the symbols
syms s t
% Define the Laplace function
Y=1/(s*(s^2+3*s+2))
% Find the time-domain function
y=ilaplace(Y)
pretty(y)
```

In MATLAB Command Window, the solution (15) is shown in the following form:

$$\frac{\exp(-2t)}{2} - \exp(-t) + \frac{1}{2} \tag{16}$$

The use of a CAS tool for this task makes it possible for the students to do *more* exercises, where the instructor can orient the activity towards *pattern discovery*, for instance recognising that $\frac{1}{s+2}$ gives the decaying exponential function e^{-2t} in the time domain, while $\frac{1}{s-2}$ gives the increasing exponential function e^{2t}. Similarly, the students can alter the values of a and b in the Laplace function $\frac{b}{(s-a)^2+b^2}$ to gain insight into the behaviour of the corresponding time-function $e^{at} \cdot \sin(bt)$: for example, if a is negative, the time function is an exponentially decreasing sinusoid, and for increasing values of b, the *damped* period of the sinusoid decreases.

Through this pattern discovery in the Laplace transform theory, the students can develop a *bridge* between skills and understanding, but also between the technique τ and the technology θ in the Laplace praxeology $(T, \tau, \theta, \cdots)$, where, as mentioned, the theory Θ is missing. Therefore, pencil-and-paper techniques in evaluating the inverse Laplace transform are *challenged* by a CAS tool, just like, for example, in finding the solution of the exponential equation (9): using a table of logarithms to find $\ln(2)$ and $\ln(5)$ is nowadays almost obsolete, since a CAS tool, such as GeoGebra, can find the *numerical solution* in just one operation:

$$x = \frac{\ln(5)}{\ln(2)} \approx 2.32 \tag{17}$$

It is interesting to note that the *modern* theory behind the compilation of a table of logarithms is also beyond the reach of upper secondary school students, since a knowledge of, for example, Maclaurin series is required:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{18}$$

where its interval of convergence is $-1 < x \leq 1$. Thus, the logarithm praxeology and the Laplace praxeology share the same structure $(T, \tau, \theta, \cdots)$, and are *analogous*. In fact, they are *epistemologically identical* in the sense that, in both, the technology θ depends on the use of either tables or CAS, and the theory Θ is absent. The question is why the Laplace praxeology is still lagging behind in the use of CAS. There may be many answers to this question, that can be addressed in future studies.

To shed more light on the issue, consider a related task from the topic of indefinite integrals. The first author asks the students to evaluate the following integral, using pencil and paper:

$$\int \frac{1}{x(x^2+3x+2)} dx \tag{19}$$

Again, the instructor anticipates that the students use partial fraction decomposition in order to match an entry in a table of indefinite integrals:

$$\int \frac{1}{x(x^2+3x+2)} dx \tag{20}$$

$$= \int \left(\frac{\frac{1}{2}}{x} + \frac{-1}{x+1} + \frac{\frac{1}{2}}{x+2} \right) dx \tag{21}$$

$$= \frac{1}{2} \int \frac{1}{x} dx - \int \frac{1}{x+1} dx + \frac{1}{2} \int \frac{1}{x+2} dx \tag{22}$$

It is only in the final step that the mathematical topic of integrals *retains* a conceptual value: an integral depends *linearly* on the integrand. The decomposition part itself does not belong to that topic and can be easily implemented in MATLAB:

```
%% Partial Fraction Decomposition in
   MATLAB
% Define the symbol
syms x
% Define the  function
f=1/(x*(x^2+3*x+2))
% Find the partial fractions
pretty(partfrac(f))
```

In MATLAB Command Window, the decomposition is shown in the following form:

$$\frac{1}{2(x+2)} - \frac{1}{x+1} + \frac{1}{2x} \tag{23}$$

Thus, even though the use of CAS requires *new techniques*, dependent on the tool, it can open up *systematic* conceptual structures more directly and quickly than the old techniques, whose *routinisation* in an *institution* is no longer necessary.

VI. Discussion and Conclusion

The impact of computers on teaching and learning activities at all levels of education is huge, and the extent of use increases as computers become more user-friendly and less expensive to buy and maintain. Using computer software, one can easily find the inverse Laplace transform of an expression, without the need to simplify in order to use the Laplace transform tables. Therefore, engineering textbooks on the subject ought to cover the use of CAS tools in their introduction to the Laplace transform method.

The authors believe that the difficulties which engineering students encounter in understanding the Laplace transform are mainly due to two *practices* in the traditional teaching of the subject:

- Training the students to apply a repetitive computational process, like a computer does, such as finding the inverse Laplace transform of an expression by hand in order to strike an entry in a Laplace transform table: for many engineering students, this is an uninteresting mechanical process to be mastered, just like struggling to evaluate an indefinite integral by hand.
- Introducing the Laplace transform by starting with its standard mathematical definition, as it is in textbooks: it may not make sense to many engineering students, since meaning and motivation are taken away from them, not to mention the reason why the Laplace transform is defined as such [26], and on top of that, the connections among concepts, formal representations, and the real world are often lacking in traditional instruction [27].

Therefore, the ability of many modern CAS tools to perform Laplace transforms and inverse Laplace transforms renders traditional Laplace tables obsolete.

Even the pragmatic and epistemological value of the Laplace transform *method* itself should be *subject to revision*, since this method is challenged by commands and mouse clicks in a computer software, that can even tackle more complex differential equations. The authors are here questioning the *relevance* of the Laplace transform *method* to solve a *narrow* type of differential equations in engineering education, given the fact that there are well-established time-domain techniques to solve such equations, beside the use of CAS. It is not, however, the intention of the authors to undermine the importance of the Laplace transform *theory* in mathematics, physics, and engineering. On the contrary, the Laplace transform is one of the essential tools used by scientists and researchers in finding the solution to their problems. In fact, the authors of the paper [28] reviewed 25 research papers in various disciplines and discussed how the Laplace transform was used to solve some research problems.

We suggest that engineering education reconsiders the study of domains and acknowledges new techniques as components of new praxeologies for these domains, and recognises that CAS tools can contribute to the students' conceptual understanding.

The dichotomy between a procedural approach and a conceptual approach to mathematics is ancient, especially in the field of algebra. For instance, teachers in the USA, even in the year 1890, were opposed to what they call an overemphasis on manipulative skills, and called for a meaningful treatment of algebra that would bring about more conceptual understanding [29]. Thus, in line with [30], when mathematical knowledge, such as the Laplace transform theory, is being *re-contextualised* to engineering science courses, a conceptual approach to mathematics is more essential than a procedural approach. For example, [31] argues that reducing engineering mathematics to procedural and algorithmic skills may obscure the role that mathematical thinking plays in engineering practices. Similarly, the employers of engineers, who are interviewed by Kent and Noss [32], put more emphasis on a *holistic* awareness of the mathematical needs for engineering work than on manipulative skills.

Thus, in light of the development of computational technologies, a need of a broader spectrum of mathematical skills for practising engineers, including conceptual understanding, rather than a narrow focus on procedural skills, is required. A concept-based instruction in undergraduate engineering mathematics education can develop conceptual knowledge, without losing out on the procedural skills.

Finally, this paper does not recommend that the students should only use CAS in their mathematics or engineering science courses. Rather, it is a call for a *new balance* in engineering education, where a CAS-assisted teaching environment is integrated with paper-and-pencil techniques so that the students get the "best of both worlds".

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