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by

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The negative bundles for complex projective spaces

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Abstract

We give a description of the negative bundles for the energy integral on $L\mathbb{C}\mathrm{P}^n$ in terms of circle vector bundles over projective Stiefel manifolds. We compute the mod p Chern classes of the associated homotopy orbit bundles.
MSC: 55N91; 55P35; 57R20; 58E05

1 Introduction

In [K2] Klingenberg studies Morse theory for the energy integral E on the free loop spaces of a projective space LP^n. (He considers complex and quaternionic projective spaces as well as the Cayley projective plane). Critical points for the energy integral are closed geodesics of various energy levels $0 = e_0 < e_1 < \dots$. Those of energy level e_q form a finite dimensional critical submanifold B_q of LP^n and there is a so-called negative vector bundle μ_q^- over B_q. The energy levels also give a filtration of the free loop space $\mathcal{F}(e_q) = E^{-1}([0, e_q])$. Morse theory in this setting states that $\mathcal{F}(e_q)$ is essentially obtained by attaching to $\mathcal{F}(e_{q-1})$ the disc bundle of μ_q^-. One of the results in Klingenberg's article is a concrete calculation of the negative bundles.

The purpose of this paper is firstly to give a simpler description of the negative bundles for the complex projective spaces as circle vector bundles over projective Stiefel manifolds (Theorem 5.10 and Definition 5.8). Secondly, we calculate the mod p Chern classes of the associated homotopy orbit bundles (Theorem 7.10).

These results are motivated by [BO] where Bökstedt and the author computes the mod p equivariant cohomology of $L\mathbb{C}\mathrm{P}^n$ with respect to the action of the unit circle group $\mathbb{T}$. The calculation uses a spectral sequence coming from the energy filtration (which is a $\mathbb{T}$-equivariant filtration). We would like to calculate the action of the Steenrod algebra in this spectral sequence, and for that purpose one needs the Chern classes of the negative bundles.

The free loop space LM of a manifold M and its homotopy orbit space $LM_{h\mathbb{T}}$ is closely related to the topological cyclic homology spectrum $TC(M, p)$. One can describe $TC(M, p)$ as a homotopy pullback in terms of these spaces [BHM]. It is likely, that a calculation of the mod p spectrum cohomology of $TC(\mathbb{C}\mathrm{P}^n, p)$ which includes the action of the Steenrod algebra would require a calculation of the Steenrod algebra action on $\mathbb{T}$-equivariant cohomology of $L\mathbb{C}\mathrm{P}^n$. An alternative method for computing these cohomology groups uses formality of $\mathbb{C}\mathrm{P}^n$ and negative cyclic homology, but this approach does not seem suited for a calculation of the Steenrod algebra action.

2 Morse theory for free loop spaces

In this section we recall some results on Morse theory for the energy integral on the Hilbert manifold model of the free loop space. For details on this we refer to [K3].

Let M be a compact Riemannian manifold equipped with the Levi-Civita connection. We use the Hilbert manifold model of the free loop space LM. Write the circle as $S^1 = [0,1]/\{0,1\}$. An element in LM is an absolutely continuous map $f : S^1 \to M$ such that f' is square integrable ie. $\int_0^1 |f'(t)|^2 dt < \infty$. The Hilbert manifold model is homotopy equivalent to the usual continuous mapping space model.

The tangent space $T_f(LM)$ is the set of absolutely continuous tangent vector fields X along f such that the covariant derivative $DX(t)/dt$ is square integrable. The free loop space LM is equipped with a Riemannian metric $\langle\langle \cdot, \cdot \rangle\rangle$ as follows:

$$\langle\langle X, Y \rangle\rangle = \int_0^1 \langle \frac{DX}{dt}(t), \frac{DY}{dt}(t) \rangle + \langle X(t), Y(t) \rangle dt,$$

where $X, Y \in T_f(LM)$.

The energy integral (or energy function) is defined by

$$E : LM \to \mathbb{R}; \quad E(f) = \frac{1}{2} \int_0^1 |f'(t)|^2 dt.$$

The critical points for E are precisely the closed geodesic on M. For a critical point f, the Hessian of E has the following form: $H_f(\cdot, \cdot) : T_f(LM) \times T_f(LM) \to \mathbb{R}$;

$$H_f(X, Y) = \int_0^1 \langle \frac{DX}{dt}(t), \frac{DY}{dt}(t) \rangle + \langle R(X(t), f'(t)) f'(t), Y(t) \rangle dt,$$

where $R(\cdot, \cdot)\cdot$ denotes the curvature tensor on M. The Hessian determines a self adjoint operator A_f on $T_f(LM)$ satisfying $H_f(X, Y) = \langle\langle A_f(X), Y \rangle\rangle$ for all X and Y. The operator A_f is the sum of the identity with a compact operator, so there are at most a finite number of negative eigenvalues, each corresponding to a finite dimensional vector space of eigenvectors of A_f. The kernel of A_f, which is also finite dimensional, consists of the periodic Jacobi fields along f.

Now let $N(e)$ be the space of critical points of E with energy level e. The negative bundle $\mu^-(e)$ over $N(e)$ is the vector bundle whose fiber at f is the vector space spanned by the eigenvectors belonging to negative eigenvalues of A_f. Similarly, $\mu^0(e)$ and $\mu^+(e)$ are the vector bundles with fibers spanned by the eigenvectors corresponding to the eigenvalue 0 and the positive eigenvalues respectively.

It is known that $-\mathrm{grad}\, E$ satisfy condition (C) of Palais and Smale so that one can do Morse theory on LM if some additional non-degeneracy condition is satisfied.

For us the so called Bott non-degeneracy condition is the relevant one. It requires firstly that for each critical value e the space $N(e)$ is a compact submanifold of LM and secondly that for each $f \in N(e)$ the restriction of the operator A_f to the complement $(T_f N(e))^{\perp}$ of $T_f N(e)$ in $T_f(LM)$ is invertible. The Bott non-degeneracy condition is a strong assumption on the metric of M, but for instance the symmetric spaces satisfy this, according to [Z, Theorem 2].

Let the critical values of the energy function be $0 = e_0 < e_1 < \dots$. Consider the filtration of LM given by $\mathcal{F}(e_i) = E^{-1}([0, e_i])$. This filtration is equivariant with respect to the action of the circle.

The tangent bundle of LM restricted to $N(e_i)$ splits $\mathbb{T}$-equivariantly into a sum of three bundles.

$$T(LM)|_{N(e_i)} \cong \mu^-(e_i) \oplus \mu^0(e_i) \oplus \mu^+(e_i).$$

Assume that the Bott non-degeneracy condition holds. Then the standard Morse theory argument can be carried through equivariantly on the Hilbert manifold LM. This was done by Klingenberg. For an account of this work see section [K1, 2.4], especially theorem 2.4.10. The statement of this theorem implies that we have an equivariant homotopy equivalence

$$\mathcal{F}(e_i)/\mathcal{F}(e_{i-1}) \simeq \mathrm{Th}(\mu^-(e_i)).$$

3 Klingenberg's calculation of negative bundles for projective spaces

We will now focus on the projective spaces $P^n(\alpha)$ over the complex numbers $\mathbb{C}$ for $\alpha = 2$, the quaternions $\mathbb{H}$ for $\alpha = 4$ and the Cayley numbers $\mathbb{O}$ for $\alpha = 8$ and $n = 1, 2$. These spaces are endowed with the Riemannian metric which makes them symmetric spaces of rank one. This metric is determined up to a positive constant, which we fix by requiring the sectional curvature to have maximal value $2\pi^2$ and minimal value $\pi^2/2$ [K2, 1.1].

Klingenberg calculates the negative bundles for $L(P^n(\alpha))$ in [K2] and we will review this calculation.

Let $B_q(P^n(\alpha)) \subseteq LP^n(\alpha)$ denote the critical submanifold of q-fold covered primitive geodesics. A geodesics $f \in B_q(P^n(\alpha))$ lies on a unique projective line $S^\alpha \cong P^1(\alpha) \subseteq P^n(\alpha)$. For each $t \in [0, 1]$ we split the tangent space at $f(t)$ into a horizontal subspace of tangent vectors to this projective line and its orthogonal complement called the vertical subspace [K2, 1.3]

$$T_{f(t)}(P^n(\alpha)) = T_{f(t)}(P^n(\alpha))_h \oplus T_{f(t)}(P^n(\alpha))_v.$$

The horizontal subspace has real dimension α and the vertical subspace has real dimension $\alpha(n-1)$. A tangent vector field $X \in T_f(P^n(\alpha))$ decompose into a horizontal component X_h and a vertical component X_v and this decomposition is compatible with the covariant derivative along f.

Lemma 3.1 (Klingenberg)**.** *Let $f \in B_q(P^n(\alpha))$ where q is a positive integer. The Hessian $H_f(\cdot, \cdot)$ on $T_f(LP^n(\alpha))$ has eigenvectors as follows:*

1.

$$X_p(t) = A\cos(2\pi pt) + B\sin(2\pi pt), \quad p \in \mathbb{N}_0,$$

where A and B are constant (i.e. parallel) horizontal vector fields along f such that $\langle A, f'(t)\rangle = \langle B, f'(t)\rangle = 0$ for all t. The eigenvalue for X_p is

$$\lambda_p = \frac{4\pi^2(p^2 - q^2)}{1 + 4\pi^2 p^2}.$$

We write $E_{h,p}$ for the vector space formed by the X_p's for a fixed p. It has real dimension $\alpha - 1$ for $p = 0$ and $2(\alpha - 1)$ for $p > 0$.

2.

$$Y_r(t) = A\cos(\pi rt) + B\sin(\pi rt), \quad r \in \mathbb{N}_0, \quad r \equiv q \ mod\ 2,$$

where A and B are constant vertical vector fields along f. The eigenvalue of Y_r is

$$\mu_r = \frac{\pi^2(r^2 - q^2)}{1 + \pi^2 r^2}.$$

We write $E_{v,r}$ for the vector space formed by Y_r. It has real dimension $\alpha(n-1)$ if $r = 0$ and $2\alpha(n-1)$ if $r > 0$.

3.

$$Z_s(t) = (a\cos(2\pi st) + b\sin(2\pi st))f'(t), \quad s \in \mathbb{N}_0,$$

where $a, b \in \mathbb{R}$. The eigenvalue for Z_s is

$$\nu_s = \frac{4\pi^2 s^2}{1 + 4\pi^2 s^2}.$$

We write $E_{t,s}$ for the vector space formed by Z_s. It has real dimension 1 for $s = 0$ and 2 for $s > 0$.

Proof. With our choice of metric, $|f'(t)|^2 = 2q^2$. Moreover, the curvature tensor for $P^n(\alpha)$ is known, and its block matrix form allows Klingenberg to decompose the Hessian into a horizontal and a vertical quadratic form [K2, 1.4]

$$\begin{aligned} H_f^h(X_h, Y_h) = \int_0^1 &\langle \frac{DX_h}{dt}(t), \frac{DY_h}{dt}(t)\rangle \\ &- 2\pi^2(2q^2\langle X_h(t), Y_h(t)\rangle - \langle f'(t), X_h(t)\rangle\langle f'(t), Y_h(t)\rangle)dt, \\ H_f^v(X_v, Y_v) = \int_0^1 &\langle \frac{DX_v}{dt}(t), \frac{DY_v}{dt}(t)\rangle - \pi^2 q^2\langle X_v(t), Y_v(t)\rangle dt. \end{aligned}$$

Consider the eigen equation $H_f^h(X_h, Y_h) = \lambda\langle\langle X_h, Y_h\rangle\rangle$ for $\lambda \in \mathbb{R}$. If X_h possess second covariant derivative, we get an equivalent equation via partial integration

$$(1 - \lambda)\frac{D^2 X_h}{dt^2} + (4\pi^2 q^2 + \lambda)X_h - 2\pi^2\langle f', X_h\rangle f' = 0. \tag{1}$$

We insert X_p in this equation. Since $\frac{D^2 X_p}{dt^2} = -4\pi^2 p^2 X_p$ we get the following:

$$((4\pi^2 p^2 + 1)\lambda - 4\pi^2(p^2 - q^2))X_p = 0.$$

Thus λ_p is an eigenvalue for $H_f^h(\cdot,\cdot)$ with eigenvector X_p.

From $H_f^v(X_v, Y_v) = \mu\langle\langle X_v, Y_v\rangle\rangle$ where $\mu \in \mathbb{R}$, we get the eigen equation

$$(1-\mu)\frac{D^2 X_v}{dt^2} + (\pi q^2 + \mu)X_v = 0. \tag{2}$$

We insert Y_r. Since $\frac{D^2 Y_r}{dt^2} = -\pi^2 r^2 Y_r$ we get

$$((\pi^2 r^2 + 1)\mu - \pi^2(r^2 - q^2))Y_r = 0.$$

Thus μ_r is an eigenvalue for $H_f^v(\cdot,\cdot)$ with eigenvector Y_r.

Finally, we insert Z_s into (1). Since f is a geodesics we have that $\frac{Df}{dt} = 0$. Thus, $\frac{D^2 Z_s}{dt^2} = -4\pi^2 s^2 Z_s$ and we obtain

$$((1 + 4\pi^2 s^2)\lambda - 4\pi^2 s^2)Z_s = 0.$$

We see that ν_s is an eigenvalue for $H_f^h(\cdot,\cdot)$ with eigenvector Z_s. □

The subspaces described in 1.-3. have trivial pairwise intersection. They also generate the full Hilbert space $T_f(P^n(\alpha))$, so we have the following result:

Corollary 3.2. *The negative subspace is the direct sum*

$$T_f(LP^n(\alpha))^- = \bigoplus_{0\le p<q} E_{h,p} \oplus \bigoplus_{0\le r<q,\, r\equiv q \ mod\ 2} E_{v,r}.$$

It has real dimension $(2q-1)(\alpha-1) + (q-1)\alpha(n-1)$. *The zero subspace is*

$$T_f(LP^n(\alpha))^0 = E_{t,0} \oplus E_{h,q} \oplus E_{v,q}.$$

It has real dimension $2\alpha n - 1$. *The positive subspace is the Hilbert direct sum*

$$T_f(LP^n(\alpha))^+ = \bigoplus_{p>q} E_{h,p} \oplus \bigoplus_{r>q,\, r\equiv q \ mod\ 2} E_{v,r} \oplus \bigoplus_{s>0} E_{t,s}.$$

Klingenberg shows that there are vector bundles over $B_q(P^n(\alpha))$ for $q \geq 1$ as follows:

Vector bundle	$\dim_{\mathbb{R}}$	Fiber over f	Condition
η_h	$\alpha - 1$	$E_{h,0}$	
$\sigma_{h,p}$	$2(\alpha-1)$	$E_{h,p}$	$p \geq 1$
$\sigma_{v,2p-1}$	$2\alpha(n-1)$	$E_{v,2p-1}$	q odd, $p \geq 1$
η_v	$\alpha(n-1)$	$E_{v,0}$	q even
$\sigma_{v,2p}$	$2\alpha(n-1)$	$E_{v,2p}$	q even

Thus, we have the following result [K2, 1.6]:

Theorem 3.3 (Klingenberg)**.** *The non-trivial critical points for the energy integral $E : L(P^n(\alpha)) \to \mathbb{R}$ decompose into the non-degenerate critical submanifolds $B_q(\alpha) = B_q(P^n(\alpha))$ consisting of the q-fold covered parametrized great circles, $q = 1, 2, \dots$; $E(B_q(\alpha)) = 2q^2$. The negative bundle μ_q^- over $B_q(\alpha)$ has the following form:*

$$\mu_q^- = \eta_h \oplus \bigoplus_{p=1}^{q-1} \sigma_{h,p} \oplus \bigoplus_{p=1}^{\frac{q-1}{2}} \sigma_{v,2p-1} \qquad \textit{for } q \textit{ odd,}$$

$$\mu_q^- = \eta_h \oplus \bigoplus_{p=1}^{q-1} \sigma_{h,p} \oplus \eta_v \oplus \bigoplus_{p=1}^{\frac{q-2}{2}} \sigma_{v,2p} \qquad \textit{for } q \textit{ even.}$$

4 Spaces of geodesics viewed as projective Stiefel manifolds

From now on, we consider the complex projective space $\mathbb{C}\mathrm{P}^n$. It has a Hermitian metric, which we now describe. References are [KN2] page 273 or [MT] page 142.

Equip $\mathbb{C}^{n+1}$ with the standard Hermitian inner product $h(v, w) = \sum_{k=1}^{n+1} v_k \overline{w}_k$. The real part $g'(v, w) = \Re h(v, w)$ is the usual inner product on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Furthermore, $h(v, w) = g'(v, w) + ig'(v, iw)$.

Let $S^{2n+1} = \{x \in \mathbb{C}^{n+1} | h(x, x) = 1\}$ be the unit sphere and write $\mathbb{T}$ for the unit circle group. Consider the Hopf projection

$$\rho : S^{2n+1} \to S^{2n+1}/\mathbb{T} = \mathbb{C}\mathrm{P}^n.$$

By restriction of h we have a Hermitian inner product on the complement $\{\mathbb{C}x\}^\perp = \{v \in \mathbb{C}^{n+1} | h(x, v) = 0\}$ and $\{\mathbb{C}x\}^\perp$ is a real subspace of the tangent space $T_x(S^{2n+1})$. One can equip $\mathbb{C}\mathrm{P}^n$ with a Hermitian metric $\tilde{h}(\cdot, \cdot)$ such that

$$\eta_x : (\mathbb{C}x)^\perp \subseteq T_x(S^{2n+1}) \xrightarrow{\rho_*} T_{\rho(x)}(\mathbb{C}\mathrm{P}^n)$$

becomes a $\mathbb{C}$-linear isometry. The following identity holds

$$\eta_{zx}(zv) = \eta_x(v) \text{ for } z \in \mathbb{T}. \tag{3}$$

The real part $\tilde{g}(\cdot, \cdot) = \Re\tilde{h}(\cdot, \cdot)$ is the Fubini-Study metric on $\mathbb{C}\mathrm{P}^n$. (In [KN2] they allow a rescaling of $\tilde{g}$ by $4/c$ for a positive constant c. We let $c = 4$.) It is known that the sectional curvature for this metric has maximal value 4 and minimal value 1 when $n > 1$. Thus the metric on $\mathbb{C}\mathrm{P}^n$ used in section 3 is $\frac{\pi^2}{2}\tilde{g}$.

For $\mathbb{C}\mathrm{P}^n$ with Riemannian metric $\tilde{g}$ and associated Levi-Civita connection, we now describe the spaces of closed geodesics $B_q(\mathbb{C}\mathrm{P}^n)$ in terms of projective Stiefel manifolds. Recall that $B_q(\mathbb{C}\mathrm{P}^n)$ is the space of constant geodesics for $q = 0$, primitive geodesics for $q = 1$ and q-fold iterated primitive geodesics for $q \geq 2$.

Write $\mathbf{V}_2(\mathbb{C}^{n+1})$ for the Stiefel manifold of complex orthonormal 2-frames in $\mathbb{C}^{n+1}$. We have a diagonal $U(1)$-action on this Stiefel manifold, and the quotient is the Projective Stiefel manifold

$$\mathbf{PV}_2(\mathbb{C}^{n+1}) = \mathbf{V}_2(\mathbb{C}^{n+1})/diag_2(U(1)).$$

Definition 4.1. Let $\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})$ denote the projective Stiefel manifold $\mathbf{PV}_2(\mathbb{C}^{n+1})$ equipped with the left $\mathbb{T}$-action

$$z * [u,v] = [(\sqrt{z})^{-1}u, \sqrt{z}v],$$

where $\sqrt{z}$ is a square root of $z \in \mathbb{T} \subseteq \mathbb{C}$. Note that the action is well-defined. We write $\mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$ for the associated $\mathbb{T}$-space, where $\mathbb{T}$ acts via $z \mapsto z^q$.

Remark 4.2. Since we mod out by a diagonal $U(1)$-action, we can also write the $\mathbb{T}$-action on $\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})$ as

$$z * [u,v] = [z^{-1}u, v] = [u, zv].$$

The $\mathbb{T}$-action is *free* since $[u,zv] = [u,v] \Rightarrow z = 1$.

Theorem 4.3. *For every positive integer q there is a $\mathbb{T}$-equivariant diffeomorphism*

$$\phi_q : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n); \quad \phi_q([u,v])(z) = \rho(\frac{(\sqrt{z})^{-q}u + (\sqrt{z})^q v}{\sqrt{2}}).$$

Proof. It is well known ([GHL] 2.110 or [KN2] page 277) that there is a bijection

$$\psi_q : \mathbf{PV}_2(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n); \quad \psi_q([a,b])(t) = \rho(\cos(q\pi t)a + \sin(q\pi t)b),$$

where $0 \le t \le 1$. We have an action of $\mathbb{T}$ on $\mathbf{V}_2(\mathbb{C}^{n+1})$ given by rotation of frames

$$R(\theta)(a,b) = (\cos(\theta)a + \sin(\theta)b, -\sin(\theta)a + \cos(\theta)b).$$

By the addition formulas for sine and cosine one finds that

$$\psi_q([a,b])(s+t) = \psi_q([R(q\pi s)(a,b)])(t).$$

Thus, ψ_q becomes equivariant when we let $\mathbb{T}$ act on $\mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$ and $\mathbf{PV}_2(\mathbb{C}^{n+1})$ by

$$(e^{2\pi i s} * f)(t) = f(s+t) \quad \text{and} \quad e^{2\pi i s} \star [a,b] = [R(q\pi s)(a,b)]$$

respectively. Write $\mathbf{PV}_{2,(q)}(\mathbb{C}^{n+1})$ for the projective Stiefel manifold equipped with this well-defined action.

We also have a diffeomorphism $\tau : \mathbf{V}_2(\mathbb{C}^{n+1}) \to \mathbf{V}_2(\mathbb{C}^{n+1})$ defined as follows:

$$\tau(u,v) = (\frac{u+v}{\sqrt{2}}, \frac{u-v}{i\sqrt{2}}), \quad \tau^{-1}(a,b) = (\frac{a+ib}{\sqrt{2}}, \frac{a-ib}{\sqrt{2}}).$$

By Euler's formulas one finds that

$$\tau(e^{-i\theta}u, e^{i\theta}v) = R(\theta)\tau(u,v).$$

Thus, τ gives us a $\mathbb{T}$-equivariant diffeomorphism $\tau_q : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{PV}_{2,(q)}(\mathbb{C}^{n+1})$. By Euler's formulas we have

$$(\psi_q \circ \tau_q)([u,v])(t) = \rho(\frac{e^{-iq\pi t}u + e^{iq\pi t}v}{\sqrt{2}}).$$

Since $t \in [0,1]/\{0,1\}$ corresponds to $e^{2\pi i t} \in \mathbb{T}$ this composite equals ϕ_q. □

Remark 4.4. We can also describe the equivariant diffeomorphism as follows:

$$\phi_q : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n); \quad \phi_q([u,v])(z) = s(z^q * [u,v]),$$

Where $s : \mathbf{PV}_2(\mathbb{C}^{n+1}) \to \mathbb{C}\mathrm{P}^n$ is given by $s([u,v]) = \rho(\frac{u+v}{\sqrt{2}})$.

5 A description of the negative bundle

In this section we will describe the negative bundles as bundles over projective Stiefel manifolds. We start by the following result regarding the constant (parallel) horizontal and vertical vector fields mentioned in Lemma 3.1.

Lemma 5.1. *Let $(u, v) \in \mathbf{V}_2(\mathbb{C}^{n+1})$ and let q be a positive integer. Define the curve*

$$c : [0,1] \to S^{2n+1}; \quad c(t) = \frac{e^{-q\pi it}u + e^{q\pi it}v}{\sqrt{2}}$$

and put $f = \rho \circ c = \phi_q([u,v])(e^{2\pi it})$. Then the horizontal and vertical subspace at $f(t)$ is given by

$$T_{f(t)}(\mathbb{C}\mathrm{P}^n)_h = \eta_{c(t)}(\mathrm{span}_{\mathbb{C}}(c'(t))), \quad T_{f(t)}(\mathbb{C}\mathrm{P}^n)_v = \eta_{c(t)}(\{u,v\}^\perp),$$

where $\perp$ is with respect to the Hermitian inner product h. Furthermore,

$$H(t) = \eta_{c(t)}(e^{-q\pi it}u - e^{q\pi it}v)$$

is a parallel and horizontal vector field along f, such that $\tilde{g}(H(t), f'(t)) = 0$ for all t, and

$$V(w)(t) = \eta_{c(t)}(w)$$

is a parallel and vertical vector field along f for all $w \in \{u,v\}^\perp$.

Proof. We have that $c'(t) = -q\pi i(e^{-q\pi it}u - e^{q\pi it}v)/\sqrt{2}$. Since u and v are orthonormal vectors it follows that $h(c'(t), c'(t)) = q^2\pi^2$ and $h(c(t), c'(t)) = 0$. Furthermore, $\{c(t), c'(t)\}^\perp = \{u,v\}^\perp$ for all t. Thus we have an orthogonal decomposition

$$\{c(t)\}^\perp = \mathrm{span}_{\mathbb{C}}(c'(t)) \oplus \{c(t), c'(t)\}^\perp = \mathrm{span}_{\mathbb{C}}(c'(t)) \oplus \{u,v\}^\perp.$$

By the chain rule $f'(t) = T_{c(t)}(\rho)(c'(t)) = \eta_{c(t)}(c'(t))$ such that

$$T_{f(t)}(\mathbb{C}\mathrm{P}^n)_h = \mathrm{span}_{\mathbb{C}}(f'(t)) = \eta_{c(t)}(\mathrm{span}_{\mathbb{C}}(c'(t)))$$

and since $\eta_{c(t)}$ is an isometry, we also obtain the desired descriptions of the vertical subspace.

Put $\tilde{H}(t) = e^{-q\pi it}u - e^{q\pi it}v$. Since $\tilde{H}$ is a rescaling of c' we see that H is a horizontal vector field.

We have equipped $S^{2n+1} \subseteq \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ with the Riemannian metric induced from $\mathbb{R}^{2n+2}$. For such a Riemannian manifold, the Levi-Civita connection is given by orthogonal projection onto the tangent space of the directional derivative. So for a smooth vector field V along a smooth curve $\gamma : [0,1] \to S^{2n+1}$, the covariant derivative is given by

$$\frac{DV}{dt}(t) = \left(\frac{dV}{dt}\right)^T,$$

where $(-)^T$ stands for orthogonal projection onto $T_{\gamma(t)}(S^{2n+1})$. The curve c has double derivative $c''(t) = -q^2\pi^2 c(t)$ so $\frac{D}{dt}\tilde{H}(t) = 0$ and since we have equipped $\mathbb{C}\mathrm{P}^n$

with the Fubini-Study metric it follows that $\frac{D}{dt}H(t) = 0$. Thus H is a parallel vector field along f.

The real part of the equation $h(\tilde{H}(t), c'(t)) = -q\pi i||\tilde{H}(t)||^2/\sqrt{2}$ gives us that $g'(\tilde{H}(t), c'(t)) = 0$. It follows that $\tilde{g}(H(t), f'(t)) = 0$ since $\eta_{c(t)}$ is an isometry.

By the first part of the lemma, $V(w)$ is a vertical vector field for all $w \in \{u, v\}^\perp$. Since w is constant, $\frac{dw}{dt} = 0$, with orthogonal projection $\frac{Dw}{dt} = 0$. It follows that $\frac{DV(w)}{dt} = 0$ such that $V(w)$ is a parallel vector field along f. $\square$

Definition 5.2. For $(u, v) \in \mathbf{V}_2(\mathbb{C}^{n+1})$ we define the *closed* geodesic

$$c(u, v) : \mathbb{T} \to S^{2n+1}; \quad c(u, v)(z) = \frac{1}{\sqrt{2}}(z^{-1}u + zv).$$

The equivariant diffeomorphism $\phi_q : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$ from Theorem 4.3 is defined by the diagram

$$\begin{tikzcd} \mathbb{T} \arrow[r, "c(u,v)"] & S^{2n+1} \arrow[d, "\rho"] \\ \mathbb{T} \arrow[u, "(\sqrt{\cdot})^q"] \arrow[r, "\phi_q({[u,v]})"] & \mathbb{C}\mathrm{P}^n. \end{tikzcd}$$

Note that

$$c(u, v)(z_1 z_2) = c(z_1^{-1}u, z_1 v)(z_2) \tag{4}$$

for all z_2, z_2 in $\mathbb{T}$. Note also that $h(c(u, v), c(u, -v)) = 0$. Thus, we can view $c(u, -v)$ as a vector field along $c(u, v)$.

Definition 5.3. Define a parallel horizontal tangent vector field along $\phi_2([u, v])$ by

$$H(u, v)(z) = \eta_{c(u,v)(z)}(c(u, -v)(z))$$

and for $w \in \{u, v\}^\perp$, where $\perp$ is with respect to h, a parallel vertical tangent vectors field by

$$V(u, v, w)(z) = \eta_{c(u,v)(z)}(w).$$

Remark 5.4. By (3) we have the following identities for all $\lambda \in U(1)$:

$$H(\lambda u, \lambda v) = H(u, v), \quad V(\lambda u, \lambda v, \lambda w) = V(u, v, w).$$

From these and (4) we see that

$$H(u, v)(-z) = H(u, v), \quad V(u, v, w)(-z) = V(u, v, -w)(z).$$

We now have sufficient information on the constant horizontal and vertical vector fields in Klingenberg's lemma. We will now define the bundles over projective Stiefel manifolds which correspond to the summands of the negative bundle. Recall that $U(1)$ acts diagonally on $\mathbf{V}_2(\mathbb{C}^{n+1})$ with quotient $\mathbf{PV}_2(\mathbb{C}^{n+1})$. We start by a construction of vector bundles over $\mathbf{PV}_2(\mathbb{C}^{n+1})$.

Proposition 5.5. *Assume that $f : \mathbf{V}_2(\mathbb{C}^{n+1}) \to X$ is a $U(1)$-map and let ξ be a complex $U(1)$-vector bundle over X. Then the quotient of the pullback formed as follows:*

$$\begin{array}{ccccc} f^*(\xi)/diag_3(U(1)) & \longleftarrow & f^*(\xi) & \longrightarrow & \xi \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{PV}_2(\mathbb{C}^{n+1}) & \longleftarrow & \mathbf{V}_2(\mathbb{C}^{n+1}) & \xrightarrow{f} & X \end{array}$$

is a complex vector bundle which we denote $\mathbf{PV}_2(f, \xi) \to \mathbf{PV}_2(\mathbb{C}^{n+1})$.

Proof. Since f is a $U(1)$-map and ξ a $U(1)$-vector bundle, the pullback $f^*(\xi)$ is a $U(1)$-vector bundle. By [tD1] I.9.4, it suffices to show that $\mathbf{V}_2(\mathbb{C}^{n+1}) \to \mathbf{PV}_2(\mathbb{C}^{n+1})$ is a principal $U(1)$-bundle.

The unitary group $U(n+1)$ acts transitively on $\mathbf{V}_2(\mathbb{C}^{n+1})$ by $A\cdot(u, v) = (Au, Av)$. Write $e_1, \dots, e_{n+1}$ for the standard basis for $\mathbb{C}^{n+1}$. The isotropy group of (e_n, e_{n+1}) is $U(n-1) \times I_2$. Thus we have a diffeomorphism

$$\frac{U(n+1)}{U(n-1) \times I_2} \to \mathbf{V}_2(\mathbb{C}^{n+1}); \quad [A] \mapsto (Ae_n, Ae_{n+1}).$$

Furthermore $U(n+1)$ acts transitively on $\mathbf{PV}_2(\mathbb{C}^{n+1})$ by $A \cdot [u, v] = [Au, Av]$. The isotropy group of $[e_n, e_{n+1}]$ is $U(n-1) \times diag_2(U(1))$. So we have a diffeomorphism

$$\frac{U(n+1)}{U(n-1) \times diag_2(U(1))} \to \mathbf{PV}_2(\mathbb{C}^{n+1}); \quad [A] \mapsto [Ae_n, Ae_{n+1}]$$

and we must show that

$$\frac{U(n+1)}{U(n-1) \times I_2} \to \frac{U(n+1)}{U(n-1) \times diag_2(U(1))}; \quad [A] \mapsto [A]$$

is a principal $U(1)$-bundle.

By [tD2] Example 14.1.16 page 334 one has the following result: If $E \to E/G$ is a principal G-bundle and H is a normal subgroup of G, then $E/H \to E/G$ is a principal G/H-bundle. We use this for $E = U(n+1)$, $G = U(n-1) \times diag_2(U(1))$ and $H = U(n-1) \times I_2$. Since G is a closed subgroup of E we have that $E \to E/G$ is a principal G-bundle. By the block matrix form of elements in G and H we see that H is a normal subgroup of G. The quotient group G/H is isomorphic to $U(1)$ by the correct isomorphism. □

We will now use $\mathbf{PV}_2(f, \xi)$ to define $\mathbb{T}$-vector bundles over $\mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$. If V is a complex vector space we use a dot to denote multiplication by a scalar in the conjugate vector space $\overline{V}$ as follows: $z \cdot v = \overline{z}v$, $z \in \mathbb{C}$, $v \in V$.

Definition 5.6. For $r = q \bmod 2$ we let $\mathbf{PV}_2(f, \xi)_{r,q} = \mathbf{PV}_2(f, \xi)$ equiped with the following $\mathbb{T}$-action on its total space:

$$z * [u, v, w] = [c^{-q}u, c^q v, c^r w],$$

where $z \in \mathbb{T}$ and $c = \sqrt{z}$ is a choice of square root of z.

Similarly for the conjugate bundle we let $\overline{\mathbf{PV}_2(f,\xi)}_{r,q} = \overline{\mathbf{PV}_2(f,\xi)}$ equipped with the $\mathbb{T}$-action

$$z * [u,v,w] = [c^{-q}u, c^q v, c^r \cdot w] = [c^{-q}u, c^q v, c^{-r}w].$$

Finally we let $(\mathbf{PV}_2(f,\xi) \otimes_{\mathbb{R}} \mathbb{C})_{r,q} = \mathbf{PV}_2(f,\xi) \otimes_{\mathbb{R}} \mathbb{C}$ equiped with the following $\mathbb{T}$-action on its total space:

$$z * ([u,v,w] \otimes_{\mathbb{R}} \lambda) = [c^{-q}u, c^q v, w] \otimes_{\mathbb{R}} c^r \lambda.$$

Note that these $\mathbb{T}$-actions are well-defined since they do not depend on whether we choose c or $-c$ as square root of z. The $\mathbb{T}$-action on the base is in all cases $z * [u,v] = [c^{-q}u, c^q v]$ so we have defined $\mathbb{T}$-vector bundles over $\mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$.

Proposition 5.7. *There is a natural isomorphism of complex $\mathbb{T}$-vector bundles*

$$(\mathbf{PV}_2(f,\xi) \otimes_{\mathbb{R}} \mathbb{C})_{r,q} \cong \mathbf{PV}_2(f,\xi)_{r,q} \oplus \overline{\mathbf{PV}_2(f,\xi)}_{r,q}.$$

Proof. Let V be a complex vector space. Recall that there is an isomorphism of complex vector spaces [MT, page 163]:

$$\phi : V \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} V \oplus \overline{V}; \qquad \phi(v \otimes z) = (zv, \overline{z}v).$$

In fact the inverse is given by

$$\phi^{-1}(x,y) = \frac{x+y}{2} \otimes 1 + \frac{x-y}{2i} \otimes i$$

as one sees by direct verification. We get a corresponding isomorphism of complex vector bundles

$$[u,v,w] \otimes \lambda \mapsto ([u,v,\lambda w], [u,v,\overline{\lambda} w]),$$

which is $\mathbb{T}$-equivariant with respect to the stated actions. □

The construction $\mathbf{PV}_2(f,\xi)$ will become useful when computing characteristic classes. But for the description of the negative bundle we only need a special case:

Definition 5.8. The complex vector bundle ν is defined by

$$\nu = \mathbf{PV}_2(\pi, \gamma_2^{\perp}),$$

where $\pi : \mathbf{V}_2(\mathbb{C}^{n+1}) \to \mathbf{G}_2(\mathbb{C}^{n+1})$ is the projection which maps a frame to its complex span and $\gamma_2^{\perp}$ is the orthorgonal complement bundle to the canonical bundle γ_2 over the Grassmannian $\mathbf{G}_2(\mathbb{C}^{n+1})$. The vector bundle $\gamma_2^{\perp}$ is viewed as a $\mathbb{T}$-vector bundle, where the $\mathbb{T}$-action on the fibers is by complex multiplication of elements in $\mathbb{T} \subseteq \mathbb{C}$. For $r = q \bmod 2$, we have associated $\mathbb{T}$-vector bundles

$$\nu_{r,q} = \mathbf{PV}_2(\pi, \gamma_2^{\perp})_{r,q}, \quad \overline{\nu}_{r,q} = \overline{\mathbf{PV}_2(\pi, \gamma_2^{\perp})}_{r,q}, \quad (\nu \otimes_{\mathbb{R}} \mathbb{C})_{r,q} = (\mathbf{PV}_2(\pi, \gamma_2^{\perp}) \otimes_{\mathbb{R}} \mathbb{C})_{r,q}.$$

Two product bundles also enter in the description. For a $\mathbb{T}$ representation V we let $\epsilon_q(V)$ denote the product bundle $pr_1 : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \times V \to \mathbf{PV}_{2,q}(\mathbb{C}^{n+1})$. Let $\mathbb{C}(s)$ for $s \in \mathbb{Z}$ denote the complex numbers $\mathbb{C}$ equipped with the $\mathbb{T}$-action $z * \lambda = z^s\lambda$, and equip the real numbers $\mathbb{R}$ with the trivial $\mathbb{T}$-action. The product bundles which enter are $\epsilon_q(\mathbb{R})$ and $\epsilon_q(\mathbb{C}(p))$. Note that $\epsilon_q(\mathbb{R})$ is a real $\mathbb{T}$ vector bundle and that the others are complex $\mathbb{T}$ vector bundles.

Finally, we need an elementary fact on the dot product. Let $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ be complex numbers written in standard form. We can view them as vectors in the plane and form the dot product $z_1 \bullet z_2 = \alpha_1\alpha_2 + \beta_1\beta_2$. Note that

$$z_1 \bullet z_2 = \frac{1}{2}(\overline{z}_1 z_2 + z_1\overline{z}_2)$$

such that $(z_1z_2) \bullet z_3 = z_1 \bullet (\overline{z}_2 z_3)$ for all z_1, z_2, z_3 in $\mathbb{C}$.

We have the following result, where the summands in Klingenbergs theorem 3.3 have been labeled by an additional index q indicating that they are vector bundles over $B_q(\mathbb{C}\mathrm{P}^n)$.

Theorem 5.9. *Let p, q and r be positive integers with $p < q$ and $r < q$. There are isomorphisms of $\mathbb{T}$-vector bundles over the $\mathbb{T}$-equivariant diffeomorphism*

$$\phi_q : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbf{B}_q(\mathbb{C}\mathrm{P}^n)$$

as follows, where h_q is defined for $q = 0$ mod 2 and $k_{r,q}$ is defined for $r = q$ mod 2:

$$\begin{aligned}
&f_q : \epsilon_q(\mathbb{R}) \to \eta_{h,q}\,; && f_q([u,v],t)(z) = tH(u,v)((\sqrt{z})^q),\\
&g_q : \epsilon_q(\mathbb{C}(p)) \to \sigma_{h,p,q}\,; && g_q([u,v],\lambda)(z) = (\lambda \bullet z^{-p})H(u,v)((\sqrt{z})^q),\\
&h_q : \nu_{0,q} \to \eta_{v,q}\,; && h_q([u,v,w])(z) = V(u,v,w)((\sqrt{z})^q),\\
&k_{r,q} : (\nu \otimes_{\mathbb{R}} \mathbb{C})_{r,q} \to \sigma_{v,r,q}\,; && k_{r,q}([u,v,w] \otimes \lambda)(z) = (\lambda \bullet (\sqrt{z})^{-r})V(u,v,w)((\sqrt{z})^q).
\end{aligned}$$

In the last formula, $\sqrt{z}$ appears twice. One must use the same choice of square root in both places.

Proof. For all four maps, the real dimension of the fiber of the domain equals the real dimension of the fiber of the codomain. So it suffices to show that each map is well-defined, surjective on fibers and $\mathbb{T}$-equivariant.

By Remark 5.4, f_q is well-defined that is independent of the choice of square root of z and choice of representative for the class $[u,v]$. By Lemma 3.1 and Lemma 5.1, f_q is surjective on fibers. By equation (4) we see that it is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned}
f_q([u,v],t)(z_1z_2) &= tH(u,v)((\sqrt{z_1z_2})^q)\\
&= tH((\sqrt{z_1})^{-q}u, (\sqrt{z_1})^q v)((\sqrt{z_2})^q) = f_q(z_1 * [u,v],t)(z_2).
\end{aligned}$$

By remark 5.4, g_q is well-defined. For $\lambda = \alpha + i\beta$ and $z = e^{-2\pi it}$ we have that

$$\lambda \bullet z^{-p} = \alpha\cos(2\pi pt) + \beta\sin(2\pi pt)$$

such that g_q is surjective on fibers by Lemma 3.1 and Lemma 5.1. Since $z_1 \in \mathbb{T}$ we have that $z_1^{-1} = \overline{z}_1$ so we see that g_q is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned}
g_q([u,v],\lambda)(z_1z_2) &= (\lambda \bullet (z_1z_2)^{-p})f_q([u,v],1)(z_1z_2)\\
&= ((\lambda z_1^p) \bullet z_2^{-p})f_q(z_1 * [u,v],1)(z_2) = g_q(z_1 * ([u,v],\lambda))(z_2).
\end{aligned}$$

By Remark 5.4, h_q is well-defined for q even. By Lemma 3.1 and Lemma 5.1, h_q is surjective on fibers. By equation (4) we get that h_q is $\mathbb{T}$-equivariant as follows:

$$\begin{aligned} h_q([u,v,w])(z_1z_2) &= V(u,v,w)((\sqrt{z_1z_2})^q) \\ &= V((\sqrt{z_1})^{-q}u, (\sqrt{z_1})^q v, w)((\sqrt{z_2})^q) = h_q(z_1 * [u,v,w])(z_2). \end{aligned}$$

By Remark 5.4, $k_{r,q}$ is well-defined for $q \equiv r \bmod 2$. For $z = e^{-2\pi i t}$ with choice of square root $\sqrt{z} = e^{-\pi i t}$ we have

$$1 \bullet (\sqrt{z})^{-r} = \cos(\pi r t), \quad i \bullet (\sqrt{z})^{-r} = \sin(\pi r t).$$

Comparing with Lemma 3.1 and using Lemma 5.1 we see that any vector in the codomain fiber is the image of an element of the form $[u,v,w_1] \otimes 1 + [u,v,w_2] \otimes i$. Thus $k_{r,q}$ is surjective on fibers. Finally, we check that it is $\mathbb{T}$-equivariant

$$\begin{aligned} k_{r,q}([u,v,w] \otimes \lambda)(z_1z_2) &= (\lambda \bullet (\sqrt{z_1z_2})^{-r})V(u,v,w)((\sqrt{z_1z_2})^q) \\ &= (\lambda(\sqrt{z_1})^r \bullet (\sqrt{z_2})^{-r})V((\sqrt{z_1})^{-q}u, (\sqrt{z_1})^q v, w)((\sqrt{z_2})^q) \\ &= k_{r,q}(z_1 * ([u,v,w] \otimes \lambda))(z_2). \end{aligned}$$

□

Combining Theorem 3.3 , Theorem 5.9 and Proposition 5.7, we obtain our first main result:

Theorem 5.10. *For every positive integer q, there are isomorphisms of $\mathbb{T}$-vector bundles as follows:*

$$\mu_q^- \cong \epsilon_q(\mathbb{R}) \oplus \bigoplus_{0<s<q} \epsilon_q(\mathbb{C}(s)) \oplus \bigoplus_{\substack{0<r<q \\ r=q \text{ mod } 2}} (\nu_{r,q} \oplus \overline{\nu}_{r,q}) \qquad \text{for } q \text{ odd,}$$

$$\mu_q^- \cong \epsilon_q(\mathbb{R}) \oplus \bigoplus_{0<s<q} \epsilon_q(\mathbb{C}(s)) \oplus \nu_{0,q} \oplus \bigoplus_{\substack{0<r<q \\ r=q \text{ mod } 2}} (\nu_{r,q} \oplus \overline{\nu}_{r,q}) \qquad \text{for } q \text{ even.}$$

6 Projective bundles and Borel constructions

In this section we establish results which are aimed at calculating characteristic classes of the Borel construction with respect to the $\mathbb{T}$-action of the negative bundle.

Proposition 6.1. *(1) Let $f : \mathbf{V}_2(\mathbb{C}^{n+1}) \to X$ be a $U(1)$-map and let ξ_1, ξ_2 be complex $U(1)$-vector bundles over X. Then there is an isomorphism of complex vector bundles*

$$\mathbf{PV}_2(f, \xi_1 \oplus \xi_2) \cong \mathbf{PV}_2(f, \xi_1) \oplus \mathbf{PV}_2(f, \xi_2).$$

(2) Write ϵ^m for the product bundle $pr_1 : X \times \mathbb{C}^m \to X$, where $\mathbb{C}^m$ is equipped whith the $U(1)$-action given by complex multiplication. Then one has

$$\mathbf{PV}_2(f, \epsilon^m) = \mathbf{PV}_2(*, \epsilon^m) =: \mathbf{PV}_2(\epsilon^m),$$

where $*$ is the map from $\mathbf{V}_2(\mathbb{C}^{n+1})$ to a point. Furthermore,

$$\mathbf{PV}_2(\epsilon^m) = \bigoplus_{i=1}^{m} \mathbf{PV}_2(\epsilon^1).$$

Proof. (1) We have a commutative diagram with a well-defined map ψ as follows:

$$\begin{tikzcd}
& \mathbf{V}_2(\mathbb{C}^{n+1}) \arrow[dd] & \\
f^*(\xi_1 \oplus \xi_2) \arrow[ur] \arrow[rr, "\cong"] \arrow[dd] & & f^*(\xi_1) \oplus f^*(\xi_2) \arrow[ul] \arrow[dd] \\
& \mathbf{PV}_2(\mathbb{C}^{n+1}) & \\
f^*(\xi_1 \oplus \xi_2)/U(1) \arrow[ur] \arrow[rr, "\psi"] & & f^*(\xi_1)/U(1) \oplus f^*(\xi_2)/U(1) \arrow[ul]
\end{tikzcd}$$

Since the back faces are pullback squares, we see that ψ is an isomorphism.

(2) The pullback of ϵ^m is the product bundle $pr_1 : \mathbf{V}_2(\mathbb{C}^{n+1}) \times \mathbb{C}^m \to \mathbf{V}_2(\mathbb{C}^{n+1})$ for any $U(1)$-map f. So the first statement holds. The second follows from (1). $\square$

Definition 6.2. Let $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ be the canonical line bundle viewed as a $U(1)$-vector bundle with action given by complex multiplication. Let π_i for $i = 1, 2$ be the the composite maps

$$\pi_i : \mathbf{V}_2(\mathbb{C}^{n+1}) \longrightarrow \mathbf{PV}_2(\mathbb{C}^{n+1}) \xrightarrow{pr_i} \mathbb{C}\mathrm{P}^n,$$

where $pr_1([u, v]) = [u]$ and $pr_2([u, v]) = [v]$. Note that π_i is a $U(1)$-maps where the action on $\mathbb{C}\mathrm{P}^n$ is trivial. Define three complex line bundles over $\mathbf{PV}_2(\mathbb{C}^{n+1})$ by

$$L_0 = \mathbf{PV}_2(\epsilon^1), \quad L_1 = \mathbf{PV}_2(\pi_1, \gamma_1), \quad L_2 = \mathbf{PV}_2(\pi_2, \gamma_1).$$

Theorem 6.3. *There is an isomorphism of complex vector bundles*

$$\nu \oplus L_1 \oplus L_2 \cong L_0^{\oplus(n+1)}$$

which gives $\mathbb{T}$-equivariant isomorphisms for $r = q$ mod 2 as follows:

$$\begin{aligned} \nu_{r,q} \oplus (L_1)_{r,q} \oplus (L_2)_{r,q} &\cong (L_0)_{r,q}^{\oplus(n+1)}, \\ \overline{\nu}_{r,q} \oplus (\overline{L}_1)_{r,q} \oplus (\overline{L}_2)_{r,q} &\cong (\overline{L}_0)_{r,q}^{\oplus(n+1)}. \end{aligned}$$

Proof. By Propositions 6.1 we have

$$\mathbf{PV}_2(\pi, \gamma_2^\perp) \oplus \mathbf{PV}_2(\pi, \gamma_2) \cong \mathbf{PV}_2(\pi, \gamma_2^\perp \oplus \gamma_2) \cong \mathbf{PV}_2(\pi, \epsilon^{n+1}) \cong L_0^{\oplus(n+1)}.$$

Thus it suffices to show that $\mathbf{PV}_2(\pi, \gamma_2) \cong L_1 \oplus L_2$ in order to establish the first isomorphism. There is an isomorphism

$$+ : \pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1) \to \pi^*(\gamma_2); \quad ((u, v, w_1), (u, v, w_2)) \to (u, v, w_1 + w_2),$$

where $w_1 \in \mathrm{span}_{\mathbb{C}}(u)$ and $w_2 \in \mathrm{span}_{\mathbb{C}}(v)$. This isomorphism is equivariant with respect to the diagonal $U(1)$-action, so we get an isomorphism

$$\phi : (\pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1))/U(1) \xrightarrow{\cong} \pi^*(\gamma_2)/U(1).$$

Furthermore, there is a commutative diagram, where the right vertical map ψ is well-defined and a bundle map over $\mathbf{PV}_2(\mathbb{C}^{n+1})$,

$$\begin{tikzcd}
\pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1) \arrow[r] \arrow[d, equal] & (\pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1))/U(1) \arrow[d, "\psi"] \\
\pi_1^*(\gamma_1) \oplus \pi_2^*(\gamma_1) \arrow[r] & \pi_1^*(\gamma_1)/U(1) \oplus \pi_2^*(\gamma_1)/U(1)
\end{tikzcd}$$

The horizontal maps are surjections so by the diagram, ψ is also a surjection and hence an isomorphism of vector bundles. Thus, $\psi \circ \phi^{-1} : \mathbf{PV}_2(\pi, \gamma_2) \to L_1 \oplus L_2$ is the desired isomorphism.

So we have an isomorphism as stated in the first part of the theorem. Note that it is given by

$$([u, v, w], [u, v, w_1], [u, v, w_2]) \mapsto [u, v, w + w_1 + w_2].$$

It follows directly from this description that the isomorphism is $\mathbb{T}$-equivariant with respect to the actions from Definition 5.6. $\square$

We will now give pullback descriptions of the three line bundles. The following notation is used: For a complex vector bundle $\xi \to X$ and integer $m \in \mathbb{Z}$ we put $\xi(m) = \xi$ where $z \in \mathbb{T} \subseteq \mathbb{C}$ acts on each fiber by multiplication with z^m. Thus, $\xi(m) \to X$ is a $\mathbb{T}$-vector bundle.

Proposition 6.4. *Let $\epsilon^1 \to \mathbb{C}\mathrm{P}^n$ be the trivial line bundle and $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ the canonical line bundle. There are pullback diagrams of $\mathbb{T}$-vector bundles as follows for $r = q$ mod 2 and $i = 1, 2$:*

$$\begin{tikzcd}
(L_i)_{r,q} \arrow[r] \arrow[d] & \epsilon^1(\frac{r+(-1)^{i+1}q}{2}) \arrow[d] \\
\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "pr_i"] & \mathbb{C}\mathrm{P}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
(L_0)_{r,q} \arrow[r] \arrow[d] & \overline{\gamma}_1(\frac{r+(-1)^{i+1}q}{2}) \arrow[d] \\
\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "pr_i"] & \mathbb{C}\mathrm{P}^n
\end{tikzcd}$$

$$\begin{tikzcd}
(\overline{L}_i)_{r,q} \arrow[r] \arrow[d] & \overline{\epsilon}^1(\frac{r+(-1)^{i}q}{2}) \arrow[d] \\
\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "pr_i"] & \mathbb{C}\mathrm{P}^n
\end{tikzcd}
\qquad
\begin{tikzcd}
(\overline{L}_0)_{r,q} \arrow[r] \arrow[d] & \gamma_1(\frac{r+(-1)^{i}q}{2}) \arrow[d] \\
\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "pr_i"] & \mathbb{C}\mathrm{P}^n
\end{tikzcd}$$

Proof. Regarding the upper left pullback diagram for $i = 1$, the bundle map over pr_1 is given by

$$f_1 : L_1 \to \mathbb{C}\mathrm{P}^n \times \mathbb{C}; \quad [u, v, w] \mapsto ([u], k(w, u)),$$

where $k(w, u) \in \mathbb{C}$ is the scalar determined by $w = k(w, u)u$. Note that $k(aw, bu) = ab^{-1}k(w, u)$ for $a, b \in \mathbb{T}$. Since $k(zw, zu) = k(w, u)$ for $z \in U(1)$, the bundle map is well-defined:

$$[zu, zv, zw] \mapsto ([zu], k(zw, zu)) = ([u], k(w, u)).$$

It is also a fiber-wise isomorphism, so we have a pullback. We check that f_1 is $\mathbb{T}$-equivariant as well: Let $c = \sqrt{z}$ be a choice of square root for $z \in \mathbb{T}$. Then,

$$f_1(z * [u, v, w]) = f_1([c^{-q}u, c^q v, c^r w]) = ([c^{-q}u], k(c^r w, c^{-q}u)) = ([u], c^{r+q}k(w, u)).$$

Similarly, the bundle map $f_2 : L_2 \to \mathbb{C}\mathrm{P}^n \times \mathbb{C}$; $[u, v, w] \mapsto ([v], k(w, v))$, where $w = k(w, v)v$, gives us the upper left pullback diagram for $i = 2$.

The bundle maps in the lower left diagram are still f_1 and f_2, but with conjugate complex structure on domain and target. For $i = 1$, we have

$$\begin{aligned} f_1(z * [u, v, w]) &= f_1([c^{-q}u, c^q v, c^r \cdot w]) = [c^{-q}u, c^q v, c^{-r}w] = ([c^{-q}u], k(c^{-r}w, c^{-q}u)) \\ &= ([u], c^{-r+q}k(w, u)) = ([u], c^{r-q} \cdot k(w, u)). \end{aligned}$$

Thus, f_1 is $\mathbb{T}$-equivariant. A similar argument gives that f_2 is also $\mathbb{T}$-equivariant so we have the stated pullback diagrams for $i = 1, 2$.

The bundle map in the upper right diagram for $i = 1$ is given by

$$g_1 : L_0 \to \overline{\gamma}_1; \quad [u, v, k] \mapsto (\mathrm{span}_{\mathbb{C}}(u), k \cdot u) = (\mathrm{span}_{\mathbb{C}}(u), \overline{k}u).$$

It is well-defined because $z\overline{z} = 1$ for $z \in U(1)$ such that

$$[zu, zv, zk] \mapsto (\mathrm{span}_{\mathbb{C}}(zu), \overline{z}\overline{k}zu) = (\mathrm{span}_{\mathbb{C}}(u), \overline{k}u).$$

Since g_1 is a fiber-wise isomorphism, we have a pullback. g_1 is also $\mathbb{T}$-equivariant:

$$\begin{aligned} g_1(z * [u, v, k]) &= g_1([c^{-q}u, c^q v, c^r k]) = (\mathrm{span}_{\mathbb{C}}(c^{-q}u), c^{-r}\overline{k}c^{-q}u) \\ &= (\mathrm{span}_{\mathbb{C}}(u), c^{-r-q}\overline{k}u) = (\mathrm{span}_{\mathbb{C}}(u), c^{r+q} \cdot \overline{k}u). \end{aligned}$$

Similarly, the bundle map $g_2 : L_0 \to \overline{\gamma}_1$; $[u, v, k] \mapsto (\mathrm{span}_{\mathbb{C}}(v), \overline{k}v)$ gives us the upper right pullback diagram for $i = 2$.

The bundle maps g_1 and g_2 with conjugate complex structure on domain and target, gives the lower right pullback diagrams. □

We are interested in the vector bundle $E\mathbb{T} \times_{\mathbb{T}} \mu_q^-$. Fortunately, forming Borel constructions of G-vector bundles is well behaved with respect to Whitney sums and pullbacks.

Proposition 6.5. *Let G be a compact Lie group and let ξ, η be G-vector bundles over a G-space X. Then there is a natural isomorphism*

$$EG \times_G (\xi \oplus \eta) \xrightarrow{\cong} (EG \times_G \xi) \oplus (EG \times_G \eta).$$

Furthermore, if $f : Y \to X$ is a G-map, then there is a natural isomorphism

$$EG \times_G f^*(\xi) \xrightarrow{\cong} (EG \times_G f)^*(EG \times_G \xi).$$

Proof. Regarding the first isomorphism, observe that

$$EG \times (\xi \oplus \eta) \xrightarrow{\cong} (EG \times \xi) \oplus (EG \times \eta)$$

as seen by the following two pullback diagrams where the bottom composite equals the diagonal map on $EG \times X$:

$$\begin{tikzcd}
EG \times (\xi \oplus \eta) \arrow[r] \arrow[d] & EG \times EG \times \xi \times \eta \arrow[r] \arrow[d] & EG \times \xi \times EG \times \eta \arrow[d] \\
EG \times X \arrow[r, "\Delta \times \Delta"] & EG \times EG \times X \times X \arrow[r, "id \times tw \times id"] & EG \times X \times EG \times X
\end{tikzcd}$$

Then consider the commutative diagram

$$\begin{tikzcd}
& EG \times X \arrow[dd] & \\
EG \times (\xi \oplus \eta) \arrow[ur] \arrow[rr, "\cong"] \arrow[dd] & & (EG \times \xi) \oplus (EG \times \eta) \arrow[ul] \arrow[dd] \\
& EG \times_G X & \\
EG \times_G (\xi \oplus \eta) \arrow[ur] \arrow[rr, "\phi"] & & (EG \times_G \xi) \oplus (EG \times_G \eta) \arrow[ul]
\end{tikzcd}$$

The map ϕ is well-defined, and the back faces are pullback squares. It follows that ϕ is an isomorphism.

The second isomorphism follows by the commutative diagram

$$\begin{tikzcd}
& EG \times f^*(\xi) \arrow[rr] \arrow[dl] \arrow[dd] & & EG \times \xi \arrow[dl] \arrow[dd] \\
EG \times_G f^*(\xi) \arrow[rr] \arrow[dd] & & EG \times_G \xi \arrow[dd] & \\
& EG \times Y \arrow[rr] \arrow[dl] & & EG \times X \arrow[dl] \\
EG \times_G Y \arrow[rr] & & EG \times_G X &
\end{tikzcd}$$

Here the front face commutes since $f^*(\xi)$ is a pullback in the category of G-vector bundles. The side faces and the back face are pullbacks. It follows that the front face is a pullback. $\square$

Lemma 6.6. *Let G be a compact Lie group and $p : \xi \to X$ a G-vector bundle over a trivial G-space X. Write $\pi : EG \to BG$ for the universal principal G-bundle, and let $i_1 : BG \to BG \times X$ be the inclusion $b \mapsto (b, x_0)$ where $x_0 \in X$. Then there is a pullback diagram*

$$\begin{tikzcd}
EG \times_G p^{-1}(x_0) \arrow[r] \arrow[d, "pr_1"] & EG \times_G \xi \arrow[d, "EG \times_G p"] \\
BG \arrow[r, "i_1"] & BG \times X
\end{tikzcd}$$

Proof. We have a pullback of G-vector bundles

$$\begin{tikzcd} p^{-1}(x_0) \arrow[r] \arrow[d] & \xi \arrow[d] \\ \{x_0\} \arrow[r] & X \end{tikzcd}$$

If we apply the functor$EG \times_G (-)$ on this diagram, we get the desired pullback by equivalence of categories [tD1, Proposition 9.4], since $EG \times X$ is a locally trivial free G-space. □

7 Characteristic classes

In this section we compute the Chern classes of the vector bundles $(\mu_q^-)_{h\mathbb{T}}$. By Theorem 6.3 and Proposition 6.4 the following result is relevant:

Proposition 7.1. *Let $x = c_1(\gamma_1)$ and $u = c_1(\gamma_1^\infty)$ be the first Chern classes of the canonical line bundles $\gamma_1 \to \mathbb{C}\mathrm{P}^n$ and $\gamma_1^\infty \to \mathbb{C}\mathrm{P}^\infty = B\mathbb{T}$ such that*

$$H^*(B\mathbb{T} \times \mathbb{C}\mathrm{P}^n; \mathbb{Z}) = \mathbb{Z}[u] \otimes \mathbb{Z}[x]/(x^{n+1}).$$

Let $\epsilon^1 \to \mathbb{C}\mathrm{P}^n$ be the trivial line bundle. Then for every $m \in \mathbb{Z}$ we have

$$\begin{aligned} c_1(E\mathbb{T} \times_{\mathbb{T}} \gamma_1(m)) &= mu \otimes 1 + 1 \otimes x, \\ c_1(E\mathbb{T} \times_{\mathbb{T}} \epsilon^1(m))) &= mu \otimes 1. \end{aligned}$$

Proof. We start by proving the following claim:

$$c_1(E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)) = mu.$$

The first Chern class defines a group homomorphism

$$c_1 : (\mathrm{Vect}^1_{\mathbb{C}}(B\mathbb{T}), \otimes, \overline{()}) \to (H^2(B\mathbb{T}; \mathbb{Z}), +, -)$$

which is in fact an isomorphism since $B\mathbb{T}$ is homotopy equivalent to the CW-complex $\mathbb{C}\mathrm{P}^\infty$ (see [H, page 250] or [Ha]). There are isomorphisms of vector bundles for every n as follows:

$$\begin{aligned} S^{2n-1} \times_{\mathbb{T}} \mathbb{C}(1) &\to \gamma_1; & [v, z] &\mapsto (\mathrm{span}_{\mathbb{C}}(v), zv), \\ S^{2n-1} \times_{\mathbb{T}} \mathbb{C}(-1) &\to \overline{\gamma}_1; & [v, z] &\mapsto (\mathrm{span}_{\mathbb{C}}(v), \overline{z}v). \end{aligned}$$

Thus, we have isomorphisms $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(1) \cong \gamma_1$ and $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(-1) \cong \overline{\gamma}_1$. Note that $\mathbb{C}(0)$ equals $\mathbb{C}$ with trivial $\mathbb{T}$-action and for $k > 0$, we have that $\mathbb{C}(k) \cong \otimes_{i=1}^k \mathbb{C}(1)$ and $\mathbb{C}(-k) \cong \otimes_{i=1}^k \mathbb{C}(-1)$. We get corresponding tensor product decompositions of the vector bundles $E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}(m)$. The claim follows.

Choose base points in $B\mathbb{T}$ and $\mathbb{C}\mathrm{P}^n$, and consider the associated inclusions

$$i_1 : B\mathbb{T} \to B\mathbb{T} \times \mathbb{C}\mathrm{P}^n, \quad i_2 : \mathbb{C}\mathrm{P}^n \to B\mathbb{T} \times \mathbb{C}\mathrm{P}^n.$$

By Lemma 6.6 the pullback of both $\gamma_1(m)_{h\mathbb{T}}$ and $\epsilon^1(m)_{h\mathbb{T}}$ along i_1 equals the line bundle $E\mathbb{T}\times_{\mathbb{T}}\mathbb{C}(m)$. Thus,

$$i_1^*(c_1(\gamma_1(m)_{h\mathbb{T}})) = i_1^*(c_1((\epsilon^1(m))_{h\mathbb{T}})) = c_1(E\mathbb{T}\times_{\mathbb{T}}\mathbb{C}(m)) = mu.$$

The pullback of $\gamma_1(m)_{h\mathbb{T}}$ along $i_2 : \mathbb{C}\mathrm{P}^n \to E\mathbb{T}\times\mathbb{C}\mathrm{P}^n \to B\mathbb{T}\times\mathbb{C}\mathrm{P}^n$ equals γ_1 and the pullback of $\epsilon^1(m)_{h\mathbb{T}}$ along i_2 is the trivial line bundle ϵ^1. Thus,

$$i_2^*(c_1(\gamma_1(m)_{h\mathbb{T}})) = x, \quad i_2^*(c_1(\epsilon^1(m)_{h\mathbb{T}})) = 0.$$

Finally, $H^2(B\mathbb{T}\times\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is generated by the two classes $u\otimes 1$, $1\otimes x$ and

$$\begin{aligned} i_1^*(u\otimes 1) &= i_1^*\circ pr_1^*(u) = u, & i_2^*(u\otimes 1) &= i_2^*\circ pr_1^*(u) = 0,\\ i_1^*(1\otimes x) &= i_1^*\circ pr_2^*(x) = 0, & i_2^*(1\otimes x) &= i_2^*\circ pr_2^*(x) = x, \end{aligned}$$

so we have the desired result. □

Remark 7.2. For any complex vector bundle ξ one has that

$$\overline{E\mathbb{T}\times_{\mathbb{T}}\xi(m)} = E\mathbb{T}\times_{\mathbb{T}}\overline{\xi}(-m),$$

since in both cases, we mod out by the equivalence relation $(ez, v)\sim(e, z^m v)$, and we have the conjugate complex structure. So by the above result

$$\begin{aligned} c_1(E\mathbb{T}\times_{\mathbb{T}}\overline{\gamma}_1(m)) &= mu\otimes 1 - 1\otimes x,\\ c_1(E\mathbb{T}\times_{\mathbb{T}}\overline{\epsilon}^1(m))) &= mu\otimes 1. \end{aligned}$$

In order to use the pullback diagrams of Proposition 6.4, we must compute the induced maps in cohomology of the two projection maps

$$(pr_i)_{h\mathbb{T}} : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1})_{h\mathbb{T}} \to (\mathbb{C}\mathrm{P}^n)_{h\mathbb{T}} = B\mathbb{T}\times\mathbb{C}\mathrm{P}^n, \quad i = 1, 2.$$

The mod p cohomology of the domain space was computed in [BO]. We will need some of the results, leading to this calculation.

Let $\pi : \mathbb{P}(\gamma_2) \to \mathbf{G}_2(\mathbb{C}^{n+1})$ denote the projective bundle of the canonical bundle $\gamma_2 \to \mathbf{G}_2(\mathbb{C}^{n+1})$. We can describe the total space as a set of flags:

$$\mathbb{P}(\gamma_2) = \{V_1 \subseteq V_2 \subseteq \mathbb{C}^{n+1} | \dim_{\mathbb{C}}(V_i) = i\}.$$

By [BO] Lemma 2.6, we have an isomorphism

$$\psi : \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \xrightarrow{\cong} \mathbb{P}(\gamma_2); \quad [u, v]\mathbb{T} \mapsto (\mathrm{span}_{\mathbb{C}}(u) \subseteq \mathrm{span}_{\mathbb{C}}(u, v) \subseteq \mathbb{C}^{n+1}).$$

There is a canonical line bundle $\lambda \to \mathbb{P}(\gamma_2)$ with orthogonal complement line bundle $\lambda^\perp \to \mathbb{P}(\gamma_2)$ as follows:

$$\lambda = \{(V_1 \subseteq V_2, v) | v \in V_1\}, \quad \lambda^\perp = \{(V_1 \subseteq V_2, w) | w \in V_1^\perp \subseteq V_2\}.$$

There are pullback diagrams

$$\begin{array}{ccc} \lambda & \longrightarrow & \gamma_1 \\ \downarrow & & \downarrow \\ \mathbb{P}(\gamma_2) & \xrightarrow{p_1} & \mathbb{C}\mathrm{P}^n \end{array} \qquad \begin{array}{ccc} \lambda^\perp & \longrightarrow & \gamma_1 \\ \downarrow & & \downarrow \\ \mathbb{P}(\gamma_2) & \xrightarrow{p_2} & \mathbb{C}\mathrm{P}^n, \end{array}$$

where $p_1(V_1 \subseteq V_2) = V_1$ and $p_2(V_1 \subseteq V_2) = V_1^\perp$. Note also that $\lambda\oplus\lambda^\perp \cong \pi^*(\gamma_2)$. We have the following slightly enhanced version of Theorem 3.2 in [BO]:

Theorem 7.3. *There is an isomorphism of graded rings*

$$H^*(\mathbb{P}(\gamma_2);\mathbb{Z}) \cong \mathbb{Z}[x_1,x_2]/(Q_n,Q_{n+1}),$$

where x_1 and x_2 have degree 2 and for positive integers k,

$$Q_k(x_1,x_2) = \sum_{i=1}^{k} x_1^i x_2^{k-i} = \frac{x_1^{k+1}-x_2^{k+1}}{x_1-x_2}.$$

Furthermore, $p_1^*(x) = x_1$ and $p_2^*(x) = x_2$.

Proof. The ring structure is given in Theorem 3.2 of [BO]. From the proof of this theorem one has that

$$x_1 = c_1(\lambda), \quad x_2 = \pi^*(c_1(\gamma_2)) - c_1(\lambda), \quad \pi^*(c_1(\gamma_2)) = x_1 + x_2.$$

Thus, $p_1^*(x) = p_1^*(c_1(\gamma_1)) = c_1(p_1^*(\gamma_1)) = c_1(\lambda) = x_1$ and

$$x_1 + x_2 = c_1(\pi^*(\gamma_2)) = c_1(\lambda \oplus \lambda^\perp) = c_1(\lambda) + c_1(\lambda^\perp) = x_1 + c_1(\lambda^\perp)$$

such that $x_2 = c_1(\lambda^\perp)$. $\square$

Recall that a left G-space X is also a right G space with action $x * g = g^{-1} * x$ for $x \in X$, $g \in G$. For the right $\mathbb{T}$-space $\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})$ we have the following result:

Lemma 7.4. *The principal $\mathbb{T}$-bundle $\rho : \mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \to \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T}$ has associated complex line bundle $\lambda \otimes_{\mathbb{C}} \overline{\lambda^\perp}$. That is, we have an isomorphism of line bundles*

$$\begin{tikzcd}
\mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{C} \arrow[r, "\cong"] \arrow[d] & \lambda \otimes_{\mathbb{C}} \overline{\lambda^\perp} \arrow[d] \\
\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[r, "\cong"] & \mathbb{P}(\gamma_2)
\end{tikzcd}$$

The Euler class of ρ is

$$e(\rho) = x_1 - x_2.$$

Proof. The bundle map over the isomorphism ψ is defined by

$$[[u,v],k] \mapsto ((\operatorname{span}_{\mathbb{C}}(u) \subseteq \operatorname{span}_{\mathbb{C}}(u,v)), k(u \otimes v)).$$

We check that this is a well-defined map. Firstly, the linear span is unchanged by a rescaling of the generators by nonzero scalars. Secondly, for $z \in U(1)$ we have $[u,v] = [zu,zv]$, but also

$$zu \otimes zv = zu \otimes \overline{z} \cdot v = z\overline{z}u \otimes v = u \otimes v.$$

Thirdly, for $z \in \mathbb{T}$ and $c^2 = z$ we have $[[u,v]*z,k] = [[cu,c^{-1}v],k] = [[u,v],zk]$ but also

$$cu \otimes c^{-1}v = cu \otimes c \cdot v = c^2 u \otimes v = zu \otimes v = z(u \otimes v).$$

The bundle map is an isomorphism on fibers.

The Euler class of ρ equals the first Chern class of the associated line bundle, which is $c_1(\lambda \otimes_{\mathbb{C}} \overline{\lambda^\perp}) = c_1(\lambda) - c_1(\lambda^\perp) = x_1 - x_2$. $\square$

Remark 7.5. By the lemma above we get a sphere bundle interpretation of the projective Stiefel manifold

$$\mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) = \mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{T} = S(\mathbf{PV}_{2,1}(\mathbb{C}^{n+1}) \times_{\mathbb{T}} \mathbb{C}) \cong S(\lambda \otimes_{\mathbb{C}} \overline{\lambda^{\perp}}).$$

Thus, there is an isomorphism of left $\mathbb{T}$-spaces for every $q \in \mathbb{Z}$:

$$\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \cong S\big((\lambda \otimes_{\mathbb{C}} \overline{\lambda^{\perp}})(-q)\big).$$

For a left $\mathbb{T}$-space X with action map $\mu : \mathbb{T} \times X \to X$, we can twist the action by the power map $\lambda_n : \mathbb{T} \to \mathbb{T}; \lambda_n(z) = z^n$ and obtain another $\mathbb{T}$-space $X^{(n)}$. Thus the underlying spaces of X and $X^{(n)}$ are equal, but the action map for $X^{(n)}$ is $\mu_n : \mathbb{T} \times X^{(n)} \to X^{(n)}$; $\mu_n(z,x) = \mu(\lambda_n(x), z)$.

Proposition 7.6. *Let X be a left $\mathbb{T}$-space and let C_n denote the cyclic group of order n. There is a vertical and horizontal pullback of fibration sequences which is natural in X as follows:*

$$\begin{tikzcd}
\star \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d] & X \arrow[d] \\
BC_n \arrow[r] \arrow[d, equal] & E\mathbb{T} \times_{\mathbb{T}} X^{(n)} \arrow[r, "E(\lambda_n)\times_{\mathbb{T}} id"] \arrow[d, "pr_1"] & E\mathbb{T} \times_{\mathbb{T}} X \arrow[d, "pr_1"] \\
BC_n \arrow[r] & B\mathbb{T} \arrow[r, "B(\lambda_n)"] & B\mathbb{T}
\end{tikzcd}$$

Assume furthermore that the right $\mathbb{T}$-space associated to X gives a principal $\mathbb{T}$-bundle $\rho : X \to X/\mathbb{T}$. Write it as a pullback of the universal bundle $E\mathbb{T} \to B\mathbb{T}$ along a map $f : X/\mathbb{T} \to B\mathbb{T}$. Then the right vertical projection map in the diagram above can be replaced by f in the following sense: There is a diagram, which commutes up to homotopy, and where pr_2 is a weak homotopy equivalence

$$\begin{tikzcd}
E\mathbb{T} \times_{\mathbb{T}} X \arrow[r, "pr_2", "\simeq"'] \arrow[d, "pr_1"] & X/\mathbb{T} \arrow[dl, "f"] \\
B\mathbb{T} &
\end{tikzcd}$$

Finally, if we let $e(\rho)$ denote the Euler class, the two maps

$$H^*(B\mathbb{T};\mathbb{Z}) \xrightarrow{pr_1^*} H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(n)};\mathbb{Z}) \xleftarrow{pr_2^*} H^*(X/\mathbb{T};\mathbb{Z})$$

satisfy that

$$pr_1^*(nu) = pr_2^*(e(\rho)).$$

Proof. A proof for the first pullback diagram can be found in [BO] Lemma 6.1. Regarding the second diagram, first note that pr_2 is a fibration with contractible fiber $E\mathbb{T}$ and hence a weak homotopy equivalence. In order to verify that the diagram commutes up to homotopy, it suffices to check, that the right triangle in the following diagram commutes up to homotopy:

$$\begin{tikzcd}
E\mathbb{T} \times_{\mathbb{T}} X \arrow[r, "E\mathbb{T}\times_{\mathbb{T}}\hat{f}"] \arrow[d, "pr_2"] & E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] \arrow[d, "pr_2"] & B\mathbb{T} \arrow[dl, equal] \\
X/\mathbb{T} \arrow[r, "f"] & B\mathbb{T} &
\end{tikzcd}$$

Both pr_1 and pr_2 in the triangle are homotopy equivalences. By the diagrams

$$\begin{tikzcd}
E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] \arrow[d, "tw"] & B\mathbb{T} \arrow[d, equal] \\
E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T} \arrow[r, "pr_1"] & B\mathbb{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T}, B\mathbb{T}]} \arrow[r, "\cong"] \arrow[d, "tw^*"] & H^2(E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T};\mathbb{Z}) = \mathbb{Z} \arrow[d, "tw^*"] \\
{[E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T}, B\mathbb{T}]} \arrow[r, "\cong"] & H^2(E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T};\mathbb{Z}) = \mathbb{Z}
\end{tikzcd}$$

it suffices to see that $tw^* = id : \mathbb{Z} \to \mathbb{Z}$. The twist gives a self map of the fibration

$$\mathbb{T} \to E\mathbb{T} \times E\mathbb{T} \to E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T}$$

which is the identity on the fiber. By the long exact sequence of homotopy groups, one sees that $tw_* = id$ on $\pi_2(E\mathbb{T} \times_{\mathbb{T}} E\mathbb{T})$. By Hurewicz and universal coefficients, the result follows for cohomology.

We have that $f^*(u) = e(\rho)$. In the second diagram of the theorem, this gives us that $pr_1^*(u) = pr_2^*(e(\rho))$. Combining this with the first diagram, the last statement follows. □

Proposition 7.7. *There is a commutative diagram for $i = 1, 2$ where π_1 and π_2 denotes projection on first and second factor:*

$$\begin{tikzcd}
\mathbb{P}(\gamma_2) \arrow[r, "p_i"] & \mathbb{C}\mathrm{P}^n \\
\mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[u, "\cong"] \arrow[r, "pr_i/\mathbb{T}"] & \mathbb{C}\mathrm{P}^n \arrow[u, equal] \\
E\mathbb{T} \times_{\mathbb{T}} \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[u, "\pi_2"] \arrow[r, "E\mathbb{T}\times_{\mathbb{T}}pr_i"] \arrow[d, "\pi_1"] & B\mathbb{T} \times \mathbb{C}\mathrm{P}^n \arrow[u, "\pi_2"] \arrow[d, "\pi_1"] \\
B\mathbb{T} \arrow[r, equal] & B\mathbb{T}
\end{tikzcd}$$

In cohomology with $\mathbb{Z}$-coefficients, one has that

$$(E\mathbb{T} \times_{\mathbb{T}} pr_i)^*(1 \otimes x) = \pi_2^*(x_i) \text{ and } (E\mathbb{T} \times_{\mathbb{T}} pr_i)^*(qu \otimes 1) = \pi_2^*(x_1 - x_2).$$

Proof. Only the top square in the diagram requires an argument and it commutes by direct verification. The first equation follows by the diagram. The second follows by Lemma 7.4 and Proposition 7.6. □

We can now prove the following enhanced version of [BO] Theorem 4.1:

Theorem 7.8. *Let p be a prime and q a positive integer. There is an isomorphism*

$$H^*_{\mathbb{T}}(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1});\mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p[x_1,x_2]/(Q_n,Q_{n+1}), & p \nmid q, \\ \mathbb{F}_p[u,x,\sigma]/(x^{n+1},\sigma^2), & p \mid q,\ p \mid (n+1), \\ \mathbb{F}_p[u,x,\overline{\sigma}]/(x^n,\overline{\sigma}^2), & p \mid q,\ p \nmid (n+1), \end{cases}$$

where the classes u, x, x_1, x_2 have degree 2 and $\deg(\sigma) = 2n-1$, $deg(\overline{\sigma}) = 2n+1$. The polynomials $Q_k \in \mathbb{F}_p[x_1,x_2]$ are defined as follows for positive integers k:

$$Q_k(x_1,x_2) = \sum_{i=0}^{k} x_1^i x_2^{k-i}.$$

The maps

$$pr_i^* : H^*(B\mathbb{T} \times \mathbb{C}\mathrm{P}^n;\mathbb{F}_p) \to H^*_{\mathbb{T}}(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1});\mathbb{F}_p)$$

are given by the following for $i = 1,2$:

$$\begin{aligned} &u \otimes 1 \mapsto \frac{1}{q}(x_1 - x_2), && 1 \otimes x \mapsto x_i, && \textit{for } p \nmid q, \\ &u \otimes 1 \mapsto u, && 1 \otimes x \mapsto x, && \textit{for } p \mid q. \end{aligned}$$

Proof. The computation of the cohomology ring is given in [BO] Theorem 4.1. We review parts of the proof in order to include the description of the projection maps.

By proposition 7.6, we have a pullback of fibration sequences

$$\begin{tikzcd} BC_q \arrow[r] \arrow[d, equal] & E\mathbb{T} \times_{\mathbb{T}} \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathbf{PV}_{2,1}(\mathbb{C}^{n+1})/\mathbb{T} \arrow[d, "f"] \\ BC_q \arrow[r] & B\mathbb{T} \arrow[r, "B(\lambda_q)"] & B\mathbb{T} \end{tikzcd}$$

Assume that $p \nmid q$. Then, $H^*(BC_q;\mathbb{F}_p) = \mathbb{F}_p$, and by the Serre spectral sequence π_2 induces an isomorphism in cohomology. The results follows by Theorem 7.3 and Proposition 7.7 via universal coefficients.

Assume that $p \mid q$. One has that $H^*(BC_q;\mathbb{F}_p) = \mathbb{F}_p[v,w]/I_{p,q}$, where the degrees are $|v| = 1$, $|w| = 2$ and $I_{p,q}$ is the ideal $(v^2 - w)$ for $p = 2$, $4 \nmid q$ and the ideal (v^2) otherwise. The E_2-page of the Serre spectral sequence for the upper fibration has the form

$$E_2^{**} = \mathbb{F}_p[x_1,x_2]/(Q_n,Q_{n+1}) \otimes \mathbb{F}_p[v,w]/I_{p,q},$$

where the bi-degrees are $||x_1|| = ||x_2|| = (2,0)$, $||v|| = (0,1)$, $||w|| = (0,2)$. Via the spectral sequence for the lower fibration sequence, one finds that $d_2(w) = 0$, $d_2(v) = x_1 - x_2$ and that w is a permanent cycle. It follows that $E_3 = E_\infty$.

We let K_n and C_n denote the kernel and cokernel of multiplication with $(x_1 - x_2)$ on $\mathbb{F}_p[x_1,x_2]/(Q_n,Q_{n+1})$. Then

$$E_\infty^{**} = E_3^{**} = (C_n \oplus vK_n) \otimes \mathbb{F}_p[w].$$

In [BO], proof of Theorem 4.1, the kernel and cokernel is analyzed further, and one obtains

$$E_\infty^{**} = \begin{cases} \mathbb{F}_p[w,x_1,\sigma]/(x_1^{n+1},\sigma^2), & p \mid (n+1), \\ \mathbb{F}_p[w,x_1,\overline{\sigma}]/(x_1^n,\overline{\sigma}^2), & p \nmid (n+1), \end{cases}$$

where σ and $\overline{\sigma}$ are represented by v multiplied with explicit polynomials in x_1 and x_2. The bidegrees are $||\sigma|| = (2n-2, 1)$ and $||\overline{\sigma}|| = (2n, 1)$.

By Proposition 7.7, $\pi_2^*(x_1) = \pi_2^*(x_2)$, and we see that this cohomology class represents x_1 in the spectral sequence. The spectral sequence gives us that $\pi_1(x_1)^{n+1} = 0$ for $p \mid (n+1)$ and $\pi_2(x_1)^n = 0$ for $p \nmid (n+1)$. By the left square in the diagram above, we get that the cohomology class $\pi_1^*(u)$ represents w in the spectral sequence.

For $p \mid (n+1)$, σ gives a well defined cohomology class since $E_\infty^{2n-1,0} = 0$. This class has $\sigma^2 = 0$ since $E_\infty^{4n-4,2} = E_\infty^{4n-3,1} = E_\infty^{4n-2,0} = 0$. Similarly, for $p \nmid (n+1)$, $\overline{\sigma}$ gives a well-defined cohomology class with $\overline{\sigma}^2 = 0$ since $E_\infty^{2n+1,0} = 0$ and $E_\infty^{4n,2} = E_\infty^{4n+1,1} = E_\infty^{4n+2,0} = 0$.

Thus for $p \mid (n+1)$ we have a homomorphism of graded rings as follows:

$$\begin{aligned} &\mathbb{F}_p[u, x, \sigma]/(x^{n+1}, \sigma^2) \to H_{\mathbb{T}}^*(\mathbf{PV}_{2,q}(\mathbb{C}^{n+1}); \mathbb{F}_p); \\ &u \mapsto \pi_1^*(u), \quad x \mapsto \pi_2^*(x_1) = \pi_2^*(x_2), \quad \sigma \mapsto \sigma \end{aligned}$$

The homomorphism induces an isomorphism on associated graded objects, and therefore it is an isomorphism of rings. By this isomorphism and Proposition 7.7 we have that $pr_i^*(1 \otimes x) = \pi_2^*(x_1) = \pi_2^*(x_2) = x$ and $pr_i^*(u \otimes 1) = \pi_1^*(u) = u$ as desired. Similarly for $p \nmid (n+1)$. $\square$

Theorem 7.9. *Let p be a prime and let q be a positive integer. Assume that r is an integer such that $r = q$ mod 2. Define two polynomials*

$$\begin{aligned} P(x_1, x_2) &= (1 + \frac{r+q}{2q}(x_1 - x_2))(1 + \frac{r-q}{2q}(x_1 - x_2)), \\ R(u) &= (1 + \frac{r+q}{2}u)(1 + \frac{r-q}{2}u). \end{aligned}$$

In mod p cohomology, we have total Chern classes as follows: If $p \nmid q$,

$$c((\nu_{r,q})_{h\mathbb{T}}) = \frac{(1 + \frac{r+q}{2q}(x_1 - x_2) - x_1)^{n+1}}{P(x_1, x_2)}, \quad c((\overline{\nu}_{r,q})_{h\mathbb{T}}) = \frac{(1 + \frac{r+q}{2q}(x_1 - x_2) + x_2)^{n+1}}{P(x_1, x_2)}$$

and if $p \mid q$,

$$c((\nu_{r,q})_{h\mathbb{T}}) = \frac{(1 + \frac{r+q}{2}u - x)^{n+1}}{R(u)}, \quad c((\overline{\nu}_{r,q})_{h\mathbb{T}}) = \frac{(1 + \frac{r+q}{2}u + x)^{n+1}}{R(u)}.$$

Proof. Put $s_i = \frac{1}{2}(r + (-1)^{i+1}q)$ for $i = 1, 2$. By Proposition 7.1 and Remark 7.2 we have that

$$c_1(\overline{\gamma}_1(s_i)_{h\mathbb{T}}) = s_i u \otimes 1 - 1 \otimes x, \quad c_1(\epsilon^1(s_i)_{h\mathbb{T}}) = s_i u \otimes 1.$$

Assume that $p \nmid q$. From the pullbacks in Proposition 6.4 and from Theorem 7.8 we get first Chern classes

$$c_1\big(((L_0)_{r,q})_{h\mathbb{T}}\big) = \frac{s_i}{q}(x_1 - x_2) - x_i, \quad c_1\big(((L_i)_{r,q})_{h\mathbb{T}}\big) = \frac{s_i}{q}(x_1 - x_2).$$

Note that since $s_1/q(x_1 - x_2) - x_1 = s_2/q(x_1 - x_2) - x_2$ there is no contradiction in the first equation. By the direct sum decomposition in Theorem 6.3, the formula for the total Chern class of $(\nu_{r,q})_{h\mathbb{T}}$ follows. By a similar argument, we get the formula for the total Chern class of $(\overline{\nu}_{r,q})_{h\mathbb{T}}$.

Assume that $p \mid q$. In this case Proposition 6.4 and Theorem 7.8 gives us first Chern classes $s_i u - x$ and $s_i u$ respectively, and via Theorem 6.3, the formula for the total Chern class of $(\nu_{r,q})_{h\mathbb{T}}$ follows. Similarly for $(\overline{\nu}_{r,q})_{h\mathbb{T}}$. $\square$

We can now prove our second main result.

Theorem 7.10. *Let p be a prime and let q be a positive integer. In cohomology with mod p coefficients, we have total Chern classes as follows: For $p \nmid q$,*

$$c((\mu_q^-)_{h\mathbb{T}}) = \prod_{0<s<q} (1 + \frac{s}{q}(x_1 - x_2)) \cdot$$
$$\prod_{\substack{0<r<q \\ r=q \bmod 2}} \frac{\big((1 + \frac{r+q}{2q}(x_1 - x_2) - x_1)(1 + \frac{r+q}{2q}(x_1 - x_2) + x_2)\big)^{n+1}}{\big((1 + \frac{r+q}{2q}(x_1 - x_2))(1 + \frac{r-q}{2q}(x_1 - x_2))\big)^2}.$$

For $p \mid q$,

$$c((\mu_q^-)_{h\mathbb{T}}) = \prod_{0<s<q} (1 + su) \prod_{\substack{0<r<q \\ r=q \bmod 2}} \frac{\big((1 + \frac{r+q}{2}u - x)(1 + \frac{r+q}{2}u + x)\big)^{n+1}}{\big((1 + \frac{r+q}{2}u)(1 + \frac{r-q}{2}u)\big)^2}.$$

Proof. We use the direct sum decomposition from Theorem 5.10 which also gives a direct sum decomposition after forming $\mathbb{T}$-homotopy orbit bundles according to Proposition 6.5.

The bundle $\epsilon_q(\mathbb{R})_{h\mathbb{T}}$ is trivial so its Chern classes are zero. The $\mathbb{T}$-vector bundle $\epsilon_q(\mathbb{C}(s))$ is the pullback of $\mathbb{C}\mathrm{P}^n \times \mathbb{C}(s) \to \mathbb{C}\mathrm{P}^n$ along $pr_i : \mathbf{PV}_{2,q}(\mathbb{C}^{n+1}) \to \mathbb{C}\mathrm{P}^n$ both for $i = 1$ and $i = 2$. So by Proposition 7.1 and Theorem 7.8 we have

$$c_1(\epsilon_q(\mathbb{C}(s))_{h\mathbb{T}}) = pr_i^*(su \otimes 1) = \begin{cases} \frac{s}{q}(x_1 - x_2), & p \nmid q, \\ su, & p \mid q. \end{cases}$$

Theorem 7.9 above gives us the Chern classes of the remaining summants. $\square$

References

[BHM] Bökstedt M., Hsiang W.C., Madsen I., The cyclotomic trace and algebraic K-theory of spaces, Invent. Math. 111 (1993), 465–539.

[BO] Bökstedt M., Ottosen I., String cohomology groups of complex projective spaces, Algebraic & Geometric Topology 7 (2007), 2165-2238.

[tD1] tom Dieck T., Transformation groups, de Gruyter Studies in Mathematics 8, Walter de Gruyter & Co., Berlin 1987.

[tD2] tom Dieck T., Algebraic Topology, EMS, 2008.

[GHL] Gallot S., Hulin D., Lafontaine J., Riemannian geometry, Second edition, Universitext, Springer-Verlag, Berlin 1990.

[Ha] Hatcher A., Vector Bundles and K-Theory, http://www.math.cornell.edu/~hatcher

[H] Husemoller D., Fibre bundles, Third edition, Graduate Texts in Mathematics 20, Springer-Verlag, New York 1994.

[K1] Klingenberg W., Lectures on closed geodesics. Grundlehren der Math. Wiss. vol. 230, Springer Verlag, Berlin Heidelberg New York 1978.

[K2] Klingenberg W., The space of closed curves on a projective space, Quart. J. Math. Oxford Ser(2) 20 (1969), 11–31.

[K3] Klingenberg W., The space of closed curves on the sphere, Topology 7 (1968), 395–415.

[KN2] Kobayashi S., Nomizu K., Foundations of Differential Geometry, Volume 2 (1969), Wiley and Sons.

[MT] Madsen I., Tornehave J., From Calculus to Cohomology, de Rham cohomology and characteristic classes, Cambridge university press, Cambridge 1997.

[Z] Ziller W., The free loop space of globally symmetric spaces, Invent. Math. 41 (1977), 1–22.