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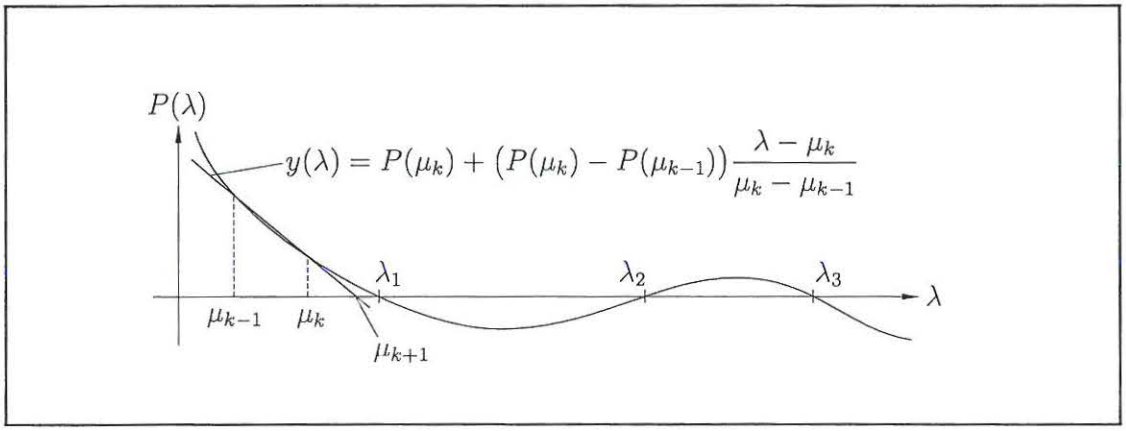

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Structural Dynamics, Vol. 9

Computational Dynamics
2nd Edition

by

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Preface

This book has been prepared for the course on Computational Dynamics given at the 8th semester at the structural engineering program in civil engineering at Aalborg University. The course presumes undergraduate knowledge of linear algebra and ordinary differential equations, as well as a basic graduate course in structural dynamics. Some of these prerequisites have been reviewed in an introductory chapter. The author wants to thank Jesper W. Larsen, Ph.D., and Ph.D. student Kristian Holm-Jørgensen for help with the preparation of figures and illustrations throughout the text.

Answers to all exercises given at the end of each chapter can be downloaded from the home page of the course at the address: www.civil.auc.dk/i5/engelsk/dyn/index/htm

Aalborg University, June 2005
Søren R.K. Nielsen

The present 2nd edition of my textbook on computational dynamic is in substance unchanged in comparison to the 1st edition. Only discovered typing errors and some formulations have been corrected.

Aalborg University, March 2007
Søren R.K. Nielsen

CHAPTER 1
INTRODUCTION

In this chapter the basic results in structural dynamics and linear algebra have been reviewed.

In Section 2.1 the relevant initial and eigenvalue problems in structural dynamics are formulated. The initial value problems form the background for the numerical integration algorithms described in Chapter 2, whereas the related undamped generalized eigenvalue problem constitute the generic problem for the numerical eigenvalue solvers described in Chapters 3-7. Formal solutions to various formulations of the initial value problem are indicated, and their shortcomings in practical applications are emphasized.

In Section 2.2 the semi-analytical solution approaches to the basic initial value problem of a multi degrees-of-freedom system in terms of expansion in various modal bases are presented. The application of these methods in relation to various reduction schemes, where typically merely the low-frequency modes are required, has been outlined.

1.1 Fundamentals of Linear Structural Dynamics

The basic equation of motion for forced vibrations of a *linear viscous* damped n degree-of-freedom system reads[1]

$$\left.\begin{array}{l} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \quad , \quad t > t_0 \\ \mathbf{x}(t_0) = \mathbf{x}_0 \quad , \quad \dot{\mathbf{x}}(t_0) = \dot{\mathbf{x}}_0 \end{array}\right\} \tag{1–1}$$

$\mathbf{x}(t)$ is the vector of *displacements* from the static equilibrium state, $\dot{\mathbf{x}}(t)$ is the *velocity vector*, $\ddot{\mathbf{x}}(t)$ is the *acceleration vector*, and $\mathbf{f}(t)$ is the *dynamic load vector*. $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$ denote the *initial value vectors* for the displacement and velocity, respectively. $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ indicate the *stiffness matrix*, *mass matrix* and *damping matrices*, all of the dimension $n \times n$. For any vector $\mathbf{a} \neq \mathbf{0}$ these fulfill the following *positive definite* and *symmetry* properties

$$\left.\begin{array}{ll} \mathbf{a}^T\mathbf{K}\mathbf{a} > 0 \quad , & \mathbf{K} = \mathbf{K}^T \\ \mathbf{a}^T\mathbf{M}\mathbf{a} > 0 \quad , & \mathbf{M} = \mathbf{M}^T \\ \mathbf{a}^T\mathbf{C}\mathbf{a} > 0 & \end{array}\right\} \tag{1–2}$$

[1] S.R.K. Nielsen: *Structural Dynamics, Vol. 1. Linear Structural Dynamics, 4th Ed.*. Aalborg tekniske Universitetsforlag, 2004.

If the structural system is not supported against stiff-body motions, the stiffness matrix is merely *positive semidefinite*, so $\mathbf{a}^T\mathbf{K}\mathbf{a} \geq 0$. Correspondingly, if some degrees of freedom are not carrying kinetic energy (pseudo degrees of freedom with zero mass or zero mass moment of inertia), the mass matrix is merely positive semidefinite, so $\mathbf{a}^T\mathbf{M}\mathbf{a} \geq 0$. The positive definite property of the damping matrix is a formal statement of the physical property that any non-zero velocity of the system should be related with energy dissipation. However, $\mathbf{C}$ needs not fulfill any symmetry properties, although energy dissipation is confined to the symmetric part of the matrix. So-called *aerodynamic damping loads* are external dynamic loads proportional to the structural velocity, i.e. $\mathbf{f}(t) = -\mathbf{C}_a\dot{\mathbf{x}}(t)$. If the *aerodynamic damping matrix* $\mathbf{C}_a$ is absorbed in the total damping matrix $\mathbf{C}$, no definite property can be stated.

The solution of the initial value problem (1-1) can be written in the following way[1]

$$\left.\begin{aligned} \mathbf{x}(t) &= \int_{t_0}^{t} \mathbf{h}(t-\tau)\mathbf{f}(\tau)d\tau + \mathbf{a}_0(t-t_0)\mathbf{x}_0 + \mathbf{a}_1(t-t_0)\dot{\mathbf{x}}_0 \\ \mathbf{a}_0(t) &= \mathbf{h}(t)\mathbf{C} + \dot{\mathbf{h}}(t)\mathbf{M} \\ \mathbf{a}_1(t) &= \mathbf{h}(t)\mathbf{M} \end{aligned}\right\} \tag{1–3}$$

$\mathbf{h}(t)$ is the *impulse response matrix* . Formally, this matrix is obtained as a solution to the initial value problem

$$\left.\begin{aligned} &\mathbf{M}\ddot{\mathbf{h}}(t) + \mathbf{C}\dot{\mathbf{h}}(t) + \mathbf{K}\mathbf{h}(t) = \mathbf{I}\,\delta(t) \\ &\mathbf{h}(0^-) = \mathbf{0} \quad , \quad \dot{\mathbf{h}}(0^-) = \mathbf{0} \end{aligned}\right\} \tag{1–4}$$

$\mathbf{I}$ is the *unit matrix* of the dimension $n \times n$, and $\delta(t)$ is *Dirac's delta function.*

The *frequency response matrix* $\mathbf{H}(i\omega)$ related to the system (1-1) is given as

$$\mathbf{H}(i\omega) = \Big((i\omega)^2\mathbf{M} + (i\omega)\mathbf{C} + \mathbf{K}\Big)^{-1} \tag{1–5}$$

where $i = \sqrt{-1}$ is the *complex unit.* The impulse response matrix is related to the frequency response matrix in terms of the *Fourier transform*

$$\mathbf{h}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{H}(i\omega)e^{i\omega t}dt \tag{1–6}$$

The convolution quadrature in (1-3) is relative easily evaluated numerically. Hence, the solution of (1-1) is available, if the impulse response matrix $\mathbf{h}(t)$ is known. In turn, the $n \times n$ components of this matrix can be calculated by the Fourier transforms (1-6). Although these transforms may be evaluated numerically, the necessary calculation efforts become excessive even for a moderate number of degrees of freedom n. Hence, more direct analytical or numerical approaches are

mandatory.

Undamped eigenvibrations $\left(\mathrm{C} = 0,\ \mathrm{f}(t) \equiv 0\right)$ are obtained as linear independent solutions to the homogeneous matrix differential equation

$$\mathrm{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{1–7}$$

Solutions are searched for on the form

$$\mathbf{x}(t) = \mathbf{\Phi}^{(j)} \mathrm{e}^{i\omega_j t} \tag{1–8}$$

Insertion of (1-8) into (1-7) provides the following homogeneous system of linear equations for the determination of the amplitude $\mathbf{\Phi}^{(j)}$ and the unknown constant ω_j

$$\left(\mathbf{K} - \lambda_j \mathrm{M}\right)\mathbf{\Phi}^{(j)} = \mathbf{0} \quad , \quad \lambda_j = \omega_j^2 \tag{1–9}$$

(1-9) is a so-called *generalized eigenvalue problem* (GEVP). If $\mathrm{M} = \mathbf{I}$, the eigenvalue problem is referred to as a *special eigenvalue problem* (SEVP).

The necessary condition for non-trivial solutions (i.e. $\mathbf{\Phi}^{(j)} \neq \mathbf{0}$) is that the determinant of the coefficient matrix is different from zero. This lead to the *characteristic equation*

$$P(\lambda) = \det\left(\mathbf{K} - \lambda \mathrm{M}\right) = 0 \tag{1–10}$$

$P(\lambda)$ indicates the *characteristic polynomial*. This may be expanded as

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n \tag{1–11}$$

The constants $a_0,\ a_1, \ldots, a_n$ are known as the *invariants* of the GEVP. This designation stems from the fact that the characteristic polynomial (1-11) is invariant under any rotation of the coordinate system. Obviously, $a_0 = (-1)^n \det(\mathrm{M})$, and $a_n = \det(\mathbf{K})$. The nth order equation (1-10) determines n solutions, $\lambda_1,\ \lambda_2, \ldots, \lambda_n$.

Assume that either M or K are positive definite. Then, all *eigenvalues* λ_j are non-negative real, which may be ordered in ascending magnitude as follows[1]

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \leq \infty \tag{1–12}$$

$\lambda_n = \infty$, if $\det(\mathrm{M}) = 0$. Similarly, $\lambda_1 = 0$, if $\det(\mathbf{K}) = 0$. The eigenvalues are denotes as *simple*, if $\lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n$. The *undamped circular eigenfrequencies* are related to the eigenvalues as follows

$$\omega_j = \sqrt{\lambda_j} \tag{1–13}$$

The corresponding solutions for the amplitude functions, $\Phi^{(1)}, \ldots, \Phi^{(n)}$, are denoted the *undamped eigenmodes* of the system, which are real as well.

The eigenvalue problems (1-9) can be assembled into following matrix formulation

$$\mathbf{K}\left[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n)}\right] = \mathbf{M}\left[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n)}\right]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \quad \Rightarrow$$

$$\mathbf{K}\Phi = \mathbf{M}\Phi\mathbf{\Lambda} \tag{1–14}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_n\end{bmatrix} \tag{1–15}$$

and Φ is the so-called *modal matrix* of dimension $n \times n$, defined as

$$\Phi = \left[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n)}\right] \tag{1–16}$$

If the eigenvalues are simple, the eigenmodes fulfill the following *orthogonality properties*[1]

$$\Phi^{(i)\,T}\mathbf{M}\Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ M_i & , \quad i = j \end{cases} \tag{1–17}$$

$$\Phi^{(i)\,T}\mathbf{K}\Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ \omega_i^2 M_i & , \quad i = j \end{cases} \tag{1–18}$$

where M_i denotes the *modal mass*.

The orthogonality properties (1-17) can be assembled in the following matrix equation

$$\left[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n)}\right]^T\mathbf{M}\left[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n)}\right] = \begin{bmatrix}M_1 & 0 & \cdots & 0\\ 0 & M_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & M_n\end{bmatrix} \quad \Rightarrow$$

$$\Phi^T\mathbf{M}\Phi = \mathbf{m} \tag{1–19}$$

where

$$\mathbf{m} = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{bmatrix} \tag{1–20}$$

The corresponding grouping of the orthogonality properties (1-18) reads

$$\Phi^T \mathbf{K} \Phi = \mathbf{k} \tag{1–21}$$

where

$$\mathbf{k} = \begin{bmatrix} \omega_1^2 M_1 & 0 & \cdots & 0 \\ 0 & \omega_2^2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n^2 M_n \end{bmatrix} \tag{1–22}$$

If the eigenvalues are all simple, the eigenmodes become linear independent, which means that the inverse Φ^{-1} exists.

In the following it is generally assumed that the eigenmodes are normalized to unit modal mass, so $\mathbf{m} = \mathbf{I}$. For the special eigenvalue problem, where $\mathbf{M} = \mathbf{I}$, it then follows from (1-19) that

$$\Phi^{-1} = \Phi^T \tag{1–23}$$

A matrix fulfilling (1-23) is known as *orthonormal* or *unitary*, and specifies a rotation of the coordinate system. All column and row vectors have the length 1, and are mutually orthogonal. It follows from (1-19) and (1-21) that in case of simple eigenvalues a so-called *similarity transformation* exists, defined by the modal matrix Φ, that reduce the mass and stiffness matrices to a diagonal form. In case of multiple eigenvalues the problem becomes considerable more complicated. For the standard eigenvalue problem with multiple eigenvalues it can be shown that the stiffness matrix merely reduces to the so-called *Jordan normal form* under the considered similarity transformation, given as follows

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{k}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{k}_m \end{bmatrix} \tag{1–24}$$

where $m \leq n$ denotes the number of different eigenvalues, and $\mathbf{k}_i$ signifies the so-called *Jordan boxes*, which are block matrices of the form

$$\omega_i^2\,,\ \begin{bmatrix}\omega_i^2 & 1\\ 0 & \omega_i^2\end{bmatrix},\ \begin{bmatrix}\omega_i^2 & 1 & 0\\ 0 & \omega_i^2 & 1\\ 0 & 0 & \omega_i^2\end{bmatrix},\ \begin{bmatrix}\omega_i^2 & 1 & 0 & 0\\ 0 & \omega_i^2 & 1 & 0\\ 0 & 0 & \omega_i^2 & 1\\ 0 & 0 & 0 & \omega_i^2\end{bmatrix},\ \ldots \tag{1–25}$$

Assume that the mass matrix is non-singular so $\mathbf{M}^{-1}$ exists. Then, the equations of motion (1-1) may be reformulated in the following *state vector form* of coupled 1st order differential equations[2]

$$\left.\begin{aligned}&\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{F}(t) \quad , \quad t > t_0\\ &\mathbf{z}(t_0) = \mathbf{z}_0\end{aligned}\right\} \tag{1–26}$$

$$\mathbf{z}(t) = \begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix},\ \mathbf{z}_0 = \begin{bmatrix}\mathbf{x}_0\\ \dot{\mathbf{x}}_0\end{bmatrix},\ \mathbf{A} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix},\ \mathbf{F}(t) = \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{f}(t)\end{bmatrix} \tag{1–27}$$

$\mathbf{z}(t)$ denotes the *state vector*. The corresponding homogeneous differential system reads

$$\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{z}(t) \tag{1–28}$$

The solution of (1-26) becomes[2]

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{A}t}\left(\mathrm{e}^{-\mathbf{A}t_0}\,\mathbf{z}_0 + \int_{t_0}^{t}\mathrm{e}^{-\mathbf{A}\tau}\mathbf{F}(\tau)d\tau\right) \tag{1–29}$$

The $n \times n$ matrix $\mathrm{e}^{\mathbf{A}t}$ is denoted the *matrix exponential function*. This forms a *fundamental matrix* to (1-28), i.e. the columns of $\mathrm{e}^{\mathbf{A}t}$ form $2n$ linearly independent solutions to (1-28). Actually, $\mathrm{e}^{\mathbf{A}t}$ is the fundamental matrix fulfilling the matrix initial value problem

$$\left.\begin{aligned}&\frac{d}{dt}\mathrm{e}^{\mathbf{A}t} = \mathbf{A}\mathrm{e}^{\mathbf{A}t} \quad , \quad t > 0\\ &\mathrm{e}^{\mathbf{A}\cdot 0} = \mathbf{I}\end{aligned}\right\} \tag{1–30}$$

where $\mathbf{I}$ denotes a $2n \times 2n$ unit matrix. Now, $\left(\mathrm{e}^{\mathbf{A}t}\right)^{-1} = \mathrm{e}^{-\mathbf{A}t}$ as shown in Box 1.1. Using this relation for $t = t_0$, (1-29) is seen to fulfil the initial value of (1-26). Since conventional differentiation rules also applies to matrix products, the fulfilment of the differential equation

[2]D.G. Zill and M.R. Cullen: *Differential Equations with Boundary-Value Problems, 6th Ed.* Brooks/Cole, 2005.

in (1-26) follows from differentiation of the right hand side of (1-29), and application of (1-30), i.e.

$$\frac{d}{dt}\mathbf{z}(t) = \frac{d}{dt}\mathrm{e}^{\mathbf{A}t}\left(\mathrm{e}^{-\mathbf{A}t_0}\mathbf{z}_0 + \int_{t_0}^{t}\mathrm{e}^{-\mathbf{A}\tau}\mathbf{F}(\tau)d\tau\right) + \mathrm{e}^{\mathbf{A}t}\Big(\mathbf{0} + \mathrm{e}^{-\mathbf{A}t}\mathbf{F}(t)\Big) =$$

$$\mathbf{A}\mathrm{e}^{\mathbf{A}t}\left(\mathrm{e}^{-\mathbf{A}t_0}\mathbf{z}_0 + \int_{t_0}^{t}\mathrm{e}^{-\mathbf{A}\tau}\mathbf{F}(\tau)d\tau\right) + \mathbf{I}\,\mathbf{F}(t) = \mathbf{A}\mathbf{z}(t) + \mathbf{F}(t) \tag{1–31}$$

The solution to (1-30) can be represented by the following infinite series of matrix products

$$\mathrm{e}^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots \tag{1–32}$$

where $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$, $\mathbf{A}^3 = \mathbf{A}\mathbf{A}\mathbf{A}$ etc. (1-32) is seen to fulfil the initial value $\mathrm{e}^{\mathbf{A}\cdot 0} = \mathbf{I}$. The fulfilment of the matrix differential equation (1-30) follows from termwise differentiation of the right-hand side of (1-32)

$$\frac{d}{dt}\mathrm{e}^{\mathbf{A}t} = \mathbf{0} + \mathbf{A} + \frac{t}{1!}\mathbf{A}^2 + \frac{t^2}{2!}\mathbf{A}^3 + \cdots = \mathbf{0} + \mathbf{A}\Big(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots\Big) = \mathbf{A}\mathrm{e}^{\mathbf{A}t} \tag{1–33}$$

The right-hand side of (1-32) converges for arbitrary values of t as the number of terms increases beyond limits on the right-hand side. Hence, $\mathrm{e}^{\mathbf{A}t}$ can in principle be calculated using this representation. However, for large values of t the convergence is very slow. In (1-29) the fundamental matrix $\mathrm{e}^{\mathbf{A}t}$ is needed for arbitrary positive and negative values of t. Hence, the use of (1-32) as an algorithm for $\mathrm{e}^{\mathbf{A}t}$ in the solution (1-29) becomes increasingly computational expensive as the integration time interval is increased. In Box 1.1 an analytical solution for $\mathrm{e}^{\mathbf{A}t}$ has been indicated, which to some extent circumvents this problem. However, this approach requires that all eigenvectors and eigenvalues of $\mathbf{A}$ are available.

Damped eigenvibrations are obtained as linear independent solutions to the homogeneous differential equation (1-28). Analog to (1-8) solutions are searched for on the form

$$\mathbf{z}(t) = \mathbf{\Psi}^{(j)}\mathrm{e}^{\lambda_j t} \tag{1–34}$$

Insertion of (1-34) into (1-28) provides the following special eigenvalue problem of the dimension $2n$ for the determination of the *damped eigenmodes* $\mathbf{\Psi}^{(j)}$ and the *damped eigenvalues* λ_j

$$\Big(\mathbf{A} - \lambda_j\mathbf{I}\Big)\mathbf{\Psi}^{(j)} = 0 \tag{1–35}$$

Since $\mathbf{A}$ is not symmetric, λ_j and $\mathbf{\Psi}^{(j)}$ are generally complex. Upon complex conjugation of (1-35), it is seen that if $(\lambda, \mathbf{\Psi})$ denotes an eigen-pair (solution) to (1-35), then $(\lambda^*, \mathbf{\Psi}^*)$ is also an eigen-pair, where * denotes *complex conjugation*. For lightly damped structures all eigenvalues are complex. In this case only n eigen-pairs $(\lambda_j, \mathbf{\Psi}^{(j)})$, $j = 1, 2, \ldots, n$ need to be considered, where no eigen-pair is a complex conjugate of another in the set.

Let the first n components of $\Psi^{(j)}$ be assembled in the n-dimensional sub-vector $\Phi^{(j)}$. Then, from (1-27) and (1-34) it follows that

$$\mathbf{x}(t) = \Phi^{(j)} e^{\lambda_j t} \quad \Rightarrow \quad \dot{\mathbf{x}}(t) = \lambda_j \Phi^{(j)} e^{\lambda_j t} \tag{1–36}$$

Consequently, the damped eigenmodes must have the structure

$$\Psi^{(j)} = \begin{bmatrix} \Phi^{(j)} \\ \lambda_j \Phi^{(j)} \end{bmatrix} \tag{1–37}$$

Hence, merely the first n components of $\Psi^{(j)}$ need to be determined.

The eigenvalue problems (1-35) can be assembled into the following matrix formulation, cf. (1-14)-(1-16)

$$\mathbf{A}\left[\Psi^{(1)}\,\Psi^{(2)}\ldots\Psi^{(2n)}\right] = \left[\Psi^{(1)}\,\Psi^{(2)}\ldots\Psi^{(2n)}\right]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2n} \end{bmatrix} \quad \Rightarrow$$

$$\mathbf{A}\Psi = \Psi\Lambda_A \tag{1–38}$$

where

$$\Lambda_A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2n} \end{bmatrix} \tag{1–39}$$

$$\Psi = \left[\Psi^{(1)}\,\Psi^{(2)}\ldots\Psi^{(2n)}\right] \tag{1–40}$$

The following representation of $\mathbf{A}$ in terms of the damped eigenmodes and eigenvalues follows from (1-38)

$$\mathbf{A} = \Psi\Lambda_A\Psi^{-1} \tag{1–41}$$

Assume that another $2n \times 2n$ matrix $\mathbf{B}$ has the same eigenvectors $\Psi^{(j)}$ as $\mathbf{A}$, whereas the eigenvalues as stored in the diagonal matrix Λ_B are different. Then, similar to (1-41), $\mathbf{B}$ has the representation

$$\mathbf{B} = \Psi\Lambda_B\Psi^{-1} \tag{1–42}$$

The matrix product of $\mathbf{A}$ and $\mathbf{B}$ becomes

$$\mathbf{AB} = \Psi\Lambda_A\Psi^{-1}\,\Psi\Lambda_B\Psi^{-1} = \Psi\Lambda_A\Lambda_B\Psi^{-1} \tag{1–43}$$

Since $\mathbf{\Lambda}_A$ and $\mathbf{\Lambda}_B$ are diagonal matrices, matrix multiplication of these is *commutative*, i.e. $\mathbf{\Lambda}_A\mathbf{\Lambda}_B = \mathbf{\Lambda}_B\mathbf{\Lambda}_A$. Then, (1-43) may be written

$$\mathbf{AB} = \mathbf{\Psi}\mathbf{\Lambda}_B\mathbf{\Lambda}_A\mathbf{\Psi}^{-1} = \mathbf{\Psi}\mathbf{\Lambda}_B\mathbf{\Psi}^{-1}\,\mathbf{\Psi}\mathbf{\Lambda}_A\mathbf{\Psi}^{-1} = \mathbf{BA} \tag{1–44}$$

Consequently, if two matrices have the same eigenvectors, matrix multiplication of two matrices is commutative. Identical eigenvectors of two matrices can also be shown to constitute the necessary condition (the "only if" requirement) for commutative matrix multiplication.

The so-called *adjoint eigenvalue problem* to (1-35) reads

$$\left(\mathbf{A}^T - \nu_i\mathbf{I}\right)\mathbf{\Psi}_{\mathrm{a}}^{(i)} = \mathbf{0} \tag{1–45}$$

Hence, $(\nu_i, \mathbf{\Psi}_{\mathrm{a}}^{(i)})$ denotes the eigenvalue and eigenvector to the *transposed matrix* $\mathbf{A}^T$. In Box 1.2 it is shown that the eigenvalues of the basic eigenvalue problem and the adjoint eigenvalue problem are identical, i.e. $\nu_j = \lambda_j$. Further, it is shown that the eigenvectors $\mathbf{\Psi}^{(j)}$ and $\mathbf{\Psi}_{\mathrm{a}}^{(j)}$ fulfill the orthogonality properties

$$\mathbf{\Psi}_{\mathrm{a}}^{(i)\,T}\mathbf{\Psi}^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ m_i & , \quad i = j \end{cases} \tag{1–46}$$

$$\mathbf{\Psi}_{\mathrm{a}}^{(i)\,T}\mathbf{A}\mathbf{\Psi}^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ \lambda_i m_i & , \quad i = j \end{cases} \tag{1–47}$$

where m_i is denoted the *complex modal mass*. Without any restriction this may be chosen as $m_j = 1$. Then, the orthogonality conditions (1-46) and (1-47) may be assembled into the following matrix relation

$$\mathbf{\Psi}_{\mathrm{a}}^T\mathbf{\Psi} = \mathbf{I} \tag{1–48}$$

$$\mathbf{\Psi}_{\mathrm{a}}^T\mathbf{A}\mathbf{\Psi} = \mathbf{\Lambda}_A \tag{1–49}$$

where

$$\mathbf{\Psi}_{\mathrm{a}} = \left[\mathbf{\Psi}_{\mathrm{a}}^{(1)}\,\mathbf{\Psi}_{\mathrm{a}}^{(2)}\ldots\mathbf{\Psi}_{\mathrm{a}}^{(2n)}\right] \tag{1–50}$$

From (1-48) follows that

$$\mathbf{\Psi}_{\mathrm{a}} = \left(\mathbf{\Psi}^{-1}\right)^T \tag{1–51}$$

Hence, the eigenvectors $\mathbf{\Psi}_{\mathrm{a}}^{(i)}$ of the adjoint eigenvalue problem (the *column* vectors in $\mathbf{\Psi}_{\mathrm{a}}$) normalized to unit modal mass are determined as the *row* vectors of $\mathbf{\Psi}^{-1}$. The eigenvectors $\mathbf{\Psi}^{(i)}$ of the direct eigenvalue problem may be arbitrarily normalized. Of course, if $(\lambda_i, \mathbf{\Psi}_{\mathrm{a}}^{(i)})$ is an eigen-solution to the adjoint eigenvalue problem, so is the complex conjugate $(\lambda_i^*, \mathbf{\Psi}_{\mathrm{a}}^{(i)*})$.

Box 1.1: Matrix exponential function

Multiple application of (1-41) provides for $k = 1, 2, \ldots$

$$\left.\begin{aligned} \mathbf{A}^2 &= \mathbf{A}\mathbf{A} &&= \boldsymbol{\Psi}\boldsymbol{\Lambda}_A\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Psi}\boldsymbol{\Lambda}_A\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\boldsymbol{\Lambda}_A^2\boldsymbol{\Psi}^{-1} \\ \mathbf{A}^3 &= \mathbf{A}\mathbf{A}^2 &&= \boldsymbol{\Psi}\boldsymbol{\Lambda}_A\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Psi}\boldsymbol{\Lambda}_A^2\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\boldsymbol{\Lambda}_A^3\boldsymbol{\Psi}^{-1} \\ &\vdots \\ \mathbf{A}^{j+1} &= \mathbf{A}\mathbf{A}^j &&= \boldsymbol{\Psi}\boldsymbol{\Lambda}_A\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Psi}\boldsymbol{\Lambda}_A^j\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\boldsymbol{\Lambda}_A^{j+1}\boldsymbol{\Psi}^{-1} \end{aligned}\right\} \tag{1–52}$$

$\boldsymbol{\Lambda}_A^{j+1}$ is a product of diagonal matrices, and then becomes a diagonal matrix itself. The diagonal elements become λ_k^{j+1}, where λ_k is the corresponding diagonal element in $\boldsymbol{\Lambda}_A$. Consider the matrix exponential function, cf. (1-32)

$$\mathrm{e}^{\boldsymbol{\Lambda}_A t} = \mathbf{I} + t\boldsymbol{\Lambda}_A + \frac{t^2}{2!}\boldsymbol{\Lambda}_A^2 + \frac{t^3}{3!}\boldsymbol{\Lambda}_A^3 + \cdots \tag{1–53}$$

Since all addends on the right-hand side of (1-53) are diagonal matrices, it follows that also $\mathrm{e}^{\boldsymbol{\Lambda}_A t}$ becomes diagonal with the diagonal elements

$$1 + t\lambda_k + \frac{t^2}{2!}\lambda_k^2 + \frac{t^3}{3!}\lambda_k^3 + \cdots = \mathrm{e}^{\lambda_k t} \tag{1–54}$$

where the Maclaurin series for the exponential function has been used in the last statement. Then, from (1-32), (1-52) and (1-53) follows

$$\mathrm{e}^{\mathbf{A}t} = \boldsymbol{\Psi}\left(\mathbf{I} + t\boldsymbol{\Lambda}_A + \frac{t^2}{2!}\boldsymbol{\Lambda}_A^2 + \frac{t^3}{3!}\boldsymbol{\Lambda}_A^3 + \cdots\right)\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\mathrm{e}^{\boldsymbol{\Lambda}_A t}\boldsymbol{\Psi}^{-1} \tag{1–55}$$

For arbitrary positive or negative t_1 and t_2 it then follows that

$$\mathrm{e}^{\mathbf{A}t_1}\mathrm{e}^{\mathbf{A}t_2} = \boldsymbol{\Psi}\mathrm{e}^{\boldsymbol{\Lambda}_A t_1}\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Psi}\mathrm{e}^{\boldsymbol{\Lambda}_A t_2}\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\mathrm{e}^{\boldsymbol{\Lambda}_A t_1}\mathrm{e}^{\boldsymbol{\Lambda}_A t_2}\boldsymbol{\Psi}^{-1} = \boldsymbol{\Psi}\mathrm{e}^{\boldsymbol{\Lambda}_A(t_1+t_2)}\boldsymbol{\Psi}^{-1} = \mathrm{e}^{\mathbf{A}(t_1+t_2)} \tag{1–56}$$

(1-56) represents the fundamental multiplication rule of matrix exponential functions. Especially for $t_1 = t$ and $t_2 = -t$ we have

$$\mathrm{e}^{\mathbf{A}t}\mathrm{e}^{-\mathbf{A}t} = \mathrm{e}^{\mathbf{A}\cdot 0} = \mathbf{I} \quad \Rightarrow \quad \mathrm{e}^{-\mathbf{A}t} = \left(\mathrm{e}^{\mathbf{A}t}\right)^{-1} \tag{1–57}$$

Further,

$$\mathbf{A}^{-n} = \mathbf{A}^{-1}\cdots\mathbf{A}^{-1} = \boldsymbol{\Psi}\boldsymbol{\Lambda}_A^{-n}\boldsymbol{\Psi}^{-1} \quad , \quad n = 1, 2, \ldots \tag{1–58}$$

(1-58) is proved by insertion of (1-52) and (1-58) into the identity $\mathbf{A}^n\mathbf{A}^{-n} = \mathbf{I}$. As seen, $\mathrm{e}^{\mathbf{A}t}$ and $\mathbf{A}^{-n}$ have identical eigenvectors. Then, from (1-44) it follows that

$$\mathbf{A}^{-n}\mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{\mathbf{A}t}\mathbf{A}^{-n} \quad , \quad n = 1, 2, \ldots \tag{1–59}$$

Box 1.2: Proof of orthogonality properties of eigenvectors and adjoint eigenvectors

(1-35) is pre-multiplied with $\Psi_{\mathrm{a}}^{(i)T}$, and (1-45) is pre-multiplied with $\Psi^{(j)T}$, leading to the identities

$$\Psi_{\mathrm{a}}^{(i)T}\mathbf{A}\Psi^{(j)} = \lambda_j \Psi_{\mathrm{a}}^{(i)T}\Psi^{(j)} \tag{1–60}$$

$$\begin{aligned} \Psi^{(j)T}\mathbf{A}^T\Psi_{\mathrm{a}}^{(i)} &= \nu_i \Psi^{(j)T}\Psi_{\mathrm{a}}^{(i)} \quad \Rightarrow \\ \Psi_{\mathrm{a}}^{(i)T}\mathbf{A}\Psi^{(j)} &= \nu_i \Psi_{\mathrm{a}}^{(i)T}\Psi^{(j)} \end{aligned} \tag{1–61}$$

The last statement follows from transposing the previous one. Withdrawal of (1-61) from (1-60) provides

$$\left(\lambda_j - \nu_i\right)\Psi_{\mathrm{a}}^{(i)T}\Psi^{(j)} = 0 \tag{1–62}$$

For $i = j$, (1-62) can only be fulfilled for $\nu_i = \lambda_i$, since $\Psi_{\mathrm{a}}^{(i)T}\Psi^{(i)} \neq 0$.

Next, presume simple eigenvalues, so $\lambda_i \neq \lambda_j$. Then, for $i \neq j$, (1-62) can only be fulfilled, if $\Psi_{\mathrm{a}}^{(i)T}\Psi^{(j)} = 0$, corresponding to (1-46).

Since the right-hand side of (1-60) is zero for $i \neq j$, this must also hold true for the left-hand side, i.e. $\Psi_{\mathrm{a}}^{(i)T}\mathbf{A}\Psi^{(j)} = 0$ for $i \neq j$. Then for $i = j$, (1-60) provides the result $\Psi_{\mathrm{a}}^{(i)T}\mathbf{A}\Psi^{(i)} = \lambda_i m_i$, which completes the proof of (1-47).

Example 1.1: Equations of motion of linear viscous damped 2DOF system

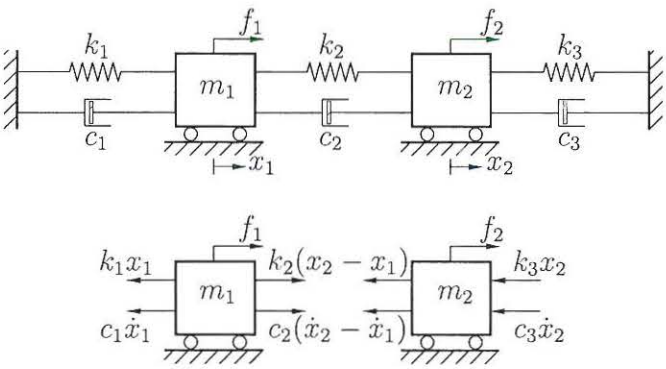

Fig. 1–1 Equation of motion of linear viscous damped 2DOF system.

The two-degree-of-freedom system shown on Fig. 1-1 consists of the masses m_1 and m_2 connected with linear elastic springs with the spring constants k_1, k_2, k_3, and linear viscous damper elements with the damper constants

c_1, c_2, c_3. The displacement of the masses from the static equilibrium state are denoted as $x_1(t)$ and $x_2(t)$. The velocities $\dot{x}_i(t)$ and accelerations $\ddot{x}_i(t)$ are considered positive in the same direction as the displacements $x_i(t)$ and the external forces $f_i(t)$. The masses are cut free from the springs and dampers in the deformed state, and the damper- and spring forces are applied as equivalent external forces. Next, Newton's 2nd law of motion is formulated for each of the masses leading to

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) + f_1(t) \\ m_2\ddot{x}_2 &= -k_3x_2 - k_2(x_2 - x_1) - c_3\dot{x}_2 - c_2(\dot{x}_2 - \dot{x}_1) + f_2(t) \end{aligned}\right\} \tag{1–63}$$

(1-63) may be formulated as the following matrix differential equations

$$\left.\begin{aligned} &\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{f}(t) \quad , \quad t > t_0 \\ &\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad , \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} \\ &\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \end{aligned}\right\} \tag{1–64}$$

For each of the masses an initial displacement $x_i(t_0) = x_{i,0}$ from the static equilibrium state and an initial velocity $\dot{x}_i(t_0) = \dot{x}_{i,0}$ are specified. These are assembled into the following initial value vectors

$$\mathbf{x}_0 = \mathbf{x}(t_0) = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \quad , \quad \dot{\mathbf{x}}_0 = \dot{\mathbf{x}}(t_0) = \begin{bmatrix} \dot{x}_{1,0} \\ \dot{x}_{2,0} \end{bmatrix} \tag{1–65}$$

The presented system will be further analyzed in various numerical examples throughout the book.

Example 1.2: Discretized equations of motion of a vibrating string

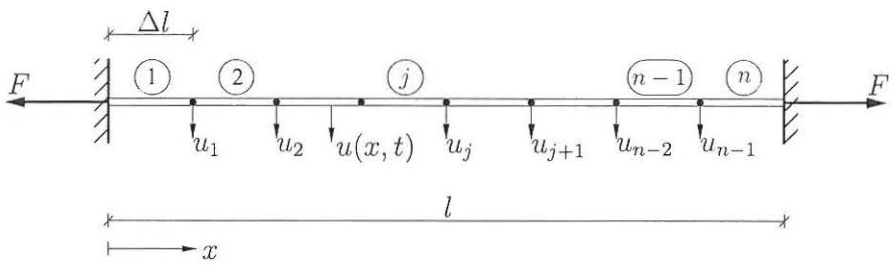

Fig. 1–2 Discretization of vibrating string.

Fig. 1-2 shows a vibrating string with the pre-stress force F, and the mass per unit length μ. The string has been divided into n identical elements, each of the length Δl. Hence, the total length of the string is $l = n\Delta l$. The displacement $u(x,t)$ of the string at the position x and time t in the transverse direction is given by the *wave equation* with homogeneous boundary conditions[1]

$$\left.\begin{aligned} &\mu\frac{\partial^2 u}{\partial t^2} - F\frac{\partial^2 u}{\partial x^2} = 0 \quad , \quad x \in]0, l[\\ &u(0,t) = u(l,t) = 0 \end{aligned}\right\} \tag{1–66}$$

where x is measured from the left support point. The spatial differential operator in (1-66) is discretized by means of a *central difference operator*,[2] i.e.

$$F\frac{\partial^2 u(x_i,t)}{\partial x^2} \simeq \frac{F}{\Delta l^2}\left(u_{i+1} - 2u_i + u_{i-1}\right) \quad , \quad i = 1,\ldots,n-1 \tag{1–67}$$

where $u_i(t) = u(x_i,t)$, $x_i = i\Delta l$. Further, let $\ddot{u}_i(t) = \frac{\partial}{\partial t^2}u(x_i,t)$. The boundary conditions imply that $u_0(t) = u_n(t) = 0$. Then, the discretized wave equation may be represented by the matrix differential equation

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{0} \tag{1–68}$$

$$\mathbf{x}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \\ u_{n-2}(t) \\ u_{n-1}(t) \end{bmatrix} , \mathbf{M} = \mu \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} , \mathbf{K} = \frac{F}{\Delta l^2} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{1–69}$$

Alternatively, the wave equation may be discretized by means of a finite element approach. Assuming linear interpolation between the nodal values stored in the vector $\mathbf{x}(t)$, and using the same interpolation for the displacement field and the variational field (*Galerkin variation*), the following mass- and stiffness matrices are obtained

$$\mathbf{M} = \frac{\mu\Delta l}{6} \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix} , \mathbf{K} = \frac{F}{\Delta l} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} \tag{1–70}$$

(1-70) represents the so-called *consistent mass matrix*, for which the same interpolation algorithm is used for discretizing the kinetic and the potential energy.[1] By contrast the diagonal mass matrix in (1-69) is referred to as a *lumped mass matrix*. As seen the central difference operator and Galerkin variation with piecewise linear interpolation leads to the same stiffness matrix. The presented system will be further analyzed in various numerical examples in what follows.

The calculated eigenvalues based on the system matrices (1-69) and (1-70) are shown in Fig. 1-3 as a function of the number of elements n. The solutions based on the lumped mass matrix (1-69) and the consistent mass (1-70) are shown with dotted and dashed signature, respectively. The numerical solutions have been given relative to the analytical solutions

$$\omega_{j,\mathrm{a}} = j\pi\sqrt{\frac{F}{\mu l^2}} \quad , \quad j = 1,\ldots,4 \tag{1–71}$$

As seen, the consistent mass matrix provides upper-bounds in accordance with the Rayleigh-Ritz principle described in Section 4.2. By contrast the lumped mass matrix provides lower bounds, when used in combination with the consistent stiffness matrix. There is no formal proof of this property, which merely is an empirical observation fulfilled in many dynamical problems. The indicated observation immediately suggest that an improvement

of the numerical solutions may be obtained by using a linear combination of the consistent and the lumped mass matrix. Typically, the mean value is used leading to the mass matrix

$$\mathbf{M} = \frac{1}{2}\frac{\mu\Delta l}{6}\begin{bmatrix} 4 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 4 \end{bmatrix} + \frac{1}{2}\mu\Delta l \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = \frac{\mu\Delta l}{12}\begin{bmatrix} 10 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 10 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 10 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 10 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 10 \end{bmatrix} \tag{1–72}$$

(1-72) is solved with the consistent stiffness matrix (1-70). The results are showed with a dashed-dotted signature on Fig. 1-3. As expected the results show a significant improvement. A theoretical argument for using the mean value of the consistent and lumped mass matrices for the combined mass matrix has been given by Krenk.[3]

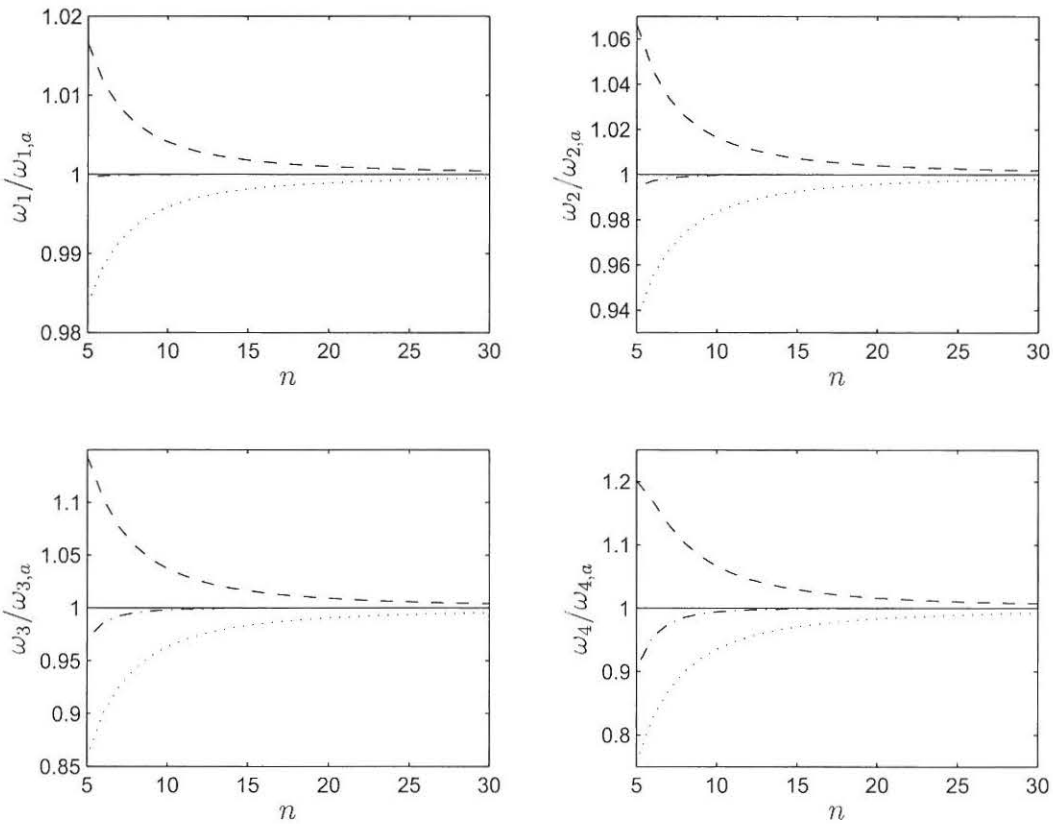

Fig. 1–3 Undamped eigenvibrations of string. —: Analytical solution. - - - : Consistent mass matrix. .... : Lumped mass matrix. -.-. : Combined mass matrix.

[3]S. Krenk: *Dispersion-corrected explicit integration of the wave equation.* Computer Methods in Applied Mechanics and Engineering, **191**, pp. 975-987, 2001.

Example 1.3: Verification of eigensolutions

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \tag{1–73}$$

Verify that the eigensolutions with modal masses normalized to 1 are given by

$$\mathbf{\Lambda} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix} \quad , \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}^{(1)} \, \mathbf{\Phi}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix} \tag{1–74}$$

Based on the proposed eigensolutions the following calculations are performed, cf. (1-14)

$$\left.\begin{aligned} \mathbf{K\Phi} &= \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ \frac{2}{5} & -\frac{24}{5} \end{bmatrix} \\ \mathbf{M\Phi\Lambda} &= \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ \frac{2}{5} & -\frac{24}{5} \end{bmatrix} \end{aligned}\right\} \tag{1–75}$$

This proofs the validity of the proposed eigensolutions. The orthonormality follows from the following calculations, cf. (1-19) and (1-21)

$$\left.\begin{aligned} \mathbf{\Phi}^T\mathbf{M\Phi} &= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix}^T \begin{bmatrix} \frac{5}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \mathbf{\Phi}^T\mathbf{K\Phi} &= \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix}^T \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12 \end{bmatrix} \end{aligned}\right\} \tag{1–76}$$

Example 1.4: M- and K-orthogonal vectors

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1–77}$$

Additionally, the following vectors are considered

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \tag{1–78}$$

From (1-78) the following matrix is formed

$$\mathbf{V} = [\mathbf{v}_1 \, \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} \tag{1–79}$$

We may then perform the following calculations, cf. (1-19) and (1-21)

$$\left.\begin{aligned}
\mathbf{V}^T\mathbf{M}\mathbf{V} &= \begin{bmatrix} 1 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
\mathbf{V}^T\mathbf{K}\mathbf{V} &= \begin{bmatrix} 1 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2.5858 & 0 \\ 0 & 5.4142 \end{bmatrix}
\end{aligned}\right\} \tag{1–80}$$

(1-80) shows that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are mutual orthogonal with weights $\mathbf{M}$ and $\mathbf{K}$, and that both have been normalized to unit modal mass. As will be shown in Example 1.5 neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ are eigenmodes, and the eigenvalues are different from 2.5858 and 5.4142. However, if *three* linear independent vectors are mutual orthogonal weighted with the three-dimensional matrices $\mathbf{M}$ and $\mathbf{K}$, they will be eigenmodes to the system.

Example 1.5: Analytical calculation of eigensolutions

The mass- and stiffness matrices defined in Example 1.4 are considered again. Now, an analytical solution of the eigenmodes and eigenvalues is wanted.

The generalized eigenvalue problem (1-9) becomes

$$\begin{bmatrix} 2-\frac{1}{2}\lambda_j & -1 & 0 \\ -1 & 4-\lambda_j & -1 \\ 0 & -1 & 2-\frac{1}{2}\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1–81}$$

The characteristic equation (1-10) becomes

$$P(\lambda) = \det\left(\begin{bmatrix} 2-\frac{1}{2}\lambda_j & -1 & 0 \\ -1 & 4-\lambda_j & -1 \\ 0 & -1 & 2-\frac{1}{2}\lambda_j \end{bmatrix}\right) =$$

$$\left(2-\frac{1}{2}\lambda_j\right)\left((4-\lambda_j)\left(2-\frac{1}{2}\lambda_j\right)-1\right)+1\cdot\left(-\left(2-\lambda_j\frac{1}{2}\right)\right) = \left(2-\frac{1}{2}\lambda_j\right)\left(6-4\lambda_j+\frac{1}{2}\lambda_j^2\right) = 0 \quad \Rightarrow$$

$$\lambda_j = \begin{cases} 2 & , \quad j=1 \\ 4 & , \quad j=2 \\ 6 & , \quad j=3 \end{cases} \tag{1–82}$$

Initially, the eigenmodes are normalized by setting an arbitrary component to 1. Here we shall choose $\Phi_3^{(j)} = 1$. The remaining components $\Phi_1^{(j)}$ and $\Phi_2^{(j)}$ are then determined from any two of the three equations (1-81). The first and the second equations are chosen, corresponding to

$$\begin{bmatrix} 2-\frac{1}{2}\lambda_j & -1 \\ -1 & 4-\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} \frac{2}{14-8\lambda_j+\lambda_j^2} \\ \frac{4-\lambda_j}{14-8\lambda_j+\lambda_j^2} \\ 1 \end{bmatrix} \tag{1–83}$$

The modal matrix with eigenmodes normalized as indicated in (1-83) is denoted as $\bar{\mathbf{\Phi}}$. This becomes

$$\bar{\Phi} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \tag{1–84}$$

The modal masses become, cf. (1-19)

$$\mathbf{m} = \bar{\Phi}^T \mathbf{M} \bar{\Phi} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{1–85}$$

$\Phi^{(1)}$ denotes the 1st eigenmode normalized to unit modal mass. This is related to $\bar{\Phi}^{(1)}$ in the following way

$$\Phi^{(1)} = \frac{1}{\sqrt{M_1}} \bar{\Phi}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1–86}$$

The other modes are treated in the same manner, which results in the following eigensolutions

$$\Lambda = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad , \quad \Phi = \left[\Phi^{(1)} \, \Phi^{(2)} \, \Phi^{(3)} \right] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{1–87}$$

Example 1.6: Undamped and damped eigenvibrations of 2DOF system

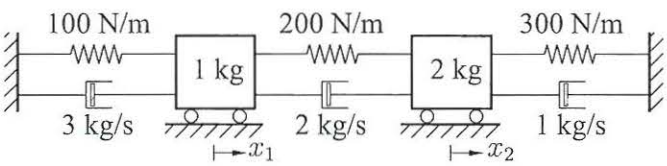

Fig. 1–4 Eigenvibrations of 2DOF system.

The system in Example 1.1 is considered again with the structural parameters defined in Fig. 1-3. The mass-damping and stiffness matrices become, cf. (1-64)

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{kg} \quad , \quad \mathbf{C} = \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix} \frac{\text{kg}}{\text{s}} \quad , \quad \mathbf{K} = \begin{bmatrix} 300 & -200 \\ -200 & 500 \end{bmatrix} \frac{\text{N}}{\text{m}} \tag{1–88}$$

The eigensolutions with modal masses normalized to 1 become

$$\Lambda = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} 131.39 & 0 \\ 0 & 418.61 \end{bmatrix} \text{s}^{-2} \quad , \quad \Phi = \left[\Phi^{(1)} \, \Phi^{(2)} \right] = \begin{bmatrix} 0.64262 & 0.76618 \\ 0.54177 & -0.45440 \end{bmatrix} \tag{1–89}$$

The matrix $\mathbf{A}$ defined by (1-27) becomes

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -300 & 200 & -5.0 & 2.0 \\ 100 & -250 & 1.0 & -1.5 \end{bmatrix} \tag{1–90}$$

The eigenvalues and eigenfunctions become

$$\mathbf{\Lambda}_A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} =$$

$$\begin{bmatrix} -0.7763 + 11.480i & 0 & 0 & 0 \\ 0 & -2.4737 + 20.231i & 0 & 0 \\ 0 & 0 & -0.7763 - 11.480i & 0 \\ 0 & 0 & 0 & -2.4737 - 20.231i \end{bmatrix} \tag{1–91}$$

$$\mathbf{\Psi} = \left[\mathbf{\Psi}^{(1)}\ \mathbf{\Psi}^{(2)} \quad \mathbf{\Psi}^{(3)}\ \mathbf{\Psi}^{(4)}\right] = \begin{bmatrix} \mathbf{\Phi}^{(1)} & \mathbf{\Phi}^{(2)} & \mathbf{\Phi}^{(1)^*} & \mathbf{\Phi}^{(2)^*} \\ \lambda_1\mathbf{\Phi}^{(1)} & \lambda_2\mathbf{\Phi}^{(2)} & \lambda_1^*\mathbf{\Phi}^{(1)^*} & \lambda_2^*\mathbf{\Phi}^{(2)^*} \end{bmatrix} =$$

$$\begin{bmatrix} 1.1693 - 0.1414i & -1.6846 - 0.3657i & 1.1693 + 0.1414i & -1.6846 + 0.3657i \\ 1 & 1 & 1 & 1 \\ 0.7153 + 13.534i & 11.565 - 33.177i & 0.7153 - 13.534i & 11.565 - 33.177i \\ -0.7763 + 11.480i & -2.4737 + 20.231i & -0.7763 - 11.480i & -2.4737 - 20.231i \end{bmatrix} \tag{1–92}$$

As seen from (1-92) the second component of the sub-vectors $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$ has been normalized to one. Hence, the entire modal matrix with 16 components is defined by merely 4 entities, namely the the first component of the sub-vectors $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$ and the eigenvalues λ_1 and λ_2.

The eigenvectors of the adjoint eigenvalue problem follows from (1-51) and (1-92)

$$\mathbf{\Psi}_{\mathrm{a}} = \left(\mathbf{\Psi}^{-1}\right)^T = \left[\mathbf{\Psi}_{\mathrm{a}}^{(1)}\ \mathbf{\Psi}_{\mathrm{a}}^{(2)} \quad \mathbf{\Psi}_{\mathrm{a}}^{(3)}\ \mathbf{\Psi}_{\mathrm{a}}^{(4)}\right] = \left[\mathbf{\Psi}_{\mathrm{a}}^{(1)}\ \mathbf{\Psi}_{\mathrm{a}}^{(2)} \quad \mathbf{\Psi}_{\mathrm{a}}^{(1)^*}\ \mathbf{\Psi}_{\mathrm{a}}^{(2)^*}\right] =$$

$$\begin{bmatrix} 0.1723 - 0.0430i & -0.1723 + 0.0388i & 0.1723 + 0.0430i & -0.1723 - 0.0388i \\ 0.3007 + 0.0411i & 0.1993 - 0.0592i & 0.3007 - 0.0411i & 0.1993 + 0.0592i \\ -0.0004 - 0.0154i & 0.0004 + 0.0087i & -0.0004 + 0.0154i & 0.0004 - 0.0087i \\ 0.0025 - 0.0260i & -0.0025 - 0.0098i & 0.0025 + 0.0260i & -0.0025 + 0.0098i \end{bmatrix} \tag{1–93}$$

As seen $\mathbf{\Psi}_{\mathrm{a}}^{(3)}$ and $\mathbf{\Psi}_{\mathrm{a}}^{(3)}$ become the complex conjugates of $\mathbf{\Psi}_{\mathrm{a}}^{(1)}$ and $\mathbf{\Psi}_{\mathrm{a}}^{(2)}$, cf. remarks subsequent to (1-51).

1.2 Solution of Initial Value Problem by Modal Decomposition Techniques

Assume that undamped eigenmodes $\Phi^{(i)}$ in addition to the orthogonality properties (1-17) and (1-18) also are orthogonal weighted with the damping matrix, i.e.

$$\Phi^{(i)T}\mathrm{C}\Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ 2\zeta_i\omega_i M_i & , \quad i = j \end{cases} \tag{1–94}$$

ζ_i denotes the *modal damping ratio*. In practice (1-94) is fulfilled, if the structure is lightly damped and the eigenfrequencies are well separated.[1] The orthogonality properties may be assembled into the following matrix relation similar to (1-19) and (1-21)

$$\Phi^T\mathrm{C}\Phi = \mathrm{c} \tag{1–95}$$

where

$$\mathrm{c} = \begin{bmatrix} 2\omega_1\zeta_1 M_1 & 0 & \cdots & 0 \\ 0 & 2\omega_2\zeta_2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\omega_n\zeta_n M_n \end{bmatrix} \tag{1–96}$$

The undamped eigenmodes are linear independent and may be used as a basis in the n-dimensional vector space. Hence, the displacement vector $\mathrm{x}(t)$ may be written as

$$\mathrm{x}(t) = \sum_{j=1}^{n} \Phi^{(j)} q_j(t) = \Phi\mathrm{q}(t) \quad , \quad \mathrm{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{1–97}$$

where $q_1(t), \ldots, q_n(t)$ represent the *undamped modal coordinates*, i.e. the coordinates in the vector basis formed by the undamped eigenmodes $\Phi^{(1)}, \ldots, \Phi^{(n)}$. Insertion of (1-95) into (1-1), followed by a pre-multiplication with Φ^T and use of (1-19), (1-21), (1-94), provides the following matrix differential equation for the modal coordinates

$$\left.\begin{aligned} &\mathrm{m}\ddot{\mathrm{q}}(t) + \mathrm{c}\dot{\mathrm{q}}(t) + \mathrm{k}\mathrm{q}(t) = \mathrm{F}(t) \quad , \quad t > t_0 \\ &\mathrm{q}(t_0) = \Phi^{-1}\mathrm{x}_0 \quad , \quad \dot{\mathrm{q}}(t_0) = \Phi^{-1}\dot{\mathrm{x}}_0 \end{aligned}\right\} \tag{1–98}$$

where

$$\mathbf{F}(t) = \mathbf{\Phi}^T \mathbf{f}(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{bmatrix} \tag{1–99}$$

$F_1(t), \ldots, F_n(t)$ are denoted the *modal loads*. Since $\mathbf{m}$, $\mathbf{c}$ and $\mathbf{k}$ are diagonal matrices the component differential equations related to (1-98) decouple completely. This is caused by the orthogonality condition (1-94) for which reason this relation is referred to as the *decoupling condition*. The differential equation for the kth modal coordinate reads

$$M_k \Big(\ddot{q}_k(t) + 2\zeta_k \omega_k \dot{q}_k(t) + \omega_k^2 q_k(t) \Big) = F_k(t) \quad , \quad k = 1, \ldots, n \tag{1–100}$$

Hence, the decoupling condition reduces the integration of a linear n degrees-of-freedom system to the integration of n single-degree-of-freedom oscillators.

Typically, the dynamic response is carried by lowest modes in the expansion (1-97). Assume that the modal response above the first $n_1 \ll n$ may be disregarded. Then (1-97) reduces to

$$\mathbf{x}(t) \simeq \sum_{j=1}^{n_1} \mathbf{\Phi}^{(j)} q_j(t) = \mathbf{\Phi}_1 \mathbf{q}_1(t) \tag{1–101}$$

where $\mathbf{\Phi}_1$ is a reduced modal matrix of dimension $n \times n_1$, and $\mathbf{q}_1(t)$ is a sub-vector of modal coordinates defined as

$$\mathbf{\Phi}_1 = \left[\mathbf{\Phi}^{(1)} \, \mathbf{\Phi}^{(2)} \ldots \mathbf{\Phi}^{(n_1)} \right] \quad , \quad \mathbf{q}_1(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_{n_1}(t) \end{bmatrix} \tag{1–102}$$

(1-101) completely ignores the influence of the higher modes. Although the *dynamic* response of these modes are ignorable, they may still influence the low-frequency components via a *quasistatic response* component. A consistent correction taken this effect into consideration reads[1]

$$\mathbf{x}(t) \simeq \sum_{j=1}^{n_1} \mathbf{\Phi}^{(j)} q_j(t) + \left(\mathbf{K}^{-1} - \sum_{j=1}^{n_1} \frac{1}{\omega_j^2 M_j} \mathbf{\Phi}^{(j)} \mathbf{\Phi}^{(j)T} \right) \mathbf{f}(t) \tag{1–103}$$

(1-103) may be represented in terms of the following equivalent matrix formulation

$$\mathbf{x}(t) \simeq \mathbf{\Phi}_1 \mathbf{q}_1(t) + \left(\mathbf{K}^{-1} - \mathbf{\Phi}_1 \, \mathbf{k}_1^{-1} \, \mathbf{\Phi}_1^T \right) \mathbf{f}(t) \tag{1–104}$$

where

$$\mathbf{k}_1 = \begin{bmatrix} \omega_1^2 M_1 & 0 & \cdots & 0 \\ 0 & \omega_2^2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{n_1}^2 M_{n_1} \end{bmatrix} \tag{1–105}$$

Both (1-101) and (1-103) requires knowledge of the first n_1 eigen-pairs $(\omega_j^2, \Phi^{(j)})$. The corresponding modal coordinates are determined from the first n_1 equations in (1-100). Correspondingly, the $2n$ eigenvectors $\Psi^{(j)}$, $j = 1, \ldots, 2n$ of the matrix $\mathbf{A}$ form a vector basis in the $2n$-dimensional vector space. Then, the state vector $\mathbf{z}(t)$ admits the representation

$$\mathbf{z}(t) = \sum_{j=1}^{2n} \Psi^{(j)} q_j(t) = \Psi \mathbf{q}(t) \tag{1–106}$$

where

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_{2n}(t) \end{bmatrix} \tag{1–107}$$

$q_1(t), \ldots, q_{2n}(t)$ represent the *damped modal coordinates*, i.e. the coordinates in the vector basis made up of the damped eigenmodes $\Psi^{(1)}, \ldots \Psi^{(2n)}$. Insertion of (1-106) into (1-26), followed by pre-multiplication of Ψ_{a}^T and use of (1-48), (1-49), the following matrix differential differential equations for the damped modal coordinates is obtained

$$\left.\begin{aligned} \dot{\mathbf{q}}(t) &= \Lambda_A \mathbf{q}(t) + \mathbf{G}(t) \quad , \quad t > t_0 \\ \mathbf{q}(t_0) &= \Psi^{-1}\mathbf{z}_0 = \Psi_{\mathrm{a}}^T \mathbf{z}_0 \end{aligned}\right\} \tag{1–108}$$

where

$$\mathbf{G}(t) = \Psi_{\mathrm{a}}^T \mathbf{F}(t) = \begin{bmatrix} G_1(t) \\ G_2(t) \\ \vdots \\ G_{2n}(t) \end{bmatrix} \tag{1–109}$$

In the initial value statement of (1-106) the relation (1-51) between the adjoint and direct modal matrices has been used. $G_j(t) = \Psi_{\mathrm{a}}^{(j)T} \mathbf{F}(t)$ denotes the jth *damped modal load.*

(1-108) indicates $2n$ decoupled complex 1st order differential equations. The differential equation for the jth modal coordinate reads

$$\dot{q}_j(t) = \lambda_j q_j(t) + G_j(t) \quad , \quad j = 1, \ldots, 2n \tag{1–110}$$

Since, $\left(\lambda_{j+n}, \Psi_{\mathrm{a}}^{(j+n)}\right) = \left(\lambda_j^*, \Psi_{\mathrm{a}}^{(j)*}\right)$ for $n = 1, \ldots, n$, it follows that $G_{j+n}(t) = G_j^*(t)$, and in turn that $q_{j+n}(t) = q_j^*(t)$. Hence, merely the first n differential equations (1-110) need to be integrated. Then, (1-104) may be written

$$\mathbf{z}(t) = 2\mathrm{Re}\left(\sum_{j=1}^{n} \Psi^{(j)} q_j(t)\right) \tag{1–111}$$

As is the case for expansion in undamped modal coordinates the response is primarily carried by the lowest n_1 modes leading to the following reduced form of (1-111)

$$\mathbf{z}(t) \simeq 2\mathrm{Re}\left(\sum_{j=1}^{n_1} \Psi_1 \mathbf{q}_1(t)\right) \tag{1–112}$$

where

$$\Psi_1 = \left[\Psi^{(1)}\, \Psi^{(2)} \ldots \Psi^{(n_1)}\right] \quad , \quad \mathbf{q}_1(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_{n_1}(t) \end{bmatrix} \tag{1–113}$$

(1-101), (1-103) and (1-112) describes the dynamic system with less coordinates than the original formulation (1-1). For this reason such formulations are referred to as *system reduction* schemes. A system reduction scheme with due consideration to the quasi-static response may also be formulated as a correction to (1-112).[1]

1.3 Conclusions

On condition that the convolution integral is evaluated numerically an analytical solution to the initial value problem (1-1) is provided by the result (1-3). Since this solution relies on the Fourier transform of the frequency matrix (1-6) for the impulse response matrix, the approach becomes computational prohibitive for a large number of degrees of freedom. Alternatively, if the initial value problem is reformulated in the state vector form (1-26) the analytical solution (1-29) is obtained. This solution relies on the fundamental matrix in terms of the matrix exponential function for the corresponding homogeneous differential system (1-28). The matrix exponential function may be calculated analytically as indicated by (1-55), but the solution requires all eigen-solutions to the system matrix $\mathbf{A}$. Again, the calculation of these becomes

prohibited for large systems. Hence, both analytical or semi-analytical solution approaches are out of the question for large degree-of-freedom systems.

The state vector formulation (1-26) directly admits the application of vectorial generalizations of standard ordinary differential equation solvers such as the Euler method, the extended Euler method, the various Runge-Kutta algorithms or the Adams-Bashforth/Adams-Moulton algorithm.[2] As is the case for all conditional stable algorithms the numerical stability of these schemes is determined by the length of the time step in proportion to the eigenperiod of highest mode of the system. Hence, in order to insure stability for large scale systems excessive small time steps becomes necessary, which means that the high accuracy of some of these algorithms cannot be utilized. Consequently, there is a need for numerical matrix differential solvers for which the length of the time step is determined from accuracy rather than stability. These algorithms predict stable although inaccurate responses for the highest modes. Instead, the time step is adjusted to predict accurate results for the lowest modes, which carry the global response of the structure. The devise of such algorithms will be the subject of Chapter 2.

System reduction schemes such as (1-101), (1-103) and (1-112) require a limited number of low-frequency eigen-pairs to be know. Since, the high frequency components have been filtered out the numerical integration of the modal coordinate differential equations (1-100) and (1-110) may be performed by standard ordinary differential solvers or by modification of the methods devised in Chapter 2. Hence, the primary obstacle in using these methods is the determination of the low frequency eigen-pairs. This problem will be the subject of the Chapters 3-7 of the book. Moreover, only solutions to the GEVP (1-9) will be considered, i.e. the involved system matrices are assumed to be symmetric and non-negative definite.

1.4 Exercises

1.1 Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a.) Calculate the eigenvalues and eigenmodes normalized to unit modal mass.

(b.) Determine two vectors that are M-orthonormal without being eigenmodes.

1.2 The eigensolutions with eigenmodes normalized to unit modal mass of a 2-dimensional generalized eigenvalue probem are given as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\, \mathbf{\Phi}^{(2)}\right] = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

(a.) Calculate **M** and **K**.

1.3 Write a MATLAB program, which solves the undamped generalized eigenvalue problem for the vibrating string problem considered in Example 1.2 for both the finite difference and the finite element discretized equations of motion. The circular eigenfrequencies should be presented in ascending order of magnitude, and the related eigenmodes should be normalized to unit modal mass.

(a.) Use the program to evaluate the 4 lowest circular eigenfrequencies of the string as a function of the number of elements n for both discretization methods, and compare the numerical results with the analytical solution (1-71).

(b.) Based on the obtained results suggest a mass matrix, which will do better.

1.4 Write a MATLAB program, which solves the undamped and damped generalized eigenvalue problems considered in Example 1.6.

CHAPTER 2
NUMERICAL INTEGRATION OF EQUATIONS OF MOTION

This chapter deals with the numerical time integration in the finite interval $[t_0, t_0 + T]$ of the initial value problem (1-1). The solution is searched for . The idea of the numerical integration scheme is to determine the solution of (1-1) approximately at the discrete instants of time $t_j = t_0 + j\Delta t,\ j = 1,\ 2,\ \ldots,\ n$, where $\Delta t = T/n$. To facilitate notations the following symbols are introduced

$$\mathbf{x}_j = \mathbf{x}(t_j) \quad , \quad \dot{\mathbf{x}}_j = \dot{\mathbf{x}}(t_j) \quad , \quad \ddot{\mathbf{x}}_j = \ddot{\mathbf{x}}(t_j) \quad , \quad \mathbf{f}_j = \mathbf{f}(t_j) \quad , \quad j = 0,\ 1,\ \ldots,\ n \qquad (2\text{–}1)$$

Singlestep algorithms in numerical time integration in structural dynamics determines the displacement vector $\mathbf{x}_{j+1}$, the velocity vector $\dot{\mathbf{x}}_{j+1}$ and the acceleration vector $\ddot{\mathbf{x}}_{j+1}$ at the new time t_{j+1}, on condition of knowledge of $\mathbf{x}_j$, $\dot{\mathbf{x}}_j$, $\ddot{\mathbf{x}}_j$ at the previous instant of time, as well as the load vectors $\mathbf{f}_j$ and $\mathbf{f}_{j+1}$ at the ends of the considered sub-interval $[t_j, t_{j+1}]$. In *multistep algorithms* the solution at the time t_{j+1} also depends on one or more solutions prior to the time t_j. Additionally, distinction will be made between *singlevalue algorithms*, which solves solely for the displacement vector $\mathbf{x}_j$, and *multivalue algorithms*, where the solution is obtained for a state vector encompassing the displacement vector $\mathbf{x}_j$, the velocity vector $\dot{\mathbf{x}}_j$, and in some cases even the acceleration vector $\ddot{\mathbf{x}}_j$. Generally, singlevalued algorithms require less computational efforts than multivalued algorithms. Classical algorithms in numerical analysis such as the vector generalization of the Runge-Kutta methods[1] may be used for the solution of (1-1). However, given that large scale structural models contain very high frequency components, these schemes may become numerical unstable unless extremely small time steps are used. For this reason the devise of useful algorithms in structural dynamics is governed by different objectives than in numerical analysis, as will be further explained below.

Newmark algorithms[2] treated in Section 2.1 are probably the most widely used algorithms in structural dynamics for solving (1-1). The derived *singlestep multivalue* formulation of the methods serves as a generic example for specification of *accuracy*, *stability*, and *numerical*

[1]D.G. Zill and M.R. Cullen: *Differential Equations with Boundary-Value Problems, 6th Ed.* Brooks/Cole, 2005.

[2]N.M. Newmark: *A Method of Computation for Structural Dynamics.* J.Eng.Mech., ASCE, **85**(EM3), 1959, 67-94.

damping in time integration.

High frequency modes of the spatially discretized equations (1-1) does not represent the behavior of the underlying physical problem very well. The corresponding modal components merely behave as numerical noise at the top of the displacement response carried by the lower frequency modes. However, the kinetic energy of these modes, which increases proportional with the squared frequency, may be significant. For this reason it is desirable to filter these components out of the response. In numerical time integrators in structural dynamics this is achieved by the introduction of numerical (artificial) damping, which are affecting merely the high frequency modes. However, it turns out that numerical damping cannot be introduced in the Newmark algorithms without compromising the accuracy of the response of the lower modes. Several suggestions to remedy this problem have been suggested. Here, we shall consider the so-called *generalized alpha algorithm* suggested by Chung and Hulbert,[3] which seems to be the most favorable single step single valued algorithm for this purpose. The outline of the text relies primarily on the monographs of Hughes [4,5] and Krenk.[6]

2.1 Newmark Algorithm

The Newmark family consists of the following equations

$$\mathbf{M}\ddot{\mathbf{x}}_{j+1} + \mathbf{C}\dot{\mathbf{x}}_{j+1} + \mathbf{K}\mathbf{x}_{j+1} = \mathbf{f}_{j+1} \quad , \quad j = 1, \ldots, n \tag{2–2}$$

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \dot{\mathbf{x}}_j \Delta t + \left(\left(\frac{1}{2} - \beta \right) \ddot{\mathbf{x}}_j + \beta \ddot{\mathbf{x}}_{j+1} \right) \Delta t^2 \tag{2–3}$$

$$\dot{\mathbf{x}}_{j+1} = \dot{\mathbf{x}}_j + \left((1-\gamma) \ddot{\mathbf{x}}_j + \gamma \ddot{\mathbf{x}}_{j+1} \right) \Delta t \tag{2–4}$$

(2-2) indicates the differential equation at the time t_{j+1}, which is required to be fulfilled for the new solution for $\ddot{\mathbf{x}}_{j+1}$, $\dot{\mathbf{x}}_{j+1}$, $\mathbf{x}_{j+1}$. (2-3) and (2-4) are approximate Taylor expansions, which have been derived in Box 2.2. The parameters β and γ determines the numerical stability and accuracy of the algorithms. The Newmark family contains several wellknown numerical algorithms as special cases. Examples are the *central difference algorithm* treated in Examples 2.2 and (2-6), which corresponds to $(\beta, \gamma) = (0, \frac{1}{2})$, the *Crank-Nicolson algorithm* treated in Example 2.3, which corresponds to $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$, and the *Fox-Goodwin algorithm*, where $(\beta, \gamma) = (\frac{1}{12}, \frac{1}{2})$.

[3] J. Chung and G.M. Hulbert: *A time Integration Algorithm for Structural Dynamics with Improved Numerical Dissipation: The Generalized α Method.* Journal of Applied Mechanics, **60**, 1993, 371-375.

[4] T.J.R. Hughes: *The Finite Element Method. Linear Static and Dynamic Finite Element Analysis.* Printice-Hall, Inc., 1987.

[5] T.J.R. Hughes: *Analysis of Transient Algorithms with Particular Reference to Stability Behavior.* Chapter 2 in *Computational Methods for Transient Analysis. Vol. 1 in Computational Methods in Mechanics*, Eds. T. Belytschko and T.J.R. Hughes, North-Holland, 1983.

[6] S. Krenk: *Dynamic Analysis of Structures. Numerical Time Integration.* Lecture Notes, Department of Mechanical Engineering, Technical University of Denmark, 2005.

There are several implementations of the methods. The most useful is the following single step single value implementation. At first, define the following *predictors*

$$\bar{\mathbf{x}}_{j+1} = \mathbf{x}_j + \dot{\mathbf{x}}_j\,\Delta t + \left(\frac{1}{2} - \beta\right)\Delta t^2\,\ddot{\mathbf{x}}_j \tag{2–5}$$

$$\dot{\bar{\mathbf{x}}}_{j+1} = \dot{\mathbf{x}}_j + \left(1 - \gamma\right)\Delta t\,\ddot{\mathbf{x}}_j \tag{2–6}$$

(2-5) and (2-6) specify predictions (preliminary solutions) for $\mathbf{x}_{j+1}$ and $\dot{\mathbf{x}}_{j+1}$ based on the information available at the time t_j. The idea of the algorithm is to insert (2-3) and (2-4) into (2-2). Given that the solution is required to fulfill the equations of motion at the time t_{j+1}, and using (2-5) and (2-6), the following equations are obtained for the new acceleration vector in terms of known solution quantities from the previous time and the load vector $\mathbf{f}_{j+1}$

$$\left(\mathbf{M} + \gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K}\right)\ddot{\mathbf{x}}_{j+1} = \mathbf{f}_{j+1} - \mathbf{C}\dot{\bar{\mathbf{x}}}_{j+1} - \mathbf{K}\bar{\mathbf{x}}_{j+1} \tag{2–7}$$

Next, based on the solution for $\ddot{\mathbf{x}}_{j+1}$ obtained from $(2-7)$, *corrected* (new) solutions for $\dot{\mathbf{x}}_{j+1}$ and $\mathbf{x}_{j+1}$ may be obtained from (2-3) and (2-4). These may be written as

$$\mathbf{x}_{j+1} = \bar{\mathbf{x}}_{j+1} + \beta\Delta t^2\,\ddot{\mathbf{x}}_{j+1} \tag{2–8}$$

$$\dot{\mathbf{x}}_{j+1} = \dot{\bar{\mathbf{x}}}_{j+1} + \gamma\Delta t\,\ddot{\mathbf{x}}_{j+1} \tag{2–9}$$

To start the algorithm the acceleration $\ddot{\mathbf{x}}_0$ at the time t_0 is needed. This is obtained from the equation of motion

$$\mathbf{M}\ddot{\mathbf{x}}_0 = \mathbf{f}_0 - \mathbf{C}\dot{\mathbf{x}}_0 - \mathbf{K}\mathbf{x}_0 \tag{2–10}$$

The algorithm has been summarized in Box 2.1. In stability and accuracy analysis a singlestep multivalue formulation for the state vector made up of the displacement and velocity vectors is preferred. In order to derived this, eqs. (2-3) and (2-4) are multiplied with $\mathbf{M}$. Next, the accelerations are eliminated by means of the differential equations at the times t_j and t_{j+1}, leading to

$$\left.\begin{aligned}
&\mathbf{M}\mathbf{x}_{j+1} = \mathbf{M}\mathbf{x}_j + \mathbf{M}\,\Delta t\dot{\mathbf{x}}_j + \\
&\left(\left(\frac{1}{2} - \beta\right)\left(\mathbf{f}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j\right) + \beta\left(\mathbf{f}_{j+1} - \mathbf{C}\dot{\mathbf{x}}_{j+1} - \mathbf{K}\mathbf{x}_{j+1}\right)\right)\Delta t^2 \\
&\mathbf{M}\dot{\mathbf{x}}_{j+1} = \mathbf{M}\dot{\mathbf{x}}_j + \\
&\left(\left(1 - \gamma\right)\left(\mathbf{f}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j\right) + \gamma\left(\mathbf{f}_{j+1} - \mathbf{C}\dot{\mathbf{x}}_{j+1} - \mathbf{K}\mathbf{x}_{j+1}\right)\right)\Delta t
\end{aligned}\right\} \Rightarrow$$

$$\begin{bmatrix} \mathbf{M} + \beta\Delta t^2\mathbf{K} & \beta\Delta t^2\mathbf{C} \\ \gamma\Delta t\,\mathbf{K} & \mathbf{M} + \gamma\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{j+1} \\ \dot{\mathbf{x}}_{j+1} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{M} - \left(\frac{1}{2} - \beta\right)\Delta t^2\mathbf{K} & \Delta t\mathbf{M} - \left(\frac{1}{2} - \beta\right)\Delta t^2\mathbf{C} \\ -(1-\gamma)\Delta t\,\mathbf{K} & \mathbf{M} - (1-\gamma)\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \end{bmatrix} + \begin{bmatrix} \left(\frac{1}{2} - \beta\right)\Delta t^2 & \beta\Delta t^2 \\ (1-\gamma)\Delta t & \gamma\Delta t \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix} \Rightarrow$$

$$\mathbf{z}_{j+1} = \bar{\mathbf{D}}\mathbf{z}_j + \mathbf{E}_j \tag{2–11}$$

where

$$\left.\begin{aligned}
\mathbf{z}_j &= \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \end{bmatrix} \\
\bar{\mathbf{D}} &= \begin{bmatrix} \mathbf{M} + \beta\Delta t^2\mathbf{K} & \beta\Delta t^2\mathbf{C} \\ \gamma\Delta t\,\mathbf{K} & \mathbf{M} + \gamma\Delta t\,\mathbf{C} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{M} - \left(\frac{1}{2} - \beta\right)\Delta t^2\mathbf{K} & \Delta t\mathbf{M} - \left(\frac{1}{2} - \beta\right)\Delta t^2\mathbf{C} \\ -(1-\gamma)\Delta t\,\mathbf{K} & \mathbf{M} - (1-\gamma)\Delta t\,\mathbf{C} \end{bmatrix} \\
\mathbf{E}_j &= \begin{bmatrix} \mathbf{M} + \beta\Delta t^2\mathbf{K} & \beta\Delta t^2\mathbf{C} \\ \gamma\Delta t\,\mathbf{K} & \mathbf{M} + \gamma\Delta t\,\mathbf{C} \end{bmatrix}^{-1} \begin{bmatrix} \left(\frac{1}{2} - \beta\right)\Delta t^2 & \beta\Delta t^2 \\ (1-\gamma)\Delta t & \gamma\Delta t \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix}
\end{aligned}\right\} \tag{2–12}$$

$\bar{\mathbf{D}}$ denotes the so-called *amplification matrix*. The bar indicates that this is an approximation to the exact amplification matrix, which has has been derived in Example 2.1.

Box 2.1: Newmark algorithm

Given the initial displacement vector $\mathbf{x}_0$ and the initial velocity vector $\dot{\mathbf{x}}_0$ at the time t_0. Calculate the initial acceleration vector $\ddot{\mathbf{x}}_0$ from

$$\mathbf{M}\ddot{\mathbf{x}}_0 = \mathbf{f}_0 - \mathbf{C}\dot{\mathbf{x}}_0 - \mathbf{K}\mathbf{x}_0$$

Repeat the following items for $j = 0, 1, \ldots, n$

1. Calculate predictors for the new displacement and velocity vectors

$$\bar{\mathbf{x}}_{j+1} = \mathbf{x}_j + \dot{\mathbf{x}}_j\,\Delta t + \left(\frac{1}{2} - \beta\right)\Delta t^2\,\ddot{\mathbf{x}}_j$$
$$\dot{\bar{\mathbf{x}}}_{j+1} = \dot{\mathbf{x}}_j + (1-\gamma)\Delta t\,\ddot{\mathbf{x}}_j$$

2. Calculate new acceleration vector from

$$\left(\mathbf{M} + \gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K}\right)\ddot{\mathbf{x}}_{j+1} = \mathbf{f}_{j+1} - \mathbf{C}\dot{\bar{\mathbf{x}}}_{j+1} - \mathbf{K}\bar{\mathbf{x}}_{j+1}$$

3. Calculate new displacement and velocity vectors

$$\mathbf{x}_{j+1} = \bar{\mathbf{x}}_{j+1} + \beta\Delta t^2\,\ddot{\mathbf{x}}_{j+1}$$
$$\dot{\mathbf{x}}_{j+1} = \dot{\bar{\mathbf{x}}}_{j+1} + \gamma\Delta t\,\ddot{\mathbf{x}}_{j+1}$$

Box 2.2: Derivation of (2-3) and (2-4)

Based on conventional integration theory the following identities may be formulated

$$\left.\begin{aligned} \mathbf{x}(t_{j+1}) &= \mathbf{x}(t_j) + \int_{t_j}^{t_{j+1}} \dot{\mathbf{x}}(\tau)d\tau \\ \dot{\mathbf{x}}(t_{j+1}) &= \dot{\mathbf{x}}(t_j) + \int_{t_j}^{t_{j+1}} \ddot{\mathbf{x}}(\tau)d\tau \end{aligned}\right\} \tag{2–13}$$

Integration by parts of the first relation provides

$$\mathbf{x}(t_{j+1}) = \mathbf{x}(t_j) - \Big[(t_{j+1}-\tau)\dot{\mathbf{x}}(\tau)\Big]_{t_j}^{t_{j+1}} + \int_{t_j}^{t_{j+1}} (t_{j+1}-\tau)\ddot{\mathbf{x}}(\tau)d\tau \quad \Rightarrow$$

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \Delta t\,\dot{\mathbf{x}}_{t_j} + \int_{t_j}^{t_{j+1}} (t_{j+1}-\tau)\ddot{\mathbf{x}}(\tau)d\tau \tag{2–14}$$

The indicated derivation is due to Krenk.[6] (2-14) may be interpreted as a truncated Taylor expansion, where the integrals represent the remainder. Correspondingly, the 2nd equation in (2-13) is written as

$$\dot{\mathbf{x}}_{j+1} = \dot{\mathbf{x}}_j + \int_{t_j}^{t_{j+1}} \ddot{\mathbf{x}}(\tau)d\tau \tag{2–15}$$

Next, the integrals in (2-14) and (2-15) are represented by the following linear combinations of the value of the acceleration vector at the end of the integration interval

$$\left.\begin{aligned} \int_{t_j}^{t_{j+1}} (t_{j+1}-\tau)\ddot{\mathbf{x}}(\tau)d\tau &\simeq \left(\frac{1}{2}-\beta\right)\Delta t^2\,\ddot{\mathbf{x}}_j + \beta\Delta t^2\,\ddot{\mathbf{x}}_{j+1} \\ \int_{t_j}^{t_{j+1}} \ddot{\mathbf{x}}(\tau)d\tau &\simeq (1-\gamma)\Delta t\,\ddot{\mathbf{x}}_j + \gamma\Delta t\,\ddot{\mathbf{x}}_{j+1} \end{aligned}\right\} \tag{2–16}$$

It is seen that the result in (2-16) becomes correct in case of constant acceleration, where $\ddot{\mathbf{x}}(\tau) \equiv \ddot{\mathbf{x}}_j = \ddot{\mathbf{x}}_{j+1}$. In any case the values of β and γ reflect the actual variation of the acceleration during the interval. If $\ddot{\mathbf{x}}(\tau)$ is assumed to be constant and equal to the mean of the end-point values, one obtains $(\beta,\gamma) = (\frac{1}{4},\frac{1}{2})$, whereas a linear variation between the end-point values provides $(\beta,\gamma) = (\frac{1}{6},\frac{1}{2})$.

The modal expansion (1-97) defines a one-to-one transformation from the physical to the modal coordinates. Hence, the time integration may equally well be performed on the differential equations for the modal coordinate equations. It follows that the synthesized motion (1-97) becomes

numerical unstable, if the integration of just one of the modal coordinates render into instability. Similarly, the accuracy of the synthesized motion is determined by the accuracy of those modal coordinates, which are retained in the truncated modal expansion (1-101). On condition of the modal decoupling condition (1-94) the time integration of the modal coordinates is reduced to the integration of n decoupled single-degrees-of-freedom systems. Since the stability and accuracy analysis of a SDOF system can be performed analytically, the important role of the modal decomposition assumption in the stability and accuracy analysis of numerical time integrators becomes clear. In this respect let $q(t)$ denote an arbitrary of the n modal coordinates, and ζ, ω and $F(t)$ the corresponding modal damping ratio, undamped circular eigenfrequency and modal load. On condition that the eigenmodes have been normalized to unit modal mass, the differential equation of the said modal coordinate reads

$$\ddot{q}(t) + 2\zeta\omega\dot{q}(t) + \omega^2 q(t) = F(t) \tag{2–17}$$

The corresponding Newmark integration of (2-17) is given by (2-11), using $\mathbf{M} = [1]$, $\mathbf{C} = [2\zeta\omega]$, $\mathbf{K} = [\omega^2]$ and $\mathbf{f}(t) = [F(t)]$ in (2-12), resulting in the system matrices

$$\left.\begin{aligned}
\mathbf{z}_j &= \begin{bmatrix} q_j \\ \dot{q}_j \end{bmatrix} \\
\bar{\mathbf{D}} &= \begin{bmatrix} 1+\beta\Delta t^2\omega^2 & 2\zeta\beta\omega\Delta t^2 \\ \gamma\omega^2\Delta t & 1+2\zeta\gamma\omega\Delta t \end{bmatrix}^{-1} \begin{bmatrix} 1-(\frac{1}{2}-\beta)\omega^2\Delta t^2 & \Delta t-(\frac{1}{2}-\beta)2\zeta\omega\Delta t^2 \\ -(1-\gamma)\omega^2\Delta t & 1-(1-\gamma)2\zeta\omega\Delta t \end{bmatrix} \\
&= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \\
\mathbf{E}_j &= \begin{bmatrix} 1+\beta\Delta t^2\omega^2 & 2\zeta\beta\omega\Delta t^2 \\ \gamma\omega^2\Delta t & 1+2\zeta\gamma\omega\Delta t \end{bmatrix}^{-1} \begin{bmatrix} (\frac{1}{2}-\beta)\Delta t^2 & \beta\Delta t^2 \\ (1-\gamma)\Delta t & \gamma\Delta t \end{bmatrix} \begin{bmatrix} F_j \\ F_{j+1} \end{bmatrix} \\
&= \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} F_j \\ F_{j+1} \end{bmatrix}
\end{aligned}\right\} \tag{2–18}$$

where

$$\left.\begin{aligned}
D_{11} &= \frac{1+2\gamma\zeta\kappa+(\beta-\frac{1}{2})\kappa^2+(2\beta-\gamma)\zeta\kappa^3}{1+2\gamma\zeta\kappa+\beta\kappa^2} \\
D_{12} &= \frac{1+(2\gamma-1)\zeta\kappa+2(2\beta-\gamma)\zeta^2\kappa^2}{1+2\gamma\zeta\kappa+\beta\kappa^2}\Delta t \\
D_{21} &= -\frac{1+\frac{1}{2}(2\beta-\gamma)\kappa^2}{1+2\gamma\zeta\kappa+\beta\kappa^2}\frac{\kappa^2}{\Delta t} \\
D_{22} &= \frac{1+2(\gamma-1)\zeta\kappa+(\beta-\gamma)\kappa^2-(2\beta-\gamma)\zeta\kappa^3}{1+2\gamma\zeta\kappa+\beta\kappa^2}
\end{aligned}\right\} \tag{2–19}$$

$$\left.\begin{aligned}
E_{11} &= -\frac{\beta - \frac{1}{2} + (2\beta - \gamma)\zeta\kappa}{1 + 2\gamma\zeta\kappa + \beta\kappa^2}\Delta t^2 \\
E_{12} &= \frac{\beta}{1 + 2\gamma\zeta\kappa + \beta\kappa^2}\Delta t^2 \\
E_{21} &= \frac{1 - \gamma + \frac{1}{2}(2\beta - \gamma)\kappa^2}{1 + 2\gamma\zeta\kappa + \beta\kappa^2}\Delta t \\
E_{22} &= \frac{\gamma}{1 + 2\gamma\zeta\kappa + \beta\kappa^2}\Delta t
\end{aligned}\right\} \tag{2–20}$$

$$\kappa = \omega\Delta t \tag{2–21}$$

Example 2.1: Exact singlestep multivalue method

Assume that the initial time in the analytical solution (1-29) is chosen as the time $t_j = j\Delta t$. Then, the initial value is changed into $\mathbf{z}(t_j) = \mathbf{z}_j$, and the solution is modified to

$$\mathbf{z}(t) = \mathrm{e}^{\mathbf{A}(t-t_j)}\left(\mathbf{z}_j + \int_{t_j}^{t} \mathrm{e}^{-\mathbf{A}(\tau-t_j)}\mathbf{F}(\tau)d\tau\right) \quad , \quad t > t_j \tag{2–22}$$

Next, (2-22) is considered at the following time $t_{j+1} = t_j + \Delta t$, which leads to following integration algorithm

$$\mathbf{z}_{j+1} = \mathrm{e}^{\mathbf{A}\Delta t}\left(\mathbf{z}_j + \int_{t_j}^{t_{j+1}} \mathrm{e}^{-\mathbf{A}(\tau-t_j)}\mathbf{F}(\tau)d\tau\right) = \mathbf{D}\mathbf{z}_j + \mathbf{E}_j \tag{2–23}$$

where

$$\mathbf{z}_j = \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \end{bmatrix} \tag{2–24}$$

$$\mathbf{D} = \mathrm{e}^{\mathbf{A}\Delta t} \tag{2–25}$$

$$\mathbf{E}_j = \mathrm{e}^{\mathbf{A}\Delta t}\int_{t_j}^{t_{j+1}} \mathrm{e}^{-\mathbf{A}(\tau-t_j)}\mathbf{F}(\tau)d\tau \simeq \mathrm{e}^{\mathbf{A}\Delta t}\Big((1-\alpha)\mathrm{e}^{-\mathbf{A}\cdot 0}\mathbf{F}_j + \alpha\mathrm{e}^{-\mathbf{A}\Delta t}\mathbf{F}_{j+1}\Big)\Delta t =$$
$$\Big((1-\alpha)\mathrm{e}^{-\mathbf{A}\Delta t}\mathbf{F}_j + \alpha\mathbf{F}_{j+1}\Big)\Delta t \tag{2–26}$$

In (2-26) the integral has been evaluated by a generalized trapezoidal rule defined by the parameter $\alpha \in [0, 1]$ in terms of a weighted average of the values $\mathrm{e}^{-\mathbf{A}\cdot 0}\mathbf{F}_j$ and $\mathrm{e}^{-\mathbf{A}\Delta t}\mathbf{F}_{j+1}$ at the ends of the integration interval $[t_j, t_{n+1}]$. $\mathbf{D}$ denotes the exact amplification matrix. Correspondingly, the evolutionary equation (2-23) determines the exact solution for the state vector to the accuracy of the approximation (2-26) for the load vector.

The matrix exponential functions entering (2-25) and (2-26) are given by, cf. (1-55)

$$\left.\begin{aligned}
\mathrm{e}^{\mathbf{A}\Delta t} &= \mathbf{\Psi}\mathrm{e}^{\mathbf{\Lambda}_\Lambda\Delta t}\mathbf{\Psi}^{-1} \\
\mathrm{e}^{-\mathbf{A}\Delta t} &= \mathbf{\Psi}\mathrm{e}^{-\mathbf{\Lambda}_\Lambda\Delta t}\mathbf{\Psi}^{-1}
\end{aligned}\right\} \tag{2–27}$$

The modal matrix $\mathbf{\Psi}$ contains the eigenmodes of the matrix $\mathbf{D}$, which are identical to the eigenmodes of the matrix $\mathbf{A}$. The diagonal matrix $e^{\mathbf{\Lambda}_A \Delta t}$ stores the eigenvalues $\lambda_{D,j} = e^{\lambda_{A,j}\Delta t}$ of $\mathbf{D}$ in the main diagonal. $\lambda_{A,j}$ denotes the corresponding eigenvalue of $\mathbf{A}$, which may be written in the form[7]

$$\lambda_{A,j} = \omega_j\Big(-\zeta_j + i\sqrt{1-\zeta_j^2}\Big) \tag{2–28}$$

ω_j and ζ_j defines the equivalent undamped circular eigenfrequency and damping ratio in case of damped of damped eigenvibration in the jth mode. These definitions correspond to the conventional definition of these quantities in case of the modal decoupling condition (1-94).[7] From (2-28) follows, that the modal damping ratio is related to the magnitude of the eigenvalues of $\mathbf{D}$ as

$$\begin{aligned} &|\lambda_{D,j}| = e^{-\zeta_j\omega_j\Delta t} \quad \Rightarrow \\ &\zeta_j = -\frac{\ln|\lambda_{D,j}|}{\omega_j\Delta t} \end{aligned} \tag{2–29}$$

The damped circular eigenfrequency $\omega_{d,j}$ is related to the eigenvalues of the amplification matrix as follows

$$\omega_{d,j} = \omega_j\sqrt{1-\zeta_j^2} = \mathrm{Im}\big[\lambda_{A,j}\big] = \frac{1}{\Delta t}\mathrm{Im}\big[\ln\lambda_{D,j}\big] = \frac{1}{\Delta t}\mathrm{Im}\Big[\ln\Big(|\lambda_{D,j}|e^{i\theta_{D,j}}\Big)\Big] = \frac{\theta_{D,j}}{\Delta t} \tag{2–30}$$

where $\theta_{D,j}$ denotes the argument of $\lambda_{D,j}$.

Example 2.2: Displacement difference equation form of the Newmark algorithm

In this example an implementation of the Newmark algorithms will be derived, where the solution for the unknown displacement vector $\mathbf{x}_{j+1}$ at the time t_{j+1} is determined as a function of the previous known displacement vectors $\mathbf{x}_j$ and $\mathbf{x}_{j-1}$, as well as the load vectors $\mathbf{f}_{j+1}$, $\mathbf{f}_j$ and $\mathbf{f}_{j-1}$. At first, (2-2) is formulated at the times t_{j+1}, t_j and t_{j-1} as follows

$$\left.\begin{aligned} \mathbf{M}\ddot{\mathbf{x}}_{j+1} + \mathbf{C}\dot{\mathbf{x}}_{j+1} + \mathbf{K}\mathbf{x}_{j+1} &= \mathbf{f}_{j+1} \\ \mathbf{M}\ddot{\mathbf{x}}_{j} + \mathbf{C}\dot{\mathbf{x}}_{j} + \mathbf{K}\mathbf{x}_{j} &= \mathbf{f}_{j} \\ \mathbf{M}\ddot{\mathbf{x}}_{j-1} + \mathbf{C}\dot{\mathbf{x}}_{j-1} + \mathbf{K}\mathbf{x}_{j-1} &= \mathbf{f}_{j-1} \end{aligned}\right\} \tag{2–31}$$

The first equation in (2-29) is multiplied with $\beta\Delta t^2$, the 2nd equation is multiplied with $\big(\frac{1}{2}-2\beta+\gamma\big)\Delta t^2$, and the third equation is multiplied with $\big(\frac{1}{2}+\beta-\gamma\big)\Delta t^2$. Finally, the resulting equations are added, leading to

$$\begin{aligned} &\Delta t^2\mathbf{M}\left[\beta\ddot{\mathbf{x}}_{j+1} - 2\beta\ddot{\mathbf{x}}_j + \beta\ddot{\mathbf{x}}_{j-1} + \left(\frac{1}{2}+\gamma\right)\ddot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\ddot{\mathbf{x}}_{j-1}\right] + \\ &\Delta t^2\mathbf{C}\left[\beta\dot{\mathbf{x}}_{j+1} - 2\beta\dot{\mathbf{x}}_j + \beta\dot{\mathbf{x}}_{j-1} + \left(\frac{1}{2}+\gamma\right)\dot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\dot{\mathbf{x}}_{j-1}\right] + \\ &\Delta t^2\mathbf{K}\left[\beta\mathbf{x}_{j+1} - 2\beta\mathbf{x}_j + \beta\mathbf{x}_{j-1} + \left(\frac{1}{2}+\gamma\right)\mathbf{x}_j + \left(\frac{1}{2}-\gamma\right)\mathbf{x}_{j-1}\right] = \\ &\beta\Delta t^2\,\mathbf{f}_{j+1} + \left(\frac{1}{2}-2\beta+\gamma\right)\Delta t^2\,\mathbf{f}_j + \left(\frac{1}{2}+\beta-\gamma\right)\Delta t^2\,\mathbf{f}_{j-1} \end{aligned} \tag{2–32}$$

Next, (2-3) and (2-4) are formulated at the time t_{j+1} and t_j as follows

[7] S.R.K. Nielsen: *Structural Dynamics, Vol. 1. Linear Structural Dynamics, 4th Ed.* Aalborg tekniske Universitetsforlag, 2004.

$$\left.\begin{aligned} \mathbf{x}_{j+1} &= \mathbf{x}_j + \Delta t\,\dot{\mathbf{x}}_j + \left(\frac{1}{2}-\beta\right)\Delta t^2\,\ddot{\mathbf{x}}_j + \beta\Delta t^2\,\ddot{\mathbf{x}}_{j+1} \\ \mathbf{x}_j &= \mathbf{x}_{j-1} + \Delta t\,\dot{\mathbf{x}}_{j-1} + \left(\frac{1}{2}-\beta\right)\Delta t^2\,\ddot{\mathbf{x}}_{j-1} + \beta\Delta t^2\,\ddot{\mathbf{x}}_j \end{aligned}\right\} \tag{2–33}$$

$$\left.\begin{aligned} \dot{\mathbf{x}}_{j+1} &= \dot{\mathbf{x}}_j + (1-\gamma)\Delta t\,\ddot{\mathbf{x}}_j + \gamma\Delta t\,\ddot{\mathbf{x}}_{j+1} \\ \dot{\mathbf{x}}_j &= \dot{\mathbf{x}}_{j-1} + (1-\gamma)\Delta t\,\ddot{\mathbf{x}}_{j-1} + \gamma\Delta t\,\ddot{\mathbf{x}}_j \end{aligned}\right\} \tag{2–34}$$

Withdrawal of the last equations in (2-33) and (2-34) from the the first equations provides the identities

$$\beta\Delta t^2\left(\ddot{\mathbf{x}}_{j+1} - 2\ddot{\mathbf{x}}_j + \ddot{\mathbf{x}}_{j-1}\right) = \mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1} - \Delta t(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_{j-1}) - \frac{1}{2}\Delta t^2(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) \tag{2–35}$$

$$\gamma\Delta t\left(\ddot{\mathbf{x}}_{j+1} - 2\ddot{\mathbf{x}}_j + \ddot{\mathbf{x}}_{j-1}\right) = \dot{\mathbf{x}}_{j+1} - 2\dot{\mathbf{x}}_j + \dot{\mathbf{x}}_{j-1} - \Delta t(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) \tag{2–36}$$

Next, $(\frac{1}{2}+\gamma)\Delta t^2\ddot{\mathbf{x}}_j + (\frac{1}{2}-\gamma)\Delta t^2\ddot{\mathbf{x}}_{j-1}$ is added on both sides of (2-35), and the last equation (2-34) is used on the resulting right hand side, which provides

$$\begin{aligned} &\Delta t^2\left(\beta\ddot{\mathbf{x}}_{j+1} - 2\beta\ddot{\mathbf{x}}_j + \beta\ddot{\mathbf{x}}_{j-1}\right) + \left(\frac{1}{2}+\gamma\right)\Delta t^2\,\ddot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\Delta t^2\,\ddot{\mathbf{x}}_{j-1} = \\ &\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1} - \Delta t(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_{j-1}) - \frac{1}{2}\Delta t^2(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) + \left(\frac{1}{2}+\gamma\right)\Delta t^2\,\ddot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\Delta t^2\,\ddot{\mathbf{x}}_{j-1} = \\ &\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1} \end{aligned} \tag{2–37}$$

(2-36) is solved for the velocity terms on the right hand side, and the resulting equation is multiplied with $\beta\Delta t^2$. Next, $(\frac{1}{2}+\gamma)\Delta t^2\dot{\mathbf{x}}_j + (\frac{1}{2}-\gamma)\Delta t^2\dot{\mathbf{x}}_{j-1}$ is added on both sides of the equation, resulting in

$$\begin{aligned} &\Delta t^2\left(\beta\dot{\mathbf{x}}_{j+1} - 2\beta\dot{\mathbf{x}}_j + \beta\dot{\mathbf{x}}_{j-1}\right) + \left(\frac{1}{2}+\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_{j-1} = \\ &\beta\gamma\Delta t^3\left(\ddot{\mathbf{x}}_{j+1} - 2\ddot{\mathbf{x}}_j + \ddot{\mathbf{x}}_{j-1}\right) + \beta\Delta t^3(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) + \left(\frac{1}{2}+\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_{j-1} = \\ &\gamma\Delta t\left(\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1}\right) - \gamma\Delta t^2\left(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_{j-1}\right) + \left(\beta - \frac{1}{2}\gamma\right)\Delta t^3(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) + \\ &\left(\frac{1}{2}+\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_j + \left(\frac{1}{2}-\gamma\right)\Delta t^2\,\dot{\mathbf{x}}_{j-1} = \gamma\Delta t\left(\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1}\right) + \Delta t\left(\mathbf{x}_j - \mathbf{x}_{j-1}\right) \end{aligned} \tag{2–38}$$

The 3rd line in (2-38) follows from the 2nd line by eliminating the term $\beta\gamma\Delta t^3(\ddot{\mathbf{x}}_{j+1} - 2\ddot{\mathbf{x}}_j + \ddot{\mathbf{x}}_{j-1})$ by means of (2-35). The final result is based on the following identity, which is obtained by a multiplication of the last equation in (2-33) with Δt, and the last equation in (2-34) with $\frac{1}{2}\Delta t^2$, following by a withdrawal of the resulting equations

$$\left(\beta - \frac{1}{2}\gamma\right)\Delta t^3(\ddot{\mathbf{x}}_j - \ddot{\mathbf{x}}_{j-1}) = \Delta t(\mathbf{x}_j - \mathbf{x}_{j-1}) - \frac{1}{2}\Delta t^2(\dot{\mathbf{x}}_j + \dot{\mathbf{x}}_{j-1}) \tag{2–39}$$

(2-37) and (2-38) are inserted into (2-32). After grouping terms with common multipliers $\mathbf{x}_{j+1}$, $\mathbf{x}_j$, $\mathbf{x}_{j-1}$ the following final result is obtained

$$\begin{aligned} &\left[\mathbf{M} + \gamma\Delta t\,\mathbf{C} + \beta\Delta t^2\,\mathbf{K}\right]\mathbf{x}_{j+1} - \left[2\mathbf{M} - (1-2\gamma)\Delta t\,\mathbf{C} - \left(\frac{1}{2} - 2\beta + \gamma\right)\Delta t^2\,\mathbf{K}\right]\mathbf{x}_j + \\ &\left[\mathbf{M} - (1-\gamma)\Delta t\,\mathbf{C} + \left(\frac{1}{2} + \beta - \gamma\right)\Delta t^2\,\mathbf{K}\right]\mathbf{x}_{j-1} = \\ &\beta\Delta t^2\,\mathbf{f}_{j+1} + \left(\frac{1}{2} - 2\beta + \gamma\right)\Delta t^2\,\mathbf{f}_j + \left(\frac{1}{2} + \beta - \gamma\right)\Delta t^2\,\mathbf{f}_{j-1} \end{aligned} \tag{2–40}$$

(2-40) represents the so-called *displacement difference equation form* of the Newmark algorithm, which constitutes a multistep singlevalue formulation of the method. At the calculation of $\mathbf{x}_1$ the previous solution $\mathbf{x}_0$ is given as one of the initial value vectors, whereas $\mathbf{x}_{-1}$ is unknown. Hence, the algorithm has a starting problem. Instead, $\mathbf{x}_1$ is calculated by the standard implementation given in Box 2.1, before the algorithm (2-40) is used for $j \geq 1$.

Next, consider the special case of $\beta = 0$ and $\gamma = \frac{1}{2}$. Then, (2-40) reduces to

$$\left[\mathbf{M} + \frac{1}{2}\Delta t\,\mathbf{C}\right]\mathbf{x}_{j+1} - \left[2\mathbf{M} - \Delta t^2\,\mathbf{K}\right]\mathbf{x}_j + \left[\mathbf{M} - \frac{1}{2}\Delta t\,\mathbf{C}\right]\mathbf{x}_{j-1} = \Delta t^2\,\mathbf{f}_j \tag{2–41}$$

Consider the *central difference*[1] approximations to the acceleration and the velocity vectors

$$\ddot{\mathbf{x}}_j \simeq \frac{\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1}}{\Delta t^2} \quad , \quad \dot{\mathbf{x}}_j \simeq \frac{\mathbf{x}_{j+1} - \mathbf{x}_{j-1}}{2\Delta t} \tag{2–42}$$

(2-41) is obtained, if the finite difference approximations (2-42) are inserted into the middlemost equation of (2-31). Hence, the central difference solution to (1-1) constitutes a special case of the Newmark family corresponding to the parameters $(\beta, \gamma) = (0, \frac{1}{2})$.

The central difference algorithm is only conditional stable. However, if $\mathbf{M}$ and $\mathbf{C}$ are diagonal it provides an explicit solution for $\mathbf{x}_{j+1}$, which makes it highly economical. In cases where the time step is controlled by accuracy rather than stability, which is often the case in many wave propagation problems, the central difference algorithm is widely used.

Example 2.3: Crank-Nicolson algorithm

For $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$, eqs. (2-11), (2-12) may be written

$$\begin{bmatrix} \mathbf{M} + \frac{1}{4}\Delta t^2\mathbf{K} & \frac{1}{4}\Delta t^2\mathbf{C} \\ \frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} + \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{j+1} \\ \dot{\mathbf{x}}_{j+1} \end{bmatrix} =$$
$$\begin{bmatrix} \mathbf{M} - \frac{1}{4}\Delta t^2\mathbf{K} & \Delta t\,\mathbf{M} - \frac{1}{4}\Delta t^2\mathbf{C} \\ -\frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} - \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\Delta t^2 & \frac{1}{4}\Delta t^2 \\ \frac{1}{2}\Delta t & \frac{1}{2}\Delta t \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix} \tag{2–43}$$

The 2nd equation in (2-43) is multiplied with $\frac{1}{2}\Delta t$, and is withdraw from the 1st equation, resulting in

$$\begin{bmatrix} \mathbf{M} & -\frac{1}{2}\Delta t\,\mathbf{M} \\ \frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} + \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{j+1} \\ \dot{\mathbf{x}}_{j+1} \end{bmatrix} = \begin{bmatrix} \mathbf{M} & \frac{1}{2}\Delta t\,\mathbf{M} \\ -\frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} - \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{2}\Delta t & \frac{1}{2}\Delta t \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix} \tag{2–44}$$

The Crank-Nicolson algorithm is singlestep multivalue method, where the amplification matrix in (2-11) is given as

$$\bar{\mathbf{D}} = \mathbf{D}_1^{-1}\mathbf{D}_2 \tag{2–45}$$

$\mathbf{D}_1$ and $\mathbf{D}_2$ are related to the system matrix $\mathbf{A}$ given by (1-27) as follows

$$\mathbf{D}_1 = \mathbf{I} - \frac{1}{2}\Delta t\mathbf{A} = \begin{bmatrix} \mathbf{I} & -\frac{1}{2}\Delta t\,\mathbf{I} \\ \frac{1}{2}\Delta t\,\mathbf{M}^{-1}\mathbf{K} & \mathbf{I} + \frac{1}{2}\Delta t\,\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{M} & -\frac{1}{2}\Delta t\,\mathbf{M} \\ \frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} + \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \tag{2–46}$$

$$\mathbf{D}_2 = \mathbf{I} + \frac{1}{2}\Delta t\mathbf{A} = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\Delta t\,\mathbf{I} \\ -\frac{1}{2}\Delta t\,\mathbf{M}^{-1}\mathbf{K} & \mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1} & 0 \\ 0 & \mathbf{M}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \frac{1}{2}\Delta t\,\mathbf{M} \\ -\frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M} - \frac{1}{2}\Delta t\,\mathbf{C} \end{bmatrix} \tag{2–47}$$

A systematic derivation of the Crank-Nicolson algorithm along with other high-accuracy methods will be given in Example 2.4.

Insertion of (2-46) and (2-47) into (2-45) provides

$$\begin{aligned}\bar{\mathbf{D}} &= \begin{bmatrix} \mathbf{M} & -\frac{1}{2}\Delta t\,\mathbf{M} \\ \frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M}+\frac{1}{2}\Delta t\,\mathbf{C}\end{bmatrix}^{-1} \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\end{bmatrix}^{-1} \begin{bmatrix} \mathbf{M}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{-1}\end{bmatrix} \begin{bmatrix} \mathbf{M} & \frac{1}{2}\Delta t\,\mathbf{M} \\ -\frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M}-\frac{1}{2}\Delta t\,\mathbf{C}\end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M} & -\frac{1}{2}\Delta t\,\mathbf{M} \\ \frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M}+\frac{1}{2}\Delta t\,\mathbf{C}\end{bmatrix}^{-1} \begin{bmatrix} \mathbf{M} & \frac{1}{2}\Delta t\,\mathbf{M} \\ -\frac{1}{2}\Delta t\,\mathbf{K} & \mathbf{M}-\frac{1}{2}\Delta t\,\mathbf{C}\end{bmatrix}\end{aligned} \tag{2–48}$$

The amplification (2-48) is identical to the one obtained from (2-44). From this is concluded that the Crank-Nicolson algorithm constitutes a special case of the Newmark family corresponding to the parameters $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$. As mentioned in Box 2.2 this parameter combination is obtained in case of a constant variation of the acceleration during the interval $[t_j, t_{j+1}]$ given by $\ddot{\mathbf{x}}(t) \equiv \frac{1}{2}(\ddot{\mathbf{x}}_j + \ddot{\mathbf{x}}_{j+1})$.

2.1.1 Numerical Accuracy

The Newmark algorithm (2-11), (2-12) provides an approximate solution $\bar{\mathbf{z}}_j$ of the state vector. By contrast the exact solution $\mathbf{z}_j$ is given by (2-23), (2-25). Assume that the load vectors $\mathbf{E}_j$ in (2-11) and (2-23) are identical, and the algorithms are started with the same state vector $\mathbf{z}_j$ at the time t_j. Then, the deviation between the exact solution $\mathbf{z}_{j+1}$ and the Newmark solution $\bar{\mathbf{z}}_{j+1}$ at the succeeding time t_{j+1} is given by

$$\mathbf{e} = \mathbf{z}_{j+1} - \bar{\mathbf{z}}_{j+1} = (\mathbf{D} - \bar{\mathbf{D}})\mathbf{z}_j \tag{2–49}$$

The error vector $\mathbf{e}$, which is denoted the *local truncation error*, determines the error during a single time step. The error is caused by the deviation $\mathbf{D} - \bar{\mathbf{D}}$ of the amplification matrices, which in turn is a function of the magnitude of the time step Δt. Hence, it may be assumed that the local truncation error has the form

$$\mathbf{e} = \mathrm{O}(\Delta t^{k+1}) \tag{2–50}$$

where k is a positive real number. The real concern is not on the error during a single time step, but rather on the accumulated errors during all the previous time steps. For this quantity the *global truncation error* is introduced as

$$\mathbf{E} = \mathbf{z}_{j+1} - \bar{\mathbf{z}}_{j+1} \tag{2–51}$$

On condition that the load vectors are the same in the two algorithms, and that the local truncation error has the form (2-50), it can be shown that the global truncation error has the form[1]

$$\mathbf{E} = \mathrm{O}(\Delta t^{k}) \tag{2–52}$$

A numerical time integrator with a global truncation error of the form (2-52) is denoted a *kth order method.* Since the local truncation error of the 4th order Runge-Kutta algorithm is of the order of magnitude $O(\Delta t^5)$ this is a 4th order method, as indicated by the naming.

The amplification matrices (2-12) and (2-25) have the spectral representation, cf. (1-41)

$$\mathbf{D} = \mathbf{\Psi}\mathbf{\Lambda}_D\mathbf{\Psi}^{-1} \tag{2–53}$$

$$\bar{\mathbf{D}} = \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1} \tag{2–54}$$

$\mathbf{\Psi}$ and $\bar{\mathbf{\Psi}}$ are complex matrices storing the eigenvectors of $\mathbf{D}$ and $\bar{\mathbf{D}}$ column-wise, and $\mathbf{\Lambda}_D$ and $\bar{\mathbf{\Lambda}}_D$ are diagonal matrices with the corresponding eigenvalues in the main diagonal. As shown in (1-55) the eigenvectors of $\mathbf{D}$ are identical to the eigenvectors of the matrix $\mathbf{A}$ given by (1-27). Hence, the columns in $\bar{\mathbf{\Psi}}$ may be considered approximations the eigenvectors of $\mathbf{A}$. The eigenvalue matrices of $\mathbf{D}$ and $\mathbf{A}$ are related as, cf. (1-55)

$$\mathbf{\Lambda}_D = \mathrm{e}^{\mathbf{\Lambda}_A\Delta t} \tag{2–55}$$

Correspondingly, the eigenvalue matrix of $\bar{\mathbf{D}}$ may be written as

$$\bar{\mathbf{\Lambda}}_D = \mathrm{e}^{\bar{\mathbf{\Lambda}}_A\Delta t} \tag{2–56}$$

where the diagonal matrix $\bar{\mathbf{\Lambda}}_A$ may be considered an approximation to $\mathbf{\Lambda}_A$. Let $\bar{\lambda}_D$ be an arbitrary eigenvalue to $\bar{\mathbf{D}}$. Then, the corresponding component in the main diagonal of $\bar{\mathbf{\Lambda}}_A$ is given as

$$\bar{\lambda}_A = \frac{\ln\bar{\lambda}_D}{\Delta t} \tag{2–57}$$

In accordance with (2-28), $\bar{\lambda}_A$ can be written on the form

$$\bar{\lambda}_A = \bar{\omega}\left(-\bar{\zeta} + i\sqrt{1-\bar{\zeta}^2}\right) \tag{2–58}$$

$\bar{\omega}$ and $\bar{\zeta}$ are approximations to the undamped circular eigenfrequency ω and damping ratio ζ of the corresponding mode. Analog to (2-29) and (2-30), $\bar{\zeta}$ and $\bar{\omega}_\mathrm{d}$ are related to the corresponding eigenvalue $\bar{\lambda}_D$ of the amplification matrix as follows

$$\bar{\zeta} = -\frac{\ln|\bar{\lambda}_D|}{\bar{\omega}\Delta t} \tag{2–59}$$

$$\bar{\omega}_\mathrm{d} = \bar{\omega}\sqrt{1-\bar{\zeta}^2} = \frac{\bar{\theta}_D}{\Delta t} \tag{2–60}$$

where $\bar{\theta}_D$ denotes the argument of $\bar{\lambda}_D$, as follows from the polar representation $\bar{\lambda}_D = |\bar{\lambda}_D| e^{i\bar{\theta}_D}$. Especially, $\bar{\mathbf{\Psi}} = \mathbf{\Psi}$ for $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$. This follows, because the Newmark algorithm in this case is identical to the Crank-Nicolson algorithm as shown in Example 2.3. Further, in Example 2.4 below it is shown that the eigenvectors of the amplification matrix of the Crank-Nicolson algorithm is identical to those of the matrix $\mathbf{A}$. Hence, for $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$ the inaccuracy of the Newmark algorithm is entirely caused by the difference between the eigenvalue matrices $\mathbf{\Lambda}_D$ and $\bar{\mathbf{\Lambda}}_D$. In all other cases the deviation of the modal matrices $\mathbf{\Psi}$ and $\bar{\mathbf{\Psi}}$ will also influence the local and global truncation errors.

In any case it is of interest to analyze the accuracy of the eigenvalues $\bar{\lambda}_D$. These are completely defined by the damping ratio (2-59) and the damped circular eigenfrequency (2-60) in the considered mode. If $\bar{\zeta}$ is larger than the exact modal damping ratio ζ as given by (2-29), the Newmark algorithm introduces artificial so-called numerical damping in the considered eigenmode. Since the numerical damping may be negative, it may overrule the structural modal damping, rendering $\bar{\zeta} < 0$. This implies that the numerical solution for the said mode becomes numerical unstable. Similarly, if $\bar{\omega}_{\mathrm{d}}$ deviates from ω_{d} the corresponding damped eigenperiods will deviate accordingly. This effect is referred to as *period errors*. The effect of period errors is that the Newmark algorithm predicts damped eigenvibrations in the considered mode with either a too long or a too short period compared to the exact damped eigenvibration. It should be noticed that numerical damping and period errors are only of importance for the low frequency modes, which determines the total response. By contrast, numerical instability in merely one mode will affect the stability of the global response. Stability of the Newmark algorithm is treated in the following Subsection 2.1.2. Numerical damping and period errors are treated in Subsection 2.1.3.

2.1.2 Numerical Stability

Numerical stability indicates the capability of the algorithm to dissipate a given disturbance. In order to analyze this property the algorithm (2-11) is considered in the homogeneous case $\mathbf{E}_j = 0$, corresponding to eigenvibrations from the initial value $\mathbf{z}_0$. Using the spectral representation (2-54) for the amplification matrix, the following sequence of solutions are obtained, cf. (1-52)

$$\left.\begin{aligned}
\mathbf{z}_1 &= \bar{\mathbf{D}}\mathbf{z}_0 &&= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 \\
\mathbf{z}_2 &= \bar{\mathbf{D}}\mathbf{z}_1 &= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1}\,\bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 &&= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D^2\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 \\
\mathbf{z}_3 &= \bar{\mathbf{D}}\mathbf{z}_2 &= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1}\,\bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D^2\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 &&= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D^3\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 \\
&\vdots \\
\mathbf{z}_{j+1} &= \bar{\mathbf{D}}\mathbf{z}_{j+1} &= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D\bar{\mathbf{\Psi}}^{-1}\,\bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D^j\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0 &&= \bar{\mathbf{\Psi}}\bar{\mathbf{\Lambda}}_D^{j+1}\bar{\mathbf{\Psi}}^{-1}\mathbf{z}_0
\end{aligned}\right\} \tag{2–61}$$

$\bar{\mathbf{\Lambda}}_D^{j+1}$ is a diagonal matrix with the components $\bar{\lambda}_D^{j+1}$ in the main diagonal, where $\bar{\lambda}_D$ is the corresponding eigenvalue of $\bar{\mathbf{D}}$. From the polar representation $\bar{\lambda}_D = |\bar{\lambda}_D| e^{i\bar{\theta}_D}$ follows

$$\bar{\lambda}_D^{j+1} = |\bar{\lambda}_D|^{j+1} e^{i(j+1)\bar{\theta}_D} \tag{2–62}$$

Hence, $\bar{\lambda}_D^{j+1}$ will grow beyond limits as $j \to \infty$, if $\bar{\lambda}_D > 1$. Consequently, the condition for stable eigenvibrations in the said mode may be formulated as

$$|\bar{\lambda}_D| \leq 1 \tag{2–63}$$

Stability of the numerical algorithm (2-11) requires that (2-63) is fulfilled for all complex modes. As follows from (2-59) the stability criteria may alternatively be indicated in terms of the modal damping ratio as

$$\bar{\zeta} \geq 0 \tag{2–64}$$

If $\bar{\zeta} < 1$, it follows from (2-58) that $\bar{\lambda}_D$ is complex. In this case motion is oscillatory, corresponding to *undercritical* damped vibrations. Alternatively, if $\bar{\zeta} > 1$, $\bar{\lambda}_D$ becomes real. In this case the motion is non-oscillatory, corresponding to *overcritical damped vibrations*.[7] If the algorithm is related with positive numerical damping in the considered mode, the numerical calculated modal response may appear as overcritical damped, although the exact solution is undercritical damped.

A numerical integration algorithm, which is stable in all modes for an arbitrary length of the time step Δt is denoted *unconditional stable*. If stability in just a single mode requires that the time step must be kept below a certain critical magnitude in order to fulfill (2-63), the algorithm is referred to as *conditional stable*. Depending on the parameters (β, γ), the Newmark algorithm may be unstable, conditional stable or unconditional stable. Generally, the magnitude of the time step should be determined by accuracy requirements. Hence, conditional stable algorithms should be avoided in cases where the time step is determined by stability requirements.

In case of modal decoupling the stability analysis of the total algorithm is reduced to a stability of a sequence of decoupled SDOF oscillators for each mode as given by (2-17). The corresponding Newmark algorithm is given by (2-18), (2-19). We will analyze this system in the undamped case corresponding to the modal damping ratio $\zeta = 0$. It should be noted that stability conditions specified for the undamped case are always conservative, i.e. the introduction of structural damping enhance the numerical stability. Then, the amplification matrix may be written as

$$\bar{\mathrm{D}} = \frac{1}{1+\beta\kappa^2}\begin{bmatrix} 1+(\beta-\frac{1}{2})\kappa^2 & \Delta t \\ -(1+\frac{1}{2}(2\beta-\gamma)\kappa^2)\frac{\kappa^2}{\Delta t} & 1+(\beta-\gamma)\kappa^2 \end{bmatrix} \tag{2–65}$$

The characteristic equation of (2-65) becomes, cf. (1-11)

$$\bar{\lambda}_D^2 - a_1\bar{\lambda}_D + a_2 = 0 \tag{2–66}$$

where the invariants a_1 and a_2, specifying the trace and the determinant of $\bar{\mathrm{D}}$, may be written on the form

$$a_1 = \mathrm{tr}(\bar{\mathbf{D}}) = \frac{2 + (2\beta - \gamma - \frac{1}{2})\kappa^2}{1 + \beta\kappa^2} = 2 - \left(\gamma + \frac{1}{2}\right)\xi^2 \tag{2–67}$$

$$a_2 = \det(\bar{\mathbf{D}}) = \frac{1 + (2\beta - \gamma + \frac{1}{2})\kappa^2 + \beta(\beta - \gamma + \frac{1}{2})\kappa^4}{(1 + \beta\kappa^2)^2} = 1 - \left(\gamma - \frac{1}{2}\right)\xi^2 \tag{2–68}$$

where

$$\xi^2 = \frac{\kappa^2}{1 + \beta\kappa^2} \tag{2–69}$$

The final formulations for the invariants, which is due to Krenk,[6] shows that these depend on $\kappa = \omega\Delta t$ and β through a single positive parameter ξ^2. Next, the solutions of (2-66) may be written as

$$\left.\begin{matrix}\bar{\lambda}_{D,1}\\ \bar{\lambda}_{D,2}\end{matrix}\right\} = 1 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2 \pm \sqrt{\frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4 - \xi^2} \tag{2–70}$$

The eigenvalues are real, corresponding to overcritical damped eigenvibrations, for

$$\begin{aligned}&\frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4 - \xi^2 > 0 \qquad \Rightarrow\\ &\gamma > \frac{2}{\xi} - \frac{1}{2}\end{aligned} \tag{2–71}$$

In case of real eigenvalues, it follows from (2-70) that $\lambda_{D,2}$ is numerical larger than $\lambda_{D,1}$. Then, instability occurs for $\lambda_{D,2} = -1$, from which the following stability condition is derived

$$\begin{aligned}&-1 \leq 1 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2 - \sqrt{\frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4 - \xi^2} \qquad \Rightarrow\\ &\gamma \leq \frac{2}{\xi^2}\end{aligned} \tag{2–72}$$

If $\gamma < \frac{2}{\xi} - \frac{1}{2}$, the eigenvalues become complex conjugated, corresponding to undercritical damped eigenvibrations. In this case

$$\left.\begin{matrix}\bar{\lambda}_{D,1}\\ \bar{\lambda}_{D,2}\end{matrix}\right\} = 1 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2 \pm i\sqrt{\xi^2 - \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4} \tag{2–73}$$

Let $\bar{\lambda}_D$ denote the eigenvalue with positive imaginary part, i.e. $\bar{\lambda}_D = \bar{\lambda}_{D,1}$. The modulus and argument of $\bar{\lambda}_D$ become

$$|\bar{\lambda}_D| = \sqrt{\left(1 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2\right)^2 + \xi^2 - \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4} = \sqrt{1 - \left(\gamma - \frac{1}{2}\right)\xi^2} \tag{2–74}$$

$$\bar{\theta}_D = \arctan\left(\frac{\sqrt{\xi^2 - \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \xi^4}}{1 - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2}\right) = \xi - \frac{1}{96}\left(12\gamma^2 - 36\gamma + 11\right)\xi^3 + \mathrm{O}(\xi^5) =$$
$$\kappa - \left(\frac{1}{96}\left(12\gamma^2 - 36\gamma + 11\right) + \frac{1}{2}\beta\right)\kappa^3 + \mathrm{O}(\kappa^5) \tag{2–75}$$

Additionally, in (2-75) low order Taylor expansions in ξ and κ have been indicated for $\bar{\theta}_D$. The last expansion follows from the first using the auxiliary expansion $\xi = \kappa - \frac{1}{2}\beta\kappa^3 + \cdots$, which follows from (2-69). Stability of oscillatory eigenvibrations requires that $|\bar{\lambda}_D| \leq 1$. As seen from (2-74) this implies that

$$\gamma \geq \frac{1}{2} \tag{2–76}$$

ξ, and hence the time step Δt, is not entering (2-76). Hence, this inequality must be fulfilled for arbitrary large time steps. By contrast, (2-72) can always be fulfilled by choosing ξ sufficiently small. Using (2-69), the inequality (2-72) may be reformulated as

$$(\gamma - 2\beta)\kappa^2 \leq 2 \tag{2–77}$$

(2-77) is fulfilled for arbitrary κ, if $\gamma \leq 2\beta$. Combining this result with (2-76), it is seen that the Newmark algorithm is unconditional stable, if the parameters β and γ fulfill

$$\frac{1}{2} \leq \gamma \leq 2\beta \tag{2–78}$$

If $\gamma > 2\beta \wedge \gamma \geq \frac{1}{2}$, the algorithm is merely conditional stable. The critical value of the time step follows from (2-77)

$$\kappa = \omega\Delta t \leq \sqrt{\frac{2}{\gamma - 2\beta}} \tag{2–79}$$

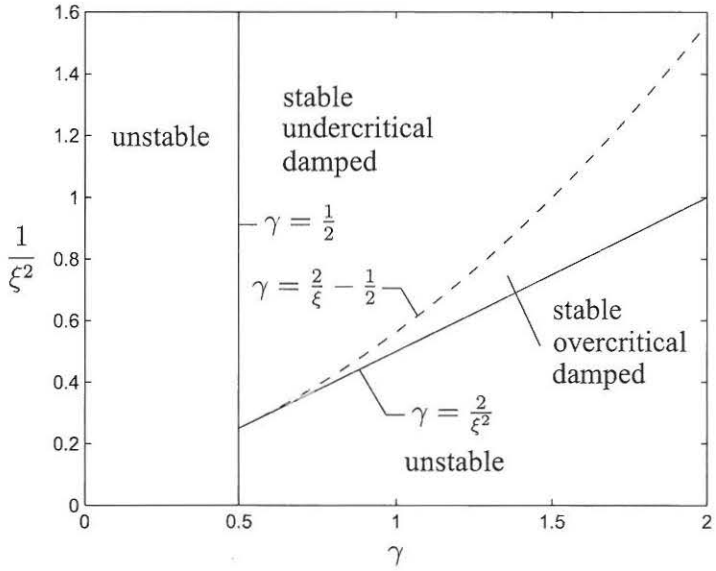

Fig. 2–1 Stability diagram of Newmark algorithms.[6]

The stability conditions of the Newmark algorithm, and the conditions for undercritical and critical damped eigenvibrations, have been summarized in Box 2-3. Additionally, these conditions have been illustrated in Fig. 2-1. The limit curve between undercritical and overcritical damped oscillations is shown with a dashed signature.

Box 2.3: Stability conditions of Newmark algorithm and qualitative behaviour of eigenvibrations

Unconditional stability:	$\frac{1}{2} \leq \gamma \leq 2\beta$
Conditional stability:	$\frac{1}{2} \leq \gamma \quad \wedge \quad \Delta t \leq \frac{1}{\omega}\sqrt{\frac{2}{\gamma - 2\beta}}$
Undercritical damped eigenvibrations:	$\gamma < \frac{2}{\xi} - \frac{1}{2} \quad , \quad \xi = \frac{\omega \Delta t}{\sqrt{1 + \beta \omega^2 \Delta t^2}}$
Overcritical damped eigenvibrations:	$\frac{2}{\xi} - \frac{1}{2} < \gamma < \frac{2}{\xi^2}$

In the undamped case the eigenvalues of the system matrix $\mathbf{A}$ becomes $\lambda_A = i\omega$, cf. (2-28), which means that the corresponding eigenvalues of the exact amplification matrix is given as $\lambda_D = \mathrm{e}^{i\kappa} = 1 + i\kappa - \frac{1}{2}\kappa^2 - i\frac{1}{6}\kappa^3 + \mathrm{O}(\kappa^4)$. Then, in case of oscillatory eigenvibrations the eigenvalue $\bar{\lambda}_D$ as given by (2-73) may be represented by the following Taylor expansion

$$\bar{\lambda}_D = 1 + i\xi - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\xi^2 - i\frac{1}{8}\left(\gamma + \frac{1}{2}\right)^2 \xi^3 + \mathrm{O}(\xi^5) =$$
$$1 + i\kappa - \frac{1}{2}\left(\gamma + \frac{1}{2}\right)\kappa^2 - i\left(\frac{1}{8}\left(\gamma + \frac{1}{2}\right)^2 + \frac{1}{2}\beta\right)\kappa^3 + \mathrm{O}(\kappa^4) =$$
$$\lambda_D - \frac{1}{2}\left(\gamma - \frac{1}{2}\right)\kappa^2 - i\left(\frac{1}{8}\left(\gamma + \frac{1}{2}\right)^2 + \frac{1}{2}\beta - \frac{1}{6}\right)\kappa^3 + \mathrm{O}(\kappa^4) \tag{2–80}$$

It follows that the error is of the order of magnitude $\mathrm{O}(\kappa^2)$, if $\gamma > \frac{1}{2}$, and of the order of magnitude $\mathrm{O}(\kappa^3)$, if $\gamma = \frac{1}{2}$. In the latter case (2-75) and (2-80) become

$$\bar{\theta}_D = \kappa + \left(\frac{1}{24} - \frac{1}{2}\beta\right)\kappa^3 + \mathrm{O}(\kappa^5) \tag{2–81}$$

$$\bar{\lambda}_D = \lambda_D + i\left(\frac{1}{24} - \frac{1}{2}\beta\right)\kappa^3 + \mathrm{O}(\kappa^4) \tag{2–82}$$

Additionally if $\beta = \frac{1}{12}$, (2-81) and (2-82) reduce to $\bar{\theta}_D = \kappa + \mathrm{O}(\kappa^5)$ and $\bar{\lambda}_D = \lambda_D + \mathrm{O}(\kappa^4)$. The Newmark algorithm with $(\beta, \gamma) = (\frac{1}{12}, \frac{1}{2})$ is known as the Fox-Goodwin algorithm. As follows from (2-79) this algorithm is conditional stable with the critical time step given by

$$\omega\Delta t = \sqrt{6} \tag{2–83}$$

It should be noticed that the error on the eigenvalue as indicated in (2-80) only contributes partly to the local truncation error (2-49). Additional errors are introduced by the difference between the eigenmodes $\bar{\mathbf{\Psi}}$ and $\mathbf{\Psi}$ of the approximate and exact multiplication matrices.

ω in (2-79) indicates the undamped circular eigenfrequency for an arbitrary mode. Hence, in case of conditional stable algorithms the critical time step insuring stability is determined by largest undamped circular eigenfrequency ω_n of the system. For systems with many degrees of freedom ω_n may be very large, and the critical time step correspondingly small. Since, the time step primarily should be determined by the accuracy of the low frequency modes, which carry the dynamic response, and not by the stability of the high frequency modes, which merely are results of the spatial discretization with no physical relevance, only unconditional stable algorithms should be used in such cases.

Example 2.4: High accuracy unconditional stable algorithms

The considered algorithms are based on the following factorization of the exact amplification matrix (2-25)

$$\mathbf{D} = \mathrm{e}^{\mathbf{A}\Delta t} = \mathrm{e}^{\mathbf{A}\frac{\Delta t}{2}}\mathrm{e}^{\mathbf{A}\frac{\Delta t}{2}} = \left(\mathrm{e}^{-\mathbf{A}\frac{\Delta t}{2}}\right)^{-1}\mathrm{e}^{\mathbf{A}\frac{\Delta t}{2}} = \mathbf{D}_1^{-1}\mathbf{D}_2 \tag{2–84}$$

where $\mathbf{A}$ is the system matrix given by (1-27). Additionally, (1-57) has been used with $t = -\frac{\Delta t}{2}$. From (1-53) follows

$$\left.\begin{aligned} \mathbf{D}_1 &= \mathrm{e}^{-\mathbf{A}\frac{\Delta t}{2}} = \mathbf{I} - \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 - \frac{1}{48}\Delta t^3\mathbf{A}^3 + \frac{1}{384}\Delta t^4\mathbf{A}^4 + \cdots \\ \mathbf{D}_2 &= \mathrm{e}^{\mathbf{A}\frac{\Delta t}{2}} = \mathbf{I} + \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 + \frac{1}{48}\Delta t^3\mathbf{A}^3 + \frac{1}{384}\Delta t^4\mathbf{A}^4 + \cdots \end{aligned}\right\} \tag{2–85}$$

Truncation of the expansions (2-85) with the same number of terms provides the following approximate multiplication matrices

$$\left.\begin{aligned} \bar{\mathbf{D}} &= \left(\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A}\right)^{-1}\left(\mathbf{I} + \frac{1}{2}\Delta t\mathbf{A}\right) \\ \bar{\mathbf{D}} &= \left(\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2\right)^{-1}\left(\mathbf{I} + \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2\right) \\ \bar{\mathbf{D}} &= \left(\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 - \frac{1}{48}\Delta t^3\mathbf{A}^3\right)^{-1}\left(\mathbf{I} + \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 + \frac{1}{48}\Delta t^3\mathbf{A}^3\right) \\ \bar{\mathbf{D}} &= \left(\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 - \frac{1}{48}\Delta t^3\mathbf{A}^3 + \frac{1}{384}\Delta t^4\mathbf{A}^4\right)^{-1} \cdot \\ &\quad \left(\mathbf{I} + \frac{1}{2}\Delta t\mathbf{A} + \frac{1}{8}\Delta t^2\mathbf{A}^2 + \frac{1}{48}\Delta t^3\mathbf{A}^3 + \frac{1}{384}\Delta t^4\mathbf{A}^4\right) \end{aligned}\right\} \tag{2–86}$$

Below, the numerical algorithms following from the approximate multiplication matrices in (2-86) are referred to as the 1st, the 2nd, the 3rd and the 4th approximation. From (1-55) follows, that the matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ alternatively may be given as

$$\left.\begin{aligned} \mathbf{D}_1 &= \mathbf{\Psi}\mathbf{\Lambda}_{D_1}\mathbf{\Psi}^{-1} \\ \mathbf{D}_2 &= \mathbf{\Psi}\mathbf{\Lambda}_{D_2}\mathbf{\Psi}^{-1} \end{aligned}\right\} \tag{2–87}$$

where $\mathbf{\Psi}$ is the modal matrix containing the eigenvectors of $\mathbf{D}_1$ and $\mathbf{D}_2$ stored column-wise. As follows from Box 1.1, this matrix is identical to the modal matrix of the system matrix $\mathbf{A}$. $\mathbf{\Lambda}_{D_1}$ and $\mathbf{\Lambda}_{D_2}$ are diagonal matrices, storing the eigenvalues of $\mathbf{D}_1$ and $\mathbf{D}_2$ in the main diagonal. These are given as, cf. (1-53)

$$\left.\begin{aligned} \mathbf{\Lambda}_{D_1} &= \mathbf{I} - \frac{1}{2}\Delta t\mathbf{\Lambda}_A + \frac{1}{8}\Delta t^2\mathbf{\Lambda}_A^2 - \frac{1}{48}\Delta t^3\mathbf{\Lambda}_A^3 + \frac{1}{384}\Delta t^4\mathbf{\Lambda}_A^4 + \cdots \\ \mathbf{\Lambda}_{D_2} &= \mathbf{I} + \frac{1}{2}\Delta t\mathbf{\Lambda}_A + \frac{1}{8}\Delta t^2\mathbf{\Lambda}_A^2 + \frac{1}{48}\Delta t^3\mathbf{\Lambda}_A^3 + \frac{1}{384}\Delta t^4\mathbf{\Lambda}_A^4 + \cdots \end{aligned}\right\} \tag{2–88}$$

where $\mathbf{\Lambda}_A$ is a diagonal matrix storing the eigenvalues of $\mathbf{A}$. From (2-66) and (2-87) follow that the approximate amplification matrices may be written as

$$\bar{\mathbf{D}} = \mathbf{\Psi}\bar{\mathbf{\Lambda}}_D\mathbf{\Psi}^{-1} \quad , \quad \bar{\mathbf{\Lambda}}_D = \mathbf{\Lambda}_{D_1}^{-1}\mathbf{\Lambda}_{D_2} \tag{2–89}$$

The eigenvectors of the approximate amplification matrices are identical to those of the exact amplification matrix, see (2-27). For this reason the local and global truncation errors are entirely related to the eigenvalues $\bar{\lambda}_D$. In order to analyze these errors consider a certain mode with the eigenvalue λ_A. Further, structural damping is ignored, in which case $\lambda_A = i\omega$. Then, the related eigenvalues of the approximate amplification matrices in (2-86) may be written on the form

$$\left.\begin{aligned}
\bar{\lambda}_D &= \frac{1+i\frac{1}{2}\kappa}{1-i\frac{1}{2}\kappa} = \lambda_D - i\frac{1}{12}\kappa^3 + \cdots \\
\bar{\lambda}_D &= \frac{1+i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2}{1-i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2} = \lambda_D + i\frac{1}{24}\kappa^3 + \cdots \\
\bar{\lambda}_D &= \frac{1+i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2-i\frac{1}{48}\kappa^3}{1-i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2+i\frac{1}{48}\kappa^3} = \lambda_D + i\frac{1}{480}\kappa^5 + \cdots \\
\bar{\lambda}_D &= \frac{1+i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2-i\frac{1}{48}\kappa^3+\frac{1}{384}\kappa^4}{1-i\frac{1}{2}\kappa-\frac{1}{8}\kappa^2+i\frac{1}{48}\kappa^3+\frac{1}{384}\kappa^4} = \lambda_D - i\frac{1}{1920}\kappa^5 + \cdots
\end{aligned}\right\} \tag{2–90}$$

where $\kappa = \omega\Delta t$ and $\lambda_D = e^{i\kappa}$. The modulus of the eigenvalue of the 1st approximation is determined from from

$$|\bar{\lambda}_D|^2 = \bar{\lambda}_D \cdot \bar{\lambda}_D^* = \frac{1+i\frac{1}{2}\kappa}{1-i\frac{1}{2}\kappa} \cdot \frac{1-i\frac{1}{2}\kappa}{1+i\frac{1}{2}\kappa} = 1 \tag{2–91}$$

Similarly, it is shown that also the eigenvalues of the other approximations in (2-90) have the modulus $|\bar{\lambda}_D| = 1$ for arbitrary value of κ. Consequently, all the approximate multiplication matrices in (2-86) result in unconditional stable numerical algorithms. A drawback of the algorithms is that the high frequency modes will not be dissipated. Further, as seen from (2-90) the 1st and 2nd approximations both have local truncation errors $O(\kappa^3)$, whereas the 3rd and 4th approximations have local truncation errors $O(\kappa^5)$. Due to the simpler amplification matrices, the algorithms based on the 1st and 3rd approximations should generally be preferred. The 1st approximation is recognized as the amplification matrix of the Crank-Nicolson algorithm given by (2-45), (2-46), (2-47). Hence, this method, which is identical to the Newmark algorithm with $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$, has the local truncation errors $-\frac{i}{12}\kappa^3 + \cdots$ in agreement with (2-82).

Example 2.5: Upper bound for largest eigenfrequency

Fig. 2–2 Bernoulli-Euler beam element with constant section.

For conditional stable algorithms an upper bound of the largest undamped circular eigenfrequency is needed in order to specify a stable time step via (2-79). It can be shown that ω_n is bounded by the maximum circular frequency of the individual elements.[4,5] Consider a plane Bernoulli-Euler element of the length l with constant mass per unit length μ, bending stiffness EI and axial stiffness AE, see Fig. 2-2. Then, the stiffness matrix and the consistent mass matrix of the element becomes[7]

$$\mathbf{k} = \begin{bmatrix}
\frac{AE}{l} & 0 & 0 & -\frac{AE}{l} & 0 & 0 \\
0 & 12\frac{EI}{l^3} & 6\frac{EI}{l^2} & 0 & -12\frac{EI}{l^3} & 6\frac{EI}{l^2} \\
0 & 6\frac{EI}{l^2} & 4\frac{EI}{l} & 0 & -6\frac{EI}{l^2} & 2\frac{EI}{l} \\
-\frac{AE}{l} & 0 & 0 & \frac{AE}{l} & 0 & 0 \\
0 & -12\frac{EI}{l^3} & -6\frac{EI}{l^2} & 0 & 12\frac{EI}{l^3} & -6\frac{EI}{l^2} \\
0 & 6\frac{EI}{l^2} & 2\frac{EI}{l} & 0 & -6\frac{EI}{l^2} & 4\frac{EI}{l}
\end{bmatrix} \tag{2–92}$$

$$\mathbf{m} = \frac{\mu l}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22l & 0 & 54 & -13l \\ 0 & 22l & 4l^2 & 0 & 13l & -3l^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13l & 0 & 156 & -22l \\ 0 & -13l & -3l^2 & 0 & -22l & 4l^2 \end{bmatrix} \tag{2–93}$$

The generalized eigenvalue problem (1-9) with the stiffness matrix (2-92) and the mass matrix (2-93) has the following eigenvalues

$$\lambda_j = \begin{cases} 0 & , \quad j = 1 \\ 0 & , \quad j = 2 \\ 0 & , \quad j = 3 \\ 12\frac{AE}{\mu l^2} & , \quad j = 4 \\ 720\frac{EI}{\mu l^4} & , \quad j = 5 \\ 8400\frac{EI}{\mu l^4} & , \quad j = 6 \end{cases} \tag{2–94}$$

The first three eigenvalues are equal to zero, because the system is unsupported and has three independent *stiff-body motions*. If all the beam elements in the structure are identical an upper bound for the undamped circular eigenfrequency then becomes

$$\omega_n \leq \max\left(\sqrt{12\frac{AE}{\mu l^2}}, \sqrt{8400\frac{EI}{\mu l^4}}\right) \tag{2–95}$$

In case of different beam elements the right hand side of (2-95) must be calculated for all element types, and the largest of these represent the true upper bound for the structure.

2.1.3 Period Errors and Numerical Damping

In case of zero structural damping the following expansion of the undamped circular eigenfrequency of a certain mode as predicted by the Newmark algorithm follows from (2-60), (2-75)

$$\bar{\omega} = \omega\left(1 - \left(\frac{1}{96}\left(12\gamma^2 - 36\gamma + 11\right) + \frac{1}{2}\beta\right)\kappa^2 + \mathrm{O}\left(\kappa^4\right)\right) \tag{2–96}$$

The corresponding period of undamped eigenvibrations is

$$\bar{T} = \frac{2\pi}{\bar{\omega}} = T\left(1 + \left(\frac{1}{96}\left(12\gamma^2 - 36\gamma + 11\right) + \frac{1}{2}\beta\right)\kappa^2 + \mathrm{O}\left(\kappa^4\right)\right) \tag{2–97}$$

where $T = \frac{2\pi}{\omega}$ denotes the exact undamped eigenperiod. As illustrated in Example 2.6 the numerical determined eigenvibrations will deviate more and more from the analytical solutions,

if the underlying eigenperiods $\bar{T}$ and T are different. As a measure of the deviations of the periods the *relative period error* is defined as

$$\frac{\Delta T}{T} = \frac{\bar{T} - T}{T} = \left(\frac{1}{96}\left(12\gamma^2 - 36\gamma + 11\right) + \frac{1}{2}\beta\right)\kappa^2 + \mathrm{O}(\kappa^4) \tag{2–98}$$

Let $\gamma = \frac{1}{2}$. Then, (2-98) implies that $\bar{T} < T$ for $\beta < \frac{1}{12}$, and $\bar{T} > T$ for $\beta > \frac{1}{12}$. Especially, $\bar{T} = T + \mathrm{O}(\kappa^4)$ for the Fox-Goodwin algorithm, where $\beta = \frac{1}{12}$.

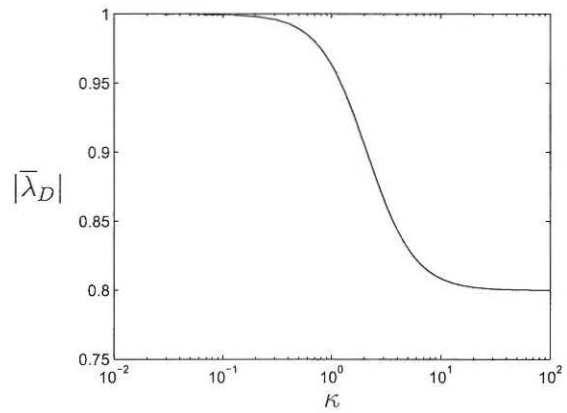

Fig. 2–3 Variation of $|\bar{\lambda}_D|$ as a function of $\kappa = \omega\Delta t$. $\alpha = 0.09$, $\beta = \frac{1}{4}$.

As mentioned in the introduction to this chapter high frequency modes does not reflect the physical reality. Apparently this seems of less importance, since these modes seldom influence the displacement solution. However, since the velocity and acceleration increase with the frequency for a given displacement amplitude, the high frequency modes may carry a substantial fictitious kinetic energy, which may be transferred to the low frequency modes via possible linear viscous coupling terms. For this reason high frequency modes should be dissipated by the numerical algorithm without affecting the accuracy of the low frequency modes. This may be obtained by choosing $\gamma = \alpha + \frac{1}{2}$, where α is a small positive number, in which case (2-74) attains the form

$$|\bar{\lambda}_D| = \sqrt{1 - \alpha\xi^2} \tag{2–99}$$

(2-99) has been illustrated in Fig. 2-3. Of special interest is the limit value $|\bar{\lambda}_D|_\infty$ approached by (2-99) in the high frequency range. From (2-69) follows that $\xi^2 \to \frac{1}{\beta}$, as κ and hence ω passes to infinity for fixed Δt. Hence,

$$|\bar{\lambda}_D|_\infty = \sqrt{1 - \frac{\alpha}{\beta}} \tag{2–100}$$

Insertion of $\gamma = \alpha + \frac{1}{2}$ into (2-80) provides

$$\bar{\lambda}_D = \lambda_D - \frac{1}{2}\alpha\,\kappa^2 - i\frac{1}{24}\Big(3\alpha^2 + 6\alpha - 1 + 12\beta\Big)\kappa^3 + \mathrm{O}\big(\kappa^4\big) \tag{2–101}$$

As seen the introduction of numerical damping reduces the local truncation error from $\mathrm{O}(\kappa^3)$ to $\mathrm{O}(\kappa^2)$. This means that the accuracy of the Newmark algorithm in this case is comparable with the those of the forward and backward Euler schemes.[1]

From (2-59), (2-69), (2-74) and (2-99) follow that the damping parameter α is related to the modal damping as follows

$$\bar{\zeta} = -\frac{1}{2\bar{\omega}\Delta t}\ln\left(1 - \alpha\xi^2\right) = \frac{\alpha\kappa^2}{2\omega\Delta t}\Big(1 + \mathrm{O}\big(\kappa^2\big)\Big) = \frac{1}{2}\alpha\kappa\Big(1 + \mathrm{O}\big(\kappa^2\big)\Big) \tag{2–102}$$

As seen $\bar{\zeta}$ is approximately proportional with α and κ. For $\frac{\Delta t}{T} = \frac{1}{20}$ and $\alpha = 0.1$ the numerical modal damping ratio becomes $\bar{\zeta} \simeq 0.0157$. This is comparable to the structural modal damping, typically of the order of magnitude $\zeta \simeq 0.01$. Hence, the algorithm will overestimate the damping properties of the system. In case of resonance under harmonic excitations, where the response is proportional to the inverse of the damping ratio, or in case of stochastic excitations, where the response is proportional to the inverse of the square root of the damping ratio, the structural response will be underestimated correspondingly. It follows, that numerical damping cannot be introduced in the Newmark algorithm without using significantly smaller time steps than suggested by the accuracy requirements for the corresponding algorithm with $\alpha = 0$. Consequently, there is a need of an algorithm, where $\bar{\zeta} = \mathrm{O}(\kappa^2)$, rather than $\bar{\zeta} = \mathrm{O}(\kappa)$.

Example 2.6: Newmark solution for a SDOF oscillator

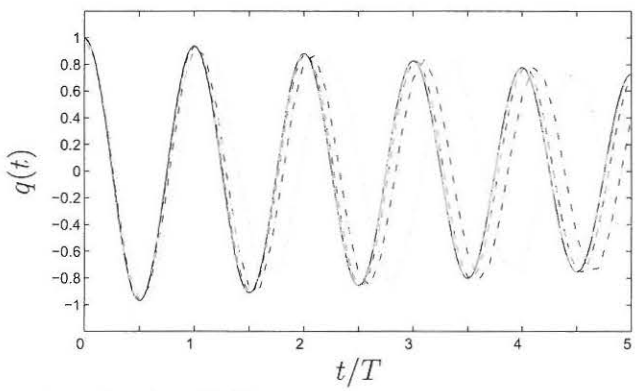

Fig. 2–4 Newmark algorithm, $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$. Damped eigenvibrations of SDOF oscillator. $\omega = 1$, $\zeta = 0.01$, $(q_0, \dot{q}_0) = (1, 0)$. Analytical solution: —, $\frac{\Delta t}{T} = 0.05$: - - -, $\frac{\Delta t}{T} = 0.10$: -.-.-, $\frac{\Delta t}{T} = 0.20$: ..... .

Fig. 2-4 shows the numerical results for eigenvibrations of a SDOF oscillator with $\omega = 1$ and $\zeta = 0.1$ obtained by the Newmark algorithm with the parameters $(\beta, \gamma) = (\frac{1}{4}, \frac{1}{2})$, and with the time steps $\frac{\Delta t}{T_k} = 0.05, 0.10, 0.20$. Since, $\gamma = \frac{1}{2}$ the algorithm is not related with numerical damping. Correspondingly, the numerical solutions are dissipated in the same rate as the analytical solution. Since, $\beta = \frac{1}{4}$ all numerical solutions are related with period elongation, which increases proportional with Δt^2 as indicated by (2-98).

2.2 Generalized Alpha Algorithm

The problem with the Newmark algorithm that numerical damping of the high frequency modes can only be treated at the cost of accuracy of the low frequency modes was soon recognized. One attempt to cure this problem is the socalled collocation methods, first proposed by Wilson.[8] Wilson's algorithm, known as the θ-method, is a modification of the Newmark algorithm with linear varyingacceleration, where the equations of motion are fulfilled at the time $t_{j+\theta}$, $\theta > 1$, rather than at the time t_{j+1}. Later the method was generalized by Hilber et al.[9] to allow for other variations of the acceleration via the introduction of an extra parameter.

Hilber et al.[10] also proposed the socalled α method as a solution to the problem. Instead of introducing the exact damping forces, stiffness forces and external forces in the equation of motion at the time t_{j+1}, a weighted sum of these forces at the times t_j and t_{j+1} is introduced as follows

$$\mathbf{M}\ddot{\mathbf{x}}_{j+1} + \alpha_f\Big(\mathbf{C}\dot{\mathbf{x}}_j + \mathbf{K}\mathbf{x}_j - \mathbf{f}_j\Big) + (1-\alpha_f)\Big(\mathbf{C}\dot{\mathbf{x}}_{j+1} + \mathbf{K}\mathbf{x}_{j+1} - \mathbf{f}_{j+1}\Big) = \mathbf{0} \qquad (2\text{–}103)$$

where α_f is an interpolation parameter fulfilling $\alpha_f \in [0,1]$. Based on the same way of thinking, Wood et al.[11] suggested an alternative α-method, where the damping forces, stiffness forces and external forces are left unchanged in the equation of motion at the time t_{j+1}, whereas the inertial forces are specified as a weighted sum of these forces at the times t_j and t_{j+1}

$$\alpha_m\mathbf{M}\ddot{\mathbf{x}}_j + (1-\alpha_m)\mathbf{M}\ddot{\mathbf{x}}_{j+1} + \mathbf{C}\dot{\mathbf{x}}_{j+1} + \mathbf{K}\mathbf{x}_{j+1} = \mathbf{f}_{j+1} \qquad (2\text{–}104)$$

Again the interpolation parameter fulfills $\alpha_m \in [0,1]$. The generalization of the indicated α-methods, where the inertial forces, and the damping, stiffness and external forces, are interpolated with different interpolation parameters, was suggested by Chung and Hulbert.[3] In this case the equations of motion for the solution at the time t_{j+1} read

$$\begin{aligned}&\alpha_m\mathbf{M}\ddot{\mathbf{x}}_j + (1-\alpha_m)\mathbf{M}\ddot{\mathbf{x}}_{j+1} + \alpha_f\Big(\mathbf{C}\dot{\mathbf{x}}_j + \mathbf{K}\mathbf{x}_j - \mathbf{f}_j\Big) + \\ &(1-\alpha_f)\Big(\mathbf{C}\dot{\mathbf{x}}_{j+1} + \mathbf{K}\mathbf{x}_{j+1} - \mathbf{f}_{j+1}\Big) = \mathbf{0} \quad , \quad j = 1\,,\,\ldots\,,\,n \qquad (2\text{–}105)\end{aligned}$$

The equations of the generalized α-method consist of (2-105), along with the Newmark equations (2-3) and (2-4). Hence, the generalized α-method contains the parameters β, γ, α_m and α_f. As for the other mentioned methods, the generalized α-method has a local truncation error

[8] E.L. Wilson: *A Computer Program for the Dynamic Stress Analysis of Underground Structures*. SESM Report No- 68-1, Division of Structural Engineering and Structural Mechanics, University of Califirnia, Berkeley, 1968.

[9] H.H. Hilber and T.J.R. Hughes and R.L. Taylor: *Collocation, Dissipation and "Overshoot" for Time Integration Schemes in Structural Dynamics*. Earthquake Engineering and St ructural Dynamics, **6**, 1978, 99-118.

[10] H.H. Hilber, T.J.R. Hughes and R.L. Taylor: *Improved Numerical Dissipation for Time Integration in Structural Dynamics*. Earthquake Engineering and Structural Dynamics, **5**, 1977, 283-292.

[11] W.L. Wood, M. Bossak and O.C. Zienkiewicz: *An Alpha Modification of Newmarks Method*. International Journal for Numerical Methods in Engineering, **15**, 1981, 1562-1566.

$O(\kappa^3)$, resulting in a numerical damping ratio $\bar{\zeta} = O(\kappa^2)$. At optimal tuning of the parameters of the method, Chung and Hulbert demonstrated that the numerical damping ratio and the relative period error are smaller than those of the competing methods. General for all the mentioned algorithms is that the equations of motion is not fulfilled at the time t_j. For this reason the singlestep multivalue formulation of the algorithms requires a state vector of dimension $3n$, encompassing the displacement vector, the velocity vector and the acceleration vector.

At first the singlestep singlevalue formulation of the generalized α-algorithm is indicated. Similar to the derivation of the Newmark algorithm, the idea is to eliminate $\mathbf{x}_{j+1}$ and $\dot{\mathbf{x}}_{j+1}$ in (2-105) by means of (2-3) and (2-4), in order to obtain an equation for the acceleration $\ddot{\mathbf{x}}_{j+1}$. This results in

$$\begin{aligned}
&\Big((1-\alpha_m)\mathbf{M} + (1-\alpha_f)(\gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K})\Big)\ddot{\mathbf{x}}_{j+1} = \\
&\alpha_f\mathbf{f}_j + (1-\alpha_f)\mathbf{f}_{j+1} - \alpha_m\mathbf{M}\ddot{\mathbf{x}}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j - \\
&(1-\alpha_f)(1-\gamma)\Delta t\mathbf{C}\ddot{\mathbf{x}}_j - (1-\alpha_f)\mathbf{K}\Big(\Delta t\,\dot{\mathbf{x}}_j + \Big(\frac{1}{2}-\beta\Big)\Delta t^2\,\ddot{\mathbf{x}}_j\Big) = \\
&\Big((1-\alpha_m)\mathbf{M} + (1-\alpha_f)(\gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K})\Big)\ddot{\mathbf{x}}_j + \\
&\alpha_f\mathbf{f}_j + (1-\alpha_f)\mathbf{f}_{j+1} - \mathbf{M}\ddot{\mathbf{x}}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j - \\
&(1-\alpha_f)\Big(\Delta t\mathbf{C}\ddot{\mathbf{x}}_j + \mathbf{K}\Big(\Delta t\dot{\mathbf{x}}_j + \frac{1}{2}\Delta t^2\,\ddot{\mathbf{x}}_j\Big)\Big) \quad \Rightarrow \\
&\ddot{\mathbf{x}}_{j+1} = \ddot{\mathbf{x}}_j + \tilde{\mathbf{M}}^{-1}\Big[\alpha_f\mathbf{f}_j + (1-\alpha_f)\mathbf{f}_{j+1} - \mathbf{M}\ddot{\mathbf{x}}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j - \\
&(1-\alpha_f)\Big(\Delta t\mathbf{C}\ddot{\mathbf{x}}_j + \mathbf{K}\Big(\Delta t\dot{\mathbf{x}}_j + \frac{1}{2}\Delta t^2\,\ddot{\mathbf{x}}_j\Big)\Big)\Big]
\end{aligned} \tag{2–106}$$

where $\tilde{\mathbf{M}}$ is the socalled *dynamic mass matrix* defined as

$$\tilde{\mathbf{M}} = (1-\alpha_m)\mathbf{M} + (1-\alpha_f)\Big(\gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K}\Big) \tag{2–107}$$

To start the algorithm the acceleration $\ddot{\mathbf{x}}_0$ at the time t_0 is needed, which is calculated from (2-10). The algorithm has been summarized in Box 2.4. The main problem left is a procedure for the optimal selection of the parameters β, γ, α_m and α_f to insure unconditional numerical stability and desirable damping of high frequency modes. This problem is addressed in the following section.

The singlestep multivalue formulation for the state vector made up of the displacement, velocity and acceleration vectors follows from (2-3), (2-4) and (2-105)

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & -\beta\Delta t^2\mathbf{I} \\ \mathbf{0} & \mathbf{I} & -\gamma\Delta t\mathbf{I} \\ (1-\alpha_f)\mathbf{K} & (1-\alpha_f)\mathbf{C} & (1-\alpha_m)\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{j+1} \\ \dot{\mathbf{x}}_{j+1} \\ \ddot{\mathbf{x}}_{j+1} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{I} & \Delta t\,\mathbf{I} & (\frac{1}{2}-\beta)\Delta t^2\mathbf{I} \\ \mathbf{0} & \mathbf{I} & (1-\gamma)\Delta t\mathbf{I} \\ -\alpha_f\mathbf{K} & -\alpha_f\mathbf{C} & -\alpha_m\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \\ \ddot{\mathbf{x}}_j \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \alpha_f\mathbf{I} & (1-\alpha_f)\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix} \quad \Rightarrow$$

$$\mathbf{z}_{j+1} = \bar{\mathbf{D}}\mathbf{z}_j + \mathbf{E}_j \tag{2–108}$$

where

$$\left.\begin{aligned} \mathbf{z}_j &= \begin{bmatrix} \mathbf{x}_j \\ \dot{\mathbf{x}}_j \\ \ddot{\mathbf{x}}_j \end{bmatrix} \\ \bar{\mathbf{D}} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\beta\Delta t^2\mathbf{I} \\ \mathbf{0} & \mathbf{I} & -\gamma\Delta t\mathbf{I} \\ (1-\alpha_f)\mathbf{K} & (1-\alpha_f)\mathbf{C} & (1-\alpha_m)\mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{I} & \Delta t\,\mathbf{I} & (\frac{1}{2}-\beta)\Delta t^2\mathbf{I} \\ \mathbf{0} & \mathbf{I} & (1-\gamma)\Delta t\mathbf{I} \\ -\alpha_f\mathbf{K} & -\alpha_f\mathbf{C} & -\alpha_m\mathbf{M} \end{bmatrix} \\ \mathbf{E}_j &= \begin{bmatrix} \mathbf{I} & \mathbf{0} & -\beta\Delta t^2\mathbf{I} \\ \mathbf{0} & \mathbf{I} & -\gamma\Delta t\mathbf{I} \\ (1-\alpha_f)\mathbf{K} & (1-\alpha_f)\mathbf{C} & (1-\alpha_m)\mathbf{M} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \alpha_f\mathbf{I} & (1-\alpha_f)\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}_j \\ \mathbf{f}_{j+1} \end{bmatrix} \end{aligned}\right\} \tag{2–109}$$

The corresponding singlestep multivalue implementation of the SDOF system (2-17) follows by specialization of (2-108) and (2-109)

$$\left.\begin{aligned} \mathbf{z}_j &= \begin{bmatrix} q_j \\ \dot{q}_j \\ \ddot{q}_j \end{bmatrix} \quad , \quad \bar{\mathbf{D}} = \mathbf{D}_1^{-1}\mathbf{D}_2 \\ \mathbf{D}_1 &= \begin{bmatrix} 1 & 0 & -\beta\Delta t^2 \\ 0 & 1 & -\gamma\Delta t \\ (1-\alpha_f)\omega^2 & (1-\alpha_f)2\zeta\omega & (1-\alpha_m) \end{bmatrix} \\ \mathbf{D}_2 &= \begin{bmatrix} 1 & \Delta t & (\frac{1}{2}-\beta)\Delta t^2 \\ 0 & 1 & (1-\gamma)\Delta t \\ -\alpha_f\omega^2 & -\alpha_f 2\zeta\omega & -\alpha_m \end{bmatrix} \\ \mathbf{E}_j &= \mathbf{D}_1^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \alpha_f & (1-\alpha_f) \end{bmatrix} \begin{bmatrix} F_j \\ F_{j+1} \end{bmatrix} \end{aligned}\right\} \tag{2–110}$$

Box 2.4: Generalized alpha algorithm

Given the initial displacement vector $\mathbf{x}_0$ and the initial velocity vector $\dot{\mathbf{x}}_0$ at the time t_0. Perform the following initial calculations

1. Calculate the initial acceleration vector $\ddot{\mathbf{x}}_0$ from

$$\ddot{\mathbf{x}}_0 = \mathbf{M}^{-1}\Big(\mathbf{f}_0 - \mathbf{C}\dot{\mathbf{x}}_0 - \mathbf{K}\mathbf{x}_0\Big)$$

2. Calculate dynamic mass matrix $\tilde{\mathbf{M}}$ from

$$\tilde{\mathbf{M}} = (1-\alpha_m)\mathbf{M} + \Big(1-\alpha_f\Big)\big(\gamma\Delta t\mathbf{C} + \beta\Delta t^2\mathbf{K}\big)$$

Repeat the following items for $j = 0, 1, \ldots, n$

1. Calculate new acceleration vector from

$$\ddot{\mathbf{x}}_{j+1} = \ddot{\mathbf{x}}_j + \tilde{\mathbf{M}}^{-1}\Bigg[\alpha_f\mathbf{f}_j + (1-\alpha_f)\mathbf{f}_{j+1} - \mathbf{M}\ddot{\mathbf{x}}_j - \mathbf{C}\dot{\mathbf{x}}_j - \mathbf{K}\mathbf{x}_j - (1-\alpha_f)\Big(\Delta t\mathbf{C}\ddot{\mathbf{x}}_j + \mathbf{K}\Big(\Delta t\dot{\mathbf{x}}_j + \frac{1}{2}\Delta t^2\,\ddot{\mathbf{x}}_j\Big)\Big)\Bigg]$$

2. Calculate new displacement and velocity vectors from

$$\mathbf{x}_{j+1} = \mathbf{x}_j + \dot{\mathbf{x}}_j\,\Delta t + \Big(\Big(\frac{1}{2}-\beta\Big)\ddot{\mathbf{x}}_j + \beta\,\ddot{\mathbf{x}}_{j+1}\Big)\Delta t^2$$

$$\dot{\mathbf{x}}_{j+1} = \dot{\mathbf{x}}_j + \Big((1-\gamma)\ddot{\mathbf{x}}_j + \gamma\,\ddot{\mathbf{x}}_{j+1}\Big)\Delta t$$

2.2.1 Numerical Stability and Selection of Parameters of the Generalized Alpha Algorithm

The special eigenvalue problem for the amplification matrix $\bar{\mathbf{D}}$ in (2-110) is reformulated into the following generalized eigenvalue problem

$$\begin{aligned}
&\bar{\mathbf{D}}\,\psi = \bar{\lambda}_D\,\psi && \Rightarrow \\
&\mathbf{D}_1^{-1}\,\mathbf{D}_2\,\psi = \bar{\lambda}_D\mathbf{I}\,\psi && \Rightarrow \\
&\mathbf{D}_2\,\psi = \bar{\lambda}_D\mathbf{D}_1\,\psi && \qquad (2\text{–}111)
\end{aligned}$$

For the undamped case, $\zeta = 0$, the characteristic equation follows from (1-10) and (2-110)

$$\det\left(\mathbf{D}_2 - \bar{\lambda}_D \mathbf{D}_1\right) =$$

$$\det\left(\begin{bmatrix} 1-\bar{\lambda}_D & \Delta t & \left(\frac{1}{2}-\beta+\beta\bar{\lambda}_D\right)\Delta t^2 \\ 0 & 1-\bar{\lambda}_D & \left(1-\gamma+\gamma\bar{\lambda}_D\right)\Delta t \\ -\left(\alpha_f+(1-\alpha_f)\bar{\lambda}_D\right)\omega^2 & 0 & -\alpha_m-(1-\alpha_m)\bar{\lambda}_D \end{bmatrix}\right) =$$

$$-\left(\alpha_m+(1-\alpha_m)\bar{\lambda}_D\right)\left(1-\bar{\lambda}_D\right)^2$$
$$-\left(\alpha_f+(1-\alpha_f)\bar{\lambda}_D\right)\left(1-\gamma+\gamma\bar{\lambda}_D-(1-\bar{\lambda}_D)\left(\frac{1}{2}-\beta+\beta\bar{\lambda}_D\right)\right)\kappa^2 = 0 \qquad (2\text{–}112)$$

Being of the 3rd order, the characteristic equation (2-112) has one real and two complex conjugated solutions. We are merely interested in unconditional stable numerical algorithms, which implies that all three eigenvalues must fulfill $|\bar{\lambda}_D| \leq 1$ for arbitrary value of κ. The parameters for which this is fulfilled can be shown to be[3,6]

$$\left.\begin{array}{l} \frac{1}{2} \leq \gamma \leq 2\beta \\ \alpha_m \leq \frac{1}{2} \quad , \quad \alpha_f \leq \frac{1}{2} \quad , \quad \gamma \geq \frac{1}{2}+\alpha_f-\alpha_m \end{array}\right\} \qquad (2\text{–}113)$$

The inequalities in the 1st line are identical to the corresponding conditions (2-78) for unconditional stability of the Newmark algorithm. The 1st and last inequality implies the condition $\alpha_f \geq \alpha_m$. The high frequency limit of the eigenvalues as $\kappa \to \infty$ may be obtained by dividing (2-112) with κ^2, and performing the limit passing. The result becomes

$$\left(\alpha_f+(1-\alpha_f)\bar{\lambda}_D\right)\left(1-\gamma+\gamma\bar{\lambda}_D-(1-\bar{\lambda}_D)\left(\frac{1}{2}-\beta+\beta\bar{\lambda}_D\right)\right) = 0 \qquad (2\text{–}114)$$

The solutions of (2-114) are

$$\bar{\lambda}_D = \begin{cases} -\dfrac{\gamma+\frac{1}{2}-2\beta+\sqrt{(\gamma+\frac{1}{2})^2-4\beta}}{2\beta} & , \quad j=1 \\ -\dfrac{\gamma+\frac{1}{2}-2\beta-\sqrt{(\gamma-\frac{1}{2})^2-4\beta}}{2\beta} & , \quad j=2 \\ -\dfrac{\alpha_f}{1-\alpha_f} & , \quad j=3 \end{cases} \qquad (2\text{–}115)$$

We shall choose β in a way that the eigenvibrations corresponding to $j=1$ and $j=2$ become *critical damped* in the high frequency limit. In this case the square root of the 1st and 2nd

eigenvalues vanish, and the roots of the characteristic equation coalesce into a double root. The condition for this is

$$\beta = \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \tag{2–116}$$

For all finite $\kappa < \infty$ the eigenvalues become complex, corresponding to oscillatory eigenvibrations. As seen from (2-69) and (2-70), the relation (2-116) is identical to the condition for critical damped eigenvibrations for the Newmark method. Insertion of (2-116) into (2-115) provides the following expression for the eigenvalues at critical dampning

$$\bar{\lambda}_D = -\frac{\gamma + \frac{1}{2} - 2\beta}{2\beta} = \frac{\gamma - \frac{3}{2}}{\gamma + \frac{1}{2}} \tag{2–117}$$

Next, γ is selected at the stability limit, cf. (2-113)

$$\gamma = \frac{1}{2} + \alpha_f - \alpha_m \tag{2–118}$$

Then, (2-117) becomes

$$\bar{\lambda}_D = \frac{\alpha_f - \alpha_m - 1}{\alpha_f - \alpha_m + 1} \tag{2–119}$$

The modulus of the eigenvalue (2-119) is denoted as $|\bar{\lambda}_D|_\infty$. Finally, the modulus of the 3rd eigenvalue in (2-115) is chosen to be equal $|\bar{\lambda}_D|_\infty$, which means that all three eigenvalues have identical modulus in the high frequency limit. Then, α_f and α_m may be expressed in terms of the $|\bar{\lambda}_D|_\infty$ as follows

$$\alpha_f = \frac{|\bar{\lambda}_D|_\infty}{|\bar{\lambda}_D|_\infty + 1} \quad , \quad \alpha_m = \frac{2|\bar{\lambda}_D|_\infty - 1}{|\bar{\lambda}_D|_\infty + 1} \tag{2–120}$$

This completes the selection of the parameters of the generalized α-algorithm. At first, the modulus of the eigenvalues of the multiplication matrix in the high frequency limit is chosen so $|\bar{\lambda}_D|_\infty \leq 1$. Values $|\bar{\lambda}_D|_\infty < 1$ introduces numerical damping of the high frequency components, whereas $|\bar{\lambda}_D|_\infty = 1$ conserve the mechanical energy of the eigenvibrations as measured by eigenvibrations. Next, α_f and α_m are determined from (2-120). Finally, γ is calculated from (2-118), and β from (2-116). The resulting algorithm will be unconditional stable. The procedure of the parameter selection has been summarized in Box 2.5.

Box 2.5: Selection of parameters of the generalized alpha algorithm

1. Select modulus of the high frequency limit of the eigenvalues of the multiplication matrix $\bar{\mathbf{D}}$ given by (2-110, so

   $|\bar{\lambda}_D|_\infty \leq 1$

2. Calculate parameters α_f and α_m from

   $$\alpha_f = \frac{|\bar{\lambda}_D|_\infty}{|\bar{\lambda}_D|_\infty + 1} \quad , \quad \alpha_m = \frac{2|\bar{\lambda}_D|_\infty - 1}{|\bar{\lambda}_D|_\infty + 1}$$

3. Calculate parameter γ from

   $$\gamma = \frac{1}{2} + \alpha_f - \alpha_m$$

4. Calculate parameter β from

   $$\beta = \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2$$

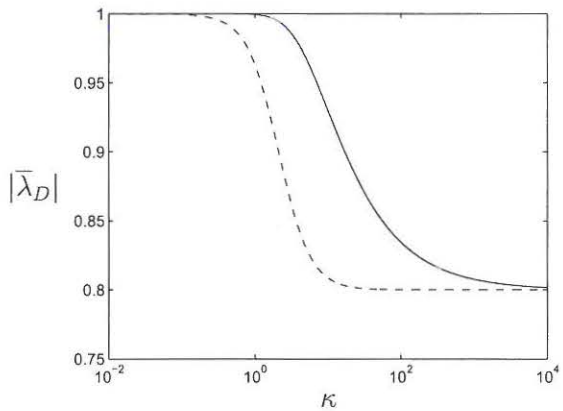

Fig. 2–5 Variation of $|\bar{\lambda}_D|$ as a function of $\kappa = \omega\Delta t$. —-: Generalized α-algorithm, $|\bar{\lambda}_D|_\infty = 0.8$. - - -: Newmark algorithm, $\alpha = 0.09$, $\beta = \frac{1}{4}$.

It turns out that the complex eigenvalues have the largest modulus. Fig. 2-5 shows the variation of the modulus of these eigenvalues as a function of κ. Additionally, the corresponding result for the Newmark algorithm calibrated to the same high frequency limit has been shown with a dashed signature. As seen the numerical damping of the low frequency modes is much smaller for the generalized α-algorithm than for the comparable Newmark algorithm.

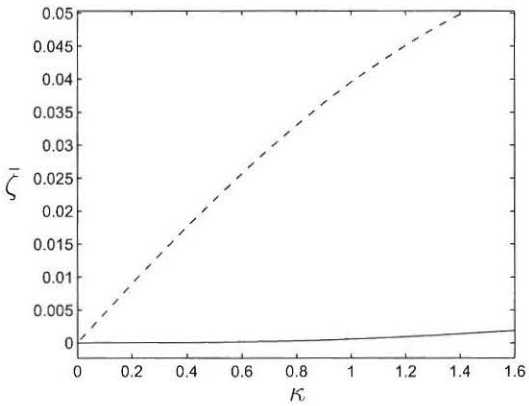

Fig. 2–6 Numerical damping ratio as a function of $\kappa = \omega\Delta t$. —: Generalized α-algorithm, $|\bar{\lambda}_D|_\infty = 0.8$. - - -: Newmark algorithm, $\alpha = 0.09$, $\beta = \frac{1}{4}$.[6]

This observation has been further illustrated in Fig. 2-6, which shows the variation of the numerical damping ratio, as calculated by (2-59). For the Newmark method an asymptotic linear variation with κ is obtained as predicted by (2-102), whereas the variation for the generalized α-method is quadratic.

Example 2.7: Generalized alpha solution for a SDOF oscillator

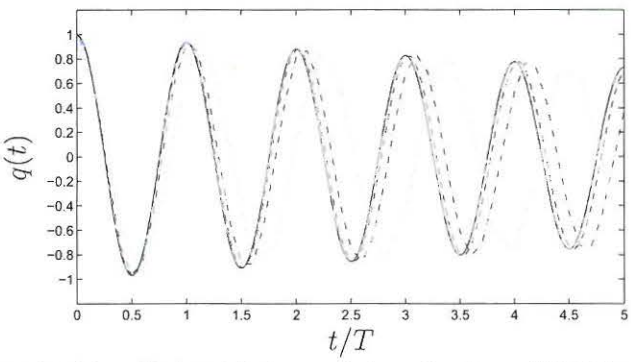

Fig. 2–7 Generalized α-algorithm, $|\bar{\lambda}_D| = 0.8$. Damped eigenvibrations of SDOF oscillator. $\omega = 1$, $\zeta = 0.01$, $(q_0, \dot{q}_0) = (1, 0)$. Analytical solution: —, $\frac{\Delta t}{T} = 0.05$: - - -, $\frac{\Delta t}{T} = 0.10$: -.-.-, $\frac{\Delta t}{T} = 0.20$: ..... .

Fig. 2-7 shows the numerical results for eigenvibrations of the same SDOF oscillator as considered in Example 2.6, obtained by the generalized α-algorithm with $|\bar{\lambda}_D| = 0.8$, and with the time steps $\frac{\Delta t}{T} = 0.05, 0.10, 0.20$. Since the numerical damping of the generalized α-algorithm is very small, the numerical solutions are dissipated in the same rate as the analytical solution. As seen the numerical solutions are related with period elongation, which are approximately of the same magnitude as for the Newmark algorithm as shown on Fig. 2-4, and hence increases proportional with Δt^2.

2.3 Exercises

2.1 Consider the damped eigenvibrations of the two-degrees-of-freedom system defined in Example 1.6 subjected to the initial values

$x_1(0) = 0.01\,m \quad , \quad x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

(a.) Write a MATLAB program, which perform Newmark integration.

(b.) Perform and compare the calculation for $(\beta, \gamma) = (0.25, 0.50)$, $(\beta, \gamma) = (0.25, 0.25)$ with the time steps $\Delta t = T_1/10$ and $\Delta t = T_1/100$, where T_1 denotes the fundamental undamped eigenperiod.

2.2 Consider the same problem as in Exercise 2.1.

(a.) Write a MATLAB program, which perform generalized alpha integration.

(b.) Perform and compare the calculation for $|\bar{\lambda}_\infty| = 0.8$, $|\bar{\lambda}_\infty| = 1.0$ with the time steps $\Delta t = T_1/10$ and $\Delta t = T_1/100$, where T_1 denotes the fundamental undamped eigenperiod.

CHAPTER 3
LINEAR EIGENVALUE PROBLEMS

In this chapter the generalized eigenvalue problem (1-9), and the related characteristic polynomial will be further analyzed. At first in Section 3.1, the *Gauss factorization* of the coefficient matrix of the generalized eigenvalue problem is treated. This factorization plays an important role in several iterative numerical eigenvalue solvers based on the characteristic polynomial. Furthermore, the factorization provides a simple way for calculating a sequence of characteristic polynomials known as the *Sturm sequence*, which makes it possible to formulate upper and lower bounds of the eigenvalues of the problem. These bounds are contained in the so-called *eigenvalue separation principle* treated in Section 3.2. Various iterative schemes for solving the indicated generalized eigenvalue problem require that the stiffness matrix is non-singular. For structures, which admit stiff-body motions, it then becomes necessary to perform a so-called *shift*, where an artificial non-singular stiffness matrix is introduced. The eigenvectors of the shifted system are identical to those of the original problem, whereas the eigenvalues of the two systems deviate with the specified shift parameter. Shifting of a generalized eigenvalue problem is treated in Section 3.3. Some iterative eigensolvers presume a special eigenvalue problem corresponding to $\mathbf{M} = \mathbf{I}$ in (1-9). In this case an introductory transformation from the anticipated generalized eigenvalue problem into an equivalent special eigenvalue problem becomes necessary. This can be achieved in several ways. In Section 3.4 a so-called *similarity transformation* has been used, which preserves the symmetry of the transformed stiffness matrix. A similarity transformation leave the eigenvalues unaffected, whereas the eigenvectors are changed in a known manner.

3.1 Gauss Factorization of Characteristic Polynomials

Since the coefficient matrix of the generalized eigenvalue problem $\mathbf{K} - \lambda\mathbf{M}$ is symmetric, it may be Gauss factorized on the form

$$\mathbf{K} - \lambda\mathbf{M} = \mathbf{L}\mathbf{D}\mathbf{L}^T \tag{3–1}$$

where $\mathbf{L}$ is a lower triangular matrix with units in the main diagonal, and $\mathbf{D}$ is a diagonal matrix, given as

$$\mathbf{L} = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix} \tag{3–2}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \tag{3–3}$$

The details of the Gauss factorization of an symmetric matrix has been given in Box. 3.1. Since, $\det(\mathbf{L}) = \det(\mathbf{L}^T) = 1$, the following representation of the characteristic polynomial (1-10) is obtained

$$P(\lambda) = \det\left(\mathbf{LDL}^T\right) = \det\left(\mathbf{L}\right)\det\left(\mathbf{D}\right)\det\left(\mathbf{L}^T\right) = \det\left(\mathbf{D}\right) = d_{11}d_{22}\cdots d_{nn} \tag{3–4}$$

At the same time (1-11) can be written on the form

$$P(\lambda) = a_0(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) \tag{3–5}$$

Despite the striking similarity between (3-4) and (3-5), d_{ii} is very different from the corresponding factor $(\lambda - \lambda_i)$ in (3-5), as demonstrated in Example 3.2 below.

Let λ be monotonously increased in the interval $[0, \infty[$. From (1-10) follows that $P(0) = a_n = \det(\mathbf{K}) \geq 0$. Since all factors in (3-5) have negative sign for $\lambda \in]0, \lambda_1[$, it follows that $P(\lambda) > 0$ throughout this interval. As λ passes λ_1 from below, the factor $(\lambda - \lambda_1)$ changes its sign from negative to positive, while the other factors remain negative. This means that the sign of $P(\lambda)$ changes from positive to negative at the passage of λ_1. Then, a similar sign change must occur in (3-4). For $\lambda < \lambda_1$ all the diagonal elements $d_{11}, d_{22}, \ldots, d_{nn}$ are positive. As λ passes λ_1, exactly one of these factors (not necessarily d_{11}) changes its sign from positive to negative, while the other factors remain positive. The sign of the characteristic polynomial remain negative for $\lambda \in]\lambda_1, \lambda_2[$. As λ passes λ_2 from below, the factor $(\lambda - \lambda_2)$ in the same way changes its sign from negative to positive, making the characteristic polynomial positive in the interval $\lambda \in]\lambda_2, \lambda_3[$. A similar sign change occurs in (3-4), meaning that an additional diagonal element has changed its sign from positive to negative. Hence, for $\lambda \in]\lambda_2, \lambda_3[$ exactly two of the diagonal elements $d_{11}, d_{22}, \ldots, d_{nn}$ are negative. Proceeding in this manner, it is seen that if λ is placed somewhere in the interval $]\lambda_m, \lambda_{m+1}[$, exactly m diagonal elements among $d_{11}, d_{22}, \ldots, d_{nn}$ are negative. This observation is contained in the following theorem.

Theorem 3.1: Let **D** be the diagonal matrix in the Gauss factorization of the coefficient matrix $\mathbf{K} - \lambda\mathbf{M}$ of a generalized eigenvalue problem, and λ is an arbitrary parameter. Then, the number of negative components in the main diagonal of **D** is equal to the number of eigenvalues of the generalized eigenvalue problem, which are smaller than the parameter λ entering the factorization.

The theorem can be used to formulate bounds for any of the eigenvalues as demonstrated below in Example 3.3. Actually, one can calculate say the jth eigenvalue λ_j with arbitrary accuracy. The method is simply to make an initial sequence of calculations of the characteristic polynomial $P(\lambda)$ as a function of λ by (3-4), until j components in the main diagonal of **D** are negative. Next, one can perform additional calculations to reduce the interval, where the jth sign change takes place. This procedure for calculation of eigenvalues is known as the *telescope method.*

Box 3.1: Gauss factorization of symmetric matrix

Gauss factorization reduces a symmetric matrix $\mathbf{K}$ of dimension $n \times n$ to an upper triangular matrix $\mathbf{S}$ in a sequence of $n-1$ matrix multiplications. After the first $(i-1)$ matrix multiplications the following matrix is considered

$$\mathbf{K}^{(i)} = \mathbf{L}_{i-1}^{-1}\mathbf{L}_{i-2}^{-1}\cdots\mathbf{L}_1^{-1}\mathbf{K} \quad , \quad i = 2, \ldots, n \tag{3–6}$$

where $\mathbf{K}^{(1)} = \mathbf{K}$. Sequentially, the indicated matrix multiplications produce zeros below the main diagonal of the columns $j = 1, \ldots, i-1$. Then, pre-multiplication of $\mathbf{K}^{(i)}$ with $\mathbf{L}_i^{-1}$ will produce zeroes below the main diagonal of the ith column without affecting the zeroes in the previous columns. $\mathbf{L}_i^{-1}$ is a lower triangular matrix with units in the principal diagonal, and where only the ith column is non-zero, given as

$$\mathbf{L}_i^{-1} = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & & & & \\ 0 & 0 & \cdots & 0 & 1 & & & \\ 0 & 0 & \cdots & 0 & -l_{i+1,i} & 1 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 & -l_{n,i} & 0 & \cdots & 1 \end{bmatrix} \tag{3–7}$$

The components $l_{j,i}$ entering the ith column are given as

$$l_{j,i} = \frac{K_{j,i}^{(i)}}{K_{i,i}^{(i)}} \quad , \quad j = i+1, \ldots, n \tag{3–8}$$

where $K_{j,i}^{(i)}$ denotes the component in the jth row and ith column of $\mathbf{K}^{(i)}$. By insertion it is proved that the inverse of (3-7) is given as

$$\mathbf{L}_i = \begin{bmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ 0 & 0 & \cdots & 1 & & & & \\ 0 & 0 & \cdots & 0 & 1 & & & \\ 0 & 0 & \cdots & 0 & l_{i+1,i} & 1 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 & l_{n,i} & 0 & \cdots & 1 \end{bmatrix} \tag{3–9}$$

Then, $\mathbf{K}^{(n)}$ obtained after the $(n-1)$th multiplication with $\mathbf{L}_{n-1}^{-1}$, has zeroes in all the first $(n-1)$ columns below the main diagonal, corresponding to an upper triangular matrix $\mathbf{S}$. Hence

$$\begin{aligned} &\mathbf{L}_{n-1}^{-1}\mathbf{L}_{n-2}^{-1}\cdots\mathbf{L}_1^{-1}\mathbf{K} = \mathbf{S} \quad \Rightarrow \\ &\mathbf{K} = \mathbf{L}\mathbf{S} \quad , \quad \mathbf{L} = \mathbf{L}_1\mathbf{L}_2\cdots\mathbf{L}_{n-1} \end{aligned} \tag{3–10}$$

Since, $\mathbf{L}$ defined by (3-10) is the product of lower triangular matrices with 1 in the main diagonal, it becomes a matrix with the same structure as indicated by (3-2).

Because $\mathbf{K}$ is symmetric, $\mathbf{S}$ must have the structure

$$\mathbf{S} = \mathbf{D}\mathbf{L}^T \tag{3–11}$$

where $\mathbf{D}$ is a diagonal matrix, given by (3-3). This proofs the validity of the factorization (3-1).

Example 3.1: Gauss factorization of a three-dimensional matrix

Given the symmetric matrix

$$\mathbf{K} = \mathbf{K}^{(1)} = \begin{bmatrix} 5 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 6 \end{bmatrix} \tag{3–12}$$

$$\mathbf{L}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ -\frac{1}{5} & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{K}^{(2)} = \mathbf{L}_1^{-1}\mathbf{K}^{(1)} = \begin{bmatrix} 5 & -4 & 1 \\ 0 & 2.8 & -3.2 \\ 0 & -3.2 & 5.8 \end{bmatrix} \\ \mathbf{L}^{(1)} = \mathbf{L}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{4}{5} & 1 & 0 \\ \frac{1}{5} & 0 & 1 \end{bmatrix} \end{cases} \tag{3–13}$$

$$\mathbf{L}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{3.2}{2.8} & 1 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{K}^{(3)} = \mathbf{L}_2^{-1}\mathbf{K}^{(2)} = \mathbf{S} = \begin{bmatrix} 5 & -4 & 1 \\ 0 & 2.8 & -3.2 \\ 0 & 0 & 2.1429 \end{bmatrix} \\ \mathbf{L}^{(2)} = \mathbf{L}_1\mathbf{L}_2 = \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.8 & 1 & 0 \\ 0.2 & -1.1429 & 1 \end{bmatrix} \end{cases} \tag{3–14}$$

From this follows that

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.8 & 1 & 0 \\ 0.2 & -1.1429 & 1 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2.8 & 0 \\ 0 & 0 & 2.1429 \end{bmatrix} \tag{3–15}$$

Example 3.2: Gauss factorization of a three-dimensional generalized eigenvalue problem

Of course for a given value of λ the matrix $\mathbf{K} - \lambda\mathbf{M}$ may be factorized according to the method explained in Box 3.1. However, for smaller problems explicit expressions may be derived, as demonstrated in the following. Given the mass- and stiffness matrices defined in Example 1.4, the components of $\mathbf{L}$ and $\mathbf{D}$ are calculated from the following identities, cf. (1-79), (3-1)

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & l & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{11}l_{21} & d_{11}l_{31} \\ d_{11}l_{21} & d_{22} + d_{11}l_{21}^2 & d_{11}l_{21}l_{31} + d_{22}l_{32} \\ d_{11}l_{31} & d_{11}l_{21}l_{31} + d_{22}l_{32} & d_{33} + d_{11}l_{31}^2 + d_{22}l_{32}^2 \end{bmatrix} =$$

$$\begin{bmatrix} 2 - \frac{1}{2}\lambda & -1 & 0 \\ -1 & 4 - \lambda & -1 \\ 0 & -1 & 2 - \frac{1}{2}\lambda \end{bmatrix} \tag{3–16}$$

Equating the corresponding components on the left and right hand sides, provides the following equations for the determination of the unknown quantities

$$\left.\begin{aligned}
&d_{11} = 2 - \frac{1}{2}\lambda = -\frac{1}{2}(\lambda - 4) \\
&d_{11}l_{21} = -1 \quad \Rightarrow \quad l_{21} = \frac{2}{\lambda - 4} \\
&d_{22} + d_{11}l_{21}^2 = 4 - \lambda \quad \Rightarrow \quad d_{22} = 4 - \lambda + \frac{1}{2}(\lambda - 4)\frac{4}{(\lambda - 4)^2} = -\frac{\lambda^2 - 8\lambda + 14}{\lambda - 4} \\
&d_{11}l_{21}l_{31} + d_{22}l_{32} = d_{22}l_{32} = -1 \quad \Rightarrow \quad l_{32} = \frac{\lambda - 4}{\lambda^2 - 8\lambda + 14} \\
&d_{33} + d_{11}l_{31}^2 + d_{22}l_{32}^2 = d_{33} - l_{32} = 2 - \frac{1}{2}\lambda \quad \Rightarrow \\
&d_{33} = 2 - \frac{1}{2} + \frac{\lambda - 4}{\lambda^2 - 8\lambda + 14} = -\frac{1}{2}\frac{\lambda^3 - 12\lambda^2 + 44\lambda - 48}{\lambda^2 - 8\lambda + 14} = -\frac{1}{2}\frac{(\lambda - 2)(\lambda - 4)(\lambda - 6)}{\lambda^2 - 8\lambda + 14}
\end{aligned}\right\} \tag{3–17}$$

Then, the following expression for the characteristic equation is obtained, in agreement with (1-80)

$$P(\lambda) = d_{11}d_{22}d_{33} = -\frac{1}{2}(\lambda - 4)\left(-\frac{\lambda^2 - 8\lambda + 14}{\lambda - 4}\right)\left(-\frac{1}{2}\frac{(\lambda - 2)(\lambda - 4)(\lambda - 6)}{\lambda^2 - 8\lambda + 14}\right) = -\frac{1}{4}(\lambda - 2)(\lambda - 4)(\lambda - 6) \tag{3–18}$$

Example 3.3: Bounds on eigenvalues

In this example bounds on the eigenvalues of the GEVP in Example 1.4 is constructed from the number of negative components in the diagonal of the matrix **D**, using Theorem 3.1.

For $\lambda = 1$ one gets:

$$\mathbf{K} - \lambda\mathbf{M} = \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & \frac{3}{2} \end{bmatrix} \quad \Rightarrow \quad \mathbf{LDL}^T = \begin{bmatrix} 1 & & \\ -\frac{2}{3} & 1 & \\ 0 & -\frac{3}{7} & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{15}{14} \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ & 1 & -\frac{3}{7} \\ & & 1 \end{bmatrix} \tag{3–19}$$

The components of the matrices **L** and **D** may be calculated by the formulas indicated in (3-17). As seen $d_{11} = \frac{3}{2} > 0$, $d_{22} = \frac{7}{3} > 0$, $d_{33} = \frac{15}{14} > 0$. Hence, all three diagonal components are positive, from which it is concluded that $\lambda_1 > \lambda = 1$.

For $\lambda = 8$ one gets:

$$\mathbf{K} - \lambda\mathbf{M} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{LDL}^T = \begin{bmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ 0 & \frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & -\frac{12}{7} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ & 1 & \frac{2}{7} \\ & & 1 \end{bmatrix} \tag{3–20}$$

As seen $d_{11} = -2 < 0$, $d_{22} = -\frac{7}{2} < 0$, $d_{33} = -\frac{12}{7} < 0$. Hence, all three diagonal components are negative, from which it is concluded that $\lambda_3 < \lambda = 8$.

For $\lambda = 5$ one gets:

$$\mathbf{K} - \lambda\mathbf{M} = \begin{bmatrix} -\frac{1}{2} & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \mathbf{LDL}^T = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & 1 & -1 \\ & & 1 \end{bmatrix} \tag{3–21}$$

As seen $d_{11} = -\frac{1}{2} < 0$, $d_{22} = 1 > 0$, $d_{33} = -\frac{3}{2} < 0$. Hence, two diagonal components are negative and one is positive, from which it is concluded that $\lambda_2 < \lambda = 5 < \lambda_3$.

For $\lambda = 3$ one gets:

$$\mathbf{K} - \lambda\mathbf{M} = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \mathbf{LDL}^T = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ & 1 & 1 \\ & & 1 \end{bmatrix} \tag{3–22}$$

As seen $d_{11} = \frac{1}{2} > 0$, $d_{22} = -1 < 0$, $d_{33} = \frac{3}{2} < 0$. Hence, two diagonal components are positive and one is negative, from which it is concluded that $\lambda_1 < \lambda = 3 < \lambda_2$.

In conclusion the following bounds prevail

$$\left.\begin{array}{l} 1 < \lambda_1 < 3 \\ 3 < \lambda_2 < 5 \\ 5 < \lambda_3 < 8 \end{array}\right\} \tag{3–23}$$

3.2 Eigenvalue Separation Principle

The matrices $\mathbf{M}^{(m)}$ and $\mathbf{K}^{(m)}$ of dimension $(n-m) \times (n-m)$ are obtained from $\mathbf{M}$ and $\mathbf{K}$, if the last m rows and columns are omitted in these matrices. Then, consider the sequence of related characteristic polynomials of the order $(n-m)$

$$P^{(m)}\left(\lambda^{(m)}\right) = \det\left(\mathbf{K}^{(m)} - \lambda^{(m)}\mathbf{M}^{(m)}\right) \quad , \quad m = 0, 1, \ldots, n-1 \tag{3–24}$$

where $\mathbf{M}^{(0)} = \mathbf{M}$, $\mathbf{K}^{(0)} = \mathbf{K}$, $\lambda^{(0)} = \lambda$ and $P^{(0)}(\lambda) = P(\lambda)$. The eigenvalues corresponding to $\mathbf{M}^{(m)}$ and $\mathbf{K}^{(m)}$ are denoted as $\lambda_1^{(m)}, \lambda_2^{(m)}, \ldots, \lambda_{n-m}^{(m)}$.

Now, for any $m = 0, 1, \ldots, n-1$ it can be proved that the roots of $P^{(m+1)}\left(\lambda^{(m+1)}\right) = 0$ are separating the roots of $P^{(m)}\left(\lambda^{(m)}\right) = 0$, i.e.

$$0 \leq \lambda_1^{(m)} \leq \lambda_1^{(m+1)} \leq \lambda_2^{(m)} \leq \lambda_2^{(m+1)} \leq \cdots \leq \lambda_{n-m-1}^{(m)} \leq \lambda_{n-m-1}^{(m+1)} \leq \lambda_{n-m}^{(m)} \leq \infty \tag{3–25}$$

A formal proof of (3-25) has been given by Bathe.[1] The sequence of polynomials $P^{(m)}(\lambda)$ with roots fulfilling the property (3-25), is denoted a Sturm sequence. (3-25) is illustrated in Example 3.4.

[1] K.-J. Bathe: *Finite Element Procedures*. Printice Hall, Inc., 1996.

Next, consider the Gauss factorization (3-1). Omitting the last m rows and columns in $\mathbf{M}$ and $\mathbf{K}$ is tantamount to omitting the last m rows and columns in $\mathbf{L}$ and $\mathbf{D}$. Then

$$P^{(m)}\left(\lambda^{(m)}\right) = \det\left(\mathbf{K}^{(m)} - \lambda^{(m)}\mathbf{M}^{(m)}\right) = \det\left(\mathbf{L}^{(m)}\mathbf{D}^{(m)}\mathbf{L}^{(\mathbf{m})^T}\right) = \det\left(\mathbf{D}^{(m)}\right) = d_{11}d_{22}\cdots d_{n-m,n-m} \tag{3–26}$$

where

$$\mathbf{L}^{(m)} = \begin{bmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ l_{31} & l_{32} & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ l_{n-m,1} & l_{n-m,2} & l_{n-m,3} & \cdots & 1 \end{bmatrix} \tag{3–27}$$

$$\mathbf{D}^{(m)} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{n-m,n-m} \end{bmatrix} \tag{3–28}$$

The bounding property explained in Theorem 3.1 for the case $m = 0$ can then easily be generalized. Let $\lambda^{(m)} = \mu$, and perform a Gauss factorization on the matrix $\mathbf{K}^{(m)} - \mu\mathbf{M}^{(m)}$. Then the number of eigenvalues, $\lambda_j^{(m)} < \mu$, will be equal to number of negative diagonal components $d_{11}, \ldots, d_{n-m,n-m}$ in the matrix $\mathbf{D}$.

The number of negative elements in main diagonal of the matrix $\mathbf{D}$ in the Gauss factorization of $\mathbf{K} - \lambda\mathbf{M} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, and hence the number of eigenvalues smaller than λ, can then be retrieved from the signs of the sequence $P^{(0)}(\lambda), P^{(1)}(\lambda), \ldots, P^{(n-1)}(\lambda)$ as seen in the following way.

Introduce $P^{(n)}(\lambda)$ as an arbitrary positive quantity. Since $P^{(n-1)}(\lambda) = d_{11}$, it follows that the sequence $P^{(n)}(\lambda), P^{(n-1)}(\lambda)$ has the sign sequence $\text{sign}(P^{(n)}(\lambda)), \text{sign}(P^{(n-1)}(\lambda)) = +, -$, if $d_{11} < 0$, and the sign sequence $+, +$, if $d_{11} > 0$. $d_{11} < 0$ indicates that at least one eigenvalue is smaller than λ, in which case one sign change, namely from $+$ to $-$, has occurred in the indicated sign sequence.

Next, $P^{(n-2)}(\lambda) = d_{11}d_{22}$ is considered. $d_{11} < 0 \wedge d_{22} < 0$ indicates that two eigenvalues are smaller than λ. This in turns implies that $P^{(n-1)}(\lambda)$ has a negative sign, and $P^{(n-2)}(\lambda)$ has a positive sign. Then, one additional sign change has occurred in the sequence of sign of the characteristic polynomials $\text{sign}(P^{(n)}(\lambda)), \text{sign}(P^{(n-1)}(\lambda)), \text{sign}(P^{(n-2)}(\lambda))$=$+, -, +$. If $d_{22} > 0$, then $P^{(n-1)}(\lambda)$ and $P^{(n-2)}(\lambda)$ have the same sign, and no additional sign change is recorded in the sequence of signs of the characteristic polynomials.

Proceeding in this way it is seen that the number of sign changes in the sequence of signs $\text{sign}(P^{(n)}(\lambda)), \text{sign}(P^{(n-1)}(\lambda)), \ldots, \text{sign}(P^{(0)}(\lambda))$ determines the total number of eigenvalues smaller than λ. This property of the sequence of characteristic polynomials is known as a *Sturm sequence check*. In Example 3.5 it is illustrated, how the sign of the components d_{11}, d_{22}, d_{33} for the case $n = 3$ can be retrieved from the sequence of signs of the Sturm sequence.

Example 3.4: Bounds on eigenvalues by eigenvalue separation principle

For the mass- and stiffness matrices defined in Example 1.4, the matrices $\mathbf{M}^{(1)}$ and $\mathbf{K}^{(1)}$ become

$$\mathbf{M}^{(1)} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{K}^{(1)} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \tag{3–29}$$

The characteristic equation (1-10) becomes

$$\det\left(\begin{bmatrix} 2 - \frac{1}{2}\lambda_j^{(1)} & -1 \\ -1 & 4 - \lambda_j^{(1)} \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1^{(1)} = 4 - \sqrt{2} = 2.59 \\ \lambda_2^{(1)} = 4 + \sqrt{2} = 5.41 \end{cases} \tag{3–30}$$

The matrices $\mathbf{M}^{(2)}$ and $\mathbf{K}^{(2)}$ become

$$\mathbf{M}^{(2)} = \left[\frac{1}{2}\right] \quad , \quad \mathbf{K}^{(2)} = \left[2\right] \quad \Rightarrow \quad \lambda_1^{(2)} = 4 \tag{3–31}$$

The relation (3-25) becomes

$$\left.\begin{array}{l} \lambda_1 \leq \lambda_1^{(1)} \leq \lambda_2 \\ \lambda_1^{(1)} \leq \lambda_1^{(2)} \leq \lambda_2^{(1)} \\ \lambda_2 \leq \lambda_2^{(1)} \leq \lambda_3 \end{array}\right\} \quad \Rightarrow \quad \begin{cases} 0 \leq \lambda_1 \leq \lambda_1^{(1)} \\ \lambda_1^{(1)} \leq \lambda_2 \leq \lambda_2^{(1)} \\ \lambda_2^{(1)} \leq \lambda_3 \leq \infty \end{cases} \quad \Rightarrow \quad \begin{cases} 0 \leq \lambda_1 \leq 2.59 \\ 2.59 \leq \lambda_2 \leq 5.41 \\ 5.41 \leq \lambda_3 \leq \infty \end{cases} \tag{3–32}$$

The exact solutions are $\lambda_1 = 2$, $\lambda_2 = 4$, and $\lambda_3 = 6$, cf. Example 1.4.

Example 3.5: Sturm sequences and correspondence to sign of components in D-matrix

Consider a generalized eigenvalue of order $n = 3$. For a given value of λ, the Sturm sequence $P^{(3)}(\lambda)$, $P^{(2)}(\lambda)$,

$P^{(1)}(\lambda)$, $P^{(0)}(\lambda)$ is calculated. Below are shown the 8 possible sign sequences of the Sturm sequence.

$$
\overset{+}{P^{(3)}}(\lambda) > 0 \begin{cases}
\overset{++}{P^{(2)}}(\lambda) > 0 \Rightarrow d_{11} > 0 \begin{cases}
\overset{+++}{P^{(1)}}(\lambda) > 0 \Rightarrow d_{22} > 0 \begin{cases} \overset{++++}{P^{(0)}}(\lambda) > 0 \Rightarrow d_{33} > 0 \\ \overset{+++-}{P^{(0)}}(\lambda) < 0 \Rightarrow d_{33} < 0 \end{cases} \\
\overset{++-}{P^{(1)}}(\lambda) < 0 \Rightarrow d_{22} < 0 \begin{cases} \overset{++-+}{P^{(0)}}(\lambda) > 0 \Rightarrow d_{33} < 0 \\ \overset{++--}{P^{(0)}}(\lambda) < 0 \Rightarrow d_{33} > 0 \end{cases}
\end{cases} \\
\overset{+-}{P^{(2)}}(\lambda) < 0 \Rightarrow d_{11} < 0 \begin{cases}
\overset{+-+}{P^{(1)}}(\lambda) > 0 \Rightarrow d_{22} < 0 \begin{cases} \overset{+-++}{P^{(0)}}(\lambda) > 0 \Rightarrow d_{33} > 0 \\ \overset{+-+-}{P^{(0)}}(\lambda) < 0 \Rightarrow d_{33} < 0 \end{cases} \\
\overset{+--}{P^{(1)}}(\lambda) < 0 \Rightarrow d_{22} > 0 \begin{cases} \overset{+--+}{P^{(0)}}(\lambda) > 0 \Rightarrow d_{33} < 0 \\ \overset{+---}{P^{(0)}}(\lambda) < 0 \Rightarrow d_{33} > 0 \end{cases}
\end{cases}
\end{cases}
\tag{3–33}
$$

With an arbitrary positive value for $P^{(3)}(\lambda)$ the first curly bracket indicate how the sign of d_{11} is retrieved from the possible signs of $P^{(2)}(\lambda)$. The sign sequences $++$ and $+-$ have been indicated atop of $P^{(2)}(\lambda)$. At the next level the sign of $P^{(1)}(\lambda)$ in combination to the previous sign sequence makes it possible to retrieve the sign of d_{22}. Finally, at the 3rd level the sign of the characteristic polynomial $P^{(0)}(\lambda)$ in combination to the previous sign sequence makes it possible to retrieve the sign of d_{33}. As an example, the sequence of signs $\text{sign}(P^{(3)}(\lambda)), \text{sign}(P^{(2)}(\lambda)), \text{sign}(P^{(1)}(\lambda)), \text{sign}(P^{(0)}(\lambda))$=$+\ -\ +-$ are obtained for the specific sign combination $d_{11} < 0$, $d_{22} < 0$ and $d_{33} < 0$. Moreover, there are three sign changes in the indicated sign sequence $+\ -\ +-$, and correspondingly all three components d_{11}, d_{22} and d_{33} are negative. The reader is encouraged to verify, that the number of sign changes in the sequence of signs at the lowest level in (3-33) always is equal to the number of negative components in the specific combination of d_{11}, d_{22} and d_{33} producing this sequence of signs.

Example 3.6: Physical interpretation of the eigenvalue separation principle

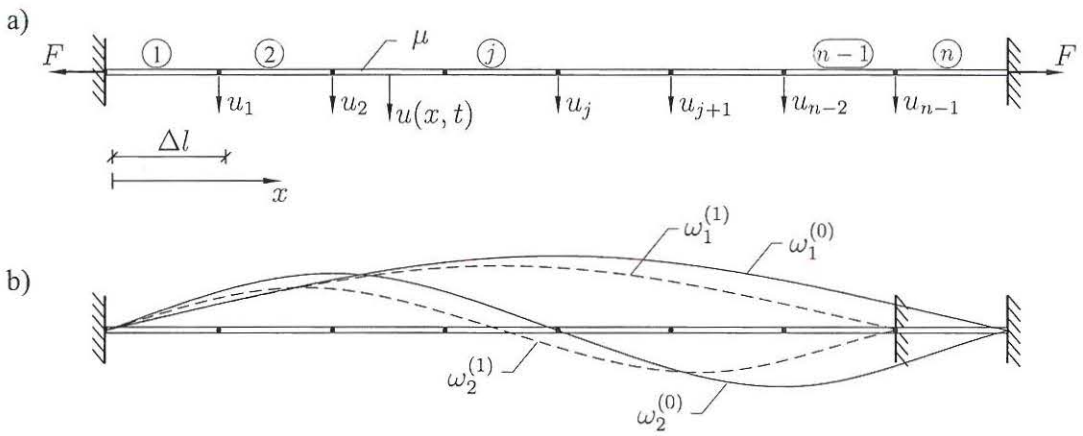

Fig. 3–1 Vibrating string. a) Definition of elements and degrees of freedom. b) Undamped eigenmodes.

The vibrating string problem in Example 1.2 is considered again. The eigenvibrations of the discretized string is given by (1-68) with $\mathbf{M}^{(0)} = \mathbf{M}$ and $\mathbf{K}^{(0)} = \mathbf{K}$ given by (1-69) or (1-70).

Next, consider the system defined by the matrices $\mathbf{M}^{(1)}$ and $\mathbf{K}^{(1)}$ of dimension $(n-2) \times (n-2)$, where the last row and column are omitted in $\mathbf{M}(0)$ and $\mathbf{K}(0)$. Physically, this corresponds to constraining the displacement $u_{n-1}(t) = 0$, as indicated by the additional support in Fig. 1.3b. The corresponding eigenmodes of the continuous system have been shown with a dashed signature. As seen in Fig. 1.3b the wave-lengths related to the circular eigenfrequencies $\omega_1^{(0)}$, $\omega_1^{(1)}$, $\omega_2^{(0)}$ and $\omega_2^{(1)}$ decreases in the indicated order. Hence, the following ordering of these eigenfrequencies prevails

$$\omega_1^{(0)} < \omega_1^{(1)} < \omega_2^{(0)} < \omega_2^{(1)} \tag{3–34}$$

Since $\lambda_j^{(m)} = \left(\omega_j^{(m)}\right)^2$, the corresponding ordering of the eigenvalues become

$$\lambda_1^{(0)} < \lambda_1^{(1)} < \lambda_2^{(0)} < \lambda_2^{(1)} \tag{3–35}$$

which corresponds to (3-25).

3.3 Shift

Occasionally, a *shift on the stiffness matrix* may be used to enhance the speed of calculation of the considered GEVP. In order to explain this the eigenvalue problem (1-9) is written in the following way

$$\left(\mathbf{K} - \rho\mathbf{M} + \rho\mathbf{M} - \lambda_j\mathbf{M}\right)\mathbf{\Phi}^{(j)} = 0 \tag{3–36}$$

Obviously, we have withdrawn and added the quantity $\rho\mathbf{M}$ inside the bracket, where ρ is a suitable real number, which will not affect neither the eigenvalues λ_j, nor the eigenvectors $\mathbf{\Phi}^{(j)}$. (3-36) is rearranged on the form

$$\left(\hat{\mathbf{K}} - \mu_j\mathbf{M}\right)\mathbf{\Phi}^{(j)} = 0 \tag{3–37}$$

where

$$\hat{\mathbf{K}} = \mathbf{K} - \rho\mathbf{M} \quad , \quad \mu_j = \lambda_j - \rho \tag{3–38}$$

Hence, instead of the original generalized eigenvalue problem defined by the matrices $(\mathbf{K}, \mathbf{M})$, the system with the matrices $(\hat{\mathbf{K}}, \mathbf{M})$ is considered in the shifted system, where $\hat{\mathbf{K}}$ is calculated as indicated in (3-38). The two systems have identical eigenvectors. However, the eigenvalues of the shifted system become $(\lambda_1 - \rho), (\lambda_2 - \rho), \ldots, (\lambda_n - \rho)$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the original system.

For non-supported systems (e.g. ships and aeroplanes) a stiffbody motion $\Phi \neq 0$ exists, which fulfills

$$\mathbf{K}\Phi = 0 \tag{3–39}$$

(3-39) shows that $\lambda = 0$ is an eigenvalue for such systems. Correspondingly, $\det(\mathbf{K}) = 0$ for systems, which possesses a stiffbody motion. However, some numerical algorithms presume that $\det(\mathbf{K}) \neq 0$. In such cases a preliminary shift on the stiffness matrix must be performed, because $\det(\mathbf{K} - \rho\mathbf{M}) \neq 0$, if $\det(\mathbf{K}) = 0$.

Example 3.7: Shift on stiffness matrix

Given the mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \tag{3–40}$$

The characteristic equation (6-6) becomes

$$\det\left(\begin{bmatrix} 3-2\lambda & -3-\lambda \\ -3-\lambda & 3-2\lambda \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 6 \end{cases} \tag{3–41}$$

$\lambda_1 = 0$, since $\det(\mathbf{K}) = 0$.

Next, a shift on the stiffness matrix with $\rho = -2$ is performed, which provides

$$\hat{\mathbf{K}} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \tag{3–42}$$

Now, the characteristic equation becomes

$$\det\left(\begin{bmatrix} 7-2\mu & -1-\mu \\ -1-\mu & 7-2\mu \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \begin{cases} \mu_1 = 2 \\ \mu_2 = 8 \end{cases} \tag{3–43}$$

3.4 Transformation of GEVP to SEVP

Some eigenvalue solvers are written for the special eigenvalue problem. Hence, their use presumes an initial transformation of the generalized eigenvalue problem (1-9). Of course, this may be performed, simply by a pre-multiplication of (1-9) with M^{-1}. However, then the resulting system matrix $\mathrm{M}^{-1}\mathbf{K}$ is no longer symmetric. In this section a similarity transformation is indicated, which preserves the symmetry of the system matrix.

Since, $\mathrm{M} = \mathrm{M}^T$ it can be factorized on the form

$$\mathrm{M} = \mathrm{S}\mathrm{S}^T \tag{3–44}$$

The generalized eigenvalue problem (1-9) may then be written in the form

$$\begin{aligned} &\mathbf{K}\left(\mathrm{S}^T\right)^{-1}\mathrm{S}^T\mathbf{\Phi}^{(j)} = \lambda_j \mathrm{S}\mathrm{S}^T\mathbf{\Phi}^{(j)} \quad \Rightarrow \\ &\mathrm{S}^{-1}\mathbf{K}\left(\mathrm{S}^{-1}\right)^T\mathrm{S}^T\mathbf{\Phi}^{(j)} = \lambda_j\,\mathrm{S}^T\mathbf{\Phi}^{(j)} \end{aligned} \tag{3–45}$$

where the identity $\left(\mathrm{S}^T\right)^{-1} = \left(\mathrm{S}^{-1}\right)^T$ has been used. (1-9) can then be formulated in terms of the following standard EVP

$$\tilde{\mathbf{K}}\tilde{\mathbf{\Phi}}^{(j)} = \lambda_j\tilde{\mathbf{\Phi}}^{(j)} \tag{3–46}$$

where

$$\tilde{\mathbf{K}} = \mathrm{S}^{-1}\mathbf{K}\left(\mathrm{S}^{-1}\right)^T \tag{3–47}$$

$$\left.\begin{aligned} \tilde{\mathbf{\Phi}}^{(j)} &= \mathrm{S}^T\mathbf{\Phi}^{(j)} \\ \mathbf{\Phi}^{(j)} &= \left(\mathrm{S}^{-1}\right)^T\tilde{\mathbf{\Phi}}^{(j)} \end{aligned}\right\} \tag{3–48}$$

(3-47) defines a similarity transformation with the transformation matrix S^{-1}, which diagonalize the mass matrix. Similarity transformations is further explained in Chapter 6. Obviously, $\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^T$. As seen from (3-46) the eigenvalues $\lambda_1, \ldots, \lambda_n$ are identical for the original and the transformed eigenvalue problem, whereas the eigenvectors $\mathbf{\Phi}^{(j)}$ and $\tilde{\mathbf{\Phi}}^{(j)}$ are related by the transformation (3-48).

The determination of a matrix S fulfilling (3-44) is not unique. Actually, infinite many solutions to this problem exist. Below, two approaches have been given. In both cases it is assumed that $\mathrm{M} = \mathrm{M}^T$ is positive definite.

Generally, *Choleski decomposition* is considered the most effective way of solving the problem. In this case a lower triangular matrix S is determined, so (3-44) is fulfilled. Obviously, S is related to the Gauss factorization as follows

$$\mathrm{S} = \mathrm{L}\mathrm{D}^{\frac{1}{2}} \quad , \quad \mathrm{D}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{d_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{d_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{d_{nn}} \end{bmatrix} \tag{3–49}$$

The diagonal matrix $\mathrm{D}^{\frac{1}{2}}$ does only exist, if the components d_{ii} of the matrix D are all positive. This is indeed the case, if M is positive definite. Although, S may be calculated from (3-49), there exists a faster and more direct algorithm for the determination of this quantity.

Alternatively, a so-called *spectral decomposition* of **M** may be used. The basis of this method is the following SEVP for **M**

$$\mathrm{M}\mathbf{v}^{(j)} = \rho_j \mathbf{v}^{(j)} \tag{3–50}$$

ρ_j and $\mathbf{v}^{(j)}$ denotes the jth eigenvalue and eigenvector of M. Both are real, since M is symmetric. The eigenvalue problems (3-50) can be assembled into the matrix formulation, cf. (1-14)

$$\mathrm{M}\boldsymbol{\mu} = \mathrm{V}\mathrm{R} \tag{3–51}$$

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} \quad , \quad \mathrm{V} = [\mathbf{v}^{(1)}\, \mathbf{v}^{(2)} \cdots \mathbf{v}^{(n)}] \tag{3–52}$$

The eigenvectors are normalized to magnitude 1, i.e. $\mathbf{v}^{(i)\,T}\mathbf{v}^{(j)} = \delta_{ij}$. Then, the modal matrix V fulfills, cf. (1-23)

$$\mathrm{V}^{-1} = \mathrm{V}^T \tag{3–53}$$

From (3-51) and (3-53) the following representation of M is obtained

$$\mathrm{M} = \mathrm{V}\boldsymbol{\mu}\mathrm{V}^T \tag{3–54}$$

Finally, from (3-44) and (3-54) the following solution for S is obtained

$$\mathrm{S} = \mathrm{V}\boldsymbol{\mu}^{\frac{1}{2}} \quad , \quad \boldsymbol{\mu}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{\mu_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\mu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\mu_n} \end{bmatrix} \tag{3–55}$$

The drawback of the spectral approach is that an initial SEVP must be solved, before the transformed eigenvalue problem (3-46) can be analyzed. Hence, the method requires the solution of two SEVP of the same dimension.

Box 3.2: Choleski decomposition of symmetric positive definite matrix

Choleski decomposition factorizes a symmetric positive definite matrix M into the matrix product of a lower triangular matrix S and its transpose, as follows

$$\mathrm{M} = \mathrm{S}\mathrm{S}^T \quad \Rightarrow$$

$$\begin{bmatrix} m_{11} & m_{21} & \cdots & m_{n1} \\ m_{21} & m_{22} & \cdots & m_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix} = \begin{bmatrix} s_{11} & 0 & \cdots & 0 \\ s_{21} & s_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{bmatrix} \begin{bmatrix} s_{11} & s_{21} & \cdots & s_{n1} \\ 0 & s_{22} & \cdots & s_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} s_{11}^2 & & & \\ s_{21}s_{11} & s_{22}^2 + s_{21}^2 & & \text{symmetric} \\ \vdots & \vdots & \ddots & \\ s_{n1}s_{11} & s_{n2}s_{22} + s_{n1}s_{21} & \cdots & s_{nn}^2 + s_{n,n-1}^2 + \cdots + s_{n2}^2 + s_{n1}^2 \end{bmatrix} \tag{3–56}$$

Equating the components of the final matrix product with the component on and below the main diagonal of **M**, equations can be formulated for the determinations of s_{ij}, which are solved sequentially. First $s_{11} = \sqrt{m_{11}}$ is calculated. Next, s_{i1}, $i = 2, \ldots, n$ are determined from $s_{i1} = m_{i1}/s_{11}$. Next, $s_{22} = \sqrt{m_{22} - s_{21}^2}$ is calculated, and s_{i2}, $i = 3, \ldots, n$ can be determined from $s_{i2} = (m_{2i} - s_{i1}s_{21})/s_{22}$. Next, the 3th column can be calculated and so forth. The general algorithm for calculating the components s_{ij} in the jth column reads

$$\begin{aligned} s_{jj} &= \sqrt{m_{jj} - s_{j,j-1}^2 - \cdots - s_{j1}^2} \quad , \quad j = 1, \ldots, n \\ s_{ij} &= (m_{ij} - s_{i,j-1}s_{j,j-1} - \cdots - s_{i1}s_{j1})/s_{jj} \quad , \quad i = j+1, \ldots, n \end{aligned} \tag{3–57}$$

3.5 Exercises

3.1 Given the same mass- and stiffness matrices as in Exercise 1.1.

(a.) Show that the eigenvalue separation principle is valid for the considered example.

3.2 Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix}$$

(a.) Calculate the eigenvalues and eigenmodes normalized to unit modal mass.

(b.) Perform a shift $\rho = 3$ on $\mathbf{K}$ and calculate the eigenvalues and eigenmodes of the new problem.

3.3 Given a symmetric matrix $\mathbf{K}$.

(a.) Write a MATLAB program, which determines the matrices $\mathbf{L}$ and $\mathbf{D}$ of a Gauss factorization as well as the matrix $(\mathbf{S}^{-1})^T$, where $\mathbf{S}$ is a lower triangular matrix fulfilling $\mathbf{S}\mathbf{S}^T = \mathbf{K}$.

3.4 Given a symmetric positive definite matrix $\mathbf{K}$.

(a.) Write a MATLAB program, which performs Choleski decomposition.

CHAPTER 4
APPROXIMATE SOLUTION METHODS

This chapter deals with various approximate solution methods for solving the generalized eigenvalue problem.

Section 4.1 consider the application of *static condensation* or *Guyan reduction*.[1] The idea of the method is to reduce the magnitude of the generalized eigenvalue problem from n to $n_1 \ll n$ degrees of freedom. Next, the reduced system is solved exact. In principle no approximation is related to the procedure.

Section 4.2 deals with the application of *Rayleigh-Ritz analysis*. Similar to static condensation this is a kind of system reduction procedure. As shown the method can be given a formulation identical to static condensation. However, exact results are no longer obtained.

Section 4.3 deals with the bounding of the error related to a certain approximate eigenvalue.

4.1 Static Condensation

The basic assumption of static condensation is that inertia is confined to the first n_1 degrees of freedom, whereas inertia effects are ignored for the remaining $n_2 = n - n_1$ degrees of freedom. The approximation of the method stems from the ignorance of these inertial couplings. This corresponds to the following partitioning of the mass and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \tag{4–1}$$

$\mathbf{M}_{11}$ and $\mathbf{K}_{11}$ are sub-matrices of dimension $n_1 \times n_1$, $\mathbf{K}_{12} = \mathbf{K}_{21}^T$ is a sub-matrix of dimension $n_1 \times n_2$, and $\mathbf{K}_{22}$ is of the dimension $n_2 \times n_2$. The eigenvalue problems for the first n_1 and the last n_2 eigenvectors can be assembled in the following partitioned matrix formulations, cf. (1-14)

[1] S.R.K. Nielsen: *Structural Dynamics, Vol. 1. Linear Structural Dynamics, 4th Ed.* Aalborg tekniske Universitetsforlag, 2004.

$$\left.\begin{aligned}
\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{11} \\ \mathbf{\Phi}_{21} \end{bmatrix} &= \begin{bmatrix} \mathbf{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{11} \\ \mathbf{\Phi}_{21} \end{bmatrix} \mathbf{\Lambda}_1 \\
\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{22} \end{bmatrix} &= \begin{bmatrix} \mathbf{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{22} \end{bmatrix} \mathbf{\Lambda}_2
\end{aligned}\right\} \tag{4–2}$$

where $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are diagonal matrices of the dimension $n_1 \times n_1$ and $n_2 \times n_2$

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_1} \end{bmatrix} \quad , \quad \mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_{n_1+1} & 0 & \cdots & 0 \\ 0 & \lambda_{n_1+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{4–3}$$

$\mathbf{\Phi}_1^{(j)}$ and $\mathbf{\Phi}_2^{(j)}$ denote sub-vectors encompassing the first n_1 and the last n_2 components of the jth eigenmode $\mathbf{\Phi}^{(j)}$. Then, the matrices $\mathbf{\Phi}_{11}$, $\mathbf{\Phi}_{12}$, $\mathbf{\Phi}_{21}$ and $\mathbf{\Phi}_{22}$ entering (4-2) are defined as

$$\left.\begin{aligned}
\mathbf{\Phi}_{11} &= \left[\mathbf{\Phi}_1^{(1)}\, \mathbf{\Phi}_1^{(2)} \cdots \mathbf{\Phi}_1^{(n_1)}\right] \quad , \quad \mathbf{\Phi}_{12} = \left[\mathbf{\Phi}_1^{(n_1+1)}\, \mathbf{\Phi}_1^{(n_1+2)} \cdots \mathbf{\Phi}_1^{(n)}\right] \\
\mathbf{\Phi}_{21} &= \left[\mathbf{\Phi}_2^{(1)}\, \mathbf{\Phi}_2^{(2)} \cdots \mathbf{\Phi}_2^{(n_1)}\right] \quad , \quad \mathbf{\Phi}_{22} = \left[\mathbf{\Phi}_2^{(n_1+1)}\, \mathbf{\Phi}_2^{(n_1+2)} \cdots \mathbf{\Phi}_2^{(n)}\right]
\end{aligned}\right\} \tag{4–4}$$

At first the solution for the first n_1 eigenmodes is considered. From the lower lower half of the first matrix equation in (4-2) follows

$$\begin{aligned}
&\mathbf{K}_{21}\mathbf{\Phi}_{11} + \mathbf{K}_{22}\mathbf{\Phi}_{21} = 0 \quad \Rightarrow \\
&\mathbf{\Phi}_{21} = -\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{\Phi}_{11}
\end{aligned} \tag{4–5}$$

From the corresponding upper half of the said matrix equation, and (4-5), follows

$$\begin{aligned}
&\mathbf{K}_{11}\mathbf{\Phi}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21}\mathbf{\Phi}_{11} = \mathbf{M}_{11}\mathbf{\Phi}_{11}\mathbf{\Lambda}_1 \quad \Rightarrow \\
&\tilde{\mathbf{K}}_{11}\mathbf{\Phi}_{11} = \mathbf{M}_{11}\mathbf{\Phi}_{11}\mathbf{\Lambda}_1
\end{aligned} \tag{4–6}$$

where

$$\tilde{\mathbf{K}}_{11} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} \tag{4–7}$$

(4-6) is a generalized eigenvalue problem of reduced dimension n_1, which is solved for (Λ_1, Φ_{11}). Next, the remaining components of the first n_1 eigenmodes are calculated from (4-5). The modal masses become

$$\mathbf{m}_1 = \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix}^T \begin{bmatrix} \mathrm{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} = \Phi_{11}^T \mathrm{M}_{11} \Phi_{11} \tag{4–8}$$

Hence, the total eigenmodes will be normalized to unit modal mass with respect to $\mathbf{M}$, if the sub-vectors Φ_{11} are normalized to unit modal mass with respect to M_{11}.

Next, the solution for the last n_2 eigenmodes are considered. From the last matrix equation in (4-2) follows

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \Lambda_2^{-1} = \begin{bmatrix} \mathrm{M}_{11}\Phi_{12} \\ 0 \end{bmatrix} \tag{4–9}$$

Obviously, (4-9) is fulfilled for $\Lambda_2^{-1} = 0 \wedge \Phi_{12} = 0$. $\Lambda_2^{-1} = 0$ implies that all n_2 eigenvalues are equal to infinity. Hence, the following eigensolutions are obtained

$$\Lambda_2 = \begin{bmatrix} \infty & 0 & \cdots & 0 \\ 0 & \infty & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \infty \end{bmatrix} \quad , \quad \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ \Phi_{22} \end{bmatrix} \tag{4–10}$$

The matrix Φ_{22} is undetermined. Any matrix with linear independent column vectors will do. Then, this quadratic matrix may simply be chosen as an $n_2 \times n_2$ unit matrix

$$\Phi_{22} = \mathbf{I} \tag{4–11}$$

The modal masses become

$$\mathbf{m}_2 = \begin{bmatrix} 0 \\ \Phi_{22} \end{bmatrix}^T \begin{bmatrix} \mathrm{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \Phi_{22} \end{bmatrix} = 0 \tag{4–12}$$

Generally, if the ith row and the ith column in $\mathbf{M}$ are equal to zero, then $\Phi^T = [0, \ldots, 0, 1, 0, \ldots, 0]$ is an eigenvector with the eigenvalue $\lambda = \infty$. The modal mass is 0.

In praxis the calculation of $\tilde{\mathbf{K}}_{11}$ is solved by means of an initial Choleski decomposition of $\mathbf{K}_{22}$, cf. Box 3.2

$$\mathbf{K}_{22} = \mathbf{S}\mathbf{S}^T \quad \Leftrightarrow \quad \mathbf{K}_{22}^{-1} = \left(\mathbf{S}^{-1}\right)^T \mathbf{S}^{-1} \tag{4–13}$$

where both $\mathbf{S}$ and $\mathbf{S}^{-1}$ are lower triangular matrices. Then, $\tilde{\mathbf{K}}_{11}$ is determined from

$$\tilde{\mathbf{K}}_{11} = \mathbf{K}_{11} - \mathbf{R}^T\mathbf{R} \tag{4–14}$$

where the $n_2 \times n_2$ matrix $\mathbf{R}$ is obtained as solution to the matrix equation

$$\mathbf{S}\mathbf{R} = \mathbf{K}_{21} \tag{4–15}$$

In principle (4-15) represent n_2 linear equations with n_2 right-hand sides. Given that $\mathbf{S}$ is a lower triangular matrix, this is relatively easily solved.

Finally, it should be noticed that the static condensation approach is only of value if $n_1 \ll n$.

Example 4.1: Static condensation

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{4–16}$$

The rows and columns are interchanged the mass and stiffness matrices, so the following eigenvalue problems are obtained

$$\left.\begin{aligned} \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \Phi_{11} \\ \\ \Phi_{21} \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{11} \\ \\ \Phi_{21} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ \\ 0 & \lambda_2 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \Phi_{12} \\ \\ \Phi_{22} \end{bmatrix} &= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_{12} \\ \\ \Phi_{22} \end{bmatrix} \begin{bmatrix} \lambda_3 & 0 \\ \\ 0 & \lambda_4 \end{bmatrix} \end{aligned}\right\} \tag{4–17}$$

A formal procedure for obtaining the mass and stiffness matrices in (4-17) by means of a similarity transformation has been demonstrated in Example 6.1. Then,

$$\mathbf{K}_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathbf{K}_{12} = \mathbf{K}_{21} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad \mathbf{K}_{22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad , \quad \mathbf{M}_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{4–18}$$

From (4-7) follows

$$\tilde{\mathbf{K}}_{11} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{4–19}$$

The reduced eigenvalue problem (4-6) becomes

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \mathbf{\Phi}_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{\Phi}_{11}\mathbf{\Lambda}_1 \tag{4–20}$$

The eigensolutions with eigenmodes normalized to modal mass 1 become

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{1}{2} + \frac{\sqrt{2}}{4} \end{bmatrix} \quad , \quad \mathbf{\Phi}_{11} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{4–21}$$

From (4-5) follows

$$\mathbf{\Phi}_{21} = -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{\sqrt{2}}{4} & -\frac{1}{4} + \frac{\sqrt{2}}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \tag{4–22}$$

From (4-10) and (4-11) follows

$$\mathbf{\Lambda}_2 = \begin{bmatrix} \lambda_3 & 0 \\ 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} \infty & 0 \\ 0 & \infty \end{bmatrix} \quad , \quad \mathbf{\Phi}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{\Phi}_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4–23}$$

After interchanging the degrees of freedom back to the original order (the 1st and 2nd components of $\mathbf{\Phi}_{11}$ and $\mathbf{\Phi}_{12}$ are placed as the 2nd and 4th component of $\mathbf{\Phi}^{(j)}$, the 1st and 2nd components of $\mathbf{\Phi}_{21}$ and $\mathbf{\Phi}_{22}$ are placed as the 3rd and 1st component $\mathbf{\Phi}^{(j)}$, the following eigensolution is obtained

$$\left.\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \infty & 0 \\ 0 & 0 & 0 & \infty \end{bmatrix} \\ \mathbf{\Phi} &= \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(3)}\ \mathbf{\Phi}^{(4)}\right] = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{4} + \frac{\sqrt{2}}{4} & -\frac{1}{4} + \frac{\sqrt{2}}{4} & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{4–24}$$

4.2 Rayleigh-Ritz Analysis

Consider the generalized eigenvalue problem (1-9). If $\mathbf{M}$ is positive definite, so $\mathbf{v}^T\mathbf{M}\mathbf{v} > 0$ for any $\mathbf{v} \neq \mathbf{0}$, the so-called *Rayleigh quotient* may be defined as

$$\rho(\mathbf{v}) = \frac{\mathbf{v}^T\mathbf{K}\mathbf{v}}{\mathbf{v}^T\mathbf{M}\mathbf{v}} \tag{4–25}$$

It can be proved that $\rho(\mathbf{v})$ fulfills the bounding, see Box 4.1

$$\lambda_1 \leq \rho(\mathbf{v}) \leq \lambda_n \tag{4–26}$$

where λ_1 and λ_n denote the smallest and the largest eigenvalues of the generalized eigenvalue problem.

Especially, if $\mathbf{v} = \mathbf{\Phi}^{(1)}$, where $\mathbf{\Phi}^{(1)}$ has been normalized to unit modal mass, it follows that $\mathbf{\Phi}^{(1)T}\mathbf{M}\mathbf{\Phi}^{(1)} = 1$ and $\mathbf{\Phi}^{(1)T}\mathbf{K}\mathbf{\Phi}^{(1)} = \lambda_1$, see (4-34) and (4-35) below. Then, $\rho(\mathbf{v}) = \lambda_1$. This property is contained in the so-called *Rayleigh's principle*

$$\lambda_1 = \min_{\mathbf{v}\in R^n} \rho(\mathbf{v}) \tag{4–27}$$

Next, assume that $\mathbf{v}$ is M-orthogonal to $\mathbf{\Phi}^{(1)}$, so $\mathbf{\Phi}^{(1)T}\mathbf{M}\mathbf{v} = 0$. Then, the following bounding of the Rayleigh quotient may be proved, see Box 4.1

$$\lambda_2 \leq \rho(\mathbf{v}) \leq \lambda_n \tag{4–28}$$

Correspondingly, λ_2 may be evaluated by the following extension of the Rayleigh principle, where the M-orthogonality of the test vector $\mathbf{v}$ to the first eigenmode $\mathbf{\Phi}^{(1)}$ has been included as a restriction

$$\lambda_2 = \begin{cases} \min\limits_{\mathbf{v}\in R^n} \rho(\mathbf{v}) \\ \\ \mathbf{\Phi}^{(1)T}\mathbf{M}\mathbf{v} = 0 \end{cases} \tag{4–29}$$

The corresponding optimal vector will be $\mathbf{v} = \mathbf{\Phi}^{(2)}$.

Generally, if $\mathbf{v}$ is M-orthogonal to the first $m-1$ eigenmodes $\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)}, \ldots, \mathbf{\Phi}^{(m-1)}$, so $\mathbf{\Phi}^{(j)T}\mathbf{M}\mathbf{v} = 0$, $j = 1, \ldots, m-1$, the following bounding of the Rayleigh quotient may be proved, see Box 4.1

$$\lambda_m \leq \rho(\mathbf{v}) \leq \lambda_n \quad , \quad m < n \tag{4–30}$$

Correspondingly, λ_m may be evaluated by the following extension of the Rayleigh variational principle, where restriction of M-orthogonal of the test vector $\mathbf{v}$ to the eigenmodes $\mathbf{\Phi}^{(j)} = 0$, $j = 1, \ldots, m-1$ are included

$$\lambda_m = \begin{cases} \min\limits_{\mathbf{v} \in R^n} \rho(\mathbf{v}) \\ \\ \mathbf{\Phi}^{(j)T}\mathbf{M}\mathbf{v} = 0 \quad , \quad j = 1, \ldots, m-1 \end{cases} \tag{4–31}$$

The corresponding optimal vector will be $\mathbf{v} = \mathbf{\Phi}^{(m)}$.

The Rayleigh quotient may be used to calculate an upper bound for the lowest eigenvalue λ_1. The quality of the estimate depends on the choice of $\mathbf{v}$. The better the qualitative and quantitative resemblance of $\mathbf{v}$ to the shape of the lowest eigenmode, the sharper will be the calculated upper bound.

Box 4.1: Proof of boundings of the Rayleigh quotient

Given the linear independent eigenmodes, normalized to unit modal mass $\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)}, \ldots, \mathbf{\Phi}^{(n)}$. Using the eigenmodes as a vector basis, any n-dimensional vector may be written as

$$\mathbf{v} = q_1\mathbf{\Phi}^{(1)} + q_2\mathbf{\Phi}^{(2)} + \cdots + q_n\mathbf{\Phi}^{(n)} \tag{4–32}$$

Insertion of (4-32) into (4-25) provides

$$\rho(\mathbf{v}) = \frac{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} q_i q_j \mathbf{\Phi}^{(i)T}\mathbf{K}\mathbf{\Phi}^{(j)}}{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} q_i q_j \mathbf{\Phi}^{(i)T}\mathbf{M}\mathbf{\Phi}^{(j)}} = \frac{q_1^2\lambda_1 + q_2^2\lambda_2 + \cdots + q_n^2\lambda_n}{q_1^2 + q_2^2 + \cdots + q_n^2} \tag{4–33}$$

where the orthonormality conditions of the eigenmodes have been used in the last statement, i.e.

$$\mathbf{\Phi}^{(i)T}\mathbf{M}\mathbf{\Phi}^{(j)} = \begin{cases} 0 \quad , \quad i \neq j \\ 1 \quad , \quad i = j \end{cases} \tag{4–34}$$

$$\mathbf{\Phi}^{(i)T}\mathbf{K}\mathbf{\Phi}^{(j)} = \begin{cases} 0 \quad , \quad i \neq j \\ \lambda_i \quad , \quad i = j \end{cases} \tag{4–35}$$

Given the following ordering of the eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \leq \infty \tag{4–36}$$

it follows directly from (4-33) that

$$\left.\begin{aligned} \rho(\mathbf{v}) &\geq \frac{q_1^2\lambda_1 + q_2^2\lambda_1 + \cdots + q_n^2\lambda_1}{q_1^2 + q_2^2 + \cdots + q_n^2} = \lambda_1 \\ \rho(\mathbf{v}) &\leq \frac{q_1^2\lambda_n + q_2^2\lambda_n + \cdots + q_n^2\lambda_n}{q_1^2 + q_2^2 + \cdots + q_n^2} = \lambda_n \end{aligned}\right\} \tag{4–37}$$

which proves the bounding (4-26).

(4-32) is pre-multiplied with $\mathbf{\Phi}^{(j)T}\mathbf{M}$. Then, use of (4-34) provides the following expression for the jth modal coordinate

$$q_j = \mathbf{\Phi}^{(j)T}\mathbf{M}\mathbf{v} \tag{4–38}$$

Hence, if $\mathbf{v}$ is $\mathbf{M}$-orthogonal to $\mathbf{\Phi}^{(j)}$, $j = 1, \ldots, m-1$ it follows that $q_1 = q_2 = \cdots = q_{m-1} = 0$. In this case (4-33) attains the form

$$\rho(\mathbf{v}) = \frac{q_m^2\lambda_m + q_{m+1}^2\lambda_{m+1} + \cdots + q_n^2\lambda_n}{q_m^2 + q_{m+1}^2 + \cdots + q_n^2} \tag{4–39}$$

Proceeding as in (4-37) it then follows that

$$\left.\begin{aligned} \rho(\mathbf{v}) &\geq \frac{q_m^2\lambda_m + q_{m+1}^2\lambda_m + \cdots + q_n^2\lambda_m}{q_m^2 + q_{m+1}^2 + \cdots + q_n^2} = \lambda_m \\ \rho(\mathbf{v}) &\leq \frac{q_m^2\lambda_n + q_{m+1}^2\lambda_n + \cdots + q_n^2\lambda_n}{q_m^2 + q_{m+1}^2 + \cdots + q_n^2} = \lambda_n \end{aligned}\right\} \tag{4–40}$$

which proves the bounding (4-30).

In the so-called *Ritz analysis* m linearly independent *base vectors*, $\mathbf{\Psi}^{(1)}, \ldots, \mathbf{\Psi}^{(m)}$, are defined, which span an m-dimensional subspace $V_m \subseteq V_n$. Often the base vectors are determined as the static deflections from m linearly independent load vectors $\mathbf{f}_1, \ldots, \mathbf{f}_m$. This is preferred, because it often is simpler to specify static load, which will produce displacements qualitatively in agreement with the eigenmodes to be determined by the analysis. The *Ritz-basis* is determined from the equilibrium equation

$$\mathbf{K}\boldsymbol{\Psi} = \mathbf{f} \quad \Rightarrow \quad \boldsymbol{\Psi} = \mathbf{K}^{-1}\mathbf{f} \tag{4–41}$$

$$\boldsymbol{\Psi} = \left[\boldsymbol{\Psi}^{(1)}\, \boldsymbol{\Psi}^{(2)} \ldots \boldsymbol{\Psi}^{(m)}\right] \quad , \quad \mathbf{f} = [\mathbf{f}_1\, \mathbf{f}_2 \cdots \mathbf{f}_m] \tag{4–42}$$

Then, any vector $\mathbf{v} \in V_m$ can be written on the form

$$\mathbf{v} = q_1\boldsymbol{\Psi}^{(1)} + q_2\boldsymbol{\Psi}^{(2)} + \cdots + q_m\boldsymbol{\Psi}^{(m)} = \left[\boldsymbol{\Psi}^{(1)}\, \boldsymbol{\Psi}^{(2)} \ldots \boldsymbol{\Psi}^{(m)}\right] \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} = \boldsymbol{\Psi}\mathbf{q}\ ,\ \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_m \end{bmatrix} \tag{4–43}$$

The idea in Ritz analysis is to insert (4-43) into the Rayleigh quotient (4-25), and determine the modal coordinates $q_1, q_2, \ldots, q_m$, which minimizes this quantity. Hence, the following reformulation of the Rayleigh quotient is considered

$$\rho(\mathbf{q}) = \frac{(\boldsymbol{\Psi}\mathbf{q})^T\mathbf{K}\boldsymbol{\Psi}\mathbf{q}}{(\boldsymbol{\Psi}\mathbf{q})^T\mathbf{M}\boldsymbol{\Psi}\mathbf{q}} = \frac{\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q}}{\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q}} \tag{4–44}$$

where

$$\left.\begin{aligned} \tilde{\mathbf{M}} &= \boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi} = [\tilde{M}_{ij}] \quad , \quad \tilde{M}_{ij} = \boldsymbol{\Psi}^{(i)\,T}\mathbf{M}\boldsymbol{\Psi}^{(j)} \\ \tilde{\mathbf{K}} &= \boldsymbol{\Psi}^T\mathbf{K}\boldsymbol{\Psi} = [\tilde{K}_{ij}] \quad , \quad \tilde{K}_{ij} = \boldsymbol{\Psi}^{(i)\,T}\mathbf{K}\boldsymbol{\Psi}^{(j)} \end{aligned}\right\} \tag{4–45}$$

$\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ are denoted as the *projected mass- and stiffness matrices* on the subspace spanned by the Ritz basis $\boldsymbol{\Psi}$.

The approximation to λ_1 then follows from (4-27)

$$\lambda_1 \leq \rho_1 = \min_{\mathbf{q}\in V_m} \rho(\mathbf{q}) = \min_{q_1,\ldots,q_m} \frac{\sum_{i=1}^{m}\sum_{j=1}^{m} q_i\tilde{K}_{ij}q_j}{\sum_{i=1}^{m}\sum_{j=1}^{m} q_i\tilde{M}_{ij}q_j} \tag{4–46}$$

Generally, ρ_1 is larger than λ_1 in agreement with (4-26). Only for $\boldsymbol{\Phi}^{(1)} \in V_m$ will modal coordinates $q_, \ldots, q_m$ exist, so $\boldsymbol{\Phi}^{(1)} = q_1\boldsymbol{\Psi}^{(1)} + \cdots + q_m\boldsymbol{\Psi}^{(m)}$, with the implication that $\rho_1 = \lambda_1$.

The necessary condition for a minimum is that

$$\frac{\partial}{\partial q_i}\left(\frac{\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q}}{\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q}}\right) = 0 \quad , \quad i = 1, \ldots, m \qquad \Rightarrow$$

$$\frac{\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q} \cdot \frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q}) - \mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q} \cdot \frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q})}{(\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q})^2} = 0 \qquad \Rightarrow$$

$$\frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q}) - \rho\frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q}) = 0 \tag{4–47}$$

Now, $\frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q}) = \frac{\partial}{\partial q_i}\sum_{j=1}^m\sum_{k=1}^m q_j\tilde{K}_{jk}q_k = 2\sum_{k=1}^m \tilde{K}_{ik}q_k$, where the symmetry property, $\tilde{K}_{jk} = \tilde{K}_{kj}$ ($\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^T$), has been applied. Similarly, $\frac{\partial}{\partial q_i}(\mathbf{q}^T\tilde{\mathbf{M}}\mathbf{q}) = 2\sum_{k=1}^m \tilde{M}_{ik}q_k$. Then, the minimum condition (4-47) reduces to

$$\sum_{j=1}^m \tilde{K}_{ij}q_j - \rho\sum_{j=1}^m \tilde{M}_{ij}q_j = 0 \tag{4–48}$$

From (4-48) follows that ρ_1 is determined as the lowest eigenvalue to the following generalized eigenvalue problem of dimension m

$$\tilde{\mathbf{K}}\mathbf{q} - \rho\tilde{\mathbf{M}}\mathbf{q} = \mathbf{0} \tag{4–49}$$

(4-49) has m eigensolutions $(\rho_i, \mathbf{q}^{(i)})$, $i = 1, \ldots, m$. ρ_i becomes an approximation to the ith eigenvalue λ_i. The corresponding approximation to the ith eigenmode is calculated from

$$\bar{\Phi}^{(i)} = q_{1,i}\Psi^{(1)} + \cdots + q_{m,i}\Psi^{(m)} = \Psi\mathbf{q}^{(i)} \quad , \quad i = 1, \ldots, m \tag{4–50}$$

where $q_{1,i}, \ldots, q_{1m,i}$ denote the components of $\mathbf{q}^{(i)}$.

The relations (4-50) can be assembled into the matrix equation

$$\bar{\Phi} = \Psi\mathbf{Q} \tag{4–51}$$

$$\bar{\Phi} = \left[\bar{\Phi}^{(1)}\,\bar{\Phi}^{(2)}\cdots\bar{\Phi}^{(m)}\right] \quad , \quad \mathbf{Q} = \left[\mathbf{q}^{(1)}\,\mathbf{q}^{(2)}\cdots\mathbf{q}^{(m)}\right] \tag{4–52}$$

We shall assume that the eigenvectors $\mathbf{q}^{(i)}$ are normalized to unit modal mass with respect to the projected mass matrix, i.e. the following orthonormality properties are fulfilled

$$\mathbf{q}^{(i)T}\tilde{\mathbf{M}}\mathbf{q}^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases} \tag{4–53}$$

$$\mathbf{q}^{(i)T}\tilde{\mathbf{K}}\mathbf{q}^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ \rho_i & , \quad i = j \end{cases} \tag{4–54}$$

Then, the modal mass of the eigenmodes $\bar{\Phi}$ become

$$\bar{\Phi}^T\mathbf{M}\bar{\Phi} = (\Psi\mathbf{Q})^T\mathbf{M}\Psi\mathbf{Q} = \mathbf{Q}^T\tilde{\mathbf{M}}\mathbf{Q} = \mathbf{I} \tag{4–55}$$

Hence, the approximate eigenmodes $\bar{\Phi}^{(i)}$ will be normalized to unit modal mass, if this is the case for the eigenvectors $\mathbf{q}^{(i)}$ with respect to the projected mass matrix. $\bar{\Phi}$ forms an alternative Ritz-basis in V^m, which in addition is **M**-orthonormal. Similarly, the approximate eigenmodes are **K**-orthogonal as follows

$$\bar{\Phi}^T\mathbf{K}\bar{\Phi} = (\Psi\mathbf{Q})^T\mathbf{K}\Psi\mathbf{Q} = \mathbf{Q}^T\tilde{\mathbf{K}}\mathbf{Q} = \mathbf{R} \tag{4–56}$$

where **R** is m-dimensional diagonal matrix with the eigenvalues $\rho_1, \ldots, \rho_m$ in the main diagonal.

Obviously, the Rayleigh quotient approach corresponds to $m = 1$. Hence, Ritz analysis is merely a multi-dimensional generalization, for which reason the name *Rayleigh-Ritz analysis* has been coined for the method.

As a generalization to (4-26) the following boundings can be proved[2]

$$\lambda_1 \leq \rho_1 \quad , \quad \lambda_2 \leq \rho_2 \quad , \quad \ldots \quad , \quad \lambda_m \leq \rho_m \leq \lambda_n \tag{4–57}$$

[2]K.-J. Bathe: *Finite Element Procedures*. Printice Hall, Inc., 1996.

Box 4.2: Rayleigh-Ritz algorithm

1. Estimate m linearly independent static load vectors $\mathrm{f}_1, \ldots, \mathrm{f}_m$, assembled column-wise in the $n \times m$ matrix $\mathrm{f} = [\mathrm{f}_1\, \mathrm{f}_2 \cdots \mathrm{f}_m]$.
2. Calculate the Ritz basis from $\Psi = \mathrm{K}^{-1}\mathrm{f}$, $\Psi = \left[\Psi^{(1)}\, \Psi^{(1)} \ldots \Psi^{(m)}\right]$.
3. Calculate projected mass and stiffness matrices in the m-dimensional subspace spanned by the Ritz basis: $\tilde{\mathrm{M}} = \Psi^T \mathrm{M} \Psi$, $\tilde{\mathrm{K}} = \Psi^T \mathrm{K} \Psi$.
4. Solve the generalized eigenvalue problem of dimension m: $\tilde{\mathrm{K}}\mathrm{Q} = \tilde{\mathrm{M}}\mathrm{Q}\mathrm{R}$.
5. Determine approximations to the lowest m eigenvector from the transformation $\bar{\Phi} = \Psi \mathrm{Q}$, $\bar{\Phi} = \left[\bar{\Phi}^{(1)}\, \bar{\Phi}^{(2)} \ldots \bar{\Phi}^{(m)}\right]$. The corresponding approximate eigenvalues are contained in the main diagonal of R.

Returning to the static condensation problem in Section 4.1, let us define a Ritz basis of the dimension $m = n_1$ as

$$\Psi_1 = \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix} \tag{4–58}$$

where I is a unit matrix of dimension $n_1 \times n_1$. Given the structure of the mass and stiffness matrices in (4-1), we may then evaluate the following projected matrices

$$\tilde{\mathrm{M}} = \Psi_1^T \mathrm{M} \Psi_1 = \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix}^T \begin{bmatrix} \mathrm{M}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix} = \mathrm{M}_{11} \tag{4–59}$$

$$\tilde{\mathrm{K}} = \Psi_1^T \mathrm{K} \Psi_1 = \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix}^T \begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix} = \mathrm{K}_{11} - \mathrm{K}_{12}\mathrm{K}_{22}^{-1}\mathrm{K}_{21} = \tilde{\mathrm{K}}_{11} \tag{4–60}$$

Hence, (4-49) reduce to the generalized eigenvalue problem (4-6), with $\mathrm{Q} = \Phi_{11}$, and $\mathrm{R} = \Lambda_1$. Consequently, static condensation may be interpreted as merely a Rayleigh-Ritz analysis with the Ritz basis (4-58).

The following identity may be proved by insertion

$$\begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix} \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix} \left(\mathrm{K}_{11} - \mathrm{K}_{12}\mathrm{K}_{22}^{-1}\mathrm{K}_{21}\right)^{-1} = \begin{bmatrix} \mathrm{I} \\ 0 \end{bmatrix} \tag{4–61}$$

Then, we may construct an alternative Ritz basis from (4-41) with the static load given as the right hand side of (4-61), i.e.

$$\Psi_2 = \mathrm{K}^{-1}\mathrm{f} = \begin{bmatrix} \mathrm{K}_{11} & \mathrm{K}_{12} \\ \mathrm{K}_{21} & \mathrm{K}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{I} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{I} \\ -\mathrm{K}_{22}^{-1}\mathrm{K}_{21} \end{bmatrix} \left(\mathrm{K}_{11} - \mathrm{K}_{12}\mathrm{K}_{22}^{-1}\mathrm{K}_{21}\right)^{-1} = \Psi_1\tilde{\mathrm{K}}_{11}^{-1} \tag{4–62}$$

Hence, the base vectors in Ψ_2 is a linear combination of the base vectors in Ψ_1. Then, Ψ_1 and Ψ_2 span the same subspace V_{n1}, for which reason both bases will determine the same eigenvalues and eigenvectors.

The projected mass and stiffness matrices become

$$\tilde{\mathrm{M}} = \Psi_2^T\mathrm{M}\Psi_2 = \tilde{\mathrm{K}}_{11}^{-1}\Psi_1^T\mathrm{M}\Psi_1\tilde{\mathrm{K}}_{11}^{-1} = \tilde{\mathrm{K}}_{11}^{-1}\mathrm{M}_{11}\tilde{\mathrm{K}}_{11}^{-1} \tag{4–63}$$

$$\tilde{\mathrm{K}} = \Psi_2^T\mathrm{K}\Psi_2 = \tilde{\mathrm{K}}_{11}^{-1}\Psi_1^T\mathrm{K}\Psi_1\tilde{\mathrm{K}}_{11}^{-1} = \tilde{\mathrm{K}}_{11}^{-1} \tag{4–64}$$

Then, the modal matrices Q_1 and Q_2, obtained as solutions to (4-49) for the respective Ritz bases, are seen to be related as

$$\mathrm{Q}_1 = \tilde{\mathrm{K}}_{11}^{-1}\mathrm{Q}_2 \tag{4–65}$$

(4-65) follows from $\bar{\Phi} = \Psi_1\mathrm{Q}_1 = \Psi_2\mathrm{Q}_2$.

Example 4.2: Rayleigh-Ritz analysis

Given the following mass- and stiffness matrices

$$\mathrm{M} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathrm{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{4–66}$$

which have the exact eigensolutions, cf. Example 1.5

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \quad , \quad \Phi = \left[\Phi^{(1)}\,\Phi^{(2)}\,\Phi^{(3)}\right] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{4–67}$$

A two dimensional Rayleigh-Ritz analysis is performed, where the static load vectors are estimated as

$$\mathrm{f} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4–68}$$

The Ritz basis becomes

$$\Psi = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 7 & 1 \\ 2 & 2 \\ 1 & 7 \end{bmatrix} \tag{4–69}$$

The projected mass and stiffness matrices become

$$\tilde{\mathbf{M}} = \frac{1}{144} \begin{bmatrix} 29 & 11 \\ 11 & 29 \end{bmatrix} \quad , \quad \tilde{\mathbf{K}} = \frac{1}{12} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} \tag{4–70}$$

The eigensolutions with modal masses normalized to 1 become

$$\mathbf{R} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} 2.4 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{Q} = \begin{bmatrix} \mathbf{q}^{(1)} \, \mathbf{q}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{5}} & 2 \\ \frac{3}{\sqrt{5}} & -2 \end{bmatrix} \tag{4–71}$$

The solutions for the eigenvectors become

$$\bar{\Phi} = \begin{bmatrix} \bar{\Phi}^{(1)} \, \bar{\Phi}^{(2)} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 7 & 1 \\ 2 & 2 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{5}} & 2 \\ \frac{3}{\sqrt{5}} & -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & 1 \\ \frac{1}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -1 \end{bmatrix} \tag{4–72}$$

As seen from (4-71) and (4-72) $\rho_2 = 4$ and $\bar{\Phi}^{(2)}$ are calculated exactly, cf. (4-67). This is so, because $\Phi^{(2)}$ is placed in the subspace spanned by the selected Ritz basis as seen from the expansion

$$\Phi^{(2)} = 2\Psi^{(1)} - 2\Psi^{(2)} = \frac{2}{12} \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{4–73}$$

Next, a new analysis is performed, where the static load vectors are estimated as

$$\mathbf{f} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{4–74}$$

The Ritz basis becomes

$$\Psi = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 \\ 4 & 2 \\ 5 & 1 \end{bmatrix} \tag{4–75}$$

The projected mass and stiffness matrices become

$$\tilde{\mathrm{M}} = \frac{1}{36}\begin{bmatrix} 41 & 13 \\ 13 & 5 \end{bmatrix} \quad , \quad \tilde{\mathrm{K}} = \frac{1}{3}\begin{bmatrix} 7 & 2 \\ 2 & 1 \end{bmatrix} \tag{4–76}$$

The eigensolutions with modal masses normalized to 1 become

$$\mathbf{R} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \quad , \quad \mathbf{Q} = \left[\mathbf{q}^{(1)}\,\mathbf{q}^{(2)}\right] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{9\sqrt{2}}{2} \end{bmatrix} \tag{4–77}$$

The solutions for the eigenvectors become

$$\bar{\mathbf{\Phi}} = \left[\bar{\mathbf{\Phi}}^{(1)}\,\bar{\mathbf{\Phi}}^{(2)}\right] = \frac{1}{6}\begin{bmatrix} 5 & 1 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{9\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{4–78}$$

In this case $(\rho_1, \bar{\mathbf{\Phi}}^{(1)}) = (\lambda_1, \mathbf{\Phi}^{(1)})$ and $(\rho_2, \bar{\mathbf{\Phi}}^{(2)}) = (\lambda_3, \mathbf{\Phi}^{(3)})$. This is so, because $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(3)}$ are placed in the sub-space spanned by the selected Ritz basis.

4.3 Error Analysis

Given a certain approximation to the jth eigen-pair $(\bar{\lambda}_j, \bar{\mathbf{\Phi}}^{(j)})$, the *error vector* is defined as

$$\varepsilon_j = \left(\mathbf{K} - \bar{\lambda}_j \mathbf{M}\right)\bar{\mathbf{\Phi}}^{(j)} \tag{4–79}$$

Presuming that the eigenvectors have been normalized to unit modal mass, it follows from (1-19) and (1-21) that

$$\mathbf{M} = \left(\mathbf{\Phi}^{-1}\right)^T \mathbf{I}\,\mathbf{\Phi}^{-1} \quad , \quad \mathbf{K} = \left(\mathbf{\Phi}^{-1}\right)^T \mathbf{\Lambda}\,\mathbf{\Phi}^{-1} \tag{4–80}$$

Insertion of (4-80) into (4-79) provides

$$\varepsilon_j = \left(\mathbf{\Phi}^{-1}\right)^T \left(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I}\right)\mathbf{\Phi}^{-1}\bar{\mathbf{\Phi}}^{(j)} \quad \Rightarrow$$

$$\bar{\mathbf{\Phi}}^{(j)} = \mathbf{\Phi}\left(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I}\right)^{-1}\mathbf{\Phi}^T \varepsilon_j \tag{4–81}$$

We shall use the Euclidean vector norm $\|\cdot\|_E$ and the Hilbert matrix norm $\|\cdot\|_H$ in the following. For a definition of these quantities, see Box 4.3. The mentioned norms are compatible, so

$$\|\bar{\mathbf{\Phi}}^{(j)}\|_E \le \left\|\mathbf{\Phi}\left(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I}\right)^{-1}\mathbf{\Phi}^T\right\|_H \|\varepsilon_j\|_E \le \|\mathbf{\Phi}\|_H \left\|\left(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I}\right)^{-1}\right\|_H \|\mathbf{\Phi}^T\|_H \|\varepsilon_j\|_E \tag{4–82}$$

The last statement of (4-82) follows from the defining properties of matrix norms, see Box 4.3.

$(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I})$ is a diagonal matrix. Then, $(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I})^{-1}$ is also a diagonal matrix with the components $(\lambda_k - \bar{\lambda}_j)^{-1}$, $k = 1, \ldots, n$ in the main diagonal. The eigenvalues of a diagonal matrix is equal to the components in the main diagonal. Since, the Hilbert norm of a symmetric matrix is equal to the numerical largest eigenvalue, it follows that

$$\left\| \left(\mathbf{\Lambda} - \bar{\lambda}_j \mathbf{I} \right)^{-1} \right\|_H = \max_{k=1,\ldots,n} \left(\frac{1}{|\lambda_k - \bar{\lambda}_j|} \right) = \frac{1}{\min\limits_{k=1,\ldots,n} |\lambda_k - \bar{\lambda}_j|} \tag{4–83}$$

The Hilbert norms of $\mathbf{\Phi}$ and $\mathbf{\Phi}^T$ are identical as stated in Box 4.3. Further, it can be shown that, see Box 4.4

$$\|\mathbf{\Phi}\|_H^2 = \frac{1}{\mu_1} \tag{4–84}$$

where μ_1 is the lowest eigenvalue of $\mathbf{M}$.

Then, (4-82), (4-83) and (4-84) provides the following bounding of the calculated eigenvalue $\bar{\lambda}_j$

$$\min_{k=1,\ldots,n} |\lambda_k - \bar{\lambda}_j| \leq \frac{1}{\mu_1} \frac{\|\varepsilon_j\|_E}{\|\bar{\mathbf{\Phi}}^{(j)}\|_E} = \frac{1}{\mu_1} \frac{|\varepsilon_j|}{|\bar{\mathbf{\Phi}}^{(j)}|} \tag{4–85}$$

(4-85) is only of value, if μ_1 can be calculated relatively easily. This is the case for the special eigenvalue problem, where $\mathbf{M} = \mathbf{I}$, which means that $\mu_1 = \cdots = \mu_n = 1$, so (4-85) reduces to

$$\min_{k=1,\ldots,n} |\lambda_k - \bar{\lambda}_j| \leq \frac{\|\varepsilon_j\|_E}{\|\bar{\mathbf{\Phi}}^{(j)}\|_E} = \frac{|\varepsilon_j|}{|\bar{\mathbf{\Phi}}^{(j)}|} \tag{4–86}$$

Box 4.3: Vector and matrix norms

A *vector norm* is a real number $\|\mathbf{v}\|$ associated to any n-dimensional vector $\mathbf{v}$, which fulfills the following conditions

1. $\|\mathbf{v}\| > 0$ for $\mathbf{v} \neq \mathbf{0}$ and $\|\mathbf{0}\| = 0$.
2. $\|c\mathbf{v}\| = |c| \cdot \|\mathbf{v}\|$ for any complex or real number c.
3. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality).

The most common vector norms are

1. p-norm ($p \in]0, \infty[$): $\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$.
2. One norm ($p = 1$): $\|\mathbf{v}\|_1 = \sum_{i=1}^{n} |v_i|$.
3. Two norm ($p = 2$, Euclidean norm): $\|\mathbf{v}\|_2 = |\mathbf{v}| = \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$.
4. Infinity norm ($p = \infty$): $\|\mathbf{v}\|_\infty = \max_{i=1,\ldots,n} |v_i|$.

where v_i denotes the components of $\mathbf{v}$. Given

$$\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \|\mathbf{v}\|_{1/2} = & \left(\sqrt{1} + \sqrt{3} + \sqrt{2}\right)^2 = 17.19 \\ \|\mathbf{v}\|_1 = & (1 + 3 + 2) = 6 \\ \|\mathbf{v}\|_2 = & \left(1^2 + 3^2 + 2^2\right)^{1/2} = 3.74 \\ \|\mathbf{v}\|_\infty = & \max(1, 3, 2) = 3 \end{cases} \tag{4–87}$$

A *matrix norm* is a real number $\|\mathbf{A}\|$ associated to any $n \times n$ matrix $\mathbf{A}$, which fulfills the following conditions

1. $\|\mathbf{A}\| > 0$ for $\mathbf{A} \neq \mathbf{0}$ and $\|\mathbf{0}\| = 0$.
2. $\|c\mathbf{A}\| = |c| \cdot \|\mathbf{A}\|$ for any complex or real number c.
3. $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ (triangle inequality).
4. $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$.

The most common matrix norms are

1. One norm: $\|\mathbf{A}\|_1 = \max\limits_{j=1,\ldots,n} \sum\limits_{i=1}^{n} |a_{ij}|$.

2. Infinity norm: $\|\mathbf{A}\|_\infty = \max\limits_{i=1,\ldots,n} \sum\limits_{j=1}^{n} |a_{ij}|$.

3. Euclidean norm: $\|\mathbf{A}\|_E = \left(\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}^2 \right)^{1/2}$.

4. Hilbert norm (spectral norm): $\|\mathbf{A}\|_H = \left(\max\limits_{i=1,\ldots,n} \lambda_i \right)^{1/2}$, where λ_i is the ith eigenvalue of $\mathbf{A}\mathbf{A}^T$ identical to the eigenvalues of $\mathbf{A}^T\mathbf{A}$, so $\|\mathbf{A}\|_H = \|\mathbf{A}^T\|_H$.

a_{ij} denotes the components of $\mathbf{A}$. Notice, if $\mathbf{A} = \mathbf{A}^T$ the eigenvalues of $\mathbf{A}\mathbf{A}^T = \mathbf{A}^2$ becomes equal to the square of the eigenvalues of $\mathbf{A}$. Given

$$\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix} \Rightarrow \mathbf{A}\mathbf{A}^T = \begin{bmatrix} 29 & 11 \\ 11 & 10 \end{bmatrix} \Rightarrow \begin{cases} \|\mathbf{A}\|_1 = & \max(2+3, 5+1) = 6 \\ \|\mathbf{A}\|_\infty = & \max(2+5, 3+1) = 7 \\ \|\mathbf{A}\|_E = & (4+25+9+1)^{1/2} = 6.24 \\ \|\mathbf{A}\|_H = & \sqrt{\frac{13}{2}(3+\sqrt{5})} = 5.83 \end{cases} \tag{4–88}$$

A matrix norm $\|\cdot\|_m$ is said to be *compatible* to a given vector norm $\|\cdot\|_v$, if

$$\|\mathbf{A}\mathbf{v}\|_v \leq \|\mathbf{A}\|_m \cdot \|\mathbf{v}\|_v \tag{4–89}$$

It can be shown that the Hilbert matrix norm is compatible to the Euclidean vector norm, that the one matrix norm is compatible to the one vector norm, and that the infinity matrix norm is compatible to the infinity vector norm. However, the Euclidean matrix norm is *not* compatible to the Euclidean vector norm.

Box 4.4: Hilbert norm of modal matrix

Presuming that the columns of the modal matrix have been normalized to unit modal mass, so $\mathbf{m} = \mathbf{I}$, it follows from (1-19) that

$$\mathbf{M} = \left(\mathbf{\Phi}^T\right)^{-1}\mathbf{\Phi}^{-1} \quad \Rightarrow \quad \mathbf{M}^{-1} = \mathbf{\Phi}\mathbf{\Phi}^T \tag{4–90}$$

From the definition of the Hilbert norm in Box 4.3 and (4-90) follows that $\|\mathbf{\Phi}\|_H^2$ becomes equal to the maximum eigenvalue of $\mathbf{M}^{-1}$. If $\mu_1, \mu_2, \ldots, \mu_n$ denote the eigenvalues of $\mathbf{M}$ in ascending order, then the eigenvalues in ascending order of $\mathbf{M}^{-1}$ become $\frac{1}{\mu_n}, \ldots, \frac{1}{\mu_2}, \frac{1}{\mu_1}$, so the maximum eigenvalue of $\mathbf{M}^{-1}$ is equal to $\frac{1}{\mu_1}$. This proves (4-84).

Example 4.3: Bound on calculated eigenvalue

Given the mass- and stiffness matrices for the following special eigenvalue problem

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \tag{4–91}$$

The eigensolutions with modal masses normalized to 1 are given as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad , \quad \mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(3)}\right] = \begin{bmatrix} \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{2\sqrt{6}}{6} & 0 & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{bmatrix} \tag{4–92}$$

Assume that the following approximate solution , $(\bar{\lambda}_2, \bar{\mathbf{\Phi}}^{(2)})$, has been calculated to the 2nd eigen-pair $(\lambda_2, \mathbf{\Phi}^{(2)})$

$$\bar{\lambda}_2 = 3.1 \quad , \quad \bar{\mathbf{\Phi}}^{(2)} = \begin{bmatrix} 1.0 \\ 0.2 \\ -1.0 \end{bmatrix} \quad \Rightarrow \quad \left|\bar{\mathbf{\Phi}}^{(2)}\right| = 1.4283 \tag{4–93}$$

Then, the error vector becomes, cf. (4-79)

$$\varepsilon_2 = \left(\begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} - 3.1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1.0 \\ 0.2 \\ -1.0 \end{bmatrix} = \begin{bmatrix} -0.30 \\ -0.22 \\ -0.10 \end{bmatrix} \quad \Rightarrow \quad |\varepsilon_2| = 0.3852 \tag{4–94}$$

Since, $\mathbf{M} = \mathbf{I}$ we may use the simplified result (4-86), which provides

$$|\lambda_2 - \bar{\lambda}_2| \leq \frac{0.3852}{1.4283} = 0.26971 \tag{4–95}$$

Actually, $|\lambda_2 - \bar{\lambda}_2| = 0.1$.

4.4 Exercises

4.1 Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a.) Perform a static condensation by the conventional procedure based on (4-5), (4-6), and next by a Rayleigh-Ritz analysis with the Ritz basis given by (4-62).

4.2 Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a.) Calculate approximate eigenvalues and eigenmodes by a Rayleigh-Ritz analysis using the following Ritz basis

$$\mathbf{\Psi} = \left[\mathbf{\Psi}^{(1)}\ \mathbf{\Psi}^{(2)}\right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

4.3 Consider the mass- and stiffness matrices in Exercise 4.2, and let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(a.) Calculate the vector $\bar{\mathbf{\Phi}}^{(1)} = \mathbf{K}^{-1}\mathbf{M}\mathbf{v}$, and next $\bar{\lambda}_1 = \rho\left(\bar{\mathbf{\Phi}}^{(1)}\right)$, as approximate solutions to the lowest eigenmode and eigenvalue.

(b.) Establish the error bound for the obtained approximation to the lowest eigenvalue.

CHAPTER 5
VECTOR ITERATION METHODS

5.1 Introduction

In structural dynamics only a small number n_1 of the lowest eigen-pairs, $\left(\lambda_1, \Phi^{(1)}\right)$, $\left(\lambda_2, \Phi^{(2)}\right)$, $\ldots$, $\left(\lambda_{n_1}, \Phi^{(n_1)}\right)$, where $n_1 \ll n$, are of structural significance. Hence, there is a need for methods, which concentrate on the determination of the low-order modes. This is the underlying motivation for most of the methods described in the following chapters.

It should be noticed that if λ_j is known, then $\Phi^{(j)}$ can be determined from the linear, homogeneous equations, cf. (1-9)

$$\left(\mathbf{K} - \lambda_j \mathbf{M}\right)\Phi^{(j)} = 0 \tag{5–1}$$

If λ_j is an eigenvalue, the coefficient matrix $\mathbf{K} - \lambda_j \mathbf{M}$ is singular. Then, $\Phi^{(j)}$ can be determined within a common factor by solving a linear system of $n-1$ equations as illustrated in Example 1.5.

On the other hand, if $\Phi^{(j)}$is known, the eigenvalue λ_j can be determined from the Rayleigh quotient, cf. (4-25)

$$\lambda_j = \frac{\Phi^{(j)T}\mathbf{K}\Phi^{(j)}}{\Phi^{(j)T}\mathbf{M}\Phi^{(j)}} \tag{5–2}$$

Since, the eigenvalues are determined as solutions to the characteristic equation (1-10), which can only be solved analytically for $n \leq 4$, all solution methods for practical problems relies implicitly or explicitly on iterative numerical schemes.

Iterative numerical solution methods may be classified in the following categories

Vector iteration methods operate directly on the generalized eigenvalue problem (5-1), so that a certain eigenvalue and associated eigenmode are determined iteratively with increasing accuracy. Vector iteration methods are considered both in Chapters 5 and 7.

Similarity transformation methods transform the generalized eigenvalue problem via a sequence of similarity transformations, so the transformed mass and stiffness matrices eventually attain a diagonal form. These methods are considered in Chapter 6.

Characteristic polynomial iteration methods operates directly or indirectly on the characteristic equation (1-10). These methods are dealt with in Section 7.4.

5.2 Inverse and Forward Vector Iteration

The principle in *inverse vector iteration* may be explained in the following way. Given a start vector, Φ_0. Based on the generalized eigenvalue problem (5-1), one may then calculate a new vector Φ_1 as follows

$$\mathrm{K}\Phi_1 = \mathrm{M}\Phi_0 \quad \Rightarrow \quad \Phi_1 = \mathrm{K}^{-1}\mathrm{M}\Phi_0 = \mathrm{A}\Phi_0 \tag{5–3}$$

where

$$\mathrm{A} = \mathrm{K}^{-1}\mathrm{M} \tag{5–4}$$

Clearly, if $\Phi_0 = \Phi^{(j)}$ is an eigenmode, then $\Phi_1 = \frac{1}{\lambda_j}\Phi_0$. If not so, we may consider Φ_1 as another, and hopefully better approximation to the eigenmode. Next, based on Φ_1 we may proceed to calculate a still better approximation Φ_2 from

$$\mathrm{K}\Phi_2 = \mathrm{M}\Phi_1 \quad \Rightarrow \quad \Phi_2 = \mathrm{A}\Phi_1 \tag{5–5}$$

This proceed may be continued until the convergence criteria $\Phi_{k+1} = \frac{1}{\lambda_j}\Phi_k$ is fulfilled with sufficient accuracy.

The inverse vector iteration algorithm may then be summarized as follows

Box 5.1: Inverse vector iteration algorithm

Given start vector Φ_0, which needs not be normalized to unit modal mass. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate $\bar{\Phi}_{k+1} = \mathrm{A}\Phi_k$.
2. Normalize solution vector to unit modal mass, so $\Phi_{k+1}^T\mathrm{M}\Phi_{k+1} = 1$:

$$\Phi_{k+1} = \frac{\bar{\Phi}_{k+1}}{\sqrt{\bar{\Phi}_{k+1}^T\mathrm{M}\bar{\Phi}_{k+1}}}$$

Obviously, the algorithm requires that the stiffness matrix is non-singular, so the inverse $\mathbf{K}^{-1}$ exists. By contrast the mass matrix needs not be non-singular as is the case in Example 5.1 below. After convergence the lowest eigenvalue is most accurately calculated from the Rayleigh quotient (4-25).

In case the lowest eigenvalue is simple, i.e. that $\lambda_1 < \lambda_2$, the inverse iteration algorithm converges towards the lowest eigenpair $(\lambda_1, \mathbf{\Phi}^{(1)})$. The solution vector obtained after the kth iteration step, $\mathbf{\Phi}_k$, is an n-dimensional vector, which may be expanded in the basis formed by the n undamped eigenmodes as follows

$$\left.\begin{aligned} &\mathbf{\Phi}_k = q_{1,k}\mathbf{\Phi}^{(1)} + q_{2,k}\mathbf{\Phi}^{(2)} + \cdots + q_{n,k}\mathbf{\Phi}^{(n)} = \mathbf{\Phi}\mathbf{q}_k \\ &\mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\,\mathbf{\Phi}^{(2)}\ldots\mathbf{\Phi}^{(n)}\right] \quad , \quad \mathbf{q}_k = \begin{bmatrix} q_{1,k} \\ q_{2,k} \\ \vdots \\ q_{n,k} \end{bmatrix} \end{aligned}\right\} \tag{5–6}$$

The components of the vector $\mathbf{q}_k$ denote the modal coordinates of the vector $\mathbf{\Phi}_k$. The expansion (5-6) should be considered as formal, since the base vectors $\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)}, \ldots, \mathbf{\Phi}^{(n)}$ are unknown. Actually, the whole analysis deals with the determination of these quantities. Similarly, the expansion for $\bar{\mathbf{\Phi}}_{k+1}$ reads

$$\bar{\mathbf{\Phi}}_{k+1} = \mathbf{\Phi}\bar{\mathbf{q}}_{k+1} \tag{5–7}$$

where $\bar{\mathbf{q}}_{k+1}$ denotes a vector of modal coordinates of $\bar{\mathbf{\Phi}}_{k+1}$. Insertion of (5-6) and (5-7) into the iteration algorithm provides

$$\begin{aligned} &\mathbf{K}\mathbf{\Phi}\bar{\mathbf{q}}_{k+1} = \mathbf{M}\mathbf{\Phi}\mathbf{q}_k \qquad \Rightarrow \\ &\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi}\,\bar{\mathbf{q}}_{k+1} = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\,\mathbf{q}_k \qquad \Rightarrow \\ &\mathbf{\Lambda}\bar{\mathbf{q}}_{k+1} = \mathbf{q}_k \end{aligned} \tag{5–8}$$

where the orthogonality properties (1-19) and (1-21) have been used, and the eigenmodes are assumed to be normalized to unit modal mass. The diagonal matrix $\mathbf{\Lambda}$ is given by (1-15). As $k \to \infty$ convergence implies that $\lambda_j\bar{\mathbf{q}}_{k+1} = \mathbf{q}_k = \mathbf{\Psi}^{(j)}$, where $\mathbf{\Psi}^{(j)}$ signifies the eigenmode in the modal space. This means that

$$\mathbf{\Lambda}\mathbf{\Psi}^{(j)} = \lambda_j \mathbf{\Psi}^{(j)} \qquad \Rightarrow$$

$$\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_n \end{bmatrix} = \lambda_j \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \vdots \\ \Psi_n \end{bmatrix} \qquad \Rightarrow$$

$$\mathbf{\Psi}^{(j)} = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_{j-1} \\ \Psi_j \\ \Psi_{j+1} \\ \vdots \\ \Psi_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5–9}$$

The jth component of $\mathbf{\Psi}^{(j)}$ is equal to 1, and the remaining components are zero.

Let the start vector be given as $\mathbf{q}_0 = [1, \ldots, 1]^T$. Then, the following sequence of results may be calculated from (5-8)

$$\left.\begin{aligned}
\mathbf{q}_1 = \mathbf{\Lambda}^{-1}\mathbf{q}_0 &= \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \\ \vdots \\ \frac{1}{\lambda_n} \end{bmatrix} \qquad \Rightarrow \\
\mathbf{q}_2 = \mathbf{\Lambda}^{-1}\mathbf{q}_1 &= \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} \\ \frac{1}{\lambda_2} \\ \vdots \\ \frac{1}{\lambda_n} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1^2} \\ \frac{1}{\lambda_2^2} \\ \vdots \\ \frac{1}{\lambda_n^2} \end{bmatrix} \qquad \Rightarrow \cdots \Rightarrow \\
\mathbf{q}_k = \mathbf{\Lambda}^{-1}\mathbf{q}_{k-1} &= \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1^{k-1}} \\ \frac{1}{\lambda_2^{k-1}} \\ \vdots \\ \frac{1}{\lambda_n^{k-1}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda_1^k} \\ \frac{1}{\lambda_2^k} \\ \vdots \\ \frac{1}{\lambda_n^k} \end{bmatrix} = \frac{1}{\lambda_1^k} \begin{bmatrix} 1 \\ \left(\frac{\lambda_1}{\lambda_2}\right)^k \\ \vdots \\ \left(\frac{\lambda_1}{\lambda_n}\right)^k \end{bmatrix}
\end{aligned}\right\} \tag{5–10}$$

If $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ it follows from (5-10) that

$$\lim_{k\to\infty} \lambda_1^k \mathbf{q}_k = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{\Psi}^{(1)} \tag{5–11}$$

Hence, the algorithm converge to $\mathbf{\Psi}^{(1)}$ in the modal space. The corresponding convergence to $\mathbf{\Phi}^{(1)}$ then takes place in the physical space.

As seen from (5-11), $|\mathbf{q}_k| \to 0$ if $\lambda_1 > 1$, and $|\mathbf{q}_k| \to \infty$ if $\lambda_1 < 1$. This is the rationale behind the normalization to unit modal mass of the iteration vector, performed at each iteration step in the algorithm in Box 5.1.

The *relative error of the iteration vector* after the kth iteration step is defined from

$$\varepsilon_{1,k} = \frac{\left|\lambda_1^k \mathbf{q}_k - \mathbf{\Psi}^{(1)}\right|}{\left|\mathbf{\Psi}^{(1)}\right|} = \left|\lambda_1^k \mathbf{q}_k - \mathbf{\Psi}^{(1)}\right| = \sqrt{\left(\frac{\lambda_1}{\lambda_2}\right)^{2k} + \left(\frac{\lambda_1}{\lambda_3}\right)^{2k} + \cdots + \left(\frac{\lambda_1}{\lambda_n}\right)^{2k}} =$$

$$\left(\frac{\lambda_1}{\lambda_2}\right)^k \sqrt{1 + \left(\frac{\lambda_2}{\lambda_3}\right)^{2k} + \cdots + \left(\frac{\lambda_2}{\lambda_n}\right)^{2k}} \tag{5–12}$$

From (5-12) follows, that the relative error at large values of k has the magnitude $\varepsilon_{1,k} \simeq \left(\frac{\lambda_1}{\lambda_2}\right)^k$. Based on the asymptotic behavior of the relative error, the *convergence rate* is defined from

$$r_1 = \lim_{k\to\infty} \frac{\varepsilon_{1,k+1}}{\varepsilon_{1,k}} = \lim_{k\to\infty} \frac{\left|\lambda_1^{k+1} \mathbf{q}_{k+1} - \mathbf{\Psi}^{(1)}\right|}{\left|\lambda_1^k \mathbf{q}_k - \mathbf{\Psi}^{(1)}\right|} =$$

$$\lim_{k\to\infty} \frac{\lambda_1}{\lambda_2} \frac{\sqrt{1 + \left(\frac{\lambda_2}{\lambda_3}\right)^{2k+2} + \cdots + \left(\frac{\lambda_2}{\lambda_n}\right)^{2k+2}}}{\sqrt{1 + \left(\frac{\lambda_2}{\lambda_3}\right)^{2k} + \cdots + \left(\frac{\lambda_2}{\lambda_n}\right)^{2k}}} = \frac{\lambda_1}{\lambda_2} \tag{5–13}$$

The last statement of (5-13) presumes that the eigenvalue λ_2 is simple, i.e. that $\lambda_2 < \lambda_3$. It follows from (5-12) that the smaller is the fraction $\frac{\lambda_1}{\lambda_2}$ the faster will the convergence to the first eigenmode be. Hence, the convergence rate as defined by (5-13) should be small (despite linguistic logics suggests the opposite). An vector iteration scheme, where the convergence rate is proportional to $\frac{\lambda_1}{\lambda_2}$ is said to have *linear convergence*. Hence, inverse vector iteration has linear convergence.

The Rayleigh quotient based on $\mathbf{\Phi}_k = \mathbf{\Phi}\mathbf{q}_k$ becomes

$$\rho(\mathbf{q}_k) = \frac{\mathbf{\Phi}_k^T \mathbf{K} \mathbf{\Phi}_k}{\mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} = \frac{\mathbf{q}_k^T \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} \mathbf{q}_k}{\mathbf{q}_k^T \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi} \mathbf{q}_k} = \frac{\mathbf{q}_k^T \mathbf{\Lambda} \mathbf{q}_k}{\mathbf{q}_k^T \mathbf{q}_k} \tag{5–14}$$

The *relative error of the Rayleigh quotient* after the kth iteration step is defined from

$$\varepsilon_{2,k} = \frac{\rho(\mathbf{q}_k) - \lambda_1}{\lambda_1} \tag{5–15}$$

From (5-10) follows that

$$\left.\begin{aligned}
\mathbf{q}_k^T \mathbf{\Lambda} \mathbf{q}_k &= \begin{bmatrix} \frac{1}{\lambda_1^k} \\ \frac{1}{\lambda_2^k} \\ \vdots \\ \frac{1}{\lambda_n^k} \end{bmatrix}^T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1^k} \\ \frac{1}{\lambda_2^k} \\ \vdots \\ \frac{1}{\lambda_n^k} \end{bmatrix} = \frac{1}{\lambda_1^{2k-1}} + \frac{1}{\lambda_2^{2k-1}} + \frac{1}{\lambda_3^{2k-1}} + \cdots + \frac{1}{\lambda_n^{2k-1}} \\
\mathbf{q}_k^T \mathbf{q}_k &= \begin{bmatrix} \frac{1}{\lambda_1^k} \\ \frac{1}{\lambda_2^k} \\ \vdots \\ \frac{1}{\lambda_n^k} \end{bmatrix}^T \begin{bmatrix} \frac{1}{\lambda_1^k} \\ \frac{1}{\lambda_2^k} \\ \vdots \\ \frac{1}{\lambda_n^k} \end{bmatrix} = \frac{1}{\lambda_1^{2k}} + \frac{1}{\lambda_2^{2k}} + \frac{1}{\lambda_3^{2k}} + \cdots + \frac{1}{\lambda_n^{2k}}
\end{aligned}\right\} \tag{5–16}$$

Then, (5-15) may be written as

$$\begin{aligned}
\varepsilon_{2,k} &= \frac{1}{\lambda_1} \frac{\frac{1}{\lambda_1^{2k-1}} + \frac{1}{\lambda_2^{2k-1}} + \frac{1}{\lambda_3^{2k-1}} + \cdots + \frac{1}{\lambda_n^{2k-1}}}{\frac{1}{\lambda_1^{2k}} + \frac{1}{\lambda_2^{2k}} + \frac{1}{\lambda_3^{2k}} + \cdots + \frac{1}{\lambda_n^{2k}}} - 1 = \\
&\frac{1 + \left(\frac{\lambda_1}{\lambda_2}\right)^{2k-1} + \left(\frac{\lambda_1}{\lambda_3}\right)^{2k-1} + \cdots + \left(\frac{\lambda_1}{\lambda_n}\right)^{2k-1}}{1 + \left(\frac{\lambda_1}{\lambda_2}\right)^{2k} + \left(\frac{\lambda_1}{\lambda_3}\right)^{2k} + \cdots + \left(\frac{\lambda_1}{\lambda_n}\right)^{2k}} - 1 = \\
&\left(\frac{\lambda_1}{\lambda_2}\right)^{2k-1} \frac{1 - \frac{\lambda_1}{\lambda_2} + \left(\frac{\lambda_2}{\lambda_3}\right)^{2k-1}\left(1 - \frac{\lambda_1}{\lambda_3}\right) + \cdots + \left(\frac{\lambda_2}{\lambda_n}\right)^{2k-1}\left(1 - \frac{\lambda_1}{\lambda_n}\right)}{1 + \left(\frac{\lambda_1}{\lambda_2}\right)^{2k} + \left(\frac{\lambda_1}{\lambda_3}\right)^{2k} + \cdots + \left(\frac{\lambda_1}{\lambda_n}\right)^{2k}} = \\
&\left(\frac{\lambda_1}{\lambda_2}\right)^{2k-1} \left(1 - \frac{\lambda_1}{\lambda_2} + \cdots\right)
\end{aligned} \tag{5–17}$$

where the dots denote terms, which converge to zero as $k \to \infty$. (5-17) shows that the relative error of the Rayleigh quotient at large values of k has the magnitude $\varepsilon_{2,k} \simeq \left(\frac{\lambda_1}{\lambda_2}\right)^{2k-1}$. Hence, the relative error on the components of the eigenmode at a certain iteration step, as measured by $\varepsilon_{1,k}$, is significantly larger than the relative error on the eigenvalue estimate, as determined by the Rayleigh quotient.

The *convergence rate of the Rayleigh quotient* is defined from

$$r_2 = \lim_{k\to\infty} \frac{\varepsilon_{2,k+1}}{\varepsilon_{2,k}} = \lim_{k\to\infty} \frac{\left(\frac{\lambda_1}{\lambda_2}\right)^{2k+1}\left(1 - \frac{\lambda_1}{\lambda_2} + \cdots\right)}{\left(\frac{\lambda_1}{\lambda_2}\right)^{2k-1}\left(1 - \frac{\lambda_1}{\lambda_2} + \cdots\right)} = \left(\frac{\lambda_1}{\lambda_2}\right)^2 \tag{5–18}$$

Hence, the Rayleigh quotient has *quadratic convergence* in inverse vector iteration.

Example 5.1: Inverse vector iteration

Consider the generalized eigenvalue problem defined by the mass and stiffness matrices in Example 4.1. Calculate the lowest eigenvalue and eigenvector by inverse vector iteration using the inverse iteration algorithm described in Box 5.1 with the start vector

$$\Phi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{5–19}$$

The matrix **A** becomes, cf. (5-5), (4-16)

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 4 \end{bmatrix} \tag{5–20}$$

At the 1st and 2nd iteration step the following calculations are performed

$$\begin{cases} \bar{\Phi}_1 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 7 \\ 8 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_1^T \mathbf{M} \bar{\Phi}_1 = 136 \\ \Phi_1 = \frac{1}{\sqrt{136}} \begin{bmatrix} 3 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 0.25725 \\ 0.51450 \\ 0.60025 \\ 0.68599 \end{bmatrix} \end{cases} \tag{5–21}$$

$$\begin{cases} \bar{\Phi}_2 = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 4 & 0 & 3 \\ 0 & 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0.25725 \\ 0.51450 \\ 0.60025 \\ 0.68599 \end{bmatrix} = \begin{bmatrix} 1.7150 \\ 3.4300 \\ 4.1160 \\ 4.8020 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_2^T \mathbf{M} \bar{\Phi}_2 = 46.588 \\ \Phi_2 = \frac{1}{\sqrt{46.588}} \begin{bmatrix} 1.7150 \\ 3.4300 \\ 4.1160 \\ 4.8020 \end{bmatrix} = \begin{bmatrix} 0.25126 \\ 0.50252 \\ 0.60302 \\ 0.70353 \end{bmatrix} \end{cases} \tag{5–22}$$

The Rayleigh quotient based on Φ_2 provides the following estimate for λ_1, cf. (4-25)

$$\rho(\Phi_2) = \frac{\begin{bmatrix}0.25126\\0.50252\\0.60302\\0.70353\end{bmatrix}^T \begin{bmatrix}2 & -1 & 0 & 0\\-1 & 2 & -1 & 0\\0 & -1 & 2 & -1\\0 & 0 & -1 & 1\end{bmatrix} \begin{bmatrix}0.25126\\0.50252\\0.60302\\0.70353\end{bmatrix}}{\begin{bmatrix}0.25126\\0.50252\\0.60302\\0.70353\end{bmatrix}^T \begin{bmatrix}0 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix}0.25126\\0.50252\\0.60302\\0.70353\end{bmatrix}} = 0.1464646 \tag{5–23}$$

The exact solutions are given as, cf. (4-24)

$$\lambda_1 = \frac{1}{2} - \frac{\sqrt{2}}{4} = 0.1464466 \quad , \quad \Phi^{(1)} = \begin{bmatrix}\frac{1}{4}\\ \frac{1}{2}\\ \frac{1}{4}+\frac{\sqrt{2}}{4}\\ \frac{\sqrt{2}}{2}\end{bmatrix} = \begin{bmatrix}0.25000\\0.50000\\0.60355\\0.70711\end{bmatrix} \tag{5–24}$$

The relative errors, ε_1 and ε_2, on the calculation of the eigenvalue and the 1st component of $\Phi^{(1)}$ becomes

$$\left.\begin{aligned}\varepsilon_{1,2} &= \frac{|\Phi_2 - \Phi^{(1)}|}{|\Phi^{(1)}|} = \frac{0.00458}{1.0848} = 4.22 \cdot 10^{-3}\\ \varepsilon_{2,2} &= \frac{\rho(\Phi_2) - \lambda_1}{\lambda_1} = \frac{0.1464646 - 0.1464466}{0.1464466} = 1.23 \cdot 10^{-4}\end{aligned}\right\} \tag{5–25}$$

As seen the relative error on the components of the eigenmode is significantly larger than the error on the Rayleigh quotient.

The generalized eigenvalue problem (1-9) may be reformulated on the form

$$\begin{aligned}&\mathrm{M}\Phi^{(1)} = \lambda_1 \mathrm{M}\mathrm{K}^{-1}\mathrm{M}\Phi^{(1)} \quad \Rightarrow\\ &\Psi^{(1)} = \lambda_1 \mathrm{M}\mathrm{K}^{-1}\Psi^{(1)} \quad \Rightarrow\\ &\mathrm{K}^{-1}\Psi^{(1)} = \lambda_1 \mathrm{K}^{-1}\mathrm{M}\mathrm{K}^{-1}\Psi^{(1)} \quad , \quad \Psi^{(1)} = \mathrm{M}\Phi^{(1)}\end{aligned} \tag{5–26}$$

From (5-26) the following Rayleigh quotient may be defined

$$\rho(\mathrm{v}) = \frac{\mathrm{v}^T \mathrm{K}^{-1}\,\mathrm{v}}{\mathrm{v}^T \mathrm{K}^{-1}\mathrm{M}\mathrm{K}^{-1}\,\mathrm{v}} \tag{5–27}$$

If $\mathrm{v} = \Psi^{(1)} = \mathrm{M}\Phi^{(1)}$ then (5-4) provides the limit λ_1. An inverse vector iteration procedure based on the formulation (5-26), (5-27) has been indicated in Box 5.2. The lowest eigenmode

$\Phi^{(1)}$ can only be retrieved after convergence, if $\mathbf{M}^{-1}$ exists.

Box 5.2: Alternative inverse vector iteration algorithm

Given start vector Ψ_0. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate $\mathbf{v}_{k+1} = \mathbf{K}^{-1}\Psi_k$.
2. Calculate $\bar{\Psi}_{k+1} = \mathbf{M}\mathbf{v}_{k+1}$.
3. Calculate the Rayleigh quotient (5-29) for the test vector Ψ_k by

$$\rho(\Psi_k) = \frac{\mathbf{v}_{k+1}^T \Psi_k}{\mathbf{v}_{k+1}^T \bar{\Psi}_{k+1}} \quad \left(= \frac{\Psi_k^T \mathbf{K}^{-1} \Psi_k}{\Psi_k^T \mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1} \Psi_k} \right)$$

4. Normalize the new solution vector, so $\Psi_{k+1}^T \mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1} \Psi_{k+1} = 1$

$$\Psi_{k+1} = \frac{\bar{\Psi}_{k+1}}{\sqrt{\mathbf{v}_{k+1}^T \bar{\Psi}_{k+1}}} \quad \left(= \frac{\bar{\Psi}_{k+1}}{\sqrt{\Psi_k^T \mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1} \Psi_k}} \right)$$

5. After convergence the lowest eigenmode at the same iteration step is calculated from $\Phi_{k+1} = \mathbf{M}^{-1}\Psi_{k+1}$.

Example 5.2: Alternative inverse vector iteration

Consider the generalized eigenvalue problem defined by the mass and stiffness matrices in Example 1.5. Calculate the lowest eigenvalue and eigenvector by inverse vector iteration using the alternative inverse vector iteration algorithm in Box 5.2 with the start vector

$$\Phi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{5–28}$$

The inverse stiffness matrix becomes, cf. (1-77)

$$\mathbf{K}^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} = \frac{1}{12}\begin{bmatrix} 7 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 7 \end{bmatrix} \tag{5–29}$$

At the 1st and 2nd iteration steps the following calculations are performed

$$
\begin{cases}
\mathbf{v}_1 = \dfrac{1}{12}\begin{bmatrix} 7 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{1}{6}\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} \\[2ex]
\bar{\mathbf{\Psi}}_1 = \dfrac{1}{6}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \dfrac{1}{12}\begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix} \quad , \quad \mathbf{v}_1^T\bar{\mathbf{\Psi}}_1 = \dfrac{1}{6\cdot 12}\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}^T\begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix} = \dfrac{41}{36} \\[2ex]
\rho(\mathbf{\Psi}_0) = \dfrac{\mathbf{v}_1^T\mathbf{\Psi}_0}{\mathbf{v}_1^T\bar{\mathbf{\Psi}}_1} = \dfrac{36}{6\cdot 41}\begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}^T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \dfrac{84}{41} = 2.0488 \\[2ex]
\mathbf{\Psi}_1 = \dfrac{\bar{\mathbf{\Psi}}_1}{\sqrt{\mathbf{v}_1^T\bar{\mathbf{\Psi}}_1}} = \dfrac{1}{12\cdot\sqrt{\frac{41}{36}}}\begin{bmatrix} 5 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 0.3904 \\ 0.6247 \\ 0.3904 \end{bmatrix}
\end{cases}
\tag{5–30}
$$

$$
\begin{cases}
\mathbf{v}_2 = \dfrac{1}{12}\begin{bmatrix} 7 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} 0.3904 \\ 0.6247 \\ 0.3904 \end{bmatrix} = \begin{bmatrix} 0.3644 \\ 0.3384 \\ 0.3644 \end{bmatrix} \\[2ex]
\bar{\mathbf{\Psi}}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0.3644 \\ 0.3384 \\ 0.3644 \end{bmatrix} = \begin{bmatrix} 0.1822 \\ 0.3384 \\ 0.1822 \end{bmatrix} \quad , \quad \mathbf{v}_2^T\bar{\mathbf{\Psi}}_2 = \begin{bmatrix} 0.3644 \\ 0.3384 \\ 0.3644 \end{bmatrix}^T\begin{bmatrix} 0.1822 \\ 0.3384 \\ 0.1822 \end{bmatrix} = 0.2473 \\[2ex]
\rho(\mathbf{\Psi}_1) = \dfrac{\mathbf{v}_2^T\mathbf{\Psi}_1}{\mathbf{v}_2^T\bar{\mathbf{\Psi}}_2} = \dfrac{1}{0.2473}\begin{bmatrix} 0.3644 \\ 0.3384 \\ 0.3644 \end{bmatrix}^T\begin{bmatrix} 0.3904 \\ 0.6247 \\ 0.3904 \end{bmatrix} = 2.0055 \\[2ex]
\mathbf{\Psi}_2 = \dfrac{\bar{\mathbf{\Psi}}_2}{\sqrt{\mathbf{v}_2^T\bar{\mathbf{\Psi}}_2}} = \dfrac{1}{\sqrt{0.2473}}\begin{bmatrix} 0.1822 \\ 0.3384 \\ 0.1822 \end{bmatrix} = \begin{bmatrix} 0.3664 \\ 0.6805 \\ 0.3664 \end{bmatrix}
\end{cases}
\tag{5–31}
$$

The lowest eigenvector at the end of the 2nd iteration step becomes

$$
\mathbf{\Phi}_2 = \mathbf{M}^{-1}\mathbf{\Psi}_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}^{-1}\begin{bmatrix} 0.3664 \\ 0.6805 \\ 0.3664 \end{bmatrix} = \begin{bmatrix} 0.7328 \\ 0.6805 \\ 0.7328 \end{bmatrix} \tag{5–32}
$$

The exact solutions are given as, cf. (1-87)

$$
\mathbf{\Phi}^{(1)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \\ 0.7071 \end{bmatrix} \quad , \quad \lambda_1 = 2 \tag{5–33}
$$

As for the simple formulation of the inverse vector iteration algorithm the convergence towards the exact eigenvalue takes place as a monotonously decreasing sequence of upper values, $\rho(\mathbf{\Psi}_0), \rho(\mathbf{\Psi}_1), \ldots$.

The principle in forward vector iteration may also be explained based on the eigenvalue problem (5-1). Given a start vector, Φ_0, a new vector Φ_1 may be calculated as follows

$$\mathrm{K}\Phi_0 = \mathrm{M}\Phi_1 \quad \Rightarrow \quad \Phi_1 = \mathrm{M}^{-1}\mathrm{K}\Phi_0 = \mathrm{B}\Phi_0 \tag{5–34}$$

where

$$\mathrm{B} = \mathrm{M}^{-1}\mathrm{K} \tag{5–35}$$

Clearly, if $\Phi_0 = \Phi^{(j)}$ is an eigenmode, then $\Phi_1 = \lambda_j \Phi_0$. If not so, a new and better approximation Φ_2 may be calculated based on Φ_1 as follows

$$\Phi_2 = \mathrm{B}\Phi_1 \tag{5–36}$$

The process may be continued until converge is obtained. The forward vector iteration algorithm may be summarized as follows

Box 5.3: Forward vector iteration algorithm

Given start vector Φ_0, which needs not be normalized to unit modal mass. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate $\bar{\Phi}_{k+1} = \mathrm{B}\Phi_k$.
2. Normalize solution vector to unit modal mass, so $\Phi_{k+1}^T \mathrm{M}\Phi_{k+1} = 1$

$$\Phi_{k+1} = \frac{\bar{\Phi}_{k+1}}{\sqrt{\bar{\Phi}_{k+1}^T \mathrm{M}\bar{\Phi}_{k+1}}}$$

Obviously, the algorithm requires that the mass matrix is non-singular, so the inverse M^{-1} exists. By contrast the stiffness matrix needs not be non-singular. After convergence the eigenvalue is calculated from the Rayleigh quotient.

In case the largest eigenvalue is simple, i.e. that $\lambda_{n-1} < \lambda_n$, the forward iteration algorithm converges towards the largest eigenpair $\left(\lambda_n, \Phi^{(n)}\right)$. The convergence rate of the eigenmode estimate is linear, and the convergence rate of the Rayleigh quotient is quadratic in the fraction $\frac{\lambda_{n-1}}{\lambda_n}$. A proof of this has been given in Section 5.3.

Example 5.3: Forward vector iteration

Consider the generalized eigenvalue problem defined by the mass and stiffness matrices in Example 1.4. Calculate the largest eigenvalue and eigenvector by forward vector iteration using the forward vector iteration algorithm in Box 5.3 with the start vector

$$\Phi_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5–37}$$

The matrix $\mathbf{B}$ becomes, cf. (5-35), (1-77)

$$\mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix} \tag{5–38}$$

At the 1st and 2nd iteration step the following calculations are performed

$$\begin{cases} \bar{\Phi}_1 = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_1^T \mathbf{M} \bar{\Phi}_1 = 9 \\ \Phi_1 = \dfrac{1}{\sqrt{9}} \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.3333 \\ -0.3333 \\ 0 \end{bmatrix} \end{cases} \tag{5–39}$$

$$\begin{cases} \bar{\Phi}_2 = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 4 & -1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1.3333 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} 6.0000 \\ -2.6667 \\ 0.6667 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_2^T \mathbf{M} \bar{\Phi}_2 = 25.333 \\ \Phi_2 = \dfrac{1}{\sqrt{25.333}} \begin{bmatrix} 6.0000 \\ -2.6667 \\ 0.6667 \end{bmatrix} = \begin{bmatrix} 1.1921 \\ -0.5298 \\ 0.1325 \end{bmatrix} \end{cases} \tag{5–40}$$

The Rayleigh quotient based on Φ_2 becomes

$$\rho(\Phi_2) = \frac{\begin{bmatrix} 1.1921 \\ -0.5298 \\ 0.1325 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1.1921 \\ -0.5298 \\ 0.1325 \end{bmatrix}}{\begin{bmatrix} 1.1921 \\ -0.5298 \\ 0.1325 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1.1921 \\ -0.5298 \\ 0.1325 \end{bmatrix}} = 5.404 \tag{5–41}$$

The results for the iteration vector and the Rayleigh quotient in the succeeding iteration steps become

$$\left.\begin{aligned}
\Phi_3 &= \begin{bmatrix} 1.0622 \\ -0.6276 \\ 0.2897 \end{bmatrix} \quad , \quad \rho(\Phi_3) = 5.697 \\
\Phi_4 &= \begin{bmatrix} 0.9584 \\ -0.6726 \\ 0.4204 \end{bmatrix} \quad , \quad \rho(\Phi_4) = 5.855 \\
\Phi_5 &= \begin{bmatrix} 0.8811 \\ -0.6923 \\ 0.5149 \end{bmatrix} \quad , \quad \rho(\Phi_5) = 5.933
\end{aligned}\right\} \tag{5–42}$$

The exact solutions becomes, cf. (6-54)

$$\Phi^{(3)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ -0.7071 \\ 0.7071 \end{bmatrix} \quad , \quad \lambda_3 = 6 \tag{5–43}$$

The relative slow convergence of the algorithm to the exact solution is because the fraction $\frac{\lambda_2}{\lambda_3} = \frac{4}{6}$ is relatively high. Theoretically the relative errors of the Rayleigh quotient after 5 iterations should be of the magnitude, cf. (5-17)

$$\varepsilon_{2,5} \simeq \left(\frac{4}{6}\right)^{2\cdot 5-1}\left(1 - \frac{4}{6}\right) = 0.0087 \tag{5–44}$$

Actually, the error is slightly larger, namely

$$\varepsilon_{2,5} = \frac{6 - 5.933}{6} = 0.0112 \tag{5–45}$$

5.3 Shift in Vector Iteration

Shift on the stiffness matrix in the eigenvalue problem (5-1) as indicated by (3-36)-(3-38) may be appropriate both in relation to inverse and forward vector iteration, either in order to obtain convergence to other eigen-pairs than $(\lambda_1, \Phi^{(1)})$ or $(\lambda_n, \Phi^{(n)})$, or to improve the convergence rate of the iteration process.

Let $\hat{\mathrm{K}} = \mathrm{K} - \rho\mathrm{M}$. denote a shift on the stiffness matrix as indicated by (3-38). The vector iteration is next performed on the shifted eigenvalue problem (3-37). The algorithms in Box 5.1 and 5.3 remain unchanged, if the matrices A and B in (5-6) and (5-35) are redefined as follows

$$\mathrm{A} = \hat{\mathrm{K}}^{-1}\mathrm{M} \tag{5–46}$$

$$\mathrm{B} = \mathrm{M}^{-1}\hat{\mathrm{K}} \tag{5–47}$$

The Rayleigh quotient estimate of the eigenvalue λ_j after the kth iteration step becomes

$$\bar{\lambda}_j = \rho(\mathbf{\Phi}_k) + \rho = \frac{\mathbf{\Phi}_k^T \hat{\mathbf{K}} \mathbf{\Phi}_k}{\mathbf{\Phi}_k^T \mathbf{M} \mathbf{\Phi}_k} + \rho \tag{5–48}$$

In the modal space the inverse vector iteration with shift on the stiffness matrix can be written as, cf. (5-8)

$$\hat{\mathbf{K}}\mathbf{\Phi}\bar{\mathbf{q}}_{k+1} = \mathbf{M}\mathbf{\Phi}\mathbf{q}_k \qquad \Rightarrow$$

$$\mathbf{\Phi}^T\Big(\mathbf{K} - \rho\mathbf{M}\Big)\mathbf{\Phi}\,\bar{\mathbf{q}}_{k+1} = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi}\,\mathbf{q}_k \qquad \Rightarrow$$

$$\Big(\mathbf{\Lambda} - \rho\mathbf{I}\Big)\bar{\mathbf{q}}_{k+1} = \mathbf{q}_k \tag{5–49}$$

(5-49) is identical to (5-8), if λ_j is replaced with $\lambda_j - \rho$. With the same start vector $\mathbf{q}_0 = [1, \ldots, 1]^T$ as in (5-10), the solution vector after the kth iteration step becomes, cf. (5-10)

$$\mathbf{q}_k = \begin{bmatrix} \frac{1}{(\lambda_1-\rho)^k} \\ \vdots \\ \frac{1}{(\lambda_{j-1}-\rho)^k} \\ \frac{1}{(\lambda_j-\rho)^k} \\ \frac{1}{(\lambda_{j+1}-\rho)^k} \\ \vdots \\ \frac{1}{(\lambda_n-\rho)^k} \end{bmatrix} = \frac{1}{(\lambda_j-\rho)^k} \begin{bmatrix} \left(\frac{\lambda_j-\rho}{\lambda_1-\rho}\right)^k \\ \vdots \\ \left(\frac{\lambda_j-\rho}{\lambda_{j-1}-\rho}\right)^k \\ 1 \\ \left(\frac{\lambda_j-\rho}{\lambda_{j+1}-\rho}\right)^k \\ \vdots \\ \left(\frac{\lambda_j-\rho}{\lambda_n-\rho}\right)^k \end{bmatrix} \tag{5–50}$$

where the jth eigenvalue fulfills

$$|\lambda_j - \rho| = \min_{i=1,\ldots,n} |\lambda_i - \rho| \tag{5–51}$$

It then follows from (5-50) that

$$\lim_{k\to\infty} \left(\lambda_j - \rho\right)^k \mathbf{q}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{\Psi}^{(j)} \tag{5–52}$$

Hence, for a value of ρ fulfilling (5-51) the algorithm converge to $\Psi^{(j)}$ in the modal space. In physical space the algorithm then converge to $\Phi^{(j)}$. The convergence rate of the eigenmode becomes, cf. (5-13)

$$r_1 = \max\left(\left|\frac{\lambda_j - \rho}{\lambda_{j-1} - \rho}\right|, \left|\frac{\lambda_j - \rho}{\lambda_{j+1} - \rho}\right|\right) \tag{5–53}$$

Then, the corresponding convergence rate of the Rayleigh quotient is given as $r_2 = r_1^2$.

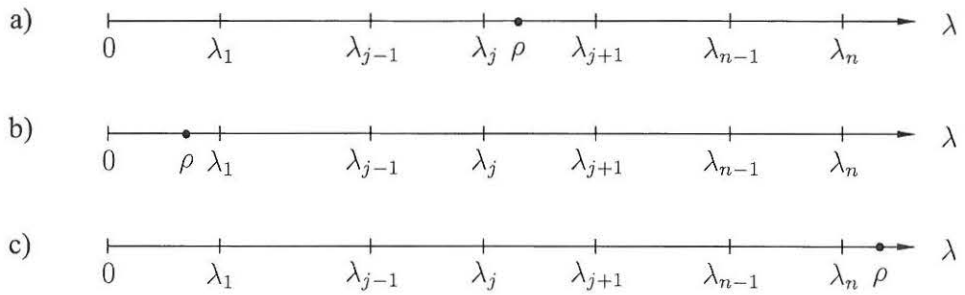

Fig. 5–1 Optimal position of shift parameter at inverse vector iteration. a) Convergence towards λ_j. b) Convergence towards λ_1. c) Convergence towards λ_n.

In case inverse vector iteration towards the jth eigenmode is attempted, the shift parameter should be place in the vicinity of λ_j as shown on Fig. 5.1a in order to obtain a small convergence rate. It should be emphasized that any inverse vector iteration with shift should be accompanied with a Sturm sequence check to insure that the calculated eigenvalue is indeed the λ_j.

At inverse vector iteration towards the lowest eigenmode the convergence rate $r_1 = |\lambda_1 - \rho|/|\lambda_2 - \rho|$ should be minimized. Hence, ρ should be placed close to but below λ_1, as shown on Fig. 5.1b.

At inverse vector iteration towards the highest eigenmode the convergence rate $r_1 = |\lambda_{n-1} - \rho|/|\lambda_n - \rho|$ should be minimized. Hence, ρ should be placed close to but above λ_n, as shown on Fig. 5.1c.

In case of forward iteration with shift, (5-49) provides the solution after k iterations

$$\mathbf{q}_k = \begin{bmatrix} (\lambda_1 - \rho)^k \\ \vdots \\ (\lambda_{j-1} - \rho)^k \\ (\lambda_j - \rho)^k \\ (\lambda_{j+1} - \rho)^k \\ \vdots \\ (\lambda_n - \rho)^k \end{bmatrix} = (\lambda_j - \rho)^k \begin{bmatrix} \left(\frac{\lambda_1 - \rho}{\lambda_j - \rho}\right)^k \\ \vdots \\ \left(\frac{\lambda_{j-1} - \rho}{\lambda_j - \rho}\right)^k \\ 1 \\ \left(\frac{\lambda_{j+1} - \rho}{\lambda_j - \rho}\right)^k \\ \vdots \\ \left(\frac{\lambda_n - \rho}{\lambda_j - \rho}\right)^k \end{bmatrix} \tag{5–54}$$

where the jth eigenvalue fulfills

$$|\lambda_j - \rho| = \max_{i=1,\ldots,n} |\lambda_i - \rho| \tag{5–55}$$

Clearly, (5-55) has the solutions $\lambda_j = \lambda_1$ or $\lambda_j = \lambda_n$. The former occurs, if ρ is closest to λ_n, and the latter if ρ is closest to λ_1. Then, it follows from (5-54) that

$$\lim_{k\to\infty} \frac{1}{(\lambda_j - \rho)^k} \mathbf{q}_k = \mathbf{\Psi}^{(j)} \quad , \quad j = 1, n \tag{5–56}$$

For a value of ρ fulfilling (5-55) the algorithm converge to $\mathbf{\Psi}^{(j)}$ in the modal space, or to $\mathbf{\Phi}^{(j)}$ in the physical space. Forward iteration with shift always converge to either the lowest or the highest eigenmode depending on the magnitude of the shift parameter. The convergence rate of the iteration vector becomes

$$r_1 = \max\left(\left|\frac{\lambda_1 - \rho}{\lambda_j - \rho}\right|, \ldots, \left|\frac{\lambda_{j-1} - \rho}{\lambda_j - \rho}\right|, \left|\frac{\lambda_{j+1} - \rho}{\lambda_j - \rho}\right|, \ldots, \left|\frac{\lambda_n - \rho}{\lambda_j - \rho}\right|\right) \tag{5–57}$$

Shift in forward vector iteration is not as useful as in inverse vector iteration, because the optimal choice of the shift parameter is more difficult to specify. At forward vector iteration towards the highest eigenmode the optimal shift parameter is typically placed somewhere in the middle of the eigenvalue spectrum. Especially for $\rho = 0$, (5-57) becomes

$$r_1 = \frac{\lambda_{n-1}}{\lambda_n} \tag{5–58}$$

as stated in Section 5.2 on forward iteration without shift.

Example 5.4: Forward vector iteration with shift

The problem in Example 5.3 is considered again. However, now a shift with $\rho = 3$ is performed on the stiffness matrix.

The matrix $\hat{\mathbf{K}}$ becomes, cf. (3-38), (1-77)

$$\hat{\mathbf{K}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \tag{5–59}$$

The matrix $\mathbf{B}$ becomes, cf. (5-47), (1-77)

$$\mathbf{B} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \tag{5–60}$$

At the 1st and 2nd iteration step the following calculations are performed

$$\begin{cases} \bar{\mathbf{\Phi}}_1 = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = 1.5 \\ \mathbf{\Phi}_1 = \dfrac{1}{\sqrt{1.5}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8165 \\ -0.8165 \\ 0 \end{bmatrix} \end{cases} \tag{5–61}$$

$$\begin{cases} \bar{\mathbf{\Phi}}_2 = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.8165 \\ -0.8165 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.4495 \\ -1.6330 \\ 1.6330 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = 7 \\ \mathbf{\Phi}_2 = \dfrac{1}{\sqrt{7}} \begin{bmatrix} 2.4495 \\ -1.6330 \\ 1.6330 \end{bmatrix} = \begin{bmatrix} 0.9258 \\ -0.6172 \\ 0.6172 \end{bmatrix} \end{cases} \tag{5–62}$$

The Rayleigh quotient estimate of λ_3 based on $\mathbf{\Phi}_2$ becomes, cf. (5-48)

$$\bar{\lambda}_3 = \rho(\mathbf{\Phi}_2) + 3 = \frac{\begin{bmatrix} 0.9258 \\ -0.6172 \\ 0.6172 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.9258 \\ -0.6172 \\ 0.6172 \end{bmatrix}}{\begin{bmatrix} 0.9258 \\ -0.6172 \\ 0.6172 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.9258 \\ -0.6172 \\ 0.6172 \end{bmatrix}} + 3 = 2.9048 + 3 = 5.9048 \tag{5–63}$$

The results for the iteration vector and the eigenvalue estimate in the succeeding iteration steps become

$$\left.\begin{aligned}
\Phi_3 &= \begin{bmatrix} 0.7318 \\ -0.7318 \\ 0.6273 \end{bmatrix} \quad , \quad \bar{\lambda}_3 = 5.9891 \\
\Phi_4 &= \begin{bmatrix} 0.7331 \\ -0.6982 \\ 0.6982 \end{bmatrix} \quad , \quad \bar{\lambda}_3 = 5.9988 \\
\Phi_5 &= \begin{bmatrix} 0.7100 \\ -0.7100 \\ 0.6983 \end{bmatrix} \quad , \quad \bar{\lambda}_3 = 5.9999
\end{aligned}\right\} \tag{5–64}$$

The results in (5-64) should be compared to those in (5-42). As seen the convergence of the shifted problem is much faster.

5.4 Inverse Vector Iteration with Rayleigh Quotient Shift

As demonstrated in Section 5.3 the convergence properties of inverse vector towards the lowest mode are improved if a shift on the stiffness matrix is performed with a shift parameter fulfilling $\rho \simeq \lambda_1$. The idea in the present section is to update the shift parameter at each iteration step with the most recent estimate of the lowest eigenvalue. Assume, that an estimate of the eigenvalue $\bar{\lambda}_1$ is known after the kth iteration step. Then, a shift with the parameter $\rho_k = \bar{\lambda}_1$ is performed, so a new un-normalized eigenmode estimate is calculated at the $(k+1)$th iteration step from

$$\bar{\Phi}_{k+1} = \left(\mathbf{K} - \rho_k \mathbf{M}\right)^{-1} \mathbf{M} \Phi_k \tag{5–65}$$

where Φ_k is the normalized estimate of the eigenmode after the kth iteration step.

A new estimate of the eigenvalue, and hence the shift parameter, then follows from (5-48)

$$\rho_{k+1} = \frac{\bar{\Phi}_{k+1}^T \left(\mathbf{K} - \rho_k \mathbf{M}\right) \bar{\Phi}_{k+1}}{\bar{\Phi}_{k+1}^T \mathbf{M} \bar{\Phi}_{k+1}} + \rho_k \tag{5–66}$$

The convergence towards $\left(\lambda_1, \Phi^{(1)}\right)$ is not safe, since the first shift determined by ρ_1 may cause convergence towards other eigen-pairs, especially if the first and second eigenvalue are close. For this reason the first couples of iteration steps are often performed without shift. When the convergence towards the first eigen-pair takes place, the convergence rate of the Rayleigh quotient estimate of the eigenvalue will be cubic, i.e. $r_2 = (\frac{\lambda_1}{\lambda_2})^3$. Additionally, the length of the converge process is very much dependent on the start vector, as demonstrated in the succeeding Example 5.5. Even though the convergence may be fast it should be realized that the process requires inversion of the matrix $\mathbf{K} - \rho_k \mathbf{M}$ at each iteration step, which may be expensive for

large systems.

Box 5.4: Algorithm for inverse vector iteration with Rayleigh quotient shift

Given start vector Φ_0, which needs not be normalized to unit modal mass, and set the initial shift to $\rho_0 = 0$. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate $\bar{\Phi}_{k+1} = \left(\mathrm{K} - \rho_k \mathrm{M}\right)^{-1} \mathrm{M} \Phi_k$.
2. Calculate new shift parameter (new estimate on the eigenvalue) from the Rayleigh quotient estimate based on $\bar{\Phi}_{k+1}$ by

$$\rho_{k+1} = \frac{\bar{\Phi}_{k+1}^T \left(\mathrm{K} - \rho_k \mathrm{M}\right) \bar{\Phi}_{k+1}}{\bar{\Phi}_{k+1}^T \mathrm{M} \bar{\Phi}_{k+1}} + \rho_k \qquad \left(\text{estimate on } \lambda_1\right)$$

3. Normalize the new solution vector to unit modal mass

$$\Phi_{k+1} = \frac{\bar{\Phi}_{k+1}}{\sqrt{\bar{\Phi}_{k+1}^T \mathrm{M} \bar{\Phi}_{k+1}}}$$

Example 5.5: Inverse vector iteration with Rayleigh quotient shift

Consider the generalized eigenvalue problem defined by the mass and stiffness matrices in Example 1.4. Calculate the lowest eigenvalue and eigenvector by inverse vector iteration with Rayleigh quotient shift with the start vector

$$\Phi_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5–67}$$

At the 1st and 2nd iteration step the following calculations are performed

$$
\left\{
\begin{aligned}
\hat{\mathbf{K}} &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} - 0 \cdot \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\
\bar{\mathbf{\Phi}}_1 &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.2917 \\ 0.0833 \\ 0.0417 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = 0.05035 \\
\rho_1 &= \frac{1}{0.05035} \begin{bmatrix} 0.2917 \\ 0.0833 \\ 0.0417 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.2917 \\ 0.0833 \\ 0.0417 \end{bmatrix} + 0 = 2.8966 \\
\mathbf{\Phi}_1 &= \frac{1}{\sqrt{0.05035}} \begin{bmatrix} 0.2917 \\ 0.0833 \\ 0.0417 \end{bmatrix} = \begin{bmatrix} 1.2999 \\ 0.3714 \\ 0.1857 \end{bmatrix}
\end{aligned}
\right. \tag{5–68}
$$

$$
\left\{
\begin{aligned}
\hat{\mathbf{K}} &= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} - 2.8966 \cdot \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.5517 & -1.0000 & 0.0000 \\ -1.0000 & 1.1034 & -1.0000 \\ 0.0000 & -1.0000 & 0.5517 \end{bmatrix} \\
\bar{\mathbf{\Phi}}_2 &= \begin{bmatrix} 0.5517 & -1.0000 & 0.0000 \\ -1.0000 & 1.1034 & -1.0000 \\ 0.0000 & -1.0000 & 0.5517 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1.2999 \\ 0.3714 \\ 0.1857 \end{bmatrix} = \begin{bmatrix} -0.0567 \\ -0.6812 \\ -1.0664 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_2^T \mathbf{M} \bar{\mathbf{\Phi}}_2 = 1.0342 \\
\rho_2 &= \frac{1}{1.0342} \begin{bmatrix} -0.0567 \\ -0.6812 \\ -1.0664 \end{bmatrix}^T \begin{bmatrix} 0.5517 & -1.0000 & 0.0000 \\ -1.0000 & 1.1034 & -1.0000 \\ 0.0000 & -1.0000 & 0.5517 \end{bmatrix} \begin{bmatrix} -0.0567 \\ -0.6812 \\ -1.0664 \end{bmatrix} + 2.8966 = 2.5206 \\
\mathbf{\Phi}_2 &= \frac{1}{\sqrt{1.0342}} \begin{bmatrix} -0.0567 \\ -0.6812 \\ -1.0664 \end{bmatrix} = \begin{bmatrix} -0.0557 \\ -0.6698 \\ -1.0486 \end{bmatrix}
\end{aligned}
\right. \tag{5–69}
$$

The results for the iteration vector and the eigenvalue estimate in the succeeding iteration steps become

$$
\left.
\begin{aligned}
\mathbf{\Phi}_3 &= \begin{bmatrix} 0.9011 \\ 0.6830 \\ 0.5049 \end{bmatrix} \quad ; \quad \rho_3 = 2.0793 \\
\mathbf{\Phi}_4 &= \begin{bmatrix} -0.6985 \\ -0.7073 \\ -0.7152 \end{bmatrix} \quad ; \quad \rho_4 = 2.0001
\end{aligned}
\right\} \tag{5–70}
$$

Despite the shifts the convergence is very slow during the 1st and 2nd iteration step. Not until the 3rd and 4th step a fast speed-up of the convergence takes place. This is due to the poor guess of the start vector.

5.5 Vector Iteration with Gram-Schmidt Orthogonalization

Inverse vector iteration or forward vector iteration with Gram-Schmidt orthogonalization is used, when other eigen-pairs than $\left(\lambda_1, \Phi^{(1)}\right)$ or $\left(\lambda_n, \Phi^{(n)}\right)$ are wanted.

Assume, that the eigenmodes $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(m)}$, $m < n$, have been determined. Next, the eigenmode $\Phi^{(m+1)}$ is wanted using inverse vector iteration by means of the algorithm in Box 5.1. In order to prevent the algorithm to converge toward $\Phi^{(1)}$ a cleansing of the vector Φ_{k+1} for information about the first m eigenmodes is performed by a so-called *Gram-Schmidt orthogonalization* . In this respect the following modified iteration vector iteration algorithm is considered

$$\hat{\Phi}_{k+1} = \bar{\Phi}_{k+1} - \sum_{j=1}^{m} c_j \Phi^{(j)} \tag{5–71}$$

Inspired by the variational problem (4-31), where the test vector **v** is chosen to be **M**-orthogonal to the previous determined eigenmodes, the modified iteration vector $\hat{\Phi}_{k+1}$ is chosen to be M-orthogonal on $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(m)}$, i.e.

$$\Phi^{(i)T}\mathrm{M}\hat{\Phi}_{k+1} = 0 \quad , \quad i = 1, \ldots, m \tag{5–72}$$

(5-71) is premultiplied with $\Phi^{(i)T}\mathrm{M}$. Assuming that the calculated eigenmodes have been normalized to unit modal mass, it follows from (1-17), (5-71) and (5-72) that the expansion coefficients $c_1, c_2, \ldots, c_m$ are determined from

$$0 = \Phi^{(i)T}\mathrm{M}\bar{\Phi}_{k+1} - \sum_{j=1}^{m} c_j \Phi^{(i)T}\mathrm{M}\Phi^{(j)} = \Phi^{(i)T}\mathrm{M}\bar{\Phi}_{k+1} - c_i \quad \Rightarrow$$

$$c_i = \Phi^{(i)T}\mathrm{M}\bar{\Phi}_{k+1} \tag{5–73}$$

After insertion of the calculated expansion coefficients into (5-71), $\hat{\Phi}_{k+1}$ is considered as the estimate to $\Phi^{(m+1)}$ at the $(k+1)$th iteration step. The convergence takes place with the linear convergence rate $r_1 = \frac{\lambda_{m+1}}{\lambda_{m+2}}$.

In principle the orthogonalization process need only to be performed after the first iteration step, since all succeeding iteration vectors then will be orthogonal to the subspace spanned by $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(m)}$. However, round-off errors inevitable introduce information about the first eigenmode. Obviously, the use of this so-called *vector deflation method* becomes increasingly cumbersome as m increases.

A similar orthogonalization process can be performed in relation to forward vector iteration to ensure convergence to eigenmodes somewhat lower than the highest.

Box 5.5: Algorithm for inverse vector iteration with Gram-Schmidt orthogonalization

Given start vector Φ_0, which needs not be normalized to unit modal mass. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate $\bar{\Phi}_{k+1} = \mathbf{K}^{-1}\mathbf{M}\Phi_k$.
2. Orthogonalize iteration vector to previous calculated eigenmodes $\Phi^{(j)}$, $j = 1, \ldots, m$

$$\hat{\Phi}_{k+1} = \bar{\Phi}_{k+1} - \sum_{j=1}^{m} c_j \Phi^{(j)} \quad , \quad c_j = \Phi^{(j)T}\mathbf{M}\bar{\Phi}_{k+1}$$

3. Normalize the orthogonalized iteration vector to unit modal mass

$$\Phi_{k+1} = \frac{\hat{\Phi}_{k+1}}{\sqrt{\hat{\Phi}_{k+1}^T \mathbf{M}\, \hat{\Phi}_{k+1}}}$$

Example 5.6: Inverse and forward vector iteration with Gram-Schmidt orthogonalization

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \tag{5–74}$$

Further, assume that the lowest and highest eigenmodes have been determined by inverse and forward vector iteration

$$\Phi^{(1)} = \begin{bmatrix} 0.31263 \\ 0.49548 \\ 0.47912 \\ 0.28979 \end{bmatrix} \quad , \quad \Phi^{(4)} = \begin{bmatrix} -0.10756 \\ 0.25563 \\ -0.72825 \\ 0.56197 \end{bmatrix} \tag{5–75}$$

Calculate $\Phi^{(2)}$ by inverse vector iteration with deflation, and $\Phi^{(3)}$ by forward vector iteration with deflation. In both cases the following start vector is used

$$\Phi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{5–76}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ become

$$\left.\begin{aligned}
\mathbf{A} = \mathbf{K}^{-1}\mathbf{M} &= \begin{bmatrix} 2.4 & 3.2 & 1.4 & 0.8 \\ 3.2 & 5.2 & 2.4 & 1.4 \\ 2.8 & 4.8 & 2.6 & 1.6 \\ 1.6 & 2.8 & 1.6 & 1.2 \end{bmatrix} \\
\mathbf{B} = \mathbf{M}^{-1}\mathbf{K} &= \begin{bmatrix} 2.5 & -2.0 & 0.5 & 0.0 \\ -2.0 & 3.0 & -2.0 & 0.5 \\ 1.0 & -4.0 & 6.0 & -4.0 \\ 0.0 & 1.0 & -4.0 & 5.0 \end{bmatrix}
\end{aligned}\right\} \tag{5–77}$$

At the 1st iteration step in the inverse iteration process towards $\mathbf{\Phi}^{(2)}$ the following calculations are performed

$$\left\{\begin{aligned}
\bar{\mathbf{\Phi}}_1 &= \begin{bmatrix} 2.4 & 3.2 & 1.4 & 0.8 \\ 3.2 & 5.2 & 2.4 & 1.4 \\ 2.8 & 4.8 & 2.6 & 1.6 \\ 1.6 & 2.8 & 1.6 & 1.2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7.8 \\ 11.2 \\ 11.8 \\ 7.2 \end{bmatrix} \quad \Rightarrow \quad c_1 = \mathbf{\Phi}^{(1)\,T}\mathbf{M}\bar{\mathbf{\Phi}}_1 = 24.7067 \\
\hat{\mathbf{\Phi}}_1 &= \begin{bmatrix} 7.8 \\ 11.2 \\ 11.8 \\ 7.2 \end{bmatrix} - 24.7067 \begin{bmatrix} 0.31263 \\ 0.49548 \\ 0.47912 \\ 0.28979 \end{bmatrix} = \begin{bmatrix} 0.07595 \\ -0.04158 \\ -0.03740 \\ 0.04016 \end{bmatrix} \quad \Rightarrow \quad \hat{\mathbf{\Phi}}_1^T\mathbf{M}\hat{\mathbf{\Phi}}_1 = 0.01801 \\
\mathbf{\Phi}_1 &= \frac{1}{\sqrt{0.01801}} \begin{bmatrix} 0.07595 \\ -0.04158 \\ -0.03740 \\ 0.04016 \end{bmatrix} = \begin{bmatrix} 0.56599 \\ -0.30989 \\ -0.27871 \\ 0.29927 \end{bmatrix}
\end{aligned}\right. \tag{5–78}$$

The results for the iteration vector in the succeeding iteration steps become

$$\left.\begin{aligned}
\mathbf{\Phi}_2 &= \begin{bmatrix} 0.61639 \\ -0.14318 \\ -0.42383 \\ -0.13960 \end{bmatrix} \\
\mathbf{\Phi}_3 &= \begin{bmatrix} 0.53412 \\ 0.02582 \\ -0.48439 \\ -0.43985 \end{bmatrix} \\
&\vdots \\
\mathbf{\Phi}_{13} &= \begin{bmatrix} 0.44527 \\ 0.12443 \\ -0.48944 \\ -0.57702 \end{bmatrix}
\end{aligned}\right\} \tag{5–79}$$

The process converged with the indicated digit after 13 iterations.

At the 1st iteration step in the forward iteration process towards $\Phi^{(3)}$ the following calculations are performed

$$\left\{\begin{array}{l}
\bar{\Phi}_1 = \begin{bmatrix} 2.5 & -2.0 & 0.5 & 0.0 \\ -2.0 & 3.0 & -2.0 & 0.5 \\ 1.0 & -4.0 & 6.0 & -4.0 \\ 0.0 & 1.0 & -4.0 & 5.0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.0 \\ -0.5 \\ -1.0 \\ 2.0 \end{bmatrix} \quad \Rightarrow \quad c_4 = \Phi^{(4)\,T}\mathbf{M}\bar{\Phi}_1 = 1.38144 \\[2ex]
\hat{\Phi}_1 = \bar{\Phi}_1 - c_4\Phi^{(4)} = \begin{bmatrix} 1.0 \\ -0.5 \\ -1.0 \\ 2.0 \end{bmatrix} - 1.38144 \begin{bmatrix} -0.10756 \\ 0.25563 \\ -0.72825 \\ 0.56197 \end{bmatrix} = \begin{bmatrix} 1.14859 \\ -0.85314 \\ 0.00604 \\ 1.22367 \end{bmatrix} \quad \Rightarrow \quad \hat{\Phi}_1^T\mathbf{M}\hat{\Phi}_1 = 5.59161 \\[2ex]
\Phi_1 = \dfrac{1}{\sqrt{5.59161}} \begin{bmatrix} 1.14859 \\ -0.85314 \\ 0.00604 \\ 1.22367 \end{bmatrix} = \begin{bmatrix} 0.48573 \\ -0.36079 \\ 0.00256 \\ 0.51748 \end{bmatrix}
\end{array}\right. \tag{5–80}$$

The results for the iteration vector in the succeeding iteration steps become

$$\left.\begin{array}{l}
\Phi_2 = \begin{bmatrix} 0.44542 \\ -0.41392 \\ -0.02891 \\ 0.50962 \end{bmatrix} \\[2ex]
\Phi_3 = \begin{bmatrix} 0.44063 \\ -0.41617 \\ -0.02534 \\ 0.51445 \end{bmatrix} \\[2ex]
\quad\vdots \\[2ex]
\Phi_9 = \begin{bmatrix} 0.43867 \\ -0.41674 \\ -0.02322 \\ 0.51696 \end{bmatrix}
\end{array}\right\} \tag{5–81}$$

The process converged with the indicated digit after 9 iterations.

Based on the Rayleigh quotient estimates of the obtained eigenmodes the following eigenvalues may be calculated, cf. (5-2)

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ = & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} 0.09654 & 0 & 0 & 0 \\ 0 & 1.39147 & 0 & 0 \\ 0 & 0 & 4.37355 & 0 \\ 0 & 0 & 0 & 10.6384 \end{bmatrix} \tag{5–82}$$

5.6 Exercises

5.1 Given the following mass- and stiffness matrices

$$\mathrm{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathrm{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a.) Perform two inverse iterations, and then calculate an approximation to λ_1.

(b.) Perform two forward iterations, and then calculate an approximation to λ_3.

5.2 Given the following mass- and stiffness matrices

$$\mathrm{M} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathrm{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

The eigenmodes $\Phi^{(1)}$ are $\Phi^{(3)}$ are known to be, cf. (1-87)

$$\Phi^{(1)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad , \quad \Phi^{(3)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

(a.) Calculate $\Phi^{(2)}$ by means of Gram-Schmidt orthogonalization, and calculate all eigenvalues.

5.3 Given the following mass- and stiffness matrices

$$\mathrm{M} = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad , \quad \mathrm{K} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(a.) Write a MATLAB program, which calculates the lowest three eigenvalues and eigenmodes of the related generalized eigenvalue problem by means of inverse vector iteration with Gram-Schmidt ortogonalization.

CHAPTER 6
SIMILARITY TRANSFORMATION METHODS

6.1 Introduction

Iterative similarity transformation methods are based on a sequence of similarity transformations of the original generalized eigenvalue problem in order to reduce this to a simpler form. The general form of a similarity transformation is defined by the following coordinate transformation of the eigenmodes

$$\Phi^{(j)} = \mathrm{P}\Psi^{(j)} \tag{6–1}$$

where P is the transformation matrix, and $\Phi^{(j)}$ and $\Psi^{(j)}$ signify the old and the new coordinates of the eigenmode. Then, the eigenvalue problem (1-9) may be written

$$\left.\begin{aligned} &\mathrm{K}\Phi^{(j)} = \lambda_j \mathrm{M}\Phi^{(j)} && \Rightarrow \\ &\mathrm{K}\mathrm{P}\Psi^{(j)} = \lambda_j \mathrm{M}\mathrm{P}\Psi^{(j)} && \Rightarrow \\ &\tilde{\mathrm{K}}\Psi^{(j)} = \lambda_j \tilde{\mathrm{M}}\Psi^{(j)} \\ &\tilde{\mathrm{K}} = \mathrm{P}^T\mathrm{K}\mathrm{P} \quad , \quad \tilde{\mathrm{M}} = \mathrm{P}^T\mathrm{M}\mathrm{P} \end{aligned}\right\} \tag{6–2}$$

The eigenvalues λ_j are unchanged under a similarity transformation, whereas the eigenmodes are related by (6-1). In the iteration process the transformation matrix $\mathbf{P}$ is determined, so this matrix converge toward the modal matrix $\Phi = [\Phi^{(1)}\ \Phi^{(2)} \ldots \Phi^{(n)}]$. Hence, after convergence of the iteration process the eigenmodes are stored column-wise in $\mathrm{P} = \Phi$. On condition that the eigenmodes have been normalized to unit modal mass it then follows from (1-19) and (1-21) that $\tilde{\mathrm{K}} = \mathrm{P}^T\mathrm{K}\mathrm{P} = \Lambda$, and $\tilde{\mathrm{M}} = \mathrm{P}^T\mathrm{M}\mathrm{P} = \mathrm{I}$, so the transformed stiffness and mass matrices become diagonal at convergence, and the eigenvalues are stored in the main diagonal of $\tilde{\mathrm{K}}$. By contrast to vector iteration methods similarity transformation methods determine all eigen-pairs $\left(\lambda_j, \Phi^{(j)}\right)$, $j = 1, \ldots, n$.

The general format of the similarity iteration algorithm has been summarized in Box 6.1.

Box 6.1: Iterative similarity transformation algorithm

Let $\mathbf{M}_0 = \mathbf{M}$, $\mathbf{K}_0 = \mathbf{K}$ and $\mathbf{\Phi}_0 = \mathbf{I}$. Repeat the following items for $k = 0, 1, \ldots$

1. Calculate appropriate transformation matrix $\mathbf{P}_k$ at the kth iteration step.
2. Calculate updated transformation matrix and transformed mass and stiffness matrices
$$\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k \mathbf{P}_k \quad , \quad \mathbf{M}_{k+1} = \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k \quad , \quad \mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k$$

After convergence:

$$\mathbf{k} = \mathbf{K}_\infty \quad , \quad \mathbf{m} = \mathbf{M}_\infty$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{j_1} & 0 & \cdots & 0 \\ 0 & \lambda_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{j_n} \end{bmatrix} = \mathbf{m}^{-1}\mathbf{k} \quad , \quad \mathbf{\Phi} = \left[\mathbf{\Phi}^{(j_1)}\, \mathbf{\Phi}^{(j_2)} \ldots \mathbf{\Phi}^{(j_n)}\right] = \mathbf{\Phi}_\infty \mathbf{m}^{-\frac{1}{2}}$$

Orthonormal transformation matrices fulfill, cf. (1-23)

$$\mathbf{P}_k^{-1} = \mathbf{P}_k^T \tag{6–3}$$

For transformation methods operating on the generalized eigenvalue problem, such as the *general Jacobi iteration method* considered in Section 6.2, the transformation matrices $\mathbf{P}_k$ are not orthonormal, in which case $\mathbf{M}_k$ and $\mathbf{K}_k$ converge towards the diagonal matrices $\mathbf{m}$ and $\mathbf{k}$ as given by (1-20) and (1-22). Then, $\mathbf{P}_k^T \mathbf{P}_k \neq \mathbf{I}$, and an original SEVP will change into a GEVP during the iteration process. The eigenvalue matrix $\mathbf{\Lambda}$ and the normalized modal matrix $\mathbf{\Phi}$ are retrieved as indicated in Box 6.1, where $\mathbf{m}^{-\frac{1}{2}}$ denotes a diagonal matrix with the components $1/\sqrt{M_j}$ in the main diagonal.

Some similarity transformation algorithms are devised for the special eigenvalue problem, as is the case for the *special Jacobi iteration method* in Section 6.1, and the *Householder-QR iteration method* in Section 6.3. Hence, application of these methods require an initial similarity transformation from a GEVP to a SEVP as explained in Section 3.4. This may be achieved by specifying the transformation matrix of the transformation $k = 0$ in Box 6.1 as, cf. (3-48)

$$\mathbf{P}_0 = \left(\mathbf{S}^{-1}\right)^T \tag{6–4}$$

where $\mathbf{S}$ fulfills (3-44). Then, $\mathbf{M}_1 = \mathbf{I}$. If the succeeding similarity transformation matrices are orthonormal, then all transformed mass matrices become identity matrices as seen by induction from $\mathbf{M}_{k+1} = \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k = \mathbf{P}_k^T \mathbf{I} \mathbf{P}_k = \mathbf{I}$. Moreover, $\mathbf{\Phi}_{k+1}$ is orthonormal at each iteration step,

as seen by induction from $\mathbf{\Phi}_{k+1}^T \mathbf{\Phi}_{k+1} = \mathbf{P}_k^T \mathbf{\Phi}_k^T \mathbf{\Phi}_k \mathbf{P}_k = \mathbf{P}_k^T \mathbf{I} \mathbf{P}_k = \mathbf{I}$.

Finally, it should be noticed that after convergence the sequence of eigenvalues in the main diagonal of $\mathbf{\Lambda}$ and the eigenmodes in $\mathbf{\Phi}$ is not ordered in ascending magnitude of the corresponding eigenvalues as indicated in Box 6.1, where the set of indices $(j_1, j_2, \ldots, j_n)$ denotes an arbitrary *permutation* of the numbers $(1, 2, \ldots, n)$.

Example 6.1: Interchange of rows and columns in GEVP by means of a similarity transformation

Interchange of rows and columns in a matrix may be performed by a similarity transformation. Assume, that the if the ith row and column are to be interchanged with the jth row and colums. Then the similarity transformation matrix is given as

$$\mathbf{P} = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & & i & & j & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ i \\ \\ j \\ \\ \\ \end{array} \tag{6–5}$$

The rule is that ones are placed at the positions (i, j) and (j, i) of $\mathbf{P}$. Consider the generalized eigenvalue problem defined by (4-16). It is easily verified that $\mathbf{P}^{-1} = \mathbf{P}^T$, so the transformation matrix is orthonormal, cf. (3-4). The intention is to interchange the 1st row and column with the 2nd row and column, and next the new 2nd row and column with the 4th row and column. This is achieved by two similarity transformations with the transformation matrices $\mathbf{P}_1$ and $\mathbf{P}_2$, given the following combined transformation matrix obtained as a product of tweo matrices of the type (6-5)

$$\mathbf{P} = \mathbf{P}_1 \mathbf{P}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{6–6}$$

The transformed stiffness and mass matrices become

$$\left.\begin{aligned} \tilde{\mathbf{M}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \tilde{\mathbf{K}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \end{aligned}\right\} \tag{6–7}$$

In the formulation (6-1 the solutions for the eigenmodes in the transformed system as given by (4-21) - (4-23) may be written as

$$\Psi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{4}+\frac{\sqrt{2}}{4} & -\frac{1}{4}+\frac{\sqrt{2}}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{bmatrix} \tag{6–8}$$

The corresponding eigenmodes of the original system is obtained from, cf. (6-1)

$$\Phi = \mathbf{P}\Psi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{4}+\frac{\sqrt{2}}{4} & -\frac{1}{4}+\frac{\sqrt{2}}{4} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{4}+\frac{\sqrt{2}}{4} & -\frac{1}{4}+\frac{\sqrt{2}}{4} & 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \tag{6–9}$$

6.2 Special Jacobi Iteration

The special Jacobi iteration algorithm operates on the special eigenvalue problem, so $\mathbf{M} = \mathbf{I}$ at the outset. The idea is to ensure during the kth transformation that the off-diagonal component $K_{ij,k}$, entering the ith and jth row and column of $\mathbf{K}_k$, becomes zero after the similarity transformation. The transformation matrix is given as

$$\mathbf{P}_k = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccccc} & & & i & & & j & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cos\theta & 0 & \cdots & -\sin\theta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sin\theta & 0 & \cdots & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\ \end{array} \tag{6–10}$$

Basically, (6-10) is a identity matrix, where only the components P_{ii}, P_{ij}, P_{ji} and P_{jj} are differing. Obviously, (6-10) is orthonormal. The components of the updated similarity transformation matrix $\Phi_{k+1} = \Phi_k \mathbf{P}_k$ and the transformed stiffness matrix $\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k$ become

$$\begin{cases} \Phi_{li,k+1} = \Phi_{li,k}\cos\theta + \Phi_{lj,k}\sin\theta \quad , \quad l = 1,\ldots,n \\ \Phi_{lj,k+1} = \Phi_{lj,k}\cos\theta - \Phi_{li,k}\sin\theta \quad , \quad l = 1,\ldots,n \end{cases} \tag{6–11}$$

$$\begin{cases} K_{ii,k+1} = K_{ii,k}\cos^2\theta + K_{jj,k}\sin^2\theta + 2K_{ij,k}\cos\theta\sin\theta \\ K_{jj,k+1} = K_{jj,k}\cos^2\theta + K_{ii,k}\sin^2\theta - 2K_{ij,k}\cos\theta\sin\theta \\ K_{ij,k+1} = \left(K_{jj,k} - K_{ii,k}\right)\cos\theta\sin\theta + K_{ij,k}\left(\cos^2\theta - \sin^2\theta\right) \\ K_{li,k+1} = K_{il,k+1} = K_{li,k}\cos\theta + K_{lj,k}\sin\theta \quad , \quad l \neq i,j \\ K_{lj,k+1} = K_{jl,k+1} = K_{lj,k}\cos\theta - K_{li,k}\sin\theta \quad , \quad l \neq i,j \end{cases} \tag{6–12}$$

The remaining components of $\mathbf{\Phi}_{k+1}$ and $\mathbf{K}_{k+1}$ are identical to those of $\mathbf{\Phi}_k$ and $\mathbf{K}_k$. Hence, only the ith and jth row and column of $\mathbf{K}_k$ are affected by the transformation.

Box 6.2: Special Jacobi iteration algorithm

Let $\mathbf{M}_0 = \mathbf{I}$, $\mathbf{K}_0 = \mathbf{K}$ and $\mathbf{\Phi}_0 = \mathbf{I}$. Repeat the following items for the sweeps $m = 1, 2, \ldots$

1. Specify omission criteria ε_m in the mth sweep.
2. Check, if the component $K_{ij,k}$ in the ith row and jth column of $\mathbf{K}_k$ fulfills the criteria

$$\sqrt{\frac{K_{ij,k}^2}{K_{ii,k}K_{jj,k}}} < \varepsilon_m$$

3. If the criteria is fulfilled, then skip to the next component in the sweep. Else perform the following calculations

   (a.) Calculate the transformation angle θ from (6-13), and then the transformation matrix $\mathbf{P}_k$ as given by (6-10).

   (b.) Calculate the components of the updated similarity transformation matrix $\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k\mathbf{P}_k$, and the transformed stiffness matrix $\mathbf{K}_{k+1} = \mathbf{P}_k^T\mathbf{K}_k\mathbf{P}_k$ from (6-11) and (6-12). Notice that k after the mth sweep is of the magnitude $\frac{1}{2}(n-1)n \cdot m$.

After convergence:

$$\mathbf{\Phi}_\infty = \mathbf{\Phi} = \left[\mathbf{\Phi}^{(j_1)}\,\mathbf{\Phi}^{(j_2)}\ldots\mathbf{\Phi}^{(j_n)}\right] \quad , \quad \mathbf{K}_\infty = \mathbf{\Lambda} = \begin{bmatrix} \lambda_{j_1} & 0 & \cdots & 0 \\ 0 & \lambda_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{j_n} \end{bmatrix}$$

Next, the angle θ is determined, so the off-diagonal component $K_{ij,k+1}$ becomes equal to zero

$$K_{ij,k+1} = -\frac{1}{2}(K_{ii,k} - K_{jj,k})\sin 2\theta + K_{ij,k}\cos 2\theta = 0 \quad \Rightarrow$$

$$\begin{cases} \theta = \dfrac{1}{2}\arctan\left(\dfrac{2K_{ij,k}}{K_{ii,k} - K_{jj,k}}\right) & , \quad K_{ii,k} \neq K_{jj,k} \\ \theta = \dfrac{\pi}{4} & , \quad K_{ii,k} = K_{jj,k} \end{cases} \tag{6–13}$$

Notice, that even though $K_{ij,k+1} = 0$ after the transformation, a subsequent transformation involving either the ith or jth row or column may reintroduce a non-zero value at this position. Optimally, $K_{ij,k}$ should be selected as the numerically largest off-diagonal component in $\mathbf{K}_k$. However, in practice the iteration process is often performed in so-called *sweeps*, where all $\frac{1}{2}(n-1)n$ components above the main diagonal in turn are selected as the critical element to become zero after the transformation. In this case the method is combined with a criteria for omission of the similarity transformation, in case the component is numerically small. The transformation is omitted, if

$$\sqrt{\frac{K_{ij,k}^2}{K_{ii,k}K_{jj,k}}} < \varepsilon_m \tag{6–14}$$

where ε_m is the omission value in the mth sweep.

Finally, it should be noticed that if $\mathbf{K}_0$ has a banded structure, so non-zero components are grouped in a band around the main diagonal, the banded structure is not preserved during the transformation process as seen from Example 6.1, where the initial matrix $\mathbf{K}_0$ is on a three diagonal form, whereas the transformed matrix $\mathbf{K}_1$ is full, see (6-16) below.

The special Jacobi iteration algorithm can be summarized as indicated in Box 6.2.

Example 6.2: Special Jacobi iteration

Given a special eigenvalue problem with the stiffness matrix

$$\mathbf{K} = \mathbf{K}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad , \quad \mathbf{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6–15}$$

In the 1st sweep the following calculations are performed for $(i,j) = (1,2)$:

$$\begin{cases} \theta = \dfrac{1}{2}\arctan\left(\dfrac{2\cdot(-1)}{2-4}\right) = 0.3927 \quad \Rightarrow \quad \begin{cases}\cos\theta = 0.9239 \\ \sin\theta = 0.3827\end{cases} \\ \mathbf{P}_0 = \begin{bmatrix} 0.9239 & -0.3827 & 0 \\ 0.3827 & 0.9239 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{\Phi}_1 = \mathbf{\Phi}_0\mathbf{P}_0 = \begin{bmatrix} 0.9239 & -0.3827 & 0 \\ 0.3827 & 0.9239 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{K}_1 = \mathbf{P}_0^T\mathbf{K}_0\mathbf{P}_0 = \begin{bmatrix} 1.5858 & 0 & -0.3827 \\ 0 & 4.4142 & -0.9239 \\ -0.3827 & -0.9239 & 2 \end{bmatrix} \end{cases} \tag{6–16}$$

Next, the calculations are performed for $(i,j) = (1,3)$:

$$\begin{cases} \theta = \dfrac{1}{2}\arctan\left(\dfrac{2\cdot(-0.3827)}{1.5858-2}\right) = 0.5374 \quad \Rightarrow \quad \begin{cases}\cos\theta = 0.8591 \\ \sin\theta = 0.5119\end{cases} \\ \mathbf{P}_1 = \begin{bmatrix} 0.8591 & 0 & -0.5119 \\ 0 & 1 & 0 \\ 0.5119 & 0 & 0.8591 \end{bmatrix} \\ \mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{P}_1 = \begin{bmatrix} 0.7937 & -0.3827 & -0.4729 \\ 0.3287 & 0.9238 & -0.1959 \\ 0.5119 & 0 & 0.8591 \end{bmatrix}, \quad \mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1 = \begin{bmatrix} 1.3578 & -0.4729 & 0 \\ -0.4729 & 4.4142 & -0.7937 \\ 0 & -0.7937 & 2.2280 \end{bmatrix} \end{cases} \tag{6–17}$$

Finally, to end the 1st sweep the calculations are performed for $(i,j) = (2,3)$:

$$\begin{cases} \theta = \dfrac{1}{2}\arctan\left(\dfrac{2\cdot(-0.7937)}{4.4142-2.2280}\right) = -0.3140 \quad \Rightarrow \quad \begin{cases}\cos\theta = 0.9511 \\ \sin\theta = -0.3089\end{cases} \\ \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9511 & 0.3089 \\ 0 & -0.3089 & 0.9511 \end{bmatrix} \\ \mathbf{\Phi}_3 = \mathbf{\Phi}_2\mathbf{P}_2 = \begin{bmatrix} 0.7937 & -0.2179 & -0.5680 \\ 0.3287 & 0.9392 & 0.0991 \\ 0.5119 & -0.2653 & 0.8171 \end{bmatrix}, \quad \mathbf{K}_3 = \mathbf{P}_2^T\mathbf{K}_2\mathbf{P}_2 = \begin{bmatrix} 1.3578 & -0.4498 & -0.1461 \\ -0.4498 & 4.6720 & 0 \\ -0.1461 & 0 & 1.9703 \end{bmatrix} \end{cases} \tag{6–18}$$

$\mathbf{\Phi}_3$ and $\mathbf{K}_3$ represents the estimates of the modal matrix $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ after the 1st sweep. As seen the $K_{12,1} = 0$, whereas $K_{12,2} = -0.4729$. This is in agreement with the statement above, that off-diagonal components set to zero in one iteration, may attain non-zero values in a later iteration. Comparison of $\mathbf{K}_0$ to $\mathbf{K}_3$ shows that the numerical maximum off-diagonal component has decreased from $|-1| = 1$ to $|-0.4498|$ after the 1st sweep. Hence, the algorithm is converging.

At the end of the 2nd and 3rd sweep the following estimates are obtained for the modal matrix and the eigenvalues

$$\begin{cases} \Phi_6 = \begin{bmatrix} 0.6276 & -0.3258 & -0.7071 \\ 0.4607 & 0.8876 & 0.0000 \\ 0.6276 & -0.3258 & 0.7071 \end{bmatrix} , \quad \mathbf{K}_6 = \begin{bmatrix} 1.2680 & 0.0039 & -0.0000 \\ 0.0039 & 4.7320 & 0 \\ -0.0000 & 0 & 2.0000 \end{bmatrix} \\ \Phi_9 = \begin{bmatrix} 0.6280 & -0.3251 & -0.7071 \\ 0.4597 & 0.8881 & 0.0000 \\ 0.6280 & -0.3251 & 0.7071 \end{bmatrix} , \quad \mathbf{K}_9 = \begin{bmatrix} 1.2679 & -0.0000 & -0.0000 \\ -0.0000 & 4.7321 & 0 \\ -0.0000 & 0 & 2.0000 \end{bmatrix} \end{cases} \tag{6–19}$$

As seen the eigenmodes are stored column-wise in Φ according to the permutation $(j_1, j_2, j_3) = (1, 3, 2)$.

6.3 General Jacobi Iteration

The general Jacobi iteration method operates on the generalized eigenvalue problem, i.e. $\mathbf{M} \neq \mathbf{I}$. The idea of the transformation is to ensure that during the kth transformation the off-diagonal component $M_{ij,k}$ and $K_{ij,k}$, entering the ith and jth row and column of $\mathbf{M}_k$ and $\mathbf{K}_k$, simultaneous become zero after the similarity transformation.

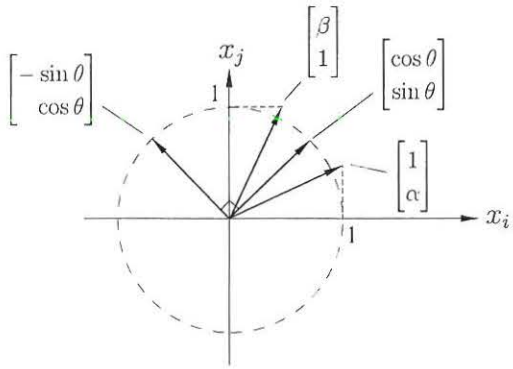

Fig. 6–1 Projection of ith and jth column vectors of similarity transformation matrix in the (x_i, x_j)-plane.

The transformation matrix is given as

$$
\mathbf{P}_k = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} & & & i & & & j & & & \end{array} \\
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & \beta & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \alpha & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{bmatrix}
\end{array}
\begin{array}{c} \\ \\ \\ i \\ \\ \\ j \\ \\ \\ \\ \end{array}
\tag{6–20}
$$

Because we have to specify requirements for both $M_{ij,k+1}$ and $K_{ij,k+1}$, we need two free parameters α and β in the transformation matrix, where only the angle θ appears in (6-10). As a consequence (6-20) is not orthonormal. Actually, the ith and jth column vectors neither have the length 1 nor are mutual orthogonal, by contrast to the corresponding vectors in (6-10), see Fig. 6-1. The components of the updated similarity transformation matrix $\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k \mathbf{P}_k$ and the transformed mass and stiffness matrices, $\mathbf{M}_{k+1} = \mathbf{P}_k^T \mathbf{M}_k \mathbf{P}_k$ and $\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k$, become

$$
\begin{cases}
\Phi_{li,k+1} = \Phi_{li,k} + \alpha\, \Phi_{lj,k} \quad , \quad l = 1, \ldots, n \\
\Phi_{lj,k+1} = \Phi_{lj,k} + \beta\, \Phi_{li,k} \quad , \quad l = 1, \ldots, n
\end{cases}
\tag{6–21}
$$

$$
\begin{cases}
M_{ii,k+1} = M_{ii,k} + \alpha^2 M_{jj,k} + 2\alpha M_{ij,k} \\
M_{jj,k+1} = M_{jj,k} + \beta^2 M_{ii,k} + 2\beta M_{ij,k} \\
M_{ij,k+1} = \beta M_{ii,k} + \alpha M_{jj,k} + M_{ij,k}(1 + \alpha\beta) \\
M_{li,k+1} = M_{il,k+1} = M_{li,k} + \alpha M_{lj,k} \quad , \quad l \neq i, j \\
M_{lj,k+1} = M_{jl,k+1} = M_{lj,k} + \beta M_{li,k} \quad , \quad l \neq i, j
\end{cases}
\tag{6–22}
$$

$$
\begin{cases}
K_{ii,k+1} = K_{ii,k} + \alpha^2 K_{jj,k} + 2\alpha K_{ij,k} \\
K_{jj,k+1} = K_{jj,k} + \beta^2 K_{ii,k} + 2\beta K_{ij,k} \\
K_{ij,k+1} = \beta K_{ii,k} + \alpha K_{jj,k} + K_{ij,k}(1 + \alpha\beta) \\
K_{li,k+1} = K_{il,k+1} = K_{li,k} + \alpha K_{lj,k} \quad , \quad l \neq i, j \\
K_{lj,k+1} = K_{jl,k+1} = K_{lj,k} + \beta K_{li,k} \quad , \quad l \neq i, j
\end{cases}
\tag{6–23}
$$

The remaining components of $\mathbf{\Phi}_{k+1}$, $\mathbf{M}_{k+1}$ and $\mathbf{K}_{k+1}$ are identical to those of $\mathbf{\Phi}_k$, $\mathbf{M}_k$ and $\mathbf{K}_k$. Hence, only the ith and the jth row and columns of $\mathbf{K}_k$ and $\mathbf{M}_k$ are affected by the transformation.

Next, the parameters α and β are determined, so the off-diagonal components $M_{ij,k+1}$ and $K_{ij,k+1}$ become equal to zero

$$\left.\begin{aligned} M_{ij,k+1} &= \beta M_{ii,k} + \alpha M_{jj,k} + M_{ij,k}\big(1+\alpha\beta\big) = 0 \\ K_{ij,k+1} &= \beta K_{ii,k} + \alpha K_{jj,k} + K_{ij,k}\big(1+\alpha\beta\big) = 0 \end{aligned}\right\} \tag{6–24}$$

The solution of (6-24) becomes, see Box 6.3

$$\left.\begin{aligned} &\alpha = \frac{1}{a}\left(\frac{1}{2} - \sqrt{\frac{1}{4} + ab}\right) \quad , \quad \beta = -\frac{a}{b}\alpha \\ &a = \frac{K_{jj,k}M_{ij,k} - M_{jj,k}K_{ij,k}}{K_{ii,k}M_{jj,k} - M_{ii,k}K_{jj,k}} \\ &b = \frac{K_{ii,k}M_{ij,k} - M_{ii,k}K_{ij,k}}{K_{ii,k}M_{jj,k} - M_{ii,k}K_{jj,k}} \end{aligned}\right\} \quad , \text{ if } \quad K_{ii,k}M_{jj,k} \neq M_{ii,k}K_{jj,k} \tag{6–25}$$

$$\alpha = \sqrt{\frac{K_{ii,k}M_{ij,k} - M_{ii,k}K_{ij,k}}{K_{jj,k}M_{ij,k} - M_{jj,k}K_{ij,k}}} \quad , \quad \beta = -\frac{1}{\alpha} \quad , \text{ if } \quad K_{ii,k}M_{jj,k} = M_{ii,k}K_{jj,k}$$

Box 6.3: Proof of equation (6-25)

From (6-19) follows

$$\frac{\beta K_{ii,k} + \alpha K_{jj,k}}{\beta M_{ii,k} + \alpha M_{jj,k}} = \frac{K_{ij,k}}{M_{ij,k}} \quad \Rightarrow \quad \beta = -\frac{K_{jj,k}M_{ij,k} - M_{jj,k}K_{ij,k}}{K_{ii,k}M_{ij,k} - M_{ii,k}K_{ij,k}}\,\alpha \tag{6–26}$$

Elimination of β in the 1st equation in (6-24) by means of (6-26) provides the following quadratic equation in α

$$M_{ij,k}\left(K_{jj,k}M_{ij,k} - M_{jj,k}K_{ij,k}\right)\alpha^2 - M_{ij,k}\left(K_{ii,k}M_{jj,k} - M_{ii,k}K_{jj,k}\right)\alpha - \\ M_{ij,k}\left(K_{ii,k}M_{ij,k} - M_{ii,k}K_{ij,k}\right) = 0 \tag{6–27}$$

If $K_{ii,k}M_{jj,k} = M_{ii,k}K_{jj,k}$ the coefficient in front of α cancels. Then, in combination to (6-26) the following solutions are obtained for α and β

$$\alpha = \pm\sqrt{\frac{K_{ii,k}M_{ij,k} - M_{ii,k}K_{ij,k}}{K_{jj,k}M_{ij,k} - M_{jj,k}K_{ij,k}}} \quad , \quad \beta = -\frac{1}{\alpha} \tag{6–28}$$

If $K_{ii,k}M_{jj,k} \neq M_{ii,k}K_{jj,k}$ solutions of the quadratic equation for α in combination to (6-26) provides

$$\alpha = \frac{1}{a}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + ab}\right) \quad , \quad \beta = -\frac{a}{b}\alpha \tag{6–29}$$

where a and b are as given in (6-25). Both sign combinations in (6-28) and (6-29) will do.

The transformations are performed in sweeps as for the special Jacobi method. In this case the criteria for omitting a transformation during the mth sweep may be formulated as

$$\sqrt{\frac{K_{ij,k}^2}{K_{ii,k}K_{jj,k}} + \frac{M_{ij,k}^2}{M_{ii,k}M_{jj,k}}} < \varepsilon_m \tag{6–30}$$

where ε_m is the omission value in the mth sweep.

The general Jacobi iteration algorithm can be summarized as indicated in Box 6.4.

Box 6.4: General Jacobi iteration algorithm

Let $\mathbf{M}_0 = \mathbf{M}$, $\mathbf{K}_0 = \mathbf{K}$ and $\mathbf{\Phi}_0 = \mathbf{I}$. Repeat the following items for the sweeps $m = 1, 2, \ldots$

1. Specify omission criteria ε_m in the mth sweep.
2. Check, if the components $M_{ij,k}$ and $K_{ij,k}$ in the ith row and jth column of $\mathbf{M}_k$ $\mathbf{K}_k$ fulfill the criteria

$$\sqrt{\frac{K_{ij,k}^2}{K_{ii,k}K_{jj,k}} + \frac{M_{ij,k}^2}{M_{ii,k}M_{jj,k}}} < \varepsilon_m$$

3. If the criteria is fulfilled, then skip to the next component in the sweep. Else perform the following calculations

   (a.) Calculate the parameters α and β as given by (6-29), and then the transformation matrix $\mathbf{P}_k$ as given by (6-20).

   (b.) Calculate the components of the updated similarity transformation matrix $\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k\mathbf{P}_k$, and the transformed mass and stiffness matrices $\mathbf{M}_{k+1} = \mathbf{P}_k^T\mathbf{M}_k\mathbf{P}_k$ and $\mathbf{K}_{k+1} = \mathbf{P}_k^T\mathbf{K}_k\mathbf{P}_k$ from (6-21), (6-22) and (6-23). Notice that k after the mth sweep is of the magnitude $\frac{1}{2}(n-1)n \cdot m$.

After convergence:

$$\mathbf{k} = \mathbf{K}_\infty \quad , \quad \mathbf{m} = \mathbf{M}_\infty$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{j_1} & 0 & \cdots & 0 \\ 0 & \lambda_{j_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{j_n} \end{bmatrix} = \mathbf{m}^{-1}\mathbf{k} \quad , \quad \mathbf{\Phi} = [\mathbf{\Phi}^{(j_1)}\,\mathbf{\Phi}^{(j_2)} \ldots \mathbf{\Phi}^{(j_n)}] = \mathbf{\Phi}_\infty \mathbf{m}^{-\frac{1}{2}}$$

Example 6.3: General Jacobi iteration

Given a generalized eigenvalue problem with the mass and stiffness matrices

$$\mathbf{M} = \mathbf{M}_0 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \mathbf{K}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad , \quad \mathbf{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6–31}$$

In the 1st sweep the following calculations are performed for $(i,j) = (1,2)$:

$$\begin{cases} \begin{cases} \alpha = \sqrt{\dfrac{2\cdot 0.5 - 0.5\cdot(-1)}{4\cdot 0.5 - 1\cdot(-1)}} & = 0.7071 \\ \beta = -\dfrac{1}{0.7071} & = -1.4142 \end{cases} \quad \left(\text{NB}: K_{11,0}M_{22,0} = K_{22,0}M_{11,0}\right) \\ \mathbf{P}_0 = \begin{bmatrix} 1 & -1.4142 & 0 \\ 0.7071 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_1 = \mathbf{\Phi}_0\mathbf{P}_0 = \begin{bmatrix} 1 & -1.4142 & 0 \\ 0.7071 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \mathbf{M}_1 = \mathbf{P}_0^T\mathbf{M}_0\mathbf{P}_0 = \begin{bmatrix} 1.7071 & 0 & 0.3536 \\ 0 & 0.5858 & 0.5 \\ 0.3536 & 0.5 & 1 \end{bmatrix}, \mathbf{K}_1 = \mathbf{P}_0^T\mathbf{K}_0\mathbf{P}_0 = \begin{bmatrix} 2.5858 & 0 & -0.7071 \\ 0 & 10.8284 & -1 \\ -0.7071 & -1 & 2 \end{bmatrix} \end{cases} \tag{6–32}$$

Next, the calculations are performed for $(i,j) = (1,3)$:

$$\begin{cases} \left.\begin{aligned} a &= \frac{2\cdot 0.3536 - 1\cdot(-0.7071)}{2.5858\cdot 1 - 1.7071\cdot 2} &&= -1.7071 \\ b &= \frac{2.5858\cdot 0.3536 - 1.7071\cdot(-0.7071)}{2.5858\cdot 1 - 1.7071\cdot 2} &&= -2.5607 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha = & 0.9664 \\ \beta = & -0.6443 \end{cases} \\ \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & -0.6443 \\ 0 & 1 & 0 \\ 0.9664 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{P}_1 = \begin{bmatrix} 1 & -1.4142 & -0.6443 \\ 0.7071 & 1 & -0.4556 \\ 0.9664 & 0 & 1 \end{bmatrix} \\ \mathbf{M}_2 = \mathbf{P}_1^T\mathbf{M}_1\mathbf{P}_1 = \begin{bmatrix} 3.3243 & 0.4832 & 0 \\ 0.4832 & 0.5858 & 0.5 \\ 0 & 0.5 & 1.2530 \end{bmatrix}, \mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1 = \begin{bmatrix} 3.0869 & -0.9664 & 0 \\ -0.9664 & 10.8284 & -1 \\ 0 & -1 & 3.9844 \end{bmatrix} \end{cases} \tag{6–33}$$

Finally, to end the 1st sweep the calculations are performed for $(i,j) = (2,3)$:

$$\begin{cases} \left.\begin{aligned} a &= \frac{3.9844\cdot 0.5 - 1.2530\cdot(-1)}{10.8284\cdot 1.2530 - 0.5858\cdot 3.9844} &&= 0.2889 \\ b &= \frac{10.8284\cdot 0.5 - 0.5858\cdot(-1)}{10.8284\cdot 1.2530 - 0.5858\cdot 3.9844} &&= 0.5341 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha = & -0.4702 \\ \beta = & 0.2543 \end{cases} \\ \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.2543 \\ 0 & -0.4702 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_3 = \mathbf{\Phi}_2\mathbf{P}_2 = \begin{bmatrix} 1 & -1.1113 & -1.0039 \\ 0.7071 & 1.2142 & -0.2012 \\ 0.9664 & -0.4702 & 1 \end{bmatrix} \\ \mathbf{M}_3 = \mathbf{P}_2^T\mathbf{M}_2\mathbf{P}_2 = \begin{bmatrix} 3.3243 & 0.4832 & 0.1229 \\ 0.4832 & 0.3926 & 0 \\ 0.1229 & 0 & 1.5452 \end{bmatrix}, \mathbf{K}_3 = \mathbf{P}_2^T\mathbf{K}_2\mathbf{P}_2 = \begin{bmatrix} 3.0869 & -0.9664 & -0.2458 \\ -0.9664 & 12.6498 & 0 \\ -0.2458 & 0 & 4.1761 \end{bmatrix} \end{cases} \tag{6–34}$$

At the end of the 2nd and 3rd sweep the following estimates are obtained for the modal matrix and the transformed mass and stiffness matrices

$$
\begin{cases}
\Phi_6 = \begin{bmatrix} 0.7494 & -1.2825 & -1.0742 \\ 0.8195 & 1.0999 & -0.2865 \\ 1.0376 & -0.6084 & 0.9213 \end{bmatrix} \\
\mathbf{M}_6 = \begin{bmatrix} 3.4931 & -0.0024 & 0.0000 \\ -0.0024 & 0.3225 & 0 \\ 0.0000 & 0 & 1.5517 \end{bmatrix} \quad , \quad \mathbf{K}_6 = \begin{bmatrix} 3.0336 & 0.0048 & -0.0000 \\ 0.0048 & 13.029 & 0 \\ -0.0000 & 0 & 4.2464 \end{bmatrix} \\
\Phi_9 = \begin{bmatrix} 0.7501 & -1.2820 & -1.0742 \\ 0.8189 & 1.1005 & -0.2865 \\ 1.0379 & -0.6076 & 0.9213 \end{bmatrix} \\
\mathbf{M}_9 = \begin{bmatrix} 3.4932 & -0.0000 & 0.0000 \\ -0.0000 & 0.3225 & 0 \\ 0.0000 & 0 & 1.5517 \end{bmatrix} \quad , \quad \mathbf{K}_9 = \begin{bmatrix} 3.0336 & 0.0000 & -0.0000 \\ 0.0000 & 13.029 & 0 \\ -0.0000 & 0 & 4.2464 \end{bmatrix}
\end{cases}
\tag{6–35}
$$

Presuming that the process has converged after the 3rd sweep the eigenvalues and normalized eigenmodes are next retrieved by the following calculations, cf. Box. 6.4

$$
\begin{cases}
\mathbf{m} = \mathbf{M}_9 = \begin{bmatrix} 3.4932 & -0.0000 & 0.0000 \\ -0.0000 & 0.3225 & 0 \\ 0.0000 & 0 & 1.5517 \end{bmatrix} \quad , \quad \mathbf{m}^{-\frac{1}{2}} = \begin{bmatrix} 0.5350 & 0 & 0 \\ 0 & 1.7608 & 0 \\ 0 & 0 & 0.8028 \end{bmatrix} \quad \Rightarrow \\
\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \mathbf{M}_9^{-1}\mathbf{K}_9 = \begin{bmatrix} 0.8684 & 0.0000 & -0.0000 \\ 0.0000 & 40.395 & -0.0000 \\ -0.0000 & -0.0000 & 2.7365 \end{bmatrix} \\
\Phi = \left[\Phi^{(1)}\, \Phi^{(3)}\, \Phi^{(2)}\right] = \Phi_9 \mathbf{m}^{-\frac{1}{2}} = \begin{bmatrix} 0.4013 & -2.2573 & -0.8623 \\ 0.4381 & 1.9378 & -0.2300 \\ 0.5553 & -1.0698 & 0.7396 \end{bmatrix}
\end{cases}
\tag{6–36}
$$

The reader should verify that the solution matrices within the indicated accuracy fulfill $\Phi^T \mathrm{M} \Phi = \mathbf{I}$ and $\Phi^T \mathrm{K} \Phi = \Lambda$.

6.4 Householder Reduction

The Householder reduction method operates on the special eigenvalue problem . Hence, a preliminary similarity transformation of a GEVP to a SEVP must be performed as explained in Section 3.4.

The Householder method reduces a symmetric matrix $\mathbf{K}_1$ to three diagonal form by totally $n-2$ consecutive similarity transformations. After the $(n-2)$th transformation the stiffness matrix has the form

$$\mathbf{K}_{n-1} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{6–37}$$

During the reduction process the numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_{n-1}$, as well as the sequence of transformation matrices $\mathbf{P}_1, \ldots, \mathbf{P}_{n-2}$ are determined. Since all transformation matrices become orthonormal all transformed mass matrices remain unit matrices.

After completing the Householder reduction process the special eigenvalue problem with the three diagonal matrix $\mathbf{K}_{n-1}$ must be solved by some kind of iteration method, which preserves the three diagonal structure of the reduced system matrix, and benefits from this reduced structure in order to improve the calculation time. As mentioned in Section 6.2 this requirement rules out the special Jacobi iteration method. Since, the inverse of a three diagonal matrix is full, inverse vector iteration with Gram-Schmidt orthogonalization must also be avoided. Of the methods discussed hitherto only forward vector iteration with Gram-Schmidt orthogonalization meets the requirement. As wee shall see the requirements are also met by the QR iteration method to be discussed in Section 6.5. Finally, an initial Householder reduction is favorable in relation to characteristic polynomial iteration methods discussed in Section 7.4.

The transformation matrix during the kth similarity transformation is given as follows

$$\mathbf{P}_k = \mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^T \quad , \quad |\mathbf{w}_k| = 1 \tag{6–38}$$

$\mathbf{w}_k$ denotes a unit column vector to be determined below. Hence, $\mathbf{w}_k^T\mathbf{w}_k = 1$.

Obviously, $\mathbf{P}_k$ is symmetric, i.e. $\mathbf{P}_k = \mathbf{P}_k^T$. Moreover, $\mathbf{P}_k$ is orthonormal as seen from the following derivation

$$\begin{aligned} &\mathbf{P}_k\mathbf{P}_k^T = \left(\mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^T\right)\left(\mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^T\right) = \\ &\mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^T - 2\mathbf{w}_k\mathbf{w}_k^T + 4\left(\mathbf{w}_k^T\mathbf{w}_k\right)\mathbf{w}_k\mathbf{w}_k^T = \mathbf{I} \quad \Rightarrow \\ &\mathbf{P}_k^T = \mathbf{P}_k^{-1} \end{aligned} \tag{6–39}$$

As mentioned, this means that the mass matrix remains an identity matrix during the Householder similarity transformations, because this is ensured in the initial transformation from a GEVP to a SEVP, as explained in the remarks subsequent to (6-4).

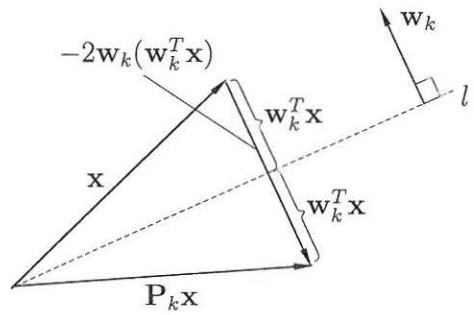

Fig. 6–2 Geometrical interpretation of the Householder transformation.

Consider a given column vector $\mathbf{x}$. Then,

$$\mathbf{P}_k\mathbf{x} = \left(\mathbf{I} - 2\mathbf{w}_k\mathbf{w}_k^T\right)\mathbf{x} = \mathbf{x} - 2\left(\mathbf{w}_k^T\mathbf{x}\right)\mathbf{w}_k \tag{6–40}$$

Notice that $\mathbf{w}_k^T\mathbf{x}$ is a scalar. The transformed vector, $\mathbf{P}_k\mathbf{x}$, may be interpreted as a reflection of $\mathbf{x}$ in the line l, which is orthogonal to the vector $\mathbf{w}_k$ and placed in the plane spanned by $\mathbf{x}$ and $\mathbf{w}_k$ as illustrated in Fig. 6-2.

At the kth transformation the applied unit vector $\mathbf{w}_k$ is taken on the following form

$$\mathbf{w}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ w_{k+1} \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{\mathbf{w}}_k \end{bmatrix} \begin{matrix} \}\ k \text{ rows} \\ \}\ n-k \text{ rows} \end{matrix} \tag{6–41}$$

where

$$\mathbf{w}_k^T\mathbf{w}_k = \bar{\mathbf{w}}_k^T\bar{\mathbf{w}}_k = w_{k+1}^2 + \cdots + w_n^2 = 1 \tag{6–42}$$

Then, the transformation matrix may be written on the following matrix form

$$\mathbf{P}_k = \begin{array}{c} \begin{array}{cc} \overbrace{\phantom{\bar{\mathbf{I}}_{n-k}}}^{k \text{ columns}} & \overbrace{}^{n-k \text{ columns}} \end{array} \\ \begin{bmatrix} \bar{\mathbf{I}}_{n-k} & 0 \\ 0 & \bar{\mathbf{P}}_k \end{bmatrix} \end{array} \begin{matrix} \}\ k \text{ rows} \\ \}\ n-k \text{ rows} \end{matrix} \quad , \quad \bar{\mathbf{P}}_k = \bar{\mathbf{I}}_k - 2\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T \tag{6–43}$$

where $\bar{\mathbf{I}}_k$ denotes a unit matrix of dimension $(n-k) \times (n-k)$.

In order to determine the sub-vector $\bar{\mathbf{w}}_k$ defining the transformation matrix, the stiffness matrix before the kth similarity transformation is considered, at which stage the stiffness matrix has been reduced to three diagonal form down to and including the $(k-1)$th row and column. Hence, the stiffness matrix has the structure

$$\mathbf{K}_k = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} & & & & & k & \overbrace{\quad}^{\substack{n-k \\ \text{columns}}} \end{array} \\ \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & \mathbf{0} \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & \mathbf{0} \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{k-1} & \beta_{k-1} & \mathbf{0} \\ 0 & 0 & 0 & \cdots & \beta_{k-1} & K_{kk} & \mathbf{k}_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{k}_k^T & \bar{\mathbf{K}}_k \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ k \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n-k \text{ rows} \end{array} \end{array} \tag{6–44}$$

$\mathbf{k}_k$ is a row vector of the dimension $(n-k)$, and $\bar{\mathbf{K}}_k$ is a symmetric matrix of the dimension $(n-k)\times(n-k)$ defined as

$$\mathbf{k}_k = [K_{k\,k+1}\, K_{k\,k+2} \cdots K_{kn}] \tag{6–45}$$

$$\bar{\mathbf{K}}_k = \begin{bmatrix} K_{k+1\,k+1} & K_{k+1\,k+2} & \cdots & K_{k+1\,n} \\ K_{k+2\,k+1} & K_{k+2\,k+2} & \cdots & K_{k+2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n\,k+1} & K_{n\,k+2} & \cdots & K_{nn} \end{bmatrix} \tag{6–46}$$

Then, with the transformation matrix given by (6-43) the stiffness matrix after the kth transformation becomes

$$\mathbf{K}_{k+1} = \mathbf{P}_k^T \mathbf{K}_k \mathbf{P}_k = \begin{array}{c} \begin{array}{ccccccc} & & & & & k & \overbrace{\quad}^{\substack{n-k \\ \text{columns}}} \end{array} \\ \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 & \mathbf{0} \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 & \mathbf{0} \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{k-1} & \beta_{k-1} & \mathbf{0} \\ 0 & 0 & 0 & \cdots & \beta_{k-1} & K_{kk} & \mathbf{k}_k\bar{\mathbf{P}}_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \bar{\mathbf{P}}_k^T\mathbf{k}_k^T & \bar{\mathbf{P}}_k^T\bar{\mathbf{K}}_k\bar{\mathbf{P}}_k \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ k \\ \left.\vphantom{\begin{matrix}0\\0\\0\end{matrix}}\right\} n-k \text{ rows} \end{array} \end{array} \tag{6–47}$$

where

$$\alpha_k = K_{kk} \tag{6–48}$$

$$\mathbf{k}_k\bar{\mathbf{P}}_k = \mathbf{k}_k\left(\bar{\mathbf{I}}_k - 2\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T\right) = \mathbf{k}_k - 2\left(\mathbf{k}_k\bar{\mathbf{w}}_k\right)\bar{\mathbf{w}}_k^T \tag{6–49}$$

$$\bar{\mathbf{P}}_k^T\bar{\mathbf{K}}_k\bar{\mathbf{P}}_k = \bar{\mathbf{K}}_k - 2\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T\bar{\mathbf{K}}_k - 2\bar{\mathbf{K}}_k\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T + 4\left(\bar{\mathbf{w}}_k^T\bar{\mathbf{K}}_k\bar{\mathbf{w}}_k\right)\bar{\mathbf{w}}_k\bar{\mathbf{w}}_k^T \tag{6–50}$$

Since, $\mathbf{k}_k$ is a row vector and $\bar{\mathbf{w}}_k$ is a column vector, $\mathbf{k}_k\bar{\mathbf{w}}_k$ is a scalar. Similarly, $\bar{\mathbf{w}}_k^T\bar{\mathbf{K}}_k\bar{\mathbf{w}}_k$ becomes a scalar.

If the kth row and column in (6-45) should be on a three-diagonal form, it is required that

$$\bar{\mathbf{P}}_k^T\mathbf{k}_k^T = \beta_k\bar{\mathbf{e}}_k \quad , \quad \bar{\mathbf{e}}_k = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6–51}$$

where $\bar{\mathbf{e}}_k$ is a unit column vector of dimension $(n-k)$. The transformation matrix is symmetric, so $\bar{\mathbf{P}}_k^T\mathbf{k}_k^T = \bar{\mathbf{P}}_k\mathbf{k}_k^T$. Moreover, $\bar{\mathbf{P}}_k\mathbf{k}_k^T$ is a reflection of the vector $\mathbf{k}_k^T$ in the line l as depicted in Fig. 6-2, and hence has the length $|\mathbf{k}_k^T|$. Hence, it follows that β_k should be selected as

$$\beta_k = \pm|\mathbf{k}_k| \tag{6–52}$$

Then it follows from (6-49) that

$$\mathbf{k}_k^T - 2\left(\mathbf{k}_k\bar{\mathbf{w}}_k\right)\bar{\mathbf{w}}_k = \pm|\mathbf{k}_k|\bar{\mathbf{e}}_k \quad \Rightarrow$$

$$\bar{\mathbf{w}}_k = a\left(\mathbf{k}_k^T \mp |\mathbf{k}_k|\bar{\mathbf{e}}_k\right) \tag{6–53}$$

where it is noticed that $2\left(\mathbf{k}_k\bar{\mathbf{w}}_k\right)$ is a scalar, which may be absorbed in the coefficient a. a is determined so the vector $\bar{\mathbf{w}}_k$ is of unit length.

Box 6.5: Householder reduction algorithm

Transform the GEVP to a SEVP by the similarity transformation matrix $\mathbf{P} = \left(\mathbf{S}^{-1}\right)^T$, where $\mathbf{S}$ is a solution to $\mathbf{M} = \mathbf{S}\mathbf{S}^T$, and define the initial transformation and stiffness matrices as

$$\mathbf{K}_1 = \mathbf{S}^{-1}\mathbf{K}\left(\mathbf{S}^{-1}\right)^T \quad , \quad \mathbf{\Phi}_1 = \left(\mathbf{S}^{-1}\right)^T$$

Next, repeat the following items for $k = 1, \ldots, n-2$

1. Calculate the similarity transformation matrix $\mathbf{P}_k$ at the kth similarity transformation by (6-43), (6-55).
2. Calculate updated transformation and stiffness matrices from (6-47), (6-57)
   $\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k\mathbf{P}_k \quad , \quad \mathbf{K}_{k+1} = \mathbf{P}_k^T\mathbf{K}_k\mathbf{P}_k$

After completion of the reduction process the following standard eigenvalue problem is solved by some iteration method

$$\mathbf{K}_{n-1}\mathbf{V} = \mathbf{V}\mathbf{\Lambda}$$

$\mathbf{\Lambda}$ is the diagonal eigenvalue matrix of the original GEVP, and $\mathbf{V}$ is the orthonormal eigenvector matrix of the three diagonal matrix $\mathbf{K}_{n-1}$. Then, the eigenmodes normalized to unit modal mass of the original GEVP are retrieved from the matrix product

$$\mathbf{\Phi} = \mathbf{\Phi}_{n-1}\mathbf{V}$$

Both sign combinations in (6-52) and (6-53) will do. However, in order to prevent numerical problems of the algorithm in the case, where $\mathbf{k}_k \simeq K_{k\,k+1}\bar{\mathbf{e}}_k$ the following choice of sign in the solutions for β_k and $\bar{\mathbf{w}}_k$ should be preferred

$$\beta_k = -\mathrm{sign}(K_{k\,k+1})\,|\mathbf{k}_k| \tag{6–54}$$

$$\bar{\mathbf{w}}_k = \frac{\mathbf{k}_k^T + \mathrm{sign}(K_{k\,k+1})|\mathbf{k}_k|\bar{\mathbf{e}}_k}{\left|\,\mathbf{k}_k^T + \mathrm{sign}(K_{k\,k+1})|\mathbf{k}_k|\bar{\mathbf{e}}_k\,\right|} \tag{6–55}$$

The updated transformation matrix before the kth transformation is partitioned as follows

$$\mathbf{\Phi}_k = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad}^{k \text{ columns}} & \overbrace{\qquad}^{n-k \text{ columns}} \end{array} \\ \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{\Phi}_{21} & \mathbf{\Phi}_{22} \end{bmatrix} \begin{array}{l} \}\ k \text{ rows} \\ \}\ n-k \text{ rows} \end{array} \end{array} \tag{6–56}$$

With the transformation matrix as given by (6-43) the transformation matrix after the kth transformation becomes

$$\Phi_{k+1} = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{k \text{ columns}} & \overbrace{\quad}^{n-k \text{ columns}} \end{array} \\ \begin{bmatrix} \Phi_{11} & \Phi_{12}\bar{\mathrm{P}}_k \\ \Phi_{21} & \Phi_{22}\bar{\mathrm{P}}_k \end{bmatrix} \begin{array}{l} \}\ k \text{ rows} \\ \}\ n-k \text{ rows} \end{array} \end{array} \tag{6–57}$$

Finally, it should be noticed that alternative algorithms for reduction to three diagonal form have been indicated by Givens[1] and Lanczos.[2]

Example 6.4: Householder reduction

Given a generalized eigenvalue problem with the mass and stiffness matrices given by (5-74). The similarity transformation matrix transforming from a GEVP to a SEVP becomes

$$\mathrm{S} = \mathrm{M}^{\frac{1}{2}} = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \left(\mathrm{S}^{-1}\right)^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6–58}$$

Then, the stiffness matrix and updated transformation matrix before the 1st Householder similarity transformation becomes, cf. (3-47), (3-48)

$$\left.\begin{array}{l} \mathrm{K}_1 = \mathrm{S}^{-1}\mathrm{K}\left(\mathrm{S}^{-1}\right)^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \\ \mathrm{K}_1 = \begin{bmatrix} \frac{5}{2} & -2 & \frac{\sqrt{2}}{2} & 0 \\ -2 & 3 & -2\sqrt{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -2\sqrt{2} & 6 & -4 \\ 0 & \frac{\sqrt{2}}{2} & -4 & 5 \end{bmatrix} \\ \Phi_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}\right\} \tag{6–59}$$

At the Householder transformation $k = 1$ one has

$$\left\{\begin{array}{l} \alpha_1 = \dfrac{5}{2} \\ \mathbf{k}_1 = \begin{bmatrix} -2 & \dfrac{\sqrt{2}}{2} & 0 \end{bmatrix} \quad , \quad |\mathbf{k}_1| = \dfrac{3\sqrt{2}}{2} \end{array}\right. \tag{6–60}$$

[1] W. Givens: *Numerical Computation of the Characteristic Values of a Real Symmetric Matrix*. Oak Ridge National Laboratory, ORNL - 1574, 1954.

[2] C. Lanczos: *An Iterative Method for the Solution of the Eigenvalue Problem of Linear Differential and Integral Operators*. Journal of Research of the National Bureau of Standards, **45**(4), 1950, 255-282.

Then, cf. (6-43), (6-54) and (6-55)

$$\begin{cases} \beta_1 = -\text{sign}(-2)\dfrac{3\sqrt{2}}{2} = \dfrac{3\sqrt{2}}{2} = 2.1213 \\ \bar{\mathbf{w}}_1 = a\left(\begin{bmatrix} -2 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} + \text{sign}(-2)\dfrac{3\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = a\begin{bmatrix} -2-\frac{3\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{w}}_1 = \begin{bmatrix} -0.9856 \\ 0.1691 \\ 0 \end{bmatrix} \\ \bar{\mathbf{P}}_1 = \bar{\mathbf{I}}_1 - 2\bar{\mathbf{w}}_1\bar{\mathbf{w}}_1^T = \begin{bmatrix} -0.9828 & 0.3333 & 0 \\ 0.3333 & 0.9428 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{cases} \tag{6–61}$$

The stiffness matrix and updated transformation matrix after the Householder transmission $k = 1$ becomes

$$\begin{cases} \mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1 = \begin{bmatrix} 2.5000 & 2.1213 & 0 & 0 \\ 2.1213 & 5.1111 & 3.1427 & -2.0000 \\ 0 & 3.1427 & 3.8889 & -3.5355 \\ 0 & -2.0000 & -3.5355 & 5.000 \end{bmatrix} \\ \mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{P}_1 = \begin{bmatrix} 0.7071 & 0 & 0 & 0 \\ 0 & -0.6667 & 0.2357 & 0 \\ 0 & 0.3333 & 0.9428 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{cases} \tag{6–62}$$

where the transformed matrices are calculated by means of (6-47) and (6-57), respectively.

At the Householder transformation $k = 2$ the following calculations are performed

$$\begin{cases} \alpha_2 = 5.1111 \\ \mathbf{k}_2 = [3.1427 \quad -2.0000] \quad , \quad |\mathbf{k}_2| = 3.7251 \end{cases} \tag{6–63}$$

$$\begin{cases} \beta_2 = -\text{sign}(3.1427)\cdot 3.7251 = -3.7251 \\ \bar{\mathbf{w}}_2 = a\left(\begin{bmatrix} 3.1427 \\ -2.0000 \end{bmatrix} + \text{sign}(3.1427)\cdot 3.7251\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = a\begin{bmatrix} 6.8678 \\ -2.0000 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{w}}_2 = \begin{bmatrix} 0.9601 \\ -0.2796 \end{bmatrix} \\ \bar{\mathbf{P}}_2 = \bar{\mathbf{I}}_2 - 2\bar{\mathbf{w}}_2\bar{\mathbf{w}}_2^T = \begin{bmatrix} -0.8436 & 0.5369 \\ 0.5369 & 0.8436 \end{bmatrix} \end{cases} \tag{6–64}$$

The stiffness matrix and updated transformation matrix after the Householder transmission $k = 2$ becomes

$$\begin{cases} \mathbf{K}_3 = \mathbf{P}_2^T \mathbf{K}_2 \mathbf{P}_2 = \begin{bmatrix} 2.5000 & 2.1213 & 0 & 0 \\ 2.1213 & 5.1111 & -3.7251 & -0.0000 \\ 0 & -3.7251 & 7.4120 & 2.0005 \\ 0 & -0.0000 & 2.0005 & 1.4769 \end{bmatrix} \\ \mathbf{\Phi}_3 = \mathbf{\Phi}_2 \mathbf{P}_2 = \begin{bmatrix} 0.7071 & 0 & 0 & 0 \\ 0 & -0.6667 & -0.1988 & 0.1265 \\ 0 & 0.3333 & -0.7954 & 0.5062 \\ 0 & 0 & 0.5369 & 0.8436 \end{bmatrix} \end{cases} \tag{6–65}$$

The reader should verify that the solution matrices within the indicated accuracy fulfill $\mathbf{\Phi}_3^T \mathbf{M} \mathbf{\Phi}_3 = \mathbf{I}$ and $\mathbf{\Phi}_3^T \mathbf{K} \mathbf{\Phi}_3 = \mathbf{K}_3$.

6.5 QR Iteration

As is the case for the Householder reduction method QR-iteration operates on the standard eigenvalue problem, so an initial similarity transformation of the GEVP to a SEVP is presumed. Let $\mathbf{K}_1 = \mathbf{S}^{-1}\mathbf{K}\left(\mathbf{S}^{-1}\right)^T$ denote the stiffness matrix after the initial similarity transformation, where $\mathbf{S}$ is a solution to $\mathbf{M} = \mathbf{S}\mathbf{S}^T$, cf. (3-44), (3-47).

QR iteration is based on the following property that any non-singular matrix $\mathbf{K}$ can be factorized on the following form

$$\mathbf{K} = \mathbf{Q}\mathbf{R} \tag{6–66}$$

where $\mathbf{Q}$ is an orthonormal matrix, and $\mathbf{R}$ is an upper triangular matrix. Hence, $\mathbf{Q}$ and $\mathbf{R}$ have the form

$$\mathbf{Q} = \left[\mathbf{q}_1\, \mathbf{q}_2 \cdots \mathbf{q}_n\right] \quad , \quad \mathbf{q}_k^T \mathbf{q}_j = \delta_{kj} \tag{6–67}$$

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & r_{24} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & r_{34} & \cdots & r_{3n} \\ 0 & 0 & 0 & r_{44} & \cdots & r_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix} \tag{6–68}$$

where δ_{ij} denotes *Kronecker's delta*. It should be noticed that the factorization (6-66) holds even for non-symmetric matrices. The orthonormality of $\mathbf{Q}$, which implies that $\mathbf{Q}^{-1} = \mathbf{Q}^T$, is

essential to the method.

Based on $\mathbf{K}_1$ a sequence of transformed stiffness matrices $\mathbf{K}_k$ are next constructed with the QR factors $\mathbf{Q}_k$ and $\mathbf{R}_k$ according to the algorithm

$$\left.\begin{aligned} \mathbf{K}_k &= \mathbf{Q}_k\mathbf{R}_k \\ \mathbf{K}_{k+1} &= \mathbf{Q}_k^T\mathbf{K}_k\mathbf{Q}_k = \mathbf{Q}_k^T\mathbf{Q}_k\mathbf{R}_k\mathbf{Q}_k = \mathbf{R}_k\mathbf{Q}_k \end{aligned}\right\} \tag{6–69}$$

Hence, $\mathbf{K}_{k+1}$ is obtained by a similarity transformation with the transformation matrix $\mathbf{Q}_k$. The transformation is reduced to an evaluation of $\mathbf{R}_k\mathbf{Q}_k$ due to the orthonormality property of $\mathbf{Q}_k$. For the same reason all transformed mass matrices remain unit matrices.

Now, it can be proved that

$$\mathbf{K}_\infty = \mathbf{R}_\infty = \mathbf{\Lambda} = \begin{bmatrix} \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{bmatrix} \quad , \quad \mathbf{\Phi}_\infty = \mathbf{\Phi} = \left[\mathbf{\Phi}^{(n)}\, \mathbf{\Phi}^{(n-1)} \cdots \mathbf{\Phi}^{(1)}\right] \tag{6–70}$$

$\mathbf{Q}_k$ converge to a unit matrix, as a consequence of $\mathbf{K}_\infty = \mathbf{R}_\infty$.

As seen, at convergence the eigen-pairs are ordered in descending order of the eigenvalues. Moreover, the algorithm converges faster to the lowest eigenmode than to the largest, as is the case for subspace iteration as describes in Section 7.3, a method which has some resemblance to QR iteration. The rate of convergence seems to be rather comparable to that of subspace iteration. These properties have been illustrated in Example 6.5 below. The proof of convergence and the associated determination of the convergence rate is rather tedious and involved, and will be omitted here.

Box 6.6: Proof of equation (6-61)

Let $\mathbf{k}_1\, \mathbf{k}_2, \ldots, \mathbf{k}_n$ denote the column vectors of the matrix $\mathbf{K}$, i.e.

$$\mathbf{K} = \left[\mathbf{k}_1\, \mathbf{k}_2 \cdots \mathbf{k}_n\right] \tag{6–71}$$

Since $\mathbf{K}$ is non-singular, $\mathbf{k}_1\, \mathbf{k}_2, \ldots, \mathbf{k}_n$ are linearly independent, and hence form a vector basis. A new orthonormal vector basis $\mathbf{q}_1\, \mathbf{q}_2 \cdots \mathbf{q}_n$ linearly dependent on $\mathbf{k}_1\, \mathbf{k}_2, \ldots, \mathbf{k}_n$ may then be constructed by a process, which resembles the Gram-Schmidt orthogonalization described in Section 5.5. (6-66) is identical to the following relations

$$\begin{cases} \mathbf{k}_1 = r_{11}\mathbf{q}_1 \\ \mathbf{k}_2 = r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}_2 \\ \vdots \\ \mathbf{k}_j = r_{1j}\mathbf{q}_1 + r_{2j}\mathbf{q}_2 + \cdots + r_{jj}\mathbf{q}_j = \displaystyle\sum_{k=1}^{j} r_{kj}\mathbf{q}_k \\ \vdots \\ \mathbf{k}_n = \displaystyle\sum_{k=1}^{n} r_{kn}\mathbf{q}_k \end{cases} \tag{6–72}$$

(6-72) is solved sequentially downwards using the properties of orthonormality of $\mathbf{q}_j$. From the 1st equation follows by scalar multiplication with $\mathbf{q}_1$

$$r_{11} = |\mathbf{k}_1| \quad \Rightarrow \quad \mathbf{q}_1 = \frac{1}{r_{11}}\mathbf{k}_1 \tag{6–73}$$

Now, $\mathbf{q}_1$ and r_{11} are known. Scalar multiplication of the 2nd equation with $\mathbf{q}_1$, and use of the orthogonality property $\mathbf{q}_1^T\mathbf{q}_2 = 0$, provides

$$r_{12} = \mathbf{q}_1^T\mathbf{k}_2 \quad \Rightarrow \quad r_{22} = |\mathbf{k}_2 - r_{12}\mathbf{q}_1| \quad \Rightarrow \quad \mathbf{q}_2 = \frac{1}{r_{22}}\left(\mathbf{k}_2 - r_{12}\mathbf{q}_1\right) \tag{6–74}$$

At the determination of $\mathbf{q}_j$, $1 < j \leq n$, the mutually ortonormal basis vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{j-1}$ have already been determined. Scalar multiplication of the jth equation with $\mathbf{q}_k$, $k = 1, 2, \ldots, j-1$, and use of the orthogonality property $\mathbf{q}_k^T\mathbf{q}_j = 0$, provides

$$r_{kj} = \mathbf{q}_k^T\mathbf{k}_j \;\Rightarrow\; r_{jj} = \left|\mathbf{k}_j - \sum_{k=1}^{j-1} r_{kj}\mathbf{q}_k\right| \quad \Rightarrow \quad \mathbf{q}_j = \frac{1}{r_{jj}}\left(\mathbf{k}_j - \sum_{k=1}^{j-1} r_{kj}\mathbf{q}_k\right) \tag{6–75}$$

Hence a solution fulfilling all requirements has been obtained for the components r_{kj} of $\mathbf{R}$ and the column vectors $\mathbf{q}_j$ of $\mathbf{Q}$, which proves the validity of the factorization (6-66).

Box 6.7: QR iteration algorithm

Transform the GEVP to a SEVP by the similarity transformation matrix $\mathbf{P} = \left(\mathbf{S}^{-1}\right)^T$, where $\mathbf{S}$ is a solution to $\mathbf{M} = \mathbf{S}\mathbf{S}^T$, and define the initial transformation and stiffness matrices as

$$\mathbf{K}_1 = \mathbf{S}^{-1}\mathbf{K}\left(\mathbf{S}^{-1}\right)^T \quad , \quad \mathbf{\Phi}_1 = \left(\mathbf{S}^{-1}\right)^T$$

Repeat the following items for $k = 1, 2, \ldots$

1. Perform a QR factorization of the stiffness matrix before the kth similarity transformation
   $\mathbf{K}_k = \mathbf{Q}_k\mathbf{R}_k$

2. Calculate updated transformation and stiffness matrices by a similarity transformation with the orthonormal transformation matrix $\mathbf{Q}_k$
   $\mathbf{\Phi}_{k+1} = \mathbf{\Phi}_k\mathbf{Q}_k \quad , \quad \mathbf{K}_{k+1} = \mathbf{Q}_k^T\mathbf{K}_k\mathbf{Q}_k = \mathbf{R}_k\mathbf{Q}_k$

After convergence:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_n & 0 & \cdots & 0 \\ 0 & \lambda_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_1 \end{bmatrix} = \mathbf{K}_\infty = \mathbf{R}_\infty \quad , \quad \mathbf{\Phi} = \left[\mathbf{\Phi}^{(n)}\, \mathbf{\Phi}^{(n-1)} \ldots \mathbf{\Phi}^{(1)}\right] = \mathbf{\Phi}_\infty$$

The general QR iteration algorithm can be summarized as indicated in Box 6.7.

Usually, the QR algorithm becomes computational expensive when applied to large full matrices, due to the time consuming orthogonalization process involved in the QR factorization. However, if $\mathbf{K}_k$ is on the three diagonal form (6-37), it can be shown that matrices $\mathbf{R}_k$ and $\mathbf{Q}_k$ have the form

$$\mathbf{R}_k = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 & 0 & \cdots & 0 \\ 0 & r_{22} & r_{23} & r_{24} & 0 & \cdots & 0 \\ 0 & 0 & r_{33} & r_{34} & r_{35} & \cdots & 0 \\ 0 & 0 & 0 & r_{44} & r_{45} & \cdots & 0 \\ 0 & 0 & 0 & 0 & r_{55} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & r_{nn} \end{bmatrix} \tag{6–76}$$

$$\mathbf{Q}_k = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & \cdots & q_{1n} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & \cdots & q_{2n} \\ 0 & q_{32} & q_{33} & q_{34} & q_{35} & \cdots & q_{3n} \\ 0 & 0 & q_{42} & q_{44} & q_{45} & \cdots & q_{4n} \\ 0 & 0 & 0 & q_{54} & q_{55} & \cdots & q_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{nn} \end{bmatrix} \tag{6–77}$$

Hence, $\mathbf{R}_k$ becomes an upper three diagonal matrix with only $3n-3$ nontrivial coefficients r_{jk} versus $\frac{1}{2}n(n+1)$ for a full matrix $\mathbf{K}_k$. Similarly, $\mathbf{Q}_k$ contains zeros below the first lower diagonal. As a consequence of the indicated structure of $\mathbf{R}_k$ and $\mathbf{Q}_k$, the matrix product $\mathbf{K}_{k+1} = \mathbf{R}_k\mathbf{Q}_k$ will again be a symmetric three diagonal matrix. Hence, this property is preserved for the transformed stiffness matrices during the iteration process. This motivates the application of QR iteration in combination to an initial Householder reduction of the initial generalized eigenvalue problem to three diagonal form, which is known as the *HOQR method.*

Example 6.5: HOQR iteration

QR iteration is performed on the stiffness matrix of Example 6.3, which has been reduced to three diagonal form by Householder reduction. Hence, the initial stiffness matrix and updated transformation matrix reads, cf. (6-65)

$$\mathbf{K}_1 = \begin{bmatrix} 2.5000 & 2.1213 & 0 & 0 \\ 2.1213 & 5.1111 & -3.7251 & -0.0000 \\ 0 & -3.7251 & 7.4120 & 2.0005 \\ 0 & -0.0000 & 2.0005 & 1.4769 \end{bmatrix} \quad , \quad \mathbf{\Phi}_1 = \begin{bmatrix} 0.7071 & 0 & 0 & 0 \\ 0 & -0.6667 & -0.1988 & 0.1265 \\ 0 & 0.3333 & -0.7954 & 0.5062 \\ 0 & 0 & 0.5369 & 0.8436 \end{bmatrix} \tag{6–78}$$

At the determination of $\mathbf{q}_1$ and r_{11} in the 1st QR iteration the following calculations are performed, cf. (6-73)

$$\begin{cases} \mathbf{k}_1 = \begin{bmatrix} 2.5000 \\ 2.1312 \\ 0 \\ 0 \end{bmatrix} \quad , \quad r_{11} = \left\| \begin{bmatrix} 2.5000 \\ 2.1312 \\ 0 \\ 0 \end{bmatrix} \right\| = 3.2787 \\ \mathbf{q}_1 = \dfrac{1}{3.2787} \begin{bmatrix} 2.5000 \\ 2.1312 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.7625 \\ 0.6470 \\ 0 \\ 0 \end{bmatrix} \end{cases} \tag{6–79}$$

$\mathbf{q}_2$ and r_{12}, r_{22} are determined from the following calculations, cf. (6-74)

$$\begin{cases} \mathbf{k}_2 = \begin{bmatrix} 2.1213 \\ 5.1111 \\ -3.7251 \\ 0 \end{bmatrix} \quad , \quad r_{12} = \begin{bmatrix} 0.7625 \\ 0.6470 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2.1213 \\ 5.1111 \\ -3.7251 \\ 0 \end{bmatrix} = 4.9244 \\ r_{22} = \left| \begin{bmatrix} 2.1213 \\ 5.1111 \\ -3.7251 \\ 0 \end{bmatrix} - 4.9244 \cdot \begin{bmatrix} 0.7625 \\ 0.6470 \\ 0 \\ 0 \end{bmatrix} \right| = 4.5001 \\ \mathbf{q}_2 = \frac{1}{4.5001} \left(\begin{bmatrix} 2.1213 \\ 5.1111 \\ -3.7251 \\ 0 \end{bmatrix} - 4.9244 \cdot \begin{bmatrix} 0.7625 \\ 0.6470 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -0.3630 \\ 0.4278 \\ -0.8278 \\ 0 \end{bmatrix} \end{cases} \tag{6–80}$$

$\mathbf{q}_3$ and r_{13}, r_{23}, r_{33} are determined from the following calculations, cf. (6-75)

$$\begin{cases} \mathbf{k}_3 = \begin{bmatrix} 0 \\ -3.7251 \\ 7.4120 \\ 2.0005 \end{bmatrix} \quad , \quad r_{13} = \mathbf{q}_1^T \mathbf{k}_3 = -2.4101 \quad , \quad r_{23} = \mathbf{q}_2^T \mathbf{k}_3 = -7.7292 \\ r_{33} = \left| \mathbf{k}_3 + 2.4101\mathbf{q}_1 + 7.7292\mathbf{q}_2 \right| = 2.6959 \\ \mathbf{q}_3 = \frac{1}{2.6959} \left(\mathbf{k}_3 + 2.4101\mathbf{q}_1 + 7.7292\mathbf{q}_2 \right) = \begin{bmatrix} -0.3590 \\ 0.4231 \\ 0.3761 \\ 0.7421 \end{bmatrix} \end{cases} \tag{6–81}$$

Finally, $\mathbf{q}_4$ and $r_{14}, r_{24}, r_{34}, r_{44}$ are determined from the following calculations, cf. (6-75)

$$\begin{cases} \mathbf{k}_4 = \begin{bmatrix} 0 \\ 0 \\ 2.0005 \\ 1.4769 \end{bmatrix} \quad , \quad r_{14} = \mathbf{q}_1^T \mathbf{k}_4 = 0 \quad , \quad r_{24} = \mathbf{q}_2^T \mathbf{k}_4 = -1.6560 \\ r_{34} = \mathbf{q}_3^T \mathbf{k}_4 = 1.8483 \quad , \quad r_{44} = \left| \mathbf{k}_4 - 0\mathbf{q}_1 + 1.6560\mathbf{q}_2 - 1.8483\mathbf{q}_3 \right| = 0.1571 \\ \mathbf{q}_4 = \frac{1}{0.1571} \left(\mathbf{k}_4 - 0\mathbf{q}_1 + 1.6560\mathbf{q}_2 - 1.8483\mathbf{q}_3 \right) = \begin{bmatrix} 0.3974 \\ -0.4684 \\ -0.4163 \\ 0.6703 \end{bmatrix} \end{cases} \tag{6–82}$$

Then, at the end of the 1st iteration the following matrices are obtained

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
\mathbf{Q}_1 = \begin{bmatrix} 0.7625 & -0.3630 & -0.3590 & 0.3974 \\ 0.6470 & 0.4278 & 0.4231 & -0.4684 \\ 0 & -0.8278 & 0.3761 & -0.4163 \\ 0 & 0 & 0.7421 & 0.6703 \end{bmatrix} \\
\mathbf{R}_1 = \begin{bmatrix} 3.2787 & 4.9244 & -2.4101 & 0 \\ 0 & 4.5001 & -7.7292 & -1.6560 \\ 0 & 0 & 2.6959 & 1.8483 \\ 0 & 0 & 0 & 0.1571 \end{bmatrix}
\end{array}
\right\} \Rightarrow \\
\\
\left.
\begin{array}{l}
\mathbf{\Phi}_2 = \mathbf{\Phi}_1 \mathbf{Q}_1 = \begin{bmatrix} 0.5392 & -0.2567 & -0.2539 & 0.2810 \\ -0.4313 & -0.1206 & -0.2629 & 0.4799 \\ 0.2157 & 0.8010 & 0.2175 & 0.5143 \\ 0 & -0.4444 & 0.8280 & 0.3420 \end{bmatrix} \\
\mathbf{K}_2 = \mathbf{R}_1 \mathbf{Q}_1 = \begin{bmatrix} 5.6860 & 2.9115 & 0 & 0 \\ 2.9115 & 8.3232 & -2.2317 & 0 \\ 0 & -2.2317 & 2.3854 & 0.1166 \\ 0 & 0 & 0.1166 & 0.1053 \end{bmatrix}
\end{array}
\right\}
\end{array}
\right.
\tag{6–83}
$$

As seen the matrices $\mathbf{R}_1$ and $\mathbf{Q}_1$ have the structure (6-76) and (6-77). Additionally, $\mathbf{K}_2$ has the same three diagonal structure as $\mathbf{K}_1$. The corresponding matrices after the 2nd and 3rd iteration become

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
\mathbf{Q}_2 = \begin{bmatrix} 0.8901 & -0.4279 & -0.1566 & 0.0117 \\ 0.4558 & 0.8356 & 0.3058 & -0.0229 \\ 0 & -0.3445 & 0.9362 & -0.0702 \\ 0 & 0 & 0.0748 & 0.9972 \end{bmatrix} \\
\mathbf{R}_2 = \begin{bmatrix} 6.3881 & 6.3850 & -1.0171 & 0 \\ 0 & 6.4780 & -2.6866 & -0.0402 \\ 0 & 0 & 1.5595 & 0.1170 \\ 0 & 0 & 0 & 0.0968 \end{bmatrix}
\end{array}
\right\} \Rightarrow \\
\\
\left.
\begin{array}{l}
\mathbf{\Phi}_3 = \mathbf{\Phi}_2 \mathbf{Q}_2 = \begin{bmatrix} 0.3629 & -0.3577 & -0.3795 & 0.3103 \\ -0.4389 & 0.1744 & -0.1796 & 0.4947 \\ 0.5570 & 0.5021 & 0.4533 & 0.4818 \\ -0.2026 & -0.6566 & 0.6648 & 0.2931 \end{bmatrix} \\
\mathbf{K}_3 = \mathbf{R}_2 \mathbf{Q}_2 = \begin{bmatrix} 8.5962 & 2.9525 & 0 & 0 \\ 2.9525 & 6.3386 & -0.5372 & 0 \\ 0 & -0.5372 & 1.4687 & 0.0072 \\ 0 & 0 & 0.0072 & 0.0966 \end{bmatrix}
\end{array}
\right\}
\end{array}
\right.
\tag{6–84}
$$

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
\mathbf{Q}_3 = \begin{bmatrix} 0.9458 & -0.3230 & -0.0345 & 0.0002 \\ 0.3248 & 0.9404 & 0.1003 & -0.0005 \\ 0 & -0.1061 & 0.9943 & -0.0051 \\ 0 & 0 & 0.0051 & 1.0000 \end{bmatrix} \\[2ex]
\mathbf{R}_3 = \begin{bmatrix} 9.0891 & 4.8514 & -0.1745 & 0 \\ 0 & 5.0643 & -0.6610 & -0.0008 \\ 0 & 0 & 1.4065 & 0.0077 \\ 0 & 0 & 0 & 0.0965 \end{bmatrix}
\end{array}
\right\} \Rightarrow \\[4ex]
\left.
\begin{array}{l}
\mathbf{\Phi}_4 = \mathbf{\Phi}_3\mathbf{Q}_3 = \begin{bmatrix} 0.2270 & -0.4134 & -0.4242 & 0.3125 \\ -0.3584 & 0.3248 & -0.1434 & 0.4954 \\ 0.6899 & 0.2442 & 0.4844 & 0.4793 \\ -0.4049 & -0.6226 & 0.6036 & 0.2900 \end{bmatrix} \\[2ex]
\mathbf{K}_4 = \mathbf{R}_3\mathbf{Q}_3 = \begin{bmatrix} 10.172 & 1.6451 & 0 & 0 \\ 1.6451 & 4.8328 & -0.1492 & 0 \\ 0 & -0.1492 & 1.3986 & 0.0005 \\ 0 & 0 & 0.0005 & 0.0965 \end{bmatrix}
\end{array}
\right\}
\end{array}
\right.
\tag{6–85}
$$

As seen from $\mathbf{R}_3$ and $\mathbf{K}_4$ the terms in the main diagonal have already after the 3rd iteration grouped in descending magnitude, corresponding to the ordering of the eigenvalues at convergence indicated in Box 6.7. Moreover, for both matrices convergence to the lowest eigenvalue $\lambda_1 = 0.0965$ has occurred, illustrating the fact that the QR algorithm converge faster to the lowest eigenmode than to the highest.

The matrices after the 14th iteration become

$$
\left\{
\begin{array}{l}
\left.
\begin{array}{l}
\mathbf{Q}_{14} = \begin{bmatrix} 1.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & -0.0000 \\ 0 & -0.0000 & 1.0000 & -0.0000 \\ 0 & 0 & 0.0051 & 1.0000 \end{bmatrix} \\[2ex]
\mathbf{R}_{14} = \begin{bmatrix} 10.638 & 0.0003 & -0.0000 & 0 \\ 0.0000 & 4.3735 & -0.0000 & -0.0008 \\ 0 & 0 & 1.3915 & 0.0077 \\ 0 & 0 & 0 & 0.0965 \end{bmatrix}
\end{array}
\right\} \Rightarrow \\[4ex]
\left.
\begin{array}{l}
\mathbf{\Phi}_{15} = \mathbf{\Phi}_{14}\mathbf{Q}_{14} = \begin{bmatrix} 0.1076 & -0.4387 & -0.4453 & 0.3126 \\ -0.2556 & 0.4167 & -0.1244 & 0.4955 \\ 0.7283 & 0.0232 & 0.4894 & 0.4791 \\ -0.5620 & -0.5170 & 0.5770 & 0.2898 \end{bmatrix} \\[2ex]
\mathbf{K}_{15} = \mathbf{R}_{14}\mathbf{Q}_{14} = \begin{bmatrix} 10.638 & 0.0001 & 0 & 0 \\ 0.0001 & 4.3735 & -0.0000 & 0 \\ 0 & -0.0000 & 1.3915 & 0.0000 \\ 0 & 0 & 0.0000 & 0.0965 \end{bmatrix}
\end{array}
\right\}
\end{array}
\right.
\tag{6–86}
$$

Presuming that convergence has occurred after the 14th iteration the following solutions are obtained for the eigenvalues and eigenmodes of the original general eigenvalue problem

$$\left.\begin{aligned}
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} = \mathbf{K}_{15} = \begin{bmatrix} 10.638 & 0 & 0 & 0 \\ 0 & 4.3735 & 0 & 0 \\ 0 & 0 & 1.3915 & 0 \\ 0 & 0 & 0 & 0.0965 \end{bmatrix} \\
\mathbf{\Phi} &= \left[\mathbf{\Phi}^{(4)}\ \mathbf{\Phi}^{(3)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(1)}\right] = \mathbf{\Phi}_{15} = \begin{bmatrix} 0.1076 & -0.4387 & -0.4453 & 0.3126 \\ -0.2556 & 0.4167 & -0.1244 & 0.4955 \\ 0.7283 & 0.0232 & 0.4894 & 0.4791 \\ -0.5620 & -0.5170 & 0.5770 & 0.2898 \end{bmatrix}
\end{aligned}\right\} \tag{6–87}$$

The reader should verify that the solution matrices within the indicated accuracy fulfill $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}$ and $\mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Lambda}$, where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices given by (5-74). (6-87) agrees with the results (5-75), (5-79), (5-81) and (5-82) in Example 5.6.

6.6 Exercises

6.1 Given a symmetric matrix $\mathbf{K}$ in a special eigenvalue problem.

(a.) Write a MATLAB program, which performs special Jacobi iteration.

6.2 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$.

(a.) Write a MATLAB program, which performs general Jacobi iteration.

6.3 Given the following mass- and stiffness matrices defined in Exercise 4.2.

(a.) Perform an initial transformation to a special eigenvalue problem, and calculate the eigenvalues and eigenvectors by means of standard Jacobi iteration.

(b.) Calculate the eigenvalues and normalized eigenvectors by means of general Jacobi iteration operating on the original general eigenvalue problem.

6.4 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$ of dimension $n \geq 3$.

(a.) Write a MATLAB program, which performs a Householder reduction to three diagonal form.

6.5 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$.

(a.) Write a MATLAB program, which performs QR iteration.

6.6 Consider the mass- and stiffness matrices defined in Exercise 4.2 after the transformation to the special eigenvalue problem as performed in Exercise 6.3.

(a.) Calculate the eigenvalues and normalized eigenvectors by means of QR iteration.

CHAPTER 7
SOLUTION OF LARGE EIGENVALUE PROBLEMS

7.1 Introduction

In civil engineering large numerical models with $n = 10^5 - 10^7$ degrees of freedom have become common practise along with the development of computer technology. However, most natural and man made loads such as wind, waves, earthquakes and traffic have spectral contents in the low frequency range. As a consequence only a relatively small number $n_1 \ll n$ of the lowest structural modes will contribute to the global structural dynamic response. In this chapter methods will be discussed, which have been devised with this specific fact in mind.

Sections 7.2 and 7.3 deals with *simultaneous inverse vector iteration* and socalled *subspace iteration*, respectively. In both cases a sequence of subspaces are defined, each of which are spanned by a specific system of basis vectors. The idea is that these subspaces at the end of the iteration process contains the n_1 lowest eigenmodes $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n_1)}$ of the general eigenvalue problem (1-9). These eigenvalue problems may be assembled on the following matrix form, cf. (1-14), (1-15), (1-16)

$$\mathbf{K}[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n_1)}] = \mathbf{M}[\Phi^{(1)}\,\Phi^{(2)}\ldots\Phi^{(n_1)}]\begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{n_1}\end{bmatrix} \Rightarrow$$

$$\mathbf{K}\Phi = \mathbf{M}\Phi\Lambda \tag{7–1}$$

$$\Lambda = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_{n_1}\end{bmatrix} \tag{7–2}$$

By contrast to the formulation in Chapter 6 the modal matrix Φ is no longer quadratic, but has the dimension $n \times n_1$, defined as

$$\Phi = \left[\Phi^{(1)}\ \Phi^{(2)} \ldots \Phi^{(n_1)}\right] \tag{7–3}$$

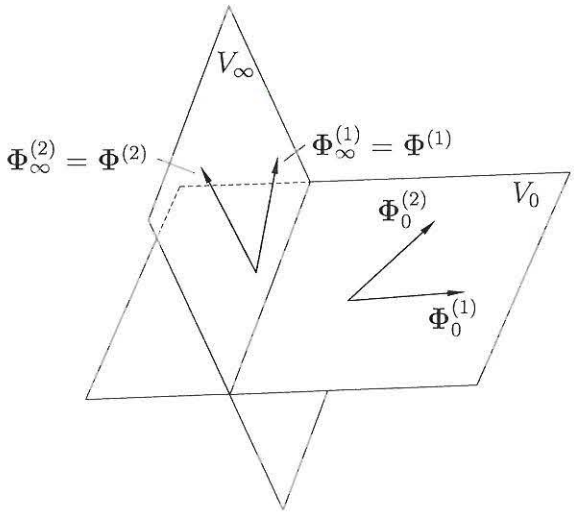

Fig. 7–1 Principle of subspace iteration.

The principle of iterating through a sequence of subspaces has been illustrated in Fig. 7-1. V_0 denotes a start subspace, which is spanned by the start basis $\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)}\right]$. The iteration process passes through a sequence of subspaces $V_1, V_2, \ldots$, where V_k is spanned by the basis $\Phi_k = \left[\Phi_k^{(1)}\ \Phi_k^{(2)}\right]$. At convergence, $\Phi_\infty = \left[\Phi_\infty^{(1)}\ \Phi_\infty^{(2)}\right] = \left[\Phi^{(1)}\ \Phi^{(2)}\right]$ is spanning the limiting subspace V_∞ containing the eigenmodes searched for.

Simultaneous inverse vector iteration is a generalization of the inverse vector iteration and inverse vector iteration with deflation described in Sections 5.2 and 5.5. The start vector basis converges towards a basis made up of the wanted eigenmodes as shown in Fig. 7-1.

The subspace iteration method and socalled *subspace iteration* described in Section 7.2 is in principle a sequence of Rayleigh-Ritz analyses, where the Ritz base vectors are forced to converge to each of the eigenmodes. Consequently, if the start basis contains the n_1 eigenmodes the subspace iteration converge in a single step as described in Section 7.2, which is generally not the case for simultaneous inverse vector iteration. Being based on a convergence of a sequence of vector bases both methods are in fact subspace iteration methods, although this name has been coined solely for the latter method. A more informative name for this method would probably be Rayleigh-Ritz iteration.

Section 7.4 deals with *characteristic polynomial iteration methods*, which operates on the characteristic equation (1-10). These methods form an alternative to inverse or forward vector iteration with deflation in case some specific eigenmode different from the smallest or largest is searched for. To be numerical effective these methods require that the generalized eigenvalue

problem has been reduced to a standard eigenvalue problem on three diagonal form, such as the Householder reduction described in Section 6.4. Polynomial methods may be based either on the numerical iteration of the characteristic polynomial directly, or based on a *Sturm sequence iteration*. Even in the first mentioned case a Sturm sequence check should be performed after the calculation to verify that the calculated n_1 eigenmodes are indeed the lowest.

It should be noticed that some problems in structural dynamics, such as acoustic transmission and noise emission, are governed by high frequency structural response. Additional to the numerical problems in calculating these modes, lack of accuracy of the underlying mechanical models in the high-frequency range adds to the problems in using modal analysis in such high frequency cases.

7.2 Simultaneous Inverse Vector Iteration

Let $\Phi_0 = \left[\Phi_0^{(1)}\, \Phi_0^{(2)} \cdots \Phi_0^{(n_1)}\right]$ denote n_1 arbitrary linearly independent vectors, which span an n_1 dimensional start subspace. Next, the algorithm for simultaneous inverse vector iteration takes place according to the algorithm

$$\bar{\Phi}_{k+1} = \mathbf{A}\Phi_k \quad , \quad k = 0, 1, \ldots \tag{7–4}$$

where $\mathbf{A} = \mathbf{K}^{-1}\mathbf{M}$, cf. (5-4). (7-4) is identical to the inverse vector iteration algorithm described by (5-4). The only difference is that now n_1 vectors are simultaneous iterated.

At convergence the iterated base vectors obtained from (7-4) will span an n_1-dimensional subspace containing the n_1 lowest eigenmodes. However, due to the inherent properties of the inverse vector iteration algorithm all the iterated base vectors tend to become mutually parallel, and parallel to the lowest eigenmode $\Phi^{(1)}$. Hence, the vector basis becomes more and more ill conditioned. For the case shown on Fig. 7-1 this means that the subspace V_k will converge to the limit plane V_∞, but the iterated base vectors $\Phi_k^{(1)}$ and $\Phi_k^{(2)}$ become more and more parallel. In order to prevent this the method is combined with a Gram-Schmidt orthogonalization procedure. Similar to the QR factorization procedure described in Box 6.6 the iterated basis $\bar{\Phi}_{k+1}$ can be written on the following factorized form

$$\bar{\Phi}_{k+1} = \Phi_{k+1}\mathbf{R}_{k+1} \tag{7–5}$$

where Φ_{k+1} is an $\mathbf{M}$-orthonormal basis in the iterated subspace, and $\mathbf{R}_{k+1}$ is an upper triangular matrix. Hence, Φ_{k+1} and $\mathbf{R}_{k+1}$ have the properties

$$\Phi_{k+1} = \left[\Phi_{k+1}^{(1)}\, \Phi_{k+1}^{(2)} \cdots \Phi_{k+1}^{(n_1)}\right] \quad , \quad \Phi_{k+1}^{(i)T}\mathbf{M}\Phi_{k+1}^{(j)} = \delta_{ij} \tag{7–6}$$

$$\mathbf{R}_{k+1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n_1} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n_1} \\ 0 & 0 & r_{33} & \cdots & r_{3n_1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{n_1n_1} \end{bmatrix} \tag{7–7}$$

The $\mathbf{M}$-orthonormal base vectors $\mathbf{\Phi}_{k+1} = \left[\mathbf{\Phi}_{k+1}^{(1)}\, \mathbf{\Phi}_{k+1}^{(2)} \cdots \mathbf{\Phi}_{k+1}^{(n_1)}\right]$ spanning the iterated subspace V_{k+1}, as well as the components of the triangular matrix $\mathbf{R}_{k+1}$, are determined sequentially in much the same way as the determination of the matrices $\mathbf{Q}$ and $\mathbf{R}$ in the QR factorization described by (6-66) - (6-69). At first, it is noticed that (7-5) is identical to the following relations

$$\begin{cases} \bar{\mathbf{\Phi}}_{k+1}^{(1)} = r_{11}\mathbf{\Phi}_{k+1}^{(1)} \\ \bar{\mathbf{\Phi}}_{k+1}^{(2)} = r_{12}\mathbf{\Phi}_{k+1}^{(1)} + r_{22}\mathbf{\Phi}_{k+1}^{(2)} \\ \vdots \\ \bar{\mathbf{\Phi}}_{k+1}^{(j)} = r_{1j}\mathbf{\Phi}_{k+1}^{(1)} + r_{2j}\mathbf{\Phi}_{k+1}^{(2)} + \cdots + r_{jj}\mathbf{\Phi}_{k+1}^{(j)} = \sum_{i=1}^{j} r_{ij}\mathbf{\Phi}_{k+1}^{(i)} \\ \vdots \\ \bar{\mathbf{\Phi}}_{k+1}^{(n_1)} = \sum_{i=1}^{n_1} r_{in_1}\mathbf{\Phi}_{k+1}^{(i)} \end{cases} \tag{7–8}$$

(7-8) is solved sequentially downwards using the $\mathbf{M}$-orthonormality of the already determined base vectors $\mathbf{\Phi}_{k+1}^{(j)}$. The details of the derivation has been given in Box 7.1.

After convergence the eigenvalues are obtained from the Rayleigh quotients evaluated with the calculated eigenvectors, cf. (4-25). Since each of the n_1 eigenmodes have been normalized to unit modal mass the quotients become

$$\lambda_j = \mathbf{\Phi}^{(j)T}\mathbf{K}\mathbf{\Phi}^{(j)} \quad , \quad j = 1, \ldots, n_1 \tag{7–9}$$

The Rayleigh quotients in (7-9) may be assembled in the following matrix equation

$$\mathbf{\Lambda} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} \tag{7–10}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_1} \end{bmatrix} \quad , \quad \mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\, \mathbf{\Phi}^{(2)} \ldots \mathbf{\Phi}^{(n_1)}\right] = \mathbf{\Phi}_\infty \tag{7–11}$$

The upper triangular matrix $\mathbf{R}_{k+1}$ converges towards the diagonal matrix Λ^{-1}. Although the Rayleigh quotients (7-10) provides more accurate estimates, the eigenvalues may then as an alternative be retrieved from

$$\Lambda = \mathbf{R}_{\infty}^{-1} \tag{7–12}$$

(7-12) follows from the following relations fulfilled at convergence

$$\begin{aligned}
&\mathbf{K}\bar{\Phi}_{\infty} = \mathbf{M}\Phi_{\infty} \quad \Rightarrow \\
&\mathbf{K}\Phi_{\infty}\mathbf{R}_{\infty} = \mathbf{M}\Phi_{\infty} \quad \Rightarrow \\
&\mathbf{K}\Phi_{\infty} = \mathbf{M}\Phi_{\infty}\mathbf{R}_{\infty}^{-1}
\end{aligned} \tag{7–13}$$

Box 7.1: M-orthonormalization of iterated basis

Evaluating the modal mass on both sides of the 1st equation of (10-8) provides

$$r_{11} = \|\bar{\Phi}_{k+1}^{(1)}\| \quad \Rightarrow \quad \Phi_{k+1}^{(1)} = \frac{1}{r_{11}}\bar{\Phi}_{k+1}^{(1)} \tag{7–14}$$

where the norm $\|\bar{\Phi}_{k+1}^{(1)}\|$ represents the square root of the modal mass of $\bar{\Phi}_{k+1}^{(1)}$ defined as

$$\|\bar{\Phi}_{k+1}^{(1)}\| = \left(\bar{\Phi}_{k+1}^{(1)T}\mathrm{M}\bar{\Phi}_{k+1}^{(1)}\right)^{\frac{1}{2}} \tag{7–15}$$

Now, $\Phi_{k+1}^{(1)}$ and r_{11} are known. Scalar pre-multiplication of the 2nd equation with $\Phi_{k+1}^{(1)T}\mathrm{M}$, and use of the orthonormality properties $\Phi_{k+1}^{(1)T}\mathrm{M}\Phi_{k+1}^{(2)} = 0$ and $\Phi_{k+1}^{(1)T}\mathrm{M}\Phi_{k+1}^{(1)} = 1$, provides

$$r_{12} = \Phi_{k+1}^{(1)T}\mathrm{M}\bar{\Phi}_{k+1}^{(2)} \quad \Rightarrow \quad r_{22} = \|\bar{\Phi}_{k+1}^{(2)} - r_{12}\Phi_{k+1}^{(1)}\| \quad \Rightarrow$$

$$\Phi_{k+1}^{(2)} = \frac{1}{r_{22}}\left(\bar{\Phi}_{k+1}^{(2)} - r_{12}\Phi_{k+1}^{(1)}\right) \tag{7–16}$$

At the determination of $\Phi_{k+1}^{(j)}$, $1 < j \leq n_1$, the mutually ortonormal basis vectors $\Phi_{k+1}^{(1)}, \Phi_{k+1}^{(2)}, \ldots, \Phi_{k+1}^{(j-1)}$ have already been determined. Scalar pre-multiplication of the jth equation with $\Phi_{k+1}^{(i)T}\mathrm{M}$, $i = 1, 2, \ldots, j-1$, and use of the orthogonality property $\Phi_{k+1}^{(i)T}\mathrm{M}\Phi_{k+1}^{(j)} = 0$ provides

$$r_{ij} = \Phi_{k+1}^{(i)T}\mathrm{M}\bar{\Phi}_{k+1}^{(j)} \quad \Rightarrow \quad r_{jj} = \left\|\bar{\Phi}_{k+1}^{(j)} - \sum_{i=1}^{j-1} r_{ij}\Phi_{k+1}^{(i)}\right\| \quad \Rightarrow$$

$$\Phi_{k+1}^{(j)} = \frac{1}{r_{jj}}\left(\bar{\Phi}_{k+1}^{(j)} - \sum_{i=1}^{j-1} r_{ij}\Phi_{k+1}^{(i)}\right) \tag{7–17}$$

It is characteristic for simultaneous inverse vector method in contrast to the subspace iteration method described in Section 7.3, that eigenmodes which at one level of the iteration process is contained in the iterated subspace, may move out of the iterated subspace at later levels as illustrated in Example 7.1.

Box 7.2: Simultaneous inverse vector iteration algorithm

Given the n_1-dimensional start vector basis $\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)} \cdots \Phi_0^{(n_1)}\right]$. The base vectors must be linearly independent, but need not be normalized to unit modal mass. Repeat the following items for $k = 0, 1, \ldots$

1. Perform simultaneous inverse vector iteration:

$$\bar{\Phi}_{k+1} = \mathbf{A}\Phi_k \quad , \quad \mathbf{A} = \mathbf{K}^{-1}\mathbf{M}$$

2. Perform Gram-Schmidt orthogonalization to obtain a new M-orthonormal iterated vector basis Φ_{k+1} as explained by (7-14) - (7-17) corresponding to the factorization:

$$\bar{\Phi}_{k+1} = \Phi_{k+1}\mathbf{R}_{k+1}$$

After convergence has been achieved the eigenvalues and eigenmodes normalized to unit modal mass are obtained from:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_1} \end{bmatrix} = \Phi_\infty^T \mathbf{K} \Phi_\infty = \mathbf{R}_\infty^{-1} \quad , \quad \Phi = \left[\Phi^{(1)}\ \Phi^{(2)} \ldots \Phi^{(n_1)}\right] = \Phi_\infty$$

As for all kind of inverse vector iteration methods the convergence rate of the iteration vector is linear in the quantity

$$r_1 = \max\left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3}, \ldots, \frac{\lambda_{n_1}}{\lambda_{n_1+1}}\right) \tag{7–18}$$

Correspondingly, the Rayleigh quotients (7-9) have quadratic convergence rate $r_2 = r_1^2$.

The simultaneous inverse vector iteration algorithm always converge towards the lowest n_1 eigenmodes. Hence, no Sturm sequence check is needed to ensure that these modes have indeed been calculated. Further, the rate of convergence seems to be comparable for all modes contained in the subspace, as demonstrated in Example 7.1 below.

The simultaneous inverse vector iteration algorithm may be summarized as indicated in Box 7.2.

Example 7.1: Simultaneous inverse vector iteration

Consider the generalized eigenvalue problem defined in Example 1.4. Calculate the two lowest eigenmodes and corresponding eigenvalues by simultaneous inverse vector iteration with the start vector basis

$$\mathbf{\Phi}_0 = \left[\mathbf{\Phi}_0^{(1)}\ \mathbf{\Phi}_0^{(2)}\right] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \tag{7–19}$$

The matrix $\mathbf{A}$ becomes, cf. (6-44)

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.2917 & 0.1667 & 0.0417 \\ 0.0833 & 0.3333 & 0.0833 \\ 0.0417 & 0.1667 & 0.2917 \end{bmatrix} \tag{7–20}$$

Then, the 1st iterated vector basis becomes, cf. (7-4)

$$\bar{\mathbf{\Phi}}_1 = \left[\bar{\mathbf{\Phi}}_1^{(1)}\ \bar{\mathbf{\Phi}}_1^{(2)}\right] = \mathbf{A}\mathbf{\Phi}_0 = \begin{bmatrix} 0.2917 & 0.1667 & 0.0417 \\ 0.0833 & 0.3333 & 0.0833 \\ 0.0417 & 0.1667 & 0.2917 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix} \tag{7–21}$$

At the determination of $\mathbf{\Phi}_1^{(1)}$ and r_{11} in the 1st vector iteration the following calculations are performed, cf. (7-14)

$$\begin{cases} \bar{\mathbf{\Phi}}_1^{(1)} = \begin{bmatrix} 0.2500 \\ 0.5000 \\ 0.7500 \end{bmatrix} \quad , \quad r_{11} = \left\|\bar{\mathbf{\Phi}}_1^{(1)}\right\| = \left(\begin{bmatrix} 0.2500 \\ 0.5000 \\ 0.7500 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.2500 \\ 0.5000 \\ 0.7500 \end{bmatrix} \right)^{\frac{1}{2}} = 0.7500 \\ \mathbf{\Phi}_1^{(1)} = \dfrac{1}{0.7500} \begin{bmatrix} 0.2500 \\ 0.5000 \\ 0.7500 \end{bmatrix} = \begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix} \end{cases} \tag{7–22}$$

$\mathbf{\Phi}_1^{(2)}$ and r_{12}, r_{22} are determined from the following calculations, cf. (7-16)

$$\begin{cases} \bar{\mathbf{\Phi}}_1^{(2)} = \begin{bmatrix} 0.7500 \\ 0.5000 \\ 0.2500 \end{bmatrix} \quad , \quad r_{12} = \begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.7500 \\ 0.5000 \\ 0.2500 \end{bmatrix} = 0.5833 \\ r_{22} = \left\| \begin{bmatrix} 0.7500 \\ 0.5000 \\ 0.2500 \end{bmatrix} - 0.5833 \cdot \begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix} \right\| = 0.4714 \\ \mathbf{\Phi}_1^{(2)} = \dfrac{1}{0.4714} \left(\begin{bmatrix} 0.7500 \\ 0.5000 \\ 0.2500 \end{bmatrix} - 0.5833 \cdot \begin{bmatrix} 0.3333 \\ 0.6667 \\ 1.0000 \end{bmatrix} \right) = \begin{bmatrix} 1.1785 \\ 0.2357 \\ -0.7071 \end{bmatrix} \end{cases} \tag{7–23}$$

Then, at the end of the 1st iteration the following matrices are obtained

$$\begin{cases} \mathbf{R}_1 = \begin{bmatrix} 0.7500 & 0.5833 \\ 0 & 0.4714 \end{bmatrix} \\ \\ \mathbf{\Phi}_1 = \begin{bmatrix} 0.3333 & 1.1785 \\ 0.6667 & 0.2357 \\ 1.0000 & -0.7071 \end{bmatrix} \end{cases} \tag{7–24}$$

The reader should verify that $\mathbf{\Phi}_1\mathbf{R}_1 = \bar{\mathbf{\Phi}}_1$. The corresponding matrices after the 2nd and 3rd iteration become

$$\begin{cases} \mathbf{R}_2 = \begin{bmatrix} 0.4787 & 0.1231 \\ 0 & 0.2611 \end{bmatrix} \\ \\ \mathbf{\Phi}_2 = \begin{bmatrix} 0.5222 & 1.1078 \\ 0.6963 & 0.1231 \\ 0.8704 & -0.8616 \end{bmatrix} \end{cases} \tag{7–25}$$

$$\begin{cases} \mathbf{R}_3 = \begin{bmatrix} 0.4943 & 0.0650 \\ 0 & 0.2529 \end{bmatrix} \\ \\ \mathbf{\Phi}_3 = \begin{bmatrix} 0.6163 & 1.0583 \\ 0.7043 & 0.0623 \\ 0.7924 & -0.9339 \end{bmatrix} \end{cases} \tag{7–26}$$

Convergence of the eigenmodes with the indicated number of digits were achieved after 14 iterations, where

$$\begin{cases} \mathbf{R}_{14} = \begin{bmatrix} 0.5000 & 0.0000 \\ 0 & 0.2500 \end{bmatrix} \\ \\ \mathbf{\Phi}_{14} = \begin{bmatrix} 0.7071 & 1.0000 \\ 0.7071 & 0.0000 \\ 0.7071 & -1.0000 \end{bmatrix} \end{cases} \tag{7–27}$$

Presuming that convergence has occurred after the 14th iteration the following eigenvalues are obtained from (7-10) and (7-12)

$$\left.\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{\Phi}_{14}^T\mathbf{K}\mathbf{\Phi}_{14} = \mathbf{R}_\infty^{-1} = \begin{bmatrix} 2.0000 & -0.0000 \\ -0.0000 & 4.0000 \end{bmatrix} \\ \\ \mathbf{\Phi} &= \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\right] = \mathbf{\Phi}_{14} = \begin{bmatrix} 0.7071 & 1.0000 \\ 0.7071 & 0.0000 \\ 0.7071 & -1.0000 \end{bmatrix} \end{aligned}\right\} \tag{7–28}$$

$\lambda_3 = 6$, see (1-87). Then, the convergence rate of the iteration vectors becomes $r_1 = \max\left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_3}\right) = \max\left(\frac{2}{4}, \frac{4}{6}\right) = \frac{2}{3}$, cf. (7-17). This is a relatively large number, which is displayed in the rather slow convergence of the iterative process. The convergence towards $\Phi^{(1)}$ and $\Phi^{(2)}$ occurred within the same iteration step. This suggests that the convergence rate is uniform to all considered modes in the subspace.

Further it is noted that

$$\left.\begin{aligned}
\Phi^{(1)} &= \frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{\sqrt{2}}{4}\cdot\begin{bmatrix}0\\1\\2\end{bmatrix} + \frac{\sqrt{2}}{4}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix} = \frac{\sqrt{2}}{4}\cdot\Phi_0^{(1)} + \frac{\sqrt{2}}{4}\cdot\Phi_0^{(2)} \\
\Phi^{(2)} &= \begin{bmatrix}1\\0\\-1\end{bmatrix} = -\frac{1}{2}\cdot\begin{bmatrix}0\\1\\2\end{bmatrix} + \frac{1}{2}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix} = -\frac{1}{2}\cdot\Phi_0^{(1)} + \frac{1}{2}\cdot\Phi_0^{(2)}
\end{aligned}\right\} \tag{7–29}$$

Hence, the 1st and 2nd eigenmode are originally in the subspace spanned by the basis Φ_0. As seen during the iteration process these eigenmodes are moving out of the iterated subspace.

7.3 Subspace Iteration

As is the case for the simultaneous inverse vector iteration algorithm the subspace iteration algorithm presumes that a start subspace V_0, spanned by the vector basis $\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)} \cdots \Phi_0^{(n_1)}\right]$, has been defined.

At the kth iteration step of the iteration process a vector basis $\Phi_k = \left[\Phi_k^{(1)}\ \Phi_k^{(2)} \cdots \Phi_k^{(n_1)}\right]$, which spans the iterated subspace V_k, has been obtained. Based on this a simultaneous inverse vector iteration is performed

$$\bar{\Phi}_{k+1} = \mathbf{A}\Phi_k \quad , \quad k = 0, 1, \ldots \tag{7–30}$$

where $\mathbf{A} = \mathbf{K}^{-1}\mathbf{M}$, cf. (5-4). Next, a Rayleigh-Ritz analysis is performed using $\bar{\Phi}_{k+1}$ as a Ritz basis, in order to obtain approximate solutions to the lowest n_1 eigenmodes and eigenvalues. This requires the solution of the following reduced generalized eigenvalue problem of the dimension n_1, cf. (1-14), (4-49)

$$\tilde{\mathbf{K}}_{k+1}\mathbf{Q}_{k+1} = \tilde{\mathbf{M}}_{k+1}\mathbf{Q}_{k+1}\mathbf{R}_{k+1} \quad , \quad k = 0, 1, \ldots \tag{7–31}$$

$\tilde{\mathbf{M}}_{k+1}$ and $\tilde{\mathbf{K}}_{k+1}$ denote the mass and stiffness matrices projected on the subspace V_{k+1}, cf. (4-45)

$$\left.\begin{aligned}\tilde{\mathbf{M}}_{k+1} &= \bar{\mathbf{\Phi}}_{k+1}^T \mathbf{M} \bar{\mathbf{\Phi}}_{k+1} \\ \tilde{\mathbf{K}}_{k+1} &= \bar{\mathbf{\Phi}}_{k+1}^T \mathbf{K} \bar{\mathbf{\Phi}}_{k+1}\end{aligned}\right\} \tag{7–32}$$

$\mathbf{Q}_{k+1} = \left[\mathbf{q}_{k+1}^{(1)} \mathbf{q}_{k+1}^{(2)} \cdots \mathbf{q}_{k+1}^{(n_1)}\right]$ of the dimension $n_1 \times n_1$ contains the eigenvectors of the eigenvalue problem (7-31). In what follows the eigenvectors $\mathbf{q}_{k+1}^{(i)}$ are assumed to normalized to unit modal mass with respect to the projected mass matrix, i.e.

$$\mathbf{q}_{k+1}^{(i)\,T} \tilde{\mathbf{M}}_{k+1} \mathbf{q}_{k+1}^{(j)} = \begin{cases} 0 \;, & i \neq j \\ 1 \;, & i = j \end{cases} \tag{7–33}$$

$\mathbf{R}_{k+1}$ is a diagonal matrix containing the corresponding eigenvalues of (7-30) in the main diagonal

$$\mathbf{R}_{k+1} = \begin{bmatrix} \rho_{1,k+1} & 0 & \cdots & 0 \\ 0 & \rho_{2,k+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \rho_{n_1,k+1} \end{bmatrix} \tag{7–34}$$

The eigenvalues $\rho_{j,k+1}$, $j = 1, \ldots, n_1$ indicates the estimate of the eigenvalues after the kth iteration. These are all upperbounds to the corresponding eigenvalues of the full problem, cf. (4-57).

At the end of the kth iteration step a new estimate of the lowest n_1 eigenvectors are determined from, cf. (4-51)

$$\mathbf{\Phi}_{k+1} = \bar{\mathbf{\Phi}}_{k+1} \mathbf{Q}_{k+1} \tag{7–35}$$

If the column vectors in $\mathbf{Q}_{k+1}$ have been normalized to unit modal mass with respect to $\tilde{\mathbf{M}}_{k+1}$, the $\mathbf{M}$-orthogonal column vectors of $\mathbf{\Phi}_{k+1}$ will automatically be normalized to unit modal mass with respect to $\mathbf{M}$, cf. (4-55).

Next, the calculations in (7-30) - (7-35) are repeated with the new estimate of the normalized eigenmodes $\mathbf{\Phi}_{k+1}$.

At convergence of the subspace iteration algorithm the lowest n_1 eigenvectors and eigenvalues are retrieved from

$$\mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\, \mathbf{\Phi}^{(2)} \ldots \mathbf{\Phi}^{(n_1)}\right] = \mathbf{\Phi}_\infty \quad , \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & \lambda_{n_1} \end{bmatrix} = \mathbf{R}_\infty = \pm \mathbf{Q}_\infty \tag{7–36}$$

At convergence, $\mathbf{Q}_\infty$ can be shown to be a diagonal matrix, where the numerical value of the components are equal to the eigenvalue of the original problem as indicated in (7-36).

It should be realized that subspace iteration involves iteration at two levels. Primary, a global simultaneous inverse vector iteration loop as defined by the index k is performed. Inside this loop a secondary iteration process is performed at the solution of the eigenvalue problem (7-31). Usually, the latter problem is solved iteratively by means of a general Jacobi iteration algorithm as described in Section 6.3. Because the applied similarity transformations in the general Jacobi algorithm are not orthonormal, the eigenvectors $\mathbf{q}_k^{(j)}$ are not normalized to unit modal mass at convergence. Hence, in order to fulfill the requirements (7-33) this normalization should be performed after convergence. Further, the eigenvalues will not be ordered in ascending order of magnitude as presumed in (7-36), cf. Box 6.4.

The convergence rate for the components in the kth eigenmode and the kth eigenvalue, $r_{1,k}$ and $r_{2,k}$, are defined as

$$\left.\begin{aligned} r_{1,k} &= \frac{\lambda_k}{\lambda_{n_1+1}} \\ r_{2,k} &= \frac{\lambda_k^2}{\lambda_{n_1+1}^2} = r_{1,k}^2 \end{aligned}\right\} \quad , \quad k = 1, \ldots, n_1 \tag{7–37}$$

Hence, convergence is achieved at first for the lowest mode and latest for mode $k = n_1$, as has been demonstrated in Example 7.2 below. This represents a marked difference from simultaneous inverse vector iteration, where as mentioned the convergence rate seems to be almost identical for all modes contained in the subspace. A rule of thumb says that approximately 10 subspace iterations are needed to obtain a solution for the components of $\mathbf{\Phi}^{(1)}$ with 6 correct digits.

Box 7.3: Subspace iteration algorithm

Given the n_1-dimensional start vector basis $\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)} \cdots \Phi_0^{(n_1)}\right]$. The base vectors must be linearly independent, but the base vectors need not be normalized to unit modal mass. Repeat the following items for $k = 0, 1, \ldots$

1. Perform simultaneous inverse vector iteration:

$$\bar{\Phi}_{k+1} = \mathrm{A}\Phi_k \quad , \quad \mathrm{A} = \mathrm{K}^{-1}\mathrm{M}$$

2. Calculate projected mass and stiffness matrices:

$$\tilde{\mathrm{M}}_{k+1} = \bar{\Phi}_{k+1}^T \mathrm{M} \bar{\Phi}_{k+1} \quad , \quad \tilde{\mathrm{K}}_{k+1} = \bar{\Phi}_{k+1}^T \mathrm{K} \bar{\Phi}_{k+1}$$

3. Solve the generalized eigenvalue problem of dimension n_1 by means of a general Jacobi iteration algorithm with the eigenvectors Q_{k+1} normalized to unit modal mass at exit:
$\tilde{\mathrm{K}}_{k+1}\mathrm{Q}_{k+1} = \tilde{\mathrm{M}}_{k+1}\mathrm{Q}_{k+1}\mathrm{R}_{k+1}$

4. Calculate new solution to eigenvectors:

$$\Phi_{k+1} = \bar{\Phi}_{k+1}\mathrm{Q}_{k+1}$$

After convergence has been achieved the eigenvalues and eigenmodes normalized to unit modal mass are obtained from:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_1} \end{bmatrix} = \mathrm{R}_\infty = \pm\mathrm{Q}_\infty \quad , \quad \Phi = \left[\Phi^{(1)}\ \Phi^{(2)} \ldots \Phi^{(n_1)}\right] = \Phi_\infty$$

Finally, a Sturm sequence check should be performed to ensure that the lowest n_1 eigenpairs have been calculated.

In order to speed up the iteration process towards the n_1 modes actually wanted, the dimension of the iterated subspace is sometimes increased to $n_2 > n_1$. Then, the convergence rate of the iteration vector the highest mode of interest decreases to

$$r_{1,n_1} = \frac{\lambda_{n_1}}{\lambda_{n_2+1}} \tag{7–38}$$

In case of an adverse choice of the start basis vector Φ_0 it may happen that one of the eigen-

modes searched for, $\Phi^{(j)}$, $j = 1, \ldots, n_1$, is **M**-orthogonal to start subspace, i.e.

$$\Phi^{(j)T}\mathrm{M}\Phi_0^{(k)} = 0 \quad , \quad k = 1, 2, \ldots, n_1 \tag{7–39}$$

In this case the subspace iteration algorithm converges towards the eigenmodes $\Phi^{(1)}, \ldots, \Phi^{(j-1)}$, $\Phi^{(j+1)}, \ldots, \Phi^{(n_1)}, \Phi^{(n_1+1)}$. In principle a similar problem occurs in simultaneous inverse vector iteration, although round-off errors normally eliminates this possibility.

Singular to subspace iteration is that eigenmodes contained in the initial basis Φ_0 remain in later iterated bases. Hence, if $\Phi^{(j)}$, $j = n_1 + 1, \ldots, n$ is contained in Φ_0, this mode will be among the calculated modes.

In both cases we are left with the problem to decide whether the calculated n_1 eigenmodes are the lowest n_1 modes of the full system. For this reason a subspace iteration should always be followed by a Sturm sequence check. This is performed in the following way. Let μ be a number slightly larger than the largest calculated eigenvalue $\rho_{n_1,\infty}$, and perform the following Gauss factorization of the matrix $\mathrm{K} - \mu\mathrm{M}$

$$\mathrm{K} - \mu\mathrm{M} = \mathrm{LDL}^T \tag{7–40}$$

where **L** and **D** are given by (3-2), (3-3). The number of eigenvalue less than μ is equal to the number of negative elements in the diagonal of the diagonal matrix **D**, cf. Section 3.1. Hence, the analysis should show exactly n_1 negative elements in **D**. Alternatively, the same information may be withdrawn from the number of sign changes in the sign sequence $\mathrm{sign}\big(P^{(n)}(\mu)\big), \mathrm{sign}\big(P^{(n-1)}(\mu)\big), \ldots, \mathrm{sign}\big(P^{(0)}(\mu)\big)$, where $P^{(n-1)}(\mu), \ldots, P^{(0)}(\mu)$ denotes the Sturm sequence of characteristic polynomials, and $P^{(n)}(\mu)$ is a dummy positive component in the sequence, cf. Section 3.2.

The marked difference between the subspace iteration algorithm and the simultaneous inverse vector iteration algorithm is that the orthonormalization process to prevent ill-conditioning of the iterated vector base in the former case is performed by an eigenvector approach related to the Rayleigh-Ritz analysis, whereas a Gram-Schmidt orthogonalization procedure is used in the latter case. There are no marked difference in the rate of convergence of the two algorithms.

Example 7.2: Subspace iteration

The generalized eigenvalue problem defined in Example 6.2 is considered again. Using the same initial start basis (7-19) as in Example 7.1, the problem is solved in this example by means of subspace iteration.

At the 1st iteration step ($k = 0$) the simultaneous inverse vector iteration produces the vector basis $\bar{\Phi}_1$, which is unchanged given by (7-21).

Based on $\bar{\Phi}_1$ the following projected mass and stiffness matrices are calculated, cf. (4-45), (7-21), (7-32)

$$\left.\begin{aligned}
\tilde{\mathbf{M}}_1 &= \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix} = \begin{bmatrix} 0.5625 & 0.4375 \\ 0.4375 & 0.5625 \end{bmatrix} \\
\tilde{\mathbf{K}}_1 &= \bar{\mathbf{\Phi}}_1^T \mathbf{K} \bar{\mathbf{\Phi}}_1 = \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix} = \begin{bmatrix} 1.2500 & 0.7500 \\ 0.7500 & 1.2500 \end{bmatrix}
\end{aligned}\right\} \tag{7–41}$$

The corresponding eigenvalue problem (7-31) becomes

$$\tilde{\mathbf{K}}_1 \mathbf{Q}_1 = \tilde{\mathbf{M}}_1 \mathbf{Q}_1 \mathbf{R}_1 \quad \Rightarrow$$

$$\begin{bmatrix} 1.2500 & 0.7500 \\ 0.7500 & 1.2500 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^{(1)} & \mathbf{q}_1^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5625 & 0.4375 \\ 0.4375 & 0.5625 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^{(1)} & \mathbf{q}_1^{(2)} \end{bmatrix} \begin{bmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,1} \end{bmatrix} \quad \Rightarrow$$

$$\mathbf{R}_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad , \quad \mathbf{Q}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & -2 \\ \frac{\sqrt{2}}{2} & 2 \end{bmatrix} \tag{7–42}$$

The estimate of the lowest eigenvectors after the 1st iteration becomes, cf. (7-35)

$$\mathbf{\Phi}_1 = \bar{\mathbf{\Phi}}_1 \mathbf{Q}_1 = \begin{bmatrix} 0.2500 & 0.7500 \\ 0.5000 & 0.5000 \\ 0.7500 & 0.2500 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -2 \\ \frac{\sqrt{2}}{2} & 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}^{(1)} & \mathbf{\Phi}^{(2)} \end{bmatrix} \tag{7–43}$$

(7-42) and (7-43) indicate the exact eigenvalues and eigenmodes, cf. (1-87). Hence, convergence is obtained in just a single iteration. This is so because the start subspace V_0, spanned by the vector basis $\mathbf{\Phi}_0$ contains the eigenmodes $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$ as shown by (7-29). This property is singular to the subspace iteration algorithm compared to the simultaneous inverse vector iteration technique.

Next, let us perform the same calculations using the start basis

$$\mathbf{\Phi}_0 = \begin{bmatrix} \mathbf{\Phi}_0^{(1)} & \mathbf{\Phi}_0^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -3 \end{bmatrix} \tag{7–44}$$

The simultaneous inverse vector iteration (7-30) provides, cf. (7-20)

$$\bar{\mathbf{\Phi}}_1 = \mathbf{A}\mathbf{\Phi}_0 = \begin{bmatrix} 0.2917 & 0.1667 & 0.0417 \\ 0.0833 & 0.3333 & 0.0833 \\ 0.0417 & 0.1667 & 0.2917 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix} \tag{7–45}$$

The projected mass and stiffness matrices become

$$\left.\begin{aligned}
\tilde{\mathbf{M}}_1 &= \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix} = \begin{bmatrix} 2.0625 & -0.0625 \\ -0.0625 & 0.2847 \end{bmatrix} \\
\tilde{\mathbf{K}}_1 &= \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix} = \begin{bmatrix} 4.2500 & -0.2500 \\ -0.2500 & 1.5833 \end{bmatrix}
\end{aligned}\right\} \tag{7–46}$$

The solution of the corresponding generalized eigenvalue problem (7-31) becomes

$$\mathbf{R}_1 = \begin{bmatrix} 2.0534 & 0 \\ 0 & 5.5656 \end{bmatrix} \quad , \quad \mathbf{Q}_1 = \begin{bmatrix} -0.6982 & -0.0254 \\ -0.0851 & -1.8784 \end{bmatrix} \tag{7–47}$$

The estimate of the lowest eigenmode after the 1st iteration becomes, cf. (7-35)

$$\mathbf{\Phi}_1 = \bar{\mathbf{\Phi}}_1 \mathbf{Q}_1 = \begin{bmatrix} 0.7500 & -0.0833 \\ 1.0000 & 0.3333 \\ 1.2500 & -0.5833 \end{bmatrix} \begin{bmatrix} -0.6982 & -0.0254 \\ -0.0851 & -1.8784 \end{bmatrix} = \begin{bmatrix} -0.5165 & 0.1375 \\ -0.7265 & -0.6516 \\ -0.8231 & 1.0640 \end{bmatrix} \tag{7–48}$$

Correspondingly, after the 2nd, 7th and 14th iteration steps the following matrices are calculated

$$\left.\begin{aligned}
\mathbf{R}_2 &= \begin{bmatrix} 2.0118 & 0 \\ 0 & 5.2263 \end{bmatrix} \quad , \quad \mathbf{Q}_2 = \begin{bmatrix} -2.0171 & 0.1513 \\ -0.0887 & -5.3145 \end{bmatrix} \\
\mathbf{\Phi}_2 &= \begin{bmatrix} 0.6195 & 0.0821 \\ 0.7241 & 0.5686 \\ 0.7535 & -1.1604 \end{bmatrix}
\end{aligned}\right\} \tag{7–49}$$

$$\left.\begin{aligned}
\mathbf{R}_7 &= \begin{bmatrix} 2.0000 & 0 \\ 0 & 4.0533 \end{bmatrix} \quad , \quad \mathbf{Q}_7 = \begin{bmatrix} -2.0000 & 0.0011 \\ -0.0007 & -4.0661 \end{bmatrix} \\
\mathbf{\Phi}_7 &= \begin{bmatrix} -0.7067 & -0.8711 \\ -0.7074 & -0.1155 \\ -0.7069 & 1.1020 \end{bmatrix}
\end{aligned}\right\} \tag{7–50}$$

$$\left.\begin{aligned}
\mathbf{R}_{14} &= \begin{bmatrix} 2.0000 & 0 \\ 0 & 4.0002 \end{bmatrix} \quad , \quad \mathbf{Q}_{14} = \begin{bmatrix} -2.0000 & 0.0000 \\ -0.0000 & -4.0002 \end{bmatrix} \\
\mathbf{\Phi}_{14} &= \begin{bmatrix} 0.7071 & 0.9931 \\ 0.7071 & 0.0068 \\ 0.7071 & -1.0068 \end{bmatrix}
\end{aligned}\right\} \tag{7–51}$$

As seen the subspace iteration process determines the 1st eigenvalue and eigenvector after 7 iteration, whereas the 2nd eigenvector has not yet been calculated with the sufficiently accuracy even after 14 iterations. By contrast the simultaneous inverse vector iteration managed to achieve convergence for this quantity after 14 iterations, see (7-27).

The 2nd calculated eigenvalue becomes $\rho_{2,14} = 4.0002$. Then, let $\mu = 4.05$ and perform a Gauss factorization of the matrix $\mathbf{K} - 4.05\mathbf{M}$, i.e.

$$\mathbf{K} - 4.05\mathbf{M} = \begin{bmatrix} -0.0250 & -1.0000 & 0.0000 \\ -1.0000 & -0.0500 & -1.0000 \\ 0.0000 & -1.0000 & -0.0250 \end{bmatrix} =$$

$$\mathbf{LDL}^T = \begin{bmatrix} 1 & 0 & 0 \\ 40 & 1 & 0 \\ 0 & -0.0250 & 1 \end{bmatrix} \begin{bmatrix} -0.0250 & 0 & 0 \\ 0 & 39.950 & 0 \\ 0 & 0 & -0.0500 \end{bmatrix} \begin{bmatrix} 1 & 40 & 0 \\ 0 & 1 & -0.0250 \\ 0 & 0 & 1 \end{bmatrix} \tag{7–52}$$

It follows that two components in the main diagonal of $\mathbf{D}$ are negative, from which is concluded that two eigenvalues are smaller than $\mu = 4.05$. In turn this means that the two eigensolutions obtained by (7-47) are indeed the lowest two eigensolutions of the original system.

Finally, consider the start vector basis

$$\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)}\right] = \begin{bmatrix} 0 & 2 \\ -1 & -1 \\ 2 & 0 \end{bmatrix} \tag{7–53}$$

Now,

$$\left.\begin{aligned} \Phi^{(1)T}\mathbf{M}\Phi_0^{(1)} &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = 0 \\ \Phi^{(1)T}\mathbf{M}\Phi_0^{(2)} &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0 \end{aligned}\right\} \tag{7–54}$$

It follows that the lowest eigenmode $\Phi^{(1)}$ is $\mathbf{M}$-orthogonal to the selected start vector basis. Hence, it should be expected that the algorithm converges towards $\Phi^{(2)}$ and $\Phi^{(3)}$. Moreover, in the present three dimensional case a start subspace, which is $\mathbf{M}$-orthogonal to $\Phi^{(1)}$, must contain $\Phi^{(2)}$ and $\Phi^{(3)}$. Actually, cf. (1-87)

$$\left.\begin{aligned} \Phi^{(2)} &= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\cdot\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \frac{1}{2}\cdot\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = -\frac{1}{2}\cdot\Phi_0^{(1)} + \frac{1}{2}\cdot\Phi_0^{(2)} \\ \Phi^{(3)} &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{\sqrt{2}}{4}\cdot\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + \frac{\sqrt{2}}{4}\cdot\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \frac{\sqrt{2}}{4}\cdot\Phi_0^{(1)} + \frac{\sqrt{2}}{4}\cdot\Phi_0^{(2)} \end{aligned}\right\} \tag{7–55}$$

Hence, convergence towards $\Phi^{(2)}$ and $\Phi^{(3)}$ should take place in a single iteration step. Actually, after the 1st subspace iteration the following matrices are calculated

$$\left.\begin{aligned} \mathbf{R}_1 &= \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \quad , \quad \mathbf{Q}_1 = \begin{bmatrix} -2.0000 & -2.1213 \\ -2.0000 & -2.1213 \end{bmatrix} \\ \Phi_2 &= \begin{bmatrix} 1.0000 & -0.7071 \\ 0.0000 & 0.7071 \\ -1.0000 & -0.7071 \end{bmatrix} \end{aligned}\right\} \tag{7–56}$$

The 2nd calculated eigenvalue becomes $\rho_{2,1} = 6$. In order to check whether $\rho_{2,1} = \lambda_2$ or $\rho_{2,1} = \lambda_3$ we choose $\mu = 6.05$, and perform a Gauss factorization of the matrix $\mathbf{K} - 6.05\mathbf{M}$, i.e.

$$\mathbf{K} - 6.05\mathbf{M} = \begin{bmatrix} -1.0250 & -1.0000 & 0.0000 \\ -1.0000 & -2.0500 & -1.0000 \\ 0.0000 & -1.0000 & -1.0250 \end{bmatrix} =$$

$$\mathbf{LDL}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0.9756 & 1 & 0 \\ 0 & 0.9308 & 1 \end{bmatrix} \begin{bmatrix} -1.0250 & 0 & 0 \\ 0 & -1.0744 & 0 \\ 0 & 0 & -0.0942 \end{bmatrix} \begin{bmatrix} 1 & 0.9756 & 0 \\ 0 & 1 & 0.9308 \\ 0 & 0 & 1 \end{bmatrix} \tag{7–57}$$

It follows that three components in the main diagonal of $\mathbf{D}$ are negative, from which is concluded that the largest of the two calculated eigenvalues must be equal to the largest eigenvalue of the original system, i.e. $\rho_{2,1} = \lambda_3$. Still, we do not know whether $\rho_{1,1} = \lambda_1$ or $\rho_{1,1} = \lambda_2$. In order to investigate this another calculation is performed with $\mu = 4.05$. The Gauss factorization of the matrix $\mathbf{K} - 4.05\mathbf{M}$ has already been performed as indicated by (7-52). Since this result shows that two eigenvalues exist, which are smaller than $\mu = 4.05$, $\rho_{1,1} = 4$ must be the largest of these, and hence the 2nd eigenvalue of the original system.

7.4 Characteristic Polynomial Iteration

In this section it is assumed that the stiffness and mass matrices have been reduced to a three diagonal form through a series of similarity transformations as explained in Section 6.4, corresponding to, cf. (6-32)

$$\mathbf{K} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 & 0 \\ 0 & \beta_2 & \alpha_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} & \alpha_n \end{bmatrix} \tag{7–58}$$

$$\mathbf{M} = \begin{bmatrix} \gamma_1 & \delta_1 & 0 & \cdots & 0 & 0 \\ \delta_1 & \gamma_2 & \delta_2 & \cdots & 0 & 0 \\ 0 & \delta_2 & \gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_{n-1} & \delta_{n-1} \\ 0 & 0 & 0 & \cdots & \delta_{n-1} & \gamma_n \end{bmatrix} \tag{7–59}$$

The Helmholz reduction in Section 6.4 results in $\mathbf{M} = \mathbf{I}$. The slightly more general case, where $\mathbf{M}$ is three diagonal has been assumed in what follows. In principle polynomial iteration methods works equally well on fully populated stiffness and mass matrices. However, the computational efforts become too extensive to make them competitive in this case.

Now, the characteristic equation of the generalized eigenvalue problem can be written in the following form, cf. (1-10)

$$P(\lambda) = P^{(0)}(\lambda) = \det\left(\mathbf{K} - \lambda\mathbf{M}\right) =$$

$$\det\left(\begin{bmatrix} \alpha_1 - \lambda\gamma_1 & \beta_1 - \lambda\delta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 - \lambda\delta_1 & \alpha_2 - \lambda\gamma_2 & \beta_2 - \lambda\delta_2 & \cdots & 0 & 0 \\ 0 & \beta_2 - \lambda\delta_2 & \alpha_3 - \lambda\gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} - \lambda\gamma_{n-1} & \beta_{n-1} - \lambda\delta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} - \lambda\delta_{n-1} & \alpha_n - \lambda\gamma_n \end{bmatrix}\right) =$$

$$\left(\alpha_n - \lambda\gamma_n\right) \cdot P^{(1)}(\lambda) - \left(\beta_{n-1} - \lambda\delta_{n-1}\right)^2 \cdot P^{(2)}(\lambda) \tag{7–60}$$

The last statement in (7-60) is obtained by expanding the determinant after the components in the last row. $P^{(1)}(\lambda)$ and $P^{(2)}(\lambda)$ denote the characteristic polynomials obtained by omitting the last row and column, and the last two rows and columns in the matrix $\mathbf{K} - \lambda\mathbf{M}$, respectively, cf. (3-26). The validity of the result (7-60) has been demonstrated for a 4-dimensional case in Example 7.3. In turn, $P^{(1)}(\lambda)$ may be expressed in terms of $P^{(2)}(\lambda)$ and $P^{(3)}(\lambda)$ by a similar expression. Actually, the complete Sturm sequence of characteristic polynomials may be calculated recursively from the algorithm

$$\left.\begin{aligned}
P^{(n-1)}(\lambda) &= \left(\alpha_1 - \lambda\gamma_1\right) \\
P^{(n-2)}(\lambda) &= \left(\alpha_1 - \lambda\gamma_1\right)\left(\alpha_2 - \lambda\gamma_2\right) - \left(\beta_1 - \lambda\delta_1\right)^2 \\
P^{(n-m)}(\lambda) &= \left(\alpha_m - \lambda\gamma_m\right) \cdot P^{(n-m+1)}(\lambda) - \left(\beta_{m-1} - \lambda\delta_{m-1}\right)^2 \cdot P^{(n-m+2)}(\lambda)\ ,\ m = 3, 4, \ldots, n
\end{aligned}\right\} \tag{7–61}$$

The effectiveness of characteristic polynomial iteration methods for matrices on three diagonal form relies on the result (7-61).

Assume that the jth eigensolution $\left(\lambda_j, \Phi^{(j)}\right)$ is wanted. At first one needs to determine two figures μ_0 and μ_1 fulfilling $\lambda_{j-1} < \mu_0 < \lambda_j < \mu_1 < \lambda_{j+1}$. This is done based on the sequence of signs $\mathrm{sign}(P^{(n)}(\mu)), \mathrm{sign}(P^{(n-1)}(\mu)), \ldots, \mathrm{sign}(P^{(1)}(\mu)), \mathrm{sign}(P^{(0)}(\mu))$, in which the number of sign changes indicates the total number of eigenvalues smaller than μ, and where $P^{(n)}(\mu)$ is a dummy positive figure, cf. Section 3.2.
Below, on Fig. 7-2 are marked two points μ_{k-1} and μ_k on the λ-axis in the vicinity of the eigenvalue searched for, which is λ_1 in the illustrated case. The values of the characteristic polynomial in these points, $P(\mu_{k-1})$ and $P(\mu_k)$, may easily be calculated by means of (7-61) (notice that $P(\mu) = P^{(0)}(\mu)$). The line through the points $\left(\mu_{k-1}, P(\mu_{k-1})\right)$ and $\left(\mu_k, P(\mu_k)\right)$ has the equation

$$y(\lambda) = P(\mu_k) + \left(P(\mu_k) - P(\mu_{k-1})\right)\frac{\lambda - \mu_k}{\mu_k - \mu_{k-1}} \tag{7–62}$$

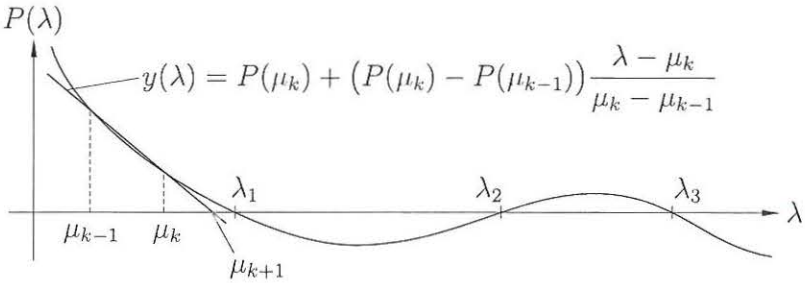

Fig. 7–2 Secant iteration of characteristic equation towards λ_1.

The line defined by (7-62) intersects the λ-axis at the point μ_{k+1}. It is clear that this point will be closer to λ_j than both μ_{k-1} and μ_k. The intersection point of the line with the λ-axis is obtained as the solution to the equation $y(\lambda) = y(\mu_{k+1}) = 0$, which is given as

$$\mu_{k+1} = \mu_k - \frac{P(\mu_k)}{P(\mu_k) - P(\mu_{k-1})}\left(\mu_k - \mu_{k-1}\right) \tag{7–63}$$

Next, the iteration index is raised to $k+1$, and a new intersection point μ_{k+2} is obtained. The sequence $\mu_0, \mu_1, \mu_2, \ldots$ converges relatively fast to the eigenvalue λ_j as demonstrated below in

Example 7.4.

Box 7.4: Characteristic polynomial iteration algorithm

In order to calculate the jth eigenvalue λ_j and the jth eigenvector $\Phi^{(j)}$ the following items are performed

1. Based on the sequence of signs $\text{sign}(P^{(n)}(\mu)), \text{sign}(P^{(n-1)}(\mu)), \ldots, \text{sign}\big(P^{(1)}(\mu)\big)$, $\text{sign}\big(P^{(0)}(\mu)\big)$ of the Sturm sequence of characteristic polynomials determine two figures μ_0 and μ_1 fulfilling the inequalities:

   $\lambda_{j-1} < \mu_0 < \lambda_j < \mu_1 < \lambda_{j+1}$

2. Perform secant iteration in search for $\lambda_j = \mu_\infty$ according to the algorithm:

   $\mu_{k+1} = \mu_k - \frac{P(\mu_k)}{P(\mu_k)-P(\mu_{k-1})}\big(\mu_k - \mu_{k-1}\big)$

3. Determine the unnormalized eigenmode $\bar{\Phi}^{(j)}$ from the algorithm (7-65).

4. Normalize the eigenmode to unit modal mass:

   $\Phi^{(j)} = \frac{\bar{\Phi}^{(j)}}{\sqrt{\bar{\Phi}^{(j)T}\mathrm{M}\bar{\Phi}^{(j)}}}$

Alternatively, the eigenvalue λ_j may be determined by means of Sturm sequence check, where the interval $]\mu_0, \mu_1[$ is increasingly narrowed around the eigenvalue λ_j by bisection of the previous interval. This algorithm, which is merely telescope method described in Section 6.2, will generally converge much slower than the secant iteration algorithm.

Finally, the components $\left[\Phi_1^{(j)}, \Phi_2^{(j)}, \ldots, \Phi_n^{(j)}\right]$ of the eigenmode $\Phi^{(j)}$ are determined as non-trivial solutions to the linear equations

$$\big(\mathrm{K} - \lambda_j \mathrm{M}\big)\Phi^{(j)} = 0 \quad \Rightarrow$$

$$\begin{bmatrix} \alpha_1 - \lambda_j\gamma_1 & \beta_1 - \lambda_j\delta_1 & 0 & \cdots & 0 & 0 \\ \beta_1 - \lambda_j\delta_1 & \alpha_2 - \lambda_j\gamma_2 & \beta_2 - \lambda_j\delta_2 & \cdots & 0 & 0 \\ 0 & \beta_2 - \lambda_j\delta_2 & \alpha_3 - \lambda_j\gamma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} - \lambda_j\gamma_{n-1} & \beta_{n-1} - \lambda_j\delta_{n-1} \\ 0 & 0 & 0 & \cdots & \beta_{n-1} - \lambda_j\delta_{n-1} & \alpha_n - \lambda_j\gamma_n \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \\ \vdots \\ \Phi_{n-1}^{(j)} \\ \Phi_n^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{7–64}$$

Let $\bar{\Phi}^{(j)} = [\bar{\Phi}_1^{(j)}, \bar{\Phi}_2^{(j)}, \ldots, \bar{\Phi}_n^{(j)}]$ denote the eigenmode with components arbitrarily normalized. Setting $\bar{\Phi}_1^{(j)} = 1$ the equations (7-64) may be solved recursively from above by the following algorithm

$$\left.\begin{aligned}
\bar{\Phi}_2^{(j)} &= -\frac{\alpha_1 - \lambda_j\gamma_1}{\beta_1 - \lambda_j\delta_1} \cdot 1 \\
\bar{\Phi}_3^{(j)} &= -\frac{\beta_1 - \lambda_j\delta_1}{\beta_2 - \lambda_j\delta_2} \cdot 1 - \frac{\alpha_2 - \lambda_j\gamma_2}{\beta_2 - \lambda_j\delta_2} \cdot \bar{\Phi}_2^{(j)} \\
\bar{\Phi}_m^{(j)} &= -\frac{\beta_{m-2} - \lambda_j\delta_{m-2}}{\beta_{m-1} - \lambda_j\delta_{m-1}} \cdot \bar{\Phi}_{m-2}^{(j)} - \frac{\alpha_{m-1} - \lambda_j\gamma_{m-1}}{\beta_{m-1} - \lambda_j\delta_{m-1}}\bar{\Phi}_{m-1}^{(j)} \quad , \quad m = 4, \ldots, n
\end{aligned}\right\} \tag{7–65}$$

Hence, the determination of the components of the vector $\bar{\Phi}^{(j)}$ is almost free. Obvious, the indicated algorithm breaks down, if any of the denominators $\beta_{m-1} - \lambda_j\delta_{m-1} = 0$. This means that the algorithm should be extended with alternatives to deal with such exceptions.

Finally, the eigenmode $\bar{\Phi}^{(j)}$ should be normalized to unit modal mass as follows

$$\Phi^{(j)} = \frac{\bar{\Phi}^{(j)}}{\sqrt{\bar{\Phi}^{(j)T}\mathbf{M}\bar{\Phi}^{(j)}}} \tag{7–66}$$

Example 7.3: Evaluation of determinant

The determinant of the following matrix on a three diagonal form of the dimension 4×4 is wanted

$$\mathbf{K} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 \\ 0 & 0 & \beta_3 & \alpha_4 \end{bmatrix} \tag{7–67}$$

Expansion of the determinant after the components in the 4th row provides

$$\det(\mathbf{K}) = P^{(0)} = \alpha_4 \cdot \det\left(\begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \beta_1 & \alpha_2 & \beta_2 \\ 0 & \beta_2 & \alpha_3 \end{bmatrix}\right) - \beta_3 \cdot \det\left(\begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \beta_1 & \alpha_2 & 0 \\ 0 & \beta_2 & \beta_3 \end{bmatrix}\right) =$$

$$\alpha_4 \cdot \det\left(\begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \beta_1 & \alpha_2 & \beta_2 \\ 0 & \beta_2 & \alpha_3 \end{bmatrix}\right) - \beta_3^2 \cdot \det\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \alpha_2 \end{bmatrix}\right) = \alpha_4 \cdot P^{(1)} - \beta_3^2 \cdot P^{(2)} \tag{7–68}$$

(7-68) has the same recursive structure as described by (7-60).

Example 7.4: Characteristic polynomial iteration

The generalized eigenvalue problem defined in Example 1.4 is considered again. Calculate the 3rd eigenvalue by secant iteration on the characteristic polynomial, and next determine the corresponding eigenvector.

At first a calculation with $\mu = 2.5$ is performed, which produces the following results

$$\left.\begin{array}{l}
\mathbf{K} - 2.5\mathbf{M} = \begin{bmatrix} 0.7500 & -1.0000 & 0.0000 \\ -1.0000 & 1.5000 & -1.0000 \\ 0.0000 & -1.0000 & 0.7500 \end{bmatrix} \quad \Rightarrow \\
\begin{cases} P^{(3)}(2.5) = 1 & , \quad \mathrm{sign}(P^{(3)}(2.5)) = + \\ P^{(2)}(2.5) = 0.7500 & , \quad \mathrm{sign}(P^{(2)}(2.5)) = + \\ P^{(1)}(2.5) = 0.7500 \cdot 1.5000 - (-1)^2 = 0.1250 & , \quad \mathrm{sign}(P^{(1)}(2.5)) = + \\ P^{(0)}(2.5) = 0.7500 \cdot 0.1250 - (-1)^2 \cdot 0.7500 = -0.6563 & , \quad \mathrm{sign}(P^{(0)}(2.5)) = - \end{cases}
\end{array}\right\} \tag{7–69}$$

Hence, the sign sequence of the Sturm sequence becomes $+++-$. One sign change occurs in this sequence from which is concluded that the lowest eigenvalue λ_1 is smaller than $\mu = 2.5$.

Next, a calculation with $\mu = 5.5$ is performed, which provided the results

$$\left.\begin{array}{l}
\mathbf{K} - 5.5\mathbf{M} = \begin{bmatrix} -0.7500 & -1.0000 & 0.0000 \\ -1.0000 & -1.5000 & -1.0000 \\ 0.0000 & -1.0000 & -0.7500 \end{bmatrix} \quad \Rightarrow \\
\begin{cases} P^{(3)}(5.5) = 1 & , \quad \mathrm{sign}(P^{(3)}(5.5)) = + \\ P^{(2)}(5.5) = -0.7500 & , \quad \mathrm{sign}(P^{(2)}(5.5)) = - \\ P^{(1)}(5.5) = (-0.7500) \cdot (-1.5000) - (-1)^2 = 0.1250 & , \quad \mathrm{sign}(P^{(1)}(5.5)) = + \\ P^{(0)}(5.5) = (-0.7500) \cdot 0.1250 - (-1)^2 \cdot (-0.7500) = 0.6563 & , \quad \mathrm{sign}(P^{(0)}(5.5)) = + \end{cases}
\end{array}\right\} \tag{7–70}$$

Now, the sign sequence of the Sturm sequence becomes $+-++$, in which two sign changes occur, from which is concluded that the lowest two eigenvalues λ_1 and λ_2 are both smaller than $\mu = 5.5$.

Finally a calculation with $\mu = 6.5$ is performed, which provided the results

$$\left.\begin{array}{l}
\mathbf{K} - 6.5\mathbf{M} = \begin{bmatrix} -1.2500 & -1.0000 & 0.0000 \\ -1.0000 & -2.5000 & -1.0000 \\ 0.0000 & -1.0000 & -1.2500 \end{bmatrix} \quad \Rightarrow \\
\begin{cases} P^{(3)}(6.5) = 1 & , \quad \mathrm{sign}(P^{(3)}(6.5)) = + \\ P^{(2)}(6.5) = -1.2500 & , \quad \mathrm{sign}(P^{(2)}(6.5)) = - \\ P^{(1)}(6.5) = (-1.2500) \cdot (-2.5000) - (-1)^2 = 2.1250 & , \quad \mathrm{sign}(P^{(1)}(6.5)) = + \\ P^{(0)}(6.5) = (-1.2500) \cdot 2.1250 - (-1)^2 \cdot (-1.2500) = -1.4063 & , \quad \mathrm{sign}(P^{(0)}(6.5)) = - \end{cases}
\end{array}\right\} \tag{7–71}$$

In this case the sign sequence of the Sturm sequence becomes $+-+-$, corresponding to three sign changes. Hence, it is concluded that all three eigenvalues λ_1, λ_2 and λ_3 are smaller than $\mu = 6.5$.

From the Sturm sequence checks it is concluded that $\lambda_2 < 5.5 < \lambda_3 < 6.5$. Then, we may use the following start values, $\mu_0 = 5.5$ and $\mu_1 = 6.5$, in the secant iteration algorithm. Moreover $P(5.5) = P^{(0)}(5.5) = 0.6563$ and $P(6.5) = P^{(0)}(6.5) = -1.4063$, cf. (7-70) and (7-71). Then, from (7-61) it follows for $k = 1$

$$\mu_2 = 6.5 - \frac{(-1.4063)}{(-1.4063) - 0.6563}(6.5 - 5.5) = 5.8182 \tag{7–72}$$

Next, $P(\mu_2) = P(5.8182) = 0.3156$ is calculated by means of the algorithm (7-61), and a new value μ_3 can be obtained from

$$\mu_3 = 5.8182 - \frac{0.3156}{0.3156 - (-1.4063)}(5.8182 - 6.5) = 5.9431 \tag{7–73}$$

During the next 5 iterations the following results were obtained

$$\left.\begin{aligned} \mu_4 &= 6.00900500472288 \\ \mu_5 &= 5.99960498912941 \\ \mu_6 &= 5.99999734553262 \\ \mu_7 &= 6.00000000078659 \\ \mu_8 &= 6.00000000000000 \end{aligned}\right\} \tag{7–74}$$

As seen the convergence of the secant iteration algorithm is very fast.

The linear equation (7-64) attains the form

$$\left(\mathbf{K} - 6.0000\mathbf{M}\right)\bar{\Phi}^{(3)} = \mathbf{0} \quad \Rightarrow$$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \bar{\Phi}_1^{(3)} \\ \bar{\Phi}_2^{(3)} \\ \bar{\Phi}_3^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7–75}$$

Setting $\bar{\Phi}_1^{(3)} = 1$ the algorithm (7-65) now provides

$$\left.\begin{aligned} \bar{\Phi}_2^{(3)} &= -\frac{(-1)}{(-1)} \cdot 1 = -1 \\ \bar{\Phi}_3^{(3)} &= -\frac{(-1)}{(-1)} \cdot 1 - \frac{(-2)}{(-1)} \cdot (-1) = 1 \end{aligned}\right\} \quad \Rightarrow \quad \bar{\Phi}^{(3)} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{7–76}$$

Normalization to unit modal mass provides, cf. (1-87)

$$\Phi^{(3)} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{7–77}$$

7.5 Exercises

7.1 Given the following mass- and stiffness matrices defined in Exercise 4.2.

(a.) Calculate the two lowest eigenmodes and corresponding eigenvalues by simultaneous inverse vector iteration with the start vector basis

$$\Phi_0 = \left[\Phi_0^{(1)}\ \Phi_0^{(2)}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$$

7.2 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$ of dimension n.

(a.) Write a MATLAB program, which for given start basis performs simultaneous inverse vector iteration for the determination of the lowest n_1 eigenmodes and eigenvalues.

7.3 Consider the general eigenvalue problem in Exercise 4.2.

(a.) Calculate the two lowest eigenmodes and corresponding eigenvalues by subspace iteration using the same start basis as in Exercise 7.1.

7.4 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$ of dimension n.

(a.) Write a MATLAB program, which for given start basis performs subspace iteration for the determination of the lowest n_1 eigenmodes and eigenvalues.

7.5 Consider the general eigenvalue problem in Exercise 4.2.

(a.) Calculate the 3rd eigenmode and eigenvalue by Sturm sequence iteration (telescope method).

7.6 Given the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$ of dimension n on three diagonal form.

(a.) Write a MATLAB program, which performs Sturm sequence check and secant iteration iteration for the determination of the jth eigenvalue, and next determines the corresponding eigenvector.

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APPENDIX A
Solutions to Exercises

A.1 Exercise 1.1

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

1. Calculate the eigenvalues and eigenmodes normalized to unit modal mass.
2. Determine two vectors that are M-orthonormal, but are not eigenmodes.

SOLUTIONS:

Question 1:

The generalized eigenvalue problem (1-9) becomes

$$\begin{bmatrix} 2-\lambda_j & -1 & 0 \\ -1 & 2-2\lambda_j & 0 \\ 0 & 0 & 3-\frac{1}{2}\lambda_j \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ \Phi_3^{(j)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Upon evaluating the determinant of the coefficient matrix after the 3rd row, the characteristic equation (1-10) becomes

$$P(\lambda) = P^{(0)}(\lambda) = \det\left(\begin{bmatrix} 2-\lambda_j & -1 & 0 \\ -1 & 2-2\lambda_j & 0 \\ 0 & 0 & 3-\frac{1}{2}\lambda_j \end{bmatrix}\right) =$$

$$\left(3-\frac{1}{2}\lambda_j\right)\left((2-\lambda_j)\left(2-2\lambda_j\right)-(-1)^2\right) = \left(3-\frac{1}{2}\lambda_j\right)\left(3-6\lambda_j+2\lambda_j^2\right) = 0 \quad \Rightarrow$$

$$\lambda_j = \begin{cases} \frac{1}{2}(3-\sqrt{3}) & , \quad j=1 \\ \frac{1}{2}(3+\sqrt{3}) & , \quad j=2 \\ 6 & , \quad j=3 \end{cases} \tag{3}$$

The largest eigenvalue $\lambda_3 = 6$ is obtained when the 1st factor in (3) is equal to 0, whereas the two lowest solutions corresponds vanishing of the 2nd factor.

Because the 3rd eigenmode is decoupled from the 1st and 2nd the solution method is slightly different in this case. As seen be inspection the solutions have the form

$$\left.\begin{aligned} \bar{\mathbf{\Phi}}^{(j)} &= \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \\ 0 \end{bmatrix} \quad , \quad j = 1,2 \\ \bar{\mathbf{\Phi}}^{(3)} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}\right\} \tag{4}$$

The 1st and 2nd components of the 1st and 2nd eigenmodes, $\Phi_1^{(j)}$ and $\Phi_2^{(j)}$ are determined from the two first equations in (2). We choose to set $\Phi_1^{(j)} = 1$, and determine $\Phi_2^{(j)}$ from the 1st equations. Notice that we may as well have determined $\Phi_2^{(j)}$ from the 2nd equation. Then

$$\left(2 - \lambda_j\right) \cdot 1 - \Phi_2^{(j)} = 0 \quad \Rightarrow \quad \Phi_2^{(j)} = \begin{cases} \frac{1}{2}\left(1 + \sqrt{3}\right) & , \quad j = 1 \\ \frac{1}{2}\left(1 - \sqrt{3}\right) & , \quad j = 2 \end{cases} \tag{5}$$

The modal masses become

$$M_j = \bar{\mathbf{\Phi}}^{(j)\,T} \mathbf{M} \bar{\mathbf{\Phi}}^{(j)} = \begin{bmatrix} 1 \\ \Phi_2^{(j)} \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \Phi_2^{(j)} \\ 0 \end{bmatrix} = 1 + 2\left(\Phi_2^{(j)}\right)^2 = \begin{cases} 3 + \sqrt{3} & , \quad j = 1 \\ 3 - \sqrt{3} & , \quad j = 2 \end{cases} \tag{6}$$

$$M_3 = \bar{\mathbf{\Phi}}^{(3)\,T} \mathbf{M} \bar{\mathbf{\Phi}}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \tag{7}$$

$\mathbf{\Phi}^{(1)}$ denotes the 1st eigenmode normalized to unit modal mass. This is related to $\bar{\mathbf{\Phi}}^{(1)}$ in the following way

$$\mathbf{\Phi}^{(1)} = \frac{1}{\sqrt{M_1}} \bar{\mathbf{\Phi}}^{(1)} = \frac{1}{\sqrt{3 + \sqrt{3}}} \begin{bmatrix} 1 \\ \frac{1}{2}\left(1 + \sqrt{3}\right) \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4597 \\ 0.6280 \\ 0 \end{bmatrix} \tag{8}$$

The other modes are treated in the same manner, which results in the following eigensolutions

$$\left.\begin{aligned}
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3-\sqrt{3}) & 0 & 0 \\ 0 & \frac{1}{2}(3+\sqrt{3}) & 0 \\ 0 & 0 & 6 \end{bmatrix} \\
\mathbf{\Phi} &= \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(3)}\right] = \begin{bmatrix} 0.4597 & 0.8881 & 0 \\ 0.6280 & -0.3251 & 0 \\ 0 & 0 & 1.4142 \end{bmatrix}
\end{aligned}\right\} \tag{9}$$

Question 2:

Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \tag{10}$$

Upon insertion the following relations are seen to be valid

$$\mathbf{v}_1^T \mathbf{M} \mathbf{v}_1 = 1 \quad , \quad \mathbf{v}_2^T \mathbf{M} \mathbf{v}_2 = 1 \quad , \quad \mathbf{v}_1^T \mathbf{M} \mathbf{v}_2 = 0 \tag{11}$$

Hence, $\mathbf{v}_1$ and $\mathbf{v}_2$ are mutually $\mathbf{M}$-orthonormal. However,

$$\left.\begin{aligned}
\mathbf{K}\mathbf{v}_1 &= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \neq \lambda_1 \mathbf{M}\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2}(3-\sqrt{3}) \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{K}\mathbf{v}_2 &= \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \sqrt{2} \\ 0 \end{bmatrix} \neq \lambda_2 \mathbf{M}\mathbf{v}_2 = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{4}(3+\sqrt{3}) \\ 0 \end{bmatrix}
\end{aligned}\right\} \tag{12}$$

Hence, neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ are eigenmodes.

A.2 Exercise 1.2

The eigensolutions with eigenmodes normalized to unit modal mass of a 2-dimensional generalized eigenvalue problem are given as

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}^{(1)} \, \mathbf{\Phi}^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{1}$$

1. Calculate $\mathbf{M}$ and $\mathbf{K}$.

SOLUTIONS:

Question 1:

From (1-19) and (1-20) follows

$$\mathbf{M} = \left(\mathbf{\Phi}^{-1}\right)^T \mathbf{m} \mathbf{\Phi}^{-1} \tag{2}$$

$$\mathbf{K} = \left(\mathbf{\Phi}^{-1}\right)^T \mathbf{k} \mathbf{\Phi}^{-1} \tag{3}$$

Since it is known that the eigenmodes have been normalized to unit modal mass it follows from (1-20) and (1-22) that

$$\mathbf{m} = \mathbf{I} \quad , \quad \mathbf{k} = \mathbf{\Lambda} \tag{4}$$

The inverse of the modal matrix becomes

$$\mathbf{\Phi}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{5}$$

Of course, (5) can be obtained by direct calculation. Alternatively, the result may be obtained from the following arguments. Notice that $\mathbf{\Phi}$ is orthonormal, so $\mathbf{\Phi}^{-1} = \mathbf{\Phi}^T$, cf. (1-23). Additionally, the modal matrix is symmetric, i.e. $\mathbf{\Phi} = \mathbf{\Phi}^T$, from which the indicated result follows.

Insertion of (4) and (5) into (1) and (2) provides

$$\mathbf{M} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$\mathbf{K} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 2.5 & -1.5 \\ -1.5 & 2.5 \end{bmatrix} \tag{7}$$

Actually, since $\mathbf{M} = \mathbf{I}$, the considered eigenvalue problem is of the special type, cf. the remarks subsequent to (1-9).

A.3 Exercise 3.1

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{1}$$

1. Show that the eigenvalue separation principle is valid for the considered example.

SOLUTIONS:

Question 1:

The eigenvalues $\lambda_j^{(0)}$, which have been calculated in Exercise 1.1, are

$$\lambda_j = \begin{cases} \frac{1}{2}(3-\sqrt{3}) & , \quad j=1 \\ \frac{1}{2}(3+\sqrt{3}) & , \quad j=2 \\ 6 & , \quad j=3 \end{cases} \tag{2}$$

Correspondingly, the eigenvalues $\lambda_j^{(1)}$ and $\lambda_j^{(1)}$ are calculated from

$$P^{(1)}(\lambda^{(1)}) = \det\left(\begin{bmatrix} 2-\lambda_j^{(1)} & -1 \\ -1 & 2-2\lambda_j^{(1)} \end{bmatrix}\right) =$$

$$\left((2-\lambda_j^{(1)})\left(2-2\lambda_j^{(1)}\right) - (-1)^2\right) = \left(3-6\lambda_j^{(1)}) + 2(\lambda_j^{(1)})^2\right) = 0 \quad \Rightarrow$$

$$\lambda_j^{(1)} = \begin{cases} \frac{1}{2}(3-\sqrt{3}) & , \quad j=1 \\ \frac{1}{2}(3+\sqrt{3}) & , \quad j=2 \end{cases} \tag{3}$$

$$P^{(2)}(\lambda^{(2)}) = \det\left(\left[2-\lambda_j^{(2)}\right]\right) \quad \Rightarrow \quad \lambda_j^{(2)} = 2 \tag{4}$$

Then, (3-25) attains the following forms for $m=0$ and $m=1$

$$0 \le \lambda_1^{(0)} \le \lambda_1^{(1)} \le \lambda_2^{(0)} \le \lambda_2^{(1)} \le \lambda_3^{(0)} \le \infty \quad \Rightarrow$$

$$0 \le \frac{1}{2}(3-\sqrt{3}) \le \frac{1}{2}(3-\sqrt{3}) \le \frac{1}{2}(3+\sqrt{3}) \le \frac{1}{2}(1+\sqrt{3}) \le 6 \le \infty \tag{5}$$

$$0 \le \lambda_1^{(1)} \le \lambda_1^{(2)} \le \lambda_2^{(1)} \le \infty \quad \Rightarrow$$

$$0 \le \frac{1}{2}(3-\sqrt{3}) \le 2 \le \frac{1}{2}(3+\sqrt{3}) \le \infty \tag{6}$$

Hence, (3-25) holds for the considered example. $\lambda_1^{(0)} = \lambda_1^{(1)}$ and $\lambda_2^{(0)} = \lambda_2^{(1)}$, because of the decoupling of the 3rd eigenmode from the 1st and 2nd eigenmode.

A.4 Exercise 3.2

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix} \tag{1}$$

1. Calculate the eigenvalues and eigenmodes normalized to unit modal mass.
2. Perform a shift $\rho = 3$ on $\mathbf{K}$ and calculate the eigenvalues and eigenmodes of the new problem.

SOLUTIONS:

Question 1:

The generalized eigenvalue problem (1-9) is written on the form

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} = \frac{1}{\lambda_j} \begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \Phi_1^{(j)} \\ \Phi_2^{(j)} \end{bmatrix} \quad , \quad j = 1, 2 \tag{2}$$

Obviously, (2) has the solution

$$\lambda_2 = \infty \quad , \quad \mathbf{\Phi}^{(2)} = \begin{bmatrix} \Phi_1^{(2)} \\ \Phi_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

Hence, $\lambda_2 = \infty$ is an eigenvalue. This is so because the mass matrix is singular, and has zeroes in the last row and column. Since, the modal mass M_2 related to eigenmode $\mathbf{\Phi}^{(2)}$ is zero, this mode cannot be normalized in the usual manner. In Section 4.1 the problem of infinite eigenvalues will be thoroughly dealt with.

The other eigensolution may be obtained by the standard approach. Then, the eigenvalue problem (2) is written on the form

$$\begin{bmatrix} 6 - 2\lambda_1 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4}$$

The characteristic equation (1-11) becomes

$$\det\left(\begin{bmatrix} 6 - 2\lambda_1 & -1 \\ -1 & 4 \end{bmatrix} \right) = 4(6 - 2\lambda_1) - (-1)^2 = 23 - 8\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{23}{8} \tag{5}$$

We choose to set $\Phi_1^{(1)} = 1$, and determine $\Phi_2^{(1)}$ from the 1st equations. Then

$$(6 - 2\lambda_1) \cdot 1 - \Phi_2^{(1)} = 0 \quad \Rightarrow \quad \Phi_2^{(1)} = \frac{1}{4} \tag{6}$$

The modal mass becomes

$$M_1 = \bar{\mathbf{\Phi}}^{(1)T}\mathbf{M}\bar{\mathbf{\Phi}}^{(1)} = \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} = 2 \tag{7}$$

Then, the eigenmode normalized to unit modal mass $\mathbf{\Phi}^{(1)}$ becomes

$$\mathbf{\Phi}^{(1)} = \frac{1}{\sqrt{M_1}}\bar{\mathbf{\Phi}}^{(1)} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.1768 \end{bmatrix} \tag{8}$$

Hence, the following eigensolutions have been obtained

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{23}{8} & 0 \\ 0 & \infty \end{bmatrix} \quad , \quad \mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}^{(1)} & \mathbf{\Phi}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.7071 & 0 \\ 0.1768 & 1 \end{bmatrix} \tag{9}$$

Question 2:

(3-38) attains the form

$$\hat{\mathbf{K}} = \mathbf{K} - 3\mathrm{M} = \begin{bmatrix} 6 & -1 \\ -1 & 4 \end{bmatrix} - 3 \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix} \tag{10}$$

The eigenvalue problem (3-37) becomes

$$\left(\begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix} - \lambda_j \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \Phi_1^{(1)} \\ \Phi_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

For the same reason as in question 1, $\lambda_2 = \infty$ is still an eigenvalue with the eigenmode given by (3). The characteristic equation for the 1st eigenvalue becomes, cf. (5)

$$\det\left(\begin{bmatrix} -2\lambda_1 & -1 \\ -1 & 4 \end{bmatrix} \right) = 4(-2\lambda_1) - (-1)^2 = -1 - 8\lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = -\frac{1}{8} \quad \left(= \frac{23}{8} - 3 \right) \tag{12}$$

Let $\Phi_1^{(1)} = 1$, and determine $\Phi_2^{(1)}$ from the 1st equations of (11)

$$(-2\lambda_1) \cdot 1 - \Phi_2^{(1)} = 0 \quad \Rightarrow \quad \Phi_2^{(1)} = \frac{1}{4} \tag{13}$$

which is identical to (6). Hence $\mathbf{\Phi}^{(1)}$ is unaffected by the shift as expected, cf. the comments following (3-38). The eigensolutions are unchanged as given by (9), save that $\lambda_1 = -\frac{1}{8}$.

A.5 Exercise 3.4: Theory

Gauss Elimination

Given a symmetric matrix $\mathbf{K}$ of the dimension $n \times n$ with the components $K_{ij} = K_{ji}$. Consider the static equilibrium equation

$$\mathbf{Kx} = \mathbf{f} \quad \Rightarrow$$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \tag{1}$$

In order to have a one at the 1st element of the main diagonal of the coefficient matrix the 1st equation is divided with K_{11} resulting in

$$\begin{bmatrix} 1 & K_{12}^{(1)} & K_{13}^{(1)} & \cdots & K_{1n}^{(1)} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} \tag{2}$$

where

$$\left.\begin{aligned} K_{1j}^{(1)} &= \frac{K_{1j}}{K_{11}} \quad , \quad j = 2, \ldots, n \\ f_1^{(1)} &= \frac{f_1}{K_{11}} \end{aligned}\right\} \tag{3}$$

In turn, the 1st equation of (2) is multiplied with K_{i1}, $i = 2, \ldots, n$, and the resulting equation is withdrawn from the ith equation. This will produce a zero in the ith row of the 1st column, corresponding to the following system of equations

$$\begin{bmatrix} 1 & K_{12}^{(1)} & K_{13}^{(1)} & \cdots & K_{1n}^{(1)} \\ 0 & K_{22}^{(1)} & K_{23}^{(1)} & \cdots & K_{2n}^{(1)} \\ 0 & K_{32}^{(1)} & K_{33}^{(1)} & \cdots & K_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & K_{n2}^{(1)} & K_{n3}^{(1)} & \cdots & K_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ \vdots \\ f_n^{(1)} \end{bmatrix} \tag{4}$$

where

$$\left.\begin{aligned} K_{ij}^{(1)} &= K_{ij} - K_{i1} K_{1j}^{(1)} \quad , \quad i = 2, \ldots, n \quad , \quad j = 2, \ldots, n \\ f_i^{(1)} &= f_i - K_{i1} f_1^{(1)} \quad , \quad i = 2, \ldots, n \end{aligned}\right\} \tag{5}$$

Next, the 2nd equation is divided with $K_{22}^{(1)}$, so the coefficient in the 2nd component in the main diagonal becomes equal to 1. In turn, the resulting 2nd equation is multiplied with $K_{i2}^{(1)}$, $i = 3, \ldots, n$, and the resulting equation is withdrawn from the ith equation. This will produce zeros in the ith row of the 2nd column below the main diagonal, corresponding to the system of equations

$$\begin{bmatrix} 1 & K_{12}^{(1)} & K_{13}^{(1)} & \cdots & K_{1n}^{(1)} \\ 0 & 1 & K_{23}^{(2)} & \cdots & K_{2n}^{(2)} \\ 0 & 0 & K_{33}^{(2)} & \cdots & K_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & K_{n3}^{(2)} & \cdots & K_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(2)} \\ f_3^{(2)} \\ \vdots \\ f_n^{(2)} \end{bmatrix} \tag{6}$$

where

$$\left.\begin{aligned} K_{2j}^{(2)} &= \frac{K_{2j}^{(1)}}{K_{22}^{(1)}} \quad , \quad j = 3, \ldots, n \\ f_2^{(2)} &= \frac{f_2^{(1)}}{K_{22}^{(1)}} \\ K_{ij}^{(2)} &= K_{ij}^{(1)} - K_{i2}^{(1)} K_{2j}^{(2)} \quad , \quad i = 3, \ldots, n \quad , \quad j = 3, \ldots, n \\ f_i^{(2)} &= f_i^{(1)} - K_{i2}^{(1)} f_2^{(2)} \quad , \quad i = 3, \ldots, n \end{aligned}\right\} \tag{7}$$

The process of producing ones in the main diagonal, and zeros below the main diagonal is continued for all n columns resulting in the following system of linear equations

$$\begin{bmatrix} 1 & K_{12}^{(1)} & K_{13}^{(1)} & \cdots & K_{1n}^{(1)} \\ 0 & 1 & K_{23}^{(2)} & \cdots & K_{2n}^{(2)} \\ 0 & 0 & 1 & \cdots & K_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(2)} \\ f_3^{(3)} \\ \vdots \\ f_n^{(n)} \end{bmatrix} \tag{8}$$

Next, (1) are solved simultaneous with n righthand sides, where the loads form the columns in a unit matrix. The n solution vectors $\mathbf{X} = [\mathbf{x}_1\, \mathbf{x}_2\, \mathbf{x}_3 \cdots \mathbf{x}_n]$ are organized in the matrix equation, i.e.

$$\mathbf{KX} = \mathbf{I} \quad \Rightarrow$$

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \cdots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \cdots & K_{nn} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{9}$$

Following the steps (2)-(8), simultaneous Gauss elimination of the coefficient matrix and the n righthand sides provides the following equivalent matrix equation

$$\begin{bmatrix} 1 & K_{12}^{(1)} & K_{13}^{(1)} & \cdots & K_{1n}^{(1)} \\ 0 & 1 & K_{23}^{(2)} & \cdots & K_{2n}^{(2)} \\ 0 & 0 & 1 & \cdots & K_{3n}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{bmatrix} = \begin{bmatrix} f_{11}^{(1)} & 0 & 0 & \cdots & 0 \\ f_{21}^{(2)} & f_{22}^{(2)} & 0 & \cdots & 0 \\ f_{31}^{(3)} & f_{32}^{(3)} & f_{33}^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1}^{(n)} & f_{n2}^{(n)} & f_{n3}^{(n)} & \cdots & f_{nn}^{(n)} \end{bmatrix} \tag{10}$$

As indicated the identity matrix on the righthand side is transformed into a lower triangular matrix $\mathbf{F}$.

In the program the triangulation of the matrix $\mathbf{K}$ and the calculation of the matrix $\mathbf{F}$ is performed in a matrix $\mathbf{A}$ of the dimension $n \times 2n$, which at the entry of the triangulation loop has the form

$$\mathbf{A} = [\mathbf{K}\,\mathbf{I}] \tag{11}$$

At exit from the triangulation loop the matrix $\mathbf{A}$ stores the triangulized stiffness at the position originally occupied by $\mathbf{K}$, and the matrix $\mathbf{F}$ at the position occupied by the unit matrix.

Calculation of $\mathbf{L}$, $\mathbf{D}$ and $(\mathbf{S}^{-1})^T$

Using the Gauss factorization of the stiffness matrix (9) may be written, cf. (3-1)

$$\begin{aligned} &\mathbf{KX} = \mathbf{LDL}^T\mathbf{X} = \mathbf{I} \quad \Rightarrow \\ &\mathbf{L}^T\mathbf{X} = \mathbf{ID}^{-1}\mathbf{L}^{-1} = \mathbf{D}^{-1}\mathbf{L}^{-1} = \mathbf{F} \end{aligned} \tag{12}$$

Upon comparison of (10) and (12) it becomes clear that $\mathbf{L}^T$ is stored as the coefficient matrix in (10), whereas the righthand sides store the matrix $\mathbf{F} = \mathbf{D}^{-1}\mathbf{L}^{-1}$. Since, $\mathbf{L}^{-1}$ is a lower triangular matrix with ones in the main diagonal, the main diagonal must contain the main diagonal of $\mathbf{D}^{-1}$. Hence,

$$\mathbf{D}^{-1} = \begin{bmatrix} f_{11}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & f_{22}^{(2)} & 0 & \cdots & 0 \\ 0 & 0 & f_{33}^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_{nn}^{(n)} \end{bmatrix} \Rightarrow \mathbf{D} = \begin{bmatrix} \frac{1}{f_{11}^{(1)}} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{f_{22}^{(2)}} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{f_{33}^{(3)}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{f_{nn}^{(n)}} \end{bmatrix} \tag{13}$$

Finally, cf. (3-49)

$$\mathbf{S} = \mathbf{LD}^{\frac{1}{2}} \quad \Rightarrow \quad \mathbf{S}^{-1} = \mathbf{D}^{-\frac{1}{2}}\mathbf{L}^{-1} = \mathbf{D}^{\frac{1}{2}}\mathbf{D}^{-1}\mathbf{L}^{-1} = \mathbf{D}^{\frac{1}{2}}\mathbf{F} \tag{14}$$

The matrices $\mathbf{D}$ and $(\mathbf{S}^{-1})^T$ are retrieved from the righthand sides of (10) as stored in the matrix $\mathbf{F}$ according to the indicated relations at the end of the program.

A.6 Exercise 4.1

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1}$$

1. Perform a static condensation by the conventional procedure based on (4-5), (4-6), and next by Rayleigh-Ritz analysis with the Ritz basis given by (4-62).

SOLUTIONS:

Question 1:

The 1st and 3rd row, and next the 1st and 3rd column of the are interchanged, which brings the matrices of the general eigenvalue problem on the following form, cf. (4-1), (4-2)

$$\left.\begin{aligned} \mathbf{M} &= \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \mathbf{K} &= \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 6 \end{bmatrix} \end{aligned}\right\} \tag{2}$$

Notice that the interchange of two rows or two columns may change the sign, but not the numerical value, of the characteristic polynomial. However, since the characteristic polynomial is zero at the eigenvalue the determination of the eigenvalues is unaffected by the sign change.

The reduced stiffness matrix (4-7) becomes

$$\tilde{\mathbf{K}}_{11} = \mathbf{K}_{11} - \mathbf{K}_{12}\mathbf{K}_{22}^{-1}\mathbf{K}_{21} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} [6]^{-1} \begin{bmatrix} 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & \frac{23}{6} \end{bmatrix} \tag{3}$$

The reduced eigenvalue problem (4-6) is solved

$$\begin{bmatrix} 2 & -1 \\ -1 & \frac{23}{6} \end{bmatrix} \mathbf{\Phi}_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{\Phi}_{11}\mathbf{\Lambda}_1 \tag{4}$$

The eigensolutions with eigenmodes normalized to modal mass 1 with respect to $\mathbf{M}_{11}$ becomes

$$\mathbf{\Lambda}_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.7325 & 0 \\ 0 & 9.1008 \end{bmatrix} \quad , \quad \mathbf{\Phi}_{11} = \left[\mathbf{\Phi}_1^{(1)}\, \mathbf{\Phi}_1^{(2)}\right] = \begin{bmatrix} 0.5320 & 1.3103 \\ 0.3892 & -0.9212 \end{bmatrix} \tag{5}$$

From (4-5) follows

$$\mathbf{\Phi}_{21} = \left[\mathbf{\Phi}_2^{(1)}\, \mathbf{\Phi}_2^{(2)}\right] = -\,[6]^{-1} \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5320 & 1.3103 \\ 0.3892 & -0.9212 \end{bmatrix} = \begin{bmatrix} 0.0649 & -0.1535 \end{bmatrix} \tag{6}$$

From (4-10) and (4-11) follows

$$\mathbf{\Lambda}_2 = [\lambda_3] = [\infty] \quad , \quad \mathbf{\Phi}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad , \quad \mathbf{\Phi}_{22} = [1] \tag{7}$$

After interchanging the degrees of freedom back to the original order (the 1st components of $\mathbf{\Phi}_{11}$ and $\mathbf{\Phi}_{12}$ are placed as the 3rd component of $\mathbf{\Phi}^{(j)}$, and the components of $\mathbf{\Phi}_{21}$ and $\mathbf{\Phi}_{22}$ are placed as the 1st component $\mathbf{\Phi}^{(j)}$, the following eigensolution is obtained

$$\left.\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.7325 & 0 & 0 \\ 0 & 9.1008 & 0 \\ 0 & 0 & \infty \end{bmatrix} \\ \mathbf{\Phi} &= \left[\mathbf{\Phi}^{(1)}\, \mathbf{\Phi}^{(2)}\, \mathbf{\Phi}^{(3)}\right] = \begin{bmatrix} 0.0649 & -0.1535 & 1 \\ 0.3892 & -0.9212 & 0 \\ 0.5320 & 1.3103 & 0 \end{bmatrix} \end{aligned}\right\} \tag{8}$$

Next, the same problem is solved by means of Rayleigh-Ritz analysis. The Ritz basis is constructed from (4-62)

$$\mathbf{\Psi}_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix} \tag{9}$$

The projected mass and stiffness matrices become, cf. (4-63), (4-64)

$$\left.\begin{aligned}
\tilde{\mathbf{M}} &= \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix}^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix} = \begin{bmatrix} 0.548125 & 0.371250 \\ 0.371250 & 0.292500 \end{bmatrix} \\
\tilde{\mathbf{K}} &= \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix}^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix} = \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \end{bmatrix}
\end{aligned}\right\} \tag{10}$$

The eigensolutions to the eigenvalue problem defined by $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ with modal masses normalized to 1 with respect to $\tilde{\mathbf{M}}$ become, cf. Box. 4.2

$$\mathbf{R} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} 0.7325 & 0 \\ 0 & 9.1008 \end{bmatrix} \quad , \quad \mathbf{Q} = \begin{bmatrix} \mathbf{q}^{(1)} \, \mathbf{q}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.6748 & 3.5418 \\ 0.9599 & -4.8415 \end{bmatrix} \tag{11}$$

The solutions for the eigenvectors become, cf. (4-51)

$$\bar{\mathbf{\Phi}} = \begin{bmatrix} \bar{\mathbf{\Phi}}^{(1)} \, \bar{\mathbf{\Phi}}^{(2)} \end{bmatrix} = \begin{bmatrix} 0.5750 & 0.1500 \\ 0.1500 & 0.3000 \\ 0.0250 & 0.0500 \end{bmatrix} \begin{bmatrix} 0.6748 & 3.5418 \\ 0.9599 & -4.8415 \end{bmatrix} = \begin{bmatrix} 0.5320 & 1.3103 \\ 0.3892 & -0.9212 \\ 0.0649 & -0.1535 \end{bmatrix} \tag{12}$$

As seen the eigenvalues (11) are identical to the lowest two eigenvalues from the static condensation procedure (8). The two lowest eigenmodes in (8) are retrieved from (12) upon interchanging the 1st and 3rd components in the latter.

A.7 Exercise 4.2

Given the following mass- and stiffness matrices

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1}$$

1. Calculate approximate eigenvalues and eigenmodes by Rayleigh-Ritz analysis using the following Ritz basis

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Psi}^{(1)} \, \mathbf{\Psi}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

SOLUTIONS:

Question 1:

The projected mass and stiffness matrices become, cf. (4-45)

$$\left.\begin{aligned}
\tilde{\mathbf{M}} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 1 & 3 \end{bmatrix} \\
\tilde{\mathbf{K}} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 16 \end{bmatrix}
\end{aligned}\right\} \tag{2}$$

The eigensolutions to the eigenvalue problem defined by $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{K}}$ with modal masses normalized to 1 with respect to $\tilde{\mathbf{M}}$ become, cf. Box. 4.2

$$\mathbf{R} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} = \begin{bmatrix} 1.0459 & 0 \\ 0 & 5.3541 \end{bmatrix} \quad , \quad \mathbf{Q} = \begin{bmatrix} \mathbf{q}^{(1)} \, \mathbf{q}^{(2)} \end{bmatrix} = \begin{bmatrix} -0.3864 & 0.0269 \\ 0.0887 & -0.5849 \end{bmatrix} \tag{3}$$

The solutions for the eigenvectors become, cf. (4-51)

$$\bar{\mathbf{\Phi}} = \begin{bmatrix} \bar{\mathbf{\Phi}}^{(1)} \, \bar{\mathbf{\Phi}}^{(2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -0.3864 & 0.0269 \\ 0.0887 & -0.5849 \end{bmatrix} = \begin{bmatrix} -0.2976 & -0.5580 \\ -0.4751 & 0.6118 \\ -0.2976 & -0.5580 \end{bmatrix} \tag{4}$$

The exact eigensolutions can be shown to be

$$\left.\begin{aligned}
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0.7245 & 0 & 0 \\ 0 & 2.9652 & 0 \\ 0 & 0 & 9.3104 \end{bmatrix} \\
\mathbf{\Phi} &= \begin{bmatrix} \mathbf{\Phi}^{(1)} \, \mathbf{\Phi}^{(2)} \, \mathbf{\Phi}^{(3)} \end{bmatrix} = \begin{bmatrix} -0.0853 & -0.6981 & -0.0458 \\ -0.3884 & -0.0486 & 0.5778 \\ -0.5251 & 0.1997 & -0.8149 \end{bmatrix}
\end{aligned}\right\} \tag{5}$$

As seen ρ_1 and ρ_2 are upperbounds to the exact eigenvalues λ_1 and λ_2, and ρ_2 is smaller than λ_3, cf. (4-57). The estimates of the eigenmodes are not useful. Not even the signs of the components of $\bar{\mathbf{\Phi}}^{(2)}$ are correctly represented. These poor results are obtained because the chosen Ritz basis is far away from the basis spanned by $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(2)}$.

A.8 Exercise 4.3

Consider the mass- and stiffness matrices in Exercise 4.2, and let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

1. Calculate the vector $\bar{\mathbf{\Phi}}^{(1)} = \mathbf{K}^{-1}\mathbf{M}\mathbf{v}$, and next $\bar{\lambda}_1 = \rho(\bar{\mathbf{\Phi}}^{(1)})$, as approximate solutions to the lowest eigenmode and eigenvalue.
2. Establish the error bound for the obtained approximation to the lowest eigenvalue.

SOLUTIONS:

Question 1:

From the given formula we calculate

$$\bar{\mathbf{\Phi}}^{(1)} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix} \tag{2}$$

The Rayleigh quotient based on $\bar{\mathbf{\Phi}}^{(1)}$ becomes, cf. (4-25)

$$\bar{\lambda}_1 = \rho(\bar{\mathbf{\Phi}}^{(1)}) = \frac{\begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix}}{\begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix}} = 0.7547 \tag{3}$$

The obtained un-normalized eigenmode $\bar{\mathbf{\Phi}}^{(1)}$ resembles $\mathbf{\Phi}^{(1)}$ much better than the corresponding approximation for $\bar{\mathbf{\Phi}}^{(1)}$ indicated in eq. (4) of Exercise 7.2. As a consequence the obtained eigenvalue $\bar{\lambda}_1$ is a much better approximation to the exact eigenvalue $\lambda_1 = 0.7245$ given in eq. (5) of Exercise 4.2, than the approximation $\rho_1 = 1.0459$ obtained by the Rayleigh-Ritz analysis. The indicated formula for obtaining $\bar{\mathbf{\Phi}}^{(1)}$ represents the 1st iteration step in the socalled inverse vector iteration algorithm described in Section 5.2

Question 2:

From (2) follows that

$$\left|\bar{\Phi}^{(1)}\right| = 2.1714 \tag{4}$$

The error vector becomes, cf. (4-79)

$$\varepsilon_1 = \left(\begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} - 0.7547 \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix} = \begin{bmatrix} 1.1698 \\ -0.2075 \\ -0.2264 \end{bmatrix} \quad \Rightarrow$$

$$\left|\varepsilon_1\right| = 1.2095 \tag{5}$$

The lowest eigenvalue of $\mathbf{M}$ can be shown to be

$$\mu_1 = 0.3820 \tag{6}$$

Then, from (4-85) the following bound is obtained

$$|\lambda_1 - \bar{\lambda}_1| \leq \frac{1}{0.3820} \cdot \frac{1.2095}{2.1714} = 2.1714 \tag{7}$$

Actually, $|\lambda_1 - \bar{\lambda}_1| = |0.7245 - 0.7547| = 0.0302$. Hence, the bounding method provides a rather crude upperbound in the present case.

A.9 Exercise 5.1

Given the following mass- and stiffness matrices defined in Exercise 4.2.

1. Perform two inverse iterations, and then calculate an approximation to λ_1.
2. Perform two forward iterations, and then calculate an approximation to λ_3.

SOLUTIONS:

Question 1:

The calculations are performed with the start vector

$$\Phi_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

The matrix **A** becomes, cf. (5-4)

$$\mathbf{A} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \tag{2}$$

At the 1st and 2nd iteration steps the following calculations are performed, cf. Box 5.1

$$\begin{cases} \bar{\Phi}_1 = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_1^T \mathbf{M} \bar{\Phi}_1 = 10.9975 \\ \Phi_1 = \dfrac{1}{\sqrt{10.9975}} \begin{bmatrix} 0.55 \\ 1.30 \\ 1.65 \end{bmatrix} = \begin{bmatrix} 0.16585 \\ 0.39201 \\ 0.49755 \end{bmatrix} \end{cases} \tag{3}$$

$$\begin{cases} \bar{\Phi}_2 = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \begin{bmatrix} 0.16585 \\ 0.39201 \\ 0.49755 \end{bmatrix} = \begin{bmatrix} 0.14436 \\ 0.53449 \\ 0.71202 \end{bmatrix} \quad \Rightarrow \quad \bar{\Phi}_2^T \mathbf{M} \bar{\Phi}_2 = 1.8812 \\ \Phi_2 = \dfrac{1}{\sqrt{1.8812}} \begin{bmatrix} 0.14436 \\ 0.53449 \\ 0.71202 \end{bmatrix} = \begin{bmatrix} 0.10526 \\ 0.38970 \\ 0.51914 \end{bmatrix} \end{cases} \tag{4}$$

Since, Φ_2 has been normalized to unit modal mass, so $\Phi_2^T \mathbf{M} \Phi_2 = 1$, an approximation is obtained from the following Rayleigh fraction, cf. (4-25)

$$\bar{\lambda}_1 = \Phi_2^T \mathbf{K} \Phi_2 = \begin{bmatrix} 0.10526 \\ 0.38970 \\ 0.51914 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.10526 \\ 0.38970 \\ 0.51914 \end{bmatrix} = 0.72629 \tag{5}$$

The exact solution is $\lambda_1 = 0.72446$.

Question 2:

The calculations are performed with the start vector given by (1).

The matrix **B** becomes, cf. (5-35)

$$\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3.0 & -0.5 & 0.0 \\ -1.0 & 5.0 & -3.0 \\ 1.0 & -6.0 & 5.0 \end{bmatrix} \tag{6}$$

At the 1st and 2nd iteration steps the following calculations are performed, cf. Box 5.3

$$\begin{cases} \bar{\mathbf{\Phi}}_1 = \begin{bmatrix} 3.0 & -0.5 & 0.0 \\ -1.0 & 5.0 & -3.0 \\ 1.0 & -6.0 & 5.0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1.0 \\ 0.0 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = 14.5 \\ \mathbf{\Phi}_1 = \dfrac{1}{\sqrt{14.5}} \begin{bmatrix} 2.5 \\ 1.0 \\ 0.0 \end{bmatrix} = \begin{bmatrix} 0.65653 \\ 0.26261 \\ 0.00000 \end{bmatrix} \end{cases} \tag{7}$$

$$\begin{cases} \bar{\mathbf{\Phi}}_2 = \begin{bmatrix} 3.0 & -0.5 & 0.0 \\ -1.0 & 5.0 & -3.0 \\ 1.0 & -6.0 & 5.0 \end{bmatrix} \begin{bmatrix} 0.65653 \\ 0.26261 \\ 0.00000 \end{bmatrix} = \begin{bmatrix} 1.83829 \\ 0.65653 \\ -0.91915 \end{bmatrix} \quad \Rightarrow \quad \bar{\mathbf{\Phi}}_1^T \mathbf{M} \bar{\mathbf{\Phi}}_1 = 7.25862 \\ \mathbf{\Phi}_1 = \dfrac{1}{\sqrt{7.25862}} \begin{bmatrix} 1.83829 \\ 0.65653 \\ -0.91915 \end{bmatrix} = \begin{bmatrix} 0.68232 \\ 0.24369 \\ -0.34116 \end{bmatrix} \end{cases} \tag{8}$$

Again, $\mathbf{\Phi}_2$ has been normalized to unit modal mass, so $\mathbf{\Phi}_2^T \mathbf{M} \mathbf{\Phi}_2 = 1$, an approximation is obtained from the following Rayleigh fraction

$$\bar{\lambda}_3 = \mathbf{\Phi}_2^T \mathbf{K} \mathbf{\Phi}_2 = \begin{bmatrix} 0.68232 \\ 0.24369 \\ -0.34116 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.68232 \\ 0.24369 \\ -0.34116 \end{bmatrix} = 3.09739 \tag{9}$$

The exact solution is $\lambda_3 = 9.31036$. The poor result is obtained because $\mathbf{\Phi}_2$ is a rather bad approximation to $\mathbf{\Phi}^{(3)}$.

A.10 Exercise 5.2

Given the following mass- and stiffness matrices

$$\mathrm{M} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1}$$

The eigenmodes $\mathbf{\Phi}^{(1)}$ are $\mathbf{\Phi}^{(3)}$ are known to be, cf. (1-87)

$$\mathbf{\Phi}^{(1)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad , \quad \mathbf{\Phi}^{(3)} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \tag{2}$$

1. Calculate $\mathbf{\Phi}^{(2)}$ by means of Gram-Schmidt orthogonalization, and calculate all eigenvalues.

SOLUTIONS:

Question 1:

$\mathbf{\Phi}^{(2)}$ may be determined either by using inverse iteration with deflation with $\mathbf{\Phi}^{(1)}$, or by forward iteration and deflation with $\mathbf{\Phi}^{(3)}$. Here an alternative strategy is used based on the knowledge of both $\mathbf{\Phi}^{(1)}$ and $\mathbf{\Phi}^{(3)}$. Consider an arbitrary vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \tag{3}$$

Since, $\mathbf{\Phi}^{(1)}$, $\mathbf{\Phi}^{(2)}$ and $\mathbf{\Phi}^{(3)}$ form a vector basis, we may write

$$\mathbf{x} = c_1 \mathbf{\Phi}^{(1)} + c_2 \mathbf{\Phi}^{(2)} + c_3 \mathbf{\Phi}^{(3)} \tag{4}$$

In order to determine the expansion coefficient c_j, (4) is premultiplied with $\mathbf{\Phi}^{(j)T}\mathrm{M}$, and the M-othonormality of the eigenmodes are used, i.e. that $\mathbf{\Phi}^{(i)T}\mathrm{M}\mathbf{\Phi}^{(j)} = \delta_{ij}$. For $j = 1, 3$ the following results are obtained

$$c_j = \mathbf{\Phi}^{(j)T}\mathbf{M}\mathbf{x} = \begin{cases} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{7\sqrt{2}}{4} \quad , \quad j = 1 \\ \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -\frac{\sqrt{2}}{4} \quad , \quad j = 3 \end{cases} \tag{5}$$

Then, from (3), (4) and (5) follows

$$c_2\mathbf{\Phi}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{7\sqrt{2}}{4} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} + \frac{\sqrt{2}}{4} \cdot \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.0 \\ 0.5 \end{bmatrix} \quad \Rightarrow$$

$$c_2^2 = \begin{bmatrix} -0.5 \\ 0.0 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.0 \\ 0.5 \end{bmatrix} = 0.25 \quad \Rightarrow$$

$$\mathbf{\Phi}^{(2)} = \frac{1}{0.5} \cdot \begin{bmatrix} -0.5 \\ 0.0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

The indicated method only works because the dimension of the problem is three, and two eigenmodes are known in advance. Hence, the modal matrix becomes, cf. (1-87)

$$\mathbf{\Phi} = \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(3)}\right] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{7}$$

Given that all eigenmodes have been normalized to unit modal mass the eigenvalues may be calculated from the Rayleigh quotient, cf. (1-21), (4-25)

$$\mathbf{\Lambda} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{8}$$

Generally, if $n-1$ eigenmodes to a general eigenvalue problem is known the remaining eigenmode can lways be determined solely from the **M**-orthonormality conditions.

A.11 Exercise 6.3

Given the mass- and stiffness matrices defined in Exercise 4.2.

1. Perform an initial transformation to a special eigenvalue problem, and calculate the eigenvalues and eigenvectors by means of standard Jacobi iteration.
2. Calculate the eigenvalues and normalized eigenvectors by means of general Jacobi iteration operating on the original general eigenvalue problem.

SOLUTIONS:

Question 1:

Initially, a Choleski decomposition of the mass matrix is performed, cf. (3-44). As indicated by the algorithm in Box 3.2 the following calculations are performed

$$\left.\begin{aligned}
s_{11} &= \sqrt{m_{11}} = \sqrt{2} \\
s_{21} &= \frac{m_{21}}{s_{11}} = \frac{0}{\sqrt{2}} = 0 \\
s_{31} &= \frac{m_{31}}{s_{11}} = \frac{0}{\sqrt{2}} = 0 \\
s_{22} &= \sqrt{m_{22} - s_{21}^2} = \sqrt{2 - 0^2} = \sqrt{2} \\
s_{32} &= \frac{1}{s_{22}}(m_{32} - s_{31} \cdot s_{21}) = \frac{1}{\sqrt{2}}(1 - 0 \cdot 0) = \frac{\sqrt{2}}{2} \\
s_{33} &= \sqrt{m_{33} - s_{32}^2 - s_{31}^2} = \sqrt{1 - \left(\frac{\sqrt{2}}{2}\right)^2 - 0^2} = \frac{\sqrt{2}}{2}
\end{aligned}\right\} \tag{1}$$

Hence, the matrices S and S^{-1} become

$$\mathrm{S} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad \Rightarrow$$

$$\mathrm{S}^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix} \tag{2}$$

The initial value of the updated similarity transformation matrix and the stiffness matrix becomes, cf. (3-48), (6-4)

$$\left.\begin{aligned}
&\mathbf{\Phi}_0 = (\mathbf{S}^{-1})^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \\
&\mathbf{K}_0 = \tilde{\mathbf{K}} = \mathbf{S}^{-1}\mathbf{K}(\mathbf{S}^{-1})^T = \\
&\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3.0 & -0.5 & 0.5 \\ -0.5 & 2.0 & -3.0 \\ 0.5 & -3.0 & 8.0 \end{bmatrix}
\end{aligned}\right\} \tag{3}$$

In the 1st sweep the following calculations are performed for $(i,j) = (1,2)$:

$$\left\{\begin{aligned}
&\theta = \frac{1}{2}\arctan\left(\frac{2\cdot(-0.5)}{3.0-2.0}\right) = -0.3927 \quad \Rightarrow \quad \begin{cases} \cos\theta = \ \ 0.9239 \\ \sin\theta = -0.3827 \end{cases} \\
&\mathbf{P}_0 = \begin{bmatrix} 0.9239 & 0.3827 & 0 \\ -0.3827 & 0.9239 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&\mathbf{\Phi}_1 = \mathbf{\Phi}_0\mathbf{P}_0 = \begin{bmatrix} 0.6533 & 0.2706 & 0 \\ -0.2706 & 0.6533 & -0.7071 \\ 0 & 0 & 1.4142 \end{bmatrix} \\
&\mathbf{K}_1 = \mathbf{P}_0^T\mathbf{K}_0\mathbf{P}_0 = \begin{bmatrix} 3.2071 & 0 & 1.6070 \\ 0 & 1.7929 & -2.5803 \\ 1.6070 & -2.5803 & 8.0000 \end{bmatrix}
\end{aligned}\right. \tag{4}$$

Next, the calculations are performed for $(i,j) = (1,3)$:

$$\left\{\begin{aligned}
&\theta = \frac{1}{2}\arctan\left(\frac{2\cdot 1.6070}{3.2071-8.0000}\right) = -0.2958 \quad \Rightarrow \quad \begin{cases} \cos\theta = \ \ 0.9566 \\ \sin\theta = -0.2915 \end{cases} \\
&\mathbf{P}_1 = \begin{bmatrix} 0.9566 & 0 & 0.2915 \\ 0 & 1 & 0 \\ -0.2915 & 0 & 0.9566 \end{bmatrix} \\
&\mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{P}_1 = \begin{bmatrix} 0.6249 & 0.2706 & 0.1904 \\ -0.0527 & 0.6533 & -0.7553 \\ -0.4122 & 0 & 1.3528 \end{bmatrix} \\
&\mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1 = \begin{bmatrix} 2.7165 & 0.7521 & 0 \\ 0.7521 & 1.7929 & -2.4682 \\ 0 & -2.4682 & 8.4906 \end{bmatrix}
\end{aligned}\right. \tag{5}$$

Finally, to end the 1st sweep the calculations are performed for $(i,j) = (2,3)$:

$$\begin{cases} \theta = \dfrac{1}{2}\arctan\left(\dfrac{2\cdot(-2.4682)}{1.7929-8.4906}\right) = 0.3176 \quad \Rightarrow \quad \begin{cases}\cos\theta = 0.9500 \\ \sin\theta = 0.3123\end{cases} \\ \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9500 & -0.3123 \\ 0 & 0.3123 & 0.9500 \end{bmatrix} \\ \mathbf{\Phi}_3 = \mathbf{\Phi}_2\mathbf{P}_2 = \begin{bmatrix} 0.6249 & 0.3165 & 0.0964 \\ -0.0527 & 0.3848 & -0.9215 \\ -0.4122 & 0.4224 & 1.2852 \end{bmatrix} \\ \mathbf{K}_3 = \mathbf{P}_2^T\mathbf{K}_2\mathbf{P}_2 = \begin{bmatrix} 2.7165 & 0.7145 & -0.2349 \\ 0.7145 & 0.9816 & 0 \\ -0.2349 & 0 & 9.3019 \end{bmatrix} \end{cases} \tag{6}$$

At the end of the 2nd and 3rd sweep the following estimates are obtained for the modal matrix and the eigenvalues

$$\begin{cases} \mathbf{\Phi}_6 = \begin{bmatrix} 0.6980 & 0.0862 & 0.0729 \\ 0.0481 & 0.3885 & -0.9202 \\ -0.2004 & 0.5249 & 1.2978 \end{bmatrix}, \quad \mathbf{K}_6 = \begin{bmatrix} 2.9652 & 0.0028 & 0.0000 \\ 0.0028 & 0.7245 & 0 \\ 0.0000 & 0 & 9.3104 \end{bmatrix} \\ \mathbf{\Phi}_9 = \begin{bmatrix} 0.6981 & 0.0853 & 0.0729 \\ 0.0486 & 0.3884 & -0.9202 \\ -0.1997 & 0.5251 & 1.2978 \end{bmatrix}, \quad \mathbf{K}_9 = \begin{bmatrix} 2.9652 & 0.0000 & 0.0000 \\ 0.0000 & 0.7245 & 0 \\ 0.0000 & 0 & 9.3104 \end{bmatrix} \end{cases} \tag{7}$$

As seen the eigenmodes are stored column-wise in $\mathbf{\Phi}$ according to the permutation $(j_1, j_2, j_3) = (2,1,3)$, cf. Box 6.2.

Question 2:

The following initializations are introduced, cf. Box 6.4

$$\mathrm{M}_0 = \mathrm{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \mathbf{K}_0 = \mathbf{K} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{\Phi}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

In the 1st sweep the following calculations are performed for $(i,j) = (1,2)$:

$$
\begin{cases}
\left.\begin{aligned} a &= \frac{4\cdot 0 - 2\cdot(-1)}{6\cdot 2 - 2\cdot 4} &= 0.5 \\ b &= \frac{6\cdot 0 - 2\cdot(-1)}{6\cdot 2 - 2\cdot 4} &= 0.5 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha = -0.4142 \\ \beta = 0.4142 \end{cases} \\[2ex]
\mathbf{P}_0 = \begin{bmatrix} 1 & 0.4142 & 0 \\ -0.4142 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_1 = \mathbf{\Phi}_0\mathbf{P}_0 = \begin{bmatrix} 1 & 0.4142 & 0 \\ -0.4142 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\[2ex]
\mathbf{M}_1 = \mathbf{P}_0^T\mathbf{M}_0\mathbf{P}_0 = \begin{bmatrix} 2.3431 & 0 & -0.4142 \\ 0 & 2.3431 & 1 \\ -0.4142 & 1 & 1 \end{bmatrix} \\[2ex]
\mathbf{K}_1 = \mathbf{P}_0^T\mathbf{K}_0\mathbf{P}_0 = \begin{bmatrix} 7.5147 & 0 & 0.4142 \\ 0 & 4.2010 & -1 \\ 0.4142 & -1 & 2 \end{bmatrix}
\end{cases} \tag{9}
$$

Next, the calculations are performed for $(i,j) = (1,3)$:

$$
\begin{cases}
\left.\begin{aligned} a &= \frac{2\cdot(-0.4142) - 1\cdot 0.4142}{7.5147\cdot 1 - 2.3431\cdot 2} &= -0.4393 \\ b &= \frac{7.5147\cdot(-0.4142) - 2.3431\cdot 0.4142}{7.5147\cdot 1 - 2.3431\cdot 2} &= -1.4437 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha = 1.0023 \\ \beta = -0.3050 \end{cases} \\[2ex]
\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & -0.3050 \\ 0 & 1 & 0 \\ 1.0023 & 0 & 1 \end{bmatrix}, \quad \mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{P}_1 = \begin{bmatrix} 1 & 0.4142 & -0.3050 \\ -0.4142 & 1 & 0.1263 \\ 1.0023 & 0 & 1 \end{bmatrix} \\[2ex]
\mathbf{M}_2 = \mathbf{P}_1^T\mathbf{M}_1\mathbf{P}_1 = \begin{bmatrix} 2.5174 & 1.0023 & 0 \\ 1.0023 & 2.3431 & 1 \\ 0 & 1 & 1.4707 \end{bmatrix} \\[2ex]
\mathbf{K}_2 = \mathbf{P}_1^T\mathbf{K}_1\mathbf{P}_1 = \begin{bmatrix} 10.3542 & -1.0023 & 0 \\ -1.0023 & 4.2010 & -1 \\ 0 & -1 & 2.4465 \end{bmatrix}
\end{cases} \tag{10}
$$

Finally, to end the 1st sweep the calculations are performed for $(i,j) = (2,3)$:

$$
\begin{cases}
\left.\begin{aligned} a &= \frac{2.4465 \cdot 1 - 1.4707 \cdot (-1)}{4.2010 \cdot 1.4707 - 2.3407 \cdot 2.4465} &= 8.7838 \\ b &= \frac{4.2010 \cdot 1 - 2.3431 \cdot (-1)}{4.2010 \cdot 1.4707 - 2.3431 \cdot 2.4465} &= 14.6745 \end{aligned}\right\} \Rightarrow \begin{cases} \alpha = -1.2369 \\ \beta = \ \ 0.7404 \end{cases} \\
\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.7404 \\ 0 & -1.2369 & 1 \end{bmatrix}, \ \mathbf{\Phi}_3 = \mathbf{\Phi}_2\mathbf{P}_2 = \begin{bmatrix} 1 & 0.7915 & 0.0016 \\ -0.4142 & 0.8437 & 0.8667 \\ 1.0023 & -1.2369 & 1 \end{bmatrix} \\
\mathbf{M}_3 = \mathbf{P}_2^T\mathbf{M}_2\mathbf{P}_2 = \begin{bmatrix} 2.5174 & 1.0023 & 0.7421 \\ 1.0023 & 2.1193 & 0 \\ 0.7421 & 0 & 4.2357 \end{bmatrix} \\
\mathbf{K}_3 = \mathbf{P}_2^T\mathbf{K}_2\mathbf{P}_2 = \begin{bmatrix} 10.3542 & -1.0023 & -0.7421 \\ -1.0023 & 10.4174 & 0 \\ -0.7421 & 0 & 3.2684 \end{bmatrix}
\end{cases} \tag{11}
$$

At the end of the 2nd and 3rd sweep the following estimates are obtained for the modal matrix and the transformed mass and stiffness matrices

$$
\begin{cases}
\mathbf{\Phi}_6 = \begin{bmatrix} 1.6779 & -0.0129 & 0.1846 \\ 0.0350 & 1.3400 & 0.8282 \\ -0.3741 & -1.9075 & 1.1188 \end{bmatrix} \\
\mathbf{M}_6 = \begin{bmatrix} 5.7469 & 0.1959 & -0.0118 \\ 0.1959 & 2.1179 & 0 \\ -0.0118 & 0 & 4.5448 \end{bmatrix}, \quad \mathbf{K}_6 = \begin{bmatrix} 17.0856 & -0.1959 & 0.0118 \\ -0.1959 & 19.6067 & 0 \\ 0.0118 & 0 & 3.2925 \end{bmatrix} \\
\mathbf{\Phi}_9 = \begin{bmatrix} 1.6780 & -0.1060 & 0.1819 \\ 0.1169 & 1.3379 & 0.8281 \\ -0.4801 & -1.8869 & 1.1195 \end{bmatrix} \\
\mathbf{M}_9 = \begin{bmatrix} 5.7769 & 0.0000 & 0.0000 \\ 0.0000 & 2.1139 & 0 \\ 0.0000 & 0 & 4.5448 \end{bmatrix}, \quad \mathbf{K}_9 = \begin{bmatrix} 17.1296 & -0.0000 & -0.0000 \\ -0.0000 & 19.6810 & 0 \\ -0.0000 & 0 & 3.2925 \end{bmatrix}
\end{cases} \tag{12}
$$

Presuming that the process has converged after the 3rd sweep the eigenvalues and normalized eigenmodes are next retrieved by the following calculations, cf. Box. 6.4

$$\left\{\begin{array}{l}
\mathbf{m} = \mathbf{M}_9 = \begin{bmatrix} 5.7769 & 0.0000 & 0.0000 \\ 0.0000 & 2.1139 & 0 \\ 0.0000 & 0 & 4.5448 \end{bmatrix} \quad , \quad \mathbf{m}^{-\frac{1}{2}} = \begin{bmatrix} 0.4161 & 0 & 0 \\ 0 & 0.6878 & 0 \\ 0 & 0 & 0.4691 \end{bmatrix} \quad \Rightarrow \\[2ex]
\mathbf{\Lambda} = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} = \mathbf{M}_9^{-1}\mathbf{K}_9 = \begin{bmatrix} 2.9652 & -0.0000 & -0.0000 \\ -0.0000 & 9.3104 & 0.0000 \\ -0.0000 & 0.0000 & 0.7245 \end{bmatrix} \\[2ex]
\mathbf{\Phi} = \left[\mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(3)}\ \mathbf{\Phi}^{(1)}\right] = \mathbf{\Phi}_9\mathbf{m}^{-\frac{1}{2}} = \begin{bmatrix} 0.6981 & -0.0729 & 0.0853 \\ 0.0486 & 0.9202 & 0.3884 \\ -0.1997 & -1.2978 & 0.5251 \end{bmatrix}
\end{array}\right. \tag{13}$$

The solutions (13) are identical to those obtained in (8) for the special Jacobi iteration algorithm. In the present case the eigenmodes are stored column-wise in $\mathbf{\Phi}$ according to the permutation $(j_1, j_2, j_3) = (2, 3, 1)$, cf. Box 6.4. The convergence rates of the special nd the general Jacobi iteration algorithm seems to be rather alike.

A.12 Exercise 6.6

Given the mass- and stiffness matrices defined in Exercise 4.2.

1. Calculate the eigenvalues and normalized eigenvectors by means of QR iteration.

SOLUTIONS:

Question 1:

At first, as indicated in Box 6.7 an initial similarity transformation of the indicated general eigenvalue problem into a special eigenvalue problem is performed with the similarity transformation matrix $\mathbf{P} = \left(\mathbf{S}^{-1}\right)^T$, where $\mathbf{S}$ is a solution to $\mathbf{M} = \mathbf{S}\mathbf{S}^T$. In case $\mathbf{S}$ is determined from an Choleski decomposition of the mass matrix the initial updated transformation and stiffness matrices have been calculated in Exercise 6.3, eq. (4). The result becomes

$$\left.\begin{array}{l}
\mathbf{\Phi}_1 = (\mathbf{S}^{-1})^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} \\[2ex]
\mathbf{K}_1 = \mathbf{S}^{-1}\mathbf{K}(\mathbf{S}^{-1})^T = \\
\begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix} \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 3.0 & -0.5 & 0.5 \\ -0.5 & 2.0 & -3.0 \\ 0.5 & -3.0 & 8.0 \end{bmatrix}
\end{array}\right\} \tag{1}$$

As seen the original three diagonal structure of $\mathbf{K}$ is destroyed by the similarity. This may be reestablished by means of a Householder transformation as described in Section 6.4. However, this will be omitted here, so the QR-iteration is performed on the full matrix $\mathbf{K}_1$.

At the determination of $\mathbf{q}_1$ and r_{11} in the 1st QR iteration the following calculations are performed, cf. (6-72)

$$\begin{cases} \mathbf{k}_1 = \begin{bmatrix} 3.0 \\ -0.5 \\ 0.5 \end{bmatrix} \quad , \quad r_{11} = \left\| \begin{bmatrix} 3.0 \\ -0.5 \\ 0.5 \end{bmatrix} \right\| = 3.0822 \\ \mathbf{q}_1 = \dfrac{1}{3.0822} \begin{bmatrix} 3.0 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.9733 \\ -0.1622 \\ 0.1622 \end{bmatrix} \end{cases} \tag{2}$$

$\mathbf{q}_2$ and r_{12}, r_{22} are determined from the following calculations, cf. (6-73)

$$\begin{cases} \mathbf{k}_2 = \begin{bmatrix} -0.5 \\ 2.0 \\ -3.0 \end{bmatrix} \quad , \quad r_{12} = \begin{bmatrix} 0.9733 \\ -0.1622 \\ 0.1622 \end{bmatrix}^T \begin{bmatrix} -0.5 \\ 2.0 \\ -3.0 \end{bmatrix} = -1.2978 \\ r_{22} = \left\| \begin{bmatrix} -0.5 \\ 2.0 \\ -3.0 \end{bmatrix} + 1.2978 \cdot \begin{bmatrix} 0.9733 \\ -0.1622 \\ 0.1622 \end{bmatrix} \right\| = 3.4009 \\ \mathbf{q}_2 = \dfrac{1}{3.4009} \left(\begin{bmatrix} -0.5 \\ 2.0 \\ -3.0 \end{bmatrix} + 1.2978 \cdot \begin{bmatrix} 0.9733 \\ -0.1622 \\ 0.1622 \end{bmatrix} \right) = \begin{bmatrix} 0.2244 \\ 0.5262 \\ -0.8202 \end{bmatrix} \end{cases} \tag{3}$$

$\mathbf{q}_3$ and r_{13}, r_{23}, r_{33} are determined from the following calculations, cf. (6-74)

$$\begin{cases} \mathbf{k}_3 = \begin{bmatrix} 0.5 \\ -3.0 \\ 8.0 \end{bmatrix} \quad , \quad r_{13} = \mathbf{q}_1^T \mathbf{k}_3 = 2.2711 \quad , \quad r_{23} = \mathbf{q}_2^T \mathbf{k}_3 = -8.0282 \\ r_{33} = \left| \mathbf{k}_3 - 2.2711\mathbf{q}_1 + 8.0282\mathbf{q}_2 \right| = 1.9080 \\ \mathbf{q}_3 = \dfrac{1}{1.9080} \left(\mathbf{k}_3 - 2.2711\mathbf{q}_1 + 8.0282\mathbf{q}_2 \right) = \begin{bmatrix} 0.0477 \\ 0.8348 \\ 0.5486 \end{bmatrix} \end{cases} \tag{4}$$

Then, at the end of the 1st iteration the following matrices are obtained

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\mathbf{Q}_1 = \begin{bmatrix} 0.9733 & 0.2244 & 0.0477 \\ -0.1622 & 0.5662 & 0.8348 \\ 0.1622 & -0.8202 & 0.5486 \end{bmatrix} \\[2ex]
\mathbf{R}_1 = \begin{bmatrix} 3.0822 & -1.2978 & 2.2711 \\ 0 & 3.4009 & -8.0282 \\ 0 & 0 & 1.9080 \end{bmatrix}
\end{array}\right\} \Rightarrow \\[4ex]
\left.\begin{array}{l}
\mathbf{\Phi}_2 = \mathbf{\Phi}_1\mathbf{Q}_1 = \begin{bmatrix} 0.6882 & 0.1587 & 0.0337 \\ -0.2294 & 0.9521 & 0.2024 \\ 0.2294 & -1.1600 & 0.7758 \end{bmatrix} \\[2ex]
\mathbf{K}_2 = \mathbf{R}_1\mathbf{Q}_1 = \begin{bmatrix} 3.5789 & -1.8540 & 0.3095 \\ -1.8540 & 8.3744 & -1.5650 \\ 0.3095 & -1.5650 & 1.0466 \end{bmatrix}
\end{array}\right\}
\end{array}\right. \tag{5}$$

The corresponding matrices after the 2nd and 3rd iteration become

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\mathbf{Q}_2 = \begin{bmatrix} 0.8853 & 0.4648 & 0.0115 \\ -0.4586 & 0.8689 & 0.1861 \\ 0.0766 & -0.1700 & 0.9825 \end{bmatrix} \\[2ex]
\mathbf{R}_2 = \begin{bmatrix} 4.0425 & -5.6020 & 1.0719 \\ 0 & 6.6809 & -1.3940 \\ 0 & 0 & 0.7405 \end{bmatrix}
\end{array}\right\} \Rightarrow \\[4ex]
\left.\begin{array}{l}
\mathbf{\Phi}_3 = \mathbf{\Phi}_2\mathbf{Q}_2 = \begin{bmatrix} 0.5391 & 0.4521 & 0.0706 \\ -0.6243 & 0.6862 & 0.3734 \\ 0.7945 & -1.0332 & 0.5489 \end{bmatrix} \\[2ex]
\mathbf{K}_3 = \mathbf{R}_2\mathbf{Q}_2 = \begin{bmatrix} 6.2303 & -3.1708 & 0.0567 \\ -3.1708 & 6.0422 & -0.1259 \\ 0.0567 & -0.1259 & 0.7275 \end{bmatrix}
\end{array}\right\}
\end{array}\right. \tag{6}$$

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\mathbf{Q}_3 = \begin{bmatrix} 0.8912 & 0.4536 & 0.0021 \\ -0.4536 & 0.8610 & 0.0219 \\ 0.0081 & -0.0205 & 0.9998 \end{bmatrix} \\[2ex]
\mathbf{R}_3 = \begin{bmatrix} 6.9910 & -5.5673 & 0.1135 \\ 0 & 3.9475 & -0.1014 \\ 0 & 0 & 0.7274 \end{bmatrix}
\end{array}\right\} \Rightarrow \\[4ex]
\left.\begin{array}{l}
\mathbf{\Phi}_4 = \mathbf{\Phi}_3\mathbf{Q}_3 = \begin{bmatrix} 0.2760 & 0.6459 & 0.0816 \\ -0.8645 & 0.3206 & 0.3871 \\ 1.1811 & -0.5714 & 0.5277 \end{bmatrix} \\[2ex]
\mathbf{K}_4 = \mathbf{R}_3\mathbf{Q}_3 = \begin{bmatrix} 8.5763 & -1.7913 & 0.0059 \\ -1.7913 & 3.5192 & -0.0148 \\ 0.0059 & -0.0148 & 0.7245 \end{bmatrix}
\end{array}\right\}
\end{array}\right. \tag{7}$$

As seen from $\mathbf{R}_3$ and $\mathbf{K}_4$ the terms in the main diagonal have already after the 3rd iteration grouped in descending magnitude, corresponding to the ordering of the eigenvalues at convergence indicated in Box 6.7. Moreover, for both matrices convergence to the lowest eigenvalue $\lambda_1 = 0.7245$ has occurred, illustrating the fact that the QR algorithm converge faster to the lowest eigenmode than to the highest.

The matrices after the 14th iteration become

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\mathbf{Q}_{14} = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.0000 \end{bmatrix} \\[2ex]
\mathbf{R}_{14} = \begin{bmatrix} 9.3104 & -0.0000 & 0.0000 \\ 0 & 2.9652 & -0.0000 \\ 0 & 0 & 0.7245 \end{bmatrix}
\end{array}\right\} \Rightarrow \\[4ex]
\left.\begin{array}{l}
\mathbf{\Phi}_{15} = \mathbf{\Phi}_{14}\mathbf{Q}_{14} = \begin{bmatrix} 0.0729 & 0.6981 & 0.0853 \\ -0.9202 & 0.0486 & 0.3884 \\ 1.2978 & -0.1997 & 0.5251 \end{bmatrix} \\[2ex]
\mathbf{K}_{15} = \mathbf{R}_{14}\mathbf{Q}_{14} = \begin{bmatrix} 9.3104 & -0.0000 & 0.0000 \\ -0.0000 & 2.9652 & -0.0000 \\ 0.0000 & -0.0000 & 0.7245 \end{bmatrix}
\end{array}\right\}
\end{array}\right. \tag{8}$$

Presuming that convergence has occurred after the 14th iteration the following solutions are obtained for the eigenvalues and eigenmodes of the original general eigenvalue problem

$$\left.\begin{aligned}
\mathbf{\Lambda} &= \begin{bmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} = \mathbf{K}_{15} = \begin{bmatrix} 9.3104 & -0.0000 & 0.0000 \\ -0.0000 & 2.9652 & -0.0000 \\ 0.0000 & -0.0000 & 0.7245 \end{bmatrix} \\
\mathbf{\Phi} &= \left[\mathbf{\Phi}^{(3)}\ \mathbf{\Phi}^{(2)}\ \mathbf{\Phi}^{(1)}\right] = \mathbf{\Phi}_{15} = \begin{bmatrix} 0.0729 & 0.6981 & 0.0853 \\ -0.9202 & 0.0486 & 0.3884 \\ 1.2978 & -0.1997 & 0.5251 \end{bmatrix}
\end{aligned}\right\} \tag{9}$$

The solution (9) agrees with the corresponding solutions for the special and general Jacobi iteration algorithms obtained in Exercise 6.3, eq. (8) and (14), respectively.

A.13 Exercise 7.1

Given the mass- and stiffness matrices defined in Exercise 4.2.

1. Calculate the two lowest eigenmodes and corresponding eigenvalues by simultaneous inverse vector iteration with the start vector basis

$$\mathbf{\Phi}_0 = \left[\mathbf{\Phi}_0^{(1)}\ \mathbf{\Phi}_0^{(2)}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$$

SOLUTIONS:

Question 1:

The matrix $\mathbf{A}$ becomes, cf. (5-4)

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \tag{1}$$

Then, the 1st iterated vector basis becomes, cf. (5-4)

$$\bar{\mathbf{\Phi}}_1 = \left[\bar{\mathbf{\Phi}}_1^{(1)}\ \bar{\mathbf{\Phi}}_1^{(2)}\right] = \mathbf{A}\mathbf{\Phi}_0 = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix} \tag{2}$$

At the determination of $\mathbf{\Phi}_1^{(1)}$ and r_{11} in the 1st vector iteration the following calculations are performed, cf. (7-14)

$$\begin{cases} \bar{\mathbf{\Phi}}_1^{(1)} = \begin{bmatrix} 0.550 \\ 1.300 \\ 1.650 \end{bmatrix}, \; r_{11} = \left\| \bar{\mathbf{\Phi}}_1^{(1)} \right\| = \left(\begin{bmatrix} 0.550 \\ 1.300 \\ 1.650 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.550 \\ 1.300 \\ 1.650 \end{bmatrix} \right)^{\frac{1}{2}} = 3.3162 \\ \mathbf{\Phi}_1^{(1)} = \dfrac{1}{3.3162} \begin{bmatrix} 0.550 \\ 1.300 \\ 1.650 \end{bmatrix} = \begin{bmatrix} 0.1659 \\ 0.3920 \\ 0.4976 \end{bmatrix} \end{cases} \tag{3}$$

$\mathbf{\Phi}_1^{(2)}$ and r_{12}, r_{22} are determined from the following calculations, cf. (7-16)

$$\begin{cases} \bar{\mathbf{\Phi}}_1^{(2)} = \begin{bmatrix} 0.275 \\ -0.350 \\ -0.675 \end{bmatrix}, \quad r_{12} = \begin{bmatrix} 0.1659 \\ 0.3920 \\ 0.4976 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.275 \\ -0.350 \\ -0.675 \end{bmatrix} = -0.9578 \\ r_{22} = \left\| \begin{bmatrix} 0.275 \\ -0.350 \\ -0.675 \end{bmatrix} + 0.9578 \cdot \begin{bmatrix} 0.1659 \\ 0.3920 \\ 0.4976 \end{bmatrix} \right\| = 0.6380 \\ \mathbf{\Phi}_1^{(2)} = \dfrac{1}{0.6380} \left(\begin{bmatrix} 0.275 \\ -0.350 \\ -0.675 \end{bmatrix} + 0.9578 \cdot \begin{bmatrix} 0.1659 \\ 0.3920 \\ 0.4976 \end{bmatrix} \right) = \begin{bmatrix} 0.6800 \\ 0.0399 \\ -0.3111 \end{bmatrix} \end{cases} \tag{4}$$

Then, at the end of the 1st iteration the following matrices are obtained

$$\begin{cases} \mathbf{R}_1 = \begin{bmatrix} 3.3162 & -0.9578 \\ 0 & 0.6380 \end{bmatrix} \\ \mathbf{\Phi}_1 = \begin{bmatrix} 0.1659 & 0.6800 \\ 0.3920 & 0.0399 \\ 0.4976 & -0.3111 \end{bmatrix} \end{cases} \tag{5}$$

The reader should verify that $\mathbf{\Phi}_1 \mathbf{R}_1 = \bar{\mathbf{\Phi}}_1$. The corresponding matrices after the 2nd and 3rd iteration become

$$\begin{cases} \mathbf{R}_2 = \begin{bmatrix} 1.3716 & -0.1507 \\ 0 & 0.3392 \end{bmatrix} \\ \mathbf{\Phi}_2 = \begin{bmatrix} 0.1053 & 0.6944 \\ 0.3897 & 0.0492 \\ 0.5191 & -0.2311 \end{bmatrix} \end{cases} \tag{6}$$

$$\begin{cases} \mathbf{R}_3 = \begin{bmatrix} 1.3798 & -0.0371 \\ 0 & 0.3374 \end{bmatrix} \\ \\ \mathbf{\Phi}_3 = \begin{bmatrix} 0.0902 & 0.6972 \\ 0.3888 & 0.0496 \\ 0.5237 & -0.2086 \end{bmatrix} \end{cases} \tag{7}$$

Convergence of the eigenmodes with the indicated number of digits were achieved after 9 iterations, where

$$\begin{cases} \mathbf{R}_{14} = \begin{bmatrix} 1.3803 & -0.0000 \\ 0 & 0.3372 \end{bmatrix} \\ \\ \mathbf{\Phi}_9 = \begin{bmatrix} 0.0853 & 0.6981 \\ 0.3884 & 0.0486 \\ 0.5251 & -0.1997 \end{bmatrix} \end{cases} \tag{8}$$

Presuming that convergence has occurred after the 9th iteration the following eigenvalues are obtained from (7-10) and (7-12)

$$\left.\begin{aligned} \mathbf{\Lambda} &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \mathbf{\Phi}_9^T \mathbf{K} \mathbf{\Phi}_9 = \mathbf{R}_\infty^{-1} = \begin{bmatrix} 0.7245 & 0.0000 \\ 0.0000 & 2.9652 \end{bmatrix} \\ \mathbf{\Phi} &= \left[\mathbf{\Phi}^{(1)}\, \mathbf{\Phi}^{(2)}\right] = \mathbf{\Phi}_9 = \begin{bmatrix} 0.0853 & 0.6981 \\ 0.3884 & 0.0486 \\ 0.5251 & -0.1997 \end{bmatrix} \end{aligned}\right\} \tag{9}$$

The solution (9) agrees with the corresponding solutions obtained in Exercises 6.2 and 6.6.

A.14 Exercise 7.3

Given the mass- and stiffness matrices defined in Exercise 4.2.

1. Calculate the two lowest eigenmodes and corresponding eigenvalues by subspace iteration with the start vector basis

$$\Phi_0 = [\Phi_0^{(1)}\ \Phi_0^{(2)}] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$$

SOLUTION:

Question 1:

The matrix **A** becomes, cf. (5-4)

$$\mathbf{A} = \mathbf{K}^{-1}\mathbf{M} = \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \tag{1}$$

Then, the 1st iterated vector basis becomes, cf. (7-4)

$$\bar{\Phi}_1 = [\bar{\Phi}_1^{(1)}\ \bar{\Phi}_1^{(2)}] = \mathbf{A}\Phi_0 = \begin{bmatrix} 0.350 & 0.125 & 0.075 \\ 0.100 & 0.750 & 0.450 \\ 0.050 & 0.875 & 0.725 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix} \tag{2}$$

In order to perform the Rayleigh-Ritz analysis in the 1st subspace iteration the following projected mass and stiffness matrices are calculated based on $\bar{\Phi}_1$, cf. (4-59), (4-60), (7-32)

$$\left.\begin{aligned} \tilde{\mathrm{M}}_1 &= \bar{\Phi}_1^T \mathrm{M} \bar{\Phi}_1 = \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix}^T \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix} = \begin{bmatrix} 10.998 & -3.1763 \\ -3.1763 & 1.3244 \end{bmatrix} \\ \tilde{\mathrm{K}}_1 &= \bar{\Phi}_1^T \mathrm{K} \bar{\Phi}_1 = \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix}^T \begin{bmatrix} 6 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix} = \begin{bmatrix} 8.300 & -1.850 \\ -1.850 & 1.575 \end{bmatrix} \end{aligned}\right\} \tag{3}$$

The corresponding eigenvalue problem (7-31) becomes

$$\tilde{\mathrm{K}}_1 \mathrm{Q}_1 = \tilde{\mathrm{M}}_1 \mathrm{Q}_1 \mathrm{R}_1 \quad \Rightarrow$$

$$\begin{bmatrix} 8.300 & -1.850 \\ -1.850 & 1.575 \end{bmatrix} [\mathrm{q}_1^{(1)}\ \mathrm{q}_1^{(2)}] = \begin{bmatrix} 10.998 & -3.1763 \\ -3.1763 & 1.3244 \end{bmatrix} [\mathrm{q}_1^{(1)}\ \mathrm{q}_1^{(2)}] \begin{bmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,1} \end{bmatrix} \quad \Rightarrow$$

$$\mathrm{R}_1 = \begin{bmatrix} 0.7246 & 0 \\ 0 & 2.9752 \end{bmatrix} \quad , \quad \mathrm{Q}_1 = \begin{bmatrix} -0.2471 & -0.4845 \\ 0.1813 & -1.5569 \end{bmatrix} \tag{4}$$

The estimate of the lowest eigenvectors after the 1st iteration becomes, cf. (7-35)

$$\mathbf{\Phi}_1 = \bar{\mathbf{\Phi}}_1 \mathbf{Q}_1 = \begin{bmatrix} 0.550 & 0.275 \\ 1.300 & -0.350 \\ 1.650 & -0.675 \end{bmatrix} \begin{bmatrix} -0.2471 & -0.4845 \\ 0.1813 & -1.5569 \end{bmatrix} = \begin{bmatrix} -0.0861 & -0.6947 \\ -0.3848 & -0.0850 \\ -0.5302 & 0.2514 \end{bmatrix} \tag{5}$$

Correspondingly, after the 2nd and 9th iteration steps the following matrices are calculated

$$\left.\begin{aligned} \mathbf{R}_2 &= \begin{bmatrix} 0.7245 & 0 \\ 0 & 2.9662 \end{bmatrix} \quad , \quad \mathbf{Q}_2 = \begin{bmatrix} -0.7245 & -0.0013 \\ 0.0004 & -2.9673 \end{bmatrix} \\ \mathbf{\Phi}_2 &= \begin{bmatrix} 0.0854 & 0.6972 \\ 0.3881 & 0.0603 \\ 0.5255 & -0.2162 \end{bmatrix} \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} \mathbf{R}_9 &= \begin{bmatrix} 0.7245 & 0 \\ 0 & 2.9652 \end{bmatrix} \quad , \quad \mathbf{Q}_9 = \begin{bmatrix} -0.7245 & -0.0000 \\ 0.0000 & -2.9652 \end{bmatrix} \\ \mathbf{\Phi}_9 &= \begin{bmatrix} -0.0853 & -0.6981 \\ -0.3884 & -0.0486 \\ -0.5251 & 0.1997 \end{bmatrix} \end{aligned}\right\} \tag{7}$$

The subspace iteration process converged with the indicated accuracy after 8 iterations.

Finally, it should be checked that the calculated eigenvalues are indeed the lowest two by a Sturm sequence or Gauss factorization check. The 2nd calculated eigenvalue becomes $\rho_{2,9} = 2.9652$. Then, let $\mu = 3.1$ and perform a Gauss factorization of the matrix $\mathbf{K} - 3.1\mathbf{M}$, i.e.

$$\mathbf{K} - 3.1\mathbf{M} = \begin{bmatrix} -0.2 & -1.0 & 0 \\ -1.0 & -2.2 & -4.1 \\ 0 & -4.1 & -1.1 \end{bmatrix} =$$

$$\mathbf{LDL}^T = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & -1.4643 & 1 \end{bmatrix} \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & 2.8 & 0 \\ 0 & 0 & -7.1036 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1.4643 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

It follows that two components in the main diagonal of $\mathbf{D}$ are negative, from which is concluded that two eigenvalues are smaller than $\mu = 3.1$. In turn this means that the two eigensolutions obtained by (8) are indeed the lowest two eigensolutions of the original system.

The solution (8) agrees with the corresponding solutions obtained in Exercises 6.3, 6.6 and 7.1.

A.15 Exercise 7.5

Given the mass- and stiffness matrices defined in Exercise 4.2.

1. Calculate the 3rd eigenmode and eigenvalue by Sturm sequence iteration (telescope method).

SOLUTION:

Question 1:

At first a calculation with $\mu = 2.5$ is performed, which produces the following results

$$\left.\begin{array}{l}
\mathbf{K} - 2.5\mathbf{M} = \begin{bmatrix} 1.0 & -1.0 & 0.0 \\ -1.0 & -1.0 & -3.5 \\ 0.0 & -3.5 & -0.5 \end{bmatrix} \quad \Rightarrow \\
\begin{cases} P^{(3)}(2.5) = 1 & , \quad \text{sign}(P^{(3)}(2.5)) = + \\ P^{(2)}(2.5) = 1.0 & , \quad \text{sign}(P^{(2)}(2.5)) = + \\ P^{(1)}(2.5) = 1.0 \cdot (-1.0) - (-1.0)^2 = -2.0 & , \quad \text{sign}(P^{(1)}(2.5)) = - \\ P^{(0)}(2.5) = -0.5 \cdot (-2.0) - (-3.5)^2 \cdot (1.0) = -11.25 & , \quad \text{sign}(P^{(0)}(2.5)) = - \end{cases}
\end{array}\right\} \tag{1}$$

Hence, the sign sequence of the Sturm sequence becomes $++--$. corresponding to the number of sign changes $n_{\text{sign}} = 1$ in the sequence. One eigenvalue is smaller than $\mu = 2.5$.

Similar calculations are performed for $\mu = 3.5,\ 4.5,\ \ldots,\ 9.5$

$$\begin{array}{lll}
\mu = 3.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 4.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 5.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 6.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 7.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 8.5: & \text{Sign sequence} = +-++ & \Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.5: & \text{Sign sequence} = +-+- & \Rightarrow \quad n_{\text{sign}} = 3
\end{array} \tag{2}$$

From this is concluded that the 3rd eigenvalue is placed somewhere in the interval $8.5 < \lambda_3 < 9.5$.

Next, similar calculations are performed for $\mu = 8.6,\ 8.7, \ldots, 9.4$

$$
\begin{aligned}
\mu = 8.6: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 8.7: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 8.8: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 8.9: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.0: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.1: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.2: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.3: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.4: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3
\end{aligned}
\tag{3}
$$

From this is concluded that the 3rd eigenvalue is confined to the interval $9.3 < \lambda_3 < 9.4$.

Next, calculations are performed for $\mu = 9.31, 9.32, \ldots, 9.39$

$$
\begin{aligned}
\mu = 9.31: &\quad \text{Sign sequence} = + - + + &\Rightarrow \quad n_{\text{sign}} = 2 \\
\mu = 9.32: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.33: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.34: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.35: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.36: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.37: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.38: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3 \\
\mu = 9.39: &\quad \text{Sign sequence} = + - + - &\Rightarrow \quad n_{\text{sign}} = 3
\end{aligned}
\tag{4}
$$

From this is concluded that the 3rd eigenvalue is confined to the interval $9.31 < \lambda_3 < 9.32$.

Proceeding in this manner, it may be shown after totally 52 Sturm sequence calculations that the 3rd eigenvalue is confined to the interval $9.31036 < \lambda_3 < 9.31037$. Each extra digit requires 9 calculations.

Setting $\lambda_3 \simeq 9.310365$, the linear equation (7-64) attains the form

$$
\left(\mathbf{K} - 9.310365\mathrm{M}\right)\bar{\Phi}^{(3)} = \mathbf{0} \quad \Rightarrow
$$

$$
\begin{bmatrix} -12.6207 & -1 & 0 \\ -1 & -14.6207 & -10.3104 \\ 0 & -10.3104 & -7.3104 \end{bmatrix}
\begin{bmatrix} \bar{\Phi}_1^{(3)} \\ \bar{\Phi}_2^{(3)} \\ \bar{\Phi}_3^{(3)} \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{5}
$$

Setting $\bar{\Phi}_1^{(3)} = 1$ the algorithm (7-65) now provides

$$
\left.
\begin{aligned}
\bar{\Phi}_2^{(3)} &= -\frac{(-12.6207)}{(-1)} \cdot 1 = -12.6207 \\
\bar{\Phi}_3^{(3)} &= -\frac{(-1)}{(-10.3104)} \cdot 1 - \frac{(-14.6207)}{(-10.3104)} \cdot (-12.6207) = 1
\end{aligned}
\right\}
\quad \Rightarrow \quad
\bar{\Phi}^{(3)} = \begin{bmatrix} 1 \\ -12.6207 \\ 17.7800 \end{bmatrix}
\tag{6}
$$

Normalization to unit modal mass provides

$$\Phi^{(3)} = \begin{bmatrix} 0.0729 \\ -0.9202 \\ 1.2978 \end{bmatrix} \tag{7}$$

The eigenvalue $\lambda_3 \simeq 9.310365$ and the corresponding eigenmode $\Phi^{(3)}$ as given by (7) agree with the corresponding results obtained in Exercises 6.3 and 6.6.