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## **Approximations to the Probability of Failure in Random Vibration by Integral Equation Methods**

Nielsen, Søren R.K.; Sørensen, John Dalsgaard

*Publication date:*  
1986

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*  
Nielsen, S. R. K., & Sørensen, J. D. (1986). *Approximations to the Probability of Failure in Random Vibration by Integral Equation Methods*. Institut for Bygningsteknik, Aalborg Universitetscenter. Structural Reliability Theory Vol. 21 No. R8618

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**STRUCTURAL RELIABILITY THEORY  
PAPER NO. 21**

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**SØREN R. K. NIELSEN & JOHN DALSGÅRD SØRENSEN  
APPROXIMATIONS TO THE PROBABILITY OF FAILURE IN RANDOM VIBRATION BY  
INTEGRAL EQUATION METHODS  
OCTOBER 1986**

**ISSN 0105-7421 R8618**

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## APPROXIMATIONS TO THE PROBABILITY OF FAILURE IN RANDOM VIBRATION BY INTEGRAL EQUATION METHODS

Søren R. K. Nielsen and John Dalsgård Sørensen  
University of Aalborg

### ABSTRACT

Close approximations to the first passage probability of failure in random vibration can be obtained by integral equation methods. A simple relation exists between the first passage probability density function and the distribution function for the time interval spent below a barrier before outcrossing. An integral equation for the probability density function of the time interval is formulated, and adequate approximations for the kernel are suggested. The kernel approximation results in approximate solutions for the probability density function of the time interval, and hence for the first passage probability density. The results of the theory agree well with simulation results for narrow banded processes dominated by a single frequency, as well as for bimodal processes with 2 dominating frequencies in the structural response.

Keywords: Random Vibration, Stochastic Processes, First Passage Failure, Bimodal Processes, Integral Equations.

## INTRODUCTION

The main theme of the present paper is the calculation of the first-passage time probability density function based on the integration of a second order Volterra integral equation with various approximations to the kernel or the inhomogeneity.

The formulation of integral equations for the first passage probability density can be traced back to Siegert [1]. Later, this work was extended by Shipley and Bernard [2], [3] where certain kernel approximations were suggested. The cited authors all assumed the considered process to be Markovian and deduced the relevant integral equation from the Chapman-Kolmogorov equation. Nielsen [4] demonstrated the nature of the integral equation as a fundamental identity, not restricted to Markov processes and derived a formal expansion of the kernel in terms of inclusion-exclusion series. Madsen and Krenk [5] used the kernel approximation of refs. [2], [3], [4], but adjusted the inhomogeneity of the integral equation to provide the exact value for the first-passage density function at time  $t = 0$ .

The starting point of the present paper is a relationship between the first-passage time probability density  $f_T$  and the distribution function  $F_L$  for the length of the time interval  $L$  spent in the safe domain before out-crossing from the safe domain at the time  $t$ , which for stationary processes with time-constant safe domain may be written

$$f_T(t) = \frac{1}{E[L]} (1 - F_L(t)) \quad (1)$$

(1) was discovered independently by Rice [6], [7] and Cook [8], [9].

Initially an integral equation for  $f_L(t)$  is derived. Further, the Cook-Rice identity (1) is generalized to non-stationary processes or time-varying safe domains. Formal inclusion-exclusion series for  $f_L(t)$ ,  $f_T(t)$  and the kernel of the integral equation are then derived. It is remarkable that the first-passage density function for non-deterministic start in this representation is expressed by unconditional joint crossing rates. The facilitation of this formulation in relation to another exact representation for the first-passage density function expressed by conditional joint crossing rates is stressed in the paper.

When the kernel in the integral equation for  $f_L(t)$  is approximated, approximate solutions for  $f_L(t)$  and hence  $f_T(t)$  are obtained. In the paper 2 such kernel approximations are suggested, one involving 3rd order joint unconditional crossing rates and another only involving 2nd order crossing statistics.

The 3rd order approximation results in highly accurate results as is demonstrated in the numerical example.

The 2nd order approximation requires computational efforts comparable to that involved in applying the integral equation for  $f_T(t)$  with the kernel approximations of refs. [2], [3], [4], [5]. The present approximation provides results at the same level of accuracy, but fits the first-passage density curve much better in the earlier stages of first-passages.

The theory has been compared with simulation studies for narrow banded processes dominated by a single frequency as well as for bimodal processes with 2 dominating frequencies in the structural response. The considered first-passage problem was Gaussian processes with a single barrier at the normalized level  $b = 2$  and stationary start at  $t = 0$ . From the numerical studies it is concluded that the kernel approximation with 3rd order crossing rates provides the best known approximations to the first-passage probability problem at low and moderate barrier levels.

#### AN INTEGRAL EQUATION FOR THE TIME INTERVALS SPENT IN THE SAFE DOMAIN BEFORE AN OUTCROSSING

The safe domain at time  $t$  is denoted  $S_t$ . The event that a sample curve crosses out of the safe domain at time  $t$  is denoted  $E_t$ .

The rate of incrossings to  $S_{t-\ell}$ ,  $\ell > 0$  on condition of an outcrossing at time  $t$  becomes

$$f_{1|E_t}^-(\ell) = \frac{f_2^{-(t-\ell, t)}}{f_1^+(t)}, \quad \ell > 0 \quad (2)$$

Similarly, the joint rate of incrossings to  $S_{t-\ell_1}, S_{t-\ell_2}, \dots, S_{t-\ell_n}$ ,  $0 < \ell_n < \dots < \ell_2 < \ell_1$  on condition of  $E_t$  becomes

$$f_{n|E_t}^{-(\ell_1, \dots, \ell_n)} = \frac{f_{n+1}^{-(t-\ell_1, \dots, t-\ell_n, t)}}{f_1^+(t)} \quad (3)$$

$f_{n+1}^{-(t-\ell_1, \dots, t-\ell_n, t)}$  signifies the unconditional joint rate of incrossings to  $S_{t-\ell_1}, \dots, S_{t-\ell_n}$  and outcrossing from  $S_t$ . This can be calculated from S.O. Rice's

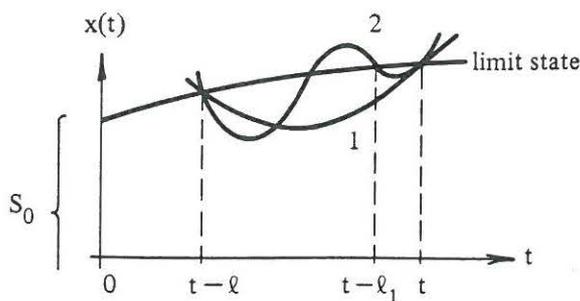


Figure 1.

formula and its multidimensional generalization [10], [11]. For one-dimensional processes in the single barrier problem the relevant formulas have been given in the appendix. In the applied notation  $-$  means incrossing into and  $+$  means outcrossings from the safe domain at times specified explicitly in the argument list.

$f_L(\ell, t)$  is the rate of incrossing to  $S_{t-\ell}$  on condition of an outcrossing from  $S_t$  and on condition that no incrossings take place in the interval  $]t - \ell, t[$ . Hence, the following identity may be formulated

$$f_L(\ell, t) = f_{1|E_t}^-(\ell) - \int_0^\ell A_{|E_t}(\ell|\ell_1) f_L(\ell_1, t) d\ell_1 \quad (4)$$

$A_{|E_t}(\ell|\ell_1)$ ,  $0 < \ell_1 < \ell$ , signifies the rate of incrossings to  $S_{t-\ell}$  on condition of  $E_t$  and on condition that the last incrossing in the open interval  $]t - \ell, t[$  took place at the time  $t - \ell_1$ .

The last term in (4) withdraws from  $f_{1|E_t}^-(\ell)$  all  $E_t$ -samples with one or more incrossings in  $]t - \ell, t[$ , e.g. samples of type 2 in figure 1 which exactly specifies the rate of samples included in  $f_L(\ell, t)$  corresponding to samples of type 1 in figure 1.

$A_{|E_t}(\ell|\ell_1) f_L(\ell_1, t)$  specifies the joint rate of incrossings to  $S_{t-\ell}$  and  $S_{t-\ell_1}$  of  $E_t$ -samples on condition of no incrossings in  $]t - \ell_1, t[$ . For this quantity the following integral equation is obtained

$$A_{|E_t}(\ell|\ell_1) f_L(\ell_1, t) = f_{2|E_t}^-(\ell, \ell_1) - \int_0^{\ell_1} B_{|E_t}(\ell, \ell_1|\ell_2) f_L(\ell_2, t) d\ell_2 \quad (5)$$

$B_{|E_t}(\ell, \ell_1|\ell_2)$ ,  $0 < \ell_2 < \ell_1 < \ell$ , signifies the joint rate of incrossings to  $S_{t-\ell}$ ,  $S_{t-\ell_1}$  on condition of  $E_t$  and on condition that the last incrossing in the interval  $]t - \ell_1, t[$  took place at the time  $t - \ell_2$ .

The last term in (5) withdraws from  $f_{2|E_t}^-(\ell, \ell_1)$  every  $E_t$ -sample with one or more incrossings in  $]t - \ell_1, t[$  which exactly specifies the joint crossing rate on the left hand side.

Inserting (5) in (4) results in

$$f_L(\ell, t) = f_{1|E_t}^-(\ell) - \int_0^\ell f_{2|E_t}^-(\ell, \ell_1) d\ell_1 + \int_0^\ell \int_0^{\ell_1} B_{|E_t}(\ell, \ell_1|\ell_2) f_L(\ell_2, t) d\ell_2 d\ell_1 \quad (6)$$

The integrals on the right hand sides of (4) and (6) are non-negative. Hence, the following bounds are obtained

$$f_L(\ell, t) \leq f_{1|E_t}^-(\ell) \quad (7)$$

$$f_L(\ell, t) \geq f_{1|E_t}^-(\ell) - \int_0^\ell f_{2|E_t}^-(\ell, \ell_1) d\ell_1 \quad (8)$$

For the integrand in (6) an expression similar to (5) can be formulated. Continuation of this process leads to the formal series for  $f_L(\ell, t)$

$$f_L(\ell, t) = f_{1|E_t}^-(\ell) - \int_0^\ell f_{2|E_t}^-(\ell, \ell_1) d\ell_1 + \int_0^\ell \int_0^{\ell_1} f_{3|E_t}^-(\ell, \ell_1, \ell_2) d\ell_2 d\ell_1 - \dots \quad (9)$$

Truncated with  $N$  terms (9) provides upper bounds when  $N$  is unequal and lower bounds when  $N$  is equal.

Similarly, a formal series for  $A_{|E_t}(\ell_1, \ell) f_L(\ell_1, t)$  is obtained during the process. The result becomes

$$A_{|E_t}(\ell|\ell_1) f_L(\ell_1, t) = f_{2|E_t}^-(\ell, \ell_1) - \int_0^{\ell_1} f_{3|E_t}^-(\ell, \ell_1, \ell_2) d\ell_2 \\ + \int_0^{\ell_1} \int_0^{\ell_2} f_{4|E_t}^-(\ell, \ell_1, \ell_2, \ell_3) d\ell_3 d\ell_2 - \dots \quad (10)$$

Truncated with  $N$  terms, (10) also provides upper bounds, when  $N$  is unequal and lower bounds when  $N$  is equal.

(9) in combination with (10) provides a formal representation of the kernel in (4)

$$A_{|E_t}(\ell|\ell_1) = \frac{f_{2|E_t}^-(\ell, \ell_1) - \int_0^{\ell_1} f_{3|E_t}^-(\ell, \ell_1, \ell_2) d\ell_2 + \dots}{f_{1|E_t}^-(\ell_1) - \int_0^{\ell_1} f_{2|E_t}^-(\ell_1, \ell_2) d\ell_2 + \dots} \quad (11)$$

When the kernel is replaced by an approximation  $A_{|E_t}^*(\ell_1|\ell)$ , an approximate solution  $f_L^*(\ell, t)$  to (4) is obtained.

Especially the following approximations are considered, retaining only the first term in the counter and the denominator

$$A_{|E_t}^*(\ell|\ell_1) = \frac{f_{2|E_t}^-(\ell, \ell_1)}{f_{1|E_t}^-(\ell_1)} = \frac{f_3^{--}(t-\ell, t-\ell_1, t)}{f_2^{--}(t-\ell_1, t)} \quad (12)$$

The approximation (12) is reasonable, because both counter and denominator are upper bounds and hence they counterbalance each other to same extent. Moreover, the approximation is asymptotically correct at high barrier levels with independent crossing rates.

If the joint crossing rates in (12) are assumed to be independent of  $E_t$ , the following approximation is obtained, only involving second order unconditional crossing rates.

$$A_{|E_t}^*(\ell|\ell_1) = \frac{f_2^-(\ell, \ell_1)}{f_1^-(\ell_1)} \quad (13)$$

For both approximations (12) and (13) the inhomogeneity  $f_{1|E_t}^-(t)$ , given by eq. (2), is applied in the integral equation.

#### INCLUSION EXCLUSION SERIES FOR THE FIRST PASSAGE DENSITY

The generalization of eq. (1) to non-stationary processes or time-varying safe domains becomes

$$P(X(0) \in S_0)f_T(t) = f_1^+(t) \left(1 - \int_0^t f_L(u, t) du\right) \quad (14)$$

In order to prove (14), consider  $N = N_0 + N_1$  realizations at time  $t = 0$ .  $N_0$  of these are in the safe domain  $S_0$  at time  $t = 0$ , whereas the remaining  $N_1$  realizations are in the unsafe domain.

In the time interval  $]t, t + \Delta t]$  totally  $\Delta N = \Delta N_0 + \Delta N_1$  outcrossings take place.  $\Delta N_0$  is the number of first passages, i.e. the number of  $N_0$ -realizations, which have not left the safe domain in  $]0, t]$ .  $\Delta N_1$  represents outcrossings of  $N_1$  realizations and  $N_0$ -realizations with one or more preceding incrossings in  $]0, t]$ .

By definition we now have

$$f_T(t)\Delta t = \frac{\Delta N_0}{N_0} \quad (15)$$

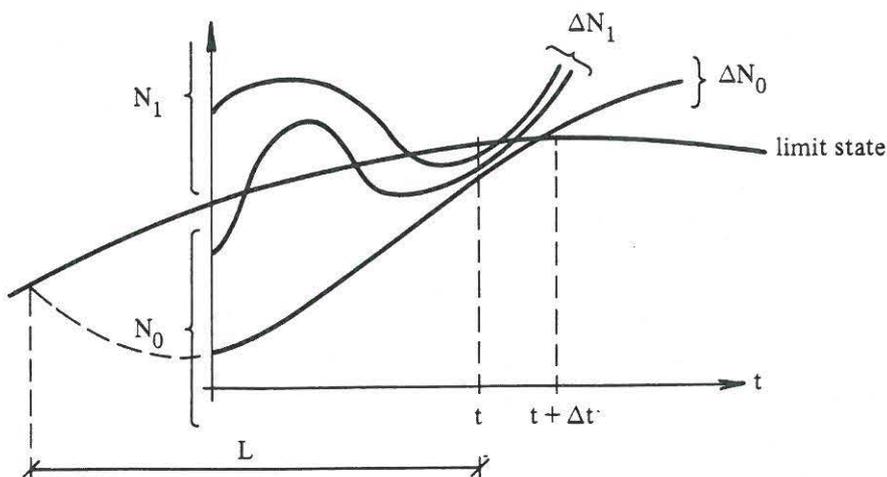


Figure 2.

$$f_1^+(t)\Delta t = \frac{\Delta N}{N} \quad (16)$$

$$P(X(0) \in S_0) = \frac{N_0}{N} \quad (17)$$

$$P(L > t) = \frac{\Delta N_0}{\Delta N} \quad (18)$$

Inserting (15) - (18) in (14) proves the identity. In the stationary case with time constant safe domain  $f_1^+(t)$  becomes constant. Integrating both sides of (14) from 0 to  $+\infty$  then provides

$$P(X(0) \in S_0) = f_1^+(t)E[L] \quad (19)$$

from which eq. (1) is derived.

From (2), (3), (9), (14), the following series is obtained for  $f_L(\ell, t)$  expressed in unconditional crossing rates

$$f_L(\ell, t) = \frac{1}{f_1^+(t)} (f_2^{--}(t-\ell, t) - \int_0^\ell f_3^{--}(t-\ell, t-\ell_1, t)d\ell_1 + \dots) \quad (20)$$

Finally, from (14) and (20) the following series for  $f_T(t)$  is obtained.

$$\begin{aligned} P(X(0) \in S_0)f_T(t) &= f_1^+(t) - \int_0^t f_2^{--}(t-\ell, t)d\ell \\ &+ \int_0^t \int_0^\ell f_3^{--}(t-\ell, t-\ell_1, t)d\ell_1 d\ell - \dots \end{aligned} \quad (21)$$

Truncation of (21) provides succeeding upper and lower bounds for  $f_T(t)$ . Alternatively, the following bounding techniques suggested in reference [4] may be applied.

Initially it is seen that if  $A_{|E_t}^*(\ell|\ell_1) \leq A_{|E_t}(\ell|\ell_1)$  for all  $\ell_1 \in ]0, \ell[$ , then  $f_L^*(\ell, t) \geq f_L(\ell, t)$ . Lower-bound kernels can be constructed by truncating the denominator in (11) with an unequal number of terms, and the counter with an equal number of terms. When the upper bound  $f_L^*(\ell, t)$  is inserted in (14), a lower bound is obtained for  $f_T(t)$ .

Similarly, if  $A_{|E_t}^*(\ell|\ell_1) \geq A_{|E_t}(\ell|\ell_1)$  for all  $\ell_1 \in ]0, \ell[$ , upper bounds for  $f_T(t)$  result. Upper bound kernels can be constructed by truncating the denominator in (11) with an equal number of terms and the counter with an unequal number of terms.

The series representation (21) is remarkable because it expresses the first-passage density function for non-deterministic start in terms of unconditional joint crossing rates.

Let  $F = \{X(0) \in S_0\}$ , i.e. the event that all samples are in the safe domain at time  $t = 0$ . Then the following integral equations can be formulated for  $f_T(t)$

$$f_T(t) = f_{1|F}^+(t) - \int_0^t K(t|t_1) f_T(t_1) dt_1 \quad (22)$$

$f_{1|F}^+(t)$  is the rate of outcrossing from  $S_t$  on condition that the samples are in the safe domain at time  $t = 0$ .

The kernel  $K(t|t_1)$  specifies the rate of outcrossing from  $S_t$  on condition that the first outcrossing of  $F$ -samples took place at time  $0 < t_1 < t$ . The argument leading to (22) is identical to the argument leading to (4).

By a derivation identical to the one leading to the results (9) and (11) the following series are obtained for  $f_T(t)$  and  $K(t|t_1)$  expressed in conditional joint outcrossing rates

$$f_T(t) = f_{1|F}^+(t) - \int_0^t f_{2|F}^{++}(t_1, t) dt_1 + \int_0^t \int_0^{t_1} f_{3|F}^{+++}(t_2, t_1, t) dt_2 dt_1 - \dots \quad (23)$$

$$K(t|t_1) = \frac{f_{2|F}^{++}(t_1, t) - \int_0^{t_1} f_{3|F}^{+++}(t_2, t_1, t) dt_2 + \dots}{f_{1|F}^+(t_1) - \int_0^{t_1} f_{2|F}^{++}(t_2, t_1) dt_2 + \dots} \quad (24)$$

(24) was derived by Nielsen [4].

The series (21) and (23) are alternative exact representations of  $f_T(t)$ . Clearly, (21) is most suitable because the calculation of cumbersome conditional joint crossing rates is omitted. Formulas for these in the one-dimensional case for the single barrier problem have been given in the appendix.

Consider the following kernel approximation, applied in refs. [2], [3], [4], [5], obtained by retaining only the first term in the counter and denominator of (24) and omitting the condition on  $F$

$$K^*(t|t_1) = \frac{f_{2|F}^{++}(t_1, t)}{f_{1|F}^+(t_1)} \quad (25)$$

Notice the symmetry in the argumentation, leading to the kernel approximations (13) and (25).

(25) and (13) are both expressed in unconditional joint crossing rates of the 2nd order. The computational effort involved in applying these approximations is then at the same level.

In order to compare the 2 formulations, (22) will be solved with the kernel approximation (25) and the inhomogeneity

$$f_{1|F}^+(t) \sim \frac{f_1^+(t)}{P(X(0) \in S_0)} \quad (26)$$

(26) which was used by Madsen and Krenk [5] will give the correct value at  $t = 0$ , where  $f_T(0) = f_{1|F}^+(0) = f_1^+(0)/P(F)$ .

### NUMERICAL RESULTS

In this section the approximations (12), (13), and (25) will be compared with simulation results and approximate results from refs. [12] and [13]. The process  $\{X(t)\}$  is assumed to be a stationary Gaussian process with zero mean and unit standard deviation. The safe area  $S$  is assumed to be time-invariant and given by  $S = ] - \infty, b [$ , where  $b = 2$  is chosen.

First, we consider a slightly damped single-degree-of-freedom (SDF) system subjected to stationary excitation of white noise. Then the autocorrelation coefficient function  $\rho$  is given by

$$\rho(t) = \exp(-\zeta\omega_0|t|) \left( \cos \omega_0 \sqrt{1-\zeta^2} t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_0 \sqrt{1-\zeta^2} |t| \right) \quad (27)$$

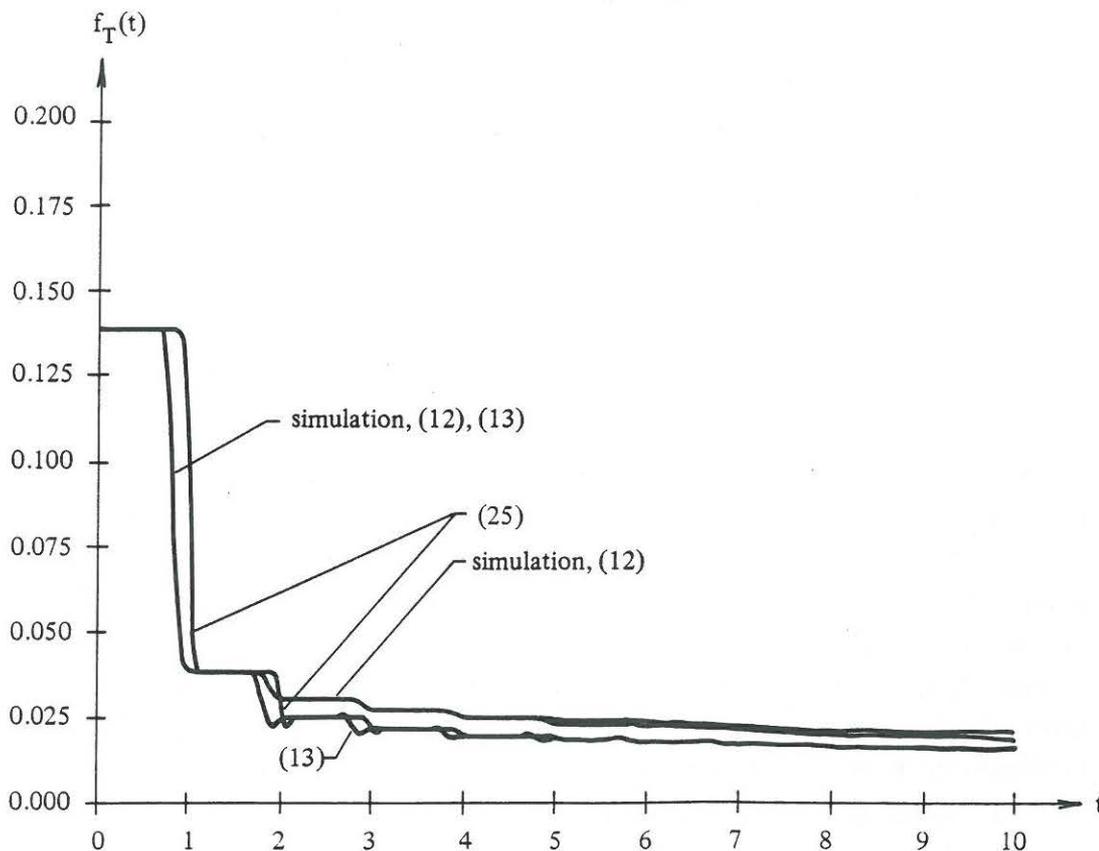


Figure 3. First passage densities for SDF system,  $\zeta = 0.01$ ,  $\omega_0 = 2\pi$ , and  $b = 2$ .

In this example we use  $\omega_0 = 2\pi$  and  $\zeta = 0.01$ . In figure 3 approximations to  $f_T$  based on eqs. (12), (13), and (25) are shown. For comparison simulation results based on eq. (1) are also shown.

From figure 3 it is seen that the approximation (12) based on third order unconditional crossing rates gives very accurate results compared with the simulation results. The approximation (13) based on second order unconditional crossing rates compares reasonably with the simulation results. The approximation (25) which is also based on second order unconditional crossing rates is seen to give the same »stair levels» as (13) but deviates considerably from the simulation result at the first downfall of the first passage curve. In table 1 the first 5 »stair levels» of the three approximations and the simulation are shown.

| t     | simulation | eq. (12) | eq. (13) | eq. (25) |
|-------|------------|----------|----------|----------|
| 0 - 1 | 0.13849    | 0.13849  | 0.13849  | 0.13849  |
| 1 - 2 | 0.03881    | 0.03821  | 0.03822  | 0.03822  |
| 2 - 3 | 0.03141    | 0.03040  | 0.02507  | 0.02507  |
| 3 - 4 | 0.02800    | 0.02715  | 0.02124  | 0.02124  |
| 4 - 5 | 0.02589    | 0.02527  | 0.01938  | 0.01938  |

Table 1. »Stair levels» of  $f_T$ .

Next, we consider a slightly damped two-degrees-of-freedom (2DF) system subjected to stationary excitation of white noise. The autocorrelation coefficient function  $\rho$  is assumed to be given by

$$\rho(t) = \sigma_1^2 \rho_1(t) + \sigma_2^2 \rho_2(t) \quad (28)$$

where

$$\rho_i(t) = \exp(-\zeta_i \omega_i |t|) \left( \cos \omega_i \sqrt{1 - \zeta_i^2} t + \frac{\zeta_i}{\sqrt{1 - \zeta_i^2}} \sin \omega_i \sqrt{1 - \zeta_i^2} |t| \right) \quad (29)$$

The correlation function (28) for  $\{X(t)\}$  corresponds to a spectrum with two dominating frequencies. In the example we use  $\sigma_1^2 = \sigma_2^2 = 0.5$ ,  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 = 2\pi$ , and  $\omega_2 = 2.5\pi$ . This corresponds to Vanmarcke's band width parameter equal to 0.11, cf. ref. [13]. In figure 4 approximations to the first-passage density  $f_T$  based on eqs. (12), (13), and (25) are compared with simulation results.

From figure 4 it is seen that also in this case the 3rd order approximation (12) gives very good results compared with simulation. The 2nd order approximations (13) and (25) fluctuate somewhat about the simulation result but reasonable results for the cumulative distribution function  $F_T$  (the probability of failure) can

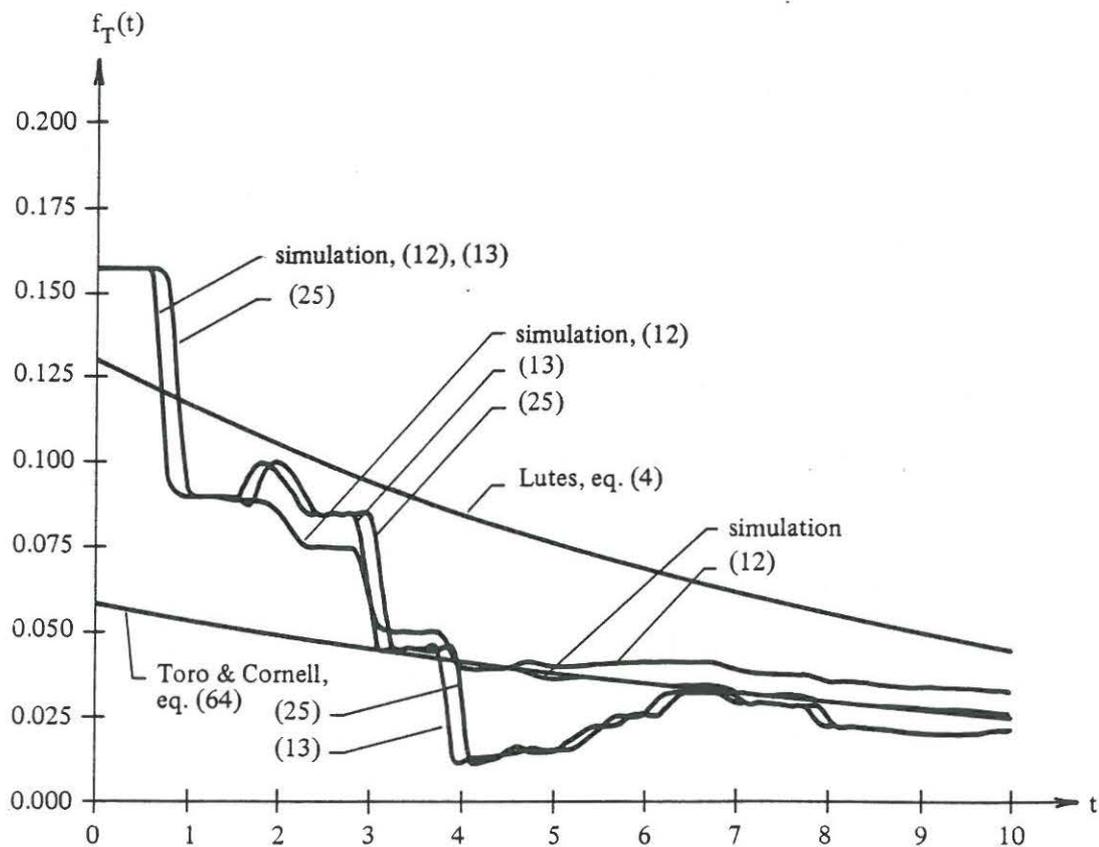


Figure 4. First passage densities for SDF system,  $\zeta_1 = \zeta_2 = 0.01$ ,  $\omega_1 = 2\pi$ ,  $\omega_2 = 2.5\pi$ ,  $\sigma_1^2 = \sigma_2^2 = 0.5$ , and  $b = 2$ .

be expected. Also shown in figure 4 are approximate results from refs. [12] and [13]. The estimate from ref. [12] is seen to overestimate the first-passage density in the whole interval  $[0, 10]$ , whereas the estimate from ref. [13] underestimates the first-passage density considerably in the interval  $[0, 4]$ .

## CONCLUSIONS

An integral equation has been derived for the time lengths spent in the safe domain before an outcrossing takes place.

The Cook-Rice identity is then generalized to non-stationary processes and a new formal series expansion of the first-passage probability density for non-deterministic start is derived expressed solely in unconditional joint crossing rates.

Two relevant approximations to the kernel are suggested based on third and second order unconditional crossing rates. In a numerical study the results from these approximations have been compared with simulation results and results obtained from other methods, for Gaussian one-dimensional processes in the single-barrier problem. The normalized barrier level was  $b = 2$ . Both a narrow banded process and a bimodal process were considered. From the examples it is concluded that the 3rd order kernel approximations provide highly accurate results even at moderate and low barrier levels. The 2nd order kernel approximation gives less accurate results, but still very good results are obtained in the earlier stages of the first passage density function.

## APPENDIX

For one-dimensional Gaussian processes in the single-barrier problem with constant barrier level, the unconditional joint crossing rate  $f_{n+1}^{- \cdots +}(t_1, \dots, t_n, t)$  and the conditional crossing rate  $f_{n+1}^{+ \cdots +}(t_1, \dots, t_n, F = \{X(0) \in S_0\})$  can be calculated by the following expressions

$$f_{n+1}^{- \cdots +}(t_1, \dots, t_n, t) = \int_{-\infty}^0 \cdots \int_{-\infty}^0 \int_0^{\infty} (-1)^n \dot{x}_1 \cdots \dot{x}_n \dot{x}_{n+1} \varphi_{2n+2}(\bar{b}, \bar{x}; \bar{\mu}, \bar{\rho}) d\dot{x}_1 \cdots d\dot{x}_n d\dot{x}_{n+1} \quad (A1)$$

where  $\varphi_{2n+2}(\bar{b}, \bar{x}; \bar{\mu}, \bar{\rho})$  is the  $2n+2$  dimensional normal density function of  $\bar{X} = (X(t_1), \dots, X(t_n), X(t))$  and  $\bar{X} = (\dot{X}(t_1), \dots, \dot{X}(t_n), \dot{X}(t))$  evaluated at  $\bar{x} = \bar{b}$  and  $\dot{x} = \bar{x}$ .  $\bar{\mu}$  is the expected value of  $\bar{X}$  and  $\bar{X}$  and  $\bar{\rho}$  is the correlation coefficient matrix for  $\bar{X}$  and  $\bar{X}$ . If  $\rho(t)$  is the autocorrelation coefficient of the stationary Gaussian process  $\{X(t)\}$ ,  $\bar{\rho}$  is given by

$$\bar{\rho} = \begin{bmatrix} \bar{\rho}_{11} & | & \bar{\rho}_{12} \\ \hline \bar{\rho}_{12}^T & | & \bar{\rho}_{22} \end{bmatrix} \quad (A2)$$

where

$$\bar{\rho}_{11} = \begin{bmatrix} 1 & \rho(t_2 - t_1) & \cdots & \rho(t - t_1) \\ & 1 & \ddots & \vdots \\ \text{symm.} & \ddots & 1 & \rho(t - t_n) \\ & & & 1 \end{bmatrix}$$

$$\bar{\rho}_{12} = \begin{bmatrix} 0 & \rho'(t_2 - t_1) & \cdots & \rho'(t - t_1) \\ -\rho'(t_2 - t_1) & 0 & \ddots & \vdots \\ \vdots & \ddots & 0 & \rho'(t - t_n) \\ -\rho'(t - t_1) & \cdots & -\rho'(t - t_n) & 0 \end{bmatrix}$$

$$\bar{\bar{\rho}}_{22} = \begin{bmatrix} \rho''(0) & \rho''(t_2-t_1) & \dots & \rho''(t-t_1) \\ & \rho''(0) & & \vdots \\ & \dots & \dots & \vdots \\ \text{symm.} & & \rho''(0) & \rho''(t-t_n) \\ & & & \rho''(0) \end{bmatrix}$$

$$f_{n|F}^{+\dots+}(t_1, \dots, t_n) = \frac{1}{\Phi(b)} \int_{-\infty}^b \int_0^\infty \dots \int_0^\infty \dot{x}_1 \dots \dot{x}_n \varphi_{2n+1}(x, \bar{b}, \bar{x}; \bar{0}, \bar{\bar{\rho}}) dx d\dot{x}_1 \dots d\dot{x}_n \quad (\text{A3})$$

where  $\varphi_{2n+1}(x, \bar{b}, \bar{x}; \bar{\mu}, \bar{\rho})$  is the  $2n+1$  dimensional normal density function of  $X(0), \bar{X} = (X(t_1), \dots, X(t_n))$  and  $\bar{X} = (\dot{X}(t_1), \dots, \dot{X}(t_n))$  evaluated at  $x(0) = x, \bar{x} = \bar{b}$  and  $\bar{x} \cdot \bar{\rho}$  is

$$\bar{\bar{\rho}} = \begin{bmatrix} 1 & \bar{\bar{\rho}}_{12} & \bar{\bar{\rho}}_{13} \\ \bar{\bar{\rho}}_{12}^T & \bar{\bar{\rho}}_{22} & \bar{\bar{\rho}}_{23} \\ \bar{\bar{\rho}}_{13}^T & \bar{\bar{\rho}}_{23} & \bar{\bar{\rho}}_{33} \end{bmatrix} \quad (\text{A4})$$

where

$$\bar{\bar{\rho}}_{12} = [\rho(t_1) \dots \rho(t_n)]$$

$$\bar{\bar{\rho}}_{13} = [\rho'(t_1) \dots \rho'(t_n)]$$

$$\bar{\bar{\rho}}_{22} = \begin{bmatrix} 1 & \rho(t_2-t_1) & \dots & \rho(t_n-t_1) \\ & 1 & & \vdots \\ & \dots & \dots & \vdots \\ \text{symm.} & & 1 & \rho(t_n-t_{n-1}) \\ & & & 1 \end{bmatrix}$$

$$\bar{\rho}_{23} = \begin{bmatrix} 1 & \rho'(t_2-t_1) & \dots & \rho'(t_n-t_1) \\ -\rho'(t_2-t_1) & 0 & \dots & \vdots \\ \vdots & \dots & 0 & \rho'(t_n-t_{n-1}) \\ -\rho'(t_n-t_1) & \dots & -\rho'(t_n-t_{n-1}) & 0 \end{bmatrix}$$

$$\bar{\rho}_{33} = \begin{bmatrix} \rho''(0) & \rho''(t_2-t_1) & \dots & \rho''(t_n-t_1) \\ & \rho''(0) & \dots & \vdots \\ \text{symm.} & \dots & \rho''(0) & \rho''(t_n-t_{n-1}) \\ & & & \rho''(0) \end{bmatrix}$$

## LIST OF SYMBOLS

|                                     |   |
|-------------------------------------|---|
| $f_T(t)$                            | first passage probability density function  |
| $f_1^+(t)$                          | unconditional rate of outcrossings from $S_t$   |
| $S_t$                               | safe domain at the time $t$   |
| $E_t$                               | the event that an outcrossing takes place from $S_t$  |
| $\{X(t)\}$                          | stochastic process  |
| $F_L(\ell, t)$                      | distribution function of $L$  |
| $f_L(\ell, t)$                      | probability density function of $L$   |
| $L$                                 | length of time interval spent in the safe domain before outcrossing from $S_t$                  |
| $b$                                 | normalized barrier level  |
| $f_{n+1}^{--+}(t_1, \dots, t_n, t)$ | unconditional joint rate of incrossings to $S_{t_1}, \dots, S_{t_2}$ and outcrossing from $S_t$ |
| $A_{ E_t}(\ell \ell_1)$             | kernel in integral equation for $f_L(\ell, t)$  |
| $B_{ E_t}(\ell, \ell_1 \ell_2)$     | kernel in integral equation   |
| $K(t t_1)$                          | kernel in integral equation for $f_T(t)$  |
| $f_L^*(\ell, t)$                    | approximate solution to $f_L(\ell, t)$  |
| $F$                                 | the event that the samples are in the safe domain at time $t = 0$                               |
| $E[ \ ]$                            | expectation operator  |
| $\zeta$                             | damping ratio   |
| $\omega_0$                          | circular eigenfrequency   |
| $[a, b]$                            | closed interval from $a$ to $b$   |
| $]a, b[$                            | open interval from $a$ to $b$   |

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