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Publication date:
2010

Document Version
Early version, also known as pre-print

ISME2010-2570

Accuracy Enhanced Stability and Structure Preserving Model Reduction Technique for Dynamical Systems with Second Order Structure

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Abstract
A method for model reduction of dynamical systems with the second order structure is proposed in this paper. The proposed technique preserves the second order structure of the system, and also preserves the stability of the original systems. The method uses the controllability and observability gramians within the time interval to build the appropriate Petrov-Galerkin projection for dynamical systems within the time interval of interest. The bound on approximation error is also derived. The numerical results are compared with the counterparts from other techniques. The results confirm that the method is more accurate than the previous counterparts.

Keywords: Model Reduction, Second Order Structure, Linear Systems, Gramians

Introduction

The ever-growing need for accurate mathematical modeling of systems for simulation, analysis and control leads to models of high complexity. There is an increasing demand for efficient computational prototyping tools to replace such complex models by approximate simpler models, which are capable of capturing dynamical behavior and respecting essential properties of the complex models. Due to this fact model reduction methods have become increasingly popular over the last decades [1]-[3]. Such methods are designed to extract a reduced order state space model that adequately describes the behavior of the system in question. On the other hand, in a lot of fields of engineering like mechanical systems, Micro-Electro-Mechanical Systems, earthquake engineering, civil engineering and aeronautics, systems are typically modeled by sets of second order ordinary differential equations. These models are usually very complicated in the sense of the large number of differential equations to describe the dynamical behavior. Therefore model reduction is a promising technique to tackle complexity of such models [4]. This problem requires a model reduction technique which respects stability and second order structure of the original system. If the reduction method does not preserve the second order structure of the original system makes the physical interpretation difficult. There are two main categories of model reduction methods, namely singular values decomposition (SVD) based and moment matching based reduction techniques. Methods from the first category are usually globally more accurate than the latter. Balanced model reduction method is an example from the class of SVD based reduction methods. These reduction methods preserve the stability of original systems and also provide bound for approximation error [1],[2] .To apply balanced reduction, first the system is represented in a basis where the states which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the controllability and the observability gramians, which are solutions to the controllability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. Methods from the family of moment matching based techniques are usually computationally more efficient than standard SVD based algorithms for reduction. In this paper we propose a method for model reduction of the dynamical systems with second order structure. In this method we build the projection matrices based on gramians within time interval. Gramians within time interval and frequency bounds was proposed[5] in and further improved in[2]. Model reduction methods based on time interval or frequency bound are more accurate than the methods based on ordinary gramians. Therefore it is desirable to devise reduction methods based on this kind of gramians. There have been some research focused on developing limited time or bounded frequency reduction methods from different aspects [2][5]-[14]. To the best of our knowledge this paper is the first attempt to devise a reduction method based on limited time gramians for dynamical systems with second order structure. In this method we use Petrov-Galerkin projection instead of balancing transformation for reduction. The numerical algorithm is more computationally stable as a result. The stability and structure preservation are also considered in the reduction procedure. The method is successfully applied to CD-player practical example. The results are more accurate than the methods based on ordinary gramians.

Model Reduction of Second order Systems Based on Limited Time Gramians

Consider a dynamical system with the following second order structure:

$$\begin{cases} \bar{E}\ddot{\bar{x}} + \bar{F}\dot{\bar{x}} + \bar{G}\bar{x} = \bar{B}u\,, \\ y = \bar{C}\bar{x}\,, \end{cases} \tag{1}$$

where $\bar{E}, \bar{F}, \bar{G} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$ and $\bar{C} \in \mathbb{R}^{n\times p}$.

The goal is to approximate the dynamical system described by (1) to lower dimensions.

It is easier to treat dynamical systems in ordinary state-space form and there are more tools available for state-space form. In first step we find the standard state-space form of system described by (1):

$$\begin{cases} E\dot{x} = \hat{A}x + \hat{B}u, \\ y = Cx, \end{cases} \quad , \quad x := \begin{bmatrix} \bar{x} \\ \dot{\bar{x}} \end{bmatrix} \in \mathbb{R}^{2n} \tag{2}$$

where :

$$E := \begin{bmatrix} I & 0 \\ 0 & \bar{E} \end{bmatrix}, \hat{A} := \begin{bmatrix} 0 & I \\ -\bar{G} & -\bar{F} \end{bmatrix}, B := \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix}$$

$$C := \begin{bmatrix} \bar{C} & 0 \end{bmatrix}$$

It is easier to deal with systems in the form:

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{3}$$

which is the standard state-space equivalent form obtained from (2).

We are ready now to go for reduction step. First we need to define gramians within limited time[1][2][5]:

$$\begin{aligned} P[t_1,t_2] &:= \int_{t_1}^{t_2} e^{A\tau} BB^T e^{A^T\tau} d\tau, \\ Q[t_1,t_2] &:= \int_{t_1}^{t_2} e^{A\tau} C^T C e^{A^T\tau} d\tau, \end{aligned} \tag{4}$$

Let $\theta_c(t) := \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau$, we have:

$$P[t_1,t_2] = \theta_c(t_2) - \theta_c(t_2) = \int_0^\infty e^{A\tau}\, V_c[t_1,t_2]\, e^{A^T\tau} d\tau, \tag{5}$$

where:

$$V_c[t_1,t_2] := e^{At_1} BB^T e^{A^T t_1} - e^{At_2} BB^T e^{A^T t_2}\,. \tag{6}$$

Similarly we have:

$$Q[t_1,t_2] = \int_0^\infty e^{A^T\tau}\, V_o[t_1,t_2]\, e^{A\tau} d\tau, \tag{7}$$

where:

$$V_o[t_1,t_2] := e^{A^T t_1} C^T C e^{At_1} - e^{A^T t_2} C^T C e^{At_2}\,. \tag{8}$$

The gramians satisfy the following Lyapunov equations[1],[2]:

$$\begin{aligned} AP[t_1,t_2] + P[t_1,t_2]A^* + V_c[t_1,t_2] &= 0 \\ A^*Q[t_1,t_2] + Q[t_1,t_2]A + V_o[t_1,t_2] &= 0 \end{aligned} \tag{9}$$

Note that in this framework $V_c[t_1,t_2]$ and $V_o[t_1,t_2]$ are not necessarily positive definite. This method is modified in [2] to guarantee stability and to provide a simple error bound. The modified version starts with EVD of $V_c[t_1,t_2]$ and $V_o[t_1,t_2]$:

$$\begin{aligned} V_c[t_1,t_2] &:= M\Lambda M^* = M\, diag(\lambda_1,...,\lambda_n)M^*, \\ V_o[t_1,t_2] &:= N\Delta N^* = N\, diag(\delta_1,...,\delta_n)N^*, \end{aligned} \tag{10}$$

where:

$$MM^* = NN^* = I_n\,, |\lambda_i| \ge |\lambda_{i+1}| \ge 0, |\delta_i| \ge |\delta_{i+1}| \ge 0\,.$$

Note that since $V_c[t_1,t_2]$ and $V_o[t_1,t_2]$ are symmetric decompositions in the form (10) exist. Let:

$$\begin{aligned} \hat{B} &:= M\ diag(|\lambda_1|^{1/2},...,|\lambda_\xi|^{1/2},0,...,0) \\ \hat{C} &:= diag(|\delta_1|^{1/2},...,|\delta_\rho|^{1/2},0,...,0)\,N^* \end{aligned} \tag{11}$$

where:

$$\begin{aligned} \xi &= rank(V_c[t_1,t_2]) \\ \rho &= rank(V_o[t_1,t_2]) \end{aligned} \tag{12}$$

The modified gramians satisfy the following Lyapunov equations instead of (9):

$$\begin{aligned} A\hat{P}[t_1,t_2] + \hat{P}[t_1,t_2]A^* + \hat{B}\hat{B}^* &= 0 \\ A^*\hat{Q}[t_1,t_2] + \hat{Q}[t_1,t_2]A + \hat{C}^*\hat{C} &= 0 \end{aligned} \tag{13}$$

The reduced order model can be obtained by simultaneously diagonalizing gramians and then by truncating the states associated to the set of the least generalized Hankel singular values. Balanced transformation can be ill-conditioned numerically when dealing with the systems with some nearly uncontrollable modes or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [15] draw some attentions to devise alternative numerical methods [16]. Balancing can be a badly conditioned even when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian based framework without balancing at all. Schur method and Square root algorithms provides projection matrices to apply balanced reduction without balanced transformation[1][16]. In our method we use the same algorithm by plugging limited time gramians into the algorithm instead of ordinary gramians and we find the Petrov-Galerkin projection matrices for reduction. Note that the results are the same as balancing algorithm. A merit of the Square Root method is that it relies on the Cholesky factors of the gramians rather than the gramians themselves, which has advantages in terms of numerical stability.

Petrov-Galerkin projection for a dynamical system[1]:

$$\begin{cases} \dot{x}(t) = f(x(t),u(t)) \quad , \quad x \in \mathbb{R}^n \quad , \\ y(t) = g(x(t),u(t)) \quad , \end{cases} \tag{14}$$

is defined as a projection $\Pi = VW^*$,where: $W^*V = I_k$, $V,W \in \mathbb{R}^{n\times k}$, $k < n$.

The reduced order model using this projection is:

$$\begin{cases} \dot{\hat{x}}(t) = W^* f(V\hat{x}(t),u(t)) \quad , \hat{x} \in \mathbb{R}^k\,, \\ y(t) = g(V\hat{x}(t),u(t)). \end{cases} \tag{15}$$

In the sequel we discuss ways for recovering the second order structure. The stability preservation is also proved. We also discussed the approximation error bound.

Second Order Structure:

There are two ways for recovering the second order structure of systems. The first approach is applicable to the reduced order systems which are of even orders with zero first Markov parameter. This method is first proposed in [4]. The equivalent second order structure for reduced order model:

$$\begin{cases} \dot{x}_r = A_r x_r + B_r u, \\ y = C_r x_r, \end{cases} \qquad x_r \in \mathbb{R}^{2k} \quad , \tag{16}$$

is:

$$\begin{cases} \bar{E}_r \ddot{z} + \bar{F}_r \dot{z} + \bar{G}_r z = \bar{B}_r u \ , \\ y = \bar{C}_r z \ , \end{cases} \tag{17}$$

where:

$$\begin{aligned} & RB_r = 0, \quad R \in \mathbb{R}^{(k-p)\times 2k}, \\ & C_z := \begin{bmatrix} C_r \\ R \end{bmatrix}, \quad T := \begin{bmatrix} C_z \\ C_z A_r \end{bmatrix}^{-1}, \end{aligned} \tag{18}$$

$$\begin{aligned} & \bar{E}_r = I, \quad \bar{B}_r = C_z A_r B_r, \\ & \begin{bmatrix} -\bar{G}_r & -\bar{F}_r \end{bmatrix} = C_z A^2{}_r T, \\ & \bar{C}_r = [I \quad 0 \quad \ldots \quad 0]. \end{aligned} \tag{19}$$

The second method is less conservative and was presented in [14]. In this method the following second order equivalent is suggested for the reduced order model:

$$\begin{aligned} & Q\ddot{x}_r + Q\dot{x}_r - (QA_r{}^2 + QA_r)x = (QA_r B_r)u, \\ & y = C_r x, \end{aligned} \tag{20}$$

where $QB = O$ **.**

More details on these two methods are available in [4][14].

Stability and Error Bound:

One of the important issues in model reduction is preservation of the stability. In other words, the question is whether the reduction method can preserve the stability of the original model in approximation. In the following proposition we show that the proposed framework for model reduction is stability preserving model reduction method.

Proposition 1. If the original system described is stable, the reduced order model by the proposed technique is guaranteed to be quadratic stable.

Proof:

Consider the standard sate-space equivalent of the second order structure.In the proposed method:

$$W^* V = I_k \ , \quad V, W \in \mathbb{R}^{n\times k}, k < n$$

$$\hat{\Sigma} : \begin{cases} \hat{A}_r = W^* A V \\ \hat{B}_r = W^* B \\ \hat{C}_r = CV \\ \hat{D} = D \end{cases} \tag{21}$$

which is projected system (reduced order model). The outcome of Square root algorithm for projection[1]: $\hat{P}[t_1, t_2]W = V\Sigma_1$ and $\hat{Q}[t_1, t_2]V = W\Sigma_1$, where $\Sigma_1 \in \mathbb{R}^{k\times k}$ is diagonal and positive definite. From (13) we have: $A\hat{P}[t_1, t_2] + \hat{P}[t_1, t_2]A^* < 0$, which implies:

$$W^*(A\hat{P}[t_1, t_2] + \hat{P}[t_1, t_2]A^*)\, W < 0.$$

On the other hand,

$$\begin{aligned} & W^*(A\hat{P}[t_1, t_2] + \hat{P}[t_1, t_2]A^*)\, W = W^* A\hat{P}[t_1, t_2]W + W^*\hat{P}[t_1, t_2]A^*\, W \\ & = W^* AV\Sigma_1 + \Sigma_1{}^* V^* A^*\, W = \hat{A}_r \Sigma_1 + \Sigma_1 \hat{A}_r^*, \end{aligned}$$

Hence:

$$\hat{A}_r \Sigma_1 + \Sigma_1 \hat{A}_r^* < 0. \tag{22}$$

where $\Sigma_1 \in \mathbb{R}^{k\times k}$ is positive definite.

Hence, reduced order model is guaranteed to be quadratic stable with the Lyapunov function: $x_r{}^T \Sigma_1 x_r$.

□

The error of approximation is bounded and is given in terms of Hankel singular values which are the diagonal elements of the limited time gramians in the balanced form. More details are discussed in [1][2].

Practical CD Player Benchmark Example

One of the well-known practical applications of model order reduction is in the control of CD player systems. The scheme of CD player mechanism is shown in Fig. 1. The control task is to achieve track following, which basically amounts to pointing the laser spot to the track of pits on the CD that is rotating.

We apply the proposed method to a strictly proper SISO practical CD player model of order 120. This CD player model is a finite element model of the dynamics between the lens actuator and radial arm position of a portable compact disc.

This model is reduced to 12th order model by the proposed method. The reduced model is more accurate than 12th order model reduced by balanced truncation. Figure 2 shows the Hankel singular values of CD player model associated to the proposed method over the time interval: $[0, 0.1]$ **.** The decay rate of Hankel singular values shows that the best approximation over this time interval is to reduce the model to 12 states. The step response error for the ordinary balanced reduction and the proposed method is presented in Figure 3. It is clear from Figure 3 that the proposed method is more accurate than the ordinary balanced truncation.

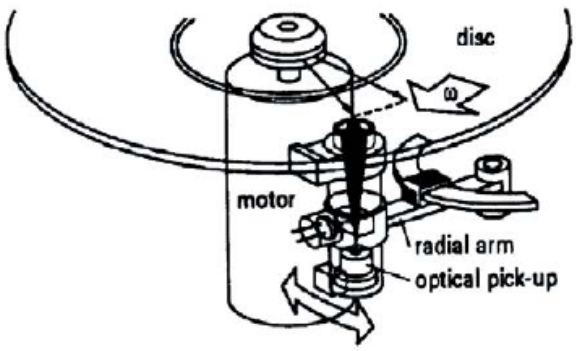

Figure 1: CD Player

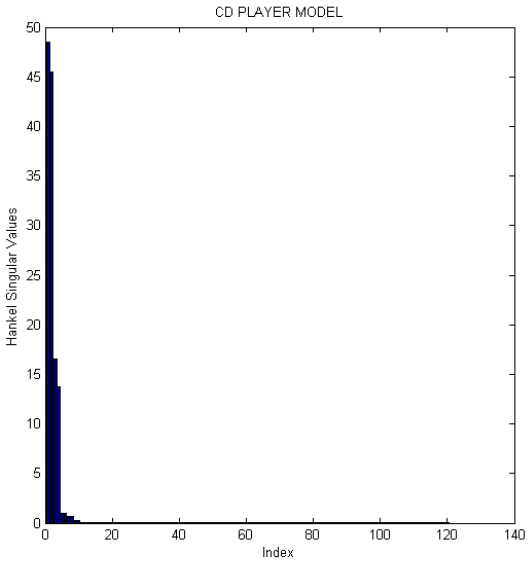

Figure 2: Hankel Singular Values (HSV) of CD Player Model

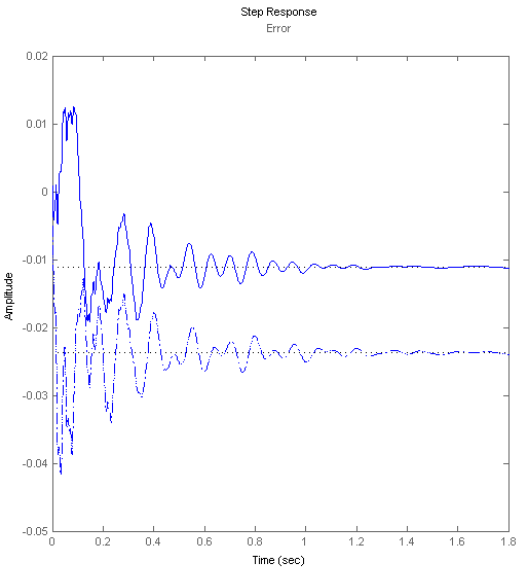

Figure 3: Step response error in ordinary balanced reduction (solid) and the proposed method (dash dotted)

Conclusions

In this paper, a framework for model reduction of dynamical systems with second order structure has been presented. The method is based on balanced reduction over time horizon. The proposed framework preserves the stability and the structure of dynamical system and it provides more accurate results than ordinary balanced reduction. The method is from gramian based reduction methods which are generally computationally expensive. Petrov-Galerkin projection based on squre root algorithm has been used to improve the efficiency and numerical stability. There are several ways to improve computational efficiency further like parallel computations and low rank gramians based computations. The method can be further extended for reduction of nonlinear systems using trajectory piece-wise linearization (TPWL).

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