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TRACE SPACES OF DIRECTED TORI WITH RECTANGULAR HOLES.

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Abstract. In [4] the trace space of parallel non-looped, non-branching processes is given as a prod-simplicial complex derived from an index category. For looped processes, the state space is a torus and the trace space is a disjoint union of tracespaces of deloopings. The index category for the trace space of the deloopings is developed from the once delooped case. When just one process is looped, the index category is generated as words in a regular language. The automaton is constructed.

1. Introduction

A simple model of concurrency is provided by Dijkstra [1]. A set of processes interact via shared objects. These objects are guarded by semaphores allowing only a certain number of processes access at the same time. The geometric model is a product of directed graphs with "holes" corresponding to the restrictions on the shared objects.
An execution is a directed path, non-decreasing in each coordinate, from an initial point to a final point.
In [4], the special case of non-looped, non-branching processes is studied. There is an algorithm for calculating the space of directed traces from an initial point to a final point. The result is a prod-simplicial complex, which is generated from an index category.
For processes with loops, the directed traces from the initial to the final point may be considered as traces in deloopings, i.e., in non-looped space. The total trace space is a disjoint union of the trace spaces of the deloopings, 2.9. And the algorithm from [4] applies to each delooping. However, there is an obvious periodicity arising from the loops, and it is this periodicity, which is made clear here.

We restrict to the case of n processes, P_j $j \in [1:n]$, where each $P_j = T_j^*$ is a loop. The general case will be developed along the same lines in another paper. Thm. 4.15 gives the index category of a delooping $T_1^{m_1}|T_2^{m_2}\dots|T_n^{m_n}$ in terms of objects in the index category of $T_1|T_2\dots|T_n$, the one time delooping of each process. And a deadlock check in the corresponding space, a cube minus hyperrectangles. In particular, when $m_r = m$ and $m_i = 1$ for $i \neq r$, the index category is words in a language. We give an alphabet consisting of allowed

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schedule restrictions, objects in the index category for $T_1|T_2\dots|T_n$. Words are concatenations, corresponding to further delooping of T_r with the concatenated schedule. Not all words are allowed, i.e., they may not support any executions. The allowed words form a *regular language*.

By Th. 5.6, the language is recognized by an automaton with at most $(2^n - 1)^l + 1$ states, where l is the number of shared objects.

2. Processes with loops

The geometric model for n looped processes in parallel is a torus with rectangular holes.

Definition 2.1. *Notation*

- For integers $l \leq m$, $[l : m] = \{n \in \mathbb{Z} | l \leq n \leq m\}$.
- For integer vectors $V = (v_1, \dots, v_n)$ and $W = (w_1, \dots, w_n)$, let $[V : W] = \{K = (k_1, \dots, k_n) | k_i \in [v_i : w_i]\}$.
- e_j is the j'th unit vector.
- For points $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$, define the closed n-interval $[a, b] = \{x \in \mathbb{R}^n | (a_i \leq x_i \leq b_i\}$, and similarly $]a, b[$, $[a, b[$ and $]a, b]$.

Definition 2.2. A d-space is a topological space Y and a subset, the dipaths, $\vec{P} \subset Y^I$ s.t

- All constant paths are in $\vec{P}$
- If $\mu, \nu \in \vec{P}$, $\nu(1) = \mu(0)$, then the concatenation $\mu \star \nu \in \vec{P}$
- For $\alpha : I \to I$ non-decreasing and $\mu \in \vec{P}$, $\mu \circ \alpha \in \vec{P}$

$\vec{I}$ is the d-space I with $\vec{P}$ the non-decrasing maps.

$\vec{I}^n$ is the n-cube with coordinate wise non-decreasing dipaths.

For a vector $M = (m_1, \dots, m_n)$, let $\vec{I}^M = \times_{i=1}^n [0, m_i]$ with coordinate wise ordering.

Definition 2.3. For $p, q \in Y$ and Y a d-space, the space of dipaths $\vec{P}(Y, p, q)$ is the set of dipaths initiating in p and ending in q. The topology is the compact open topology. The trace space $\vec{T}(Y, p, q)$ is the quotient of the path space under reparametrization, see [5].

Remark 2.4. In the following, d-spaces are state spaces and dipaths are (partial) execution paths.

Definition 2.5. The geometric model of n non-looped processes, T_j, $j \in [1 : n]$ in parallel with conflicts at l shared objects: The *state space* is $X = \vec{I}^n \setminus F$, where $F = \cup_{i=1}^l R^i$, $R^i =]a^i, b^i[$ and ${a^i}_j, {b^i}_j \in]0, 1[$ for all $i \in [1 : l]$ and all $j \in [1 : n]$.

We keep the setup, except each process T_j is now a loop:

Definition 2.6. Let $X = \vec{I}^n \setminus F$, where $F = \cup_{i=1}^{l} R^i$ as above. Consider n parallel processes T_j^*, i.e., each of them a loop, conflicts as above. The state space is the torus $X/\sim$. Here $(x_1, \ldots, x_{i-1}, 0, x_{i+1} \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, 1, x_{i+1} \ldots, x_n)$, and $X/\sim$ has the quotient topology.
A d-path is a continuous path $\gamma : I \to X/\sim$ s.t. γ is *locally non-dencreasing* in the following sense:
For each coordinate $\gamma_i : I \to \vec{I}/(0 \sim 1)$, let $\gamma_i^{-1}(0) = \cup_{j=0}^{m_i}[t_j^i, s_j^i]$, where $t_0^i = 0$ and $s_{m_i}^i = 1$. Then γ is locally non-decreasing, if for all i, k the restriction $\gamma_i :]s_k^i, t_{k+1}^i[\to \vec{I}$ is order preserving. A d-path is an *execution* if $\gamma(0) = 0$, $\gamma(1) = 1$.

Definition 2.7. With notation from above, for a vector $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$ the state space of the M delooping X^M of $X/\sim$ is the cube $\vec{I}^M \setminus F^M$ where $F^M = \cup\{R^{i,K} | i \in [1 : l], K \in [0 : M-1]\}$, where $(M-1)_i = m_i - 1$ and $R^{i,K} =]a^i + K, b^i + K[$.
Executions are order preserving paths $\gamma : \vec{I} \to X^M$ with $\gamma(0) = 0, \gamma(1) = 1$
. The projection $\Pi_M : X^M \to X/\sim$ is $\Pi_M(y_1, \ldots, y_n) = (\bar{y}_1, \ldots, \bar{y}_n)$, where $\bar{y}$ is the fractional part of y.

Lemma 2.8. *The projection $\Pi_M : X^M \to X/\sim$ is continuous and maps execution paths to execution paths.*

Proof. For each coordinate, this is the covering map $t \to e^{i2\pi t}$ from $\mathbb{R}$ to S^1, so Π_M is a restriction of the covering map from $\mathbb{R}^n$ to the torus. Hence, it is continuous. A directed path $\gamma : \vec{I} \to X^M$ is increasing in each coordinate. $(\Pi_M \circ \gamma)_j^{-1}(1) = \gamma_j^{-1}(\mathbb{N})$. On a connected component C of $\vec{I} \setminus \gamma_j^{-1}(\mathbb{N})$, the integer component of $\gamma_j(t)$ is constant. Hence, for $t_1, t_2 \in C$ $\gamma_j(t_1) \leq \gamma_j(t_2)$ if and only if this holds for the fractional part. □

Proposition 2.9. *The trace space $\vec{T}(X/\sim)(0,1)$ is a disjoint union of trace spaces of the deloopings $\bigsqcup_{M \in \mathbb{N}^n} \Pi_M(\vec{T}(X^M)(0, M))$.*

Proof. Let $\gamma : \vec{I} \to X/\sim$, $\gamma(0) = 0 = \gamma(1)$.
Let $i : X/\sim \to I^n/\sim$ be the inclusion. $i \circ \gamma$ represents a class $[\gamma] \in \pi_1(I^n/\sim)$ Let the directed loops $\mu_r(t) = te_r$ be generators of $\pi_1(I^n/\sim)$ and let $M = (m_1, \ldots, m_n) \in \oplus_{r=1}^n \mathbb{Z}$ be the homotopy class represented by γ. Since γ is directed, all m_i are positive and there is a dipath $\bar{\gamma} : \vec{I} \to X^M$, s.t., $\pi \circ \bar{\gamma} = \gamma$. I.e., M is the index of the corresponding delooping.

If $\mu : \vec{I} \to X/\sim$ is dihomotopic to γ, then $i \circ \mu$ is homotopic to $i \circ \gamma$, and the delooping corresponding to μ is X^M. Hence, trace spaces corresponding to different deloopings are disconnected.

□

3. Traces and delooping

In [4], trace spaces in the non-looped case are given as the geometric realization of a prod-simplicial complex. This provides a description of $\vec{T}(X^M, 0, M)$, and there is an algorithm for determining this prodsimplicial complex and calculate topological invariants of the trace space. However, in the case of a delooping, there is an obvious periodicity. This may be exploited to simplify the algorithm in this case. The reader is referred to [4] for a full definition of the prod-simplicial complex. Here, we focus on the index category and periodicity.

Definition 3.1. A *schedule restriction* or *schedule* for $X = I^n \setminus F$, $F = \cup_{i=1}^{l} R^i$ is a (set of) vectors $J = (j_1 \dots, j_l)$, $j_i \in [1 : n]$. The restricted space X_J is $X \setminus F_J$, where $F_J = \cup_{i=1}^{l} R^i_{j_i}$ and $R^i_{j_i} = [0, b^i_1[\times \dots [0, b^i_{j_i-1}[\times]a^i_{j_i}, b^i_{j_i}[\times[0, b^i_{j_i+1}[\times \dots \times [0, b^i_n[$, the rectangle R^i extended to the lower boundary along all coordinates except the $j_i' th$.
For a set of vectors $J = \{J1, \dots Jr\}$, the restricted space is $X_J = X_{J1} \cap X_{J2} \cap \dots \cap X_{Jr}$ This is defined similarly for deloopings $X^M = \vec{I}^M \setminus F^M$.

Remark 3.2. As in [4], observe that $X_J = X \setminus F_J$ where $F_J = \cup_{i=1}^{r} F_{Ji}$ and

$$\cup_{i=1}^{r} F_{Ji} = \cup_{i=1}^{l} \cup_{k=1}^{r} R^k_{(Ji)_k}$$

Hence, the l-tuple of sets, $(J_1, \dots, J_l)$ $J_k = \{(Ji)_k | i = 1, \cdot, r]\}$ defines X_J. Since such an l-tuple represents several J, the l-tuples of sets give a smaller indexing category. In the following *a schedule restriction is such an l-tuple* $(J_1, \dots, J_l)$, where $J_i \subset [1 : n]$ and $J_i \neq \emptyset$. i.e., all rectangles are extended.

Proposition 3.3. *$\vec{T}(X_J)(0, 1)$ is either empty or contractible.*

Proof. This is [4] Prop 2.8 (2) □

The index category defining the prod-simplicial set is

Definition 3.4. Let $\mathcal{C}(X^M)$ be the poset category with objects all non-empty $\vec{T}(X^M_J)(0, M)$ for $J = \{J_{i,K} | i \in [1 : l], K \in [0, M-1]\}$, a *non-empty* subset $J_{i,K} \subset [1 : n]$ for each of the $l|M|$ forbidden cubes. The partial order, the morphisms in the poset category, $J \leq J'$ if $J_{i,K} \subset J'_{i,K}$ for all i, K. Hence, $J \leq J'$, if $X^M_J \supseteq X^M_{J'}$

In [4], a pasting scheme is given to build the trace space $\vec{T}(X^M)(0, M)$ as a prod-simplicial complex from this poset category. An element J is represented very efficiently as an $l|M| \times n$ binary matrix B_J, where $B_{st} = 1$ if $t \in J_{f(s)}$, where $f : [1 : l|M|] \to \{(i, K) i \in [1 : l], K \in [0, M-1]\}$ is a fixed bijection. Moreover, by [4], section 4, $\vec{T}(X^M_J)(0, M)$ is empty if and only if the extended rectangles intersect to give a *deadlock*. $p \in X^M_J$ is a deadlock if all dipaths initiating in p are empty. Hence, p is the infimum of the intersection of n extended rectangles. Since rectangles producing the deadlock are extended to 0 in all directions except

one, all points in the hyperrectangle $[0,p]$ are *unsafe* in the sense that no dipath initiating there will leave the hyperrectangle and in particular, they will not reach 1. The hyperrectangle $[0,p]$ is the *unsafe region* associated to the deadlock p. For more on deadlocks and unsafe areas, see [4] and [2].

For large M, this gets out of hand. In the following, the obvious periodicity in a delooping is explored.

4. Periodicity in the trace algorithm

Definition 4.1. Given a schedule $J = \{J_{i,K} | i \in [1:l], K \in [[0, M-1]\}$ for a delooping X^M and $K_0 \in [0:M-1]$. The restriction $J|_{K_0}$ is the schedule $\{J_1, \ldots, J_r\}$ for X^1 with $J_i = J_{i,K_0}$ for $i \in [1,l]$. $R^{i,K}$, $i = 1, \ldots, l$

Remark 4.2. In $J|_{K_0}$, the rectangle R^i is extended according to the schedule of R^{i,K_0}, $i = 1, \ldots, l$. However, $X_{J|_{K_0}} \neq X_J^M \cap X^{K_0}$ since in the latter, rectangles $R^{i,K}$ with $K > K_0$ may be extended to intersect X^{K_0}.

Proposition 4.3. *If $\overrightarrow{Tr}(X_{J1}, 0, 1) = \emptyset$, and J is a schedule for X^M such that $J|_K = J1$ for some $K \in [0:M-1]$, then $\vec{T}(X_J^M, 0, M) = \emptyset$*

Proof. Since $\vec{T}(X_{J1}, 0, 1) = \emptyset$, there is a deadlock in X_{J1} which is the intersection of (at least) n extended rectangles $R^r_{J1_r}$. Inserting X_{J1} in X^M, the schedule requirement causes the rectangle R^r to be extended to a product of intervals $]a^r_{J1_r}, b^r_{J1_r}[$ and $[0, b^r_i]$ for $i \neq J1_r$, in X^M. Thus, $[0, b^r_i[$ is extension to the lower i'th boundary in X^M. The unsafe region of that deadlock will then contain the minimum, 0 of X^M □

The schedule restriction in X^M is given by schedule restrictions J for all $K \in [1:M]$, and Prop. 4.3 gives a necessary condition for, when J is allowed. It is not a sufficient condition:

Example 4.4. Let $X = \vec{I}^3 \setminus R$ where $R =]1/4, 3/4[\times]1/4, 3/4[\times]1/4, 3/4[$. Consider the three schedules $J = 1$, $J = 2$ and $J = 3$. They are all allowed - there are no deadlocks. In the following, these are the schedules considered. I.e., unions such as $J = \{1, 2\}$ are not considered.

Let $M = (3, 1, 1)$, then $X^M = [0,3]\times[0,1]\times[0,1]\setminus(R^1 \cup R^2 \cup R^3)$, where $R^1 = R$ $R^2 =]5/4, 7/4[\times]1/4, 3/4[\times]1/4, 3/4[$ and $R^3 =]9/4, 11/4[\times]1/4, 3/4[\times]1/4, 3/4[$.

There are 27 schedules for X^M, namely the vectors $v \in [(1,1,1) : (3,3,3)]$, where (v_1, v_2, v_3) assigns v_1 to R^1, v_2 to R^2 and v_3 to R^3.
There is no deadlock if $v_i = v_j$ and $j \neq i$, since the corresponding extended rectangles intersect trivially - one is contained in the other.
The 6 remaining schedules are $(1,2,3)$,$(1,3,2)$,$(2,1,3)$,$(2,3,1)$, $(3,1,2)$ and $(3,2,1)$.

- $(1,2,3)$ and $(1,3,2)$ give a deadlock at $(1/4, 1/4, 1/4)$:
  $R^1_1 =]1/4, 3/4[\times[0, 3/4[\times[0, 3/4[$, $R^2_2 = [0, 7/4[\times]1/4, 3/4[\times]0, 3/4[$ and $R^3_3 = [0, 11/4[\times[0, 3/4[\times]1/4, 3/4[$. Hence, $(1/4, 1/4, 1/4) \in X^M_{(1,2,3)}$ and it

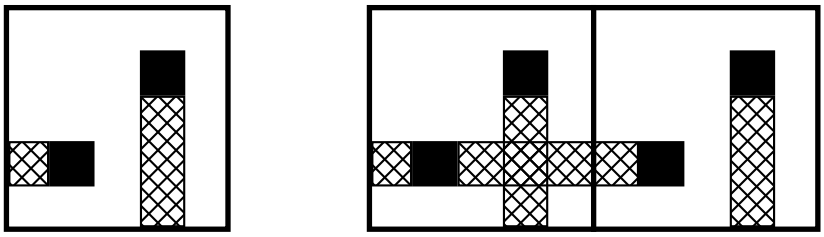

FIGURE 1. Not all schedules can be repeated.

is a deadlock. Moreover, $(0,0,0)$ is unsafe, as may be seen by inspection here, or in general by the deadlock algorithm. Similarly for $(1,3,2)$.

- $(2,1,3)$, $(2,3,1)$, $(3,1,2)$ and $(3,2,1)$ do not produce deadlocks. Let $v = (2,1,3)$. $R^1_2 = [0,3/4[\times]1/4,3/4[\times[0,3/4[$, $R^2_1 =]5/4,7/4[\times[0,3/4[\times[0,3/4[$ and $R^3_3 = [0,11/4[\times[0,3/4[\times]1/4,3/4[$. Since $R^1_2 \cap R^2_1 = \emptyset$, there is no deadlock. Similarly for the other cases.

In the example, allowed schedules on X are combined to give a schedule on X^M. Rectangles are extended in the global setting X^M producing deadlocks "below". This explains why the condition in 4.3 is not sufficient. Notice also the non-commutativity, that $(1,2,3)$ gives a deadlock, but $(2,1,3)$ does not.

An allowed schedule may not be repeatable:

Example 4.5. Let $X = \vec{I}^2 \setminus F$, where $F = R^1 \cup R^2$ and $R^1 =]1/8,3/8[\times]1/8,3/8[$ and $R^2 =]5/8,7/8[\times]5/8,7/8[$, two diagonal holes. The schedule $J = (\{2\},\{1\})$ extends the forbidden area to $R^1_2 = [0,3/8[\times]1/8,3/8[$ and $R^2 =]5/8,7/8[\times[0,7/8[$. This is an allowed schedule. see Fig.1. But the concatenated schedule $(2,1,2,1)$ for the 4 forbidden rectangles in $[0,2]\times[0,1]$ has a deadlock at $(5/8,1/8)$.

Definition 4.6. The replacement operator ρ *replaces* symbols. E.g.

$$\rho(\prod_{j=1}^{n}[c_j,d_j];([c_k,d_k],[f,g])) =$$

$$[c_1,d_1]\times[c_2,d_2]\times\ldots[c_{k-1},d_{k-1}]\times[f,g]\times[c_{k+1},d_{k+1}]\times\ldots[c_n,d_n].$$

When a rectangle is extended to the maximal or minimal point, the replacement is denoted

$$\rho^k_m(\times^n_{j=1}]a_j,b_j[) := \rho(\times^n_{j=1}]a_j,b_j[,(]a_k,b_k[,]a_k,1])(]a_m,b_m[,[0,b_m[)$$

Iteratively, for subsets $K,M \in \{1,\ldots,n\}$ ρ^K_M is the replacement of $]a_k,b_k[$ by $[a_k,1[$ for all $k\in K$ and $]a_m,b_m[$ by $]0,b_m]$ for all $m\in M$. If $s\in M\cap K$ $]a_s,b_s[$ is replaced by $[0,1]$.

Example 4.7. When $Ji = \{j_i\}$, the i'th rectangle R^i is extended to $\rho_{\{j\neq j_i\}}(R^i)$.

Definition 4.8. *The downward effect*: Let S be the set of subsets of I^n and χ the set of pairs (F, J), where $F = \cup_{i=1}^{l} R^i$, R^i is an n-rectangle and $J = (J1, \dots, Jl)$ is a schedule restriction. Let $\emptyset \neq r \subsetneq [1:n]$. Somtimes r is denoted by a binary vector $r \in \{0,1\}^n$ via the correspondence $k \in r$ if and only if $r_k = 1$. The downward effect $D_r : \chi \to map(S, S)$ is defined:
Let $Y \subset I^n$, $J = (j_1, \dots, j_l)$ and X_J as in 3.1. Then

$$D_r(X_J)(Y) = Y \setminus \bigcup_{j_i \notin r} \rho^{\{k \in r\}}_{\{j \neq j_i\}} (\times_{j=1}^{n}]a^i_j, b^i_j[).$$

The rectangles removed are generalized cylinders:

$$Cyl(i, j_i, r) = \rho^{k \in r}_{\{j \neq j_i\}} (\times_{j=1}^{n}]a^i_j, b^i_j[) = [w, v]$$

where $w_{j_i} = a^i_{j_i}$ and $w_j = 0$ otherwise. $v_k = 1$ for $k \in r$ and $v_j = b^i_j$ otherwise. In other words.

$$Cyl(i, j_i, r) = \times_{j=1}^{n} S_j$$

where $S_k = [0, 1]$ for $k \in r$, $S_{j_i} =]a^i_{j_i}, b^i_{j_i}[$ and $S_j = [0, b^i_j[$ otherwise.
$D_r(X_J)(Y)$ is the *downwards effect from direction* r of X_J on Y.
For a string $X_{J1} X_{J2} \cdots X_{Jk}$, the combined down effect is the composition: $D_r(X_{J1} X_{J2} \cdots X_{Jk})(Y) = D_r(X_{J1}) \circ D_r(X_{J2}) \circ \cdot \circ D_r(X_{Jk})(Y)$

Remark 4.9. The cylinders are extensions of the extended rectangles;

$$Cyl(i, j_i, r) = \rho^{\{k \in r\}}(R^i_{j_i}).$$

If $X = I^n \setminus F$, F a union of rectangles and J a schedule, $D(X_J)$ means $D(F, J)$.

The combined down effects is the effect of a union of schedule restrictions:

Proposition 4.10. *The combined down effect of $X_{J1} \dots X_{Js}$ is the effect of the union of schedules: $D_r(X_{J1} X_{J2} \dots X_{Js}) = D_r(X_{J1 \vee J2 \dots \vee Js})$ were $(J_1, \dots J_l) \vee (J'_1, \dots, J'_l) = (J_1 \cup J'_1, \dots, J_l \vee J'_l)$ is the least upper bound in the poset $\mathcal{C}(X)$*

Proof. Let $Y \subset I^n$. $D_r(X_{J1} X_{J2} \dots X_{Js})(Y) = Y \setminus F$ where

$$F = \cup \{Cyl(i, j_i, r) | j \neq j_i, j_i \notin r, j_i \in Jk_i \text{ for some } k \in [1:s]\}$$

Now $j_i \in Jk_i$ for some $k \in [1:s]$ if and only if $j_i \in (J1 \vee J2 \dots \vee Js)_i$. □

Lemma 4.11. *Let X^M be the M delooping and let J be a schedule restriction. Suppose $J_{i,K} = \{\alpha\}$ is a schedule restriction for $R^{i,K}$. Then $R^{i,K}_\alpha \cap I^n$ is empty if $k_\alpha \neq 0$. If $k_\alpha = 0$, then $I^n \setminus R^{i,K}_\alpha = D_{r(K)}(J_{i,K})(I^n)$, where $r(K)_j = \inf\{K_j, 1\}$.*

Proof. The extended rectangle is

$$R^{i,K}_\alpha = \times_{j=1}^{n} S_j.$$

where $S_\alpha =]a^i_\alpha + k_\alpha, b^i_\alpha + k_\alpha[$ and $S_j = [0, b^i_\alpha + k_\alpha[$ else. The intersection is $R^{i,K}_{j_0} \cap I^n = \times_{j=1}^{n} (S_j \cap [0, 1])$. If $k_\alpha \neq 0$, then $S_\alpha \cap [0, 1] = \emptyset$ so the intersection is empty.

If $k_\alpha = 0$, then

- $S_\alpha \cap [0,1] = S_\alpha$,
- $S_j \cap [0,1] = [0, b^i_j[$ if $k_j = 0$, i.e.,if $r(K) = 0$ (and $j \neq \alpha$)
- $S_j \cap [0,1] = [0,1]$ if $k_j \geq 1$, i.e., if $r(K)_j = 1$.

Hence, $R^{i,K}_\alpha \cap I^n = Cyl(i, \alpha, r(K)$. □

Similarly, if $J_{i,K}$ has more than one element, there are several extended rectangles. Then $I^n \setminus \cup_{\alpha \in J_{i,K}} R^{i,K}_\alpha$ is $D_{r(K)}(J_{i,K})(I^n)$

Example 4.12. When r has just one element, there is a *Concatenation* of X_{J1} and X_{J2} along the r'th direction: Let $r \in \{1, \cdots, n\}$, then the subset $X_{J1} \star_r X_{J2} \subset I^v$, where $v \in \mathbb{R}^n$, $v_r = 2$ and $v_j = 1$ else, is defined by $x = (x_1, \ldots, x_n) \in X_{J1} \star_r X_{J2}$ if either $x_r \geq 1$ and $x - e_r \in X_{J2}$ or
$x_r \leq 1$ and $x \in D_r(X_{J2})(X_{J1})$.
Iteratively, $X_{J1} \star_r X_{J2} \star_r \cdots \star_r X_{Jk} \subset I^w$, $w_r = k$ and $w_j = 1$ otherwise. It is defined by $x \in X_{J1} \star_r X_{J2} \star_r \cdots \star_r X_{Jk}$ if $x_r \geq 1$ and $x - e_r \in X_{J2} \star_r X_{J3} \star_r \cdots \star_r X_{Jk}$ or $x_r \leq 1$ and $x \in D_r(X_{J2} \star_r X_{J3} \star_r \cdots \star_r X_{Jk})(X_{J1})$

Remark 4.13. The concatenated object $X_{J1} \star_r X_{J2}$ may also be obtained by gluing X to X along the r'th face, enumerating the now $2l$ forbidden rectangles consecutively - $R^1, \ldots, R^l$ as before, in the lower copy of X (with $0 \leq x_r \leq 1$) and R^{l+m}, the copy of R^m in the upper copy of X (where $1 \leq x_r \leq 2$). Then $X_{J1} \star_r X_{J2}$ is obtained by extending rectangles according to the schedule restriction $J = (j_{11}, \ldots, j_{1l}, j_{21} \ldots, j_{2l})$

Definition 4.14. For a delooping X^M $M = (m_1, \ldots, m_n)$ and $K \in [0 : M-1]$, X_{JK} denotes X with extensions according to the restriction $J|_K$. The *combined down effect* on X_{JK_0} is the composition

$$D(X^M_J, K_0)(X_{JK_0}) = \circ_{K \in]K_0, M]} D_{r(K-K_0)}(X_{JK})(X_{JK_0})$$

Here $r(K - K_0)$ is the binary vector $r(K - K_0)_k = \inf((K - K_0)_k, 1)$.

Theorem 4.15. *Let X^M be the delooping given by $M = (m_1, \ldots, m_n)$. A schedule J for X^M is allowed if and only if for all $K \in [0 : M-1]$*

- *If p is a deadlock in $D(X^M_J, K)(X_{JK})$, then $p_k = 1$ for some k.*

Proof. J is allowed if and only if there are no deadlocks on intersections of extended rectangles. Such deadlocks have coordinates $(a^{i_1}_1 + k_1, \ldots, a^{i_n}_n + k_n)$ where $k_i \in \mathbb{N}$ and $a_j \in]0,1[$.
As a consequence, if p is a deadlock X^M and $p \in X^M \cap X_K$, then it is in the interior of X_K.
By Lem. 4.11 extended rectangles $R^{i,K}_j$ will intersect X_{K_0} only if $(K - K_0)_j = 0$ and the intersection with X_{K_0} is $D_{r(K-K_0)}(R^i_j)(X_{K_0})$. Therefore, p corresponds to an interior deadlock in $D(X^M)(X_{JK})$.
If q is a deadlock in $D(X^M)(X_{JK})$ with $q_k = 1$, the corresponding point $\tilde{q} = q + K$ is not a deadlock in X^M, since it has an integer coordinate. □

Remark 4.16. In particular, as noted before, all X_{JK} should be allowed.

Corollary 4.17. *A schedule J for X^M is allowed if and only if for all $K \in [0 : M-1]$ There is a k s.t. $\vec{T}(D(X^M_J, K)(X_{JK}), 0, \partial^+_k) \neq \emptyset$, where $\partial^+_k = \{x \in D(X^M_J, K)(X_{JK}) | x_k = 1\}$*

Proof. If there is a dipath to ∂^+_k, then 0 is not in the unsafe area for an interior deadlock, and hence there are no such deadlocks. If there is an interior deadlock, then 0 is in the associated unsafe area and $\vec{T}(D(X^M_J, K)(X_{JK}), 0, \partial^+_k) = \emptyset$ for all k. □

5. An alphabet of building blocks

A schedule for a delooping X^M is a concatenation of allowed schedules for X, the delooping once of every loop of the torus, as above. The concatenations are allowed, if no deadlocks are produced. Hence, the allowed schedules for X are the fundamental building blocks for the schedules for X^M. The allowed schedules for X^M, where $M = (1, 1, \ldots, m, 1, \ldots, 1)$ is delooping in a fixed direction, are words in a regular language over the allowed schedules on X.

Remark 5.1. *Notation* In the following, J is a schedule for the rectangles R^i, $i = 1, \ldots, l$, X_J is the corresponding subset of X. In concatenations $X_{J1} \star_r \ldots \star_r X_{Js}$ and also in X^M, 0 denotes the minimal point and 1 the maximal point.

Definition 5.2. Let $\mathcal{A} = \{X_J | \vec{T}(X_j, 0, 1) \neq \emptyset\}$.

$$\mathcal{A}_{r,m} = \{w = a_1 \star_r a_2 \star_r \ldots \star_r a_m | a_i \in \mathcal{A}_1 \text{ and } \vec{T}(w, 0, 1) \neq \emptyset\}$$

is the set of allowed r-words of length m.
Let $w = w_1 w_2 \ldots w_m$ be a word on an alphabet $\mathcal{A}$. Then $Alph(w) = \{a \in \mathcal{A} | a \in w\}$, the letters used in w.

Lemma 5.3. *Properties of the operators D_r. Let $w, z \in \mathcal{A}^*$, $w = w_1 w_2 \ldots w_k$, $c \in \mathcal{A}$.*

1. *$D_r(w)(c) = D_r(\vee_{i=1}^k w_i)(c)$, where $\vee$ is as in Prop. 4.10*
2. *D_r is commutative in the following sense: $D_r(w) \circ D_s(z) = D_s(z) \circ D_r(w)$*
3. *D_r is idempotent: $D_r(w) \circ D_r(w) = D_r(w)$*

Proof. 1) and 3) follows from Prop. 4.10. The statement in 2) is $(C \setminus A) \setminus B = (C \setminus B) \setminus A$ for sets, A, B, C. □

Lemma 5.4. *Let $w \in \mathcal{A}_{r,m}$ and let $a \in \mathcal{A}$. Then $a \star_r w \in \mathcal{A}_{r,m+1}$ if and only if all deadlocks in $D_r(w)(a)$ are on the upper boundary.*

1. *If $w \in \mathcal{A}_{r,m}$ and $\hat{w}$ is a subsequence of the letters in w, then $\hat{w}$ is allowed.*
2. *If $w \star_r w$ is allowed, then all cyclic permutations of w are allowed.*
3. *If $w \star_r w$ is allowed, then $w^{r,n} = w \star_r w \star_r w \star_r \ldots \star_r w$, concatenation of n copies of w is allowed for all n.*

Proof. $a \star_r w$ is allowed, if $\vec{T}(a \star_r w, 0, 1) \neq \emptyset$. The geometric object representing $a \star_r w$ is the subset of $C_{(m+1)r} = \rho(\times_{j=1}^{n}[0,1]_j, ([0,1]_r, [0, m+1])$ given by $x = (x_1, \ldots, x_n) \in a \star_r w$ if $x_r \geq 1$ and $x - e_r \in w$, or $x_r \leq 1$ and $x \in D_r(w)(a)$.

Hence, since w is allowed and thus has no deadlocks, it follows from Thm. 4.15 that $\vec{T}(a \star_r w, 0, 1) \neq \emptyset$ if and only if $D_r(w)(a)$ has no interior deadlocks. 1), 2) and 3) are then a consequence of Prop. 4.10

□

Corollary 5.5. *$\mathcal{A}_{r+1,m}$ is the set of $a \star_r w$ for which $(a, w) \in \mathcal{A}_1 \times \mathcal{A}_{r,m}$ and $D_r(\vee_{b \in alph(w)} b)(a)$ has no interior deadlocks.*

Hence, since $\mathcal{A}$ is finite, the problem is now finite. A consequence of finiteness is

Theorem 5.6. *Let $\mathcal{A}$ be the alphabet of allowed schedule restrictions for X. The language $L_r \subset \mathcal{A}^*$, $r \in [1 : n]$ consists of words $w = a_1 a_2 \cdot a_m$ such that $a_1 \star_r a_2 \star_r \cdots \star_r a_m$ is allowed. Then L_r is a regular language. It is recognized by an automaton with at most $2^n - 1)^l + 1$ state.*

Proof. This follows from the Myhill-Nerode theorem. For $w_1, w_2 \in \mathcal{A}^*$, let $w_1 \sim w_2$ if for all $u \in \mathcal{A}^*$ $uw_1 \in L \Leftrightarrow uw_2 \in L$.
The number of equivalence classes is finite:

For $w \in \mathcal{A}^* \setminus L$, $u \in \mathcal{A}^*$, $uw \in \mathcal{A}^* \setminus L$, so $\mathcal{A}^* \setminus L$ is an equivalence class.

Let $w \in L$. Then $uw \in L$ if and only if $D_r(\vee_{a \in Alph(w)} a)(u) \in L$. Hence, if $\vee\{Alph(w_1)\} = \vee\{Alph(w_2)\}$, then $w_1 \sim w_2$.

For all w, $\vee\{Alph(w)\} = (J_1, \ldots, J_l)$, where $\emptyset \neq J_i \subset [1 : n]$.

Hence $|\mathcal{A}^*/\sim| \leq (2^n - 1)^l + 1$. The Myhill-Nerode Theorem [3] chapter 3 says, that L is a regular language if and only if $|\mathcal{A}^*/\sim|$ is finite. And there is an automaton with $|\mathcal{A}^*/\sim|$ states which recognizes it. □

Definition 5.7. An $l \times n$ binary matrix B represents the schedule $J_B = (J1, \ldots, Jl)$ where $s \in Jk \subset [0 : n]$ if and only if $B_{ks} = 1$.

Corollary 5.8. *The automaton generating L has*

- *States: All $l \times n$ binary matrices B for which there is a word $w \in L$ with $\vee\{a \in Alph(w)\} = J_B$*
- *Transitions $a : B \to B \vee a$ if $D_r(B)(a)$ has no interior deadlocks.*

Remark 5.9. The automaton may be generated as follows:

- The initial state is $\{\varepsilon\}$
- Add $a : \{\varepsilon\} \to \{a\}$ the transition a and state $\{a\}$ for all $a \in \mathcal{A}$
- For all $b \in \mathcal{A}$ and all states B add a transition (and a state, if $B \vee b$ is not already there) $b : B \to b \vee B$ if $D_r(B)(b)$ has no interior deadlocks.

Now iterate. This stops, since $\mathcal{A}$ is finite.

Proposition 5.10. *Let $a_1, a_2 \in \mathcal{A}$, $a_i = X_{J^i}$, $J^i = (J1^i, J2^i, \ldots, Jl^i)$ where $Jk^i \subset [1:n]$ for $k \in [1:l]$. The partial order on the poset category $C(X)$ is $a_1 \leq a_2$ if $Jk^1 \subset Jk^2$ for all $k \in [1:l]$. Let $m = (1, \ldots, 1) + (m-1)e_r$. The objects in the poset category $C(X^m)$ for the m'th delooping are r-words of length m in the regular language L_r over $\mathcal{A}$. The partial order on words is $w_i = a_1^i a_2^i \ldots a_m^i$, $w_1 \leq w_2$ if $a_j^1 \leq a_j^2$ for all j.*

Proof. Objects in the poset are allowed schedule restrictions. Words in the language L_r are the allowed schedule restrictions. The partial order is the set inclusion order on restrictions corresponding to individual rectangles. □

Example 5.11. For the 3-cube in Ex.4.4, the allowed schedule restrictions are $a = \{1\}$, $b = \{2\}$, $c = \{3\}$, $d = \{1, 2\}$, $e = \{1, 3\}$ and $f = \{2, 3\}$. So $\mathcal{A} = \{a, b, c, d, e, f\}$. The possible states are the 8 binary 1×3 matrices, where (000) is the initial state. They are all allowed states, since the word *bca* is allowed and hence allows the matrix (111). However, not all transitions are allowed. The transition $a : (011) \to (111)$ is not in the automaton, since the words *abc* and *acb* are not allowed.

Example 5.12. The square with two diagonal holes, see Ex. 4.5 the allowed schedules are $a = \{1\}\{1\}$, $b = \{2\}\{1\}$,$c = \{1\}\{2\}$,$d = \{2\}\{2\}$. So this is the alphabet. Given as 2×2 matrices.

$$a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \; b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; d = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Alph(db) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \; Alph(da) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Alph(ca) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \; Alph(ba) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

All possible states are allowed. But not all transactions, e.g. $b : \{b\} \to \{b\}$ is not allowed as we saw earlier.

For deloopings in general $M = (m_1, \ldots, m_n)$, Thm. 4.15 implies, that all the n automata generating L_r for $r \in [1:n]$ should be combined. This will not be developed here.

There are algorithmic gains in the following observations:

- If A, A' are states, $A' \leq A$ (as matrices) and $a : A \to A \vee A$ is allowed, then $a : A' \to a \vee A'$ is allowed, since $D_r(A)(a) \subset D_r(A')(a)$, so if there is a dipath to an upper face in $D_r(A)(a)$, it is a dipath in $D_r(A')(a)$.
- The possible deadlocks are intersections of cylinders and extended rectangles. Finding them is a check of inequalities as in [4]. Some of these inequalities are trivial - for intervals $[0, 1]$.

Remark 5.13. There is a dual of the index category $\mathcal{C}(X)$ where rectangles are extended upwards. This corresponds to 1) Invert the order on $\vec{I}$ and hence on X to get X^*. 2) Calculate $\vec{T}(X^*, 1, 0)$ as in [4]. The resulting automaton will generate words iteratively $w \to wa$, i.e., the letter a is a suffix, not a prefix.

6. Conclusion and further work.

The periodicity implies, that the index category $\mathcal{C}(X^M)$ is described in terms of deadlocks in X between extended rectangles and cylinders. For a delooping in one direction, the index category is described in terms of words in a regular language, and the automaton is given.

For general deloopings X^M, the allowed schedules and the corresponding index category $\mathcal{C}(X^M)$ involves checking for deadlocks in $D_r(X_K)(X_J)$ $r \in \{0,1\}^n$, i.e., a check of inequalites, some of which are trivial. This will be explored in another paper. Moreover, for general processes, which are not pure loops, deadlocks in the deloopings appear in a combination of down effects from rectangles after the loops and rectangles within the loop. The down effects from rectangles after the loops give rise to cylinders penetrating the whole delooping, and hence there is still a periodicity, which may be used for better algorithms.

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