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ON TRACES OF GENERAL DECOMPOSITION SPACES

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ABSTRACT. The decomposition space approach is a general method to construct smoothness spaces on $\mathbb{R}^d$ that include Besov, Triebel-Lizorkin, modulation, and α-modulation spaces as special cases. This method also handles isotropic and anisotropic spaces within the same framework.

In this paper we consider a trace theorem for general decomposition type smoothness spaces. The result is based on a simple geometric estimate related to the structure of coverings of the frequency space used in the construction of decomposition spaces.

1. INTRODUCTION

Smoothness spaces such as the Besov, Modulation, and Triebel-Lizorkin spaces play an important role in approximation theory and harmonic analysis. Important applications of such spaces, and of smoothness spaces in general, include the study of (partial) differential equation and signal processing (compression, de-noising etc.).

Decomposition spaces were introduced by Feichtinger and Gröbner [7] and Feichtinger [6], and are based on structured coverings of the frequency space $\mathbb{R}^d$. In this context, the classical Triebel-Lizorkin and Besov spaces correspond to dyadic coverings, see [20]. However, many other covering of the frequency space can be considered leading to new smoothness spaces. For example, by choosing structured "polynomial" coverings we obtain so-called α-modulation spaces, see [11].

An approach to constructing stable frames for decomposition smoothness spaces was introduced in [2, 3]. The approach in [2, 3] is quite general and a multitude of isotropic and anisotropic spaces are covered by the construction, which is based on simple geometric coverings.

One important application of smoothness spaces is to the study of partial-differential and pseudo-differential operators. Such operators on Besov and Triebel-Lizorkin spaces have been studied by many authors. For example, the Besov case was considered by Gibbons [10] and Bourdaud [4], while the Triebel-Lizorkin case was studied by Päivärinta [15] and Bui [17]. The anisotropic case was considered by Yamazaki [21, 22]. PDOs have also been studied on spaces of Besov type based on non-dyadic frequency splittings. In particular, boundedness of such operators on modulation spaces has been considered by many authors, see e.g. [5, 12, 16, 19].

Key words and phrases. Trace operator, α-modulation space, Besov space, Triebel-Lizorkin space.

Partial differential equations are often studied in the context of boundary value problems. For such problems, the so-called trace operator plays an important role, see [13, 20]. In this paper we study the trace operator acting on multivariate smoothness spaces constructed by the decomposition method.

Let us formally define the trace operator. Since the trace operator acts pointwise, we have to be somewhat careful with the definition. Here we follow standard procedure and use a denseness argument to make sure that everything is well-defined. Let $d \geq 2$ and for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, we denote $x' = (x_1, \ldots, x_{d-1})$. For any (quasi-) Banach function space $X(\mathbb{R}^d)$ defined on $\mathbb{R}^d$, with the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ dense in X, the trace operator is first defined for $f \in \mathscr{S}(\mathbb{R}^d)$ by restricting f to the hyperplane $\{x \in \mathbb{R}^d : x = (x', 0)\}$. That is, $(\mathrm{Tr} f)(x') = f(x', 0)$, $x' \in \mathbb{R}^{d-1}$. Now, suppose Y is a (quasi-) Banach function space defined on $\mathbb{R}^{d-1}$, and we have a constant $C > 0$ such that

$$\|\mathrm{Tr} f\|_Y \leq C\|f\|_X, \qquad f \in \mathscr{S}(\mathbb{R}^d).$$

Then we simply extend Tr to all of X by a denseness argument.

The trace operator on various smoothness spaces has been studied by many authors. For Besov and Triebel-Lizorkin spaces, see [1, 9, 20], and for modulation, α-modulation and Besov spaces the operator has been considered in [8].

The structure of this paper is as follows. In Section 2 we define the decomposition type smoothness spaces following a review of the machinery needed to construct such spaces. It is necessary to go into some details about the construction of the smoothness spaces in order to be able to analyze the action of the trace operator. Section 3 contains our main result on the trace operator. The result is based on a simple geometric estimate derived directly from the very construction of the spaces. Examples are given throughout Sections 2 and 3, where we focus on the class of α-modulation spaces that include Besov and modulation spaces as special cases.

2. Decomposition-type smoothness spaces

In this section we give a brief description of the decomposition type smoothness spaces of Triebel-Lizorkin and modulation type introduced in [3]. A more detailed study of such spaces and associated discrete decompositions can be found in [2, 3], see also [14] for a construction of compactly supported frames for such spaces.

The construction is based on the following three steps.

- Construct a structured covering of the frequency space $\mathbb{R}^d$. The structure of the covering then determines the nature of the associated smoothness space. For example, coverings obtained by translating a fixed set generate so-called modulation spaces, while dyadic type coverings generate Besov and Triebel-Lizorkin spaces.
- Define a suitable resolution of the identity on $\mathbb{R}^d$ compatible with the covering of $\mathbb{R}^d$. That is, a countable collection of smooth functions $\{\varphi_k\}$ with $\sum_k \varphi_k = 1$.

- Used the resolution of identity to define spaces of modulation/Besov type and Triebel-Lizorkin type.

First we start by introducing a method to generate (possibly anisotropic) coverings of $\mathbb{R}^d$ that eventually will serve as support sets for the resolution of identity. The starting point is to introduce an anisotropic quasi-distance on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively.

2.1. **Anisotropies and quasi-distances on $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$.** Throughout this paper we assume that the dimension $d \geq 2$ is fixed. An anisotropy on $\mathbb{R}^d$ is defined to be a vector $\mathbf{a} = (a_1, a_2, \dots, a_d)$ of strictly positive numbers, which we assume is normalize such that $\sum_{j=1}^d a_i = d$. For $t \geq 0$, the $d \times d$-anisotropic dilation matrix $\delta_{\mathbf{a}}(t)$ is defined by $\delta_{\mathbf{a}}(t) = \operatorname{diag}(t^{a_1}, \dots, t^{a_d})$.

We now introduce the standard quasi-norm $|\cdot|_{\mathbf{a}}$ associated with $\mathbf{a}$.

Definition 2.1. Define the function $|\cdot|_{\mathbf{a}} : \mathbb{R}^d \to \mathbb{R}_+$ by $|0|_{\mathbf{a}} := 0$, and for $\xi \in \mathbb{R}^d \backslash \{0\}$, by letting $|\xi|_{\mathbf{a}}$ be the unique solution t to the equation $|\delta_{\mathbf{a}}(t)| = 1$.

One can verify (see [18]) that there is a constant $C_{\mathbf{a}} \geq 1$ such that

$$|\xi + \eta|_{\mathbf{a}} \leq C_{\mathbf{a}} [|\xi|_{\mathbf{a}} + |\eta|_{\mathbf{a}}], \qquad \forall \xi, \eta \in \mathbb{R}^d. \tag{2.1}$$

Moreover, it can be verified that

$$|\xi|_{\mathbf{a}} \asymp \sum_{j=1}^{d} |\xi_j|^{1/a_j}, \qquad \xi \in \mathbb{R}^d.$$

The bracket associated with $|\cdot|_{\mathbf{a}}$ is defined by

$$\langle \xi \rangle_{\mathbf{a}} := 1 + |\xi|_{\mathbf{a}}, \qquad \xi \in \mathbb{R}^d. \tag{2.2}$$

Finally, we define the balls (essentially ellipsoids) $\mathscr{B}_{\mathbf{a}}(\xi, r) := \{\zeta \in \mathbb{R}^d : |\xi - \zeta|_{\mathbf{a}} < r\}$. It can be verified that $|\mathscr{B}_{\mathbf{a}}(\xi, r)| = r^d \omega_d^{\mathbf{a}}$, where $\omega_d^{\mathbf{a}} := |\mathscr{B}_{\mathbf{a}}(0,1)|$, so $(\mathbb{R}^d, |\cdot|_A, \dot{\xi})$ is a space of homogeneous type with homogeneous dimension d.

Remark 2.2. An even more general setup can be considered where δ_t is the one-parameter group generated by a matrix A with positive eigenvalues, see [3]. However, the present slightly less general setup turns out to be well-suited for our proposed study of the trace operator.

Since our focus is on the trace operator, we also introduce the truncated anisotropy $\mathbf{a}' = (a_1, a_2, \dots, a_{d-1})$. In a similar fashion, $\mathbf{a}'$ defines a quasi-norm $|\cdot|_{\mathbf{a}'}$ on $\mathbb{R}^{d-1}$, and writing $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, we notice that uniformly in $\xi \in \mathbb{R}^d$,

$$|\xi|_{\mathbf{a}} \asymp |\xi'|_{\mathbf{a}'} + |\xi_d|^{1/a_d}.$$

In particular, for any $\xi = (\xi', 0)$ with $\xi' \in \mathbb{R}^{d-1}$, and $t \in \mathbb{R}$, we have uniformly that

$$|\xi|_{\mathbf{a}} \asymp |\xi'|_{\mathbf{a}'}, \qquad |(0,t)|_{\mathbf{a}} \asymp |t|^{1/a_d}. \tag{2.3}$$

2.2. **Anisotropic structured coverings.** Now we first introduce so-called admissible coverings and a method to generate them, see [2, 6]. These coverings are then used to construct a suitable resolution of the identity which are then used to define the Triebel-Lizorkin type spaces and the associated modulation spaces.

Definition 2.3. A set $\mathcal{Q} := \{Q_k\}_{k\in\mathbb{Z}^d}$ of measurable subsets $Q_k \subset \mathbb{R}^d$ is called an admissible covering if $\mathbb{R}^d = \cup_{k\in\mathbb{Z}^d} Q_k$ and there exists $n_0 < \infty$ such that $\#\{j \in \mathbb{Z}^d : Q_k \cap Q_j \neq \emptyset\} \le n_0$ for all $k \in \mathbb{Z}^d$.

To generate an admissible covering we will use a suitable collection of $\mathscr{B}_{\mathbf{a}}$-balls, where the radius of a given ball is a so-called moderate function of its center.

Definition 2.4. A function $h : \mathbb{R}^d \to [\varepsilon_0, \infty)$ for $\varepsilon_0 > 0$ is called moderate if there exists constants $\rho_0, R_0 > 0$ such that $|\xi - \zeta|_{\mathbf{a}} \le \rho_0 h(\xi)$ implies $R_0^{-1} \le h(\zeta)/h(\xi) \le R_0$.

In this paper we shall always assume that any moderate function is *increasing* in the sense that

$$h(\xi) \ge h(\zeta) \text{ whenever } |\xi|_{\mathbf{a}} \ge |\zeta|_{\mathbf{a}}. \tag{2.4}$$

With a moderate function h, it is then possible to construct an admissible covering by using balls, see [6, Lemma 4.7] and [3, Lemma 5].

Lemma 2.5. *Given a moderate function h with constants $\rho_0, R_0 > 0$, there exists a countable admissible covering $\mathscr{C} := \{\mathscr{B}_{\mathbf{a}}(\xi_k, \rho h(\xi_k))\}_{k\in\mathbb{N}}$ for $\rho < \rho_0/2$, and there exists a constant $0 < \rho' < \rho$ such that the sets in $\mathscr{C}$ are pairwise disjoint.*

The proof of the Lemma is straightforward. One picks a maximally disjoint covering of the type $\mathscr{C} := \{\mathscr{B}_{\mathbf{a}}(\xi_k, \rho h(\xi_k))\}_{k\in\mathbb{N}}$ with $\rho < \rho_0/2$ (using, e.g., Zorn's Lemma) and checks that this covering satisfies the requirements.

Example 2.6. Let $0 \le \alpha \le 1$. Then

$$h(\xi) := \langle \xi \rangle_{\mathbf{a}}^{\alpha} \tag{2.5}$$

is moderate on $\mathbb{R}^d$. The covering $\{Q_k\}_k$ given by Lemma 2.5 satisfies the geometric constraint

$$|Q_k| \asymp \langle \xi \rangle_{\mathbf{a}}^{d\alpha}, \qquad \xi \in Q_k,$$

uniformly in k. Therefore, we refer to $\{Q_k\}_k$ as an α-covering. For $\alpha = 0$ we obtain a uniform covering, while for $\alpha = 1$ we obtain a dyadic covering. We obtain polynomial type coverings for $0 < \alpha < 1$ that form the foundation for defining the so-called α-modulation spaces. We shall return to this example several times below.

By using that $\mathscr{B}_A(\xi_k, \rho' h(\xi_k))$ are disjoint it can be shown that $\mathscr{B}_{\mathbf{a}}(\xi_k, 2\rho h(\xi_k))$ also give an admissible covering. Notice that the covering $\mathscr{C}$ from Lemma 2.5 is generated by a family of invertible affine transformations applied to $\mathscr{B}_{\mathbf{a}}(0, \rho)$ in the sense that

$$\mathscr{B}_{\mathbf{a}}(\xi_k, \rho h(\xi_k)) = T_k \mathscr{B}_{\mathbf{a}}(0, \rho), \; T_k := \delta_{\mathbf{a}}(t_k) \cdot + \xi_k,$$

where $t_k := h(\xi_k)$. This observation is essential for the construction since the Fourier transform is well-behaved under an affine change of variable.

Remark 2.7. Geometrically, a covering of the type given by Lemma 2.5 consists of ellipsoids. In some cases, a covering by generalized rectangles may be preferred. This can easily be obtained by observing that there exist $0 < a < b < \infty$ such that

$$[-a,a]^d \subseteq \mathscr{B}_{\mathbf{a}}(0,\rho) \subseteq [-b,b]^d.$$

Then $\{T_k([-b,b]^d)\}_k$ is such a covering by rectangles.

Given a structured covering $\mathscr{C}$ on $\mathbb{R}^d$, there is an easy way to construct a derived covering $\mathscr{C}'$ on $\mathbb{R}^{d-1}$. We simply put

$$\mathscr{C}' = \{Q' := \operatorname{supp}(\chi_Q(x',0)) : Q \in \mathscr{C}, \chi_Q(\cdot,0) \neq 0\}. \tag{2.6}$$

We notice that the structure of $\mathscr{C}$ is preserved in the following sense. If $p : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the projection operator $p((x',t)) = x'$, and $Q = \delta_{\mathbf{a}}(t_k)([-b,b]^d) + \xi_k$, then

$$Q' = \delta_{\mathbf{a}'}(t_k)([-b,b]^{d-1}) + p(\xi_k). \tag{2.7}$$

2.3. **Bounded admissible partition of unity.** We can now generate our resolution of the identity, and for technical reasons we shall require it to satisfy Definition 2.8.

Let us mention that throughout this paper, the Fourier transform $\mathscr{F}$ is defined and normalized as follows, $(\mathscr{F}f)(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\,dx$, $f \in L_1(\mathbb{R}^d)$.

Definition 2.8. Let $\mathscr{C} := \{T_k\mathscr{B}_A(0,\rho)\}_{k\in\mathbb{N}}$ be an admissible covering of $\mathbb{R}^d$ from Lemma 2.5. A corresponding bounded admissible partition of unity (BAPU) is a family of functions $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^d)$ satisfying:

(a) $\operatorname{supp}(\varphi_k) \subseteq T_k\mathscr{B}_{\mathbf{a}}(0,2\rho)$, $k \in \mathbb{N}$.
(b) $\sum_{k\in\mathbb{N}} \varphi_k(\xi) = 1$, $\xi \in \mathbb{R}^d$.
(c) $\sup_{k\in\mathbb{N}} |T_k\mathscr{B}_{\mathbf{a}}(0,2\rho)|^{1/p-1}\|\mathscr{F}^{-1}\varphi_k\|_{L_p(\mathbb{R}^d)} < \infty$, $\forall p \in (0,1]$.

A standard trick for generating a BAPU for $\mathscr{C}$ is to pick $\Phi \in C^\infty(\mathbb{R}^d)$ non-negative with $\operatorname{supp}(\Phi) \subseteq \mathscr{B}_{\mathbf{a}}(0,2\rho)$ and $\Phi(\xi) = 1$ for $\xi \in \mathscr{B}_{\mathbf{a}}(0,\rho)$. One can then show that

$$\Phi_k(\xi) := \frac{\Phi(T_k^{-1}\xi)}{\sum_{j\in\mathbb{Z}^d}\Phi(T_j^{-1}\xi)}$$

defines a BAPU for $\mathscr{C}$.

Since the objective is to study the trace operator, we also need to introduce function spaces on $\mathbb{R}^{d-1}$ that are suitable for capturing the action of the trace operator. The following simple observation provides us with a BAPU on $\mathbb{R}^{d-1}$ that will be suitable for such a purpose. Writing $\xi = (\xi',\xi_d) \in \mathbb{R}^{d-1}\times\mathbb{R}$, we notice that

$$\{\varphi_k(\xi') := \Phi_k(\xi',0) | \Phi_k(\cdot,0) \neq 0\}_{k\in\mathbb{N}} \tag{2.8}$$

defines an associated BAPU on $\mathbb{R}^{d-1}$, with support set $\mathscr{C}' := \{\operatorname{supp}(\varphi_k)\}_k$ that is subordinate to the covering $\mathscr{C}'$ considered in (2.6). That is, we pick the subset of the resolution of the identity with support that intersects the hyperplane $\{(x',0) : x' \in \mathbb{R}^{d-1}\}$.

With a BAPU in hand we can now define the Triebel-Lizorkin type spaces and the associated modulation spaces.

Definition 2.9. Let h be a moderate function satisfying

$$C_1\langle\xi\rangle_{\mathbf{a}}^{\gamma_1} \le h(\xi) \le C_2\langle\xi\rangle_{\mathbf{a}}^{\gamma_2}, \qquad \xi \in \mathbb{R}^d, \tag{2.9}$$

for some $0 < \gamma_1 \le \gamma_2 < \infty$. Let $\mathscr{C}$ be a admissible covering of $\mathbb{R}^d$ from Lemma 2.5, $\{\varphi_k\}_{k\in\mathbb{N}}$ a corresponding BAPU and $\varphi_k(D)f := \mathscr{F}^{-1}(\varphi_k\mathscr{F}f)$.

- For $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \le \infty$, we define $F^s_{p,q}(\mathbb{R}^d)$ as the set of distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ satisfying
$$\|f\|_{F^s_{p,q}(\mathbb{R}^d)} := \Big\|\Big(\sum_{k\in\mathbb{N}} |h(\xi_k)^s\Phi_k(D)f|^q\Big)^{1/q}\Big\|_{L_p} < \infty.$$
- For $s \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q < \infty$, we define $M^s_{p,q}(\mathbb{R}^d)$ as the set of distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ satisfying
$$\|f\|_{M^s_{p,q}(\mathbb{R}^d)} := \Big(\sum_{k\in\mathbb{N}} \big\|h(\xi_k)^s\Phi_k(D)f\big\|^q_{L_p}\Big)^{1/q} < \infty.$$

If $q = \infty$, then the l_q-norm is replaced by the l_∞-norm.

It can be shown that $F^s_{p,q}(\mathbb{R}^d)$ depends only on h (and not on $\mathscr{C}$) up to equivalence of the norms (see [3, Proposition 5.3]), so the Triebel-Lizorkin type spaces are well-defined and similar for the modulation spaces. Furthermore, they both constitute quasi-Banach spaces, and for $p, q < \infty$, $\mathscr{S}$ is dense in both (see [3, Proposition 5.2]). The fact that $\mathscr{S}$ is dense in the respective spaces is essential for us to be able to define the trace operator in Section 3.

The modulation and Triebel-Lizorkin spaces with comparable parameters are quite similar as the following embedding property shows for $0 < p < \infty$, see [3, Proposition 5.7],

$$M^s_{p,\min\{p,q\}}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow M^s_{p,\max\{p,q\}}(\mathbb{R}^d), \qquad 0 < q \le \infty. \tag{2.10}$$

We will use (2.10) in the sequel to derive trace results for the Triebel-Lizorkin type spaces from the results we obtain for the modulation spaces.

2.4. Derived smoothness spaces on $\mathbb{R}^{d-1}$.

Suppose we have a moderate function h on $\mathbb{R}^d$ with structured admissible covering $\mathscr{C}$ and associated BAPU $\{\Phi_k\}$. Then we can easily introduce derived smoothness spaces $F^s_{p,q}(\mathbb{R}^{d-1})$ and $M^s_{p,q}(\mathbb{R}^{d-1})$ on $\mathbb{R}^{d-1}$ by considering the restricted moderate function $h'(x') := h(x', 0)$ associated with the derived covering $\mathscr{C}'$ of $\mathbb{R}^{d-1}$, see (2.6). An associated BAPU $\{\phi_k\}$ on $\mathbb{R}^{d-1}$ is given by (2.8). The spaces associated with this particular setup are denoted $F^s_{p,q}(\mathbb{R}^{d-1})$ and $M^s_{p,q}(\mathbb{R}^{d-1})$, respectively.

Example 2.10. Given an anisotropy $\mathbf{a}$ on $\mathbb{R}^d$, consider $h(\xi) = \langle\xi\rangle^\alpha_{\mathbf{a}}$ for some $0 \le \alpha \le 1$, with an associated covering $\mathscr{C}$ of $\mathbb{R}^d$ given by Lemma 2.5. This defines the so-called anisotropic α-modulation spaces denoted by $F^{\alpha,s}_{p,q}(\mathbb{R}^d)$ and $M^{\alpha,s}_{p,q}(\mathbb{R}^d)$, respectively.

Now, clearly we have $h'(\xi') = (1+|\xi'|_{\mathbf{a}'})^\alpha$, $\xi' \in \mathbb{R}^{d-1}$. Let $Q' \in \mathscr{C}'$, with Q' derived from $Q \in \mathscr{C}$, see (2.6). We claim that uniformly for $\xi' \in Q'$

$$|Q'| \asymp (1+|\xi'|_{\mathbf{a}'})^{(d-a_d)\alpha} = (1+|\xi'|_{\mathbf{a}'})^{(d-1)\frac{d-a_d}{d-1}\alpha}.$$

First notice that $(\xi',0) \in Q$, so

$$|Q| \asymp (1+|(\xi',0)|_{\mathbf{a}})^{d\alpha} \asymp (1+|\xi'|_{\mathbf{a}'})^{d\alpha}.$$

Then we use the observation (2.7) to conclude. Hence the derived covering $\mathscr{C}'$ is a β-covering on $\mathbb{R}^{d-1}$ with $\beta = \alpha\cdot(d-a_d)/(d-1)$. So in cases where $a_d = 1$, such as the isotropic setup, we have $\beta = \alpha$ and there is no "corrective" factor.

Hence, with this particular setup, the derived function spaces on $\mathbb{R}^{d-1}$ are $F^{\beta,s}_{p,q}(\mathbb{R}^{d-1})$ and $M^{\beta,s}_{p,q}(\mathbb{R}^{d-1})$, respectively.

3. The trace operator on decomposition spaces

With the decomposition type smoothness defined in the previous section, we now turn our attention to the trace operator defined for $f \in \mathscr{S}(\mathbb{R}^d)$ by $(\mathrm{Tr})f(x') = f(x',0)$, $x' \in \mathbb{R}^{d-1}$.

Our goal is to obtain a general trace result that applies to (almost) all decomposition smoothness spaces. However, for this to be possible, we can obviously only rely on properties that are common to all of these spaces.

Of particular interest to us is the construction of structured coverings of $\mathbb{R}^d$. For our trace results, we need the following simple geometric observation. Suppose the covering of $\mathbb{R}^d$ is generated by the affine transforms $\{\delta_{\mathbf{a}}(t_k)\cdot+\xi_k\}_{k\in\mathbb{N}}$. Then for a fixed dilation parameter t_l, $l \in \mathbb{N}$, we let

$$T_l = |\xi|_{\mathbf{a}}, \text{ for } \xi \in \mathbb{R}^d \text{ chosen such that } h(\xi) = t_l. \tag{3.1}$$

We claim that there is a constant C such that uniformly in l,

$$\sum_{k:t_k\le t_l} t_k^d \le C T_l^d. \tag{3.2}$$

To verify this claim, we pick any $t_k \le t_l$. Let ξ_k be the center of the $B_{\mathbf{a}}$-ball B_k associated with t_k. Since h is increasing, see (2.4), we have $|\xi_k|_{\mathbf{a}} \le |\xi|_{\mathbf{a}} = T_l$. It follows by the moderation of h that for any $\zeta \in B_k$, we have

$$|\zeta|_{\mathbf{a}} \le C_{\mathbf{a}}(|\zeta-\xi_k|_{\mathbf{a}} + |\xi_k|_{\mathbf{a}}) \le C(\rho' t_k + T_l) \le C' T_l.$$

Hence, the pairwise disjoint union $\cup_{\{k:t_k\le t_l\}}\mathscr{B}_{\mathbf{a}}(\xi_k,\rho' t_k)$ is contained in $\mathscr{B}_{\mathbf{a}}(0, C'T_l)$, and (3.2) follows immediately from this fact by a volume argument.

Example 3.1. Let $0 < \alpha \le 1$. For $h(\xi) := (1+|\xi|_{\mathbf{a}})^\alpha$, we observe that $\xi \in \mathbb{R}^d$ with $|\xi|_{\mathbf{a}} = t_l^{1/\alpha}$ implies that $h(\xi) \asymp t_l$. Hence, uniformly in l,

$$\sum_{k:t_k\le t_l} t_k^d \le C T_l^d \asymp t_l^{d/\alpha}.$$

This simple observation will be put to use in Corollary 3.4.

Before we state the main result, we need to review one result on a Peetre maximal function adapted to the anisotropy **a**.

Suppose $\Omega \subseteq \mathbb{R}^d$. Then we let

$$L_p^{\Omega}(\mathbb{R}^d) := \{f \in L_p(\mathbb{R}^d) \mid \mathrm{supp}(\hat{f}) \subseteq \Omega\}.$$

We have the following result.

Proposition 3.2. *([3, Corollary 3.4]) Suppose* $0 < p < \infty$ *and* $0 < q \leq \infty$, *and let* $\Omega = \{T_k\mathscr{C}\}_{k\in\mathbb{N}}$ *be a sequence of compact subsets of* $\mathbb{R}^d$ *generated by a family* $\{T_k = \delta_{t_k}\cdot + \xi_k\}_{k\in\mathbb{N}}$ *of invertible affine transformations on* $\mathbb{R}^d$, *with* $\mathscr{C}$ *a fixed compact subset of* $\mathbb{R}^d$. *If* $0 < r < p$, *then there exists a constant* K *such that*

$$\Big\| \sup_{z\in\mathbb{R}^d} \langle \delta_{\mathbf{a}}(t_k)z\rangle_{\mathbf{a}}^{-d/r} |f_k(\cdot - z)| \Big\|_{L_p(\mathbb{R}^d)} \leq K \|f_k\|_{L_p(\mathbb{R}^d)}, \qquad \forall k, f_k \in L_p^{T_k(\Omega)}(\mathbb{R}^d). \tag{3.3}$$

We can now state our main result.

Proposition 3.3. *Let* $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{R}_+^d$ *be an anisotropy on* $\mathbb{R}^d$, *and let* h *be an associated moderate function satisfying* (2.9). *We let* $\mathscr{C}$ *be an admissible covering of* $\mathbb{R}^d$ *associated with* h *that is generated by the affine transformations* $\{T_k := \delta\mathbf{a}(t_k)\cdot + \xi_k\}_{k\in\mathbb{N}}$. *Suppose there exist constants* $M, \beta > 0$ *such that* $T_l \leq M t_l^{\beta/d}$ *for* $l \in \mathbb{N}$, *where* T_l *is defined by* (3.1). *Then for* $0 < p, q < \infty$, *let* $s_p = max\{0, (d - a_d)(1/p - 1)\}$, *and put* $s = d/q - s_p > 0$. *Then we have the continuous embedding*

$$Tr : M_{p,\min\{1,p,q\}}^{\beta/q + a_d/p}(\mathbb{R}^d) \to M_{p,q}^{s}(\mathbb{R}^{d-1}), \tag{3.4}$$

where the smoothness space on $\mathbb{R}^{d-1}$ *is defined in Section 2.4.*

Proof. Let us fix a BAPU $\{\Phi_k\}_{k\in\mathbb{N}}$ on $\mathbb{R}^d$ associated with $\mathscr{C}$. First we notice that for $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with $(0, x_d) \in \mathscr{B}_{\mathbf{a}}(0, t_k^{-1})$,

$$|\Phi_k(D)f(x', 0)| \leq \sup_{z\in\mathbb{R}^d} \langle \delta_{\mathbf{a}}(t_k)z\rangle_{\mathbf{a}}^{-d/r} |\Phi_k(D)f(x - z)|. \tag{3.5}$$

Now, $(0, x_d) \in B(0, t_k^{-1})$ implies that $|x_d| \leq c t_k^{-a_d}$, see (2.3). Hence, with $0 < r < p$,

$$\begin{aligned}
\|\Phi_k(D)f(x', 0)\|_{L_p(\mathbb{R}^{d-1})}^p &\leq c' t_k^{a_d} \int_{|x_d|\leq c t_k^{-a_d}} \int_{\mathbb{R}^{d-1}} |\sup_{z\in\mathbb{R}^d} \langle \delta_{\mathbf{a}}(t_k)z\rangle_{\mathbf{a}}^{-d/r} |\Phi_k(D)f(x - z)||^p \, dx' dx_d \\
&\leq c' t_k^{a_d} \| \sup_{z\in\mathbb{R}^d} \langle \delta_{\mathbf{a}}(t_k)z\rangle_{\mathbf{a}_B}^{-d/r} |\Phi_k(D)f(x - z)| \|_{L_p(\mathbb{R}^d)}^p \\
&\leq c'' t_k^{a_d} \|\Phi_k(D)f)\|_{L_p(\mathbb{R}^d)}^p, \qquad (3.6)
\end{aligned}$$

where we used Proposition 3.2 in the last step.

Now we pick $f \in \mathscr{S}(\mathbb{R}^d)$. The strategy is first to make a Littlewood-Paley decomposition of f using the BAPU $\{\Phi_k\}_k$ and then we use this decomposition in the definition of modulation type smoothness space, see Definition 2.9, to study the action of the trace operator. Notice that $f = \sum_k \Phi_k(D)f$ with convergence in e.g. $L_2(\mathbb{R}^d)$. Now, let

$\{\phi_k\}_k$ be the associated BAPU on $\mathbb{R}^{d-1}$. We let $\mathscr{F}_{x'}$ denote the (partial) Fourier transform acting on the variable $x' \in \mathbb{R}^{d-1}$, and $\mathscr{F}_{\xi'}^{-1}$ denotes its inverse. Then,

$$\begin{aligned}(\mathscr{F}_{\xi'}^{-1}\phi_{k'}\mathscr{F}_{x'}f)(x',0) &= \sum_{l\in\mathbb{N}} \mathscr{F}_{\xi'}^{-1}(\phi_{k'}\mathscr{F}_{x'}\Phi_l(D)f)(x',0)\\ &= \sum_{l\in\mathbb{N}} (\mathscr{F}_{\xi'}^{-1}\phi_{k'}) * (\Phi_l(D)f)(x',0)\end{aligned}$$

Notice that $\operatorname{supp}(\Phi_{k'}) \subset B \times I$, with $B \subset \mathbb{R}^{d-1}$ and $I \subset \mathbb{R}$, implies that

$$\operatorname{supp}[\mathscr{F}_{x'}\Phi_k(D)(x',0)] \subset B.$$

This observation implies that there is a constant $K < \infty$ such that

$$(\mathscr{F}_{\xi'}^{-1}\phi_{k'}) * (\Phi_l(D)f)(\cdot,0) = 0$$

whenever $t_l \le K t_{k'}$. Therefore,

$$(\mathscr{F}_{\xi'}^{-1}\phi_{k'}\mathscr{F}_{x'}f)(x',0) = \sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} (\mathscr{F}_{\xi'}^{-1}\phi_{k'}) * (\Phi_l(D)f)(x',0). \tag{3.7}$$

Now we have to consider a number of cases depending on the particular values of p and q.

Case 1. $1 \le p < \infty$. In this case $s_p = 0$ and $s = d/q$. We notice that by construction, $\sup_n \|\mathscr{F}_{\xi'}^{-1}\phi_n\|_{L_1(\mathbb{R}^{d-1})} < \infty$. Hence, for $1 \le p < \infty$ we can apply Young's inequality to (3.7) to conclude that

$$\begin{aligned}\|(\mathscr{F}_{\xi'}^{-1}\phi_{k'}\mathscr{F}_{x'}f)(x',0)\|_{L_p(\mathbb{R}^{d-1})} &\le \sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} \|\Phi_l(D)f)(x',0)\|_{L_p(\mathbb{R}^{d-1})}\\ &\le \sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_l^{a_d/p} \|\Phi_l(D)f)\|_{L_p(\mathbb{R}^{d})},\end{aligned}$$

where we used (3.6) in the last estimate. Next we have to consider two different arguments depending on the value of q. First suppose $0 < q \le 1$. Then we have,

$$\begin{aligned}\|f(\cdot,0)\|_{M^s_{p,q}(\mathbb{R}^{d-1})} &\asymp \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq} \|\phi_{k'}(D)f(x',0)\|^q_{L_p(\mathbb{R}^{d-1})}\Big)^{1/q}\\ &\lesssim \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq} \Big(\sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_l^{a_d/p} \|\Phi_l(D)f)\|_{L_p(\mathbb{R}^{d})}\Big)^q\Big)^{1/q}\\ &= \Big(\sum_{l\in\mathbb{N}}\sum_{k'\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_{k'}^{sq} (t_l^{a_d/p} \|\Phi_l(D)f)\|_{L_p(\mathbb{R}^{d})})^q\Big)^{1/q}\\ &\lesssim \Big(\sum_{l\in\mathbb{N}} \big(t_l^{\beta/q+a_d/p} \|\Phi_l(D)f)\|_{L_p(\mathbb{R}^{d})}\big)^q\Big)^{1/q}\\ &\asymp \|f\|_{M^{\beta/q+a_d/p}_{p,q}(\mathbb{R}^d)}.\end{aligned} \tag{3.8}$$

where we used that $sq = d$, and the estimate (3.2), in (3.8). Now we turn to the case, $1 < q < \infty$, We have, using Minkowski's inequality,

$$\begin{aligned}
\|f(\cdot,0)\|_{M^s_{p,q}(\mathbb{R}^{d-1})} &\asymp \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq}\|\phi_{k'}(D)f(x',0)\|^q_{L_p(\mathbb{R}^{d-1})}\Big)^{1/q}\\
&\lesssim \Big(\sum_{k'\in\mathbb{N}}\Big(\sum_{l\in\mathbb{N}} t_{k'}^{s}\chi_{\{t_{k'}\le Kt_l\}}t_l^{a_d/p}\|\Phi_l(D)f)\|_{L_p(\mathbb{R}^d)}\Big)^q\Big)^{1/q}\\
&\lesssim \sum_{l\in\mathbb{N}}\Big(\sum_{k'\in\mathbb{N}}\Big(t_{k'}^{s}\chi_{\{t_{k'}\le Kt_l\}}t_l^{a_d/p}\|\Phi_l(D)f)\|_{L_p(\mathbb{R}^d)}\Big)^q\Big)^{1/q}\\
&\lesssim \sum_{l\in\mathbb{N}} t_l^{a_d/p+\beta/q}\|\Phi_l(D)f)\|_{L_p(\mathbb{R}^d)}\\
&\asymp \|f\|_{M^{\beta/q+a_d/p}_{p,1}(\mathbb{R}^d)},
\end{aligned}$$

where we again used that $sq = d$ and the estimate (3.2). This concludes the case $1 \le p < \infty$.

Case 2. $0 < p < 1$. We now turn to the case $0 < p < 1$ which has to be treated differently. For $0 < p < 1$, we notice that $\|\mathscr{F}^{-1}_{\xi'}\phi_{k'}\|_{L_p(\mathbb{R}^{d-1})} \asymp |\mathrm{supp}(\phi_{k'})|^{1-1/p}$ uniformly in k', which follows from Definition 2.8.(c), adjusted to the derived covering $\{\phi_k\}$ for $\mathbb{R}^{d-1}$. Moreover, $\mathrm{supp}(\phi_{k'}) \subseteq \delta_{\mathbf{a}'}(t_{k'})Q + \zeta_{k'}$, with Q a fixed compact subset of $\mathbb{R}^{d-1}$ independent of k'. Hence, by these facts and by calling on [20, §1.5.3], the generic term in (3.7) can be estimated by

$$\|(\mathscr{F}^{-1}_{\xi'}\phi_{k'}) * (\Phi_l(D)f)(x',0)\|_{L_p(\mathbb{R}^{d-1})} \lesssim t_{k'}^{(d-a_d)(1/p-1)}\|(\Phi_l(D)f)(x',0)\|_{L_p(\mathbb{R}^{d-1})}.$$

Hence, from (3.7) we obtain

$$\begin{aligned}
\|(\mathscr{F}^{-1}_{\xi'}\phi_{k'}\mathscr{F}_{x'}f)(x',0)\|^p_{L_p(\mathbb{R}^{d-1})} &\lesssim \sum_{l\in\mathbb{N}}\chi_{\{t_{k'}\le Kt_l\}}t_{k'}^{(d-a_d)(1-p)}\|\Phi_l(D)f)(x',0)\|^p_{L_p(\mathbb{R}^{d-1})}\\
&\lesssim \sum_{l\in\mathbb{N}}\chi_{\{t_{k'}\le Kt_l\}}t_{k'}^{(d-a_d)(1-p)}\,t_l^{a_d}\|\Phi_l(D)f)\|^p_{L_p(\mathbb{R}^d)},
\end{aligned}$$

where we used (3.6) in the last estimate. It follows that

$$\begin{aligned}
\|f(\cdot,0)\|_{M^s_{p,q}(\mathbb{R}^{d-1})} &\asymp \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq}\|\phi_{k'}(D)(x',0)\|^q_{L_p(\mathbb{R}^{d-1})}\Big)^{1/q}\\
&\lesssim \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq}\Big\{\sum_{l\in\mathbb{N}}\chi_{\{t_{k'}\le Kt_l\}}t_{k'}^{(d-a_d)(1-p)}\,t_l^{a_d}\|\Phi_l(D)f)\|^p_{L_p(\mathbb{R}^d)}\Big\}^{q/p}\Big)^{1/q}. \qquad (3.9)
\end{aligned}$$

We consider two cases. First suppose $q \le p$. Then, using that $\ell_q \hookrightarrow \ell_p$ in (3.9),

$$\begin{aligned}
\|f(\cdot,0)\|_{M^s_{p,q}(\mathbb{R}^{d-1})} &\lesssim \Big(\sum_{k'\in\mathbb{N}} t_{k'}^{sq} \sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} [t_{k'}^{(d-a_d)(1/p-1)} t_l^{a_d/p} \|\Phi_l(D)f\|_{L_p(\mathbb{R}^d)}]^q\Big)^{1/q} \\
&\le \Big(\sum_{l\in\mathbb{N}} \Big[\sum_{k'\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_{k'}^{sq} t_{k'}^{q(d-a_d)(1/p-1)}\Big] [t_l^{a_d/p} \|\Phi_l(D)f\|_{L_p(\mathbb{R}^d)}]^q\Big)^{1/q} \\
&\lesssim \Big(\sum_{l\in\mathbb{N}} [t_l^{\beta/q+a_d/p} \|\Phi_l(D)f\|_{L_p(\mathbb{R}^d)}]^q\Big)^{1/q} \\
&\asymp \|f\|_{M^{\beta/q+a_d/p}_{p,q}(\mathbb{R}^d)},
\end{aligned}$$

where we have used that $sq + q(d-a_d)(1/p-1) = d$ and the estimate (3.2).

Now, suppose $q > p$. Then by Minkowski's inequality

$$\begin{aligned}
\|f(\cdot,0)\|_{M^s_{p,q}(\mathbb{R}^{d-1})} &\lesssim \Big(\sum_{k'\in\mathbb{N}} \Big\{t_{k'}^{sp} \sum_{l\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_{k'}^{(d-a_d)(1-p)} t_l^{a_d} \|\Phi_l(D)f\|^p_{L_p(\mathbb{R}^d)}\Big\}^{q/p}\Big)^{\frac{p}{q}\frac{1}{p}} \\
&\le \Big(\sum_{l\in\mathbb{N}} \Big[\sum_{k'\in\mathbb{N}} \chi_{\{t_{k'}\le K t_l\}} t_{k'}^{sq} t_{k'}^{q(d-a_d)(1/p-1)}\Big]^{p/q} [t_l^{a_d/p} \|\Phi_l(D)f\|_{L_p(\mathbb{R}^d)}]^p\Big)^{1/p} \\
&\le \Big(\sum_{l\in\mathbb{N}} [t_l^{\beta/q} t_l^{a_d/p} \|\Phi_l(D)f\|_{L_p(\mathbb{R}^d)}]^p\Big)^{1/p} \\
&\asymp \|f\|_{M^{\beta/q+a_d/p}_{p,p}(\mathbb{R}^d)},
\end{aligned}$$

where we again used that $sq + q(d-a_d)(1/p-1) = d$ and the estimate (3.2).

This completes the proof since $\mathscr{S}(\mathbb{R}^d)$ is dense in $M^s_{p,q}(\mathbb{R}^d)$ and Tr by definition is defined by an extension argument. □

We now apply Proposition 3.3 to the example $h = \langle\cdot\rangle^{\alpha}_{\mathbf{a}}$.

Corollary 3.4. *Let* $\mathbf{a} = (a_1,\dots,a_d)$ *be an anisotropy on* $\mathbb{R}^d$*. For* $0<\alpha\le 1$ *and* $0<p,q<\infty$*, let* $s_p = max\{0,(d-a_d)(1/p-1)\}$ *and put* $s = d/q - s_p$*. Suppose* $h = \langle\cdot\rangle^{\alpha}_{\mathbf{a}}$*. Then for the* α*-modulation spaces, we have*

$$Tr : M^{\alpha, s/\alpha + a_d/p}_{p,\min\{1,p,q\}}(\mathbb{R}^d) \to M^{\beta,s}_{p,q}(\mathbb{R}^{d-1}), \tag{3.10}$$

where $\beta = \alpha\cdot(d-a_d)/(d-1)$.

Proof. Follows from Proposition 3.3 together with Examples 2.10 and 3.1. □

Remark 3.5. The Corollary does not cover the Modulation space case since our geometric argument breaks down for $\alpha = 0$. For trace results for modulation spaces, see [8].

4. Embeddings and the trace operator

The result given by Proposition 3.3 imposes a strong link between the parameters s and q. This is due to the technique we apply in the proof and the fact that we only know how to handle sums of the type (3.2). However, we can use an embedding result to relax the dependence between s and q but at the cost of an increased smoothness parameter.

The following embedding result was proved in [3]

Lemma 4.1. *Let* $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}_+^d$ *be an anisotropy on* $\mathbb{R}^d$*, and let h be an associated moderate function satisfying* (2.9)*. Suppose the admissible covering used to define* $M^s_{p,q_0}(\mathbb{R}^d)$ *is generated by* $\mathcal{T} = \{T_k = \delta_{\mathbf{a}}(h(\xi_k)) \cdot + \xi_k\}_k$*. Let* $0 < q_1 \le \infty$ *and suppose* s_0 *is a constant such that* $\{h(\xi_k)^{-s_0}\}_k \in \ell_{q_1}$*. Then*

$$M^{s+s_0}_{p,q_0}(\mathbb{R}^d) \hookrightarrow M^s_{p,q_1}(\mathbb{R}^d)$$

for all $0 < q_0 \le \infty$*.*

Remark 4.2. One possible concern is the applicability of Lemma 4.1 in the general setting. However, it is not difficult to verify that any set of affine transformations $\{T_k = \delta_{h(\xi_k)} \cdot + \xi_k\}_k$ generated by a moderate function h using Lemma 2.5 satisfies

$$0 < \inf_{m \neq n} |\xi_m - \xi_n|.$$

It follows that for any moderate function h satisfying $h(\xi) \ge C\langle\xi\rangle_{\mathbf{a}}^{\kappa}$ for some $\kappa > 0$, $\xi \in \mathbb{R}^d$, there exists a constant β such that $\{h(\xi_k)^{-\beta}\}_k \in \ell_1$. The exact value of β depends strongly on h. For example, for $h = \langle\cdot\rangle_{\mathbf{a}}^{\alpha}$, we have $\beta \to 0$ as $\alpha \to 1^-$ and $\beta \to \infty$ as $\alpha \to 0^+$.

We can now combine the embedding result with Proposition 3.3 to obtain the following.

Corollary 4.3. *Let* $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}_+^d$ *be an anisotropy on* $\mathbb{R}^d$*, and let h be an associated moderate function satisfying* (2.9)*. We let* $\mathscr{C}$ *be an admissible covering of* $\mathbb{R}^d$ *associated to h that is generated by the affine transformations* $\{T_k := \delta_{\mathbf{a}}(t_k) \cdot + \xi_k\}_{k \in \mathbb{N}}$*. Suppose there exist constants* $M, \beta > 0$ *such that* $T_l \le M t_l^{\beta/d}$ *for* $l \in \mathbb{N}$*, where* T_l *is defined by* (3.1)*. Then for* $0 < p, q < \infty$*, let* $s_p = max\{0, (d - a_d)(1/p - 1)\}$*, and put* $s = d/q - s_p > 0$*. Suppose* t_0 *is a constant such that* $\{h(\xi_k)^{-t_0}\}_k \in \ell_{\min\{1,p,q\}}$*. Then for* $0 < q_0 \le \infty$*, we have the continuous embedding*

$$Tr : M^{\beta/q + a_d/p + t_0}_{p,q_0}(\mathbb{R}^d) \to M^s_{p,q}(\mathbb{R}^{d-1}), \tag{4.1}$$

where the smoothness space on $\mathbb{R}^{d-1}$ *is defined in Section 2.4.*

Proof. Follows from Proposition 3.3 and Lemma 4.1. □

The trace operator can obviously also be considered on the related Triebel-Lizorkin type spaces. Here we appeal to embedding results of the type (2.10) combined with Proposition 3.3 to obtain results of this type. With the same notation as in Corollary 4.3, and $0 < p, q < \infty$, we notice that (2.10) combined with Lemma 4.1 gives the continuous embeddings,

$$F^{\beta/q + a_d/p + t_0}_{p,q}(\mathbb{R}^d) \hookrightarrow M^{\beta/q + a_d/p + t_0}_{p,\max\{p,q\}}(\mathbb{R}^d) \hookrightarrow M^{\beta/q + a_d/p}_{p,\min\{1,p,q\}}(\mathbb{R}^d),$$

where t_0 is a constant such that $\{h(\xi_k)^{-t_0}\}_k \in \ell_{\min\{1,p,q\}}$. This leads to the following trace result, which concludes the paper.

$$Tr : F^{\beta/q+a_d/p+t_0}_{p,q_0}(\mathbb{R}^d) \to M^s_{p,q}(\mathbb{R}^{d-1}), \tag{4.2}$$

with parameters as in Corollary 4.3.

Remark 4.4. An alternative approach to results of the type (4.2) is to repeat the proof of Proposition 3.3 and use the pointwise estimate (3.5) directly in conjunction with a vector valued version of Proposition 3.2. A vector valued estimate of this type can be found in [3, Corollary 3.4].

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