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On the Existence of Friendship Hypergraphs

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Abstract

A 3-uniform friendship hypergraph is a 3-uniform hypergraph in which, for all triples of vertices x, y, z there exists a unique vertex w, such that xyw, xzw and yzw are edges in the hypergraph. Sós showed that such 3-uniform friendship hypergraphs on n vertices exist with a so called universal friend if and only if a Steiner triple system, $S(2,3,n-1)$ exists. Hartke and Vandenbussche used integer programming to search for 3-uniform friendship hypergraphs without a universal friend and found one on 8, three non-isomorphic on 16 and one on 32 vertices. So far, these five hypergraphs are the only known 3-uniform friendship hypergraphs. In this paper we construct an infinite family of 3-uniform friendship hypergraphs on 2^k vertices and $2^{k-1}(3^{k-1}-1)$ edges. We also construct 3-uniform friendship hypergraphs on 20 and 28 vertices using a computer. Furthermore, we define r-uniform friendship hypergraphs and state that the existence of those with a universal friend is dependent on the existence of a Steiner system, $S(r-1,r,n-1)$. As a result hereof, we know infinitely many 4-uniform friendship hypergraphs with a universal friend. Finally we show how to construct a 4-uniform friendship hypergraph on 9 vertices and with no universal friend.

Keywords: friendship graph, 3-uniform friendship hypergraph, r-uniform friendship hypergraph

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1. Introduction

A *hypergraph* is a pair $\mathcal{H} = (\mathcal{V}(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ where $\mathcal{V}(\mathcal{H}) = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices* of $\mathcal{H}$ and $\mathcal{E}(\mathcal{H})$ is a subset of the powerset of V without the empty set, each element in $\mathcal{E}(\mathcal{H})$ denoted as an *edge*. The number of vertices n, is also denoted as the *order* of the hypergraph. If each edge in $\mathcal{E}(\mathcal{H})$ contains exactly r vertices, then we say the hypergraph is r-uniform. If the hypergraph is 2-uniform, we just denote it as a *graph* and write $H = (V(H), E(H))$. Two vertices x and y are said to be neighbours in a graph H, if and only if $\{x, y\} \in E(H)$. When possible, we will denote an edge $\{x_1, x_2, \ldots, x_r\}$ as $x_1x_2 \ldots x_r$ for short. When we do not specify a hypergraph to be r-uniform in this paper, we will assume it to be 3-uniform.

A *friendship graph* is a graph in which every pair of vertices has a unique common neighbour. The Friendship Theorem states, that if G is a friendship graph, then there exists a single vertex joined to all others. Also, friendship graphs exists only for odd number of vertices, and they are unique in the sense that the graphs consisting of $(n-1)/2$ triangles joined at a single vertex, so called windmill graphs, are the only type of friendship graphs, which was proved by Erdős, Rényi and Sós in 1966, see [1].

In this paper we will consider a known generalization of the friendship graphs, which concerns 3-uniform hypergraphs. We say that a 3-uniform hypergraph is a 3-*uniform friendship hypergraph*, if it satisfies the *friendship property* that

Definition 1 (Friendship Property)**.** *For every three vertices x, y and z, there exists a unique vertex w such that xyw, xzw and yzw are edges in the hypergraph.*

In the remaining part of this paper, we will denote such a w as the *completion* of x, y, z. Sós was the first one to consider this generalization in 1976, see [2]. She actually just considered 3-uniform hypergraphs with edge set $\{v_i, v_j, v_n\}$ for all $1 \leq i < j < n$ and a Steiner triple system on the vertices $\{v_1, v_2, \ldots, v_{n-1}\}$ and observed that they satisfy the friendship property. The vertex v_n will be denoted as a *universal friend*. As Steiner triple systems, $S(2, 3, n-1)$, are known to exist if and only if $n \equiv 2 \mod 6$ or $n \equiv 4 \mod 6$, we see that these are the only orders for which there exist friendship hypergraphs with a universal friend.

Sós then asked, whether there exist other 3-uniform hypergraphs satisfying the friendship property other than the ones with a universal friend. This

was answered by Hartke and Vandenbussche in 2008, see [3], where they used integer programming to prove that for $n = 8, 16$ and 32 there exist hypergraphs satisfying the friendship property without containing a universal friend. The integer programming also showed that for $n \leq 10$ and $n \neq 8$ the only 3-uniform friendship hypergraphs are the hypergraphs with a universal friend. They also succeeded in showing that the 3-uniform friendship hypergraph with $n = 8$ vertices they constructed is the only one of its kind without a universal friend. For $n = 16$ they showed there are at least three nonisomorphic constructions.

All friendship hypergraphs can be characterized by using complete 3-uniform hypergraphs on four vertices. Such a complete 3-uniform hypergraph on four vertices we will denote as a *quad*. We see why we can use quads to describe the friendship hypergraphs in stead of edges in the following lemma, for which we include the proof from [3].

Lemma 1. *[3] The following is true for every 3-uniform friendship hypergraph $\mathcal{H}$.*

(a) Every pair of vertices appears in at least one edge together.

(b) Every edge must be contained in a unique quad.

Proof. (a) Let $x, y \in \mathcal{V}(\mathcal{H})$ and $z \neq x, y$. Then the triple x, y, z has a completion w such that xyw, xzw and yzw are edges in $\mathcal{H}$. Hence x, y are in an edge together.

(b) Let $xyz \in \mathcal{E}(\mathcal{H})$, then the triple x, y, z has a completion w such that xyw, xzw and yzw are also edges in $\mathcal{H}$. These four edges form a quad. The uniqueness of the quad follows from the uniqueness of the completion w.

□

Observation (b) implies that the edges of the friendship hypergraph can be partitioned into quads. This allows one to solely use quads to describe the friendship hypergraph, as the quad structure tells us everything about the edge structure. As the number of edges are at most $\binom{n}{3}$, this also gives an upper bound on the number of quads in a friendship hypergraph as $\binom{n}{3}/4$. A lower bound on the number of edges was proved to be $n(n-2)/2$ in [3].

With the exception of one of the friendship hypergraphs on 16 vertices, all of the friendship hypergraphs found in [3] also satisfy three properties,

which where actually used in describing the IP and later slacked to try to obtain more friendship hypergraphs. The computation time where greatly improved by having specified these properties, which are the *inductive, pair and automorphism property.* The friendship hypergraphs on $n = 16$ and 32 vertices satisfy that they contain two disjoint copies of a friendship hypergraph on $n/2$ vertices, also called the inductive property. Except for one of the friendship hypergraphs on 16 vertices, they also satisfy the pair property, which is that the vertices can be divided into pairs, such that each pair appears in a quad with each other pair. Finally if we view the vertices as binary $\log(n)$-tuples, then any map that flips a fixed subset of the $\log(n)$ bits is an automorphism, hence the friendship hypergraph satisfies the automorphism property. Note that this means that the friendship hypergraph is regular and vertex-transitive and also that the number of quads is a multiple of $n/4$. Hartke and Vandenbussche [3] conjectured that for all positive integers $k \geq 4$ friendship hypergraphs with $n = 2^k$ vertices satisfying the three properties exist.

Hartke and Vandenbussche's results were improved upon by Li, van Rees, Seo and Singhi in [4], where it was stated that no 3-uniform friendship hypergraph on 11 or 12 vertices exists. They also showed that the three friendship hypergraphs on 16 vertices which where found by Hartke and Vandenbussche are the only friendship hypergraphs on 16 vertices which satisfy that the vertex-set can be partitioned into groups of disjoint quads and for which the corresponding friendship design can be embedded into an affine geometry. The lower bound on the number of edges were also improved, as they showed that if n is odd, then there are at least (roughly) $2n^2/3$ edges and if n is even there are at least $n^2/2$ edges. Also the upper bound was improved, to obtain that there are at most $\binom{n}{3}(2n-6)/(3n-10)$ edges.

In this paper we first construct an infinite family of 3-uniform hypergraphs given in the following definition.

Definition 2. *Let $k \geq 2$ and $H = (V, E)$ be a hypercube on $n = 2^k$ vertices, where the vertices are labelled with the k-bit binary strings from 0 to $2^k - 1$, such that two neighbouring vertices differs in exactly one bit. Then we define* ***the cubeconstructed hypergraph*** *$\mathcal{H}$ as the 3-uniform hypergraph with vertex-set V and with the triple xyz in the edge-set if and only if $dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2k$.*

In Figure 1 we see the hypercube on $8 = 2^3$ vertices and in Table 1 the

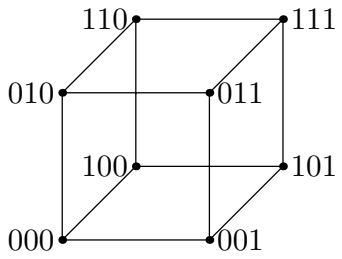

Figure 1: The hypercube of dimension $k = 3$ with the described labels.

quads of the corresponding cubeconstructed hypergraph on 8 vertices.

{000,001,110,111}
{000,010,101,111}
{000,100,011,111}
{000,110,101,011}
{010,100,101,011}
{010,001,110,101}
{001,100,011,110}
{001,100,010,111}

Table 1: Quads in the cubeconstructed hypergraph on 8 vertices.

In Section 2 we show that the cubeconstructed hypergraphs are in fact 3-uniform friendship hypergraphs without a universal friend on $n = 2^k$ vertices for all $k \geq 2$, and that they satisfy the conjecture of [3] for all $k \geq 4$. We also show that they have $2^{k-1}(3^{k-1} - 1)$ edges.

Furthermore, in Section 3 we construct friendship hypergraphs on 20 and on 28 vertices, hence dismiss a conjecture of [4] saying that all friendship hypergraphs without a universal friend must be on 2^k vertices.

In Section 4 we generalize the concept of friendship graphs and 3-uniform friendship hypergraphs to r-uniform friendship hypergraphs and observe that the existence of those with a universal friend are dependent on the existence of a Steiner system $S(r-1, r, n-1)$ similar to the case of the 3-universal friendship hypergraphs. Finally we construct a 4-uniform friendship hypergraph on 9 vertices using the Steiner system $S(5, 6, 12)$.

2. 3-Uniform Friendship Hypergraphs on 2^k vertices

Theorem 1. *The cubeconstructed hypergraphs are* 3*-uniform friendship hypergraphs.*

Proof. To prove the theorem, we only need to prove that the friendship property is satisfied in the cubeconstructed hypergraphs, hence that for all three vertices x, y and z exists a unique vertex w such that

$$\begin{aligned} dist_H(x,y) + dist_H(x,w) + dist_H(y,w) &= 2k, \\ dist_H(x,z) + dist_H(x,w) + dist_H(z,w) &= 2k, \\ dist_H(y,z) + dist_H(y,w) + dist_H(z,w) &= 2k. \end{aligned} \tag{1}$$

Due to vertex-transitivity of H, we only need to consider sets containing the vertex $0\ldots0$. So let $x = 0\ldots0$ and let y and z be two arbitrary vertices in H. Let a, b, c and d be the non-negative integers, such that there is a bits where y has a 1 and z has a 0, b bits where they both have a 1, c bits where y has a 0 and z has a 1 and finally d bits where they both have 0. Without loss of generality, we can assume x, y and z to be as in (2), as the distribution of the corresponding bits in y and z are irrelevant.

$$\begin{aligned} x &= 0\ldots0\,0\ldots0\,0\ldots0\,0\ldots0, \\ y &= 1\ldots1\,1\ldots1\,0\ldots0\,0\ldots0, \\ z &= \underbrace{0\ldots0}_{a}\underbrace{1\ldots1}_{b}\underbrace{1\ldots1}_{c}\underbrace{0\ldots0}_{d}. \end{aligned} \tag{2}$$

Now let r, s, t and u be non-negative integers, such that w has r bits of value 1 among the first a bits, s bits of value 1 among the next b bits, t bits of value 1 among the following c bits and finally u bits of value 1 among the last d bits, hence w consists of $r+s+t+u$ bits of value 1 and $k-(r+s+t+u)$ bits of value 0.

As we wish to determine w, we need to solve (1) with respect to r, s, t and u.

Given the method we used to label the vertices in the hypercube, we get

$$dist_H(x,y) = a + b,$$

$$dist_H(x,z) = b + c,$$

$$dist_H(y,z) = a + c,$$

$$dist_H(x,w) = r+s+t+u,$$
$$dist_H(y,w) = a-r+b-s+t+u$$

and

$$dist_H(z,w) = r+b-s+c-t+u$$

and hence, we get from (1) that

$$\begin{aligned} a+b+r+s+t+u+a-r+b-s+t+u \\ = 2a+2b+2t+2u = 2k, \\ b+c+r+s+t+u+r+b-s+c-t+u \\ = 2b+2c+2r+2u = 2k, \\ a+c+a-r+b-s+t+u+r+b-s+c-t+u \\ = 2a+2b+2c-2s+2u = 2k. \end{aligned}$$

Clearly $k = a+b+c+d$, so from the above we get

$$\begin{aligned} t+u &= c+d, \\ r+u &= a+d, \\ -s+u &= d \end{aligned}$$

and as $r \leq a$, $s \geq 0$, $t \leq c$ and $u \leq d$, the unique solution is $r = a$, $s = 0$, $t = c$ and $u = d$.

So

$$w = \underbrace{1\ldots1}_{a}\underbrace{0\ldots0}_{b}\underbrace{1\ldots1}_{c}\underbrace{1\ldots1}_{d}$$

is the unique solution to (1) we were looking for, and hence the cubeconstructed hypergraphs satisfy the friendship property. □

We wish to determine the number of edges in a cubeconstructed hypergraph, but before doing so, we need to do the following observations.

Lemma 2. *Let x, y and z be vertices in a cubeconstructed hypergraph $\mathcal{H}$. Then xyz is an edge of $\mathcal{H}$ if and only if in each bit at most two of the three vertices share the same value.*

Proof. First, assume xyz is an edge, then we know

$$dist_H(x,y) + dist_H(x,z) + dist_H(y,z) = 2k. \tag{3}$$

As stated in Lemma 1 each pair of vertices appears in at least one edge together, so without loss of generality, we can assume that x and y are as below, with $0 \leq a, b, c, d \leq k$,

$$\begin{aligned} x =& 1\ldots1\,1\ldots1\,0\ldots0\,0\ldots0 \\ y =& \underbrace{0\ldots0}_{a}\underbrace{1\ldots1}_{b}\underbrace{1\ldots1}_{c}\underbrace{0\ldots0}_{d} \end{aligned}$$

which means $dist_H(x,y) = a + c$. Now, no matter which value we have for z in each of the bits corresponding to the $a + c$ bits in x and y, we will get a contribution of exactly $a + c$ to the left-hand-side of (3), as z in each bit differs either from x or y. We are now $2k - 2(a + c) = 2b + 2d$ short of satisfying (3), and the only possibility to obtain this, is if z differs from both x and y in the corresponding $b+d$ bits, hence in the bits where x and y have the same value, z has to be the other value, proving the implication.

Now assume x, y and z are vertices such that in each bit at most two of them have the same value. Thus there is a contribution of two for each bit to $dist_H(x,y) + dist_H(x,z) + dist_H(y,z)$, hence a total of $2k$ is obtained, which proves that xyz is an edge. □

From Lemma 2 we get the following Corollary.

Corollary 1. *The four vertices x, y, z and w form a quad in the cube-constructed hypergraph if and only if in each bit the vertices agree two and two.*

Lemma 3. *Each pair x, y of vertices with $dist_H(x,y) = i$ is in exactly 2^i edges if $i < k$ and in exactly $2^k - 2$ edges if $i = k$.*

Proof. Without loss of generality, let x and y be as in the proof of Lemma 2, so $i = a + c$. If $i < k$ we can choose a z, such that xyz is an edge, in 2^i ways according to the proof of Lemma 2. If $i = k$, then the only vertices we cannot choose as z are x and y, hence there are $2^k - 2$ vertices to choose from. □

Theorem 2. *The number of edges in the cubeconstructed hypergraph on $n = 2^k$ vertices is $2^{k-1}(3^{k-1} - 1)$.*

Proof. Due to Lemma 3 we just need to calculate how many pairs of distance i there are in $\mathcal{H}$.

First assume $i = k$, then for each vertex, there is exactly one vertex in distance k, hence the total number of pairs in distance k is $2^k/2 = 2^{k-1}$. Now let $i < k$, then every pair in distance i has exactly i bits where they differ in value, and $k - i$ bits where they are equal in value. In each of these bits we have two choices, as in the i bits where they differ, we can choose which one of the vertices should have a 1 and in the bits where they are equal, can choose if they should be both 1 or both 0. Also the placement of the i bits where they differ can be done in $\binom{k}{i}$ ways, so in total we get $2^i 2^{k-i}\binom{k}{i}/2 = 2^{k-1}\binom{k}{i}$ pairs in distance i, as we have divided by 2 to avoid counting the pairs twice.

Combining this with Lemma 3 and the fact that we count an edge three times, once for each of the three pairs in the edge, we get the total number of edges in the hypergraph as

$$\frac{1}{3}\left(\sum_{i=1}^{k-1} 2^{k-1}2^i\binom{k}{i} + 2^{k-1}(2^k - 2)\right) = 2^{k-1}(3^{k-1} - 1). \tag{4}$$

□

As each edge is contained in a unique quad (Lemma 1) and there are four edges in each quad, we obtain the following corollary.

Corollary 2. *The number of quads in the cubeconstructed hypergraph on $n = 2^k$ vertices is $2^{k-3}(3^{k-1} - 1)$.*

The next theorem states, that if we know the cubeconstructed hypergraph on 8 vertices, then the remaining cubeconstructed hypergraphs can be constructed inductively.

Theorem 3. *Let $k > 3$, then the cubeconstructed hypergraph on 2^k vertices is isomorphic to the union of $k^2 - k$ copies of a cubeconstructed hypergraph on 2^{k-1} vertices.*

Proof. First, let H be a hypercube of dimension $k - 1$. Then we define a new labeling on H with k-bit binary labels from the old labeling of H to obtain a copy of H which we denote by H', and we show there are $k^2 - k$ different labelings of this kind. We then show that from this H' we get a

hypergraph from the cubeconstructed hypergraph on 2^{k-1} vertices, which is in fact a subhypergraph of the cubeconstructed hypergraph on 2^k vertices.

In the vertices of H we know that in each bit, half of the vertices have a 0 and the other half have a 1. We split the vertex set into two equal size parts by fixing a bit and letting H_1 be the vertices with a 0 in this bit and H_2 be the vertices with a 1 in this bit. For example, if $k = 4$, then if we fix the second bit, we get a division into $H_1 = \{000, 001, 100, 101\}$ and $H_2 = \{111, 110, 011, 010\}$. We can choose the bit in $k - 1$ ways. Notice, that according to Lemma 2 all edges in the corresponding cubeconstructed hypergraph will have at most two vertices in H_1 and H_2 respectively. Now we add an extra bit to each vertex in H, to obtain H' with H'_1 and H'_2 such that the value of the bit is distinct in H'_1 and H'_2 respectively. Using the example from before, and placing the new bit between the second and third bit, we get, by letting the value of the new bit be 1 in H_1 and 0 in H_2, that $H_1 = \{0010, 0011, 1010, 1011\}$ and $H_2 = \{1101, 1100, 0101, 0100\}$. Choosing which position to place the new bit in, can be done in k ways, and choosing the value can be done in two different ways. Then the vertices in H' corresponds to half of the vertices in the hypercube of dimension k, namely two disjoint subcubes of dimension $k-2$ where one is just the vertices in the other with all bits flipped. Going through every possible choice of the bit that splits H and added bit, we obtain the same H' two times, as in the new labeling we cannot distinguish between whether a bit has been chosen in the splitting of H or whether it has been added to obtain H'. Therefore we get that there are a total of $(k-1)k = k^2 - k$ copies of a H' in the hypercube of dimension k.

Let x, y, z be vertices in H. If xyz is an edge in the corresponding cubeconstructed hypergraph, we know

$$dist_H(x, y) + dist_H(x, z) + dist_H(y, z) = 2(k - 1)$$

and that at most two of x, y and z are in H_1 and H_2 respectively. Hence according to the construction of H' above, we add a bit to x, y and z to obtain x', y' and z', such that one, let's say x', differs from the others, which is then y' and z'. Hence in the hypercube of dimension k we will get

$$\begin{aligned} dist_{H'}(x', y') &+ dist_{H'}(x', z') + dist_{H'}(y', z') \\ &= dist_H(x, y) + 1 + dist_H(x, z) + 1 + dist_H(y, z) \\ &= 2(k - 1) + 2 \\ &= 2k. \end{aligned}$$

Thus $x'y'z'$ is an edge in the cubeconstructed hypergraph with 2^k vertices. Similarly, if xyz is not an edge, we know

$$dist_H(x,y) + dist_H(x,z) + dist_H(y,z) < 2(k-1),$$

and we get the largest addition to the sum of the distances, when one is in say H_1' and the two others in H_2'. So in total

$$\begin{aligned} dist_{H'}(x',y') &+ dist_{H'}(x',z') + dist_{H'}(y',z') \\ &\leq dist_H(x,y) + dist_H(x,z) + dist_H(y,z) + 2 \\ &< 2(k-1) + 2 \\ &= 2k \end{aligned}$$

and hence $x'y'z'$ is not an edge in the cubeconstructed hypergraph with 2^k vertices.

So clearly, the $k^2 - k$ copies of the cubeconstructed hypergraphs on 2^{k-1} vertices are isomorphic to a subhypergraph of the cubeconstructed hypergraph on 2^k vertices.

Now assume we have a cubeconstructed hypergraph on 2^k vertices, and let xyz be an edge therein. Then we wish to prove that xyz corresponds to an edge in a cubeconstructed hypergraph with 2^{k-1} vertices as the ones above. First we see that there are at least two of the three vertices which are in distance no more than $k-2$ from each other, otherwise we would have

$$\begin{aligned} 2k &= dist_H(x,y) + dist_H(x,z) + dist_H(y,z) \\ &\geq 3(k-1) \\ &= 3k-3 \end{aligned}$$

a contradiction as $k > 3$. Let's assume $dist_H(x,y) \leq k-2$, this means x and y share at least two bits, which determines whether they are in a copy of some H_1 or H_2 as given above, let us say without loss of generality they are in a copy of H_1. As xyz is an edge, we have according to Lemma 2 that z must differ in these two bits, which means z must be in a copy of the corresponding H_2.

So the cubeconstructed hypergraph on 2^k vertices is isomorphic to a subhypergraph of the $k^2 - k$ copies of a cubeconstructed hypergraphs on 2^{k-1} vertices, proving the theorem. □

Notice that the friendship hypergraph on 8 vertices and one of the ones on 16 vertices ($\mathcal{F}_1^{16}$ found in [3]) is actually just cubeconstructed hypergraphs. The cubeconstructed hypergraph on 32 vertices and 320 quads is however not isomorphic to the friendship hypergraph on 32 vertices found in [3], as this contains 344 quads.

The following corollary gives a affirmation of the conjecture from [3].

Corollary 3. *The cubeconstructed hypergraphs satisfy the inductive, pair and automorphism property.*

Proof. According to Theorem 3 and [3] the inductive property is satisfied. Also the automorphism property is satisfied, due to the hypercube being vertex-transitive.

Regarding the pair property, we see that the division of the vertex set of the cubeconstructed hypercube on 2^k vertices into pairs where each pair contains vertices in distance k, results in this property being satisfied as well. □

3. Other friendship hypergraphs

In [4] it was conjectured, that no other friendship hypergraphs than the ones with a universal friend and the ones on 2^k vertices exists. Our next theorem will show, that this is not true. But first we state a necessity for a friendship hypergraph to be vertex transitive.

Lemma 4. *Let $\mathcal{H}$ be a friendship hypergraph on n vertices which is vertex-transitive. Then $(n-1)$ or $(n-2)$ must be divisible by 3.*

Proof. Due to the friendship property, we know that for each set of three vertices, there exists a unique completion of these. As $\mathcal{H}$ is vertex-transitive, we know that each vertex must be a completion to such a set, the same number of times.

Hence this number is given by

$$\frac{\binom{n}{3}}{n} = \frac{(n-1)(n-2)}{3 \cdot 2},$$

and as 2 divides either $n-1$ or $n-2$, 3 must also be a prime divisor of one of them. □

Theorem 4. *There exist at least three non-isomorphic friendship hypergraphs on* 20 *vertices and* 420 *edges and at least one friendship hypergraph on* 28 *vertices and* 1036 *edges.*

Proof. Let the n vertices be represented by the elements in $\mathbb{Z}_n$. The constructions arise from some fixed quads $\{a, b, c, d\}$ where $a, b, c, d \in \mathbb{Z}_n$ and given these, all the quads of the type $\{a, b, c, d\} + i = \{a + i \mod n, b + i \mod n, c + i \mod n, d + i \mod n\}$ where $i \in \mathbb{Z}_n$.

By computer search, we have found three non-isomorphic friendship hypergraphs on $n = 20$ vertices, all of which contain the following five fixed quads:

$$\{0, 1, 10, 11\}, \{0, 2, 10, 12\}, \{0, 3, 10, 13\}, \{0, 4, 10, 14\}, \{0, 5, 10, 15\}.$$

Except for $\{0, 5, 10, 15\}$ which represents a total of 5 quads, all these fixed quads represent 10 quads each.

The remaining fixed quads are specific to the different friendship hypergraphs as given below:

a) $\{0, 1, 3, 14\}, \{0, 1, 9, 15\}, \{0, 2, 4, 7\}$,

b) $\{0, 1, 4, 13\}, \{0, 1, 6, 12\}, \{0, 2, 4, 7\}$,

c) $\{0, 1, 4, 13\}, \{0, 1, 9, 15\}, \{0, 2, 4, 17\}$.

These fixed quads in a)-c) all represent 20 quads each, so all three hypergraphs contain a total of 105 quads. Hence, they all contain 420 edges.

For $n = 28$ we also found a friendship hypergraph using computer search. It has the following seven fixed quads:

$$\{0, 1, 14, 15\}, \{0, 2, 14, 16\}, \{0, 3, 14, 17\}, \{0, 4, 14, 18\},$$
$$\{0, 5, 14, 19\}, \{0, 6, 14, 20\}, \{0, 7, 14, 21\},$$

which share similarities with the five fixed quads in all the found friendship hypergraphs on 20 vertices. Expect for {0,7,14,21} which represents 7 quads, they all represent 14 quads each.

Furthermore it contains the six fixed quads

$$\{0, 1, 4, 17\}, \{0, 1, 5, 20\}, \{0, 1, 7, 13\}, \{0, 2, 4, 23\}, \{0, 3, 8, 19\}, \{0, 3, 10, 20\},$$

which all represent 28 quads each. Hence a total of 259 quads and 1036 edges. □

Inspired by Theorem 4 and the other known vertex-transitive friendship hypergraphs presented in this paper, we conjecture the following.

Conjecture 1. *For all n which is divisible by 4 and not divisible by 3, there exists a vertex-transitive friendship hypergraph.*

4. r-uniform friendship hypergraphs

In this section we will give another generalization of the friendship graphs, the so called *r-uniform friendship hypergraphs* for $r \geq 2$, which satisfy the following property.

Definition 3 (Friendship property for r-uniform hypergraphs)**.** *For every r vertices $x_1, x_2, \ldots, x_r$, there exists a unique vertex w such that*

$$\{x_1, x_2, \ldots, x_r, w\} - \{x_i\}$$

is an edge in the hypergraph for all $i = 1, 2, \ldots, r$.

Notice that for $r = 2$ the above definition corresponds to that of friendship graphs and for $r = 3$ it corresponds to Definition 1. Similar to before, we will denote an r-uniform hypergraph which satisfies the friendship property for r-uniform hypergraphs as a *r-uniform friendship hypergraph.* Also w will be denoted as the *completion* of $x_1, x_2, \ldots, x_r$.

Similarly to Lemma 1, we have the following observations for r-uniform friendship hypergraphs.

Lemma 5. *The following is true for every r-uniform friendship hypergraph $\mathcal{H}$.*

(a) Every set of at most $r-1$ vertices appears in at least one edge together.

(b) Every edge must be contained in a unique complete r-uniform hypergraph on $r+1$ vertices.

The proof is similar to that of Lemma 1.

We also observe that a friendship hypergraph on n vertices with a universal friend exists, if and only if a Steiner system, $S(r-1, r, n-1)$, exists, as it has edges $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{r-1}}, v_n\}$ for all $1 \leq i_1 < i_2 < \ldots < i_{r-1} < n$

and the remaining edges described by a Steiner system on $\{v_1, v_2, \ldots, v_{n-1}\}$. From this we know it has

$$\binom{n-1}{r-1} + \frac{\binom{n-1}{r-1}}{\binom{r}{r-1}} = (1 + \frac{1}{r})\binom{n-1}{r-1}$$

edges.

The only $r \geq 4$, for which we know in general that Steiner systems exist, is $r = 4$ and the Steiner systems $S(3, 4, n-1)$ is referred to as a Steiner quadruple systems. They exist if and only if $n \equiv 3 \mod 6$ or $n \equiv 5 \mod 6$. For all other values of r we only know a finite number of Steiner systems, see [5] for an overview of which Steiner systems are known, and some of the properties of Steiner systems.

The last r-uniform friendship hypergraph we will present in this paper, is the one given in the following definition.

Definition 4 (4-uniform hypergraph on 9 vertices)**.** *Let a, b, c be three of the elements in the Steiner system $S(5, 6, 12)$ and let the remaining 9 elements represent the vertex set V in a 4-uniform hypergraph. The quadruple $v_1v_2v_3v_4$ is an edge in the 4-uniform hypergraph if and only if $\{a, v_1, v_2, v_3, v_4, x\} \in S(5, 6, 12)$ for some $x \in V$.*

In the next lemma, we will use the block intersection numbers $\lambda_{i,j}$ of a Steiner system $S(t, k, v)$ which, given two disjoint sets I of size i and J of size j, determines the number of blocks which contain I but do not contain J. Due to the properties of Steiner systems, this number only depends on i and j, and hence we can calculate any $\lambda_{i,j}$ from the following:

$$\lambda_{i,0} = \begin{cases} \binom{v-i}{t-i} / \binom{k-i}{t-i} & \text{for } 0 \leq i \leq t \\ 1 & \text{for } t < i \leq k \end{cases}$$
$$\lambda_{i,j} = \lambda_{i,j-1} - \lambda_{i+1,j-1}.$$

Lemma 6. *The 4-uniform hypergraph given in Definition 4 has 90 edges*

Proof. For each 6-block in $S(5, 6, 12)$ which contains a and neither b or c, we get a total of 5 edges due to the definition of the 4-uniform hypergraph. So we wish to determine this number of blocks, which can be done by using the

block intersection number, $\lambda_{1,2}$. According to the above we get

$$\begin{aligned}\lambda_{1,2} &= \lambda_{1,0} - 2\lambda_{2,0} + \lambda_{3,0} \\ &= \frac{\binom{9}{2}}{3} - 2 \cdot \frac{\binom{10}{3}}{4} + \frac{\binom{11}{4}}{5} \\ &= 18.\end{aligned}$$

So the number of edges in the hypergraph is $5 \cdot 18 = 90$. □

Theorem 5. *The hypergraph given in Definition 4 is a 4-uniform friendship hypergraph without a universal friend.*

Proof. Due to the fact that the 4-uniform friendship hypergraph with a universal friend on 9 vertices has $(1 + 1/4)\binom{8}{3} = 70$ edges and the 4-uniform hypergraph given in Definition 4 has 90 edges, the one from Definition 4 cannot have a universal friend.

It remains to show that the friendship property is in fact satisfied in the 4-uniform hypergraph given in Definition 4, hence that for all quadruples v_1, v_2, v_3, v_4 there exists a unique vertex w (the completion) such that $\{a, v_1, v_2, v_3, w, x_4\}$, $\{a, v_1, v_2, v_4, w, x_3\}$, $\{a, v_1, v_3, v_4, w, x_2\}$ and $\{a, v_2, v_3, v_4, w, x_4\} \in S(5, 6, 12)$ for some $x_i \in V$ for all $i = 1, 2, 3, 4$.

As we know $S(5, 6, 12)$ is a Steiner system, we know that there exists some unique element x such that $\{a, v_1, v_2, v_3, v_4, x\} \in S(5, 6, 12)$. If $x \in V$, then this x is the completion of v_1, v_2, v_3, v_4. Now assume $x \notin V$, without loss of generality we can assume that $x = b$, so $\{a, b, v_1, v_2, v_3, v_4\} \in S(5, 6, 12)$. Now we know that $\{a, c, v_1, v_2, v_3, y_4\} \in S(5, 6, 12)$ for some element y_4, and we see that $y_4 \neq b$ as otherwise it would be a contradiction to $S(5, 6, 12)$ being a Steiner system. So we must have $y_4 \in V$. Similarly we see that $\{a, c, v_1, v_2, v_4, y_3\}, \{a, c, v_1, v_3, v_4, y_2\}, \{a, c, v_2, v_3, v_4, y_1\} \in S(5, 6, 12)$ where $y_i \neq y_j$ for $i \neq j$ and $y_i \neq b$ for $i = 1, 2, 3$.

Now we have a unique vertex $w \in V$ given by $w \neq y_i, v_i$ for all $i = 1, 2, 3, 4$ which satisfies $\{a, v_1, v_2, v_3, w, x_4\}$, $\{a, v_1, v_2, v_4, w, x_3\}$, $\{a, v_1, v_3, v_4, w, x_2\}$ and $\{a, v_2, v_3, v_4, w, x_4\} \in S(5, 6, 12)$ for some $x_i \in V$ for all $i = 1, 2, 3, 4$. So this w is the completion, hence proving the friendship property for a 4-uniform hypergraph is satisfied. □

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