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DISTANCES IN GENERALIZED DOUBLE RINGS AND DEGRE THREE CHORDAL RINGS

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ABSTRACT
Generalized Double Rings ($N2R$) are compared to Degree Three Chordal Rings (CR) in terms of average distance, diameter, k-average distance and k-diameter. For each number of nodes, structures of each class are chosen to minimize diameter and average distance, an approach which is shown to result in all other parameters being either minimized or nearly minimized. Average distance and diameter are compared for all structures with up to 1000 nodes, and k-average distances and k-diameters for all structures with up to 400-900 nodes. $N2R$ are shown to be superior with regard to these parameters, especially for large structures.

KEY WORDS
Interconnection Networks, Broadband Networks, Planning, Interconnection Topologies, Network Structures.

1 Introduction

When designing a network or parallel computing system, interconnection topologies are important. Much research has been conducted in order to compare different topologies, in particular for multiprocessor systems[1], but comparison parameters have been determined also for large-scale communication networks[2]. Most of the compared topologies contain nodes of degree four or more.

In general, it is important to keep the cost and thus the node degrees as low as possible, and for communication infrastructures this in particular so; nodes are often placed in different physical locations, spread over large geographical areas. Not only node equipment but also digging is costly and should be kept to a minimum, and a limited set of potential ducts such as roads often limit the number of possible physical paths. Therefore it is not surprising that most communication infrastructures have until now been based on trees and rings. Trees offer no redundancy, while rings offer connectivity in case of any one arbitrary failure. Unfortunately, two failures will split the network, and even a single failure leads to notably larger distances and thus higher transmission delays and traffic load. Currently, the convergence of communications is leading to an increasing dependency on the Internet[3], a trend supported by the fact that a large number of applications are being developed requiring both Quality of Service and reliability. These include home automation[4], tele operations[5] and tele robotics[6]. These needs for reliability can to some extent be satisfied by using wireless back-up as a supplement to Fiber To The Home solutions for the last mile access networks[7], but there is an urgent need for developing more robust topologies for the higher layers, in particular local and regional backbones. In order to increase reliability while keeping the costs down, 3-regular 3-connected topologies are interesting. They have the smallest possible node degree, while still providing connectivity in case of any two independent failures. Double rings[8] are simple 3-regular 3-connected topologies, which offer easy routing, restoration and protection schemes, but suffer from large distances. Two alternative classes of 3-regular 3-connected topologies are the Generalized Double Rings also known as $N2R$[8], which is a subset of the Generalized Petersen Graphs[9], and the Degree Three Chordal Rings, e.g. [1][10] (for simplicity we write $N2R$ and CR throughout the paper). CR and $N2R$ share many properties: in addition to being 3-regular and 3-connected they are not in general planar, they can be expanded in similar ways and they have fairly short diameters[11]. However, there are also a few important differences: CR are node symmetric, while $N2R$ contain either one or two classes of nodes with symmetry within each class. Another difference is that CR are based on one main ring while $N2R$ are based on two main rings. Since they perform comparable with regard to those quantitative and qualitative parameters, the distances are important when selecting which topology should be used for some network or parallel computing system. In this paper, we compare the two classes of structures with regard to a number of different distance parameters. The results apply to physical as well as logical level networks, making them interesting in a broad context.

2 Preliminaries

A structure is a set of nodes and a set of lines, where each line interconnects two nodes. Lines are bi-directional, so if a pair of nodes (u, v) is connected, so is (v, u). A structure can be considered a model of a network, abstracting from specific physical conditions such as node equipment, media and wiring, and the definition is similar to that of a simple graph: a path between two distinct nodes u and v is a sequence of nodes and lines: $(u = u_0), e_1, u_1, e_2, u_2, \ldots, u_{n-1}, e_n, (u_n = v)$, such that every line e_i connects the nodes u_{i-1} and u_i. The length of a path equals the number of lines it contains, so in the

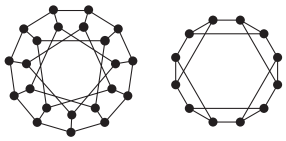

Figure 1. $N2R(11;3)$ and $CR(12;3)$

case above the path is of length n. The distance between a pair of distinct nodes (u, v) equals the length of the shortest path between them and is written $d(u, v)$. This paper considers only connected structures, i.e. between every pair of distinct nodes there exists a path. Two paths between a pair of nodes (u, v) are said to be independent if they share no lines or nodes except for u and v, and a set of paths is said to be independent if the paths are pair wise independent. The size of a structure equals the number of nodes it contains.

$N2R$ structures are defined as follows. Let p and q be positive integers, such that $p \geq 3$, $q < \frac{p}{2}$ and $gcd(p, q) = 1$. p and q then define a structure $N2R(p; q)$, which consists of two rings, an outer ring and an inner ring, each containing p nodes. The nodes of the outer ring are labeled $o_0, o_1, \ldots, o_{p-1}$ and the nodes of the inner ring labeled $i_0, i_1, \ldots, i_{p-1}$. Thus, it contains $2p$ nodes. For each i such that $0 \leq i \leq p - 1$ there exists a line between each of the following pairs of nodes:

- $(o_i, o_{i+1(mod\ p)})$ (lines of the outer ring)
- $(i_i, i_{i+q(mod\ p)})$ (lines of the inner ring)
- (o_i, i_i) (lines connecting the two rings)

The classical double ring with $2p$ nodes obviously corresponds to $N2R(p; 1)$. An example of a $N2R$ is shown in Figure 1. One more restriction to q given p applies throughout the paper: given p, let $q_1 < q_2$ fulfill for $i = 1, 2$ that $q_i < \frac{p}{2}$ and $gcd(q_i, p) = 1$. Then $N2R(p; q_1)$ is isomorphic to $N2R(p; q_2)$ if $q_1 q_2 = 1(mod\ p)$ or $q_1 q_2 = p - 1(mod\ p)$. For such two isomorphic structures q_2 is discarded and only q_1 considered a permitted value.

CR structures are defined as follows. Let w be an even integer such that $w \geq 6$, and let s be an odd integer, such that $3 \leq s \leq \frac{w}{2}$. w and s then define $CR(w, s)$ with w nodes labeled $u_0, \ldots, u_{w-1}$. For $0 \leq i \leq w - 1$ there exists a line between each of the following pairs of nodes:

- $(u_i, u_{i+1(mod\ w)})$
- $(u_i, u_{i+s(mod\ w)})$, for i even.

An example of a CR is shown in Figure 1.

2.1 Evaluation parameters

Widely used distance measures for network topologies are average distance and diameter, indicating transmission delays as well as traffic load[2].

- Average distance: The average of $d(u, v)$ taken over all pairs of distinct nodes.
- Diameter: The maximum of $d(u, v)$ taken over all pairs of distinct nodes.

For real-time applications where even short transmission outages are not acceptable, protection schemes are used. For this, k paths are established when the connection is set up. Traffic can be sent simultaneously along all these k paths, or along only one path, keeping the last $k-1$ path(s) ready for immediate use whenever a failure is detected. In both cases, long restoration times are avoided. The k-measures k-average distance and k-diameter reflect the considerations of average distance and diameter:

- k-average distance: For every pair of distinct nodes (u, v), k independent paths between u and v are constructed such that the sum of the lengths of these paths is smallest possible. The k-average distance is the average of these sums over all pairs of distinct nodes.
- k-diameter: For every pair of distinct nodes (u, v) k independent paths between u and v are constructed such that the longest of these paths is shortest possible. The k-diameter is the maximum over the lengths of these longest paths, over all pairs of distinct nodes.

Since both $N2R$ and CR are 3-regular, these parameters are considered for $k = 2, 3$. 1-average distance and 1-diameter equal average distance and diameter. Where not confusing, we will simply write k-average instead of k-average distance.

3 Methods

The first step is to determine which structures to compare. In order to facilitate a comparison it is desirable to have only one $N2R$ and one CR of a given size, or a limited number with parameters close to each other. This is especially so for general-purpose networks where more parameters are used for selection; assume that a network structure is to be chosen, which should have short average distance and diameter. It is little interesting if one structure performs well with regard to average distance and another structure belonging to the same class performs well with regard to diameter, if no structure of that class perform satisfactory with regard to both. For both CR and $N2R$ there can exist several structures of the same size. For $N2R$ a policy was introduced for selecting q given p[12]:

- Select the values of q such that the diameter is minimum.
- Among those values of q, select those such that the average distance is smallest possible.

It was shown for $p \leq 2000$ that this leads to structures, which are close to optimal with regard to average

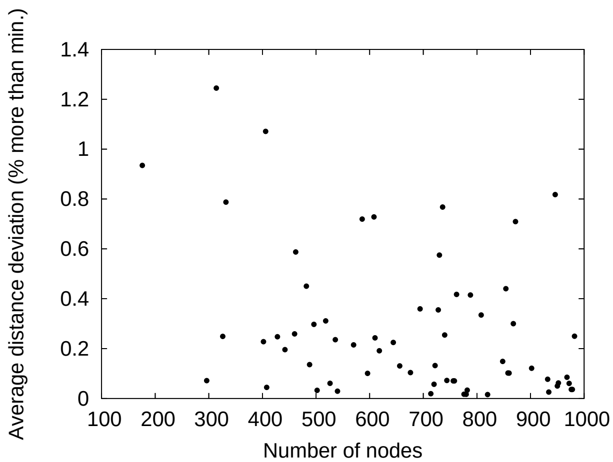

Figure 2. The difference between minimum average distances and minimum average distances when diameters are minimized.

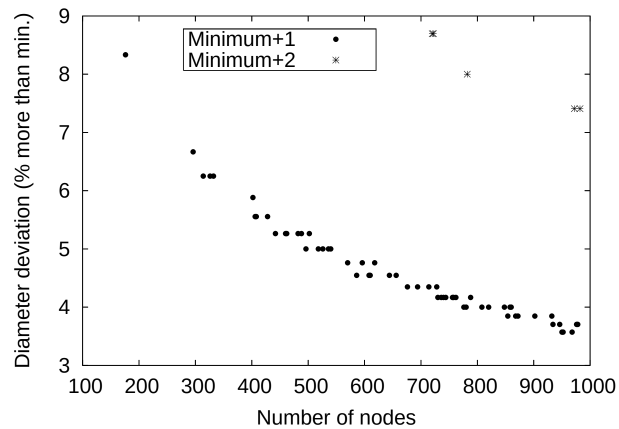

Figure 3. The difference between minimum diameters and minimum diameters when average distances are minimized. In these 64 cases, the diameter is either 1 or 2 higher than minimum.

distance, and for $p \leq 82$ they were also shown to be optimal or close to optimal for k-averages and k-diameters. k-averages and k-diameters were not evaluated for larger structures. In this paper, we evaluate the same selection policy for CR and compare it to the related policy, where average distance is minimized first, and s then chosen among these possible values to minimize the diameter. For average distance and diameter, this comparison is performed for all structures with $w \leq 1000$. Average distances and diameters are determined by simply calculating these values for all structures, using standard shortest-path algorithms and making use of the symmetries. Since no efficient algorithms are known for determining k-average and k-diameter, they are calculated using brute-force algorithms. Therefore, the policies were only evaluated for $w \leq 100$ with regard to those parameters. Since the policies result in the same structures for $w \leq 100$, no policy comparison were made here.

Based on the results obtained, the policy of selecting first diameter and then average distance is used in the rest of the paper, providing a base for comparison of all parameters. First, diameter and average distance are compared. Due to the selection policy, for each number of nodes all selected structures within each class have the same average distance and diameter. The results are derived from the calculations carried out in order to compare the selection policies.

For the k-measures, the parameters can be determined for significantly larger structures if they are calculated only for good values of q and s rather than for all permitted values. This also reduces the number of structures for which the k-measures are evaluated. Therefore, they are for each value of p or w determined only for the values of q or s determined by the selection policy. While it is not guaranteed for any k-measure that the minimum value is obtained, any other choice would imply a trade-off between average distance/diameter and one or more k-measures. For general purpose networks, average distance and diameter would in many cases be considered most important, and the presented selection policy therefore used anyway. Given this selection, 2-average, 3-average, 2-diameter and 3-diameter were evaluated for structures with up to 900, 500, 800 and 400 nodes respectively. In some cases multiple $N2R$ and CR exist for each number of nodes, but as the k-measures for these structures turn out to be close to each other, no further selection is done.

4 Results

4.1 Selection policies for CR

4.1.1 Average distance and diameter

Two approaches were evaluated for choosing s given w. In the first approach, for every value of w, all values of s minimizing the diameter are selected, and among these, s is chosen to minimize average distance. In the second approach all values of s minimizing the average distance are selected, and among these, the values resulting in the smallest diameter are selected.

In 434 of the 498 cases, there exist structures minimizing both average distance and diameter. Figures 2-3 illustrate the resulting average distances and diameters compared to the optimal values in the remaining 64 cases. With minimized diameters, the average distances are on average over these 64 cases 0.27% higher than minimum, and with minimized average distances, the diameters are on average 4.90% higher. Over all 498 cases the differences are on average 0.035% and 0.63%. We choose to use the first approach for our studies, but both nearly minimize the parameters.

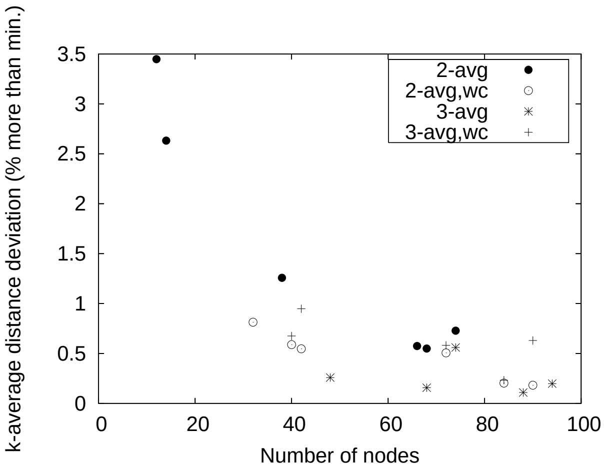

Figure 4. 2-average and 3-average compared to minimum values in the 12 respectively 10 cases when they are not always minimized. wc indicates that some but not all choices of s minimize the parameter.

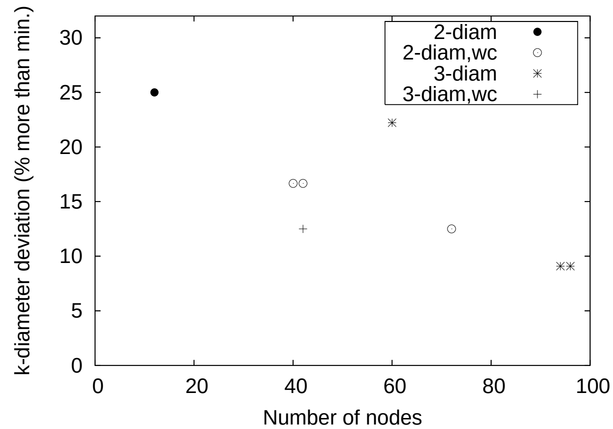

Figure 5. 2-diameter and 3-diameter compared to minimum values in the 4 respectively 4 cases when they are not always minimized. wc indicates that some but not all choices of s minimize the parameter.

4.1.2 k-average and k-diameter

We evaluate to what extent the optimal k-average and k-diameter is obtained when for each value of w, the values of s are selected using the chosen selection policy. This is done for $6 \leq w \leq 100$, a total of 48 values. For 31 of these, all chosen values of s minimized all parameters, and for additionally 5 values of w, at least one of the chosen values of s minimizes all parameters. In the remaining cases, the choices of s result in structures, which are close to minimal as can be seen in Figures 4-5. They show how much larger the k-measures are for the selected values of s compared to minimum values. Only values of w and s not minimizing the respective parameters are shown; if for w some but not all selected values of s minimize a parameter, the values not minimizing s are marked wc (for worst-case).

Over all 48 values of w, the 2-average is on average 0.19% higher than the minimum values in the best-case and 0.25% higher in the worst-case. For the 3-average the corresponding values are 0.027% and 0.091%. For 2-diameter the values are 0.52% and 1.48%, and for 3-diameter 0.84% and 1.10%. For $w = 60$, the 3-diameter is 2 higher than the minimum value, but in all other cases the difference in k-diameter does not exceed one. We conclude that the proposed selection policy nearly minimizes the k-measures, and so this approach can be used. Through the rest of the paper, when referring to CR and $N2R$ structures, they are implicitly assumed to be selected in this manner.

4.2 Comparison of $N2R$ and CR

4.2.1 Average distance and diameter

In general $N2R$ have lower average distances and diameters than CR, as can be seen in Figures 6-7. For small structures the differences are limited, and in 2 cases (12 and

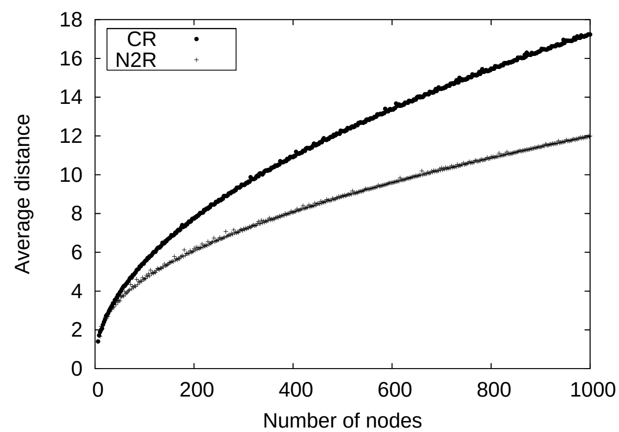

Figure 6. Average distances in $N2R$ and CR.

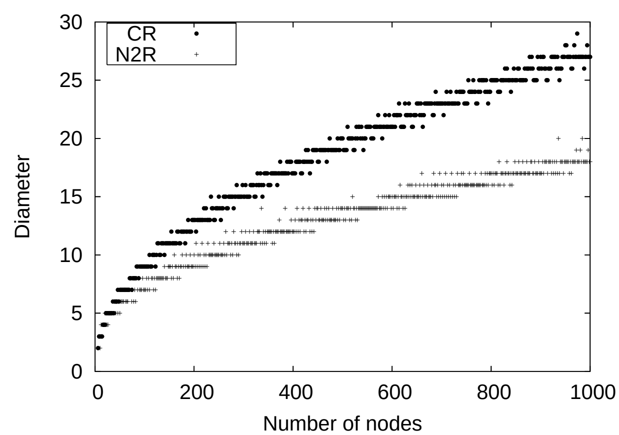

Figure 7. Diameters in $N2R$ and CR.

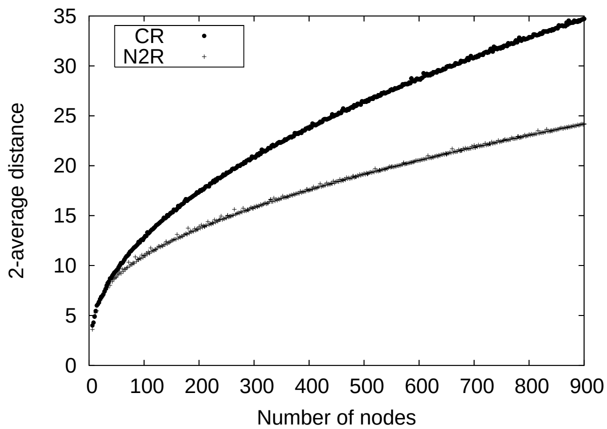

Figure 8. 2-average of $N2R$ and CR.

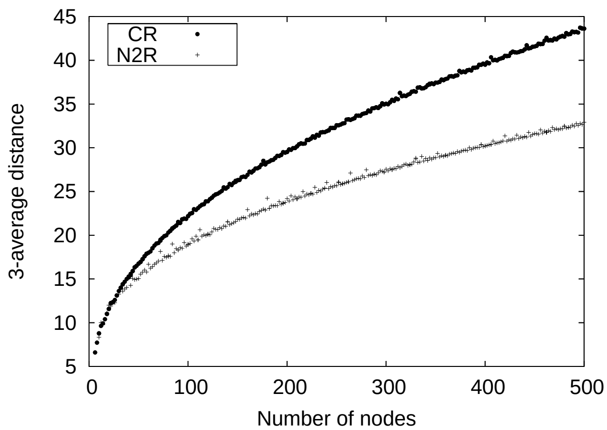

Figure 9. 3-average of $N2R$ and CR.

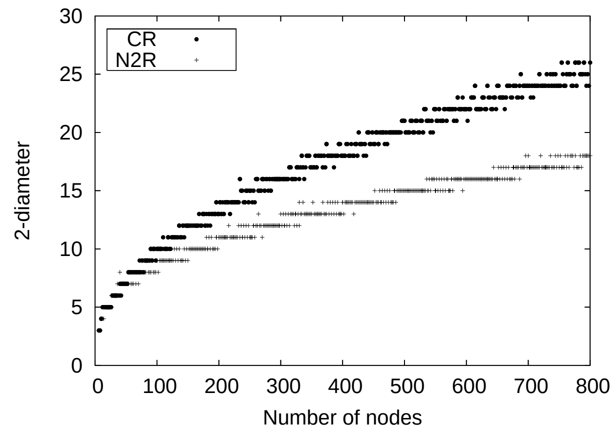

Figure 10. 2-diameters of $N2R$ and CR.

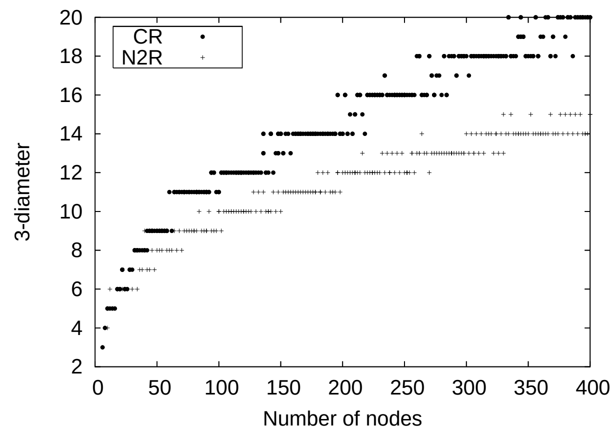

Figure 11. 3-diameters of $N2R$ and CR.

20 nodes), CR are better than $N2R$ with regard to both average distance and diameter. In all other cases, $N2R$ are better than or equal to CR with regard to both parameters. The differences generally increase with the size of the structures, but are significant even for some rather small structures. For example, average distances and diameters of $N2R(25;7)$ and $CR(50;9)$, both minimizing both parameters within each class of structures, are 3.49 respectively 5 for the former and 3.94 respectively 7 for the latter. For $N2R(400;91)$ and $CR(800;109)$ the average distances are 10.9 respectively 15.5, and the diameters 17 respectively 25. Thus, using $N2R$ instead of CR in these cases reduces the average distance by 29.4% and the diameter by 32.0%.

4.2.2 k-average and k-diameter

Figures 8-11 show the k-averages and k-diameters of all selected CR compared to all selected $N2R$. The chosen selection policy can result in several different structures of the same size, in which case they are all shown in the plots. For $w \leq 900$ there are on average 1.91 CR and 1.23 $N2R$ for each number of nodes, but the k-measures only differ slightly. Over all considered values of w and p, the maximum k-measures are on average 0.0078% - 0.17% larger than the minimum for CR and 0.0024%-0.078% larger than the minimum for $N2R$. The highest differences are found for the k-diameters.

$N2R$ structures are in general superior with respect to all four parameters. For all structures with 44 or more nodes, $N2R$ are better than CR in terms of at least one parameter, and equal to or better than CR with respect to all parameters. For structures with fewer than 44 nodes, the picture is more mixed, and the differences generally small: in 8 cases $N2R$ are better with regard to at least one parameter and better than or equal with regard to all other parameters. In 2 cases $N2R$ and CR are equal in terms of all parameters, and in 7 cases $N2R$ are best with regard to some parameters and CR best with regard to others. 2 cases remain. In each of these there exist one $N2R$ but multiple CR. For $w = 32$, $s = 13$ implies that all parameters are equal, while for $s = 7$ and $s = 9$ CR have slightly lower 2-average than $N2R$. This choice does not affect the other parameters. For $w = 40$, $CR(40; 9)$ has the same 2-average as $N2R$, but is better in terms of all other parameters. $CR(40; 11)$ has 2-average slightly higher, but is also superior to the $N2R$ in terms of all other parameters.

As for the average distance and diameter, the differences increase with the size of the structures. For 400 nodes, $CR(400; 47)$ and $N2R(200; 19)$ minimize both average distance and diameter within each class of structures. CR has 2-average 23.72, 3-average 39.5, 2-diameter 18 and 3-diameter 20. The corresponding values for $N2R$ are 17.56, 30.2, 14 and 15. Thus, choosing $N2R$ instead of CR in this case reduces the parameters by 26.0%, 23.6%, 22.2% and 25.0% respectively.

5 Conclusion and Discussion

Two important results were obtained. First, a policy for selecting the best Degree Three Chordal Rings (CR) given the number of nodes was devised and evaluated. It was shown that selecting s to minimize diameter and to the largest possible extent also average distance leads to structures, which are close to optimal with regard to average distance, k-average distance and k-diameter, $k = 1, 2$. This selection policy facilitates the comparison of CR to other structures, and this was applied to obtain the most interesting result, namely the comparison of CR to Generalized Double Rings($N2R$), which share many properties with CR. As the selection policy was previously shown to be good also for $N2R$, they are selected in the same way.

Average distance and diameter were compared for structures with up to 1000 nodes. For 2 small structures, CR performed better than $N2R$ with regard to both of these parameters, but for large structures $N2R$ performed considerably better than CR. For structures with 800 nodes the average distance of $N2R$ is 29.4% and the diameter 32.0% lower than for CR. 2-average distance, 3-average distance, 2-diameter and 3-diameter were calculated for all structures with up to 900, 500, 800 and 400 nodes respectively. For structures with less than 44 nodes, the differences between $N2R$ and CR were small, even though $N2R$ generally performed better than CR. For structures with 44 or more nodes, $N2R$ performed better in all cases, with the differences between each of the four parameters generally increasing with the size of the structures. For instance, for structures with 400 nodes the four parameters are 22.2-26.0% lower for $N2R$ than for CR.

Using $N2R$ instead of CR in logical or physical networks may require more careful planning because simple rings are so easily extended to CR, but if this careful planning is done, the perspectives are promising and facilitate the design of networks with shorter distances and thus shorter transmission delays as well as lower traffic loads. While our results indicate that $N2R$ are superior to CR, it must also be taken into consideration that CR are more extensively studied with regard to routing and transmission abilities. Therefore, we would like to encourage further research to deal with these aspects of $N2R$.

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