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Publication date:
2013

Document Version
Early version, also known as pre-print

Link to publication from Aalborg University

Citation for published version (APA):
Johnsen, J., Hansen, S. M., & Sickel, W. (2013). *Anisotropic Lizorkin-Triebel spaces with mixed norms -- traces on smooth boundaries*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2013-07

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by

J. Johnsen, S. Munch Hansen and W. Sickel

R-2013-07 August 2013

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 99 40 ▪ Telefax: +45 99 40 35 48
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

ANISOTROPIC LIZORKIN–TRIEBEL SPACES WITH MIXED NORMS — TRACES ON SMOOTH BOUNDARIES

J. JOHNSEN, S. MUNCH HANSEN, W. SICKEL

AUGUST 23, 2013

ABSTRACT. This article deals with trace operators on anisotropic Lizorkin–Triebel spaces with mixed norms over cylindrical domains, where the boundary is sufficiently smooth. As a preparation we include a rather self-contained exposition of Lizorkin–Triebel spaces on manifolds and extend these results to mixed-norm Lizorkin–Triebel spaces on cylinders in Euclidean space.

1. INTRODUCTION

The present paper departs from the work [JS08] of the first and third author dealing with traces on hyperplanes of anisotropic Lizorkin–Triebel spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with mixed norms.

The application of such spaces to parabolic differential equations is to some extent known. It was outlined in the introduction to [JS08] how they apply to fully inhomogeneous initial and boundary value problems: for such problems the $F^{s,\vec{a}}_{\vec{p},q}$-spaces are in general *inevitable* for a correct description of the boundary data. Previously, a somewhat similar conclusion had been obtained in works of Weidemaier [Wei98, Wei02, Wei05] (and also by Denk, Hieber and Prüss [DHP07]). He discovered the necessity of isotropic Lizorkin–Triebel spaces (for vector-valued functions) for an optimal description of the time regularity of the boundary data. However, with integral exponents p_x and p_t in the space and time directions, respectively, Weidemaier worked under the technical restriction that $p_x \le p_t$.

For the reader's sake, it is recalled that the main purpose of [JS08] was to extend the classical theory of trace operators to the $F^{s,\vec{a}}_{\vec{p},q}$-scales. However, because the mixed norms do not allow a change of integration order, this meant that the techniques had to be worked out both for the 'inner' and 'outer' traces given on, say smooth functions as

$$u(x_1,x'') \mapsto u(0,x''), \quad \text{resp.} \quad u(x',x_n) \mapsto u(x',0).$$

When $u \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, then in the first case the trace was proved to be surjective on the mixed-norm Lizorkin–Triebel space $F^{s-a_1/p_1,a''}_{p'',p_1}(\mathbb{R}^{n-1})$ having the specific sum exponent $q = p_1$, while in the second case the trace space is (as usual) a Besov space, namely $B^{s-a_n/p_n,a'}_{p',p_n}(\mathbb{R}^{n-1})$.

As indicated, only traces on hyperplanes were covered in [JS08]; but the study included (almost) necessary and sufficient conditions on s in relation to $\vec{a}$, $\vec{p}$ and q, also in combination with normal derivatives (Cauchy traces), and existence and continuity of right-inverses. Furthermore, Weidemaier's restriction on the integral exponents was never encountered with the framework and methods adopted in [JS08].

These investigations in [JS08] are in this work followed up with a general study of trace operators and their right-inverses in the scales $F^{s,\vec{a}}_{\vec{p},q}$ of anisotropic Lizorkin–Triebel spaces with mixed norms defined on smooth cylinders $\Omega \times I$ and their curved boundaries $\partial\Omega \times I$.

Key words and phrases. Trace operators, mixed norms, cylindrical domains, parabolic boundary problems.

In doing so, it is a main technical question to obtain invariance of the spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ under the map $f \mapsto f \circ \sigma$, when $U \subset \mathbb{R}^n$ is open and σ is a C^∞-bijection. We addressed this question in our joint paper [JSH13b], where we proved invariance e.g. under the restriction that σ only affects groups of coordinates x_j for which the corresponding p_j are equal in the vector of integral exponents $\vec{p} = (p_1, \dots, p_n)$; and similarly for the a_j.

This was done by generalising Triebel's method in [Tri92, 4.3.2]. Indeed, having reduced to large s using a lift operator, it relies on Taylor expansion of the inner and outer functions, whereby most terms are manageable when the $F^{s,\vec{a}}_{\vec{p},q}$-spaces are normed via kernels of local means developed in [JSH13a]; an underlying parameter-dependent estimate obtained in [JSH13a] finally gives control over the effects of the Jacobian matrices.

In this paper, we develop the consequences for trace operators. E.g. the trace r_0 at $\{t = 0\}$ of $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I)$, where $I :=]0, T[$, is given a meaning in a pedestrian way using an arbitrary extension of u to $\mathbb{R}^{n+1}$ and applying the trace at $\{t = 0\}$ from [JS08]. We obtain, using the splitting $\vec{p} = (p', p_t)$ with all entries in p' being equal and likewise for $\vec{a}$, the following, cf. Theorem 6.1 below:

Theorem. *For s sufficiently large (cf.* (61) *below) the operator* r_0 *is a bounded* surjection*,*

$$r_0 : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega).$$

Furthermore, r_0 *has a right-inverse* K_0 *going the opposite way and it is bounded for every* $s \in \mathbb{R}$*,*

$$K_0 : \overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I).$$

The process of giving meaning to the curved trace γ of u is more involved, since it requires to first work locally and then observing that the local pieces define a global trace. After this has been done, we obtain using the splitting $\vec{p} = (p_1, p'')$, where $p_1 = \ldots = p_n$ and likewise for $\vec{a}$, cf. Theorem 6.9 below:

Theorem. *When* $\partial\Omega$ *is compact and s is sufficiently large (cf.* (76)*), the operator* γ *is a bounded* surjection*,*

$$\gamma : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I) \to \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I).$$

Furthermore, γ *has a right-inverse* K_γ *going the opposite way and it is bounded for every* $s \in \mathbb{R}$*,*

$$K_\gamma : \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma \times I) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega \times I).$$

The operator K_γ is constructed using the right-inverse in [JS08, Thm. 2.6] to the trace at $\{x_1 = 0\}$ and Rychkov's universal extension operator in [Ryc99], which is modified such that it applies to anisotropic, mixed-norm Lizorkin–Triebel spaces over half-spaces, cf. Section 5.

We also give in Theorem 6.8 an *explicit* construction of a right-inverse Q_Ω (of r_0 on $\mathbb{R}^{n+1}$) having the support preserving property

$$Q_\Omega : \overset{\circ}{B}{}^{s,a'}_{p',p_t}(\overline{\Omega}) \to \overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\overline{\Omega} \times \mathbb{R}) \quad \text{for all } s \in \mathbb{R}.$$

Finally, we analyse in Sections 6.4–6.5 traces at the curved corner $\Gamma \times \{0\}$ associated to $\Omega \times I$ and follow up by giving the resulting compatibility properties for solutions of the heat equation.

Contents. Section 2 contains a review of our notation and the definition of anisotropic Lizorkin–Triebel spaces with mixed norms is recalled, together with some needed properties and a pointwise multiplier assertion. Moreover, a basic lemma for elements in $F^{s,\vec{a}}_{\vec{p},q}$ with compact support on cross sections of the cylindrical domain is proved.

In Section 3 sufficient conditions for $f\mapsto f\circ\sigma$ to leave the spaces $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ invariant for a certain range of the parameters, including negative values of s, are recalled.

Section 4 contains first a preparatory treatment of unmixed Lizorkin–Triebel spaces on general C^∞-manifolds and these results are then extended to $F^{s,\vec{a}}_{\vec{p},q}$-spaces on the curved boundary of a cylinder.

Rychkov's universal extension operator in [Ryc99] is modified to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)$ in Section 5. Moreover, its properties on temperate distributions are analysed in addition.

Finally, Section 6 contains a discussion of the trace at the flat as well as at the curved boundary of a cylindrical domain, including some applications to e.g. the Dirichlet boundary problem for the heat equation.

2. Preliminaries

2.1. **Notation.** The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ consists of the rapidly decreasing C^∞-functions and it is equipped with the family of seminorms, using $D^\alpha:=(-\mathrm{i}\partial_{x_1})^{\alpha_1}\cdots(-\mathrm{i}\partial_{x_n})^{\alpha_n}$ for each multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, $\mathrm{i}^2=-1$ and $\langle x\rangle^2:=1+|x|^2$,

$$p_M(\varphi):=\sup\big\{\,\langle x\rangle^M|D^\alpha\varphi(x)|\,\big|\,x\in\mathbb{R}^n,|\alpha|\le M\big\},\quad M\in\mathbb{N}_0.$$

By duality, the Fourier transformation $\mathscr{F}g(\xi)=\widehat{g}(\xi)=\int_{\mathbb{R}^n}e^{-\mathrm{i}x\cdot\xi}g(x)\,dx$ for $g\in\mathscr{S}(\mathbb{R}^n)$ extends to the dual space $\mathscr{S}'(\mathbb{R}^n)$ of temperate distributions.

Throughout, inequalities for vectors $\vec{p}=(p_1,\dots,p_n)$ are understood componentwise; likewise for functions, e.g. $\vec{p}\,!=p_1!\cdots p_n!$, while $t_+:=\max(0,t)$ for $t\in\mathbb{R}$. For $0<\vec{p}\le\infty$ the space $L_{\vec{p}}(\mathbb{R}^n)$ consists of the Lebesgue measurable functions such that

$$\|\,u\,|L_{\vec{p}}(\mathbb{R}^n)\|:=\bigg(\int_{-\infty}^{\infty}\bigg(\dots\bigg(\int_{-\infty}^{\infty}|u(x_1,\dots,x_n)|^{p_1}dx_1\bigg)^{p_2/p_1}\dots\bigg)^{p_n/p_{n-1}}dx_n\bigg)^{1/p_n}<\infty;$$

in case $p_j=\infty$, the essential supremum over x_j is used. When equipped with this quasi-norm, it is a quasi-Banach space (normed if $\vec{p}\ge1$).

In addition, we shall for $0<q\le\infty$ denote by $L_{\vec{p}}(\ell_q)(\mathbb{R}^n)$ the space of sequences $(u_k)_{k\in\mathbb{N}_0}$ of Lebesgue measurable functions $u_k:\mathbb{R}^n\to\mathbb{C}$ such that

$$\|\,(u_k)_{k\in\mathbb{N}_0}\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|:=\bigg\|\,\Big(\sum_{k=0}^{\infty}|u_k|^q\Big)^{1/q}\bigg|L_{\vec{p}}(\mathbb{R}^n)\bigg\|<\infty;$$

with the supremum over k in case $q=\infty$. For brevity, $\|\,(u_k)_{k\in\mathbb{N}_0}\,|L_{\vec{p}}(\ell_q)(\mathbb{R}^n)\|$ is written $\|\,u_k\,|L_{\vec{p}}(\ell_q)\|$ and when $\vec{p}=(p,\dots,p)$, $L_{\vec{p}}$ is simplified to L_p etc. We recall that sequences of C_0^∞-functions are dense in $L_{\vec{p}}(\ell_q)$ if $\max(p_1,\dots,p_n,q)<\infty$.

Generic constants will be denoted by c or C, with their dependence on certain parameters explicitly stated when relevant.

Lastly, the closure of an open set $U\subset\mathbb{R}^n$ is denoted $\overline{U}$ and $B(0,r)$ is the ball centered at 0 with radius $r>0$; the dimension of the surrounding Euclidean space will be clear from the context or otherwise stated explicitly.

2.2. **Anisotropic Lizorkin–Triebel Spaces with Mixed Norms.** This section only contains the Fourier-analytic definition of the mixed-norm Lizorkin–Triebel spaces and a few essential properties used in this paper; for an actual introduction to these spaces we refer the reader to [JS07] and [JS08, Sec. 3].

First we recall the definition of the anisotropic distance function $|\cdot|_{\vec{a}}$, where $\vec{a} = (a_1, \dots, a_n) \in [1, \infty[^n$, on $\mathbb{R}^n$ and some of its properties. Using the quasi-homogeneous dilation $t^{\vec{a}}x := (t^{a_1}x_1, \dots, t^{a_n}x_n)$ for $t \geq 0$, $|x|_{\vec{a}}$ is for $x \in \mathbb{R}^n \setminus \{0\}$ defined as the unique $t > 0$ such that $t^{-\vec{a}}x \in S^{n-1}$ ($|0|_{\vec{a}} := 0$), i.e.

$$\frac{x_1^2}{t^{2a_1}} + \cdots + \frac{x_n^2}{t^{2a_n}} = 1.$$

For basic properties of $|\cdot|_{\vec{a}}$ we refer to [JS07, Sec. 3].

The Fourier-analytic definition also relies on a Littlewood–Paley decomposition, i.e. $1 = \sum_{j=0}^{\infty} \Phi_j(\xi)$, which is based on a (for convenience fixed) $\psi \in C_0^\infty$ such that $0 \leq \psi(\xi) \leq 1$ for all ξ, $\psi(\xi) = 1$ if $|\xi|_{\vec{a}} \leq 1$ and $\psi(\xi) = 0$ if $|\xi|_{\vec{a}} \geq 3/2$. Setting $\Phi = \psi - \psi(2^{\vec{a}}\cdot)$, we define

$$\Phi_0(\xi) = \psi(\xi), \qquad \Phi_j(\xi) = \Phi(2^{-j\vec{a}}\xi), \quad j = 1, 2, \dots \tag{1}$$

Definition 2.1. The Lizorkin–Triebel space $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < \vec{p} < \infty$ and $0 < q \leq \infty$ consists of the $u \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\| u \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| := \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} \left|\mathscr{F}^{-1}\left(\Phi_j(\xi)\mathscr{F}u(\xi)\right)(\cdot)\right|^q \right)^{1/q} \middle| L_{\vec{p}}(\mathbb{R}^n) \right\| < \infty.$$

The number q is called a sum exponent and the entries in $\vec{p}$ integral exponents, while s is a smoothness index. In case $\vec{a} = (1, \dots, 1)$, the parameter $\vec{a}$ is omitted.

When studying traces on the flat boundary of a cylinder, Besov spaces are inevitable:

Definition 2.2. The Besov space $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < \vec{p}, q \leq \infty$ consists of the $u \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\| u \,|B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| := \left(\sum_{j=0}^{\infty} 2^{jsq} \| \mathscr{F}^{-1}\left(\Phi_j \mathscr{F}u\right) |L_{\vec{p}}(\mathbb{R}^n)\|^q \right)^{1/q} < \infty.$$

Both $F^{s,\vec{a}}_{\vec{p},q}$ and $B^{s,\vec{a}}_{\vec{p},q}$ are quasi-Banach spaces (normed if $\min(p_1, \dots, p_n, q) \geq 1$) and the quasi-norm is subadditive when raised to the power $d := \min(1, p_1, \dots, p_n, q)$,

$$\| u + v \,|F^{s,\vec{a}}_{\vec{p},q}\|^d \leq \| u \,|F^{s,\vec{a}}_{\vec{p},q}\|^d + \| v \,|F^{s,\vec{a}}_{\vec{p},q}\|^d, \quad u, v \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n). \tag{2}$$

Different choices of anisotropic decomposition of unity give the same space (with equivalent quasi-norms) and there are continuous embeddings

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n), \tag{3}$$

where $\mathscr{S}$ is dense in $F^{s,\vec{a}}_{\vec{p},q}$ for $q < \infty$.

Lemma 2.3. *For $\lambda > 0$ so large that $\lambda\vec{a} \geq 1$, the spaces $B^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ coincide with $B^{\lambda s,\lambda\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$, respectively $F^{\lambda s,\lambda\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ and the corresponding quasi-norms are equivalent.*

The proof of this lemma for Besov spaces follows that of Lizorkin–Triebel spaces, which can be found in [JS08, Lem. 3.24]. Indeed, the only exception is that [JS08, Lem. 3.23] needs to be adapted to Besov spaces, but this is easily done using the modifications indicated just above Lemma 3.21 there.

In view of Lemma 2.3, most results obtained for the scales when $\vec{a} \geq 1$ can be extended to the range $0 < \vec{a} < \infty$ (for details we refer to [JSH13a, Rem. 2.6]).

The Banach space $C_{\mathrm{b}}(\mathbb{R}^n)$ of continuous, bounded functions is equipped with the sup-norm, while the subspace $L_{1,\mathrm{loc}}(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n)$ of locally integrable functions is endowed with the Fréchet space topology defined from the seminorms $u \mapsto \int_{|x| \leq j} |u(x)|\, dx$, $j \in \mathbb{N}$.

Lemma 2.4 ([JSH13b, Lem. 1]). *Let $s\in\mathbb{R}$ and $\alpha\in\mathbb{N}_0^n$ be arbitrary.*

(i) *The differential operator D^α is bounded $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)\to F^{s-\alpha\cdot\vec a,\vec a}_{\vec p,q}(\mathbb{R}^n)$.*

(ii) *For $s>\sum_{\ell=1}^n\big(\frac{a_\ell}{p_\ell}-a_\ell\big)_+$ there is an embedding $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)\hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^n)$.*

(iii) *The embedding $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)\hookrightarrow C_{\mathrm b}(\mathbb{R}^n)$ holds for $s>\frac{a_1}{p_1}+\dots+\frac{a_n}{p_n}$.*

Next, we recall a paramultiplication result and refer to [JSH13b, Sec. 2.4] for details,

Lemma 2.5. *Let $s\in\mathbb{R}$ and take $s_1>s$ such that also*

$$s_1>\sum_{\ell=1}^n\Big(\frac{a_\ell}{\min(1,q,p_1,\dots,p_\ell)}-a_\ell\Big)-s. \tag{4}$$

Then each $u\in B^{s_1,\vec a}_{\infty,\infty}(\mathbb{R}^n)$ defines a multiplier of $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ and

$$\|\,u\cdot v\,|F^{s,\vec a}_{\vec p,q}\|\le c\,\|\,u\,|B^{s_1,\vec a}_{\infty,\infty}\|\cdot\|\,v\,|F^{s,\vec a}_{\vec p,q}\|,\quad v\in F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n).$$

In particular, it holds for $u\in C^\infty_{L_\infty}(\mathbb{R}^n):=\{\,g\in C^\infty(\mathbb{R}^n)\,|\,\forall\alpha\in\mathbb{N}_0^n: D^\alpha g\in L_\infty(\mathbb{R}^n)\}$.

The characterisation of $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ by kernels of local means as developed in [JSH13a, Thm. 5.2] is utilised below, hence it is included here for convenience, using the notation

$$\varphi_j(x)=2^{j|\vec a|}\varphi(2^{j\vec a}x),\quad \varphi\in\mathscr{S},\ \ j\in\mathbb{N}. \tag{5}$$

Theorem 2.6. *Let $k_0,k^0\in\mathscr{S}(\mathbb{R}^n)$ such that $\int k_0(x)\,dx\neq0\neq\int k^0(x)\,dx$ and set $k(x)=\Delta^Nk^0(x)$ for some $N\in\mathbb{N}$. When $0<\vec p<\infty$, $0<q\le\infty$, and $s<2N\min(a_1,\dots,a_n)$, then a distribution $f\in\mathscr{S}'(\mathbb{R}^n)$ belongs to $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ if and only if*

$$\|\,f\,|F^{s,\vec a}_{\vec p,q}\|^*:=\|\,k_0*f\,|L_{\vec p}\|+\|\{2^{sj}k_j*f\}_{j=1}^\infty\,|L_{\vec p}(\ell_q)\|<\infty. \tag{6}$$

Furthermore, $\|\,f\,|F^{s,\vec a}_{\vec p,q}\|^*$ is an equivalent quasi-norm on $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$.

We also recall the definition of $F^{s,\vec a}_{\vec p,q}$-spaces over open sets. Here we use the notation introduced by Hörmander [Hör07, App. B.2] and place a bar over F etc., to indicate that it is a space of restricted distributions.

Definition 2.7. Let $U\subset\mathbb{R}^n$ be open. The space $\overline{F}^{s,\vec a}_{\vec p,q}(U)$ is defined as the set of $u\in\mathscr{D}'(U)$ such that there exists a distribution $f\in F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ satisfying

$$f(\varphi)=u(\varphi)\quad\text{for all}\quad\varphi\in C_0^\infty(U). \tag{7}$$

We equip $\overline{F}^{s,\vec a}_{\vec p,q}(U)$ with the quotient quasi-norm

$$\|\,u\,|\overline{F}^{s,\vec a}_{\vec p,q}(U)\|:=\inf_{r_Uf=u}\|\,f\,|F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)\|,$$

which is a norm if $\vec p,q\ge1$. (Besov spaces over open sets are defined analogously.)

The space $\overset{\circ}{F}{}^{s,\vec a}_{\vec p,q}(\overline{U})$ consists of the distributions in $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$, which are supported in the closed set $\overline{U}$.

Recall that since $F^{s,\vec a}_{\vec p,q}(\mathbb{R}^n)$ is a quasi-Banach space, $\overline{F}^{s,\vec a}_{\vec p,q}(U)$ is so too by the usual arguments for quotient spaces modified to exploit the subadditivity in (2).

In (7) it is tacitly understood that on the left-hand side φ is extended by 0 outside U. For this we henceforth use the operator notation $e_U\varphi$. Likewise r_U denotes restriction to U, whereby $u=r_Uf$ in (7). We shall refer to such f as an extension of u.

Remark 2.8. Theorem 2.6 induces an equivalent quasi-norm $\| u\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)\|^*$ on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ by taking the infimum of $\| f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|^*$ for $r_U f = u$.

As a preparation we include a slightly modified version of [JSH13b, Lem. 8]:

Lemma 2.9. *Let $U \subset \mathbb{R}^n$ be open. When $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})$ is normed as in Remark 2.8 using kernels of local means with $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$ for an $r>0$, and when $K \subset U$ is a compact set fulfilling*

$$\operatorname{dist}(K, \mathbb{R}^n \setminus U) > 2r, \tag{8}$$

then it holds for every $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})$ with $\operatorname{supp} f \subset K\times\mathbb{R}$ that

$$\| f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U\times\mathbb{R})\|^* = \| e_{U\times\mathbb{R}} f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^*.$$

That is, the infimum is for such f attained at $e_{U\times\mathbb{R}} f$.

Proof. For an arbitrary extension $\widetilde{f}$ of f, it holds for $g := \widetilde{f} - e_{U\times\mathbb{R}} f$ that $\operatorname{supp} e_{U\times\mathbb{R}} f \cap \operatorname{supp} g = \emptyset$, hence by (8),

$$\operatorname{supp}(k_j * e_{U\times\mathbb{R}} f) \cap \operatorname{supp}(k_j * g) = \emptyset, \quad j \in \mathbb{N}_0.$$

When $g \neq 0$, there exists $j \in \mathbb{N}_0$ such that $\operatorname{supp}(k_j * g) \neq \emptyset$, thus $k_j * g(x) \neq 0$ on an open set disjoint from $\operatorname{supp}(k_j * e_{U\times\mathbb{R}} f)$. This term therefore effectively contributes to the $L_{\vec{p}}$-norm in the local means characterisation, yielding $\| \widetilde{f}\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^* > \| e_{U\times\mathbb{R}} f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^*$. $\square$

For temperate distributions vanishing in the time direction, we let $e_{I\to I'}$ denote extension by 0 from $\mathbb{R}^{n-1}\times I$ to $\mathbb{R}^{n-1}\times I'$ for open intervals $I \subset I'$. Then we similarly get

Lemma 2.10. *Let $I =]b,c[$ and $I' =]a,c[$ where $-\infty \le a < b < c \le \infty$. When $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1}\times I)$ is normed as in Remark 2.8 using kernels of local means with $\operatorname{supp} k_0, \operatorname{supp} k \subset B(0,r)$ for an $r>0$, then it holds for every $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1}\times I)$ satisfying $f(\cdot,t) = 0$ for $t \in]b, b+2r[$ that*

$$\| f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1}\times I)\|^* = \| e_{I\to I'} f\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1}\times I')\|^*. \tag{9}$$

A similar equality holds for extension from $I =]a,b[$ to I', when $f(\cdot,t) = 0$ for $t \in]b-2r, c[$.

Proof. The inequality $\le$ follows immediately, since the distributions considered in the infimum on the right-hand side in (9) also are considered on the left-hand side.

To prove equality we assume that $<$ holds. Then there exists an extension $\widetilde{f}$ of f which is not among the distributions considered in the infimum on the right-hand side, and which, with an infimum over $r_{\mathbb{R}^{n-1}\times I'} h = e_{I\to I'} f$, moreover fulfils

$$\| \widetilde{f}\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| < \inf \| h\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|. \tag{10}$$

Actually, it suffices to consider those h for which $h \equiv 0$ on $\mathbb{R}^{n-1}\times]-\infty, b+2r[$. Indeed, for any other h the distribution $(1-\chi(t))h(\cdot,t)$, where $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $t \in]-\infty, a[$ and $\chi(t) = 0$ for $t \in]b, \infty[$, has a smaller quasi-norm than h. This can be verified similarly to the proof of Lemma 2.9, using that the distance between $\operatorname{supp} h \cap (\mathbb{R}^{n-1}\times]-\infty, a[)$ and $\operatorname{supp}(1-\chi)h$ is at least $2r$.

Now for such h we have $\operatorname{supp} h \subset \mathbb{R}^{n-1}\times [b+2r, \infty[$, and since $\widetilde{f}(t) \not\equiv 0$ for $a < t < b$ it is easily seen by the proof strategy of Lemma 2.9 that $\| \widetilde{f}\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| > \| h\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\|$, which contradicts (10). $\square$

For simplicity of notation the * on the quasi-norm is omitted in the following.

3. Invariance under Diffeomorphisms

To introduce Lizorkin–Triebel spaces on manifolds, it is essential that the spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(U)$ for certain open subsets $U \subset \mathbb{R}^n$ are invariant under suitable C^∞-bijections σ. An extensive treatment of this subject can be found in [JSH13b], but for convenience we recall the needed results. These hold for $0 < \vec{p} < \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$ unless additional requirements are specified. First a result pertaining to isotropic spaces,

Theorem 3.1. *When $\sigma : U \to V$ is a C^∞-bijection between open sets $U, V \subset \mathbb{R}^n$ and $f \in \overline{F}^s_{p,q}(V)$ has compact support, then $f \circ \sigma \in \overline{F}^s_{p,q}(U)$ and*

$$\| f \circ \sigma \,|\overline{F}^s_{p,q}(U)\| \le c \| f \,|\overline{F}^s_{p,q}(V)\| \tag{11}$$

holds for a constant c depending only on σ and the set $\operatorname{supp} f$.

In the anisotropic situation it cannot be expected, e.g. if σ is a rotation, that $f \circ \sigma$ has the same regularity as f, nor that $f \circ \sigma \in L_{\vec{p}}$ when $f \in L_{\vec{p}}$. We therefore restrict to

$$\vec{p} = (\underbrace{p_1, \dots, p_1}_{N_1}, \underbrace{p_2, \dots, p_2}_{N_2}, \dots, \underbrace{p_m, \dots, p_m}_{N_m}), \quad N_1 + \cdots + N_m = n, \ \ m \ge 2, \tag{12}$$

and $\vec{a}$ having the same structure.

Theorem 3.2. *Let $\sigma_j : U_j \to V_j$, $j = 1, \dots, m$, be C^∞-bijections, where $U_j, V_j \subset \mathbb{R}^{N_j}$ are open. When $\vec{a}$, $\vec{p}$ fulfil* (12) *and $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U_1 \times \cdots \times U_m)$ has compact support, then* (11) *holds for $U = U_1 \times \cdots \times U_m$ and $V = V_1 \times \cdots \times V_m$.*

For traces at the curved boundary of cylinders, the next special case is useful:

Theorem 3.3. *Let $U, V \subset \mathbb{R}^{n-1}$ be open and let $\sigma : U \times \mathbb{R} \to V \times \mathbb{R}$ be a C^∞-bijection on the form*

$$\sigma(x) = (\sigma'(x_1, \dots, x_{n-1}), x_n) \quad \textit{for all} \quad x \in U \times \mathbb{R}.$$

When $\vec{a}, \vec{p}$ satisfy (12) *with $m = 2$, $N_1 = n-1$, $N_2 = 1$ and $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(V \times \mathbb{R})$ has* $\operatorname{supp} f \subset K \times \mathbb{R}$, *whereby $K \subset V$ is compact, then $f \circ \sigma \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})$ and*

$$\| f \circ \sigma \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U \times \mathbb{R})\| \le c(\operatorname{supp} f, \sigma) \| f \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|.$$

The above three theorems can be found with proofs as Theorems 6–8, respectively, in [JSH13b]. As needed, we shall tacitly apply these results in situations with $n+1$ variables, the last of which is interpreted as time. Then we let $t = x_{n+1}$.

4. Function Spaces on Manifolds

To develop Lizorkin–Triebel spaces over cylinders and to settle the necessary notation, we first review distributions on manifolds.

4.1. Distributions on Manifolds.

To allow comparison with existing literature on partial differential equations, we follow [Gru09, Sec. 8.2] and [Hör90, Sec. 6.3]. E.g. a diffeomorphism is in the following a bijective C^∞-map between open sets, and we recall

Definition 4.1. An n-dimensional manifold X is a second countable Hausdorff space which is locally homeomorphic to $\mathbb{R}^n$. The manifold X is C^∞ (or smooth), if it is equipped with a C^∞-structure, i.e. a family $\mathscr{F}$ of homeomorphisms κ mapping open sets $X_\kappa \subset X$ onto open sets $\widetilde{X}_\kappa \subset \mathbb{R}^n$, with $X = \bigcup_{\kappa\in\mathscr{F}} X_\kappa$, such that the maps

$$\kappa \circ \kappa_1^{-1} : \kappa_1(X_\kappa \cap X_{\kappa_1}) \to \kappa(X_\kappa \cap X_{\kappa_1}), \quad \kappa, \kappa_1 \in \mathscr{F}, \tag{13}$$

are diffeomorphisms, and $\mathscr{F}$ contains every homeomorphism $\kappa_0 : X_{\kappa_0} \to \widetilde{X}_{\kappa_0}$, for which the compositions in (13) with $\kappa = \kappa_0$ are diffeomorphisms.

A subfamily of $\mathscr{F}$ where the X_κ cover X is called a (compatible) atlas, and $\mathscr{F}_1 \subset \mathscr{F}_2$ means that every chart κ in $\mathscr{F}_1$ is also a member of $\mathscr{F}_2$. (The definition of a C^∞-manifold X means that a maximal atlas has been chosen on the set X.)

Unless otherwise stated, X denotes an n-dimensional C^∞-manifold and $\mathscr{F}$ is the maximal atlas. A partition of unity $1 = \sum_{j\in\mathbb{N}} \psi_j(x)$ with $\psi_j \in C_0^\infty(X)$ and $\psi_j(x) \geq 0$ for $x \in X$ is said to be subordinate to $\mathscr{F}$ (instead of to the covering $X = \bigcup_{\kappa\in\mathscr{F}} X_\kappa$), when for each $j \in \mathbb{N}$ there exists a chart $\kappa(j) \in \mathscr{F}$ such that $\operatorname{supp}\psi_j \subset X_{\kappa(j)}$. It is locally finite, when $1 = \sum \psi_j(x)$ for every $x \in X$ has only finitely many non-trivial terms in some neighbourhood of x. Note that for each compact set $K \subset X$, this finiteness extends to an open set $U \supset K$.

We recall the definition of a distribution on a C^∞-manifold, using the notation $\varphi^* u$ for the pullback of a distribution u by a function φ; when u is a function then $\varphi^* u = u \circ \varphi$.

Definition 4.2. The space $\mathscr{D}'(X)$ consists of the families $\{u_\kappa\}_{\kappa\in\mathscr{F}}$, where $u_\kappa \in \mathscr{D}'(\widetilde{X}_\kappa)$ and which for all $\kappa, \kappa_1 \in \mathscr{F}$ fulfil

$$u_{\kappa_1} = (\kappa \circ \kappa_1^{-1})^* u_\kappa \quad \text{on} \quad \kappa_1(X_\kappa \cap X_{\kappa_1}). \tag{14}$$

($\mathscr{D}'(X)$ only identifies with the dual of $C_0^\infty(X)$ if there is a positive density on X; cf. [Hör90, Ch. 6].)

Each $u \in C^k(X)$, $k \in \mathbb{N}_0$, can be identified with the family $u_\kappa := u \circ \kappa^{-1}$ of functions in $C^k(\widetilde{X}_\kappa)$, which evidently transform as in (14). Thus $C^k(X) \subset \mathscr{D}'(X)$ is obvious. For any $u \in \mathscr{D}'(X)$, the notation $u \circ \kappa^{-1}$ is also used to denote u_κ.

In (14) restriction of e.g. u_κ to $\kappa(X_\kappa \cap X_{\kappa_1})$ is tacitly understood. To ease notation we will in the rest of the paper, when composing with a chart, suppress such restriction to the chart's co-domain.

Lemma 4.3 ([Hör90, Thm. 6.3.4])**.** *For any atlas $\mathscr{F}_1 \subset \mathscr{F}$, each family $\{u_\kappa\}_{\kappa\in\mathscr{F}_1}$ of elements $u_\kappa \in \mathscr{D}'(\widetilde{X}_\kappa)$ fulfilling* (14) *for $\kappa, \kappa_1 \in \mathscr{F}_1$ is obtained from a unique $v \in \mathscr{D}'(X)$ by "restriction" to $\mathscr{F}_1$, i.e. $v \circ \kappa^{-1} = u_\kappa$ for every $\kappa \in \mathscr{F}_1$.*

So if an open set $U \subset \mathbb{R}^n$ is seen as a manifold X, then $\mathscr{F}_1 = \{\mathrm{id}_U\}$ at once gives $\mathscr{D}'(U) \hookrightarrow \mathscr{D}'(X)$; the surjectivity of this map follows by gluing together, cf. [Hör90, Thm. 2.2.4].

For $Y \subset X$ open, the restriction of $u \in \mathscr{D}'(X)$ to Y is $r_Y u := \{r_{\kappa(Y\cap X_\kappa)} u_\kappa\}$, where κ runs through the charts in $\mathscr{F}$ for which $X_\kappa \cap Y \neq \emptyset$. If instead of $\mathscr{F}$ we only consider an atlas $\mathscr{F}_1 \subset \mathscr{F}$, then the corresponding subfamily identifies with a distribution $u_Y \in \mathscr{D}'(Y)$, cf. Lemma 4.3, and since this is unique, $r_Y u = u_Y$, i.e. it is sufficient to consider an arbitrary atlas when determining the restriction of a distribution.

A distribution $u \in \mathscr{D}'(X)$ is said to be 0 on an open set $Y \subset X$ if $r_Y u = 0$. Using this,

$$\operatorname{supp} u := X \setminus \bigcup \{\, Y \subset X \text{ open} \mid u = 0 \text{ on } Y \,\}, \tag{15}$$

and it is easily seen that for any atlas $\mathscr{F}_1 \subset \mathscr{F}$,

$$\operatorname{supp} u = \bigcup_{\kappa_1\in\mathscr{F}_1} \kappa_1^{-1}(\operatorname{supp} u_{\kappa_1}). \tag{16}$$

The space $\mathscr{E}'(X)$ consists of the distributions $u \in \mathscr{D}'(X)$ having compact support, while $\mathscr{E}'(K)$ for an arbitrary $K \subset X$ consists of the $u \in \mathscr{E}'(X)$ with $\operatorname{supp} u \subset K$. Any $u \in \mathscr{E}'(Y)$, where $Y \subset X$ is open, has an "extension by 0"; even locally:

Corollary 4.4. *When $Y \subset X$ is open and $u \in \mathscr{E}'(Y)$, then there exists $v \in \mathscr{E}'(X)$ such that $r_Y v = u$ and* $\operatorname{supp} v = \operatorname{supp} u$. *Moreover, when given $u_\kappa \in \mathscr{E}'(\widetilde{X}_\kappa)$ for a single $\kappa \in \mathscr{F}$, then there exists $v \in \mathscr{E}'(X)$ such that $v_\kappa = u_\kappa$ and* $\operatorname{supp} v = \kappa^{-1}(\operatorname{supp} u_\kappa)$.

Proof. In the case that $\operatorname{supp} u \subset X_\kappa \subset Y$ for some $\kappa \in \mathscr{F}$, then there exists an open set $U \subset X$ such that $\operatorname{supp} u \subset U \subset \overline{U} \subset X_\kappa$ (X is normal). The family $\mathscr{F}_1 := \{\kappa\} \cup \{\kappa_1 \in \mathscr{F} \mid \overline{U} \cap X_{\kappa_1} = \emptyset\}$ is an atlas, since its domains covers X. Setting $v_\kappa = u_\kappa$ and $v_{\kappa_1} = 0$ for the other $\kappa_1 \in \mathscr{F}_1$, the family $\{v_{\kappa_1}\}_{\kappa_1 \in \mathscr{F}_1}$ clearly transforms as in (14), hence defines a $v \in \mathscr{D}'(X)$, cf. Lemma 4.3. From (16) it is clear that $\operatorname{supp} v = \operatorname{supp} u$; and $r_Y v = u$ is evident in the atlas $\mathscr{F}_1$.

In the general case, we use that any $u \in \mathscr{E}'(Y)$ can be written as a finite sum $u = \sum \psi_j u$, where $1 = \sum \psi_j$ is a locally finite partition of unity subordinate to the atlas $\{\, \kappa|_{Y \cap X_\kappa} \mid \kappa \in \mathscr{F} : Y \cap X_\kappa \neq \emptyset\}$ on Y. Since for each summand, $\operatorname{supp} \psi_j u \subset Y \cap X_{\kappa(j)}$ is compact, the above gives the existence of a $v_j \in \mathscr{D}'(X)$ such that $r_Y v_j = \psi_j u$ and $\operatorname{supp} v_j = \operatorname{supp} \psi_j u$. Because the restriction operator is linear, taking $v = \sum v_j$ proves the statement.

For the last part, consider X_κ as a manifold with the atlas containing only the chart κ. Lemma 4.3 gives a $w \in \mathscr{D}'(X_\kappa)$ such that $w_\kappa = u_\kappa$, hence the special case above applied to w and $Y = X_\kappa$ gives, that there exists a $v \in \mathscr{E}'(X)$ such that $v_\kappa = w_\kappa$, and $\operatorname{supp} v = \operatorname{supp} w$. □

4.2. **Isotropic Lizorkin–Triebel Spaces on Manifolds.** Since we later need a few isotropic results, and since the proofs are much cleaner for isotropic spaces, we shall fix ideas in this section by working with arbitrary $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let us add that most references on isotropic spaces over manifolds just describe the outcome without referring directly to the general definitions in [Hör90, Ch. 6], thus being inadequate for our generalisations here.

4.2.1. *Manifolds in General.* We first recall that when $U \subset \mathbb{R}^n$ is open, $u \in \mathscr{D}'(U)$ is said to belong to the Lizorkin–Triebel space $\overline{F}^s_{p,q}(U)$ locally, if $\varphi u \in \overline{F}^s_{p,q}(U)$ for all $\varphi \in C_0^\infty(U)$; the set of such elements is denoted $F^s_{p,q;\mathrm{loc}}(U)$. Here we use the notation without bar, since φ has compact support in U. This can be generalised to

Definition 4.5. The local Lizorkin–Triebel space $F^s_{p,q;\mathrm{loc}}(X)$ consists of the $u \in \mathscr{D}'(X)$ such that $u_\kappa \in F^s_{p,q;\mathrm{loc}}(\widetilde{X}_\kappa)$ for every $\kappa \in \mathscr{F}$.

For $u \in \mathscr{D}'(X)$ to belong to $F^s_{p,q;\mathrm{loc}}(X)$, it suffices that u_{κ_1} is in $F^s_{p,q;\mathrm{loc}}(\widetilde{X}_{\kappa_1})$ for each κ_1 in an atlas $\mathscr{F}_1 \subset \mathscr{F}$. Indeed given $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$, a partition of unity yields a reduction to the case where $\operatorname{supp}(\varphi \circ \kappa) \subset X_\kappa \cap X_{\kappa_1}$, and the transition rule in (14) gives

$$(\kappa \circ \kappa_1^{-1})^*(\varphi u_\kappa) = \varphi \circ (\kappa \circ \kappa_1^{-1}) u_{\kappa_1}.$$

Since $\varphi \circ (\kappa \circ \kappa_1^{-1})$ is in $C_0^\infty(\kappa_1(X_\kappa \cap X_{\kappa_1}))$, the product is in $\overline{F}^s_{p,q}(\kappa_1(X_\kappa \cap X_{\kappa_1}))$ by assumption on $\mathscr{F}_1$; so by Theorem 3.1 one has $\varphi u_\kappa \in \overline{F}^s_{p,q}(\widetilde{X}_\kappa)$.

For example, when X is an open set $U \subset \mathbb{R}^n$, the identification $\mathscr{D}'(X) \simeq \mathscr{D}'(U)$ implies that $F^s_{p,q;\mathrm{loc}}(X) \simeq F^s_{p,q;\mathrm{loc}}(U)$ as it according to the above suffices to consider the atlas $\{\mathrm{id}_U\}$.

For a partition of unity $1 = \sum_{j=1}^\infty \psi_j$ subordinate to $\mathscr{F}$, we shall for brevity use

$$\widetilde{\psi}_j := \psi_j \circ \kappa(j)^{-1}.$$

The partition is of course already subordinate to $\mathscr{F}_1 := \{\, \kappa(j) \mid j \in \mathbb{N}\}$, which by the above suffices for determining $F^s_{p,q;\mathrm{loc}}(X)$. This is moreover true, when the cut-off functions $\widetilde{\psi}_j$ of a locally finite partition of unity are invoked:

Lemma 4.6. *A distribution $u \in \mathscr{D}'(X)$ belongs to $F^s_{p,q;\mathrm{loc}}(X)$ if and only if*

$$\widetilde{\psi}_j u_{\kappa(j)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)}), \quad j \in \mathbb{N}. \tag{17}$$

Proof. Since $\widetilde{\psi}_j \in C_0^\infty(\widetilde{X}_{\kappa(j)})$, this condition is necessary for u to be in $F^s_{p,q;\mathrm{loc}}(X)$.

Conversely, for an arbitrary $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$ we obtain $\varphi u_\kappa = \sum_{j\in I} \psi_j \circ \kappa^{-1} \varphi u_\kappa$ with summation over a finite index set $I \subset \mathbb{N}$, because $(\psi_j)_{j\in\mathbb{N}}$ is locally finite. As $\operatorname{supp}(\psi_j \circ \kappa^{-1}\varphi) \subset \kappa(X_\kappa \cap X_{\kappa(j)})$, [JSH13b, Lem. 8] and then Theorem 3.1 applied to $\kappa(j) \circ \kappa^{-1}$ yields, cf. (14),

$$\| \varphi u_\kappa \,|\overline{F}^s_{p,q}(\widetilde{X}_\kappa)\| \le c_\kappa \sum_{j\in I} \big\| \widetilde{\psi}_j \cdot (\varphi \circ \kappa \circ \kappa(j)^{-1}) \cdot u_{\kappa(j)} \,\big|\overline{F}^s_{p,q}(\kappa(j)(X_\kappa \cap X_{\kappa(j)}))\big\|. \tag{18}$$

After multiplication with $\chi_j \in C_0^\infty(\widetilde{X}_{\kappa(j)})$ chosen such that $\chi_j \equiv 1$ on $\operatorname{supp}\widetilde{\psi}_j$, we obtain by applying Lemma 2.5 with some $s_1 > s$ satisfying (4) and suppressing extension by 0 to $\mathbb{R}^n$ that

$$\| \varphi u_\kappa \,|\overline{F}^s_{p,q}(\widetilde{X}_\kappa)\| \le c_\kappa \sum_{j\in I} \| \varphi \circ \kappa \circ \kappa(j)^{-1} \chi_j \,|B^{s_1}_{\infty,\infty}(\mathbb{R}^n)\| \| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\|. \tag{19}$$

The right-hand side is by (17) finite, hence $u_\kappa \in F^s_{p,q;\mathrm{loc}}(\widetilde{X}_\kappa)$ for each $\kappa \in \mathscr{F}$. □

The space $F^s_{p,q;\mathrm{loc}}(X)$ can be topologised through a separating family of quasi-seminorms,

$$\mu_j(u) := \| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\|, \quad j \in \mathbb{N}. \tag{20}$$

Indeed, if u is non-zero in $F^s_{p,q;\mathrm{loc}}(X)$, then there exists $\kappa \in \mathscr{F}$ and $\varphi \in C_0^\infty(\widetilde{X}_\kappa)$ such that $\varphi u_\kappa \neq 0$, i.e. $\| \varphi u_\kappa \,|\overline{F}^s_{p,q}(\widetilde{X}_\kappa)\| > 0$. So (19) gives that $\mu_j(u) > 0$ for at least one $j \in \mathbb{N}$.

Going a step further, one obtains an equivalent family of quasi-seminorms even for a "restricted" family $\{v_{\kappa_1}\}_{\kappa_1\in\mathscr{F}_1}$:

Lemma 4.7. *Let $1 = \sum \varphi_k$ be a locally finite partition of unity subordinate to some atlas $\mathscr{F}_1 \subset \mathscr{F}$ and let $\widetilde{\varphi}_k = \varphi_k \circ \kappa_1(k)^{-1}$. When a family of distributions $v_{\kappa_1} \in \mathscr{D}'(\widetilde{X}_{\kappa_1})$, $\kappa_1 \in \mathscr{F}_1$, transforms as in* (14) *and $\widetilde{\varphi}_k v_{\kappa_1(k)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa_1(k)})$ for every $k \in \mathbb{N}$, then there exists a unique $u \in F^s_{p,q;\mathrm{loc}}(X)$ such that $u_{\kappa_1} = v_{\kappa_1}$ for all $\kappa_1 \in \mathscr{F}_1$ and*

$$\| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\| \le c_j \max \| \widetilde{\varphi}_k u_{\kappa_1(k)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa_1(k)})\|, \quad j \in \mathbb{N}, \tag{21}$$

with maximum over $k \in \mathbb{N}$ for which $\operatorname{supp}\psi_j \cap \operatorname{supp}\varphi_k \neq \emptyset$, cf. (15).

Proof. There exists a unique distribution $u \in \mathscr{D}'(X)$ such that $u_{\kappa_1} = v_{\kappa_1}$ for all $\kappa_1 \in \mathscr{F}_1$, cf. Lemma 4.3, and using (19) with $\varphi = \widetilde{\psi}_j$ and $1 = \sum \varphi_k$ as the partition of unity readily shows (21). Consequently $u \in F^s_{p,q;\mathrm{loc}}(X)$. □

Since the opposite inequality of (21) can be shown similarly from (18)–(19), we obtain

Corollary 4.8. *The space $F^s_{p,q;\mathrm{loc}}(X)$ can be equivalently defined from any atlas $\mathscr{F}_1 \subset \mathscr{F}$. Lemma 4.6 holds for any locally finite partition of unity subordinate to $\mathscr{F}_1$, and the resulting system of quasi-seminorms is equivalent to* (20).

As a preparation we include an obvious consequence of the proof of Corollary 4.4:

Corollary 4.9. *When given $u_\kappa \in \mathscr{E}'(\widetilde{X}_\kappa) \cap \overline{F}^s_{p,q}(\widetilde{X}_\kappa)$ for a single $\kappa \in \mathscr{F}$, then there exists $v \in \mathscr{E}'(X) \cap F^s_{p,q;\mathrm{loc}}(X)$ such that $v_\kappa = u_\kappa$ and $\operatorname{supp} v = \kappa^{-1}(\operatorname{supp} u_\kappa)$.*

When an open set $U \subset \mathbb{R}^n$ is seen as a manifold X, then $F^s_{p,q;\mathrm{loc}}(U)$ obviously coincides with $F^s_{p,q;\mathrm{loc}}(X)$, since it by Corollary 4.8 suffices to consider $\mathscr{F}_1 = \{\mathrm{id}_U\}$ and any partition of unity $1 = \sum_{j=1}^{\infty} \psi_j$ on U. On $F^s_{p,q;\mathrm{loc}}(U)$, the family in (20) gives the usual structure of a Fréchet space if $p,q \geq 1$, and in general we have:

Theorem 4.10. *The space $F^s_{p,q;\mathrm{loc}}(X)$ is a complete topological vector space with a translation invariant metric; for $p,q \geq 1$ it is locally convex, hence a Fréchet space.*

Proof. It follows straightforwardly from [Gru09, Thm. B.5], which is based on a separating family of seminorms, that the separating family $(\mu_j^d)_{j\in\mathbb{N}}$, whereby $d := \min(1,p,q)$, of subadditive functionals can be used to construct a topology, which turns $F^s_{p,q;\mathrm{loc}}(X)$ into a topological vector space. Indeed, only a minor modification in the proof of continuity of scalar multiplication is needed, since the μ_j^d are not positive homogeneous — unless $p,q \geq 1$, and in this case the positive homogeneity implies that $F^s_{p,q;\mathrm{loc}}(X)$ is locally convex.

A translation invariant metric can be defined as in [Gru09, Thm. B.9], i.e.

$$d'(u,v) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\mu_j(u-v)^d}{1+\mu_j(u-v)^d}, \tag{22}$$

and the arguments there immediately yield that d' defines the same topology as $(\mu_j^d)_{j\in\mathbb{N}}$.

For an arbitrary Cauchy sequence (u_m) in $F^s_{p,q;\mathrm{loc}}(X)$, the sequence $(\widetilde{\psi}_j u_{m,\kappa(j)})$, where $u_{m,\kappa(j)} := u_m \circ \kappa(j)^{-1}$, is Cauchy in $\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$ for each $j \in \mathbb{N}$. Since this space is complete, there exists $\widetilde{v}_{\kappa(j)} \in \overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$ such that

$$\| \widetilde{\psi}_j u_{m,\kappa(j)} - \widetilde{v}_{\kappa(j)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\| \to 0 \quad \text{for} \quad m \to \infty. \tag{23}$$

Clearly $\widetilde{v}_{\kappa(j)} \in \mathscr{E}'(\widetilde{X}_{\kappa(j)})$, hence it follows from Corollary 4.9 that there exists a $v^{(\kappa(j))} \in \mathscr{E}'(X) \cap F^s_{p,q;\mathrm{loc}}(X)$ so that $\operatorname{supp} v^{(\kappa(j))} = \kappa(j)^{-1}(\operatorname{supp} \widetilde{v}_{\kappa(j)}) \subset \operatorname{supp} \psi_j$ and $v^{(\kappa(j))}_{\kappa(j)} = \widetilde{v}_{\kappa(j)}$.

To find a limit for (u_m), we note that $\widetilde{u}_{\kappa(j)} := \sum_{l\in\mathbb{N}} v^{(\kappa(l))}_{\kappa(j)}$ is well defined in $\mathscr{D}'(\widetilde{X}_{\kappa(j)})$, since on every compact set $K \subset \widetilde{X}_{\kappa(j)}$ there are only finitely many non-trivial terms. This family transforms as in (14), for in $\mathscr{D}'(\kappa(j)(X_{\kappa(j)} \cap X_{\kappa(k)}))$,

$$\widetilde{u}_{\kappa(k)} \circ \kappa(k) \circ \kappa(j)^{-1} = \sum_l v^{(\kappa(l))}_{\kappa(k)} \circ \kappa(k) \circ \kappa(j)^{-1} = \sum_l v^{(\kappa(l))}_{\kappa(j)} = \widetilde{u}_{\kappa(j)}. \tag{24}$$

Since $\widetilde{\psi}_j \widetilde{u}_{\kappa(j)} = \sum_l \widetilde{\psi}_j v^{(\kappa(l))}_{\kappa(j)}$ has finitely many summands, hence yields an element of $\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})$, existence of $u \in F^s_{p,q;\mathrm{loc}}(X)$ with $u_{\kappa(j)} = \widetilde{u}_{\kappa(j)}$ for all j follows from Lemma 4.7.

To show the convergence of u_m to u in $F^s_{p,q;\mathrm{loc}}(X)$, we rely on extra copies of the locally finite partition of unity to estimate by finitely many terms,

$$\mu_j(u_m - u)^d \leq \sum_{\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k \neq \emptyset} \big\| \widetilde{\psi}_j \big(\psi_k \circ \kappa(j)^{-1} u_{m,\kappa(j)} - v^{(\kappa(k))}_{\kappa(j)}\big) \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(j)})\big\|^d.$$

For $k \neq j$ the domains can clearly be changed to $\kappa(j)(X_{\kappa(j)} \cap X_{\kappa(k)})$, since $v^{(\kappa(k))}$ and the ψ_k have compact support in $X_{\kappa(k)}$. Using Theorem 3.1, each term can then be estimated by,

$$c\big\| \psi_j \circ \kappa(k)^{-1} (\widetilde{\psi}_k u_{m,\kappa(k)} - \widetilde{v}_{\kappa(k)}) \,|\overline{F}^s_{p,q}\big(\kappa(k)(X_{\kappa(j)} \cap X_{\kappa(k)})\big)\big\|^d.$$

By means of a cut-off function equal to 1 on the compact supports, one can extend by 0 to $\mathbb{R}^n$ and apply Lemma 2.5 and [JSH13b, Lem. 8], which yields

$$\mu_j(u_m - u)^d \le c \sum_{\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k \neq \emptyset} \| \widetilde{\psi}_k u_{m,\kappa(k)} - \widetilde{v}_{\kappa(k)} \,|\overline{F}^s_{p,q}(\widetilde{X}_{\kappa(k)})\|^d.$$

Each term converges to 0, cf. (23), hence $F^s_{p,q;\mathrm{loc}}(X)$ is complete. □

4.2.2. *Compact Manifolds.* For trace operators on cylinders, compact manifolds are of special interest, since the intersection of the curved and the flat boundary is often of such nature.

When X is compact there exists a finite atlas $\mathscr{F}_0$ and a partition of unity $1 = \sum_{\kappa\in\mathscr{F}_0} \psi_\kappa$ such that $\operatorname{supp}\psi_\kappa \subset X_\kappa$ is compact for each $\kappa \in \mathscr{F}_0$. The space $F^s_{p,q;\mathrm{loc}}(X)$ is in this case just denoted $F^s_{p,q}(X)$, since the elements satisfy a global condition according to

Theorem 4.11. *When X is a compact C^∞-manifold, then $F^s_{p,q}(X)$ is a quasi-Banach space (normed if $p,q \ge 1$) when equipped with*

$$\| u \,|F^s_{p,q}(X)\| := \Big(\sum_{\kappa\in\mathscr{F}_0} \| \widetilde{\psi}_\kappa u_\kappa \,|\overline{F}^s_{p,q}(\widetilde{X}_\kappa)\|^d \Big)^{1/d}, \quad d := \min(1,p,q), \tag{25}$$

and $\| \cdot \,|F^s_{p,q}(X)\|^d$ is subadditive.

Proof. Positive homogeneity and subadditivity are inherited from the quasi-norms on the $\overline{F}^s_{p,q}(\widetilde{X}_\kappa)$ and then the quasi-triangle inequality follows for $d < 1$ by using dual exponents $\frac{1}{d}, \frac{1}{1-d}$,

$$\| u + v \,|F^s_{p,q}(X)\| \le 2^{\frac{1-d}{d}} \big(\| u \,|F^s_{p,q}(X)\| + \| v \,|F^s_{p,q}(X)\| \big), \quad u,v \in F^s_{p,q}(X).$$

For any $u \in F^s_{p,q}(X)$ with $\| u \,|F^s_{p,q}(X)\| = 0$, clearly $\widetilde{\psi}_\kappa u_\kappa = 0$ on $\widetilde{X}_\kappa$, $\kappa \in \mathscr{F}_0$. Also $\psi_\kappa \circ \kappa_1^{-1} u_{\kappa_1} = 0$ for $\kappa, \kappa_1 \in \mathscr{F}_0$ with $X_\kappa \cap X_{\kappa_1} \neq \emptyset$, as (14) applies on $\kappa_1(X_\kappa \cap X_{\kappa_1})$. Therefore $u_{\kappa_1} = \sum_{\kappa\in\mathscr{F}_0} (\psi_\kappa \circ \kappa_1^{-1}) u_{\kappa_1} = 0$ for all $\kappa_1 \in \mathscr{F}_0$, hence $u = 0$.

Completeness follows from Theorem 4.10, since we for X compact have a partition of unity with only finitely many non-zero elements, hence the topology there is equal to the one defined from (25). □

4.3. **Isotropic Besov Spaces on Manifolds.** For later reference, it is briefly mentioned that all the definitions and results in Section 4.2 can be adapted to Besov spaces $B^s_{p,q;\mathrm{loc}}(X)$. E.g. they are complete, when endowed with the quasi-seminorms

$$\mu_j(u) := \| \widetilde{\psi}_j u_{\kappa(j)} \,|\overline{B}^s_{p,q}(\widetilde{X}_{\kappa(j)})\|, \quad j \in \mathbb{N},$$

and for $p,q \ge 1$ even Fréchet spaces. Moreover, when X is compact, $B^s_{p,q}(X)$ is a quasi-Banach space under the norm

$$\| u \,|B^s_{p,q}(X)\| := \Big(\sum_{\kappa\in\mathscr{F}_0} \| \widetilde{\psi}_\kappa u_\kappa \,|\overline{B}^s_{p,q}(\widetilde{X}_\kappa)\|^d \Big)^{1/d}, \quad d := \min(1,p,q). \tag{26}$$

Indeed, this results since the arguments in Section 4.2 merely rely on Lemma 2.5 and Theorem 3.1. For one thing, the paramultiplication result in Lemma 2.5 is simply replaced by a Besov version, cf. [Joh95], [RS96] or [Tri92, 4.2.2], while we now indicate the needed modifications of the invariance result in Theorem 3.1:

The proof of [JSH13b, Thm. 6], i.e. Theorem 3.1, was divided into two steps. For large s, the arguments carry over to $B^s_{p,q}$ using [Tri83, Sec. 2.7.1, Rem. 2] instead of [JSH13b, Lem. 1(iii)] and also using the characterisation of isotropic Besov spaces by kernels of local means, cf. [Ryc99a, Thm. BPT] or [Tri06, Thm. 1.10]. This characterisation also readily gives a variant of [JSH13b, Lem. 8] for $B^s_{p,q}$.

Then [JSH13b, Lem. 2] is replaced by [JSH13a, Cor. 3.3] and it is noted that [JSH13a, Thm. 4.4] carries over to the quasi-norm $\|\cdot|\ell_q(L_p)\|$. Indeed, the only modification is to apply the inequality in [Ryc99a, (21)] instead of [JSH13a, Lem. 2.7] in the last line of the proof.

Finally, the reference to [JSH13b, Thm. 2] is changed to [Ryc99a, (23)]. However, Rychkov's starting point [Ryc99a, (34)] was flawed, as mentioned in [JSH13a, Rem. 1.1], but it can be derived from our anisotropic version in [JSH13a, Prop. 4.6], as the elementary inequality $\prod(1+|2^{ja_l}z_l|)^{r_0} \ge (1+|2^{j\vec{a}}z|)^{r_0}$ brings us back at once to the isotropic maximal functions. Our anisotropic dilations by $2^{j\vec{a}}$ disappear when invoking the majorant property of the maximal function (cf. its proof in [Ste93, p. 57]).

For small s, the lift operator

$$I_r u = \mathscr{F}^{-1}(\langle\xi\rangle^r \mathscr{F}u) \tag{27}$$

is used instead of [JSH13b, (22)], because application of [Tri83, 2.3.8] then readily gives an $h \in B^{s+r}_{p,q}(\mathbb{R}^n)$ for some even integer $r > s_1 - s$ such that $e_V f = I_r h$. Since

$$I_r h = (1-\Delta)^{\frac{r}{2}} h,$$

the rest of the proof is easily carried over to a full proof of the fact that a C^∞-bijection $\sigma : U \to V$ sends $\overline{B}^s_{p,q}(V)$ boundedly into $\overline{B}^s_{p,q}(U)$.

4.4. **Mixed-Norm Lizorkin–Triebel Spaces on Curved Boundaries.** As a motivation, we first note that in case of evolution equations, the function $u(x,t)$, depending on the location x in space and the time t, describes to each t in an open interval $I \subset \mathbb{R}$ the state of a system (as a function of x in an open subset $\Omega \subset \mathbb{R}^n$). Thus solutions are sought in $C_{\mathrm{b}}(\mathbb{R}, L_{\vec{r}}(\Omega))$, say for some $\vec{r} \ge 1$, equipped with the norm

$$\sup_{t\in I} \| u(x,t) \,|L_{\vec{r}}(\Omega)\|.$$

Thus it should be natural to work in the scale of mixed-norm Lizorkin–Triebel spaces $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, in which t is taken as the outer integration variable in the norm of $L_{\vec{p}}$; i.e. we take $t = x_{n+1}$ with associated weight a_t and integral exponent p_t (when it eases notation, they will be written with $n+1$ as index).

The results in Section 4.2 can be carried over to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ under the assumptions that

$$a_0 := a_1 = \ldots = a_n, \qquad p_0 := p_1 = \ldots = p_n, \tag{28}$$

and that Ω is C^∞ in the sense adopted e.g. by [Gru09]:

Definition 4.12. An open set $\Omega \subset \mathbb{R}^n$ with boundary Γ is C^∞ (or smooth), when for each boundary point $x \in \Gamma$ there exists a diffeomorphism λ defined on an open neighbourhood $U_\lambda \subset \mathbb{R}^n$ such that $\lambda : U_\lambda \to B(0,1) \subset \mathbb{R}^n$ is surjective and

$$\begin{aligned}\lambda(x) &= 0,\\ \lambda(U_\lambda\cap\Omega) &= B(0,1)\cap\mathbb{R}^n_+,\\ \lambda(U_\lambda\cap\Gamma) &= B(0,1)\cap\mathbb{R}^{n-1},\end{aligned}$$

whereby $\mathbb{R}^n_+ := \{(x_1,\ldots,x_n)\in\mathbb{R}^n \,|\, x_n > 0\}$ and $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n-1}\times\{0\}$.

The unit ball in $\mathbb{R}^n$ will below be denoted by B and in $\mathbb{R}^{n-1}$ by B'.

4.4.1. *Curved Boundaries in General.* Let $I \subset \mathbb{R}$ be an open interval. As $\Gamma \times I$ is a C^∞-manifold, $\mathscr{D}'(\Gamma \times I)$ is a special case of Definition 4.2 and therefore the results regarding distributions on manifolds in Section 4.1 are applicable. The manifold can be equipped with e.g. the atlas $\mathscr{F} \times \mathscr{N}$, where $\mathscr{F} = \{\kappa\}$ and $\mathscr{N} = \{\eta\}$ are maximal atlases on Γ, respectively on I.

Locally finite partitions of unity $1 = \sum \psi_j(x)$ and $1 = \sum \varphi_l(t)$ subordinate to $\mathscr{F}$, respectively to $\mathscr{N}$ give a locally finite partition of unity $1 = \sum \psi_j \otimes \varphi_l$ on $\Gamma \times I$. Note that we formally should sum with respect to a fixed enumeration of the pairs (j,l) in $\mathbb{N} \times \mathbb{N}$, but for simplicity's sake we avoid this. (The sums are locally finite anyway.) As above, we use the notation $\widetilde{\psi_j \otimes \varphi_l} = (\psi_j \otimes \varphi_l) \circ (\kappa(j)^{-1} \times \eta(l)^{-1})$.

Since the maximal atlas on $\Gamma \times I$ contains charts that do not respect the splitting into t and the x-variables, it is not obviously useful for the anisotropic spaces. We have therefore chosen to adopt Lemma 4.6 as our point of departure for the $F^{s,\vec{a}}_{\vec{p},q}$-spaces on the curved boundary. Because Γ is of dimension $n-1$, it is noted that the parameters $\vec{a}$, $\vec{p}$ for these spaces only contain n entries.

Definition 4.13. The space $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ consists of all the $u \in \mathscr{D}'(\Gamma \times I)$ for which

$$\widetilde{\psi_j \otimes \varphi_l u_{\kappa(j)\times\eta(l)}} \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_{\kappa(j)} \times \widetilde{I}_{\eta(l)}), \quad j,l \in \mathbb{N}.$$

The family in (20) and Corollary 4.8 adapted to this set-up, cf. Theorem 3.2, give that

$$\mu_{j,l}(u) := \big\| \widetilde{\psi_j \otimes \varphi_l u_{\kappa(j)\times\eta(l)}} \,\big|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_{\kappa(j)} \times \widetilde{I}_{\eta(l)})\big\|, \quad j,l \in \mathbb{N}, \tag{29}$$

is a separating family of quasi-seminorms and that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ can be equivalently defined from any atlas $\mathscr{F}_1 \times \mathscr{N}_1$, where $\mathscr{F}_1 \subset \mathscr{F}$ and $\mathscr{N}_1 \subset \mathscr{N}$; with the same topology.

Theorem 4.14. *The space $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ is a complete topological vector space with a translation invariant metric; for $p_0, p_t, q \geq 1$ it is locally convex, hence a Fréchet space.*

Proof. For $d := \min(1, p_0, p_t, q)$ the separating family $(\mu^d_{j,l})_{j,l\in\mathbb{N}}$, cf. (29), is used to construct a topology as in Theorem 4.10. This immediately gives that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma \times I)$ is a topological vector space and even locally convex, when $d \geq 1$.

The metric is in this case obtained by letting the $\mu_{j,l}$ enter the summation formula for $d'(u,v)$, cf. (22), as any enumeration of the (j,l) gives the same sum; adapting the arguments in the proof of [Gru09, Thm. B.9] to two summation indices is straightforward.

Completeness follows as in the isotropic case, but with application of Theorem 3.2 instead of Theorem 3.1 when showing the convergence. $\square$

4.4.2. *Curved Boundaries in the Compact Case.* When Γ is compact, a finite atlas on the boundary can e.g. be obtained from the composite maps $\kappa = \widetilde{\gamma}_{0,n} \circ \lambda$, where $\widetilde{\gamma}_{0,n} \colon (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, 0)$ in local coordinates. Indeed, according to Definition 4.12 and the compactness of Γ there exists on Γ a finite open cover $\{U_\lambda\}$, where λ runs in an index set Λ, which together with Ω gives an open cover of $\overline{\Omega}$. Each $\lambda \in \Lambda$ induces a diffeomorphism $\kappa : \Gamma_\kappa \to B'$ on $\Gamma_\kappa := U_\lambda \cap \Gamma$ by $\kappa = \widetilde{\gamma}_{0,n} \circ \lambda$. These maps form an atlas $\mathscr{F}_0$ on Γ and thereby an atlas $\{\kappa \times \mathrm{id}_{\mathbb{R}}\}_{\kappa\in\mathscr{F}_0}$ on $\Gamma \times \mathbb{R}$.

A partition of unity is obtained by using a function $\chi \in C^\infty(\mathbb{R}^n)$ such that $\chi \equiv 1$ on $\Omega \setminus \bigcup_\lambda U_\lambda$ to slightly generalise [Gru09, Thm. 2.16]. This yields a family of functions $\{\psi_\lambda\} \cup \{\psi\}$ with $\psi_\lambda \in C_0^\infty(U_\lambda)$ and $\psi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset \Omega$ such that $\sum_\lambda \psi_\lambda(x) + \psi(x) = 1$ for $x \in \overline{\Omega}$. (Existence of such χ is similar to [Gru09, Cor. 2.14], where K need not be compact.)

In addition, the functions $\psi_\kappa := \psi_\lambda|_\Gamma \in C_0^\infty(\Gamma_\kappa)$ constitute a finite partition of unity of Γ subordinate to $\mathscr{F}_0$. Hence $1 = \sum_{\kappa\in\mathscr{F}_0} \psi_\kappa \otimes \mathbb{1}_{\mathbb{R}}$, with $\mathbb{1}_{\mathbb{R}}$ denoting the characteristic function of $\mathbb{R}$, is a partition of unity on $\Gamma \times \mathbb{R}$.

Recalling that $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times I)$ is equivalently defined from any atlas $\mathscr{F}_1\times\mathscr{N}_1$, where $\mathscr{F}_1\subset\mathscr{F}$ and $\mathscr{N}_1\subset\mathscr{N}$, we obtain

Theorem 4.15. *Let Γ be compact and $J\subset\mathbb{R}$ be a compact interval. The space*

$$\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J):=\{\,u\in F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R})\,|\,\operatorname{supp}u\subset\Gamma\times J\}\tag{30}$$

is closed and a quasi-Banach space (normed if $\vec{p},q\geq 1$), when equipped with the quasi-norm

$$\|\,u\,|\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)\|:=\Big(\sum_{\kappa\in\mathscr{F}_0}\big\|\,\widetilde{\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}}}u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\,\big|\overline{F}^{s,\vec{a}}_{\vec{p},q}(B'\times\mathbb{R})\big\|^d\Big)^{1/d},\tag{31}$$

where $d:=\min(1,p_0,p_t,q)$. Furthermore, $\|\cdot|\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)\|^d$ is subadditive.

The support condition in (30) means $\bigcup_{\kappa\in\mathscr{F}_0}(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})(\operatorname{supp}u_{\kappa\times\mathrm{id}_{\mathbb{R}}})\subset\Gamma\times J$, cf. (16), hence

$$\operatorname{supp}u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\subset B'\times J.\tag{32}$$

This implies that each summand in (31) is finite, since the factor $\mathbb{1}_{\mathbb{R}}$ can be replaced by some $\chi\in C_0^\infty(\mathbb{R})$ where $\chi=1$ on J; and this χ can be chosen as a finite sum of the φ_l from Definition 4.13.

Proof. By the same arguments as in Theorem 4.11, the expression in (31) is a quasi-norm. It gives the same topology on $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ as the family $(\mu^d_{j,l})_{j,l\in\mathbb{N}}$, since there exist $c_1,c_2>0$ such that for each $u\in\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$, cf. (29),

$$c_1\mu_{j,l}(u)^d\leq\|\,u\,|\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)\|^d\leq c_2{\sum_{j',l'\in\mathbb{N}}}'\mu_{j',l'}(u)^d,\tag{33}$$

where the prime indicates that the summation is over finitely many integers.

Indeed, Theorem 3.2 yields that $\mu_{j,l}(u)^d$ is bounded from above by

$$\begin{aligned}&\sum_{\kappa\in\mathscr{F}_0}\big\|\,(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\kappa(j)^{-1}\times\eta(l)^{-1})\widetilde{\psi_j\otimes\varphi_l}u_{\kappa(j)\times\eta(l)}\,\big|\overline{F}^{s,\vec{a}}_{\vec{p},q}\big(\kappa(j)\times\eta(l)(\Gamma_{\kappa(j)}\cap\Gamma_\kappa\times\mathbb{R}_{\eta(l)})\big)\big\|^d\\&\quad\leq c\sum_{\kappa\in\mathscr{F}_0}\big\|\,\widetilde{\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}}}(\psi_j\otimes\varphi_l)\circ(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\,\big|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\widetilde{\Gamma}_\kappa\times\mathbb{R})\big\|^d.\end{aligned}$$

Using for each $\kappa\in\mathscr{F}_0$ some function $\chi_\kappa\in C^\infty_{L_\infty}(\mathbb{R}^n)$ chosen such that $\chi_\kappa=1$ on $\operatorname{supp}\widetilde{\psi}_\kappa\cap\operatorname{supp}(\psi_j\circ\kappa^{-1})$ and $\operatorname{supp}\chi_\kappa\subset\kappa(\Gamma_{\kappa(j)}\cap\Gamma_\kappa)$, we extend by 0 to $\mathbb{R}^{n+1}$ and apply Lemma 2.5 to obtain the left-hand inequality in (33).

The right-hand inequality can be shown similarly by first replacing $\mathbb{1}_{\mathbb{R}}$ in (31) with some $\chi\in C_0^\infty(\mathbb{R})$ where $\chi=1$ on J, as discussed above.

To prove that $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ is closed, we consider an arbitrary sequence $(u_m)_{m\in\mathbb{N}}$, which belongs to $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ and converges in $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R})$ to some u. Since $u_{m,\kappa\times\mathrm{id}_{\mathbb{R}}}$ converges to $u_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ in $\mathscr{D}'(B'\times\mathbb{R})$ and (32) holds for each $u_{m,\kappa\times\mathrm{id}_{\mathbb{R}}}$, it follows that $\operatorname{supp}u_{\kappa\times\mathrm{id}_{\mathbb{R}}}\subset B'\times J$, whence $\operatorname{supp}u\subset\Gamma\times J$.

Completeness follows immediately, since each Cauchy sequence in $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ converges to some u in $F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R})$ and closedness of $\mathring{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ then gives that $\operatorname{supp}u\subset\Gamma\times J$. □

5. RYCHKOV'S UNIVERSAL EXTENSION OPERATOR

A key ingredient in the construction of right-inverses to the trace operators is a modification of Rychkov's extension operator, introduced in [Ryc99] for bounded or special Lipschitz domains $\Omega \subset \mathbb{R}^n$,

$$\mathscr{E}_{u,\Omega} : \overline{F}^s_{p,q}(\Omega) \to F^s_{p,q}(\mathbb{R}^n). \tag{34}$$

The linear and bounded operator $\mathscr{E}_{u,\Omega}$ works for all $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, cf. [Ryc99, Thm 4.1]; and it also applies to Besov spaces ($p = \infty$ included). Thus it was termed a *universal* extension operator.

In Section 6.3 below it will be clear that we for $\Omega = \mathbb{R}^n_+$ also need an extension operator for anisotropic spaces with mixed norms. We therefore modify $\mathscr{E}_{u,\Omega}$ accordingly, relying on the proof strategy in [Ryc99], yet we present significant simplifications in the proof of Proposition 5.2 and add e.g. Proposition 5.3. The reader may choose to skip the proofs in a first reading.

We take another approach than Rychkov when defining $\overline{\mathscr{S}}'(\mathbb{R}^n_+)$; this can be justified by [Ryc99, Prop. 3.1] and the remark prior to it. Similarly to [Gru96, App. A.4] we use the following:

Definition 5.1. For any open set $U \subset \mathbb{R}^n$, the space $\overline{\mathscr{S}}'(U)$ is defined as the set of $f \in \mathscr{D}'(U)$ for which there exists $\widetilde{f} \in \mathscr{S}'(\mathbb{R}^n)$ such that $r_U \widetilde{f} = f$.

The spaces $\mathring{\mathscr{S}}(\overline{U})$ and $\mathring{\mathscr{S}}'(\overline{U})$ consist of the functions in $\mathscr{S}(\mathbb{R}^n)$, respectively the distributions in $\mathscr{S}'(\mathbb{R}^n)$ supported in $\overline{U}$.

We define the convolution $\varphi * f(x)$ for $x \in \mathbb{R}^n_+$, when $f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+)$, cf. Definition 5.1, and when $\varphi \in \mathscr{S}(\mathbb{R}^n)$ has its support in the opposite half-space $\overline{\mathbb{R}}^n_-$, that is $\varphi \in \mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_-)$. This is done by using an arbitrary extension $\widetilde{f} \in \mathscr{S}'(\mathbb{R}^n)$ of f, i.e.

$$\varphi * f(x) := \langle \widetilde{f}, \varphi(x - \cdot) \rangle, \quad x \in \mathbb{R}^n_+, \tag{35}$$

which is well defined, since it as a function on $\mathbb{R}^n_+$ clearly does not depend on the choice of extension $\widetilde{f}$.

This is used in a variant of Calderón's reproducing formula (cf. Proposition 5.2 below),

$$f = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n_+), \tag{36}$$

to give meaning to each $\psi_j * (\varphi_j * f)$; cf. (5) for the subscript notation. Indeed, $\varphi_j * \widetilde{f} \in C^\infty(\mathbb{R}^n) \cap \mathscr{S}'(\mathbb{R}^n)$ is an extension of $\varphi_j * f$ by (35), so (35) also yields

$$\psi_j * (\varphi_j * f)(x) := \psi_j * (\varphi_j * \widetilde{f})(x), \quad x \in \mathbb{R}^n_+. \tag{37}$$

The idea in Rychkov's extension operator $\mathscr{E}_u$ is to use *another* extension of $\varphi_j * f$, namely

$$e_+(\varphi_j * f)(x) := \begin{cases} 0 & \text{for } x \in \overline{\mathbb{R}}^n_-, \\ \varphi_j * \widetilde{f}(x) & \text{for } x \in \mathbb{R}^n_+; \end{cases}$$

for brevity, we use $e_+ = e_{\mathbb{R}^n_+}$ and $r_+ = r_{\mathbb{R}^n_+}$. Indeed, $e_+(\varphi_j * f)$ is C^∞ for $x_n \neq 0$, hence measurable, and in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Moreover $e_+(\varphi_j * f)$ is in $\mathscr{S}'(\mathbb{R}^n)$, because it is $O((1+|x|^2)^N)$ for a large N. Using (35), we obtain the alternative formula

$$\psi_j * (\varphi_j * f)(x) = \psi_j * e_+(\varphi_j * f)(x), \quad x \in \mathbb{R}^n_+. \tag{38}$$

Here we can exploit that $\psi_j * e_+(\varphi_j * f)$ is defined on all of $\mathbb{R}^n$, hence by substituting this into the right-hand side of (36), $\mathscr{E}_u$ is obtained simply by letting x run through not just $\mathbb{R}^n_+$, but $\mathbb{R}^n$, i.e.

$$\mathscr{E}_u(f) := \sum_{j=0}^{\infty} \psi_j * e_+(\varphi_j * f) \quad \text{for } f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+). \tag{39}$$

To make this description more precise, we first justify (36). So we recall that a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ fulfils moment conditions of order L_φ, when

$$D^\alpha(\mathscr{F}\varphi)(0) = 0 \quad \text{for} \quad |\alpha| \le L_\varphi.$$

Proposition 5.2. *There exist 4 functions $\varphi_0, \varphi, \psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ supported in $\mathbb{R}^n_-$ and with $L_\varphi, L_\psi = \infty$ such that* (36) *holds for all $f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+)$.*

Proof. We shall exploit the existence of a real-valued function $g \in \mathscr{S}(\mathbb{R})$ with

$$\int g(t)\,dt \ne 0, \qquad \int t^k g(t)\,dt = 0 \quad \text{for all} \quad k \in \mathbb{N},$$

and $\operatorname{supp} g \subset [1,\infty[$. (This may be obtained as in [Ryc99, Thm. 4.1(a)].)

With $\varphi_0(x) := g(-x_1)\cdots g(-x_n)/c^n$ for $c = \int g\,dt$, the properties of g immediately give

$$\operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n \mid x_k < 0, k = 1,\dots,n\},$$
$$\int \varphi_0\,dx = 1, \quad \int x^\alpha \varphi_0(x)\,dx = 0 \quad \text{for} \quad |\alpha| > 0.$$

Thus the support of $\varphi := \varphi_0 - 2^{-|\vec{a}|}\varphi_0(2^{-\vec{a}}\cdot)$ lies in $\mathbb{R}^n_-$, and $L_\varphi = \infty$ since for $|\alpha| \ge 0$,

$$\int x^\alpha \varphi(x)\,dx = \int x^\alpha \varphi_0(x)\,dx - 2^{\vec{a}\cdot\alpha}\int x^\alpha \varphi_0(x)\,dx = 0.$$

The functions $\psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ are conveniently defined via $\mathscr{F}$,

$$\begin{aligned} \widehat{\psi}_0(\xi) &= \widehat{\varphi}_0(\xi)(2 - \widehat{\varphi}_0(\xi)^2), \\ \widehat{\psi}(\xi) &= \big(\widehat{\varphi}_0(\xi) + \widehat{\varphi}_0(2^{\vec{a}}\xi)\big)\big(2 - \widehat{\varphi}_0(\xi)^2 - \widehat{\varphi}_0(2^{\vec{a}}\xi)^2\big). \end{aligned} \tag{40}$$

Since $\widehat{\varphi}_j(\xi) = \widehat{\varphi}(2^{-j\vec{a}}\xi) = \widehat{\varphi}_0(2^{-j\vec{a}}\xi) - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)$ for $j \ge 1$, we obtain by basic algebraic rules,

$$\begin{aligned} \widehat{\psi}_j(\xi)\widehat{\varphi}_j(\xi) &= \big(2 - \widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big)\big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big) \\ &= 2\big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^2 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^2\big) - \big(\widehat{\varphi}_0(2^{-j\vec{a}}\xi)^4 - \widehat{\varphi}_0(2^{(1-j)\vec{a}}\xi)^4\big). \end{aligned}$$

This gives a telescopic sum:

$$\sum_{j=0}^\infty \widehat{\psi}_j(\xi)\widehat{\varphi}_j(\xi) = 2\lim_{N\to\infty}\widehat{\varphi}_0(2^{-N\vec{a}}\xi)^2 - \lim_{N\to\infty}\widehat{\varphi}_0(2^{-N\vec{a}}\xi)^4 = 1, \tag{41}$$

using that $\widehat{\varphi}_0(0) = 1$. As the convergence is in $\mathscr{S}'(\mathbb{R}^n)$, the inverse Fourier transformation yields

$$\sum_{j=0}^\infty \psi_j * \varphi_j = \delta. \tag{42}$$

The fact that $L_\psi = \infty$ is obvious from (40), since $D^\alpha\widehat{\varphi}_0(0) = 0$ for all $\alpha \in \mathbb{N}_0^n$. The inclusion $\operatorname{supp}\psi_0 \subset \mathbb{R}^n_-$ is clear, because $\psi_0 = \varphi_0 * (2\delta - \varphi_0 * \varphi_0)$. Similarly $\operatorname{supp}\psi \subset \mathbb{R}^n_-$, since ψ is a sum of convolutions of functions with such support.

To show (36), we note that when $\widetilde{f} \in \mathscr{S}'(\mathbb{R}^n)$ fulfils $r_+\widetilde{f} = f$, then (42) entails

$$\widetilde{f} = \sum_{j=0}^\infty \psi_j * (\varphi_j * \widetilde{f}) \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n). \tag{43}$$

More precisely, to circumvent that the summands in (42) need not have compact supports, one can show that $\sum_{j<N} \widehat{\psi}_j \widehat{\varphi}_j \mathscr{F}\widetilde{f}$ converges to $\mathscr{F}\widetilde{f}$ in $\mathscr{S}'(\mathbb{R}^n)$ by using (41) and a function in $\mathscr{S}(\mathbb{R}^n)$. Then (37) gives,

$$f = r_+\widetilde{f} = \sum_{j=0}^{\infty} r_+\big(\psi_j * (\varphi_j * \widetilde{f})\big) = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n_+),$$

in view of the continuity of $r_+ : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n_+)$. □

As a novelty, one can now show directly that $\mathscr{E}_u$ has nice properties on the space $\overline{\mathscr{S}}'(\mathbb{R}^n_+)$ of restricted temperate distributions:

Proposition 5.3. *The series for $\mathscr{E}_u(f)$ in* (39) *converges in $\mathscr{S}'(\mathbb{R}^n)$ for very $f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+)$, and the induced map $\mathscr{E}_u : \overline{\mathscr{S}}'(\mathbb{R}^n_+) \to \mathscr{S}'(\mathbb{R}^n)$ is w^*-continuous.*

Remark 5.4. The space $\overline{\mathscr{S}}'(\mathbb{R}^n_+)$ is endowed with the seminorms $f \mapsto |\langle \widetilde{f}, \varphi\rangle|$ for $\varphi \in \mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_+)$ and $r_+\widetilde{f} = f$, using the well-known fact that it is the dual of $\mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_+)$. (I.e. $f_\nu \to 0$ means that for some (hence every) net $\widetilde{f}_\nu$ of extensions, one has $\langle \widetilde{f}_\nu, \varphi\rangle \to 0$ for all $\varphi \in \mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_+)$.)

Proof. It suffices according to the limit theorem for $\mathscr{S}'$ to obtain convergence of the series

$$\sum_{j=0}^{\infty} \langle e_+(\varphi_j * f), \check{\psi}_j * \eta\rangle \quad \text{for } \eta \in \mathscr{S}(\mathbb{R}^n), \tag{44}$$

where $\check{\psi}(x) = \psi(-x)$ as usual. Since $L_\psi = \infty$, it follows at once from [JSH13a, Lem. 4.2] that the second entry tends rapidly to zero, i.e. for any seminorm p_M one has

$$p_M(\check{\psi}_j * \eta) = O(2^{-jN}) \quad \text{for every } N > 0. \tag{45}$$

For the first entries, a test against an arbitrary $\phi \in \mathscr{S}(\mathbb{R}^n)$ gives, for some M,

$$\begin{aligned} |\langle e_+(\varphi_j * f), \phi\rangle| &= \Big| \int \langle \tilde{f}(y), \varphi_j(x-y)\rangle 1_{\mathbb{R}^n_+}(x)\phi(x)\,dx\Big| \\ &= |\langle 1_{\mathbb{R}^n_+} \otimes \tilde{f}(x, x-y), \phi \otimes \varphi_j)\rangle_{\mathbb{R}^n\times\mathbb{R}^n}| \\ &\le c p_M(\phi \otimes \varphi_j) \le c' p_M(\phi) p_M(\varphi_j). \end{aligned} \tag{46}$$

Here $p_M(\varphi_j) = p_M(2^{j|\vec{a}|}\varphi(2^{j\vec{a}}\cdot)) = O(2^{j(|\vec{a}|+Ma^0)})$ grows at a fixed rate. Therefore the choice $\phi = \check{\psi}_j * \eta$ shows via (45) that the series has rapidly decaying terms, hence converges.

To obtain continuity of $\mathscr{E}_u$, it clearly suffices to show that $T\eta := \sum_{j=0}^{\infty} \check{\varphi}_j * \big(\mathbb{1}_{\mathbb{R}^n_+}(\check{\psi}_j * \eta)\big)$ defines a transformation $T : \mathscr{S}(\mathbb{R}^n) \to \mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_+)$ satisfying

$$\langle \mathscr{E}_u(f), \eta\rangle = \langle \widetilde{f}, T\eta\rangle \quad \text{for all } \eta \in \mathscr{S}(\mathbb{R}^n). \tag{47}$$

To this end we may let $\mathbb{1}_{\mathbb{R}^n_+}$ act first in (46), which via (44) gives

$$\langle \mathscr{E}_u(f), \eta\rangle = \sum_{j=0}^{\infty} \Big\langle \widetilde{f}, \int \check{\psi}_j * \eta(x) \mathbb{1}_{\mathbb{R}^n_+}(x)\varphi_j(x-y)\,dx\Big\rangle. \tag{48}$$

The integral is in $\mathscr{S}(\mathbb{R}^n)$ as a function of y (cf. the theory of tensor products), and since $\operatorname{supp}\varphi_j \subset \overline{\mathbb{R}}^n_-$ it is only non-zero for $y_n \ge x_n > 0$. Hence the summands in $T\eta$ belong to $\mathring{\mathscr{S}}(\overline{\mathbb{R}}^n_+)$, so T has range in this subspace, if its series converges in $\mathscr{S}(\mathbb{R}^n)$. But by the completeness, this follows since any seminorm p_M applied to $\int \check{\psi}_j * \eta(x)\mathbb{1}_{\mathbb{R}^n_+}(x)\varphi_j(x-y)\,dx$ is estimated by $cp_M(\check{\varphi}_j)p_{M+n+1}(\check{\psi}_j*\eta)$, which tends rapidly to 0 as above.

Finally, (48) now yields (47) by summation in the second entry. $\square$

In the next convergence result, the familiar dyadic corona condition, cf. e.g. [JS08, Lem. 3.20], has been weakened to one involving convolution with a function ψ satisfying a moment condition of infinite order. It appeared implicitly in [Ryc99].

Lemma 5.5. *Let $(g^j)_{j\in\mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$ such that*

$$\| (g^j) \| := \| 2^{js} G^j \,|L_{\vec{p}}(l_q)\| < \infty,$$

where for some $\vec{r} > 0$,

$$G^j(x) = \sup_{y\in\mathbb{R}^n} \frac{|g^j(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}}, \quad x\in\mathbb{R}^n.$$

When $\psi_0,\psi\in\mathscr{S}(\mathbb{R}^n)$ with $L_\psi=\infty$, then $\sum_{j=0}^\infty \psi_j * g^j$ converges in $\mathscr{S}'(\mathbb{R}^n)$ for any such $(g^j)_{j\in\mathbb{N}_0}$ and

$$\Big\| \sum_{j=0}^\infty \psi_j * g^j \,\Big| F^{s,\vec{a}}_{\vec{p},q} \Big\| \le c_{q,s} \| (g^j) \| \tag{49}$$

with a constant $c_{q,s}$ independent of $(g^j)_{j\in\mathbb{N}_0}$.

Proof. By assumption $\|(g^j)\| < \infty$, hence $G^j(\widetilde{x}) < \infty$ for an $\widetilde{x}\in\mathbb{R}^n$, $j\in\mathbb{N}_0$, implying $|g^j(x)| \le G^j(\widetilde{x})\prod_{l=1}^n (1+2^{ja_l}|\widetilde{x}_l - x_l|)^{r_l}$. Thereby, g^j belongs to $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and grows at most polynomially, thus g^j and therefore also $\psi_j * g^j$ are in $\mathscr{S}'(\mathbb{R}^n)$.

Using Φ_l from (1), the following estimate holds for $l\in\mathbb{N}_0$, $x\in\mathbb{R}^n$,

$$|\mathscr{F}^{-1}\Phi_l * \psi_j * g^j(x)| \le \int |\mathscr{F}^{-1}\Phi_l * \psi_j(z)||g^j(x-z)|\,dz \le I_{j,l}\cdot G^j(x), \tag{50}$$

where

$$I_{j,l} = \int |\mathscr{F}^{-1}\Phi_l * \psi_j(z)| \prod_{l=1}^n (1+2^{ja_l}|z_l|)^{r_l}\,dz.$$

Since $L_\psi = \infty = L_{\mathscr{F}^{-1}\Phi}$, a straightforward application of [JSH13a, Lem. 4.5] yields the following estimate of the anisotropic dilations in $I_{j,l}$: for every $M>0$ there is some $C_M>0$ such that

$$I_{j,l} \le C_M 2^{-|l-j|M} \quad \text{for all } \ j,l\in\mathbb{N}_0.$$

For $M = \varepsilon + |s|$, where $\varepsilon>0$ is arbitrary, we obtain from (50),

$$2^{ls}|\mathscr{F}^{-1}\Phi_l * \psi_j * g^j(x)| \le c_s 2^{js}2^{-|l-j|\varepsilon}G^j(x), \tag{51}$$

which implies, using $|j-l|\ge j-l$,

$$\| \psi_j * g^j \,|F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}\| \le c_s\Big(\sum_{l=0}^\infty 2^{(-|j-l|-2l)\varepsilon}\Big)\| 2^{js}G^j\,|L_{\vec{p}}\| \le c_s 2^{-j\varepsilon}\|(g^j)\|.$$

This yields for $d := \min(1,p_1,\dots,p_n)$,

$$\sum_{j=0}^\infty \| \psi_j * g^j \,|F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}\|^d \le c_s^d \|(g^k)\|^d \sum_{j=0}^\infty 2^{-j\varepsilon d} < \infty,$$

hence $\sum_{j=0}^\infty \psi_j * g^j$ converges in the quasi-Banach space $F^{s-2\varepsilon,\vec{a}}_{\vec{p},1}$ and thus in $\mathscr{S}'$.

Finally, by (51) and [JSH13a, Lem. 2.7] applied to $(2^{js}G^j)_{j\in\mathbb{N}_0}$,

$$\Big\| \sum_{j=0}^{\infty} \psi_j * g^j \,\Big|F^{s,\vec{a}}_{\vec{p},q}\Big\| \le c_{q,s} \Big\| \Big(\sum_{j=0}^{\infty} 2^{-|l-j|\varepsilon} 2^{js} G^j \Big)_{l\in\mathbb{N}_0} \Big| L_{\vec{p}}(\ell_q) \Big\| \le c_{q,s} \| 2^{js} G^j \,| L_{\vec{p}}(\ell_q) \|,$$

which shows (49). □

We recall a variant φ_j^+ of the Peetre-Fefferman-Stein maximal operators induced by $(\varphi_j)_{j\in\mathbb{N}_0}$, where $\varphi_0, \varphi \in \mathscr{S}(\mathbb{R}^n)$ are supported in $\mathbb{R}^n_-$; i.e. for $f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+)$ and $\vec{r} > 0$,

$$\varphi_j^+ f(x) = \sup_{y\in\mathbb{R}^n_+} \frac{|\varphi_j * f(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}}, \quad x \in \mathbb{R}^n_+, \ \ j \in \mathbb{N}_0. \tag{52}$$

Now we are ready to state the main theorem of this section:

Theorem 5.6. *When $\varphi_0, \varphi, \psi_0, \psi \in \mathscr{S}(\mathbb{R}^n)$ are functions as in Proposition 5.2, then*

$$\mathscr{E}_u(f) := \sum_{j=0}^{\infty} \psi_j * e_+(\varphi_j * f) \tag{53}$$

is a linear extension operator from $\overline{\mathscr{S}}'(\mathbb{R}^n_+)$ to $\mathscr{S}'(\mathbb{R}^n)$, i.e. $r_+\mathscr{E}_u f = f$ in $\mathbb{R}^n_+$ for every $f \in \overline{\mathscr{S}}'(\mathbb{R}^n_+)$. Moreover, $\mathscr{E}_u : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is bounded for all $s \in \mathbb{R}$, $0 < \vec{p} < \infty$ and $0 < q \le \infty$.

Proof. First it is shown using (52) that for any $f \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+)$ and $\vec{r} > \min(q, p_1, \dots, p_n)^{-1}$,

$$\| 2^{js} \varphi_j^+ f \,| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \| \le c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \|. \tag{54}$$

Besides $\varphi_j^+ f$, we shall use the well-known maximal operator $\varphi_j^* f$, where the supremum in (52) is replaced by supremum over $\mathbb{R}^n$. Hence for every $g \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ such that $r_+ g = f$, we get from (35) that

$$\varphi_j^+ f(x) = \sup_{y\in\mathbb{R}^n_+} \frac{|\varphi_j * g(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}} \le \varphi_j^* g(x), \quad x \in \mathbb{R}^n_+. \tag{55}$$

This yields (54) when combined with the following, obtained from techniques behind [JSH13a, Thm. 5.1]:

$$\inf_{r_+g=f} \| 2^{js} \varphi_j^* g \,| L_{\vec{p}}(\ell_q)(\mathbb{R}^n) \| \le c \inf_{r_+g=f} \| g \,| F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n) \| = c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \|. \tag{56}$$

More precisely, since we only have $L_\varphi = \infty$ available, it is perhaps simplest to exploit that the Tauberian conditions are fulfilled by the functions $\mathscr{F}^{-1}\Phi_0$, $\mathscr{F}^{-1}\Phi$ appearing in the definition of $F^{s,\vec{a}}_{\vec{p},q}$, cf. (1). Therefore [JSH13a, Thm. 4.4] yields that the quasi-norm on the left-hand side in (56) is estimated by $\| 2^{js} (\mathscr{F}^{-1}\Phi_j)^* g \,| L_{\vec{p}}(\ell_q) \|$, which in turn is estimated by $\| g \,| F^{s,\vec{a}}_{\vec{p},q} \|$ using [JSH13a, Thm. 4.8].

To apply Lemma 5.5, we estimate $\| (e_+(\varphi_j * f)) \|$ using the extension of (52) to $\mathbb{R}^n$, that is

$$\widetilde{\varphi}_j^+ f(x) := \sup_{y\in\mathbb{R}^n_+} \frac{|\varphi_j * f(y)|}{\prod_{l=1}^n (1+2^{ja_l}|x_l-y_l|)^{r_l}}, \quad x \in \mathbb{R}^n, \ \ j \in \mathbb{N}_0,$$

with which it is immediate to see that

$$\| (e_+(\varphi_j * f)) \| = \| 2^{js} \widetilde{\varphi}_j^+ f \,| L_{\vec{p}}(\ell_q) \|.$$

A splitting of the integral on the right-hand side in one over $\mathbb{R}^n_+$, respectively one over $\mathbb{R}^n_-$ yields, using the obvious inequality $\widetilde{\varphi}_j^+ f(x', x_n) \le \varphi_j^+ f(x', -x_n)$ for $x \in \mathbb{R}^n_-$ and (54), cf. Lemma 5.5,

$$\| \mathscr{E}_u f \,| F^{s,\vec{a}}_{\vec{p},q} \| \le c \| (e_+(\varphi_j * f)) \| \le 2c \| 2^{js} \varphi_j^+ f \,| L_{\vec{p}}(\ell_q)(\mathbb{R}^n_+) \| \le 2c \| f \,| \overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n_+) \|.$$

Finally, continuity of $r_+ : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n_+)$ together with (38) and Proposition 5.2 give

$$r_+(\mathscr{E}_u f) = \sum_{j=0}^{\infty} r_+\big(\psi_j * e_+(\varphi_j * f)\big) = \sum_{j=0}^{\infty} \psi_j * (\varphi_j * f) = f,$$

hence $\mathscr{E}_u f$ is an extension of f. □

In the study of trace operators, it will be necessary to extend from more general domains. Indeed, using the splitting $x = (x', x_n)$ on $\mathbb{R}^n$ and writing $f(x', C - x_n)$ as $f(\cdot, C - \cdot)$, the fact that $x \mapsto (x', C - x_n)$ is an involution easily gives a universal extension from the half-line $]-\infty, C[$:

Corollary 5.7. *For any $C \in \mathbb{R}$, the operator*

$$\mathscr{E}_{u,C} f(x) := \mathscr{E}_u\big(f(\cdot, C - \cdot)\big)(x', C - x_n)$$

is a linear and bounded extension from $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[)$ to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$.

Proof. The quasi-norm on $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ is invariant under translations $\tau_h u = u(\cdot - h)$, cf. [JS08, Prop. 3.3], and under the reflection $\mathscr{R}u = u(\cdot, -\cdot)$, when Φ_0, Φ are invariant under $\mathscr{R}$, as we may assume up to equivalence. So, clearly $u(x', C - x_n)$ is in $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)$ with the same quasi-norm as u.

By Definition 2.7 this readily implies that the change of coordinates is also continuous from the space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[)$ to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]0, \infty[)$. Thus

$$\begin{aligned} \| \mathscr{E}_{u,C} f \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| &\le c \| \mathscr{E}_u(f(\cdot, C - \cdot)) \,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n)\| \\ &\le c \| f(\cdot, C - \cdot) \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]0, \infty[)\| \le c \| f \,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n-1} \times]-\infty, C[)\|, \end{aligned}$$

and the linearity of $\mathscr{E}_{u,C}$ follows directly from the linearity of $\mathscr{E}_u$. □

In comparison with the well-known half-space extension by Seeley [See64] we note that the above construction is applicable for all $s \in \mathbb{R}$, even in the mixed-norm case. Also it has the advantage that several results from [JSH13a] can be utilised, making the argumentation less cumbersome.

6. Trace Operators

Under the assumption in (28), we study the trace at the flat boundary of a cylinder $\Omega \times I$, where $\Omega \subset \mathbb{R}^n$ is C^∞ and $I :=]0, T[$, possibly $T = \infty$. The trace at the curved boundary is studied only for $T < \infty$ and under the additional assumption that $\partial\Omega$ is compact. The associated operators are

$$\begin{aligned} r_0 :\ & f(x_1, \dots, x_n, t) \mapsto f(x_1, \dots, x_n, 0), \\ \gamma :\ & f(x_1, \dots, x_n, t) \mapsto f(x_1, \dots, x_n, t)|_\Gamma. \end{aligned}$$

As a preparation (for a discussion of compatibility conditions), the chapter ends with a discussion of traces on both the flat and the curved boundary at the corner $\partial\Omega \times \{0\}$ of the cylinder.

For the reader's sake, we recall some notation from [JS08], namely that the trace at the hyperplane where $x_k = 0$ is denoted by $\gamma_{0,k}$:

$$\gamma_{0,k} :\ f(x_1, \dots, x_n, t) \mapsto f(x_1, \dots, 0, \dots, x_n, t). \tag{57}$$

It will be convenient for us to use $p' := (p_1, \dots, p_{k-1})$, $p'' := (p_{k+1}, \dots, p_n, p_t)$, analogously for $\vec{a}$, and $r_l := \max(1, p_l)$. Furthermore, we recall that $x_{n+1} = t$, $a_{n+1} = a_t$, $p_{n+1} = p_t$, hence we shall work with $\vec{a}, \vec{p}$ of the form, cf. (28),

$$\vec{a} = (a_0, \dots, a_0, a_t), \qquad \vec{p} = (p_0, \dots, p_0, p_t) < \infty, \tag{58}$$

where the finiteness of $\vec{p}$ is assumed in order to apply the results in [JS08].

6.1. **The Trace at the Flat Boundary.** The trace r_s, defined by evaluation at $t=s$, is for each $s\in I$ well defined on the subspace,
$$C(I,\mathscr{D}'(\Omega))\subset\mathscr{D}'(\Omega\times I), \tag{59}$$
where the embedding can be seen by modifying the proof of [Joh00, Prop. 3.5]. On the smaller subspace $C(\bar{I},\mathscr{D}'(\Omega))$ consisting of the elements having a continuous extension in t to $\mathbb{R}$, even the trace r_0 is well defined (and it induces a similar operator also denoted r_0). Indeed, for $u\in C(\bar{I},\mathscr{D}'(\Omega))$ all extensions f are equal in $\Omega\times I$ and by continuity therefore also at $t=0$, hence
$$r_0u:=f(\cdot,0). \tag{60}$$

Now, it was shown in [JS08, Thm. 2.4] that
$$F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\hookrightarrow C_{\mathrm{b}}(\mathbb{R},L_{r'}(\mathbb{R}^n))\quad\text{when}\quad s>\frac{a_t}{p_t}+n\Big(\frac{a_0}{\min(1,p_0)}-a_0\Big), \tag{61}$$
and this induces an embedding $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\hookrightarrow C(\bar{I},L_{r'}(\Omega))$, so the trace r_0 can be applied to u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, i.e. for an arbitrary extension f in $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$,
$$r_0u=r_\Omega f(\cdot,0). \tag{62}$$

To define a right-inverse of r_0 when applied to $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, we recall that a bounded right-inverse K_{n+1} of the analogous trace $\gamma_{0,n+1}$ on Euclidean space, cf. [JS08, Thm. 2.6],
$$K_{n+1}:B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n)\to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}),\quad s\in\mathbb{R}, \tag{63}$$
is given by the following, where $\psi\in C^\infty(\mathbb{R})$ such that $\psi(0)=1$ and $\operatorname{supp}\mathscr{F}\psi\subset[1,2]$,
$$K_{n+1}v(x):=\sum_{j=0}^{\infty}\psi(2^{ja_{n+1}}x_{n+1})\mathscr{F}^{-1}(\Phi_j(\xi',0)\mathscr{F}v(\xi'))(x'). \tag{64}$$

Theorem 6.1. *When $\vec{a},\vec{p}$ fulfil* (58) *and s satisfies the inequality in* (61)*, then*
$$r_0:\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\to\overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)$$
is a bounded surjection *and it has a right-inverse K_0. More precisely, the operator K_0 can be chosen so that $K_0:\overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)\to\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ is bounded for all $s\in\mathbb{R}$.*

Proof. The analogue of this theorem on Euclidean spaces, cf. [JS08, Thm. 2.5], yields for any $f\in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$ the existence of a constant c (only depending on $s,\vec{p},q,\vec{a}$) such that
$$\big\|\gamma_{0,n+1}f\,|B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n)\big\|\le c\|f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|.$$
Choosing f in (62) so the right-hand side is bounded by $2c\|u\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|$, we obtain boundedness of r_0, since $r_\Omega(\gamma_{0,n+1}f)=r_0u$, cf. (57).

A right-inverse K_0 is constructed using K_{n+1} in (63) and Rychkov's extension operator in (34):
$$K_0:=r_{\Omega\times I}\circ K_{n+1}\circ\mathscr{E}_{u,\Omega}:\overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)\to\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I). \tag{65}$$
(Since (34) applies only to isotropic spaces over $\Omega\subset\mathbb{R}^n$, one can exploit (58) to make rescalings $(s,a')\leftrightarrow s/a_0$, cf. Lemma 2.3.)

It is bounded for all $s\in\mathbb{R}$, because K_{n+1} and $\mathscr{E}_{u,\Omega}$ are so. Finally, (62) yields for any $v\in\overline{B}^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\Omega)$,
$$r_0\circ K_0v=r_\Omega(K_{n+1}\circ\mathscr{E}_{u,\Omega}v)(x_1,\dots,x_n,0)=r_\Omega\circ\gamma_{0,n+1}\circ K_{n+1}\circ\mathscr{E}_{u,\Omega}v=v,$$

hence K_0 is a right-inverse of r_0. □

6.2. **A Support Preserving Right-Inverse.** As a further preparation for a discussion of parabolic boundary problems we now present a support preserving right-inverse to the trace at $\{t=0\}$. It is useful in reduction to problems with homogeneous boundary conditions. At no extra cost, general $\vec{a}$ and $\vec{p}$ are treated in most of this section.

It is known from [JS08] that whenever $s > \frac{a_t}{p_t} + \sum_{k\le n}\Big(\frac{a_k}{\min(1,p_1,\dots,p_k)} - a_k\Big)$, then r_0 is bounded,

$$r_0 : F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^n\times\mathbb{R}) \to B^{s-\frac{a_t}{p_t},a'}_{p',p_t}(\mathbb{R}^n).$$

The particular right-inverse in (64) shall now be replaced by a finer construction of a right-inverse Q having the useful property that

$$\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \implies \operatorname{supp} Qu \subset \overline{\mathbb{R}}^n_+ \times \mathbb{R}. \tag{66}$$

Roughly speaking the idea is to replace the use of Littlewood–Paley decompositions by the kernels of local means $(k_j)_{j\in\mathbb{N}_0}$. That is, we tentatively take Q of the form

$$Qu(x,t) = \sum_{j=0}^{\infty} \eta(2^{ja_t}t)k_j * u(x). \tag{67}$$

Hereby the auxiliary function $\eta\in\mathscr{S}(\mathbb{R})$ is again chosen with $\eta(0)=1$ and such that $\operatorname{supp}\widehat{\eta}\subset[1,2]$.

The main reason for this choice of Qu is that the property (66) will eventually result when the kernels k_j are so chosen that

$$\operatorname{supp} u \subset \overline{\mathbb{R}}^n_+ \implies \operatorname{supp} k_j * u \subset \overline{\mathbb{R}}^n_+. \tag{68}$$

By the support rule for convolutions, this follows if $\operatorname{supp} k_j \subset \overline{\mathbb{R}}^n_+$. However, in order to choose the k_j, we shall first take functions $\varphi_0,\varphi,\psi_0,\psi$ in $\mathscr{S}(\mathbb{R}^n)$ with support in $\overline{\mathbb{R}}^n_+$ and satisfying

$$\int \varphi_0\,dx = 1 = \int \psi_0\,dx, \qquad L_\varphi = \infty = L_\psi, \tag{69}$$

in such a way that by setting e.g. $\psi_j(x) = 2^{j|\vec{a}|}\psi(2^{j\vec{a}}x)$ one has Calderon's reproducing formula

$$u = \sum_{j=0}^{\infty} \psi_j * \varphi_j * u \quad \text{ for } u\in\mathscr{S}'(\mathbb{R}^n). \tag{70}$$

Existence of these functions may be obtained as in the proof of Proposition 5.2, simply by omitting the reflection in the definition of φ_0 and proceeding with the argument for (43) in the proof there.

Now we can simply obtain $\operatorname{supp} k_j \subset \overline{\mathbb{R}}^n_+$ by choosing

$$k_0 = \psi_0 * \varphi_0, \quad k = \psi * \varphi.$$

Then (70) states that $u = \sum_{j\ge 0} k_j * u$, which together with the condition $\eta(0)=1$ will imply that Q is a right-inverse of r_0.

Since the supports of the k_j are only confined to be in the half-space $\overline{\mathbb{R}}^n_+$, we refer to the k_j as kernels of *localised* means. (Triebel termed them local in case the supports are compact.)

In addition we need to recall an $\mathscr{S}'$-version of [Joh00, Prop. 3.5].

Lemma 6.2. *There is an (algebraic) embedding $C_{\mathrm{b}}(\mathbb{R},\mathscr{S}'(\mathbb{R}^n))\subset\mathscr{S}'(\mathbb{R}^n\times\mathbb{R})$ given by*

$$\langle \Lambda_f,\psi\rangle = \int_{\mathbb{R}} \langle f(t),\psi(\cdot,t)\rangle_{\mathbb{R}^n}\,dt$$

for each continuous, bounded map $f:\mathbb{R}\to\mathscr{S}'(\mathbb{R}^n)$ and $\psi\in\mathscr{S}(\mathbb{R}^n\times\mathbb{R})$.

Proof. By the boundedness, the family $\{f(t)\}_{t\in\mathbb{R}}$ is equicontinuous, so for some $M>0$ we have $|\langle f(t),\phi\rangle| \le cp_M(\phi)$ for all $t\in\mathbb{R}$ and $\phi\in\mathscr{S}(\mathbb{R}^n)$. Hence the integrand is continuous and estimated crudely by $cp_{M+2}(\psi)/(1+t^2)$, so Λ_f makes sense and $|\langle\Lambda_f,\psi\rangle| \le c\pi p_{M+2}(\psi)$. □

Using this lemma, we can now improve on (67) by giving Qu a more precise meaning as an element of $C_{\mathrm{b}}(\mathbb{R}_t,\mathscr{S}'(\mathbb{R}^n_x))$. Namely, $Qu(\cdot,t)$ is the distribution given on $\phi\in\mathscr{S}(\mathbb{R}^n)$ by

$$\langle Qu(\cdot,t),\phi\rangle = \sum_{j=0}^{\infty}\eta(2^{ja_t}t)\langle k_j*u,\phi\rangle. \tag{71}$$

This will be clear from the proof of

Proposition 6.3. *The operator Q is a well-defined w*-continuous linear map $\mathscr{S}'(\mathbb{R}^n)\to\mathscr{S}'(\mathbb{R}^n\times\mathbb{R})$ having range in $C_{\mathrm{b}}(\mathbb{R}_t,\mathscr{S}'(\mathbb{R}^n_x))$. It is a right-inverse of r_0 preserving supports in $\overline{\mathbb{R}}^n_+$ in the strong form*

$$\operatorname{supp}u\subset\overline{\mathbb{R}}^n_+ \implies \forall t:\operatorname{supp}Qu(\cdot,t)\subset\overline{\mathbb{R}}^n_+. \tag{72}$$

In particular, $Q:\mathring{\mathscr{S}}'(\overline{\mathbb{R}}^n_+)\to\mathring{\mathscr{S}}'(\overline{\mathbb{R}}^n_+\times\mathbb{R})$, cf. Definition 5.1.

Remark 6.4. We can of course add that (72) $\implies$ (66), for we may apply Lemma 6.2 to $f=Qu$ and consider the $\psi(x,t)$ that vanish for $x_n\ge 0$: when (72) holds, the integrand is identically 0. (Unlike (72), property (66) is meaningful also without continuity of Qu with respect to t.)

Proof. It is first noted that $\sum\langle k_j*u,\phi\rangle$ converges absolutely for each test function $\phi\in\mathscr{S}'(\mathbb{R}^n)$. In fact, using the notation $\check{k}_j(x)=k_j(-x)$, the estimate $|\langle u,\check{k}_j*\phi\rangle|\le cp_M(k_j*\phi)\le c2^{-jN}$ holds for any $N>0$; this follows from the infinitely many vanishing moments, i.e. $L_k=\infty$, cf. [JSH13a, Lem. 4.2].

Hence $\sum\langle k_j*u,\phi\rangle\eta(2^{ja_t}t)$ is a Cauchy series for each $\phi\in\mathscr{S}(\mathbb{R}^n)$ as $\eta(2^{ja_t}t)$ is a bounded sequence for fixed t. Since it converges, Qu is defined in $\mathscr{S}'(\mathbb{R}^n)$ for each t.

The convergence is absolute and uniform in t, so $t\mapsto\langle Qu(t),\phi\rangle$ is continuous; and bounded by $c\sum|\langle k_j*u,\phi\rangle|$. Therefore Qu is in the subspace $C_{\mathrm{b}}(\mathbb{R}_t,\mathscr{S}'(\mathbb{R}^n_x))$; cf. Lemma 6.2.

Consequently r_0Qu is defined by evaluation at $t=0$, which gives $\sum\eta(0)k_j*u(x)$, hence gives back u because of (70). Using the convergence in $\mathscr{S}'(\mathbb{R}^n)$, the support preservation in (72) is immediate from (68) by test against any $\phi\in C_0^\infty(\mathbb{R}^n)$ vanishing for $x_n\ge 0$.

Finally, continuity of Q follows at once if $\langle Qu,\psi\rangle=\langle u,T\psi\rangle$ for $\psi\in\mathscr{S}(\mathbb{R}^{n+1})$, i.e. if Q is the transpose of $T:\mathscr{S}(\mathbb{R}^{n+1})\to\mathscr{S}(\mathbb{R}^n)$ given by

$$(T\psi)(x)=\int_{\mathbb{R}}\sum_{j=0}^{\infty}\check{k}_j*\psi(x,t)\eta(2^{ja_t}t)\,dt.$$

This series is Cauchy in the space $\mathscr{S}(\mathbb{R}^{n+1})$, for a seminorm p_M applied to the general term is less than $p_M(\eta(2^{ja_t}t))=O(2^{ja_tM})$ times $p_M(\check{k}_j*\psi)$, which decays rapidly as $L_k=\infty$. Denoting the sum by $S(x,t)$, also $x\mapsto\int S(x,t)\,dt$ is a Schwartz function, so $T\psi$ is well defined and by the definition of tensor products we get

$$\langle u,T\psi\rangle=\langle u\otimes 1,S\rangle=\int\langle u,S(\cdot,t)\rangle\,dt=\int\langle Qu(\cdot,t),\psi(\cdot,t)\rangle\,dt=\langle Qu,\psi\rangle,$$

using (71) and Lemma 6.2. □

Before we go deeper into the boundedness of Q in the scales of mixed-norm Lizorkin–Triebel spaces, we first sum up the fundamental estimate in the next result. In the isotropic case it goes back at least to the trace investigations of Triebel [Tri83, p. 136].

Proposition 6.5. *For $\vec{p}=(p_1,\dots,p_n,r)$ in $]0,\infty[^{n+1}$, a real number $a>0$ and $0<q\le\infty$ there is a constant c with the property that*

$$\big\| \{v_j\otimes 2^{j\frac{a}{r}}f(2^{ja}\cdot)\}_{j=0}^\infty \,\big|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1})\big\| \le c\Big(\sum_{j=0}^\infty \|v_j\,|L_{p'}(\mathbb{R}^n)\|^r\Big)^{1/r}$$

whenever (v_j) is a sequence of measurable functions on $\mathbb{R}^n$ and $f\in C(\mathbb{R})$ is such that $t^N f(t)$ is bounded for some $N>0$ satisfying $Nr>1$.

Proof. To save a page of repetition from [JS08, Sec. 4.2.3], we leave it to the reader to carry over the proof given there with a few notational changes. (Note that f itself is bounded, so the arguments there extend to our case without any Schwartz class assumptions on f.) $\square$

Theorem 6.6. *The operator Q is for $0<\vec{p}<\infty$, $0<q\le\infty$ and $\vec{a}\ge 1$ a bounded map*

$$Q: B^{s,a'}_{p',p_t}(\mathbb{R}^n)\to F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^n\times\mathbb{R})\quad \textit{for all } s\in\mathbb{R}.$$

Proof. By means of an auxiliary function $\mathscr{F}\tilde\eta\in C_0^\infty(\mathbb{R})$ fixed such that $\mathscr{F}\tilde\eta=1$ on $[1,2]\supset\operatorname{supp}\widehat{\eta}$ and $\operatorname{supp}\mathscr{F}\tilde\eta\subset]0,\infty[$ we rewrite Qu in terms of convolutions on $\mathbb{R}^{n+1}$, using that $k_j=\psi_j*\varphi_j$,

$$Qu=\sum_{j=0}^\infty \tilde\eta_j*\eta(2^{ja_t}\cdot)(t)k_j*u(x)=\sum_{j=0}^\infty(\psi\otimes\tilde\eta)_j*(\varphi_j*u\otimes\eta(2^{ja_t}\cdot)).$$

Hereby it is understood for $j=0$ that the first factor is $\psi_0\otimes\tilde\eta$.

Now we may invoke Lemma 5.5 as the function $\psi\otimes\tilde\eta$ has all its moments equal to 0, because its Fourier transformed function is supported in a half-plane disjoint from the origin in $\mathbb{R}^{n+1}$. This gives an estimate of the Lizorkin–Triebel norm as follows,

$$\big\|Qu\,\big|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\big\|\le c\big\|\{2^{(s+\frac{a_t}{p_t})j}(\varphi_j*u\otimes\eta(2^{ja_t}\cdot))^*_j\}_{j=0}^\infty\,\big|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1})\big\|.$$

Here the maximal function $(\cdot)^*_j$ considered in the lemma allow us to estimate the j^{th} term by

$$\sup_{y,y_t}\big|2^{sj}\varphi_j*u(y)2^{j\frac{a_t}{p_t}}\eta(2^{ja_t}y_t)\big|\prod_{l=1}^{n+1}(1+2^{ja_l}|x_l-y_l|)^{-r_l}\le v_j(x)2^{j\frac{a_t}{p_t}}f(2^{ja_t}t)$$

if we set

$$\begin{aligned} v_j&=\sup_y|2^{sj}\varphi_j*u(y)|\prod_{l=1}^n(1+2^{ja_l}|x_l-y_l|)^{-r_l},\\ f(t)&=\sup_{y_t}|\eta(y_t)|(1+|t-y_t|)^{-r_t}.\end{aligned}$$

To invoke Proposition 6.5, we note that v_j, f are continuous (by an argument similar to e.g. [Joh11, (6)–(7)]) and, moreover, $\sup|t^Nf(t)|<\infty$ for $0<N\le r_t$. We therefore apply the proposition for $r=p_t$, $a=a_t$ and note that if we fix the above parameter r_t such that $r_tp_t>1$, then $Np_t>1$ is fulfilled at least for $N=r_t$. This gives

$$\big\|Qu\,\big|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\big\|\le c\big\|\{v_j\otimes 2^{j\frac{a_t}{p_t}}f(2^{ja_t}\cdot)\}_{j=0}^\infty\,\big|L_{\vec{p}}(\ell_q)(\mathbb{R}^{n+1})\big\|\le c\Big(\sum_{j=0}^\infty\|v_j\,|L_{p'}(\mathbb{R}^n)\|^{p_t}\Big)^{1/p_t}.$$

So by writing the v_j in terms of the Peetre–Fefferman–Stein maximal function $\varphi_j^*u(x)$,

$$\big\|Qu\,\big|F^{s+\frac{a_t}{p_t},\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\big\|\le c\Big(\sum_{j=0}^\infty\|2^{sj}\varphi_j^*u\,|L_{p'}(\mathbb{R}^n)\|^{p_t}\Big)^{1/p_t}\le c\|u\,|B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|.$$

The last inequality is essentially known from [Ryc99a, (4)], but to account for effects of the flaws pointed out in [JSH13a, Rem. 1.1], let us briefly note the following: if we apply [Ryc99a, (21)] to the very last formula in the proof of [JSH13a, Thm. 4.4], then we get an estimate of the above sum by $\| 2^{sj}(\mathscr{F}^{-1}\Phi)_j^* u \,|\, \ell_{p_t}(L_{p'})\|$. This can be controlled by the $\ell_{p_t}(L_{p_0})$-norm of the convolutions $2^{sj}\mathscr{F}^{-1}\Phi_j * u$ (i.e. by the stated $\| u \,|\, B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|$) by following the argument for [Ryc99a, (23)], after the remedy discussed in Section 4.3, say for simplicity with $r_0 := r_1 = \ldots = r_n$ and $r_0 p_0 > n$. □

Remark 6.7. By combining Proposition 6.3 and Theorem 6.6, one directly obtains

$$Q : \overset{\circ}{B}{}^{s,a'}_{p',q}(\overline{\mathbb{R}}^n_+) \to \overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}) \quad \text{for all } s \in \mathbb{R}.$$

The operator Q is now used to replace the particular right-inverse to r_0 in (65) by an operator Q_Ω that preserves support in $\overline{\Omega}$.

The construction uses the partition of unity $1 = \sum_\lambda \psi_\lambda + \psi$ on $\overline{\Omega}$ constructed in Section 4.4 as well as cut-off functions $\eta_\lambda \in C_0^\infty(\mathbb{R}^n)$, $\lambda \in \Lambda$, chosen such that $\operatorname{supp}\eta_\lambda \subset B$ and $\eta_\lambda = 1$ on $\operatorname{supp}\widetilde{\psi}_\lambda$. Moreover, $\eta_\Omega \in C^\infty_{L_\infty}(\mathbb{R}^n)$, cf. Lemma 2.5 for the definition of $C^\infty_{L_\infty}$, and $\operatorname{supp}\eta_\Omega \subset \Omega$ with $\eta_\Omega = 1$ on $\operatorname{supp}\psi$.

Theorem 6.8. *When $\vec{a},\vec{p}$ satisfy (58), $0 < q \le \infty$ and $s \in \mathbb{R}$, then the operator Q_Ω defined by*

$$Q_\Omega u := \sum_\lambda e_{U_\lambda \times \mathbb{R}}\big((\eta_\lambda Q u_\lambda)\circ(\lambda \times \mathrm{id}_\mathbb{R})\big) + \eta_\Omega Q(\psi u), \quad u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n), \tag{73}$$

where $u_\lambda := e_B\big((\psi_\lambda u)\circ\lambda^{-1}\big)$, is bounded,

$$Q_\Omega : B^{s,a'}_{p',p_t}(\mathbb{R}^n) \to F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}),$$

and $r_0 Q_\Omega u = u$ whenever $u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n)$ fulfils $\operatorname{supp} u \subset \overline{\Omega}$.

Moreover, Q_Ω has range in $C(\mathbb{R}_t, \mathscr{D}'(\mathbb{R}^n_x))$ and preserves supports in $\overline{\Omega}$ in the strong form

$$\operatorname{supp} u \subset \overline{\Omega} \implies \forall t : \operatorname{supp} Q_\Omega u(\cdot,t) \subset \overline{\Omega}. \tag{74}$$

Proof. For the terms in the sum over λ in (73), we note that the multiplication result in [Tri92, 4.2.2] together with the Besov version of Theorem 3.1, cf. Section 4.3, imply

$$u_\lambda = e_B\big((\psi_\lambda u)\circ\lambda^{-1}\big) \in B^{s,a'}_{p',p_t}(\mathbb{R}^n). \tag{75}$$

(These results apply to isotropic Besov spaces, so we use Lemma 2.3 to rescale $(s,a') \leftrightarrow s/a_0$, cf. (58).)

Theorem 6.6 and the paramultiplication result [JSH13b, Lem. 7] now gives that $\eta_\lambda Q u_\lambda$ belongs to $F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$, hence according to Theorem 3.3,

$$(\eta_\lambda Q u_\lambda)\circ(\lambda \times \mathrm{id}_\mathbb{R}) \in \overline{F}^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(U_\lambda \times \mathbb{R}).$$

As $\operatorname{supp}\eta_\lambda \subset B$, Lemma 2.9 gives that extension of this composition by 0 belongs to $F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$.

For the last term in (73), it is an immediate consequence of [Tri92, 4.2.2] that ψu belongs to $B^{s,a'}_{p',p_t}(\mathbb{R}^n)$, since $\psi \in C^\infty_{L_\infty}$ (as $\psi = 1 - \sum_\lambda \psi_\lambda$ on $\overline{\Omega}$ and $\partial\Omega$ is compact).

This shows that $Q_\Omega u \in F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$ and by applying the quasi-norm estimates in the theorems and lemmas referred to above, we obtain

$$\| Q_\Omega u \,|\, F^{s+a_t/p_t,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le c \| u \,|\, B^{s,a'}_{p',p_t}(\mathbb{R}^n)\|.$$

Furthermore, it follows from Proposition 6.3 that $Q_\Omega u \in C(\mathbb{R}_t, \mathscr{D}'(\mathbb{R}^n_x))$ and therefore the effect of r_0 on $Q_\Omega u$ is simply restriction to $t=0$, cf. (60). Hence for $u \in B^{s,a'}_{p',p_t}(\mathbb{R}^n)$,

$$r_0 Q_\Omega u = \sum_\lambda e_{U_\lambda}\big((\eta_\lambda Q u_\lambda)(\cdot,0)\circ\lambda\big) + \eta_\Omega Q(\psi u)(\cdot,0).$$

Since Q according to Proposition 6.3 is a right-inverse of r_0, this sum equals the following by using (75) as well as the properties of $\eta_\lambda, \eta_\Omega$, and in the final step that $\operatorname{supp} u \subset \overline{\Omega}$,

$$\sum_\lambda e_{U_\lambda}\big((\eta_\lambda u_\lambda)\circ\lambda\big) + \psi u = \sum_\lambda e_{U_\lambda}\big(\eta_\lambda\circ\lambda\cdot\psi_\lambda u\big) + \psi u = \sum_\lambda \psi_\lambda u + \psi u = u.$$

Finally, the support preserving property in (74) follows from (72). Indeed, when $\operatorname{supp} u \subset \overline{\Omega}$, then the support of each u_λ is contained in $\overline{\mathbb{R}}^n_+$ and therefore $\operatorname{supp}(\eta_\lambda Q u_\lambda)\circ(\lambda(\cdot),t) \subset \overline{\Omega}$ for all $t \in \mathbb{R}$, which immediately gives that $\operatorname{supp} Q_\Omega u(\cdot,t) \subset \overline{\Omega}$. □

6.3. **The Trace at the Curved Boundary.** We now address the trace γ of distributions in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, where for simplicity $I =]0,T[$, $T<\infty$, and Ω is smooth as in Definition 4.12 with compact boundary Γ.

6.3.1. *Preliminaries.* The trace is first worked out locally and then it is observed that the local pieces define a global trace. In this process we use that the trace $\gamma_{0,1}$ is a bounded surjection, cf. [JS08, Thm. 2.2],

$$\begin{aligned}\gamma_{0,1} : &F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}) \to F^{s-\frac{a_0}{p_0},a''}_{p'',p_0}(\mathbb{R}^n)\\ &\text{for}\quad s > \frac{a_0}{p_0} + (n-1)\Big(\frac{a_0}{\min(1,p_0,q)} - a_0\Big) + \Big(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Big).\end{aligned} \tag{76}$$

This is also valid for $\gamma_{0,n}$ in view of (58) and we prefer to work with this, for locally the boundary Γ is defined by the equation $x_n = 0$, as usual. For the s in (76), we have by [JS08, Thm. 2.1], since $r_k := \max(1,p_k)$,

$$F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}) \hookrightarrow C_{\mathrm{b}}(\mathbb{R}, L_{r''}(\mathbb{R}^n)) \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}^{n+1}). \tag{77}$$

So when $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ for such s, an extension f in the corresponding space on $\mathbb{R}^n$ is a function and for this we right away get

$$f\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}}) \in L_{1,\mathrm{loc}}(B\times\mathbb{R}). \tag{78}$$

Moreover, if we work locally with cut-off functions $\psi \in C_0^\infty(U_\lambda)$, $\varphi \in C_0^\infty(\mathbb{R})$, then Lemma 2.5 yields $\psi\otimes\varphi f \in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$. Changing coordinates, Theorem 3.3 implies that $(\psi\otimes\varphi f)\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}})$ is in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(B\times\mathbb{R})$, hence it extends by 0 to $F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$. By (76),

$$\gamma_{0,n}\big((\psi\otimes\varphi f)\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}})\big) \in F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n).$$

Strictly speaking, we should have inserted the extension by 0, namely $e_{B\times\mathbb{R}}$, before applying $\gamma_{0,n}$, but we have chosen not to burden notation with this. Now restriction to $B'\times\mathbb{R}$ gives an element in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B'\times\mathbb{R})$, and since it is easily seen using (77) that restriction to $\{x_n = 0\}$ and $e_{B\times\mathbb{R}}$ can be interchanged, we obtain

$$(\psi\otimes\varphi f)\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}}) \in \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B'\times\mathbb{R}). \tag{79}$$

Furthermore, to describe the range of γ, we introduce for an open interval $I' \supset I$ the restriction (with notation as in Section 4.4)

$$r_I : F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times I') \to F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times I),$$

which for any $v \in F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times I')$ is defined as the distribution arising from the family $\{r_{B'\times I} v_{\kappa\times\mathrm{id}_{I'}}\}_{\kappa\in\mathscr{F}_0}$ of distributions on $B'\times I$, cf. the paragraph on restriction just below Lemma 4.3.

Using r_I, we also introduce a space of restricted distributions (in the time variable only),

$$\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times I) := r_I F^{s,\vec{a}}_{\vec{p},q;\mathrm{loc}}(\Gamma\times\mathbb{R}) = r_I \overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J) \tag{80}$$

valid for any compact interval $J\supset I$. Since $\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)$ is a quasi-Banach space, cf. Theorem 4.15, the space $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times I)$ is so too when equipped with

$$\| u\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times I)\| := \inf_{r_I v=u} \| v\,|\overset{\circ}{F}{}^{s,\vec{a}}_{\vec{p},q}(\Gamma\times J)\|. \tag{81}$$

6.3.2. *The Definition.* To give sense to γu in $\mathscr{D}'(\Gamma\times I)$, it is first observed that (78) induces invariantly defined functions. Indeed, in view of the identity $\kappa^{-1}(\cdot)=\lambda^{-1}(\cdot,0)$, we set

$$f_\kappa = f\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}}) \in L_{1,\mathrm{loc}}(B'\times\mathbb{R})$$

and as distributions they transform as in (14), since

$$f_\kappa\circ(\kappa\circ\kappa_1^{-1}\times\mathrm{id}_{\mathbb{R}}) = f_{\kappa_1} \quad \text{on} \quad \kappa_1(\Gamma_{\kappa_1}\cap\Gamma_\kappa)\times\mathbb{R}. \tag{82}$$

Hence by Lemma 4.3 there exists a unique $v\in\mathscr{D}'(\Gamma\times\mathbb{R})$ with

$$v_{\kappa\times\mathrm{id}_{\mathbb{R}}} = f_\kappa. \tag{83}$$

That v is in $F^{s-a_0/p_0,a''}_{p'',p_0;\mathrm{loc}}(\Gamma\times\mathbb{R})$ is a special case of (79), cf. Definition 4.13.

Note that the distribution v does not depend on the atlas $\mathscr{F}_0$, for when another atlas $\mathscr{F}_1$ in the same way induces a distribution v_1, then formula (82) read with κ running through $\mathscr{F}_0$ and κ_1 running through $\mathscr{F}_1$ implies that both v and v_1 result by "restriction" from the distribution w induced by $\mathscr{F}_0\cup\mathscr{F}_1$.

Now we define the trace γu in $\mathscr{D}'(\Gamma\times I)$ by

$$\gamma u = r_I v. \tag{84}$$

Indeed, to verify that γu is independent of the chosen f, it suffices to derive that for any two extensions $f_1,f_2\in F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})$, the following identity holds for each $\lambda\in\Lambda$ and $(x',t)\in B'\times I$:

$$f_1\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t) = f_2\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t). \tag{85}$$

To do so, we choose $\psi\in C_0^\infty(U_\lambda)$, $\varphi\in C_0^\infty(\mathbb{R})$ such that $\psi(\lambda^{-1}(x',0))\neq 0$ and $\varphi(t)\neq 0$. Since f_1,f_2 coincide in $\Omega\times I$, the functions

$$e_{B\times\mathbb{R}}\big((\psi\otimes\varphi f_j)\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}})\big)(x,t), \quad j=1,2$$

are identical for $(x,t)\in B\times I$ with $x_n>0$. Letting $x_n\to 0^+$ therefore gives the same limits in $L_{r''}(\mathbb{R}^{n-1}\times I)$, cf. (77), in particular they coincide in $L_{r''}(B'\times I)$. As $(\psi\otimes\varphi)\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(x',t)\neq 0$, this yields (85).

Furthermore, (85) can be used to show that γ does not depend on the Lizorkin–Triebel space satisfying (76). For when u belongs to two different Lizorkin–Triebel spaces, we can take f_1 above to be an extension in one of the spaces and f_2 to be an extension in the other. The identity in (85) then gives that γu belongs to the intersection of the corresponding Lizorkin–Triebel spaces over the curved boundary.

We also note that the trace γ has the natural property that $r_I\circ\gamma=\gamma\circ r_I$ on $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I')$ for any open interval $I'\supset I$.

Finally, γ applied to any $u \in r_{\Omega\times I}C(\mathbb{R}^{n+1})$ gives the expected, namely $r_{\Gamma\times I}\widetilde{u}$ for any extension $\widetilde{u} \in C(\mathbb{R}^{n+1})$ of u. Indeed using (84),

$$(\gamma u)_{\kappa\times\mathrm{id}_I} = r_{B'\times I}\big(\widetilde{u}\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})\big) = (r_{\Gamma\times I}\widetilde{u})\circ(\kappa^{-1}\times\mathrm{id}_I) = (r_{\Gamma\times I}\widetilde{u})_{\kappa\times\mathrm{id}_I},$$

which shows that γu equals a restriction, $r_{\Gamma\times I}\widetilde{u}$, of the continuous function $\widetilde{u}$.

6.3.3. *The Theorem.* To construct a right-inverse of γ, we use a bounded right-inverse K_n of $\gamma_{0,n}$, where because of (58) we may refer to [JS08, Thm. 2.6] for a right-inverse of the similar trace $\gamma_{0,1}$ in (76),

$$K_n : F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n) \to F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1}), \quad s\in\mathbb{R}, \tag{86}$$

given by, cf. just above (64) for the ψ,

$$K_n v(x) := \sum_{j=0}^{\infty} \psi(2^{ja_n}x_n)\mathscr{F}^{-1}(\Phi_j(\xi',0,\xi_{n+1})\mathscr{F}v(\xi',\xi_{n+1}))(x',x_{n+1}).$$

Hereby we have set $p'' = (p_0,\dots,p_0,p_t) \in]0,\infty[^n$, which results when $p_n = p_0$ is left out; cf. (58).

Theorem 6.9. *When Γ is compact, $\vec{a},\vec{p}$ satisfy* (58) *and (s,q) fulfils the inequality in* (76)*, then*

$$\gamma : \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I) \to \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)$$

is a bounded surjection*, which has a right-inverse K_γ. More precisely, the operator K_γ can be chosen such that $K_\gamma : \overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I) \to \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$ is bounded for every $s\in\mathbb{R}$.*

Proof. Since the space $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)$, cf. (80), does not depend on how the compact interval $J\supset I$ is chosen, it is fixed in the following. Moreover, γu does not depend on the extension f of u, thus we take f such that $\operatorname{supp} f\subset\mathbb{R}^n\times J$. By (83) and (80), $\gamma u = r_I v$ is in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)$.

To prove boundedness, note that v according to (16) belongs to $\overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$, since

$$\operatorname{supp} v \subset \bigcup_{\lambda\in\Lambda}(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})(B'\times J) = \Gamma\times J. \tag{87}$$

Hence it can be inferred from Theorem 4.15 that

$$\begin{aligned}\|\gamma u\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)\|^d \\ \le \inf_{\substack{r_{\Omega\times I}f=u\\ \operatorname{supp} f\subset\mathbb{R}^n\times J}} \sum_{\lambda\in\Lambda} \|(\psi_\lambda\otimes\mathbb{1}_{\mathbb{R}}f)\circ(\lambda^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B'\times\mathbb{R})\|^d.\end{aligned} \tag{88}$$

By choosing first a cut-off function on $\mathbb{R}$, we can use the infimum norm to fix f such that $\|f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le 2\|u\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|$. Using the arguments leading up to (79) and the boundedness of $\gamma_{0,n}$, cf. (76), each summand in (88) can be estimated by

$$c\|(\psi_\lambda\otimes\mathbb{1}_{\mathbb{R}}f)\circ(\lambda^{-1}\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(B\times\mathbb{R})\|^d.$$

Finally, applying Theorem 3.3 and Lemma 2.5, since $\psi_\lambda\otimes\mathbb{1}_{\mathbb{R}}\in C^\infty_{L_\infty}(\mathbb{R}^{n+1})$, we obtain

$$\|\gamma u\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)\| \le c\|f\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\| \le 2c\|u\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|.$$

The construction of a right-inverse K_γ uses that for any $w\in\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times I)$ there exists a $v\in \overset{\circ}{F}{}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times J)$ such that $r_I v = w$. It is easily verified that

$$w^\kappa := r_{\mathbb{R}^{n-1}\times I}\big(e_{B'\times\mathbb{R}}(\widetilde{\psi}_\kappa\otimes\mathbb{1}_{\mathbb{R}}v_{\kappa\times\mathrm{id}_{\mathbb{R}}})\big) \tag{89}$$

is independent of the extension v; and obviously w^κ is in $\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^{n-1}\times I)$ with support in $B'\times I$. For $\chi_1,\chi_2\in C_0^\infty(\mathbb{R})$ such that $\chi_1+\chi_2\equiv 1$ on a neighbourhood of I and such that χ_1, χ_2 vanish before the right, respective the left end point of I, we let, cf. Theorem 5.6 and Corollary 5.7,

$$w^\kappa_{\mathrm{ext}}=\mathscr{E}_u(\chi_1 w^\kappa)+\mathscr{E}_{u,T}(\chi_2 w^\kappa),$$

where extension by 0 to $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}\times\,]-\infty,T[$ before application of $\mathscr{E}_u$, respectively $\mathscr{E}_{u,T}$ is understood. Lemma 2.10 gives that this extension does not change the regularity of the elements, hence w^κ_{ext} belongs to $F^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^n)$; and furthermore $r_{\mathbb{R}^{n-1}\times I}w^\kappa_{\mathrm{ext}}=w^\kappa$.

Now using K_n in (86) as well as functions $\eta_\lambda\in C_0^\infty(\mathbb{R}^n)$, $\lambda\in\Lambda$, such that $\operatorname{supp}\eta_\lambda\subset B$ and $\eta_\lambda=1$ on $\operatorname{supp}\widetilde{\psi}_\lambda$, we define (using the v-independence of w^κ_{ext})

$$K_\gamma w=r_{\Omega\times I}\sum_{\lambda\in\Lambda}e_{U_\lambda\times\mathbb{R}}(\eta_\lambda K_n w^\kappa_{\mathrm{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}}).\tag{90}$$

Boundedness of K_γ is a consequence of first using Lemma 2.9 and Theorem 3.3, $d:=\min(1,p_0,p_t,q)$,

$$\begin{aligned}\|K_\gamma w\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|^d&\le\sum_{\lambda\in\Lambda}\|(\eta_\lambda K_n w^\kappa_{\mathrm{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(U_\lambda\times\mathbb{R})\|^d\\&\le c\sum_{\lambda\in\Lambda}\|\eta_\lambda K_n w^\kappa_{\mathrm{ext}}\,|F^{s,\vec{a}}_{\vec{p},q}(\mathbb{R}^{n+1})\|^d,\end{aligned}$$

and then Lemma 2.5, Lemma 2.10 and the mapping properties of K_n, $\mathscr{E}_u$, $\mathscr{E}_{u,T}$,

$$\begin{aligned}\|K_\gamma w\,|\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)\|^d&\le c\sum_{\substack{\kappa\in\mathscr{F}_0\\ j=1,2}}\|\chi_j w^\kappa\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(\mathbb{R}^{n-1}\times I)\|^d\\&\le c\sum_{\kappa\in\mathscr{F}_0}\|(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}}v)\circ(\kappa^{-1}\times\mathrm{id}_{\mathbb{R}})\,|\overline{F}^{s-a_0/p_0,a''}_{p'',p_0}(B'\times\mathbb{R})\|^d.\end{aligned}$$

The extension v is chosen arbitrarily among those in $\mathring{F}^{s-a_0/p_0,a''}_{p'',p_0}(\Gamma\times J)$ satisfying $r_I v=w$, thus taking the infimum over all such v yields the boundedness of K_γ, cf. (81) and (31).

To verify that K_γ is indeed a right-inverse, we use that an extension of $K_\gamma w$ is

$$f=\sum_{\lambda\in\Lambda}e_{U_\lambda\times\mathbb{R}}(\eta_\lambda K_n w^\kappa_{\mathrm{ext}})\circ(\lambda\times\mathrm{id}_{\mathbb{R}}).$$

Hence the definition of γ, cf. (84), gives that $\gamma(K_\gamma w)=r_I h$, where $h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}=f\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})$. We shall now prove that $r_{B'\times I}h_{\kappa_1\times\mathrm{id}_R}=w_{\kappa_1\times\mathrm{id}_I}$ for each $\kappa_1\in\mathscr{F}_0$. Indeed,

$$r_{B'\times I}h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}=r_{B'\times I}\sum_{\lambda\in\Lambda}(\eta_\lambda K_n w^\kappa_{\mathrm{ext}})\circ(\lambda\circ\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}}),\tag{91}$$

where extension by 0 from $\kappa_1(\Gamma_{\kappa_1}\cap\Gamma_\kappa)\times\mathbb{R}$ to $B'\times\mathbb{R}$ in each term is understood. Using that K_n is a right-inverse of $\gamma_{0,n}$ and that $w^\kappa_{\mathrm{ext}}=w^\kappa$ on $\kappa(\Gamma_{\kappa_1}\cap\Gamma_\kappa)\times I$, each summand in (91) equals, cf. also (89), (14),

$$(\eta_\lambda w^\kappa)\circ(\lambda\circ\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})=(\eta_\lambda\circ\lambda\cdot\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}.$$

As $\eta_\lambda\circ\lambda\equiv 1$ on $\operatorname{supp}\psi_\kappa$ and $\sum\psi_\kappa\equiv 1$ on Γ, we finally obtain, using that $r_I v=w$,

$$r_{B'\times I}h_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}=r_{B'\times I}\Big(v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}\sum_{\lambda\in\Lambda}(\psi_\kappa\otimes\mathbb{1}_{\mathbb{R}})\circ(\lambda_1^{-1}(\cdot,0)\times\mathrm{id}_{\mathbb{R}})\Big)=w_{\kappa_1\times\mathrm{id}_I},$$

hence K_γ is a right-inverse of γ. $\square$

6.4. **The Traces at the Corner.** The trace from either the flat or the curved boundary to the corner $\Gamma\times\{0\}\simeq\Gamma$ cannot simply be obtained by applying r_0 and then γ, or vice versa, since these operators are defined on spaces over the whole cylinder.

In the following, under the assumptions that $I=]0,T[$ is finite and Γ compact, the trace operators $r_{0,\Gamma}$, γ_Γ will therefore be introduced (the subscript Γ indicates that we end up at $\Gamma\times\{0\}\simeq\Gamma$). We note that focus will not be on optimality regarding the co-domains, since the purpose of this section merely is to prepare for a discussion of compatibility conditions in connection with PDEs; and from this point of view the interesting question is whether the following identity holds in $\mathscr{D}'(\Gamma)$,

$$r_{0,\Gamma}\circ\gamma u=\gamma_\Gamma\circ r_0 u. \tag{92}$$

Recall that when working with spaces on the boundary, the anisotropy and the vector of integral exponents only have n entries. Since it is different entries that need to be left out, depending on whether we are studying $\Gamma\times I$ or Ω, it will in the following be convenient to use $a''=(a_1,\dots,a_{n-1},a_t)$ as well as $a'=(a_1,\dots,a_n)$; and likewise for p', p''. Moreover, (58) is a standing assumption on $\vec{a}$, $\vec{p}$.

We assume that s satisfies the inequality in (61) adapted to vectors of n entries, i.e. for the trace from the curved boundary $\Gamma\times I$,

$$s>\frac{a_t}{p_t}+(n-1)\Big(\frac{a_0}{\min(1,p_0)}-a_0\Big), \tag{93}$$

and for the trace from the flat boundary Ω,

$$s>\frac{a_0}{p_0}+(n-1)\Big(\frac{a_0}{\min(1,p_0)}-a_0\Big). \tag{94}$$

Remark 6.10. When $v\in\overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$ for a compact interval J and s fulfils (93), then

$$v_{\kappa\times\mathrm{id}_\mathbb{R}}\in C_\mathrm{b}(\mathbb{R}_t,L_{1,\mathrm{loc}}(B'))\quad\text{for each }\kappa\in\mathscr{F}_0.$$

This follows if for every compact set $K\subset B'$, the map $t\mapsto v_{\kappa\times\mathrm{id}_\mathbb{R}}(\cdot,t)$ is continuous with values in $L_1(K)$. In Theorem 4.15 we may, if necessary, change the partition of unity (using some $\varphi\in C_0^\infty(B')$ equalling 1 on K) such that $\psi_\kappa\equiv1$ on $\kappa^{-1}(K)$. Then $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_\mathbb{R}}$ is in $\overline{F}{}^{s,a''}_{p'',q}(B'\times\mathbb{R})$, which because of (61) and (93) is contained in $C_\mathrm{b}(\mathbb{R}_t,L_1(B'))$. Hence $v_{\kappa\times\mathrm{id}_\mathbb{R}}$ is in $L_1(K)$, continuously in time.

6.4.1. *The Curved Boundary.* For $w\in\overline{F}{}^{s,a''}_{p'',q}(\Gamma\times I)$ there exists a $v\in\overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$, where $J\supset I$ is any compact interval, such that $r_I v=w$, cf. (80). By exploiting that $v_{\kappa\times\mathrm{id}_\mathbb{R}}$ is continuous with respect to t, cf. Remark 6.10, we define for $x\in\Gamma$,

$$r_{0,\Gamma}w(x)=\sum_{\kappa\in\mathscr{F}_0}\psi_\kappa(x)v_{\kappa\times\mathrm{id}_\mathbb{R}}(\kappa(x),0) \tag{95}$$

with the understanding that the product $\psi_\kappa(x)v_{\kappa\times\mathrm{id}_\mathbb{R}}(\kappa(x),0)$ is defined to be 0 outside Γ_κ. On Γ_κ the product is meaningful, since $v_{\kappa\times\mathrm{id}_\mathbb{R}}$ is in $C_\mathrm{b}(\mathbb{R},L_{1,\mathrm{loc}}(B'))$.

The trace $r_{0,\Gamma}$ in (95) is independent of the chosen $v\in\overset{\circ}{F}{}^{s,a''}_{p'',q}(\Gamma\times J)$, since for any two extensions v_1,v_2 in this space, $\widetilde{\psi}_\kappa\cdot r_{B'\times I}v_{j,\kappa\times\mathrm{id}_\mathbb{R}}$, $j=1,2$, coincide on $B'\times I$, hence by continuity also on $B'\times\{0\}$.

Moreover, the trace depends neither on the atlas nor on the subordinate partition of unity. Indeed, considering another atlas $\mathscr{F}_1$ with a subordinate partition of unity $1=\sum_{\kappa_1\in\mathscr{F}_1}\varphi_{\kappa_1}$, we have on Γ, cf. (14) for the atlas $\mathscr{F}_0\cup\mathscr{F}_1$,

$$\sum_\kappa\psi_\kappa v_{\kappa\times\mathrm{id}_\mathbb{R}}(\kappa(\cdot),0)=\sum_{\kappa,\kappa_1}\psi_\kappa\varphi_{\kappa_1}v_{\kappa_1\times\mathrm{id}_\mathbb{R}}(\kappa_1(\cdot),0)=\sum_{\kappa_1}\varphi_{\kappa_1}v_{\kappa_1\times\mathrm{id}_\mathbb{R}}(\kappa_1(\cdot),0).$$

In the following theorem the co-domain of the trace is $B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)$; the definition and properties of this space follow from Section 4.3, since it coincides with an isotropic space in view of (58) and Lemma 2.3. Note that we have abbreviated the $(n-1)$-vector $(a_0,\dots,a_0)$ to a_0, and similarly for p_0.

Theorem 6.11. *When a'',p'' are as above with $0<p''<\infty$ and s satisfies* (93)*, then $r_{0,\Gamma}$ is bounded,*

$$r_{0,\Gamma}:\overline{F}^{s,a''}_{p'',q}(\Gamma\times I)\to B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma).$$

Proof. From Remark 6.10 we have that $v_{\kappa\times\mathrm{id}_{\mathbb{R}}}$ is in $C_{\mathrm{b}}(\mathbb{R},L_{1,\mathrm{loc}}(B'))$, hence using the bounded trace operator, cf. [JS08, Thm. 2.5] and (93),

$$\gamma_{0,n}:F^{s,a''}_{p'',q}(\mathbb{R}^n)\to B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\mathbb{R}^{n-1}),\tag{96}$$

it is readily seen that

$$\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\cdot,0)=r_{B'}\gamma_{0,n}e_{B'\times\mathbb{R}}(\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}).$$

Since $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}\in F^{s,a''}_{p'',q}(B'\times\mathbb{R})$, we therefore have by (96) that $\widetilde{\psi}_\kappa v_{\kappa\times\mathrm{id}_{\mathbb{R}}}(\cdot,0)$ belongs to $\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(B')$. Now Corollary 4.9 adapted to Besov spaces, cf. Section 4.3, implies that $r_{0,\Gamma}w\in B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)$.

To prove $r_{0,\Gamma}$ is bounded, we use (26) to estimate $\|\,r_{0,\Gamma}w\,|B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)\|^d$, $d:=\min(1,p_0,p_t)$, by

$$\sum_{\kappa,\kappa_1\in\mathscr{F}_0}\Big\|\,\psi_{\kappa_1}\circ\kappa^{-1}\cdot\widetilde{\psi}_\kappa v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\kappa_1\circ\kappa^{-1}(\cdot),0)\,\big|\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\kappa(\Gamma_\kappa\cap\Gamma_{\kappa_1}))\Big\|^d.$$

After a change of coordinates $x\mapsto\kappa\circ\kappa_1^{-1}(x)$ and a slight restriction of the domain to a suitable open subset W such that $\overline{W}\subset\kappa_1(\Gamma_{\kappa_1}\cap\Gamma_\kappa)$, and finally multiplication by a $\chi_{\kappa_1}\in C_0^\infty(B')$ where $\chi_{\kappa_1}\equiv 1$ on $\operatorname{supp}\widetilde{\psi}_{\kappa_1}$, this can be estimated by, cf. [Tri92, 4.2.2] for an s_1 large enough,

$$c\sum_{\kappa,\kappa_1\in\mathscr{F}_0}\Big(\sum_{|\alpha|\le s_1}\|\,e_{B'}(\psi_\kappa\circ\kappa_1^{-1}\chi_{\kappa_1})\,|L_\infty\|\Big)^d\,\Big\|\,e_{B'}(\widetilde{\psi}_{\kappa_1}v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}(\cdot,0))\,\big|B^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\mathbb{R}^{n-1})\Big\|^d;$$

the constant c contains $\sup_{\overline{W}}|\det J(\kappa\circ\kappa_1^{-1})|^d$ as a finite factor (J denotes the Jacobian matrix). Now boundedness of $\gamma_{0,n}$ in (96) gives

$$\Big\|\,r_{0,\Gamma}w\,\big|\overline{B}^{s-\frac{a_t}{p_t},a_0}_{p_0,p_t}(\Gamma)\Big\|^d\le c\sum_{\kappa_1\in\mathscr{F}_0}\|\,\widetilde{\psi}_{\kappa_1}v_{\kappa_1\times\mathrm{id}_{\mathbb{R}}}\,|\overline{F}^{s,a''}_{p'',q}(B'\times\mathbb{R})\|^d,$$

hence taking the infimum over all admissible v (as we may since $r_{0,\Gamma}$ is independent of the extension) proves that $r_{0,\Gamma}$ is bounded. □

6.4.2. *The Flat Boundary.* In this section we consider the trace operator γ_Γ, which simply is the trace at Γ of distributions defined on Ω. In view of (92) and Theorem 6.1, the domain of interest for γ_Γ is the unmixed Besov space $\overline{B}^{s,a'}_{p',q}(\Omega)$, which according to Lemma 2.3 even equals an isotropic space, cf. (58),

The operator is defined by carrying over the definition and results for γ in Section 6.3. Indeed, we remove the time dependence and use the Besov space result in [FJS00, Thm. 1] for $\gamma_{0,n}$. An embedding similar to (77) also holds in the case of Besov spaces, cf. [FJS00, Prop. 1] and (94), and Theorem 4.15, 3.3 are replaced by the Besov versions, cf. Section 4.3, of Theorem 4.11, 3.1 respectively. Recalling that the $(n-1)$-vector $(a_0,\dots,a_0)$ is abbreviated a_0, and likewise for p_0, this yields

Theorem 6.12. *When $a'=(a_0,\dots,a_0)\in[1,\infty[^n$, $p'=(p_0,\dots,p_0)\in]0,\infty[^n$ and s satisfies* (94)*, then γ_Γ is a bounded operator,*

$$\gamma_\Gamma:\overline{B}^{s,a'}_{p',q}(\Omega)\to B^{s-a_0/p_0,a_0}_{p_0,q}(\Gamma).$$

We note that, as usual for Besov spaces, the sum exponent is not changed and, moreover, a formula similar to the one in (95) for $r_{0,\Gamma}$ holds for γ_Γ. I.e. for any extension f of $w \in \overline{B}^{s,a'}_{p',q}(\Omega)$, with (83)–(84) adapted to γ_Γ for the f_κ, we have when extension by 0 outside Γ_κ is suppressed,

$$\gamma_\Gamma w = \sum_{\kappa \in \mathscr{F}_0} \psi_\kappa \cdot f_\kappa \circ \kappa. \tag{97}$$

Indeed, $\big(\sum_{\kappa\in\mathscr{F}_0} \psi_\kappa \cdot f_\kappa \circ \kappa\big)_{\kappa_1} = \sum_{\kappa\in\mathscr{F}_0} \psi_\kappa \circ \kappa_1^{-1} \cdot f_{\kappa_1} = f_{\kappa_1} = (\gamma_\Gamma w)_{\kappa_1}$ for each $\kappa_1 \in \mathscr{F}_0$. This formula is convenient in a discussion of compatibility conditions, cf. the next section.

6.5. **Applications.** Without proof, we now indicate, by merely adapting [GS90, Ch. 6] to the present set-up, what the above considerations yield in a study of e.g. the heat equation. That is, for $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$ we consider

$$\partial_t u - \Delta u = g \quad \text{in} \quad \Omega \times I, \tag{98}$$

$$\gamma u = \varphi \quad \text{on} \quad \Gamma \times I, \tag{99}$$

$$r_0 u = u_0 \quad \text{on} \quad \Omega \times \{0\}. \tag{100}$$

Under the assumption that $\vec{a} = (1,\dots,1,2)$ and $\vec{p} = (p_0,\dots,p_0,p_t) < \infty$, we give in the theorem below necessary conditions for the existence of a solution u in $\overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, when γ and r_0 in (99), (100) make sense, i.e. when s fulfils the two conditions

$$\begin{aligned} &s > \frac{1}{p_0} + (n-1)\Big(\frac{a_0}{\min(1,p_0,q)} - a_0\Big) + \Big(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Big) \quad \text{and} \\ &s > \frac{2}{p_t} + n\Big(\frac{1}{\min(1,p_0)} - 1\Big). \end{aligned} \tag{101}$$

Theorem 6.13. *Let $\vec{a}$, $\vec{p}$ and s satisfy the requirements above. When the boundary value problem in* (98)–(100) *has a solution $u \in \overline{F}^{s,\vec{a}}_{\vec{p},q}(\Omega\times I)$, then the data (g,φ,u_0) necessarily satisfy*

$$g \in \overline{F}^{s-2,\vec{a}}_{\vec{p},q}(\Omega\times I), \quad \varphi \in \overline{F}^{s-\frac{1}{p_0},a''}_{p'',p_0}(\Gamma\times I), \quad u_0 \in \overline{B}^{s-\frac{2}{p_t}}_{p_0,p_t}(\Omega).$$

Moreover, for all $l \in \mathbb{N}_0$ fulfilling both

$$\begin{aligned} &2l < s - \frac{1}{p_0} - \frac{2}{p_t} - (n-1)\Big(\frac{1}{\min(1,p_0)} - 1\Big) \quad \textit{and} \\ &2l < s - \frac{1}{p_0} - (n-1)\Big(\frac{a_0}{\min(1,p_0,q)} - a_0\Big) - \Big(\frac{a_t}{\min(1,p_0,p_t,q)} - a_t\Big), \end{aligned} \tag{102}$$

the data are compatible in the sense that

$$r_{0,\Gamma}\partial_t^l \varphi = \gamma_\Gamma\Big(\Delta^l u_0 + \sum_{j=0}^{l-1} \Delta^j r_0(\partial_t^{l-1-j} g)\Big), \tag{103}$$

which reduces to $r_{0,\Gamma}\varphi = \gamma_\Gamma u_0$ for $l = 0$ (the sum is void).

We recall that the corrections containing the minima in (101), (102) amount to 0 in the classical case in which $\vec{p}, q \geq 1$.

Remark 6.14. In the construction of solutions to e.g. (98)–(100), it is well known from [GS90, Thm. 6.3] that the problem for $p_0 = 2 = p_t$ is solvable, when the data (g,φ,u_0) are subjected to the compatibility *conditions* in (103). For general p_0, p_t a first step could be to reduce to the case in which $\varphi \equiv 0$, $u_0 \equiv 0$. This can be achieved by combining the surjectivity of γ in Theorem 6.9 with the support preserving right-inverse Q_Ω (of r_0) analysed in Theorem 6.8.

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DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, DK-9220 AALBORG ØST, DENMARK
E-mail address: JJOHNSEN@MATH.AAU.DK

DEPARTMENT OF MATHEMATICAL SCIENCES, AALBORG UNIVERSITY, FREDRIK BAJERS VEJ 7G, DK-9220 AALBORG ØST, DENMARK
E-mail address: SABRINA@MATH.AAU.DK

MATHEMATISCHES INSTITUT, ERNST-ABBE-PLATZ 2, D-07740 JENA, GERMANY
E-mail address: WINFRIED.SICKEL@UNI-JENA.DE