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by

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A short update on equipackable graphs

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Abstract

A graph is called equipackable if every maximal packing in it is also maximum. This generalizes randomly packable graphs. We survey known results both on randomly packable graphs and on equipackable graphs. As a new result is given a characterization of P_3-equipackable graphs with all valencies at least two.

1 Definitions

Let H be a subgraph of G. A collection $H_1, H_2, \ldots, H_k$ of edge disjoint subgraphs of G, each isomorphic to H, is called an H-packing of G, and it is *maximal* if $G - \cup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to H. It is maximum if no H-packing in G with more than k copies of H exists. Bosák

[3] wrote a book on packing, he used the equivalent term *decomposition*, and there are many papers on the topic. G is called *H-packable* if there exists an H-packing of G which uses all edges in G and G is called *randomly H-packable* if every maximal H-packing in G uses all edges in G, i.e., if every H-packing can be extended to a decomposition of the edges of G into copies of H. Note that not every graph H produces a family of randomly H-packable graphs, if e.g. H is the disjoint union of K_2 and K_3, no other graph than H itself is randomly H-packable.

As a relaxation of random H-packability we define G to be *H-equipackable* if every maximal H-packing is also a maximum H-packing. So the randomly H-packable graphs is contained as a subclass in the class of H-equipackable graphs. This paper focuses on $H = P_3$.

2 Notation

A graph G has *order* $|V(G)|$ and *size* $|E(G)|$. The *path* and *circuit* on k vertices is denoted by P_k and C_k, respectively. By $C_m \bullet C_n$ we denote the graph of order $n+m-1$ obtained from two circuits C_m and C_n by identifying one vertex from each. $S^{(r)}_{2k+1}$ denotes the graph obtained from r paths P_{2k+1} by identifying their center vertices. The *corona* $H \circ K_1$ on H is is the graph of order $2|H|$ obtained by adding for each vertex x of H one new vertex x' and a new edge xx'. By $H + G$ we denote the graph obtained from two disjoint graphs H and G by adding edges joining each vertex of H to each vertex of G. A *matching* in the graph G is a set of independent edges in G, it is *perfect* if it covers all vertices of G. By M_t, $t \geq 1$, we denote a matching having t edges. The union of k disjoint copies of a graph G is denoted by nG, e.g. $M_t = tK_2$.

3 Results

An early result by Caro and Schönheim is

Lemma 1 ([4]) *A connected graph G is P_3-packable if and only if G has even size.*

Observation *A connected graph G of odd size contains an edge whose deletion leaves a connected graph, which necessarily is of even size and therefore by Lemma 1 is P_3-packable.*

It is clear that a maximum P_3-packing in a connected P_3-equipackable graph contains either all or all but one edge of the graph.

It follows that a P_3-equipackable connected graph of even size is also randomly P_3-packable. Another useful observation is that if there is a maximal P_3-packing of a connected graph G which omits at least two edges, then G is not P_3-equipackable.

Ruiz characterized randomly P_3-packable graphs.

Theorem 1 ([11]) *A connected graph G is randomly P_3-packable if and only if $G \cong C_4$ or $G \cong K_{1,2k}, k \geq 1$.*

Thus P_3-equipackable graphs of even size are quadrilaterals or stars. It remains to characterize P_3-equipackable graphs of odd size. Hartnell and Vestergaard did that for graphs of girth at least five.

Theorem 2 ([5]) *A connected graph G of girth 5 or more is P_3-equipackable if and only if G satisfies one of the following:*

(i) G is a tree consisting of a single star (i.e., $K_{1,n}$)

(ii) G is a tree which has two stems that are at distance 3, where the vertices on this shortest path are w_1 and w_2. Furthermore the stems are of odd parity and have no neighbours other than leaves and w_1 or w_2. In addition w_1 and w_2 are of degree two.

(iii) G is a tree which has two stems that are at distance two where w is the common neighbour of the stems. The two stems must be of different parity and neither stem has other neighbours than its leaves and w. Furthermore, the vertex w must be of degree two.

(iv) G is a tree which has two stems that are adjacent where these stems are of the same parity and these stems have only each other and their leaves as neighbours.

(v) G is either C_7, C_5 or has $5{+}2m$ vertices where G consists of a circuit of length 5 along with $2m$ leaves attached to exactly one node on the 5-cycle.

The remaining problem is to characterize P_3-equipackable graphs of girth 3 and 4. We shall consider graphs with $\delta(G) \geq 2$.

Theorem 3 *Let G be a graph with $\delta(G) \geq 2$, which is connected and has a cutvertex. Then G is P_3-equipackable if and only if either (1) G is of order eight and is obtained from two vertex disjoint quadrilaterals together with an edge joining a vertex in one quadrilateral to a vertex in the other or (2) G is a $C_3 \bullet C_4$.*

Proof. The two graphs described can be checked to be P_3-equipackable. Conversely, let x be a cutvertex in the P_3-equipackable graph G and let $A'_1, \dots, A'_m$ be the components of $G - x$, while $A_1, \dots, A_m$ are the graphs spanned in G by $A'_i \cup \{x\}$. We first observe that $|E(A_i)|$ can be odd for at most one i. Assume namely A'_i is a component with $|E(A_i)|$ odd. We shall by P_3-removals isolate a subgraph of A'_i having odd size. If there is an even number of edges from x to A'_i we P_3-remove them in pairs. This isolates A'_i with an odd number of edges. If there is an odd number of edges from x to A'_i we can P_3-remove all but one of them, say xy, and as $\delta(G) \geq 2$ we have that $y \in V(A'_i)$ is incident with an edge yz, $z \in A'_i \setminus \{y\}$. By removal of $\{xy, yz\}$ we also in this case have isolated a subgraph of A'_i with an odd number of edges and $|E(A_i)|$ can by the observation after Lemma 1 be odd for at most one index i.

If all $|E(A_i)|$ are even, then G has even size and, by the remark after Lemma 1, G is randomly P_3-packable and hence by Theorem 1 is C_4 or a star, but C_4 has no cutvertex and the star violates $\delta(G) \geq 2$, so this case cannot occur.

Thus $|E(A_1)|$, say, is odd and all other $|E(A_i)|, i \geq 2$, are even.

Subcase 1. Assume a bridge xy of G joins x to A_1'. P_3-remove yx, xz, $z \in N(x) \cap V(A_2')$. Then A_1' is isolated and as a connected graph of even size, it is therefore randomly P_3-packable and by Theorem 1 isomorphic to C_4 as $\delta(G) \geq 2$. The graph spanned in G by the union of $A_2, \ldots, A_m$ is connected, of even size and it is also randomly P_3-packable. By Theorem 1 it is isomorphic to C_4 and G consists of two disjoint C_4's joined by an edge as claimed in (1).

Subcase 2. Assume x is joined to A_1' by more than one edge. Again the graph spanned by the union of $A_2, \ldots, A_m$ is connected and of even size. It is randomly P_3-packable because we can by P_3-removals inside A_1 make sure that the unique non-removed edge is isolated inside A_1'. Thus all of $A_2, \ldots, A_m$ spans a C_4. We can deduce that A_1 is isomorphic to C_3, because A_1 minus an edge must be randomly P_3-packable and C_4 cannot be fitted in to ensure P_3-equipackability of G, so A_1 minus an edge must be a star and only P_3 with its two ends joined to x will do. In this case G is the disjoint union of a quadrilateral and a triangle with one vertex from each identified as claimed in (2). This proves Theorem 3.

Theorem 4 *A connected graph G with $\delta(G) \geq 2$ is P_3-equipackable if and only if G is one of of the graphs listed in Figure 1.*

Proof. We can by inspection verify that the graphs in Fig. 1 all are P_3-equipackable. If G has a cutvertex we know by Theorem 3 that $G \cong C_3 \bullet C_4$ or G can be obtained by joining two quadrilaterals by an edge. So assume that G is 2-connected and P_3-equipackable, we must prove that G is one of the remaining graphs listed on Fig. 1. Let C with length ℓ be a longest circuit in G. If $\ell = 3$ necessarily $G \cong C_3$, a graph in the family on top of Fig. 1. If $\ell = 4$ and C has no diagonal the only possibility is $G \cong C_4$ since by 2-connectivity any x in $V(G) \setminus V(C)$ must be joined to C by two independent paths. Each path must be an edge, otherwise G would contain a circuit longer than four. Also x must have valency 2. Thus G is of even size, that implies that G is randomly P_3-packable and hence by Theorem 1 must be a C_4. If $\ell = 4$ and C has a diagonal we can by a similar argument obtain that G is a graph in the family on top of Fig. 1. If $\ell = 5$ and G has order 5 we obtain the four graphs of Fig. 1. If $\ell = 5$ and G has order > 5 each x in $V(G) \setminus V(C)$ is by 2-connectivity joined by two independent paths to C.

Again, not to produce a longer circuit, x is joined by edges to two vertices at distance two on C, and x has valency two. But now it is easy to see that we by P_3-deletions can isolate two edges, so this case cannot occur. If $\ell = 6$ we can find that C must contain a triangular diagonal. If G has order 6, either G is this graph, included in Fig. 1, or we can by P_3-removals isolate two edges, a contradiction. In the remaining cases either G is C_7, included in Fig. 1, or we can for each vertex x on C pairwise P_3-remove its adjacent edges in $E(G) \setminus E(C)$, so at most one edge besides the two circuit edges remain at x. If the end vertex y of such an edge has a neighbour z such that $yz \notin E(C)$, we P_3-remove xy, yz. We have thus isolated a component consisting of C and possibly at some C-vertices one other edge in $E(G) \setminus E(C)$, a pendent edge or a diagonal. At most one vertex on C has a pendent edge, otherwise we could by P_3-removals on C isolate the two pendent edges, a contradiction. We can now see that we by further P_3-removals can isolate two edges, a contradiction. This proves the theorem.

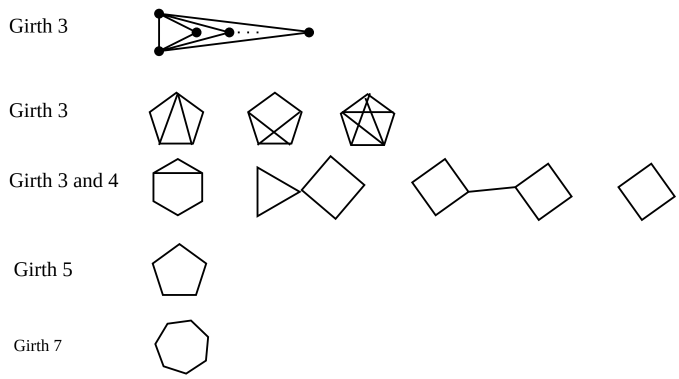

Figure 1: All P_3-equipackable graphs which are connected and have all valencies at least two.

4 Randomly packable graphs

Equipackability is a relaxation of random packability, so let us mention a few results from packings, all have potential for generalizations to equipackability.

Theorem 1 by Ruiz [11] was later generalized from P_3 to $K_{1,r}$ by Barrientos, Bernasconi, Jeltsch, Tronisco and Ruiz:

Theorem 5 ([1]) *For $r \geq 2$ a connected graph is randomly $K_{1,r}$-packable if and only if it is $K_{r,r}$ or it is bipartite with all valencies in one partite set being multiples of r and all valencies in the other set being less than r.*

4.1 Randomly path-packable graphs

Beineke, Goddard, Hamburger [2] generalized Theorem 1 from P_3 to P_k, $3 \leq k \leq 6$, and Molina with coauthors [9, 10] extended to $k \leq 10$:

Characterization of connected, randomly P_k-packable graphs		
[11]	Randomly P_2-packable	Trivially every graph
[11]	Randomly P_3-packable	C_4 and stars of even size
[2] [9, 10]	Randomly P_4-packable	$P_4, K_4, K_{2,3}, C_6, C_3 \bullet C_3$
[2] [9, 10]	Randomly P_5-packable	$P_5, K_{2,4}, C_4 \bullet C_4, C_8, S_5^{(k)}, k \geq 2$
[2] [9, 10]	Randomly P_6-packable	$P_6, C_{10}, C_5 \bullet C_5, K_4 \circ K_1$ and the graph obtained by joining two new vertices by an edge to the same valency 2 vertex of a $K_{2,4}$
[9, 10]	Randomly P_7, P_8, P_9,- and P_{10}-packable	Families of graphs whose descriptions become increasingly complex with growing k

4.2 Randomly matching-packable graphs

Ruiz characterized randomly M_2-packable graphs:

Theorem 6 ([11]) *A graph is randomly M_2-packable if and only if it one of the following: $C_4, K_4, 2K_3, K_3 \cup K_{1,3}, 2K_{1,n}$ or $2nK_2, n \geq 1$.*

This was generalized to randomly M_t-packable graphs by Beineke, Goddard, Hamburger, but only for graphs with sufficiently many edges:

Theorem 7 ([2] Th. 2.5) *For a given integer $t \geq 2$, a graph with at least $2t^3 - t^2$ edges is randomly M_t-packable if and only if it is isomorphic to tH, where H is either nK_2 or $K_{1,n}$ for some $n \geq 1$.*

Those graphs in which each matching can be extended to a perfect matching are called *randomly matchable* and if each matching extends to a maximum matching, which is not necessecarily perfect, they are called *equimatchable*. Sumner characterized randomly matchable graphs:

Theorem 8 ([12]) *A connected graph G is randomly matchable if and only if $G \cong K_{2n}$ or $G \cong K_{n,n}$, where n is a positive integer.*

Lesk, Plummer and Pulleyblank [6] gave a characterization of equimatchable graphs in terms of the Gallai-Edmonds structure theorem (described in [7]). Define a *total matching* to be a subset X of $E(G) \cup V(G)$ such that no two elements of X are adjacent or incident. Topp and Vestergaard characterized totally equimatchable graphs:

Theorem 9 ([13]) *A connected graph G is totally equimatchable if and only if G is $K_n, K_{n,n}$ or $K_1 + \cup_{i=1}^{n} K_{2m_i}$, where n and $m_1, m_2, \ldots, m_n$ are any positive integers.*

4.3 Randomly K_n-packable graphs

Beineke, Hamburger and Goddard proved

Theorem 10 ([2] Th. 3.1) *A graph G is randomly K_n-packable if and only if every edge is in precisely one copy of K_n in G.*

McSorley and Porter ([8]) have considered a vertex variant. They let

(*) $\{G_{\alpha n}\}_{\alpha=1}^{\infty} \;=\; G_n, G_{2n}, \ldots, G_{\alpha n}, \ldots$

be a sequence of graphs such that $G_1 \cong K_n$, $G_{\alpha n}$ has order αn and $G_{\alpha n} - K_n \cong G_{(\alpha-1)n}$ for any complete subgraph K_n in $G_{\alpha n}$, $\alpha \geq 2$.

For $0 \leq \lambda \leq n$ they call (*) a (K_n, λ)-removable sequence if $G_{\alpha n}$ is $(n-1)+(\alpha-1)\lambda$-regular, and they prove that for a fixed $n \geq 2$ there is a unique (K_n, λ)-removable sequence for $\lambda = 0, n-1$ or n.

Theorem 11 ([8]) *For a fixed $n \geq 2$,*

$\{\alpha K_n\}_{\alpha=1}^{\infty}$ *is the unique $(K_n, 0)$-removable sequence.*

$\{K\underbrace{\alpha, \alpha, \alpha, \ldots, \alpha}_{n}\}_{\alpha=1}^{\infty}$ *is the unique $(K_n, n-1)$-removable sequence.*

$\{K_{\alpha n}\}_{\alpha=1}^{\infty}$ *is the unique (K_n, n)-removable sequence.*

5 Open problems

Molina and coauthors in [10] posed a still unsolved problem: Does the characterizations become easier if only 2-connected graphs are considered? Their examples of 2-connected randomly P_k-packable graphs contain only two copies of P_k and with three exceptions have a vertex of odd degree, and they asked whether that holds generally.

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