Beyond Multiplexing Gain in Large MIMO Systems

Cakmak, Burak; Müller, Ralf R.; Fleury, Bernard Henri

Publication date:
2013

Document Version
Early version, also known as pre-print

Beyond Multiplexing Gain in Large MIMO Systems

Burak Çakmak, AAU Denmark, Ralf R. Müller, FAU Germany, Bernard H. Fleury, AAU Denmark

***Abstract*—Given the common technical assumptions in the literature on MIMO channel modeling, we derive generic results for MIMO systems in the large system limit LSL. Consider a $\phi T \times T$ MIMO system with T tending to infinity. By increasing the antenna ratio ϕ when $\phi \geq 1$, the amount of capacity increase per receive antenna converges to the binary entropy function of the antenna ratio $1/\phi$ at high SNR. We also show this "binary entropy increase" for $\phi < 1$. Furthermore, we define the deviation of the effective capacity growth from the traditionally assumed linear growth (multiplexing gain). Even when the channel entries are i.i.d. the deviation from the linear growth is significant. We also find an additive property of the deviation for a concatenated MIMO system. Finally, we quantify the deviation of the large SNR capacity from the exact capacity and find an accurate approximation of it that is easy to calculate.**

I. Preliminary Notations

NOTATION 1 *We denote the binary entropy function (BEF) as*

$$H(p) = (p-1)\log_2(1-p) - p\log_2 p, \quad p \in [0,1]$$

with the convention $0\log_2 0 = 0$.

NOTATION 2 *Consider a $N \times \beta N$ random matrix $\boldsymbol{X}$. Assume that as $N \to \infty$ with the ratio β fixed, $\boldsymbol{X}\boldsymbol{X}^\dagger$ has a limiting eigenvalue distribution (LED) which is denoted by $\mu_{\boldsymbol{X}}$.*

DEFINITION 1 *A projector $\boldsymbol{P}_\beta$ is a diagonal matrix $\boldsymbol{P}_\beta \in \{0,1\}^{N\times N}$ with the ratio $\beta = tr(\boldsymbol{P}_\beta)/N$ fixed as $N \to \infty$.*

II. Introduction

Consider an $\phi T \times T$ multiple-input–multiple-output (MIMO) system with independent identically distributed (i.i.d.) inputs. Assume that the empirical singular value distribution of channel matrix $\boldsymbol{H}$ converges to a limit distribution as T tends to infinity[1]. We define the normalized rank measure of $\boldsymbol{H}\boldsymbol{H}^\dagger$ as $\alpha \triangleq 1 - \mu_{\boldsymbol{H}}(0)$ and introduce the *non-zero probability distribution*

$$\tilde{\mu}_{\boldsymbol{H}}(x) \triangleq (1-1/\alpha)u(x) + \mu_{\boldsymbol{H}}(x)/\alpha \tag{1}$$

with $u(x)$ denoting the unit-step function. Further, let γ denote the signal-to-noise ratio (SNR). With channel state information known only at the receiver, the mutual information (MU) per receive antenna can be expressed as

$$\underbrace{\alpha\int_0^\infty \log_2(\gamma x)d\tilde{\mu}_{\boldsymbol{H}}(x)}_{\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}})} + \underbrace{\alpha\int_0^\infty \log_2\left(1+\frac{1}{x\gamma}\right)d\tilde{\mu}_{\boldsymbol{H}}(x)}_{\mathcal{I}_\sim(\gamma;\mu_{\boldsymbol{H}})}. \tag{2}$$

The first summand $\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}})$ is a lower bound of the MU; it gives exact capacity at high SNR, so we call it the high SNR lower bound. The second summand $\mathcal{I}_\sim(\gamma;\mu_{\boldsymbol{H}})$ is the deviation of the lower bound from the exact MU.

[1]For the sake of brevity, throughout the paper we sacrifice the specification of the probability measures for the finite size matrices.

III. Binary Entropy Increase

RESULT 1 *Consider a MIMO system with ϕT receive and T transmit antennas. Let the $M \times M$ matrix $\boldsymbol{P}_\beta$ be a projector with $M \triangleq \max(T, \phi T)$ and $\beta \triangleq \min(\phi, 1/\phi)$. Assume that the channel matrix $\boldsymbol{H}$ has full rank with probability one. Furthermore let the dimension T tend to infinity with $\boldsymbol{H}$ having a limiting singular value distribution and being free of $\boldsymbol{P}_\beta$. Then we have*

$$\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}}) - \mathcal{I}_-(\gamma;\mu_{\boldsymbol{P}_\beta\boldsymbol{H}}) = H(\beta); \quad \phi \geq 1$$

$$\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}^\dagger}) - \mathcal{I}_-(\gamma;\mu_{\boldsymbol{P}_\beta\boldsymbol{H}^\dagger}) = H(\beta); \quad \phi < 1.$$

In other words, in the LSL, when $\phi \geq 1$ ($\phi < 1$) increasing the antenna ratio ϕ ($1/\phi$) for a given number of transmit (receive) antennas, the amount of capacity increase per receive (transmit) antenna converges to the BEF of the antenna ratio β at high SNR.

IV. Deviation from the Linear Capacity Growth

It is commonly admitted that at high SNR, with the number of receive antenna kept constant the capacity of a MIMO system grows linearly with the number of transmit antennas[2], see e.g. [1]. This statement is obvious when the channel matrix has orthogonal columns. However, when the matrix has i.i.d. entries for instance, a significant crosstalk arises due to the lack of orthogonality of its columns and cross-talk is a quite non-linear phenomenon. The example in Section V shows that in this case (i.i.d. entries) the deviation from the linear growth is significant. In this section, we investigate the behavior of the deviation from the linear growth.

Let the channel matrix $\boldsymbol{H} \in \mathbb{C}^{R\times R}$ have a limiting singular value distribution and be asymptotically free of an $R \times R$ projector $\boldsymbol{P}_\beta$ as $R \to \infty$. Then we introduce the deviation of the linear capacity growth in the LSL as

$$\Delta\mathcal{L}_{\boldsymbol{H}\boldsymbol{P}_\beta} \triangleq \mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}\boldsymbol{P}_\beta}) - \beta\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}}). \tag{3}$$

Notice that $\mathcal{I}_-(\gamma;\mu_{\boldsymbol{H}})$ in (3) corresponds to the growth rate of the multiplexing gain β (per receive antenna).

RESULT 2 *Consider an almost surely full-rank random matrix $\boldsymbol{H} = \boldsymbol{X}\boldsymbol{Y}$ with $\boldsymbol{X} \in \mathbb{C}^{R\times R}$ and $\boldsymbol{Y} \in \mathbb{C}^{R\times R}$. Let the matrices $\boldsymbol{X}$, $\boldsymbol{Y}$ and the $R \times R$ projector $\boldsymbol{P}_\beta$ have a LED each and be asymptotically free of each other as $R \to \infty$. Then we have*

$$\Delta\mathcal{L}_{\boldsymbol{H}\boldsymbol{P}_\beta} = \Delta\mathcal{L}_{\boldsymbol{X}\boldsymbol{P}_\beta} + \Delta\mathcal{L}_{\boldsymbol{Y}\boldsymbol{P}_\beta}.$$

[2]For the sake of brevity, in this section we assume the number of transmit antenna less than the number of receive antennas. The generalization is straightforward.

V. Example

We consider the random matrix

$$\boldsymbol{H} = \prod_{n=1}^{N} \boldsymbol{A}_n \tag{4}$$

where the $R \times R$ matrices $\boldsymbol{A}_n$, $n = 1, \ldots, N$, have i.i.d entries with zero mean and variance $1/R$. Then as R tends to infinity the deviation from the linear capacity growth reads

$$\Delta\mathcal{L}_{\boldsymbol{H}\boldsymbol{P}_\beta} = N\left(H(\beta) + \beta \log_2 \beta\right). \tag{5}$$

Indeed, the entries of the product of two matrices with i.i.d. entries are not i.i.d. anymore, but correlated. Furthermore, as N in (4) increases, so does the correlation between the entries of $\boldsymbol{H}$, implying that the cross-talk increases as well.

VI. The Non-High SNR Deviation

Calculation of the high SNR lower bound for a given channel model is often analytically tractable. However the high SNR lower bound in itself is a crude approximation of the capacity. On the other hand, calculation of the deviation of the high SNR lower bound from the exact capacity, i.e. $\mathcal{I}_\sim(\gamma; \mu_{\boldsymbol{H}})$ in (2), is usually, though not always, analytically intractable. We derive in this section an analytical approximation of this deviation that can then be used in combination with the high SNR lower bound to obtain an approximation of the capacity. To this end we introduce the parameters

$$m \triangleq \int x d\tilde{\mu}_{\boldsymbol{H}}(x); \quad \hat{m} \triangleq \frac{1}{\int x^{-1} d\tilde{\mu}_{\boldsymbol{H}}(x)} \tag{6}$$

where $\hat{m}$ is known in the literature as the harmonic mean of the distribution $\tilde{\mu}_{\boldsymbol{H}}$. We further note that, due to Lemma 2&4 in [2], we have $\hat{m} < m$ as $\tilde{\mu}_{\boldsymbol{H}}$ is not a Dirac measure.

Result 3 *Let $\mu_{\boldsymbol{H}}$ in (1) be not a Bernoulli distribution and have a fixed mean. Define $\lambda = \frac{m}{m-\hat{m}}$ and $\gamma' = (m - \hat{m})\gamma$. Furthermore let μ_{MP} be the Marčenko-Pastur distribution with the rate parameter λ. Then we have*

$$\mathcal{I}_\sim(\gamma; \mu_{\boldsymbol{H}}) \approx \alpha \mathcal{I}_\sim(\gamma'; \mu_{\text{MP}}) \tag{7}$$

such that

$$\mathcal{I}_\sim(\gamma; \mu_{\text{MP}}) = \mathcal{I}(\gamma; \mu_{\text{MP}}) - \log_2 \gamma - \lambda H\left(\frac{1}{\lambda}\right) + 1. \tag{8}$$

The term $\mathcal{I}(\gamma; \mu_{\text{MP}})$ in (8) is the well-known large system capacity expression for i.i.d. zero-mean fading coefficients. We refer to [3, Eq. (9)] for a closed form expression of it. In addition, further analytical expositions by using some facts presented in [2] shows that under some technical conditions (not reported here for the sake of the space), (7) gives a lower bound of $\mathcal{I}_\sim(\gamma; \mu_{\boldsymbol{H}})$. Finally we define the approximation of the capacity in (2) as

$$\mathcal{I}_{\cong}(\gamma; \mu_{\boldsymbol{H}}) \triangleq \mathcal{I}_-(\gamma; \mu_{\boldsymbol{H}}) + \alpha \mathcal{I}_\sim(\gamma'; \mu_{\text{MP}}) \tag{9}$$

with $\mathcal{I}_\sim(\gamma'; \mu_{\text{MP}})$ given in (7).

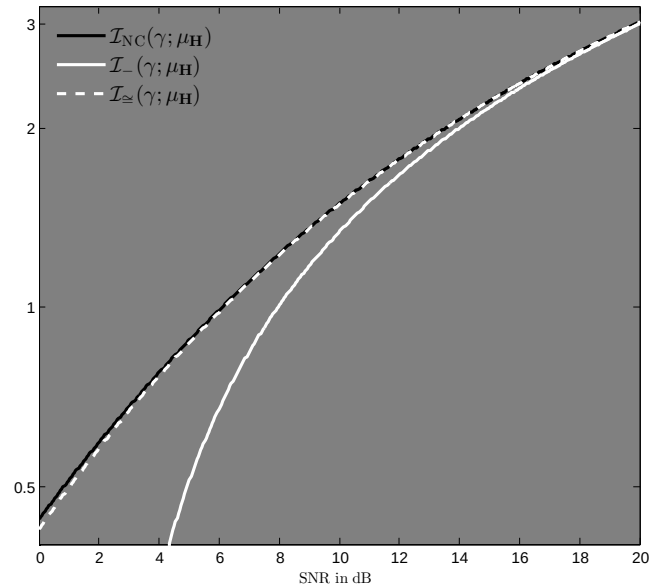

Fig. 1. Approximation of the capacity of the concatenated scattering channels [4] with the parameter setting $\rho_0 = 1/2$, $\rho_1 = 2$ so that $m = 1, \hat{m} = 0.338$.

A. Analytical and Numerical Comparison

To validate the analytical results, we consider the concatenated scattering channel model proposed in [4], $\boldsymbol{H} = \boldsymbol{X}\boldsymbol{Y}$, where the entries of the $R \times S$ matrix $\boldsymbol{X}$ and the $S \times T$ matrix $\boldsymbol{Y}$ are i.i.d with zero mean and variance $1/\sqrt{RS}$. We define the system parameters $\rho_0 = T/R$, $\rho_1 = S/R$ and $\rho_2 = 1$, the latter parameter being auxiliary. We calculate the high SNR lower bound in the LSL (as $R, S, T \to \infty$) to be

$$\begin{aligned} \mathcal{I}_-(\gamma; \mu_{\boldsymbol{H}}) &= H(\alpha) + \alpha(\log_2 \gamma - 2) + \\ &\quad \sum_{n=1}^{2} \rho_{n-1} H\left(\frac{\alpha}{\rho_{n-1}}\right) + \alpha \log_2\left(\frac{\alpha}{\rho_n}\right) \end{aligned}$$

with $\alpha = \min(\rho_0, \rho_1, \rho_2)$. To approximate $I_\sim(\gamma; \mu_{\boldsymbol{H}})$ by using Result 3, we compute m and $\hat{m}$ for $\boldsymbol{H}$ to be

$$m = \frac{\rho_0}{\alpha}; \quad \hat{m} = \frac{\rho_0}{\alpha} \prod_{n \in \mathcal{N}_\alpha} \frac{\rho_n - \alpha}{\rho_n}$$

with $\mathcal{N}_\alpha \triangleq \{\{0, 1, 2\} \backslash \{i\} : \rho_i = \alpha\}$. In Figure 1, we compare the capacity approximation $\mathcal{I}_{\cong}$ with the exact one, denoted as $\mathcal{I}_{\text{NC}}$, obtained by *numerical computation* [4, Eq. (54)]. We chose a parameter setting that has a practical relevance, see the caption of Figure 1. Note that $\mathcal{I}_{\cong}$ and $\mathcal{I}_{\text{NC}}$ are visually indistinctive. In general, $\mathcal{I}_{\cong}$ gives a tight lower bound of $\mathcal{I}_{\text{NC}}$.

References

[1] C. Chuah, D. Tse, J. Kahn, and R. Valenzuela, "Capacity scaling in MIMO wireless systems under correlated fading," *IEEE Transactions on Information Theory*, vol. 48(3), 2002.

[2] U. Haagerup and S. Möller, "The law of large numbers for the free multiplicative convolution," *arXiv:1211.4457v1*, November 2012.

[3] S. Verdú and S. Shamai, "Spectral efficiency of CDMA with random spreading," *IEEE Transactions on Information Theory*, vol. 45, pp. 622–640, March 1999.

[4] R. R. Müller, "A random matrix model for communication via antenna arrays," *IEEE Transactions on Information Theory*, vol. 48, no. 9, pp. 2495–2506, September 2002.