Welch's Method for PSD Estimation - Revisited

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Welch's method for PSD Estimation - Revisited

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Outline

- Introduction
- Random errors
  - Bendat/Welch
  - Simulations
- Bias errors
  - Bendat
  - Damping bias
- Conclusions

Introduction – basic idea

$$R_{XY}(\tau) \leftrightarrow G_{XY}(f)$$

$$\hat{R}_{XY}(\tau) = x(\tau) * y(-\tau)$$

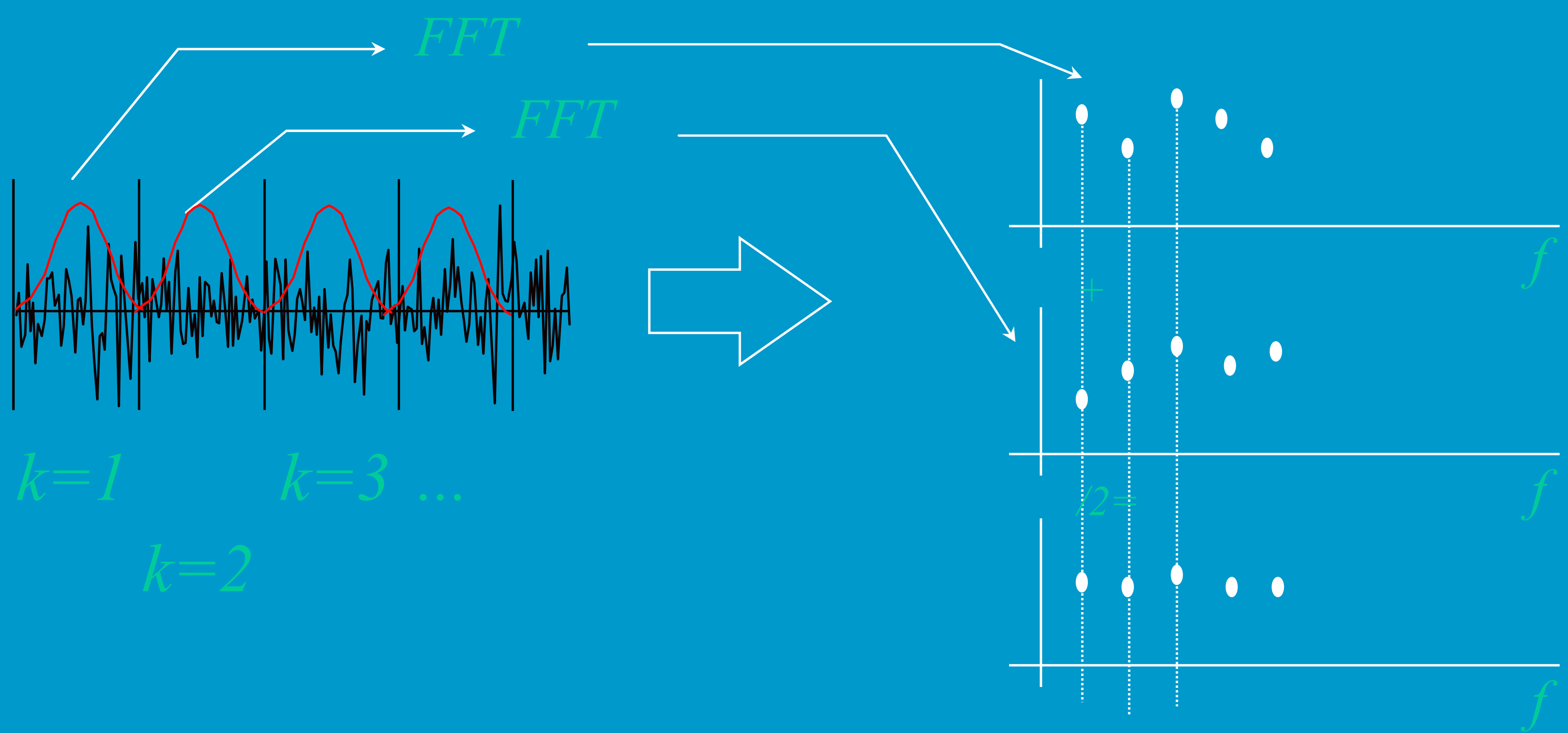
$$T = N\Delta t$$

$$\hat{G}_{XYs} = X_s Y_s^*$$

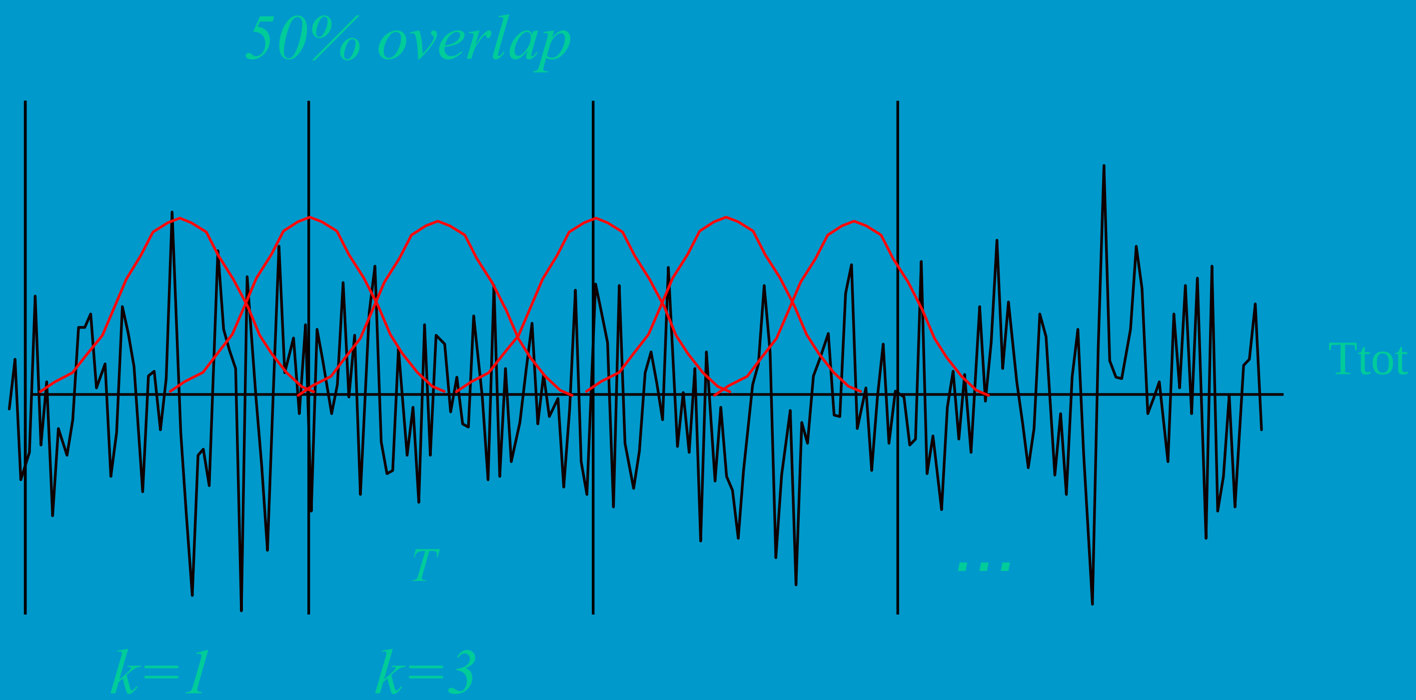
$$\hat{G}_{XY} = \frac{1}{M}\sum_{s=1}^{M} X_s Y_s^*$$

- Definition
- Estimation
- Choose data segment
- One spectral estimate
- Average them all

Introduction – Windows



Introduction – Overlapping



Introduction - sampling

- Data sampling

$$f_v = \frac{1}{2\Delta t}$$

- Nyquist

- Segment sampling

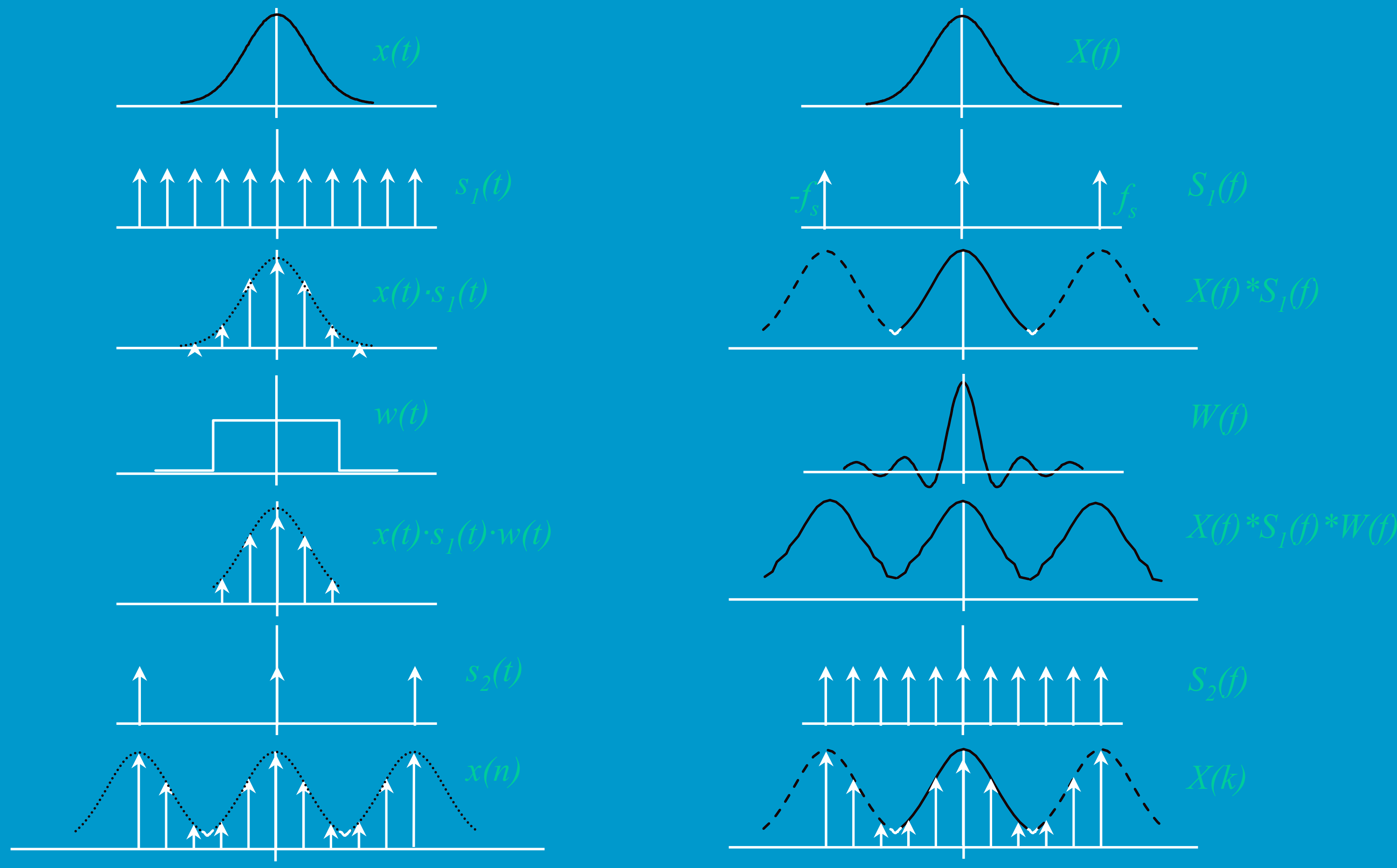
N data points

M segments

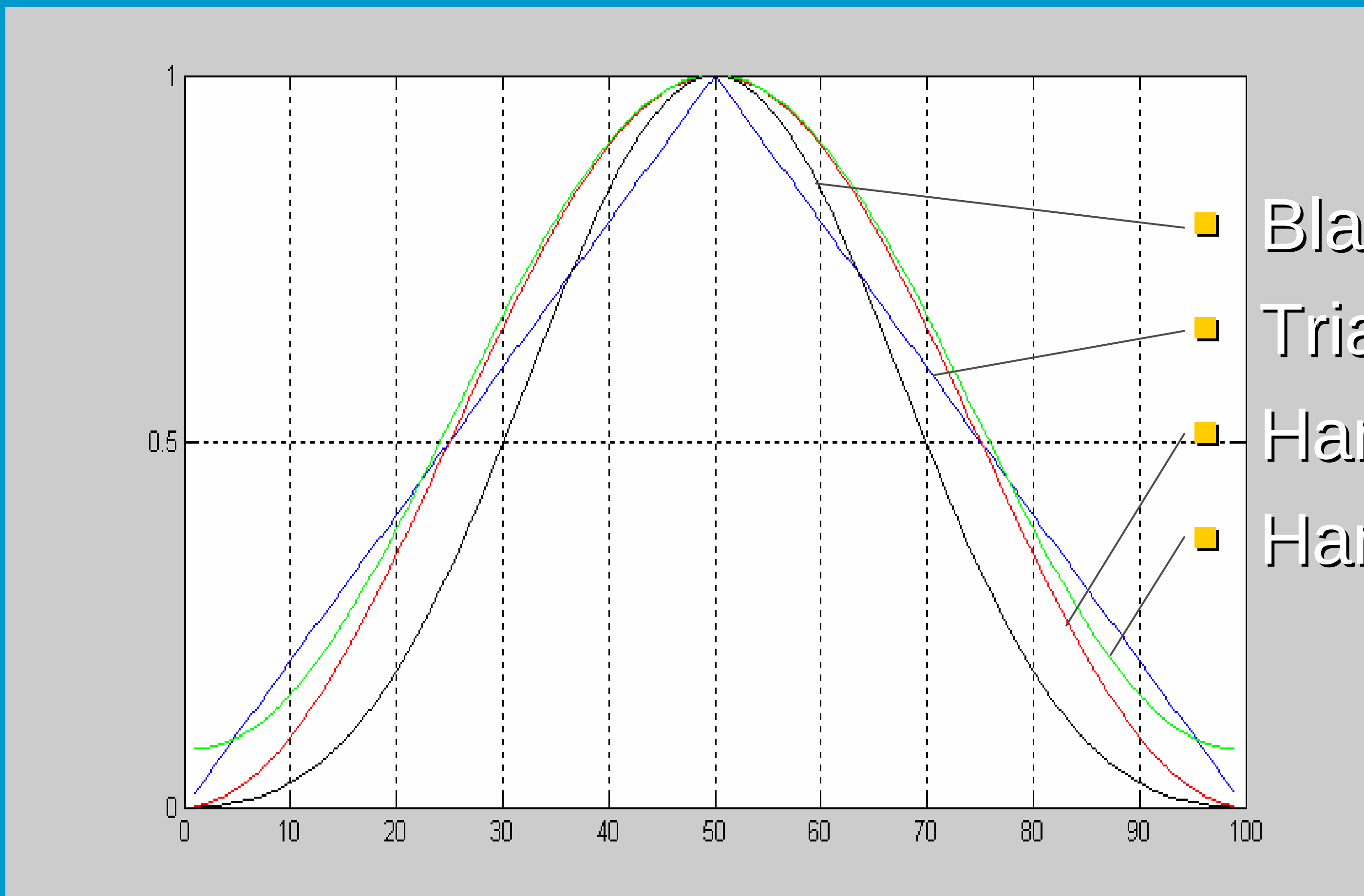
$$\Delta f = \frac{f_v}{N/2} = \frac{1}{N\Delta t} = \frac{1}{T}$$

- Frequency resolution

Introduction - DFT



Introduction – Windows



- Blackman
- Triangular
- Hanning
- Hamming

Introduction - problems

- Periodic assumption => Bias
- Reduce bias => Windows
- Reduce variance => Overlapping

Good old questions:

1. What is the random error?
2. What is the bias?

Error definition

- Random error

$$\sigma_{\hat{\phi}} = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left\{\hat{\phi}_n - E[\hat{\phi}]\right\}^2}$$

- Bias error

$$b_{\hat{\phi}} = E[\hat{\phi}] - \phi$$

$$= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\hat{\phi}_n - \phi$$

- Normalized random error

$$\varepsilon_r = \frac{\sigma_{\hat{\phi}}}{\phi}$$

- Normalized bias error

$$\varepsilon_b = \frac{b_{\hat{\phi}}}{\phi}$$

Random error - Bendat

$$\varepsilon_r = \frac{1}{\sqrt{B_e T_t}}$$

$$B_e = \frac{1}{T}$$

$$T_{tot} = MT$$

$$\varepsilon_r = \frac{1}{\sqrt{MT / T}} = \frac{1}{\sqrt{M}}$$

- Approximate error
- Effective bandwidth
- Total data length
- Aproximate error

Random error - Welch

- The variance of M averages of dependent variables is

$$\varepsilon_r^2 = \frac{1}{M}\left[1 + 2\sum_{q=1}^{M-1}\frac{M-q}{M}\rho(q)\right]$$

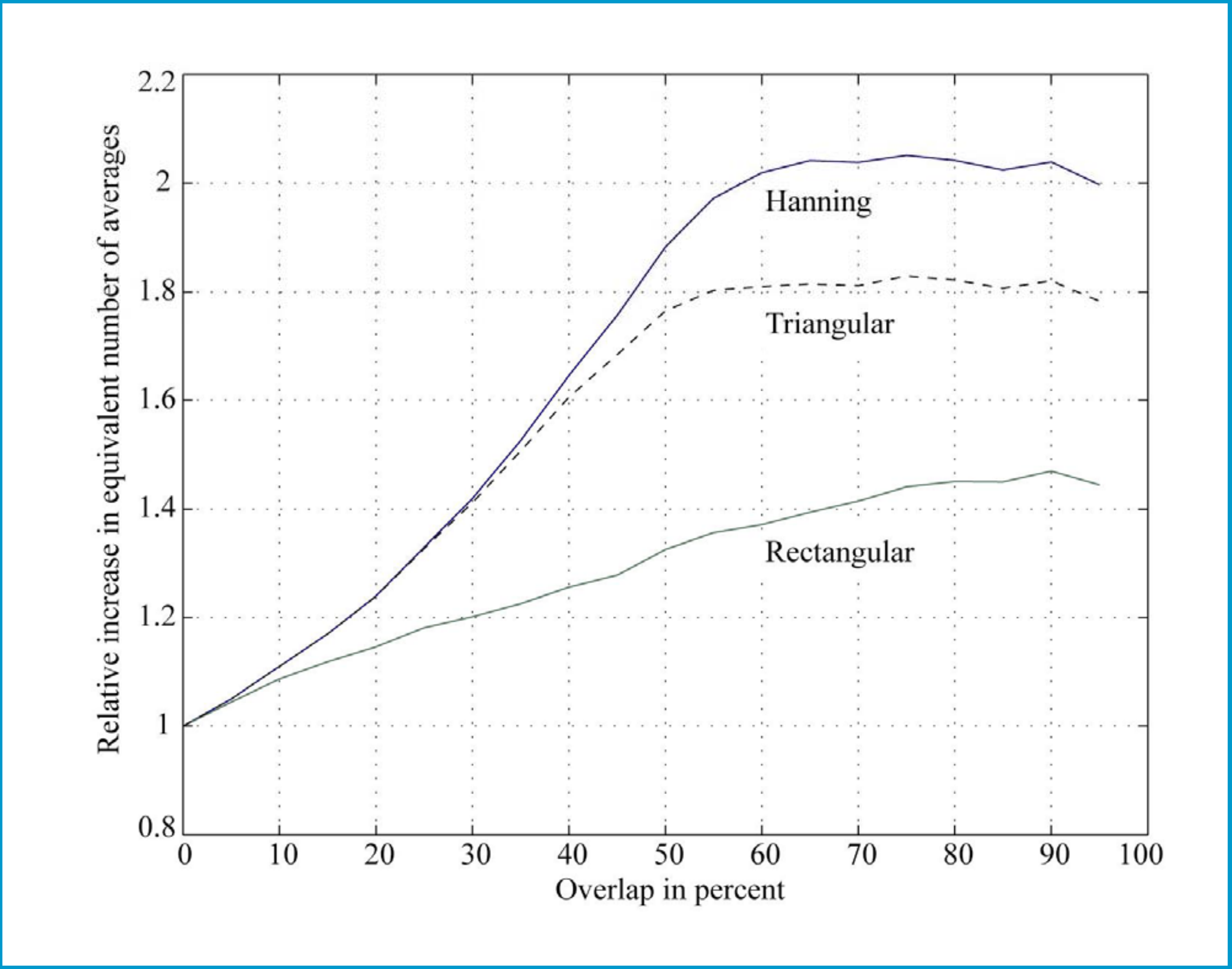
- Where, for an overlap of D samples

$$\rho(q) = \frac{\left[\sum_{n=0}^{N-1} w(n)w(n+qD)\right]^2}{\sum_{n=0}^{N-1} w^2(n)}$$

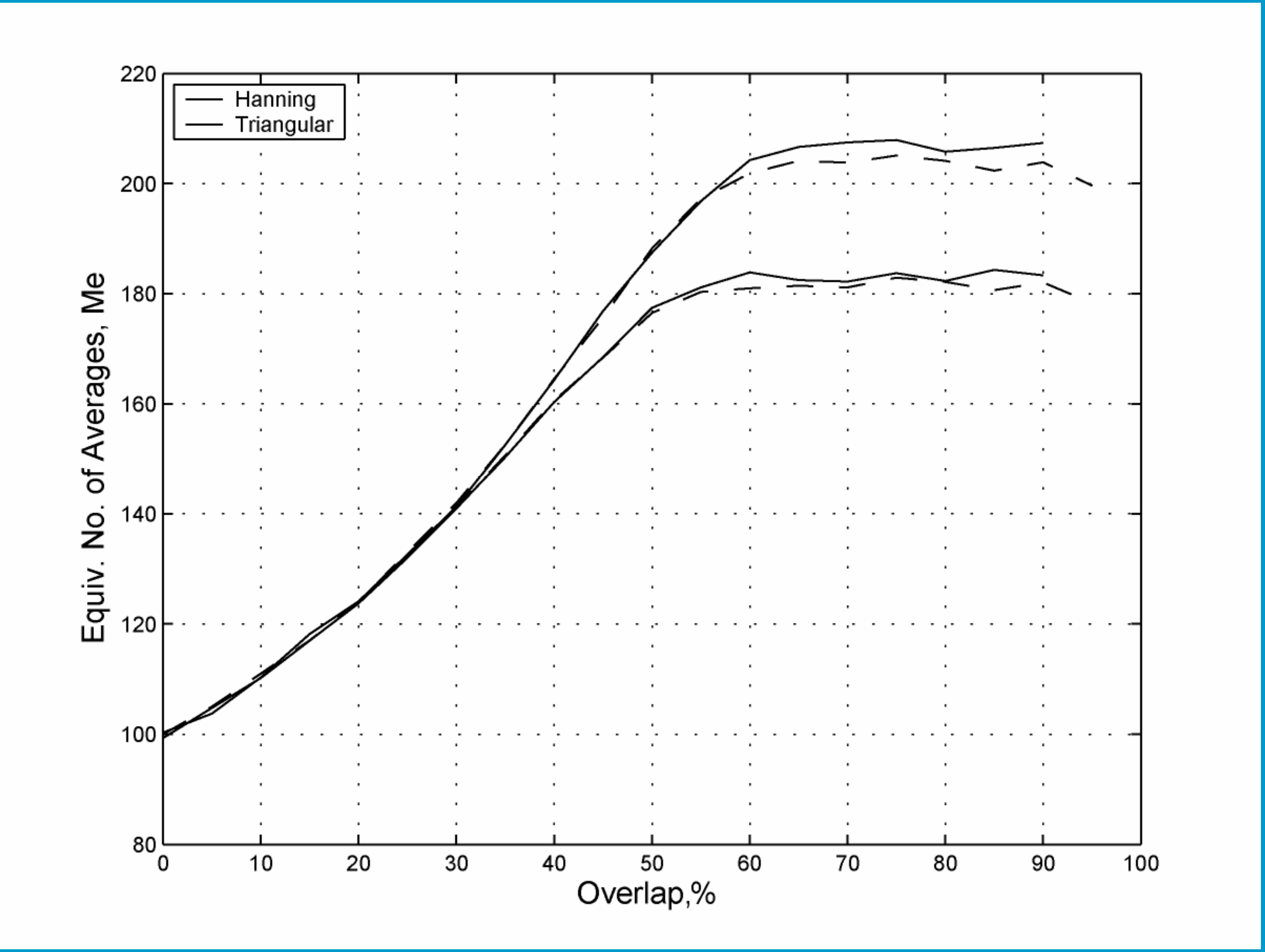
Random error - Welch

$$\varepsilon_r = \frac{1}{\sqrt{M_e}}$$

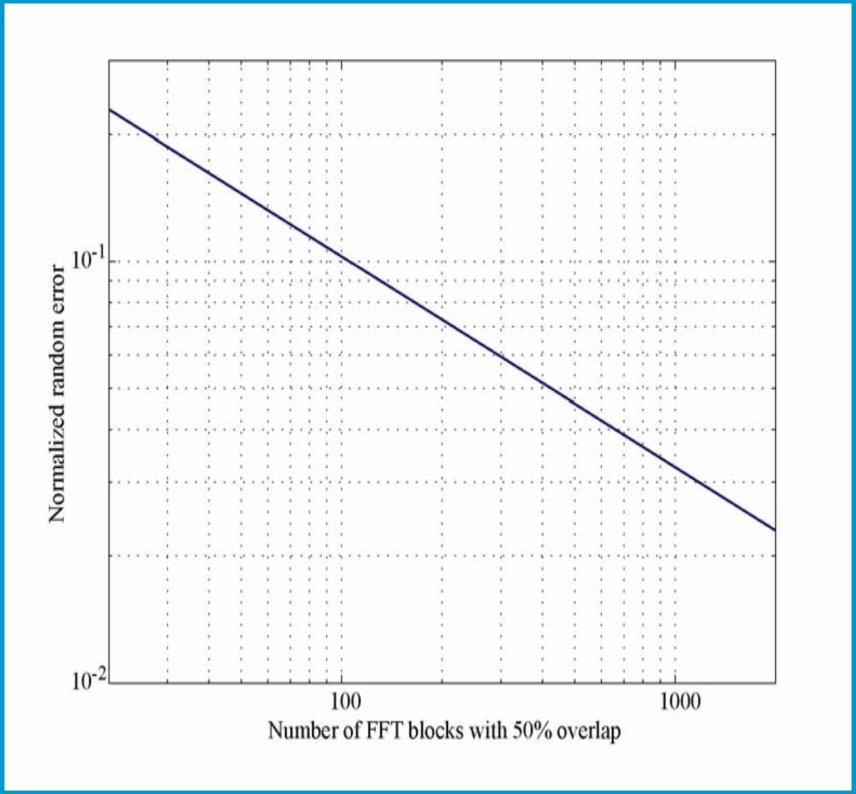
$$M_e = \alpha M$$



Random error - Simulation



Random error - Conclusions



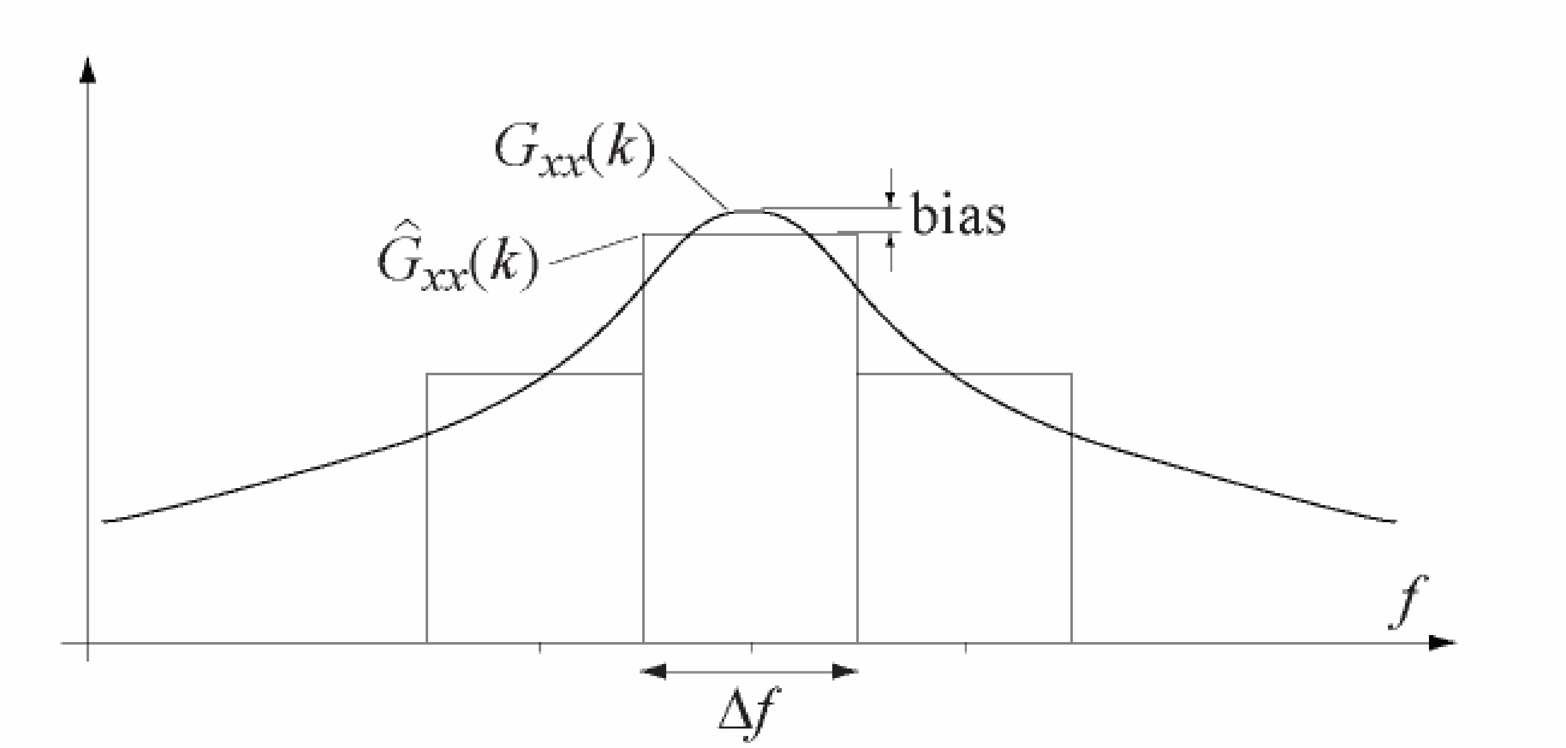
Welch error formula for Hanning window with 50 % overlap

- Proper windowing
- Proper overlapping

=>

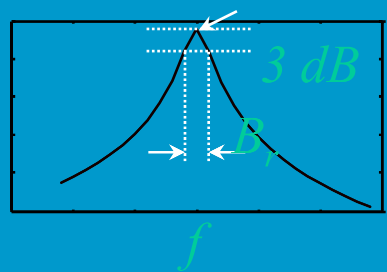
- minimum variance
- ”Same” as Bendat

Bias error - Definition



Bias error - Bendat

$$\varepsilon_b = \frac{1}{3}\left(\frac{B_e}{B_r}\right)^2$$



$$B_e = \Delta f$$

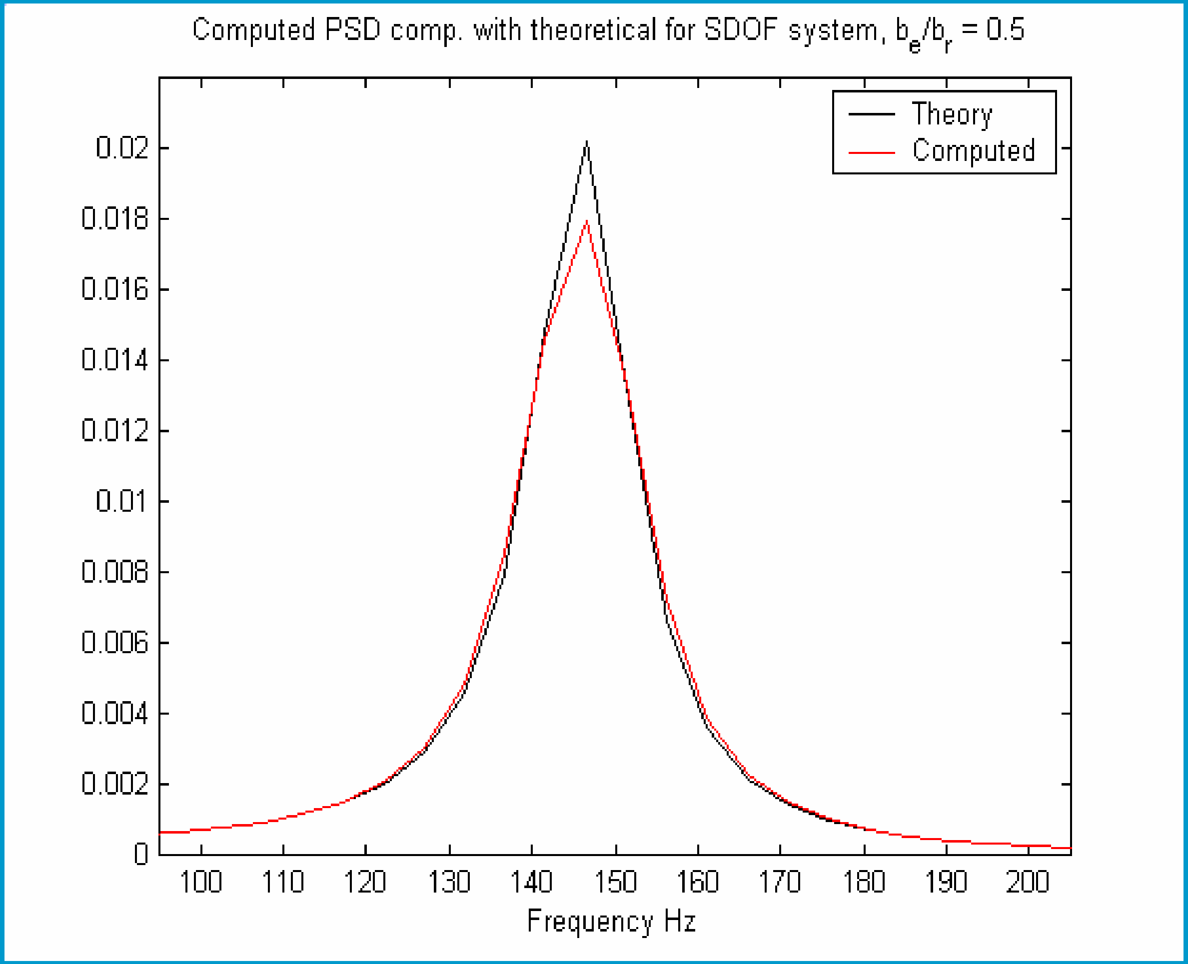
$$B_r = 2\varsigma f$$

$$\varepsilon_b = \frac{1}{3}\left(\frac{f_v}{f}\right)^2\left(\frac{1}{\varsigma N}\right)^2$$

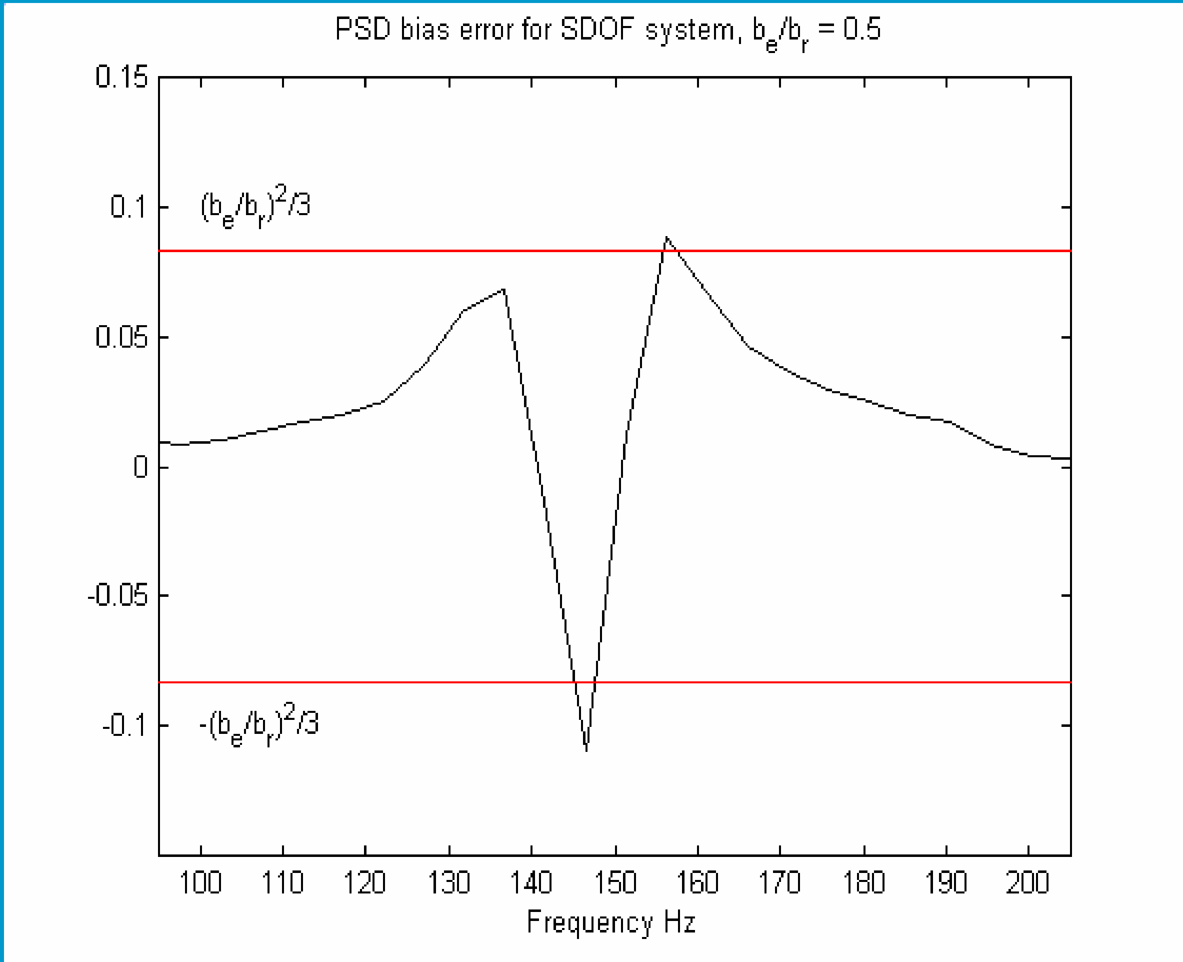
- The lower the frequency
- The smaller the damping
- The smaller the data segment

... The larger is the bias

Bias error - Leakage



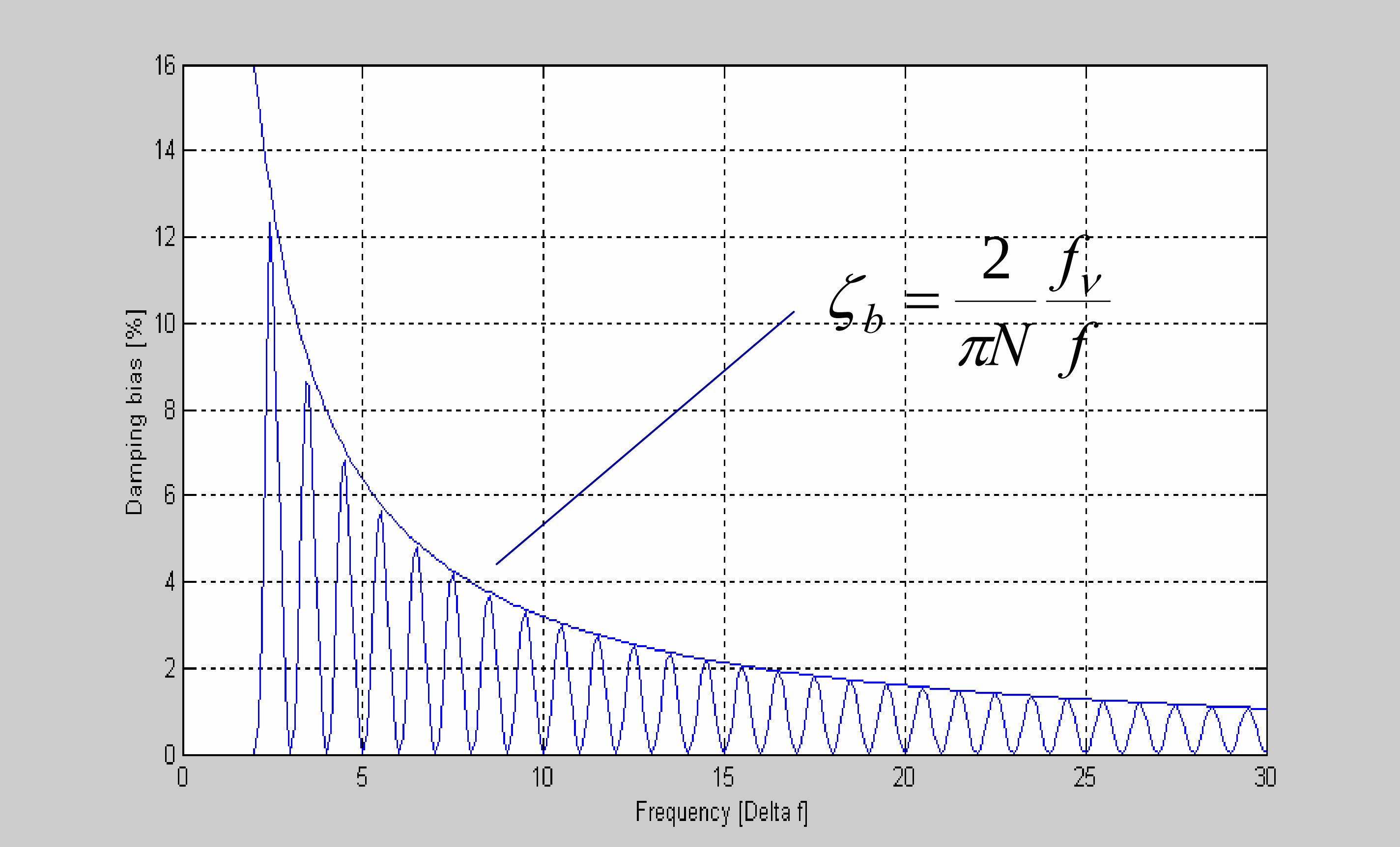
Bias error - illustrated



Bias error - Damping

- Consider the frequency of the harmonic as a multiplum of the frequency resolution
- Calculate spectral density of harmonic
- Convert to correlation function by IFFT
- Calculate the bias as the decay = damping

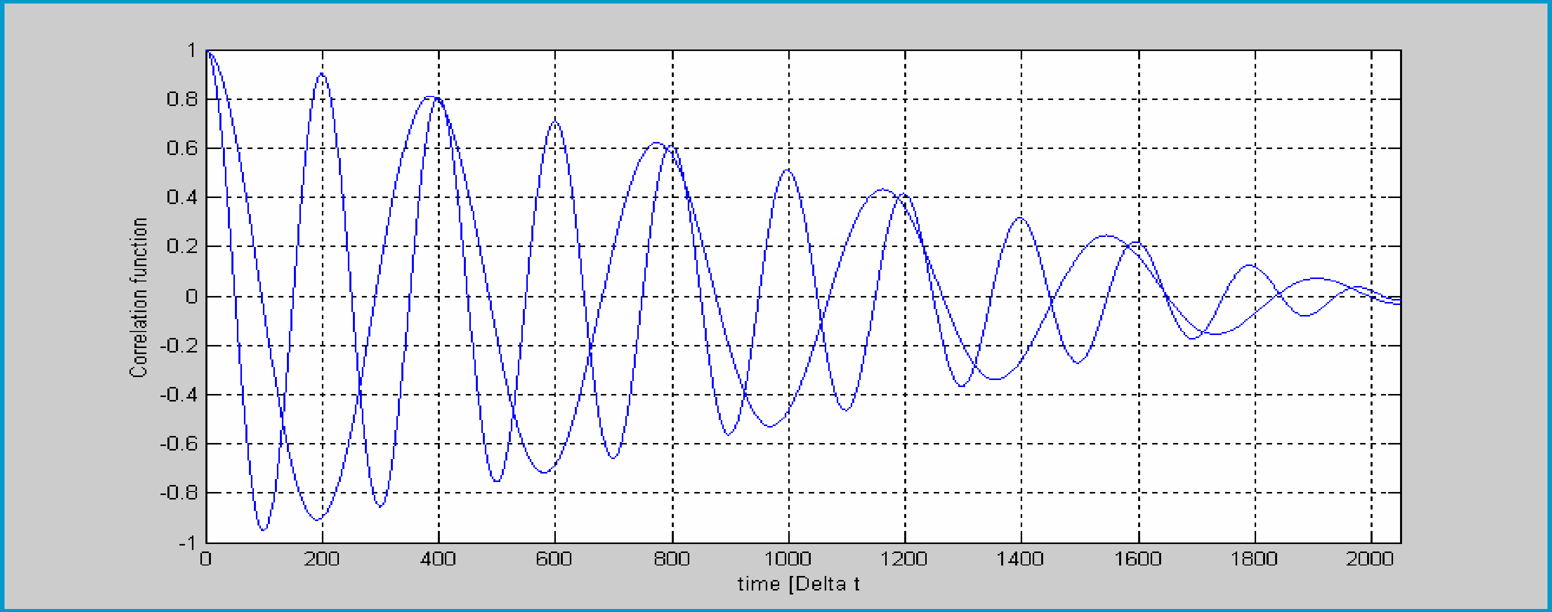
Bias error - Damping



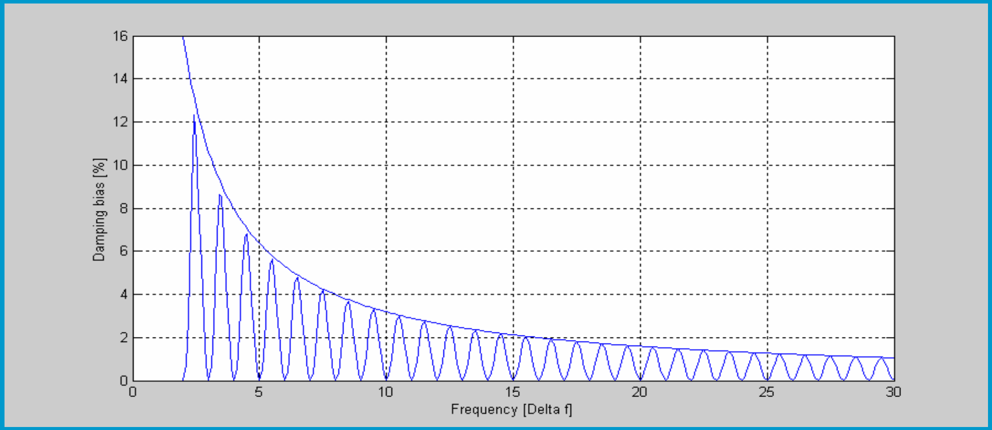
The absolute error on the damping in % (damping bias) as a function of the dimensionless frequency $\alpha = f / \Delta f$

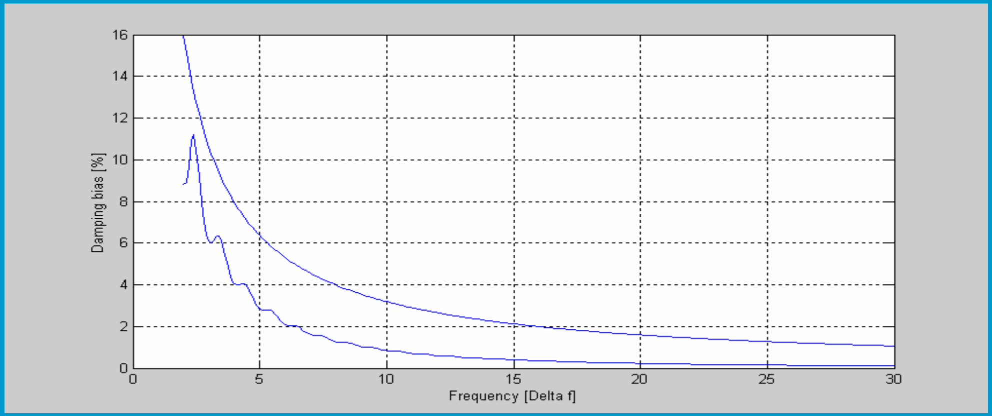
Bias error - Damping

- At maximum bias the correlation function has a linear decay
- Minimum damping is at beginning (smallest relative error)



Bias error - Windows





- No window
- Triangular window

Bias error - Windows

Damping bias in % for different spectral windows as a function of the dimensionless frequency

$f / \Delta f$	Boxcar	Blackman	Hanning	Triang.	Hamming
10	3.18	1.36	1.01	0.83	0.81
20	1.59	0.35	0.26	0.22	0.21
30	1.06	0.16	0.12	0.10	0.09
40	0.80	0.09	0.07	0.06	0.05
50	0.64	0.06	0.04	0.04	0.03
60	0.53	0.04	0.03	0.03	0.03
70	0.45	0.04	0.03	0.03	0.02
80	0.40	0.03	0.02	0.02	0.02
90	0.35	0.03	0.02	0.02	0.02
100	0.32	0.02	0.02	0.02	0.02

Conclusions

Random errors

- Overlapping essential
- With proper overlap-ping the error is ”the same” as given by classical analysis

Bias errors

- Windows essential
- Choice of window less important
- Low frequency is dangerous

May Anders and Kjell forgive me...

Thank you