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On Optimal Filter Designs for Fundamental Frequency Estimation

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***Abstract*— Recently, we proposed using Capon's minimum variance principle to find the fundamental frequency of a periodic waveform. The resulting estimator is formed such that it maximises the output power of a bank of filters. We present an alternative optimal single filter design, and then proceed to quantify the similarities and differences between the estimators using asymptotic analysis and Monte Carlo simulations. Our analysis shows that the single filter can be expressed in terms of the optimal filterbank, and that the methods are asymptotically equivalent, but generally different for finite length signals.**

I. Introduction

Bandlimited periodic waveforms can be decomposed into a finite set of sinusoids having frequencies that are integer multiples of a so-called fundamental frequency. Much research has been devoted to the problem of finding the fundamental frequency, and rightfully so. It is an important problem in many applications in, for example, speech and audio processing, and the problem has become no less relevant with the many interesting new applications in music information retrieval. The fundamental estimation problem can be mathematically defined as follows: a signal consisting of a set of harmonically related sinusoids related by the fundamental frequency ω_0 is corrupted by an additive white complex circularly symmetric Gaussian noise, $w(n)$, having variance σ^2, for $n = 0, \ldots, N-1$, i.e.,

$$x(n) = \sum_{l=1}^{L} \alpha_l e^{j\omega_0 l n} + w(n), \tag{1}$$

where $\alpha_l = A_l e^{j\psi_l}$, with $A_l > 0$ and ψ_l being the amplitude and the phase of the lth harmonic, respectively. The problem of interest is to estimate the fundamental frequency ω_0 from a set of N measured samples $x(n)$. Some representative examples of the various types of methods that are commonly used for fundamental frequency estimation are: linear prediction [1], correlation [2], subspace methods [3], frequency fitting [4], maximum likelihood (e.g., [5]), Bayesian estimation [6], and comb filtering [7]. The basic idea of the comb filtering approach is that when the teeth of the comb filter coincide with the frequencies of the individual harmonics, the output power of the filter is maximized. This idea is conceptually related to our approach derived in [5]; however, here we design optimal signal-adaptive filters reminiscent of beamformers for coherent signals, e.g. [8], for the estimation of the fundamental frequency. In particular, we consider two fundamental frequency estimators based on the well-known minimum variance principle [9]. The two estimators are based on different filter design formulations with one being based on a bank of filters and the other on only a single filter. The first of these estimators was recently proposed [5], while the second one is novel. The estimators are compared and the asymptotic properties of the estimators are analyzed and their finite length performance is investigated and compared in Monte Carlo simulations. For simplicity, we will here consider only the single pitch estimation problem but the presented methods can easily be applied to multi pitch estimation as well (see [5]).

The remainder of this paper is organized as follows. First, we introduce the two filter designs and the associated estimators in Section II. Then, we analyze and compare the estimators and their asymptotic properties in Section III. Their finite length performance is investigated in Section IV, before we conclude on our work in Section V.

II. Optimal Filter Designs

A. Filterbank Approach

We begin by introducing some useful notation, definitions and review the fundamental frequency estimator proposed in [5]. First, we construct a vector from M consecutive samples of the observed signal, i.e., $\mathbf{x}(n) = [\, x(n)\ x(n-1)\ \cdots\ x(n-M+1)\,]^T$ with $M \leq N$ and with $(\cdot)^T$ denoting the transpose. Next, we introduce the output signal $y_l(n)$ of the lth filter having coefficients $h_l(n)$ as $y_l(n) = \sum_{m=0}^{M-1} h_l(m)x(n-m) = \mathbf{h}_l^H \mathbf{x}(n)$, with $(\cdot)^H$ denoting the Hermitian transpose and $\mathbf{h}_l = [\, h_l(0)\ \ldots\ h_l(M-1)\,]^H$. Introducing the expected value $\mathrm{E}\{\cdot\}$ and defining the covariance matrix as $\mathbf{R} = \mathrm{E}\left\{\mathbf{x}(n)\mathbf{x}^H(n)\right\}$, the output power of the lth filter can be written as

$$\mathrm{E}\left\{|y_l(n)|^2\right\} = \mathrm{E}\left\{\mathbf{h}_l^H \mathbf{x}(n)\mathbf{x}^H(n)\mathbf{h}_l\right\} = \mathbf{h}_l^H \mathbf{R}\mathbf{h}_l. \tag{2}$$

The total output power of all the filters is $\sum_{l=1}^{L} \mathrm{E}\left\{|y_l(n)|^2\right\} = \sum_{l=1}^{L} \mathbf{h}_l^H \mathbf{R}\mathbf{h}_l$. Defining a matrix $\mathbf{H}$ consisting of the filters $\mathbf{h}_l$ as $\mathbf{H} = [\, \mathbf{h}_1\ \cdots\ \mathbf{h}_L\,]$, we can write the total output power as a sum of the power of the subband signals, i.e., $\sum_{l=1}^{L} \mathrm{E}\left\{|y_l(n)|^2\right\} = \mathrm{Tr}\left[\mathbf{H}^H \mathbf{R}\mathbf{H}\right]$. The filter design problem can now be stated. We seek to find a set of filters that pass power undistorted at specific frequencies, here the harmonic frequencies, while minimizing

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the power at all other frequencies. This problem can be formulated mathematically as the optimization problem:

$$\min_{\mathbf{H}} \operatorname{Tr}\left[\mathbf{H}^H\mathbf{R}\mathbf{H}\right] \quad \text{s.t.} \quad \mathbf{H}^H\mathbf{Z} = \mathbf{I}, \tag{3}$$

where $\mathbf{I}$ is the $L \times L$ identity matrix. Furthermore, the matrix $\mathbf{Z} \in \mathbb{C}^{M\times L}$ has a Vandermonde structure and is constructed from L complex sinusoidal vectors as

$$\mathbf{Z} = [\ \mathbf{z}(\omega_0)\ \cdots\ \mathbf{z}(\omega_0 L)\ ], \tag{4}$$

with $\mathbf{z}(\omega) = [\ 1\ e^{-j\omega}\ \cdots\ e^{-j\omega(M-1)}\ ]^T$. Or in words, the matrix contains the harmonically related complex sinusoids. The filter bank matrix $\mathbf{H}$ solving (3) is given by (see, e.g., [10])

$$\mathbf{H} = \mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}. \tag{5}$$

This data and frequency dependent filter bank can then be used to estimate the fundamental frequencies by maximizing the power of the filter's output, yielding

$$\hat{\omega}_0 = \arg\max_{\omega_0} \operatorname{Tr}\left[\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\right], \tag{6}$$

which depends only on the covariance matrix and the Vandermonde matrix constructed for different candidate fundamental frequencies.

B. An Alternative Approach

We proceed to examine an alternative formulation of the filter design problem and state its optimal solution. Suppose that we wish to design a single filter, $\mathbf{h}$, that passes the signal undistorted at the harmonic frequencies and suppresses everything else. This filter design problem can be stated as

$$\min_{\mathbf{h}} \mathbf{h}^H\mathbf{R}\mathbf{h} \quad \text{s.t.} \quad \mathbf{h}^H\mathbf{z}(\omega_0 l) = 1, \quad \text{for} \quad l = 1,\ldots,L. \tag{7}$$

It is worth stressing that the single filter in (7) is designed subject to L constraints, whereas in (3) the filter bank is formed using a matrix constraint. Clearly, these two formulations are related; we will return to this relation in detail in the following section. Introducing the Lagrange multipliers $\boldsymbol{\lambda} = [\ \lambda_1\ \ldots \lambda_L\ ]$, the Lagrangian dual function associated with the problem stated above can be written as

$$L(\mathbf{h},\boldsymbol{\lambda}) = \mathbf{h}^H\mathbf{R}\mathbf{h} - \left(\mathbf{h}^H\mathbf{Z} - \mathbf{1}^T\right)\boldsymbol{\lambda} \tag{8}$$

with $\mathbf{1} = [\ 1\ \ldots\ 1\ ]^T$. Taking the derivative with respect to the unknown filter impulse response, $\mathbf{h}$ and the Lagrange multipliers, we get

$$\nabla L(\mathbf{h},\boldsymbol{\lambda}) = \begin{bmatrix} \mathbf{R} & -\mathbf{Z} \\ -\mathbf{Z}^H & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}. \tag{9}$$

By setting this expression equal to zero, i.e., $\nabla L(\mathbf{h},\boldsymbol{\lambda}) = \mathbf{0}$, and solving for the unknowns, we obtain the optimal Lagrange multipliers for which the equality constraints are satisfied as $\boldsymbol{\lambda} = \left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}$ and the optimal filter as $\mathbf{h} = \mathbf{R}^{-1}\mathbf{Z}\boldsymbol{\lambda}$. By combining the last two expressions, we get the optimal filter expressed in terms of the covariance matrix and the Vandermonde matrix $\mathbf{Z}$, i.e.,

$$\mathbf{h} = \mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}. \tag{10}$$

The output power of this filter can then be expressed as

$$\mathbf{h}^H\mathbf{R}\mathbf{h} = \mathbf{1}^H\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}, \tag{11}$$

which, as for the first design, depends only on the inverse of $\mathbf{R}$ and the Vandermonde matrix $\mathbf{Z}$. By maximizing the output power, we readily obtain an estimate of the fundamental frequency as

$$\hat{\omega}_0 = \arg\max_{\omega_0} \mathbf{1}^H\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}. \tag{12}$$

III. Analysis

We will now relate the two filter design methods and the associated estimators in (6) and (12). It is perhaps not clear whether the two methods are identical or if there are some subtle differences. On one hand, the optimization problem in (3) allows for more degrees of freedom, since L filters of length M are designed while (7) involves only a single filter. On the other hand, the former design is based on L^2 constraints as opposed to the latter approach only involving L. Comparing the optimal filters in (5) and (10), we observe that the latter can be written in terms of the former as

$$\mathbf{h} = \mathbf{R}^{-1}\mathbf{Z}\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1} = \mathbf{H}\mathbf{1} = \sum_{l=1}^{L}\mathbf{h}_l, \tag{13}$$

so, clearly, the two methods are related. Using this to rewrite the output power in (11), we get

$$\mathbf{h}^H\mathbf{R}\mathbf{h} = \left(\sum_{l=1}^{L}\mathbf{h}_l^H\right)\mathbf{R}\left(\sum_{m=1}^{L}\mathbf{h}_m\right) \tag{14}$$

as opposed to $\operatorname{Tr}\left[\mathbf{H}^H\mathbf{R}\mathbf{H}\right] = \sum_{l=1}^{L}\mathbf{h}_l^H\mathbf{R}\mathbf{h}_l$ for the filterbank approach. It can be seen that the single-filter approach includes the cross-terms $\mathbf{h}_l^H\mathbf{R}\mathbf{h}_m$ for $l \neq m$, while these do not appear in the filterbank approach. From this it follows that the cost functions are generally different, i.e.,

$$\mathbf{1}^H\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1} \neq \operatorname{Tr}\left[\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\right] \tag{15}$$

$$\mathbf{h}^H\mathbf{R}\mathbf{h} \neq \operatorname{Tr}\left[\mathbf{H}^H\mathbf{R}\mathbf{H}\right]. \tag{16}$$

This means that the two filters will result in different output powers and thus possibly different estimates. Next, we will analyze the asymptotic properties of the cost function

$$\lim_{M\to\infty} M\mathbf{1}^H\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}. \tag{17}$$

In doing so we will make use of the following result (see, e.g., [11])

$$\lim_{M\to\infty}(\mathbf{A}\mathbf{B}) = \left(\lim_{M\to\infty}\mathbf{A}\right)\left(\lim_{M\to\infty}\mathbf{B}\right) \tag{18}$$

where it is assumed that the limits $\lim_{M\to\infty}\mathbf{A}$ and $\lim_{M\to\infty}\mathbf{B}$ exist for the individual elements of $\mathbf{A}$ and $\mathbf{B}$. Using (18) to rewrite the limit of $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, we get

$$\lim_{M\to\infty}\mathbf{I} = \left(\lim_{M\to\infty}\mathbf{A}\right)\left(\lim_{M\to\infty}\mathbf{A}^{-1}\right). \tag{19}$$

Next, suppose we have an analytic expression for the limit of $\lim_{M\to\infty}\mathbf{A}$, say, $\bar{\mathbf{A}}$, then we have $\mathbf{I} = \bar{\mathbf{A}}\left(\lim_{M\to\infty}\mathbf{A}^{-1}\right)$

from which we conclude that $(\lim_{M\to\infty} \mathbf{A}^{-1}) = \bar{\mathbf{A}}^{-1}$ and thus

$$\left(\lim_{M\to\infty} \mathbf{A}^{-1}\right) = \left(\lim_{M\to\infty} \mathbf{A}\right)^{-1}. \tag{20}$$

Applying (18) and (20) to the cost function in (23), yields

$$\lim_{M\to\infty} M\mathbf{1}^H \left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1} = \mathbf{1}^H \left(\lim_{M\to\infty}\left(\frac{1}{M}\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)\right)^{-1}\mathbf{1} \tag{21}$$

We are now left with the problem of determining the limit $\lim_{M\to\infty} \frac{1}{M}\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)$. In doing so, we will make use of the asymptotic equivalence of Toeplitz and circulant matrices. For a given Toeplitz matrix, here $\mathbf{R}$, we can construct an asymptotically equivalent circulant $M \times M$ matrix $\mathbf{C}$, under certain conditions, in the sense that [12] $\lim_{M\to\infty} \frac{1}{\sqrt{M}}\|\mathbf{C} - \mathbf{R}\|_F = 0$, where $\|\cdot\|_F$ is the Frobenius norm and the limit is taken over the dimensions of $\mathbf{C}$ and $\mathbf{R}$. A circulant matrix $\mathbf{C}$ has the eigenvalue decomposition $\mathbf{C} = \mathbf{Q}\mathbf{\Gamma}\mathbf{Q}^H$ where $\mathbf{Q}$ is the Fourier matrix. Thus, the complex sinusoids in $\mathbf{Z}$ are asymptotically eigenvectors of $\mathbf{R}$. This allows us to determine the limit as (see [12], [13])

$$\lim_{M\to\infty} \frac{1}{M}\left(\mathbf{Z}^H\mathbf{R}\mathbf{Z}\right) = \operatorname{diag}\left([\ \Phi(\omega_0)\ \cdots\ \Phi(\omega_0 L)\ ]\right) \tag{22}$$

with $\Phi(\omega)$ being the power spectral density of $x(n)$. Similarly, an expression for the inverse of $\mathbf{R}$ can be obtained as $\mathbf{C}^{-1} = \mathbf{Q}\mathbf{\Gamma}^{-1}\mathbf{Q}^H$ (again, see [12] for details). We now arrive at the following (see also [13] and [14]):

$$\lim_{M\to\infty} \frac{1}{M}\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right) = \operatorname{diag}\left([\ \Phi^{-1}(\omega_0)\ \cdots\ \Phi^{-1}(\omega_0 L)\ ]\right). \tag{23}$$

Asymptotically, (12) can therefore be written as

$$\lim_{M\to\infty} M\mathbf{1}^H \left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1} = \sum_{l=1}^{L} \Phi(\omega_0 l), \tag{24}$$

which is simply the sum over the power spectral density evaluated at the harmonic frequencies. Similar derivations for the filterbank formulation yield

$$\lim_{M\to\infty} M \operatorname{Tr}\left[\left(\mathbf{Z}^H\mathbf{R}^{-1}\mathbf{Z}\right)^{-1}\right] = \sum_{l=1}^{L} \Phi(\omega_0 l). \tag{25}$$

which is the same as (24). Note that for a finite M the above expression still involves only the diagonal terms (due to the trace), only the diagonal terms are not the power spectral density $\Phi(\omega)$ evaluated in certain points. From the above derivations, we conclude that the two cost functions are different for finite M and may yield different estimates, but are asymptotically equivalent.

IV. Experimental Results

The question remains to be answered whether there are any important differences for finite length covariance matrices and filters, and we will now seek to answer that question with some experiments, specifically using Monte Carlo simulations with synthetic signals generated according to (1). For each realization, the sample covariance matrix is estimated as $\widehat{\mathbf{R}} = \frac{1}{N-M+1}\sum_{n=0}^{N-M} \mathbf{x}(n)\mathbf{x}^H(n)$ which is used in place of the true covariance matrix. Since both methods require that $\widehat{\mathbf{R}}$ is invertible, we obviously have that $M < \frac{N}{2}$ and in practice we use $M = \left\lfloor \frac{2}{5}N \right\rfloor$, a value that has been determined empirically to yield good results. First, we will investigate the accuracy of the obtained fundamental frequency estimates measured in terms of the root mean square estimation error (RMSE). We do this for $\omega_0 = 0.6364$ with $L = 3$, unit amplitudes, and random phases drawn from a uniform probability density function. In Figure 1, the RMSE is plotted for $N = 50$ as a function of the signal-to-noise ratio (SNR) (as defined in [3] for the problem in (1)). The RMSE was estimated using 200 different realizations. Similarly, the RMSE is shown as a function of the number of samples, N, in Figure 2 for an SNR of 20 dB., again for 200 realizations. In both figures, the Cramér-Rao lower bound (CRLB), as derived in [3], is also shown. Both figures suggest that, all things considered, there is very little difference in terms of accuracy for the estimated parameters, with both estimators performing well. The methods seem to have different thresholding behaviour, though. We note that our simulations also show that the methods perform similarly as a function of ω_0, but in the interest of brevity, this figure has not been included herein. Next, we will measure the differences

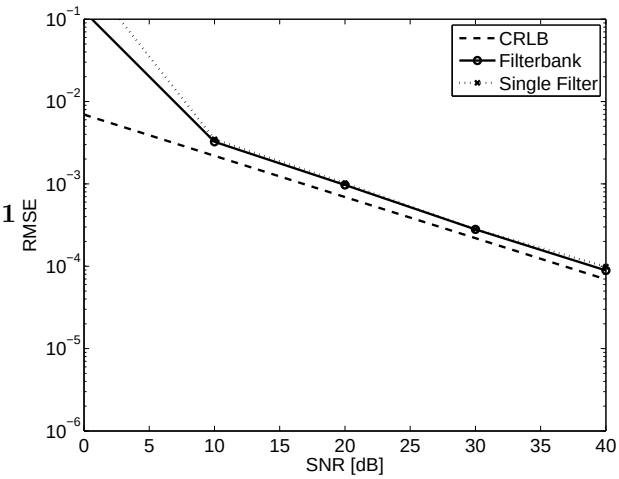

Fig. 1. RMSE as a function of the SNR for $N = 50$.

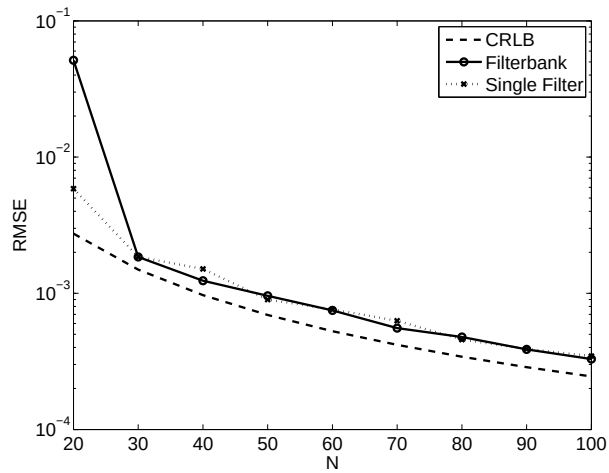

Fig. 2. RMSE as a function the number of samples N for $SNR = 20$ dB.

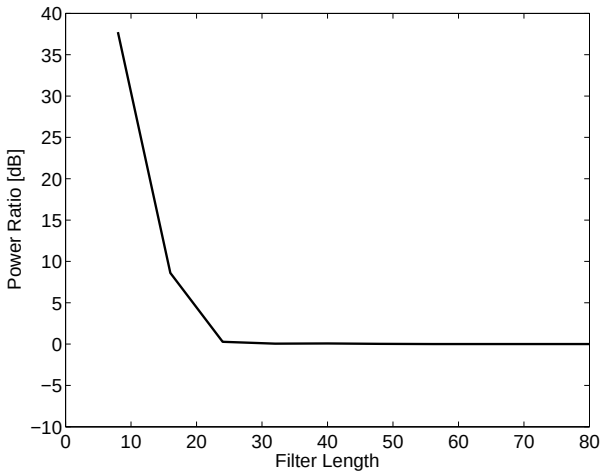

Fig. 3. Power ratio in dB as a function of the filter length M.

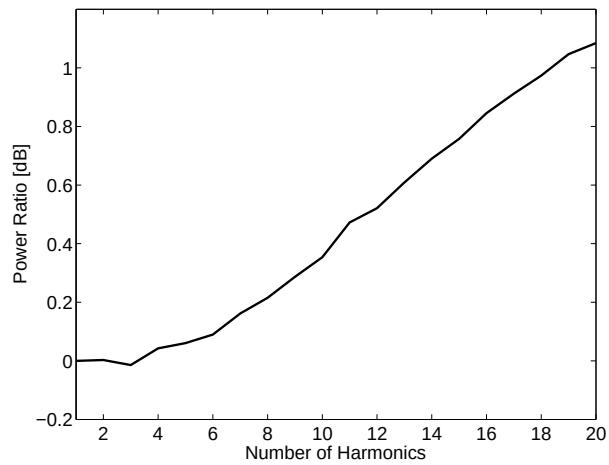

Fig. 4. Power ratio in dB as a function of the number of harmonics L.

of the estimated output powers. We measure this using the following power ratio (PR):

$$PR = 10\log_{10}\frac{\mathrm{E}\left\{\mathrm{Tr}\left[\left(\mathbf{Z}^H\widehat{\mathbf{R}}^{-1}\mathbf{Z}\right)^{-1}\right]\right\}}{\mathrm{E}\left\{\mathbf{1}^H\left(\mathbf{Z}^H\widehat{\mathbf{R}}^{-1}\mathbf{Z}\right)^{-1}\mathbf{1}\right\}} \quad [\mathrm{dB}], \qquad (26)$$

which is positive if the output power of the filterbank exceeds that of the single filter and vice versa. It should be noted that the expectation is taken over the realizations of the sample covariance matrix $\widehat{\mathbf{R}}$. The power ratio (averaged over 1000 realizations) is shown in Figure 3 as a function of the filter length M for an SNR of 10 dB. The filter length is related to the number of samples as $M = \left\lfloor \frac{2}{5}N \right\rfloor$. The fundamental frequency was drawn from a uniform distribution in the interval $[0.1571; 0.3142]$ with $L = 5$ in this experiment to avoid any biases due to special cases. The true fundamental frequency was used in obtaining the optimal filters. In Figure 4, the same is plotted for $N = 100$, this time as a function of the number of harmonics L with all other conditions being the same as before. Interestingly, both Figures 3 and 4 paint a rather clear picture: for low filter lengths and high number of harmonics, the single filter design method actually leads to a better estimate of the signal power while for high filter orders and few harmonics, the methods tend to perform identically. This suggests that the single filter design method is preferable.

V. Conclusion

We have presented two different optimal filter designs that can be used for finding high-resolution estimates of the fundamental frequency of periodic signals. The two designs differ in that one is based on the design of a filterbank while the other is based on a single filter. We have shown that the optimal single filter can in fact be obtained from the optimal filters of the filterbank and that the methods are in fact different for finite lengths, but are asymptotically equivalent. Experiments indicate that the single filter leads to superior results in terms of estimating the output power.

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