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Multiple Description Coding for Closed Loop Systems over Erasure Channels

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Abstract

In this paper, we consider robust source coding in closed-loop systems. In particular, we consider a (possibly) unstable LTI system, which is to be stabilized via a network. The network has random delays and erasures on the data-rate limited (digital) forward channel between the encoder (controller) and the decoder (plant). The feedback channel from the decoder to the encoder is assumed noiseless. Since the forward channel is digital, we need to employ quantization. We combine two techniques to enhance the reliability of the system. First, in order to guarantee that the system remains stable during packet dropouts and delays, we transmit quantized control vectors containing current control values for the decoder as well as future predicted control values. Second, we utilize multiple description coding based on forward error correction codes to further aid in the robustness towards packet erasures. In particular, we transmit M redundant packets, which are constructed such that when receiving any J packets, the current control signal as well as J-1 future control signals can be reliably reconstructed at the decoder. We prove stability subject to quantization constraints, random dropouts, and delays by showing that the system can be cast as a Markov jump linear system.

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I. Introduction

Source coding is essential in any communication system with digital channels. Traditionally, one considers open-loop source coding, where the source is encoded subject to a fidelity criterion and perhaps by taking the channel into account, i.e., joint source-channel coding. However, there is a recent trend towards controlling systems via wireless networks [1]. In these cases, the source coder must also guarantee that the overall system remains stable. Thus, it becomes a joint source-channel-control problem, which defines a very challenging problem.

In this paper, we consider the situation depicted in Fig. 1. A controller is connected to a plant via a forward channel (from the controller to the plant) and a backward channel (from the plant to the controller). The forward channel is a data-rate limited (digital) channel, which is subject to random packet delays and packet erasures. The backward channel is assumed noiseless.

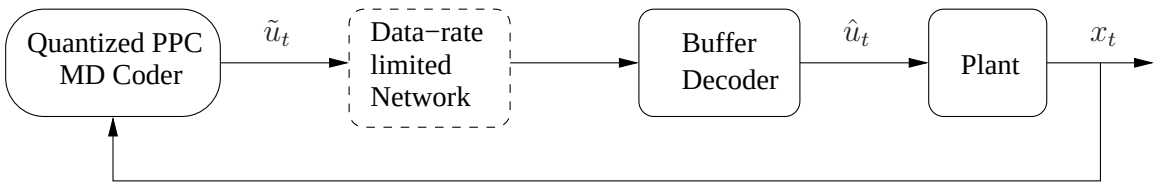

Figure 1: The encoder (packetized predictive controller (PPC) and MD coder) communicates with the decoder (plant) via a data-rate limited erasure channel with delays.

We will make use of quantized packetized predictive control (PPC) over the forward channel [2]–[4]. In quantized PPC, a control vector with the current and $N-1$ future predicted quantized plant inputs are constructed at the controller side to compensate for random delays and packet dropouts in the forward channel. We will also be using multiple descriptions (MDs) in order to further increase the robustness towards packet dropouts and packet delays. MDs is traditionally used as a joint source-channel coding technique, where the source is encoded into several descriptions [5]. The descriptions are individually good and are furthermore able to improve upon each other when combined. Thus, with MDs it is possible to provide several quality layers, where the reconstruction performance depends upon the channel conditions, i.e., the performance depends upon which packets that are received. In this work, we combine PPC with MDs and thereby obtain a joint source-channel-control framework. Our idea is to construct M packets with control information. The packets are constructed such that if any single packet is received, only the control value for the current time will be available at the plant side. Moreover, if any two packets out of the M packets are received, the current control value and the future predicted control value for the next time instance will be available. In general, when receiving any J descriptions, the current and $J-1$ future controls values are available at the plant.

We will consider the case of discrete-time noisy LTI systems and i.i.d. packet dropouts. In a previous work, we focused on the case where the packets are always received in-order

and without delays, i.e., out-of-order and/or delayed packets are discarded [6]. The current paper, generalizes the results of [6] and solves the case, where packets can be delayed and furthermore be received out-of-order. This is a much more challenging situation. For example, control information constructed for the current time t could be applied on time (i.e., at time t) at the plant. Then, at time $t+1$, the plant could potentially apply control information constructed at time $t-1$ due to receiving packets out-of-order. The challenge is to construct the joint PPC and MD coder so that one can guarantee stability in case of random packet dropouts, packet delays, and out-of-order packets. Towards that end, we show that the complete system can be cast as a Markov jump linear system (MJLS), where system stability can be addressed via linear matrix inequalities.

The paper is organized as follows: In Section II, we provide background information on quantized PPC. Then in Section III, we present the main idea, which is combining MDs with PPC. Sections IV provides a system analysis for the case with out-of-order packet reception.

II. Packetized and Quantized Control over Erasure Channels

This section contains the preliminaries regarding the system model, network, and quantized PPC.

A. System Model

As previously mentioned, we consider the plant to be discrete-time and linear time invariant (LTI). The model of the dynamical system is given by state $x_t \in \mathbf{R}^z$, $z \geq 1$ and scalar input $u_t \in \mathbf{R}$:

$$x_{t+1} = Ax_t + B_1 u_t + B_2 w_t, \quad t \in \mathbb{N}, \tag{1}$$

where $w_t \in \mathbf{R}^{z'}$, $z' \geq 1$, is an unmeasured disturbance, which can be arbitrarily distributed with finite mean and variance. We further assume that the pair (A, B_1) is stabilizable. Finally, the initial state x_0 is arbitrarily distributed with finite mean and variance.

B. Cost Function

The quantized control law should be designed so that it minimizes an appropriate objective. In this work, we consider the case, where at each time instant t and for a given plant state x_t, the following objective is minimized:

$$J(\bar{u}\,', x_t) \triangleq \|x'_N\|_P^2 + \sum_{\ell=0}^{N-1} \left(\|x'_\ell\|_Q^2 + \lambda (u'_\ell)^2\right), \tag{2}$$

where $N \geq 1$ is the horizon length and $\|x\|_Q^2 \triangleq x^T Q x$ is a weighted ℓ_2 norm. We consider the design variables $P \succeq 0$, $Q \succeq 0$, and $\lambda > 0$, which allow one to trade-off control performance versus control effort, to be given. See [4] for more information about how to choose these variables.

The objective in (2) takes into account the predicted future evolution of the dynamical system. In particular, the objective minimizes the sum of the weighted ℓ_2 norms of the nominal state vector $x'_{t+l}, l = 0, \ldots, N$. Clearly, at time t, the plant state x_{t+1} cannot be known exactly since the external disturbances w_t, the initial plant state x_0, and the buffer

state b_t (which depends upon the actual realizations of the packet droputs and delays) are not known to the controller, when forming the control signal u_t. However, the controller may form a qualified estimate x'_{t+l} of x_{t+l} by ignoring the unknows. Specifically, the controller can generate a sequence of control signals $\bar{u}'_t = \begin{bmatrix} u'_t & \dots & u'_{t+N-1} \end{bmatrix}^T$, which represents possible current and future plant input signals, based on the nominal plant state evolution given by

$$x'_{\ell+1} = Ax'_\ell + B_1 u'_\ell, \quad \text{and} \quad x'_0 = x_t, \tag{3}$$

which does not take into account the buffer contents at the decoder, the dropout probabilities, or the external disturbances.

The idea in PPC is at each time instant t to send the whole control vector $\bar{u}'_t$ to the plants buffer. If the packet is received, the first control signal $\bar{u}'_t(1)$ is applied to the plant input. If it is not received, but the packet with $\bar{u}'_{t-1}$ has been received, then the signal $\bar{u}'_{t-1}(2)$ is applied and so forth. At the next time slot, say $t+1$, the controller uses information about the new plant state x_{t+1} as a basis for finding another optimizing sequence $\bar{u}'_{t+1}$, and so on.

C. Network Effects

The backward channel of the network is assumed noiseless and instantaneous. The forward channel is a packet erasure channel, where packets can be delayed and also be received out-of-order. In MD coding, it is common to assume the availability of either M separate and independent channels or a single (compound) channel where the M packets can be sent simultaneously and yet be subject to independent erasures and delays. Thus, at time t, we assume that the M transmitted packets are subject to erasures and delays independently of each other. Moreover, these erasures and delays are also assumed independent over time. In particular, we model transmission effects via the discrete processes $\{d^i_{t,t'}\}_{t'=0}^{\infty}$, where $0 \le t \le t'$ and $i = 1, \dots, M$, defined via:

$$d^i_{t,t'} \triangleq \begin{cases} 1, \text{ if packet } i \text{ generated for time } t \text{ is in the buffer at time } t' \ge t, \\ 0, \text{ else.} \end{cases}$$

These processes are generally not independent. If a packet has been received at time instant t' it is still received at time instances $t'+n, n \ge 1$. However, for $t' = t$, the outcomes $d^i_{t,t}, i = 1, \dots, M, t \ge 0$, are mutually independent.

D. Quantization Constraints

Due to the digital channel between the controller and the plant input, the control signals need to be quantized. A closed form solution to this problem was derived in [7]. Furthermore, in [4] the problem was cast into the framework of entropy-constrained (subtractively) dithered (lattice) quantization (ECDQ) [8]. For completeness, we briefly repeat key results of [4], [7] that we will be needing in the sequel:

Let $\bar{Q} \triangleq \mathrm{diag}(Q, \ldots, Q, P)$ and let Φ, Υ be defined by

$$\Phi \triangleq \begin{bmatrix} B_1 & 0 & \ldots & 0 \\ AB_1 & B_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_1 & A^{N-2}B_1 & \ldots & B_1 \end{bmatrix}, \Upsilon \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}. \tag{4}$$

Theorem 2.1 (Quantized Predictive Control [7]): Consider any quantized set $\mathcal{U} \subset \mathbb{R}^{\mathcal{N}}$, the matrices $\overline{Q}$, Φ, and Υ given in (4), and define:

$$\xi_t \triangleq \Gamma x_t, \quad \Gamma \triangleq -\Psi^{-T}\Phi^T\overline{Q}\Upsilon, \tag{5}$$

where $\Psi \in \mathbb{R}^{N\times N}$ is obtained from the factorization $\Psi^T\Psi = \Phi^T\overline{Q}\Phi + \lambda I$, where λ is as in (2).

Then the constrained optimizer $\tilde{u}_t = \arg\min_{\bar{u}'\in\mathcal{U}} J(\bar{u}', x_t)$, see (2), satisfies:

$$\tilde{u}_t = \Psi^{-1}\mathcal{Q}(\xi_t), \tag{6}$$

where $\mathcal{Q}(\cdot)$ is a (nearest neighbour) vector quantizer with alphabet $\Psi\mathcal{U}$. □

It follows from (5) and (6) that the optimal quantized control signal $\tilde{u}_t$ is obtained by vector quantizing the signal ξ_t. When the quantizer is an ECDQ, it was shown in [4] that the quantized (and reconstructed) control variable $\bar{u}_t$ can be modelled as

$$\bar{u}_t = \Psi^{-1}(n_t + \xi_t), \tag{7}$$

where n_t (the quantization noise) and ξ_t are mutually independent and $\xi_t = \Gamma x_t$. As done in [6], we will use $\bar{u}_t$ to denote the quantized (and reconstructed) control vector, which has been found by using an ECDQ on ξ_t. Thus, $\bar{u}_t$ denotes a vector of continuous-alphabet variables whereas $\tilde{u}_t$ is the vector of corresponding quantized discrete-alphabet variables, which is entropy coded and thereby converted into a bit-stream (to be transmitted over the network).

III. Multiple Descriptions

MDs based on vector quantization and with many packets have been considered in e.g. [9]. Using the MD approach of [9] would provide several quantized predicted control vectors, which are able to refine each other. However, we are here interested in receiving an accurate control signal for the current time instance, independently of which packet is received. To achieve this ability, we will instead be using MD based on forward error correction (FEC) codes [10], [11]. The idea we first presented in [6] turns out to be applicable to present situation as well and we will briefly describe the key points below. For more details, we refer the reader to [6]. Recall that the quantized control vector $\tilde{u}_t$ contains N elements, i.e., the current control value and $N-1$ future control values. Let us now split this N-dimensional vector in to $M = N$ sub sequences, each consisting of a single (scalar) control value. The first discrete control value $\tilde{u}_t(1)$ is repeated in each of the M packets. The second control signal $\tilde{u}_t(2)$ is split into two equal sized bit sequences. Then, an $(M, 2)$-erasure code is applied and the M resulting outputs are evenly distributed to the M packets. The third control $\tilde{u}_t(3)$ is split into three equal sized bit sequences, and an $(M, 3)$-erasure code is applied. This process continues until all M signals are

used. This results in M partially redundant packets, which are constructed in such a way that upon reception of any single packet, the first discrete control signal $\tilde{u}_t(1)$ can be reliably reconstructed. Moreover, from $\tilde{u}_t(1)$, we can generate $\bar{u}_t(1)$ simply by subtracting the known (pseudo-random) dither signal. Upon reception of any $0 < J \leq M$ packets the first J control signals $\bar{u}_t(1), \ldots, \bar{u}_t(J)$, can be recovered errorlessly. To simplify the notation, we will only consider the case where $M = N$, i.e., the number of packets equals the horizon length. It is, however, straight-forward to generalize this construction to any $M < N$ packets. For example, if $M < N$ then the remaining $N - M$ control signals can be treated together as input to the last (M, M)-erasure code.

IV. System Analysis with Time-Delays and Dropouts

This section considers the case where the network could introduce time-delays, packet erasures, and where packets could be received out-of-order.

A. Buffering at the Decoder and Reconstruction of Control Signals

As shown in Fig. 1, there is a receive buffer at the plant side, which keeps track of the received packets. The buffer is finite, i.e., at time t it contains all received packets, which are not older than $t - N + 1$. The current control signal, say $\hat{u}_t$, will be generated based upon the contents of buffer.

The control signals are generated in the following way. Let us denote the M packets at time t as $\bar{c}_t^i, i = 1, \ldots, M,$ and let us assume that the buffer is empty prior to time t. Moreover, let us assume that $M = 3$, so that three packets are generated at each time instance. At time t, we could receive any subset of the three packets (including the empty set). For the sake of example, assume that we receive a single packet, say $\bar{c}_t^2$. Since the first control signal can be obtained from any of the three packets, the buffer is able to reconstruct the control signal as $\hat{u}_t = \bar{u}_t(1)$. At time $t + 1$, we could e.g., receive two packets, say $\bar{c}_{t+1}^1$ and $\bar{c}_t^3$. The latter packet is a delayed packet. However, from the former packet $\bar{c}_{t+1}^1$, we obtain the control signal $\hat{u}_{t+1} = \bar{u}_{t+1}(1)$ and momentarily simply ignore the late packet $\bar{c}_t^3$. Finally, at time $t + 2$, we only receive the late packet $\bar{c}_t^1$. This means that, at time $t + 2$, the buffer contains all three packets from time t, a single packet from time $t + 1$, and no packets from time $t + 2$. We would thus construct the current control signal as $\hat{u}_{t+2} = \bar{u}_t(3)$. All the possible combinations for the case of $M = N = 3$ are illustrated in Table I.

For the case where $M = N \geq 1$, we use the idea first presented in [6] and define $\mathcal{I}_{t,t'}^k \in \{0, 1\}$ as an indicator function, which is "1" if at least k out of M packets of time stamp t are in the buffer at time t' and "0" otherwise. In particular,

$$\begin{aligned}\mathcal{I}_{t,t}^1 = &\prod_{i=1}^{M} d_{t,t}^i + \sum_{j=1}^{M}(1 - d_{t,t}^j) \prod_{i=1, i\neq j}^{M} d_{t,t}^i \\ &+ \sum_{j=1}^{M-1} \sum_{l=j+1}^{M} (1 - d_{t,t}^j)(1 - d_{t,t}^l) \prod_{i=1, i\neq j,l}^{M} d_{t,t}^i \\ &+ \cdots + \sum_{j=1}^{M} d_{t,t}^j \prod_{i=1, i\neq j}^{M} (1 - d_{t,t}^i)\end{aligned}$$

$\hat{u}_t$	$\bar{c}_t^1$	$\bar{c}_t^2$	$\bar{c}_t^3$	$\bar{c}_{t-1}^1$	$\bar{c}_{t-1}^2$	$\bar{c}_{t-1}^3$	$\bar{c}_{t-2}^1$	$\bar{c}_{t-2}^2$	$\bar{c}_{t-2}^3$
$\bar{u}_t(1)$	1	x	x	x	x	x	x	x	x
$\bar{u}_t(1)$	x	1	x	x	x	x	x	x	x
$\bar{u}_t(1)$	x	x	1	x	x	x	x	x	x
$\bar{u}_{t-1}(2)$	0	0	0	1	1	x	x	x	x
$\bar{u}_{t-1}(2)$	0	0	0	1	x	1	x	x	x
$\bar{u}_{t-1}(2)$	0	0	0	x	1	1	x	x	x
$\bar{u}_{t-2}(3)$	0	0	0	x	0	0	1	1	1
$\bar{u}_{t-2}(3)$	0	0	0	0	x	0	1	1	1
$\bar{u}_{t-2}(3)$	0	0	0	0	0	x	1	1	1

Table I: Control value $\hat{u}_t$ at time t from available buffer contents. "1" indicates that the packet is in the buffer and "0" indicates that it is not. "x" indicates that the control value does not depend on the given packet. In all other cases, we set $\hat{u}_t = 0$.

$$\begin{aligned}
\mathcal{I}_{t-1,t}^2 &= \prod_{i=1}^{M} d_{t-1,t}^i + \sum_{j=1}^{M}(1-d_{t-1,t}^j) \prod_{i=1,i\neq j}^{M} d_{t-1,t}^i \\
&\quad + \sum_{j=1}^{M-1}\sum_{l=j+1}^{M}(1-d_{t-1,t}^j)(1-d_{t-1,t}^l)\prod_{i=1,i\neq j,l}^{M} d_{t-1,t}^i \\
&\quad + \cdots + \sum_{j=1}^{M-1}\sum_{l=j+1}^{M} d_{t-1,t}^j d_{t-1,t}^l \prod_{o=1,i\neq j,l}^{M} d_{t-1,t}^o \\
&\vdots \\
\mathcal{I}_{t-N+1,t}^M &= \prod_{i=1}^{M} d_{t-N+1,t}^i.
\end{aligned}$$

Recall that with FEC based MD, the vector $(\bar{u}_t(1), \bar{u}_t(2), \ldots, \bar{u}_t(k))$ is available at the buffer at time t' if and only if $\mathcal{I}_{t,t'}^k = 1$. With this, the control signal $\hat{u}_t$ to be used at time t is given by:

$$\begin{aligned}
\hat{u}_t = \bar{u}_t(1)\mathcal{I}_{t,t}^1 &+ (1-\mathcal{I}_{t,t}^1)[\mathcal{I}_{t-1,t}^2 \bar{u}_{t-1}(2) + (1-\mathcal{I}_{t-1,t}^2) \\
&\times [\mathcal{I}_{t-3,t}^3 \bar{u}_{t-2}(3) + \cdots \\
&+ (1-\mathcal{I}_{t-N+2,t}^{M-1})\mathcal{I}_{t-N+1,t}^M \bar{u}_{t-N+1}(N)] \cdots],
\end{aligned} \tag{8}$$

where it follows that $\hat{u}_t = 0$ in the event that $\mathcal{I}_{t,t}^1 = \mathcal{I}_{t-1,t}^2 = \cdots = \mathcal{I}_{t-N+1,t}^M = 0$, which is the case if no packets have arrived in $N = M$ consecutive time instances.

B. Markov Jump Linear System with Delay and Packet Dropouts

At this point, we define $\tilde{\mathcal{I}}_{t',t}^k$ to be the indicator function, which is one if k of the $M = N$ packets constructed at time t' are in the receive buffer at time t but less than k

of these packets are in the buffer at times prior to t. Specifically,

$$\tilde{\mathcal{I}}^k_{t',t} \triangleq \begin{cases} 1, & \text{if } \mathcal{I}^k_{t',t} = 1 \text{ and } \mathcal{I}^k_{t',t-j} = 0, \forall j > 0, \\ 0, & \text{else,} \end{cases} \tag{9}$$

where it follows that $\tilde{\mathcal{I}}^k_{t,t} = \mathcal{I}^k_{t,t}, \forall t, k$. Finally, let $\bar{\mathcal{I}}_t$ denote the indicator matrix given by:

$$\bar{\mathcal{I}}_t(i,j) \triangleq \begin{cases} \tilde{\mathcal{I}}^i_{t,t}, & \text{if } i = j, \\ \tilde{\mathcal{I}}^j_{t+i-j,t} \prod_{\ell=i}^{j-1}(1 - \tilde{\mathcal{I}}^\ell_{t+i-\ell,t}), & \text{if } j > i, \\ 0, & \text{if } i > j. \end{cases} \tag{10}$$

Note that each row of the indicator matrix $\bar{\mathcal{I}}_t$ is either the all zero vector or it contains a single one and $M-1$ zeros. The position of such a one is always in the upper triangluar region where $j \geq i$. For example, if all M packets generated at time t are also received at time t then $\bar{\mathcal{I}}_t$ is the identity matrix. To further clarify, consider the following scenario. Assume that $N = M = 3$ and that at time $t-2$, the only packets received are two out of the three packets constructed at time $t-2$. The third missing packet generated at time $t-2$ is not received until at time t. In addition, at time $t-1$, a single packet constructed at time $t-1$ is received. Finally, at time t, no additional packets are received. This yields the following sequence of indicator matrices:

$$\bar{\mathcal{I}}_{t-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{I}}_{t-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\mathcal{I}}_t = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

Lemma 4.1: The number K of distinct indicator matrices $\bar{\mathcal{I}}_t$ is

$$K = \frac{M}{6}(5 + M^2) + 1. \tag{12}$$

Proof: See Appendix A. ■

Before we present the main result of this paper, we introduce $\bar{f}_t = [f_t(1), \ldots, f_t(N)]^T$, which is an N-length vector representing the buffer at the plant input side holding present and future control values. In particular, $f_t(1)$ is the control value to be applied at current time t, and $f_t(i)$ is to be applied at time $t+i$, unless the buffer is changed in the mean time.

Below we present the main theorem. Since the proof is quite lengthy it has been omitted due space considerations and will instead appear in a forthcoming journal version of this paper.

Theorem 4.1: Let $\bar{x}_t = [x_t, \ldots, x_{t+1-N}]^T$, and $\bar{n}_t = [n_t, \ldots, n_{t+1-N}]^T$, be the N past and present system state vectors and quantization noise vectors, respectively. Moreover, let Ξ_t be the augmented state variable given by

$$\Xi_t \triangleq \begin{bmatrix} \bar{x}_t \\ \bar{f}_{t-1} \end{bmatrix}. \tag{13}$$

Then, in the presence of i.i.d. packet dropouts (and random delays), (13) satifies the

MJLS given by

$$\Xi_{t+1} = \bar{A}(\bar{\mathcal{I}}_t)\Xi_t + \bar{B}(\bar{\mathcal{I}}_t)\begin{bmatrix} w_t \\ \mathbf{0} \\ \bar{n}_t \end{bmatrix}, \tag{14}$$

where $\bar{A}(\bar{\mathcal{I}}_t)$ and $\bar{B}(\bar{\mathcal{I}}_t)$ are the system's switching matrices having $\frac{M}{6}(5+M^2)+1$ different realizations (or jump states), which are indexed by $\bar{\mathcal{I}}_t$. $\triangle$

From a stability assessment point of view, it is very convenient that the system can be cast as a finite state MJLS, since this allows one to use e.g., linear matrix inequalities to obtain some sufficient and necessary conditions for stability. In particular, we include the following well-known results:

Corollary 4.1 (Sufficient Condition for MSS): The system (14) is mean square stable (MSS) if there exists $\Gamma \succ 0$ such that

$$\Gamma - \sum_{\bar{\mathcal{I}}_t} \tilde{p}(\bar{\mathcal{I}}_t)\bar{A}^T(\bar{\mathcal{I}}_t)\Gamma\bar{A}^T(\bar{\mathcal{I}}_t) \succ 0, \tag{15}$$

where $\tilde{p}(\bar{\mathcal{I}}_t)$ denotes the probability associated with the jump state indexed by $\bar{\mathcal{I}}_t$.

Proof: Follows immediately from [12, Corollary 3.26] since the jump-state sequence $\{\bar{\mathcal{I}}_t\}$ is ergodic and the number of jump-states is finite. ■

Corollary 4.2 (Necessary Condition for Stability): Let λ_t^i denote the ith eigenvalue of the matrix $\bar{A}(\bar{\mathcal{I}}_t)$. Then, a necessary condition for MSS is that

$$\max_i\{|\lambda_t^i|\} < \sqrt{1/\tilde{p}(\bar{\mathcal{I}}_t)}, \text{ for all } \bar{\mathcal{I}}_t. \tag{16}$$

Proof: Follows immediately from [13]. ■

Appendix A
Proof of Lemma 4.1

The first row of $\bar{\mathcal{I}}_t$ can be any of $M+1$ distinct patterns, where at most one of the M elements is one and the others are zero. In general, the ith row can take on $M+2-i$ distinct patterns. The maximum number of possible distinct patterns is therefore $(M+1)!$. Notice, however, that these rows are not independent, since we always use the newest control information available. Thus, if we at time t have two packets from time $t-1$ and three packets from time $t-2$, we will use the control value $\bar{u}_{t-1}(2)$ from time $t-1$. From this we deduce that if element (i,j) is one, then no element $(i-i', j+j'), i'>0, j'\geq 0$, can be one, since otherwise an older control signal is used for $f_t(i-i')$ than the one used for $f_t(i)$, which is not allowed. It therefore also follows that the column sum is at most one.

Consider an $M' \times M'$ diagonal matrix with ones on the first i entries on its main diagonal. It is possible to up-shift this matrix at most $i-1$ times and end up with distinct non-zero matrices. Indeed, after $i-1$ up shifts, there is a single one at position $(1,i)$. Summing over all i's from one to M' yields $0+1+\cdots+M'-1 = \frac{1}{2}M'(M'+1)$ possible shifts, which results in distinct non-zero matrices. We also have the original M distinct diagonal matrices having $i=1,\ldots,M$, ones on main the diagonal, and in addition we have the all zero matrix. Thus, in total we have $M+1+\frac{1}{2}M'(M'+1)$ distinct matrices.

Having ones on the jth diagonal and zero elsewhere, corresponds to packets constructed at the same time $t-j$. Thus, the above up-shifting operations account for all the possible combinations of receiving subsets of packets that are constructed at the same time.

It is also possible to receive packets constructed at different times. For example, at time t one may receive packets constructed at time $t-i$ as well as packets constructed at time $t-j, i \neq j, 0 \leq i, j \leq N-1$. To take all the admissible mixing possibilities into account, we split the $M \times M$ indicator matrix into a sequence of $M-1$ nested submatrices. Specifically, we form the first submatrix as the $(M-1) \times (M-1)$ matrix, which do not include the first row or first column of the original matrix. The jth submatrix is the $(M-j) \times (M-j)$ matrix, which do not include the first j rows or j columns of the original matrix. For each of the submatrices, we form all possible up-shifts that lead to non-zero matrices. Combining all this results in the total number, say K, of distinct admissible indicator matrices, i.e.,

$$K = (M+1) + \sum_{M'=1}^{M-1} \frac{1}{2}(M'(M'+1) = \frac{M}{6}(5+M^2)+1. \tag{17}$$

■

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