



Welch's Method for PSD Estimation - Revisited

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Welch's method for PSD Estimation - Revisited

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Outline

Introduction

Random errors

- Bendat/Welch
- Simulations

Bias errors

- Bendat
- Damping bias

Conclusions

Introduction – basic idea

$$R_{XY}(\tau) \leftrightarrow G_{XY}(f)$$

$$\hat{R}_{XY}(\tau) = x(\tau) * y(-\tau)$$

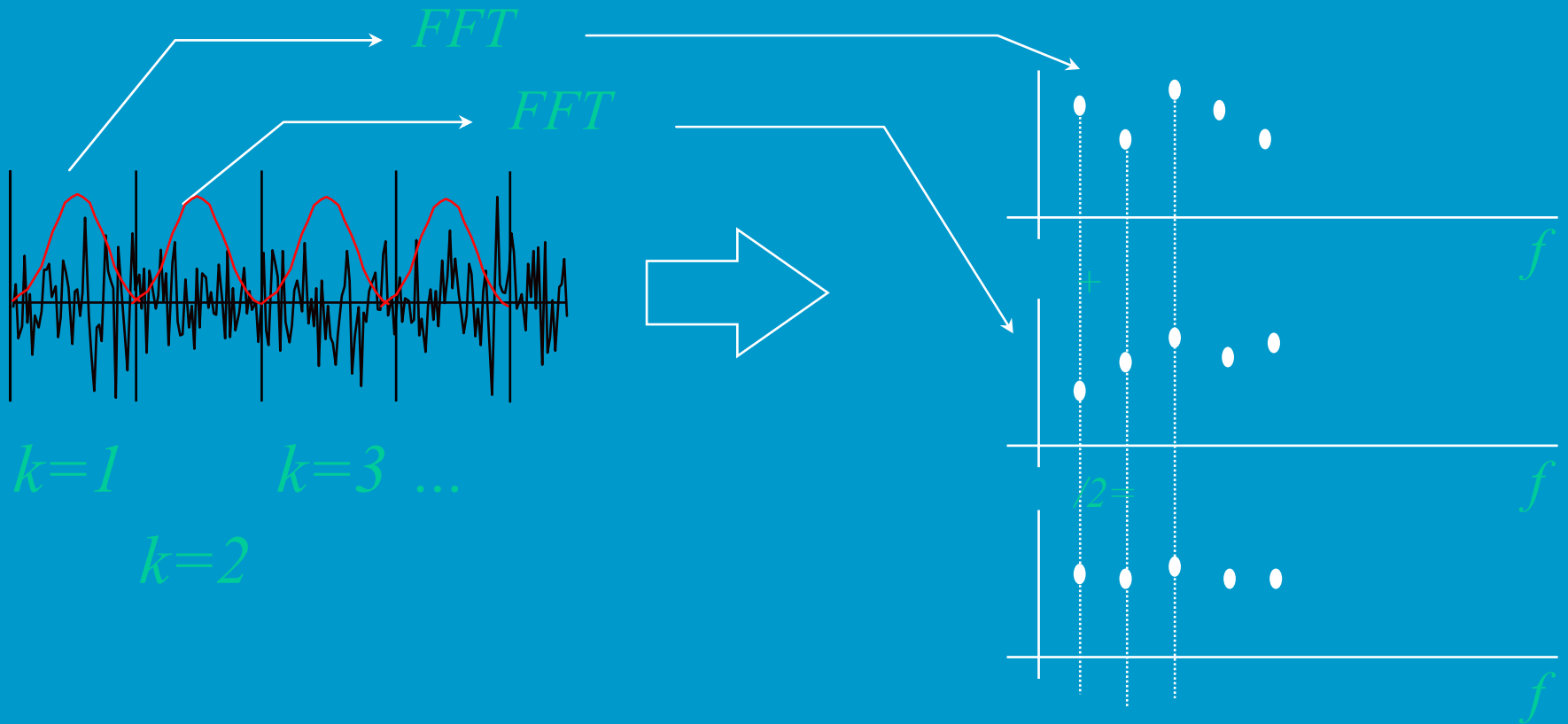
$$T = N\Delta t$$

$$\hat{G}_{XY_s} = X_s Y_s^*$$

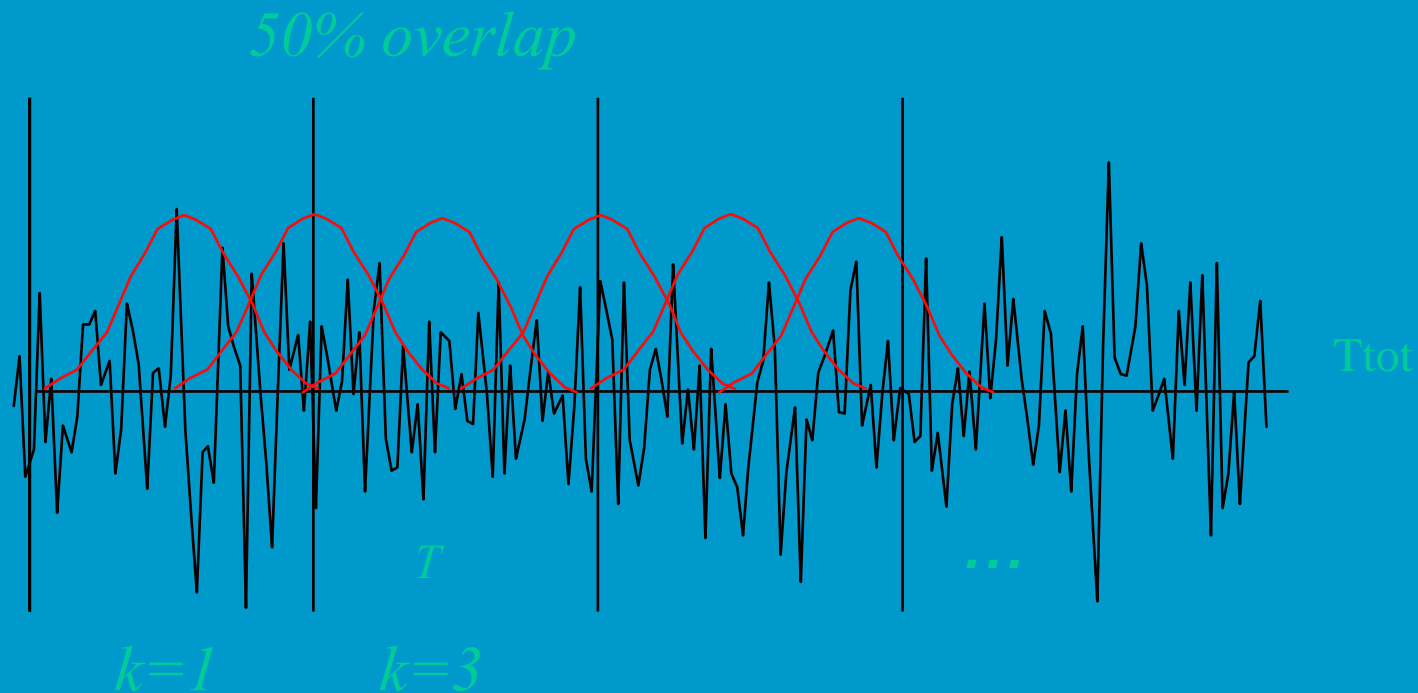
$$\hat{G}_{XY} = \frac{1}{M} \sum_{s=1}^M X_s Y_s^*$$

- Definition
- Estimation
- Choose data segment
- One spectral estimate
- Average them all

Introduction – Windows



Introduction – Overlapping



Introduction - sampling

- Data sampling

$$f_v = \frac{1}{2\Delta t}$$

- Nyquist

- Segment sampling

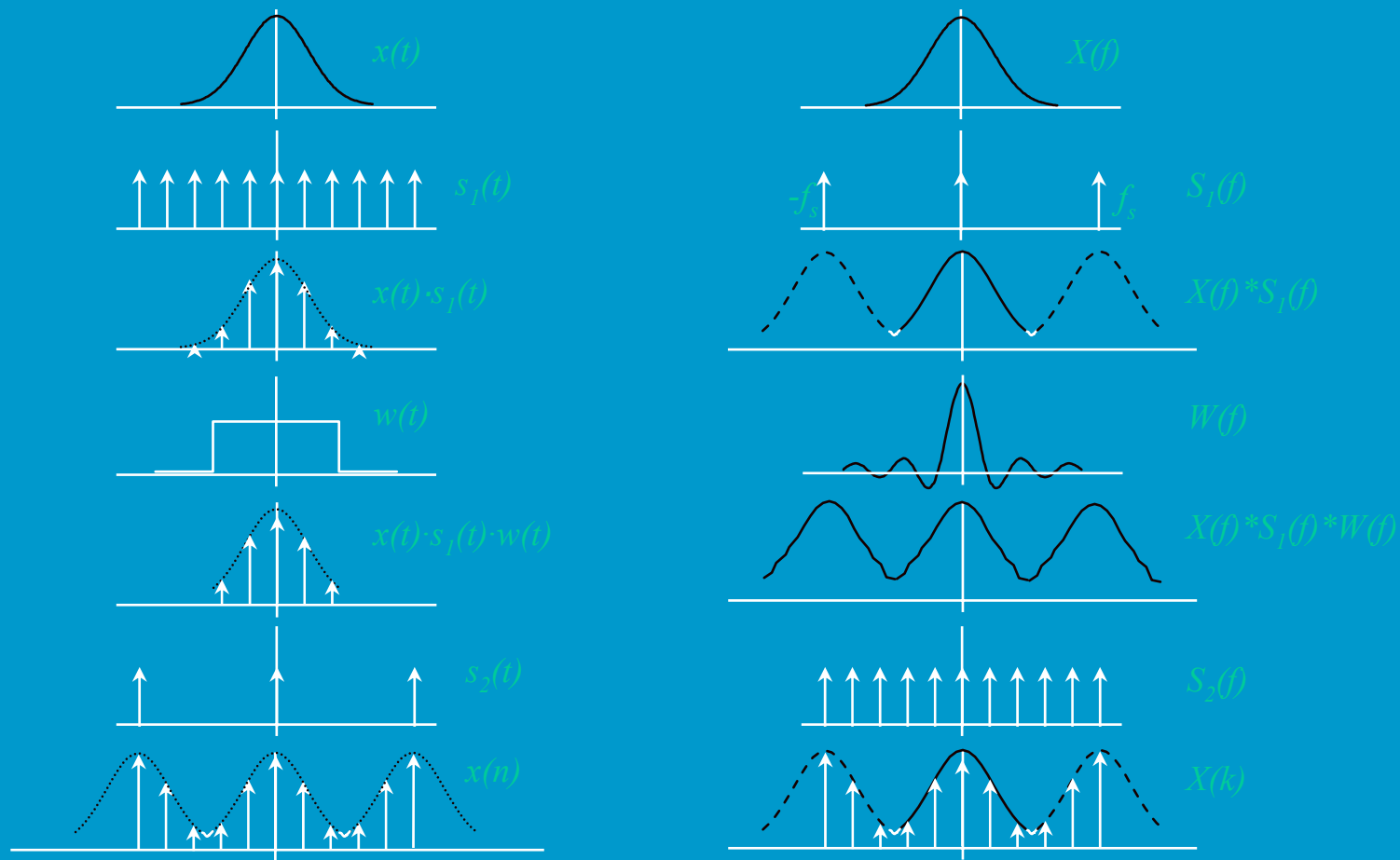
N data points

M segments

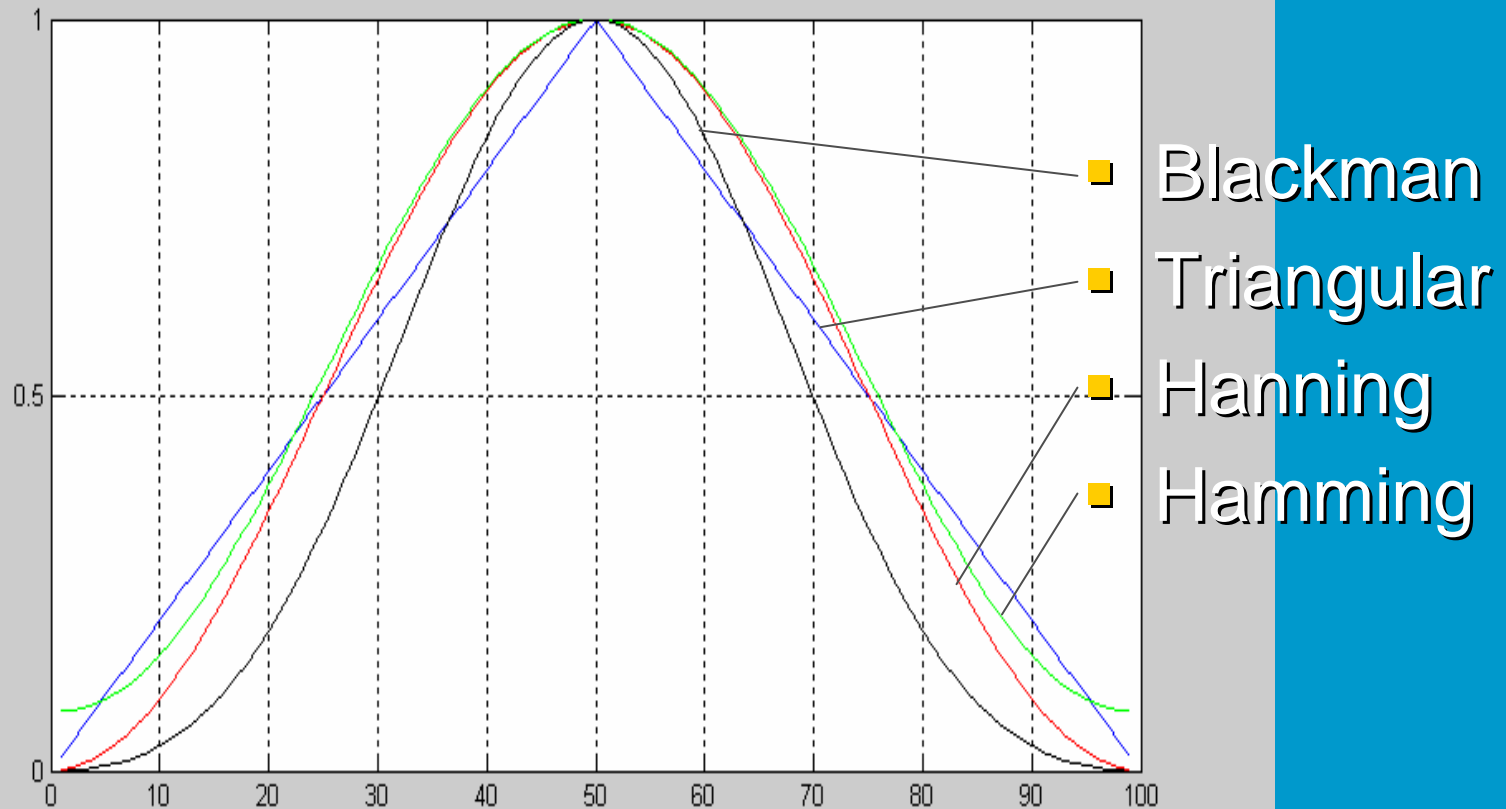
$$\Delta f = \frac{f_v}{N/2} = \frac{1}{N\Delta t} = \frac{1}{T}$$

- Frequency resolution

Introduction - DFT



Introduction – Windows



Introduction - problems

- Periodic assumption \Rightarrow Bias
- Reduce bias \Rightarrow Windows
- Reduce variance \Rightarrow Overlapping

Good old questions:

1. What is the random error?
2. What is the bias?

Error definition

- Random error

$$\sigma_{\hat{\phi}} = \sqrt{\frac{1}{N-1} \sum_{n=1}^N \left\{ \hat{\phi}_n - E[\hat{\phi}] \right\}^2}$$

- Bias error

$$\begin{aligned} b_{\hat{\phi}} &= E[\hat{\phi}] - \phi \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \hat{\phi}_n - \phi \end{aligned}$$

- Normalized random error

$$\varepsilon_r = \frac{\sigma_{\hat{\phi}}}{\phi}$$

- Normalized bias error

$$\varepsilon_b = \frac{b_{\hat{\phi}}}{\phi}$$

Random error - Bendat

$$\varepsilon_r = \frac{1}{\sqrt{B_e T_t}}$$

$$B_e = \frac{1}{T}$$

$$T_{tot} = MT$$

$$\varepsilon_r = \frac{1}{\sqrt{MT/T}} = \frac{1}{\sqrt{M}}$$

- Approximate error
- Effective bandwidth
- Total data length
- Approximate error

Random error - Welch

- The variance of M averages of dependent variables is
- Where, for an overlap of D samples

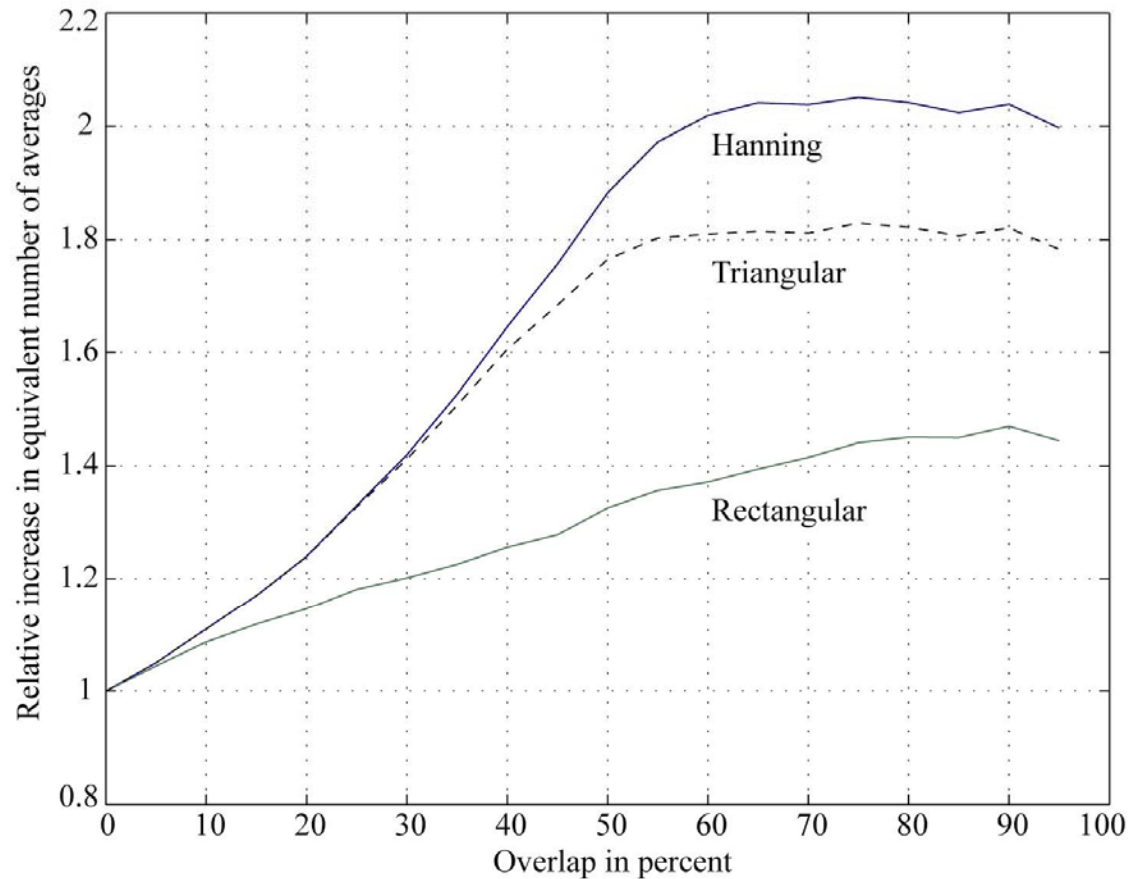
$$\varepsilon_r^2 = \frac{1}{M} \left[1 + 2 \sum_{q=1}^{M-1} \frac{M-q}{M} \rho(q) \right]$$

$$\rho(q) = \frac{\left[\sum_{n=0}^{N-1} w(n)w(n+qD) \right]^2}{\sum_{n=0}^{N-1} w^2(n)}$$

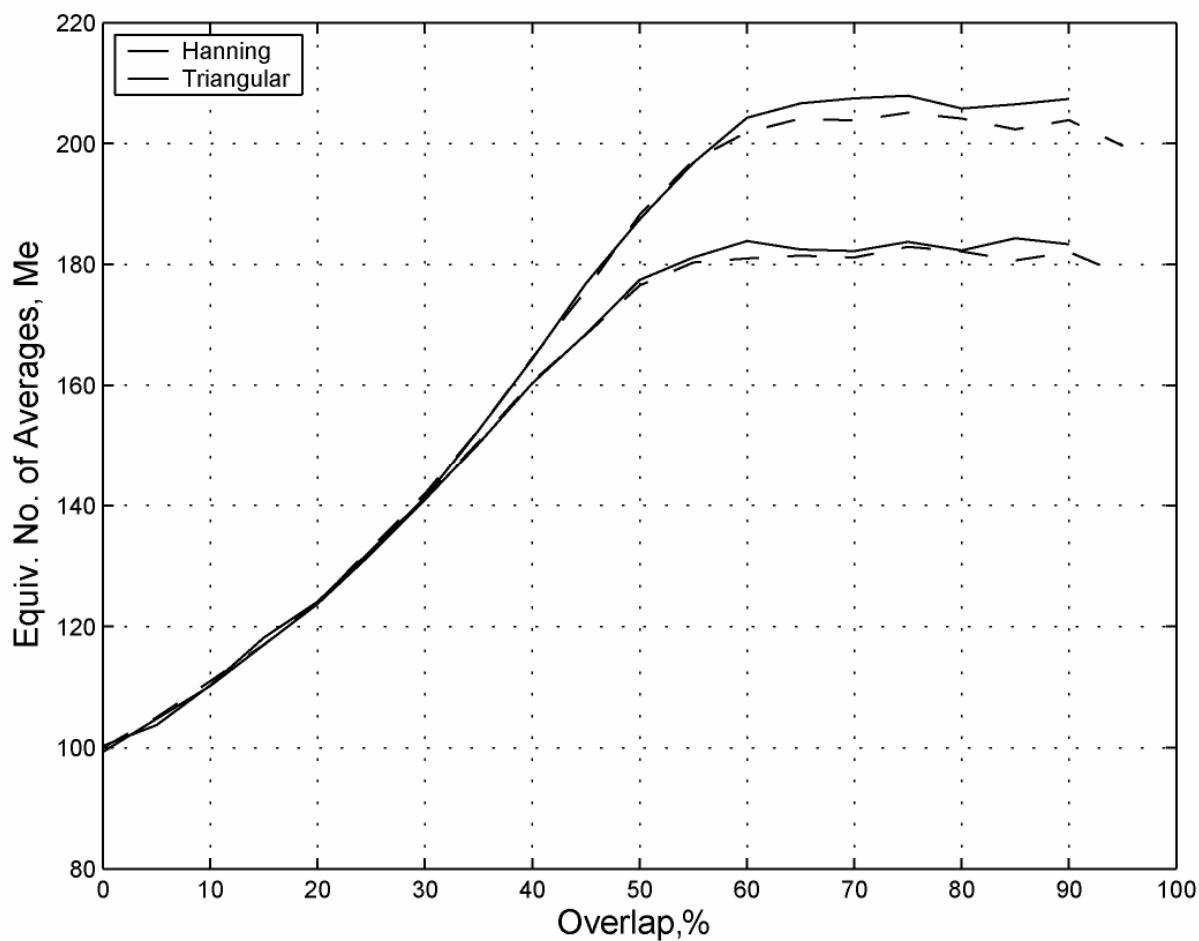
Random error - Welch

$$\varepsilon_r = \frac{1}{\sqrt{M_e}}$$

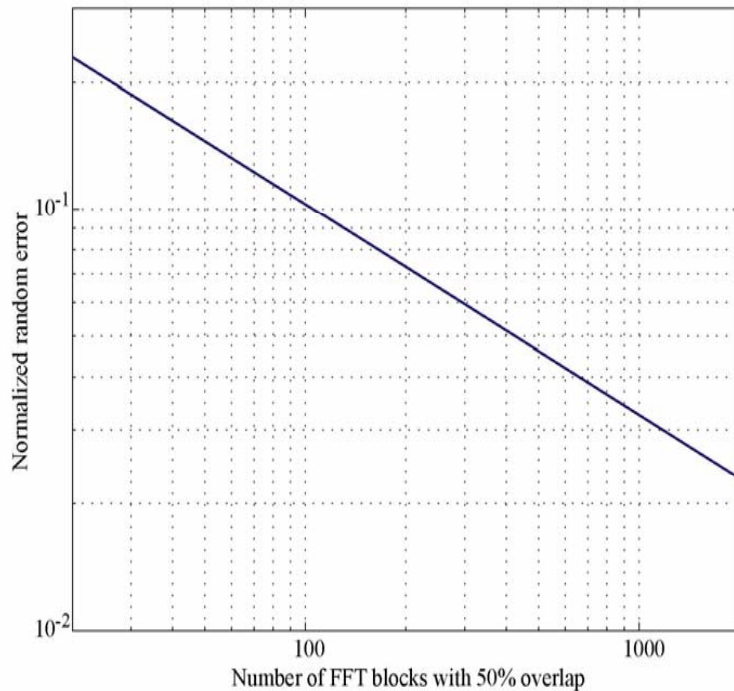
$$M_e = \alpha M$$



Random error - Simulation



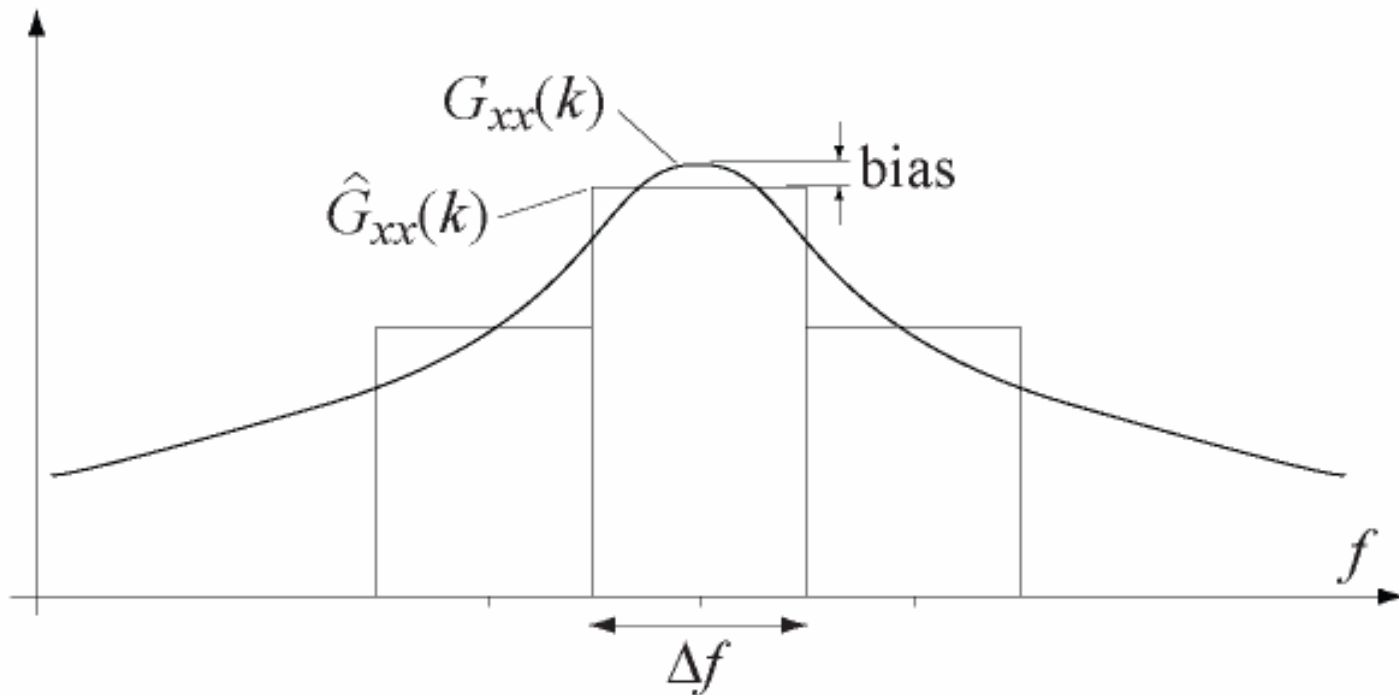
Random error - Conclusions



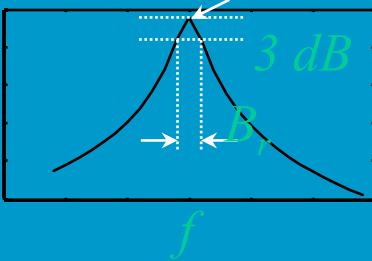
- Proper windowing
 - Proper overlapping
- =>
- minimum variance
 - "Same" as Bendat

Welch error formula for Hanning window with 50 % overlap

Bias error - Definition



Bias error - Bendat

$$\varepsilon_b = \frac{1}{3} \left(\frac{B_e}{B_r} \right)^2$$


$$B_e = \Delta f$$

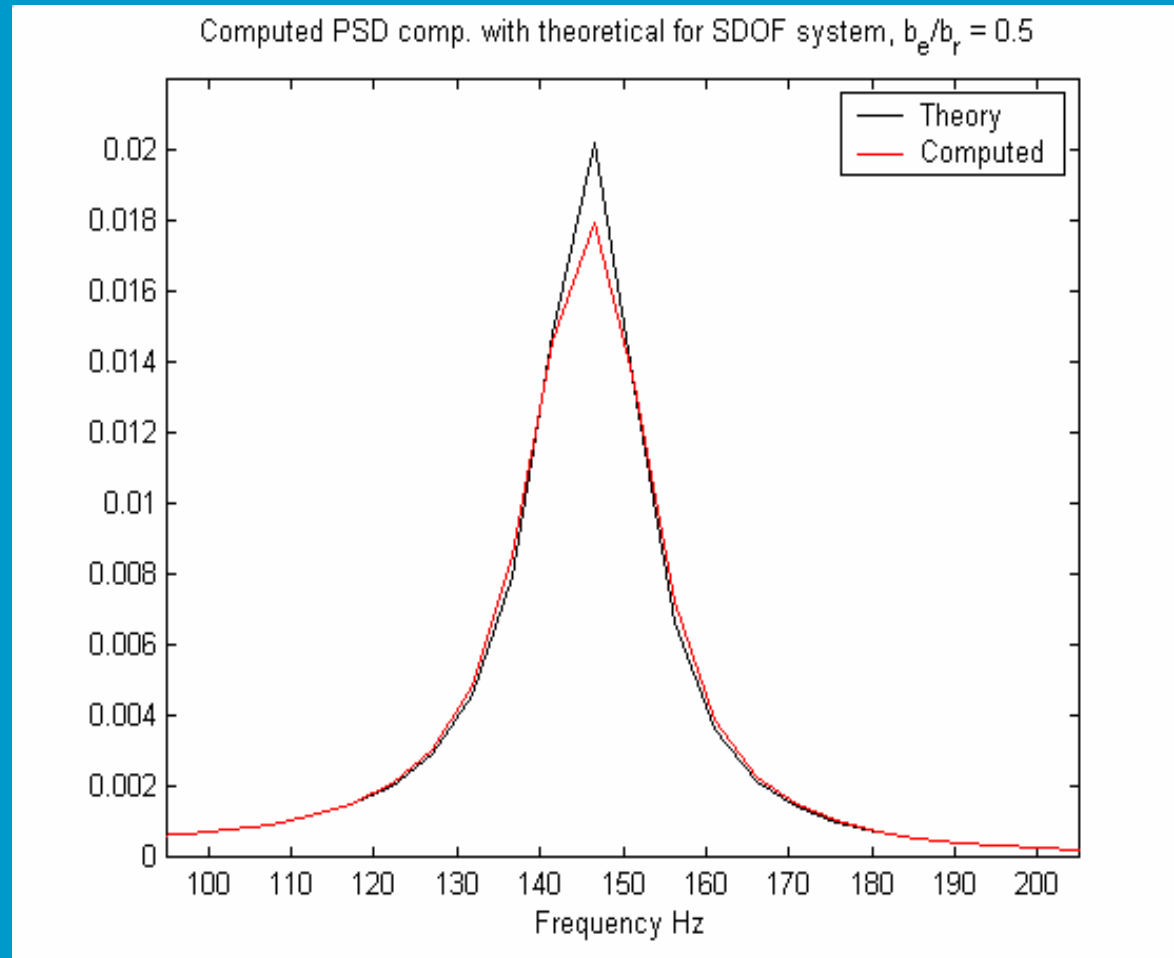
$$B_r = 2\zeta f$$

$$\varepsilon_b = \frac{1}{3} \left(\frac{f_v}{f} \right)^2 \left(\frac{1}{\zeta N} \right)^2$$

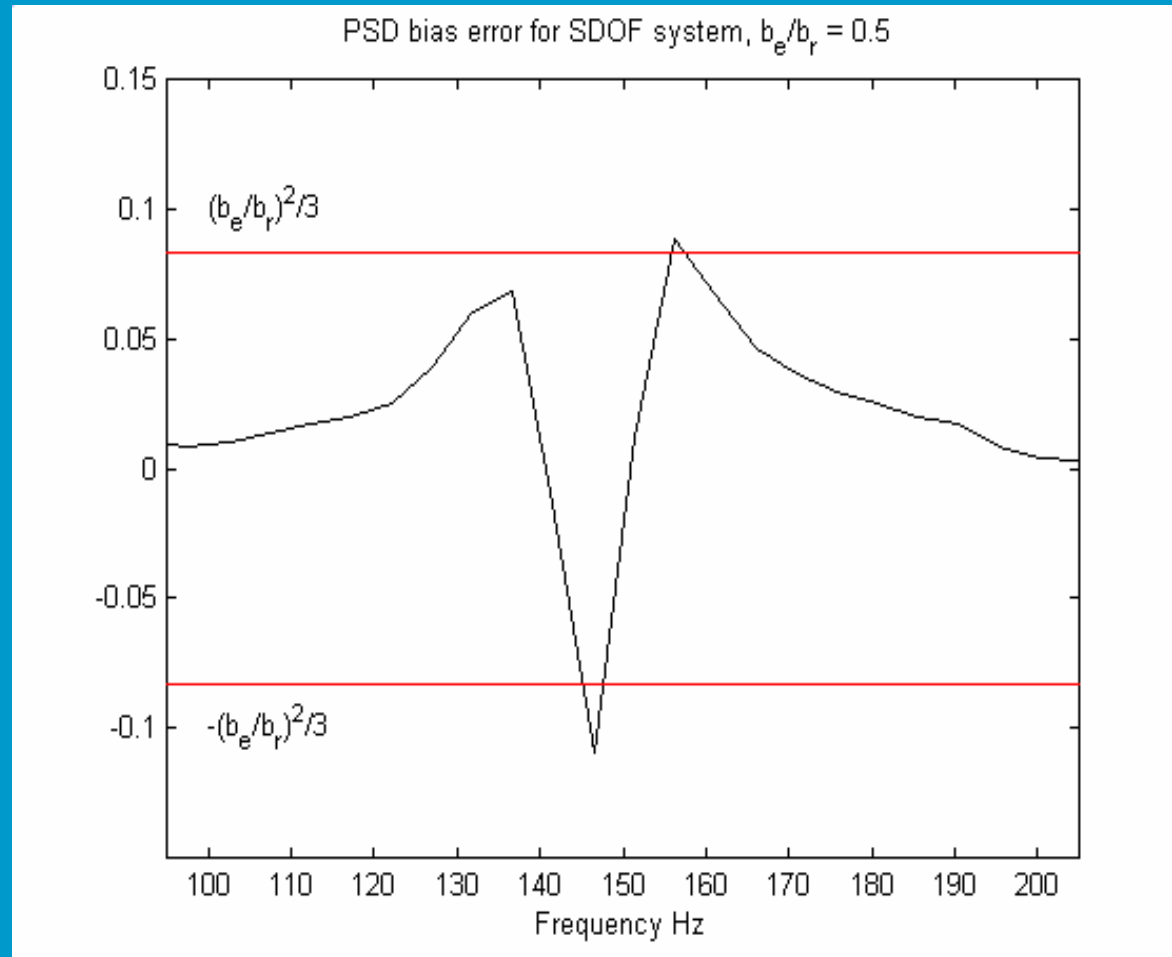
- The lower the frequency
- The smaller the damping
- The smaller the data segment

... The larger is the bias

Bias error - Leakage



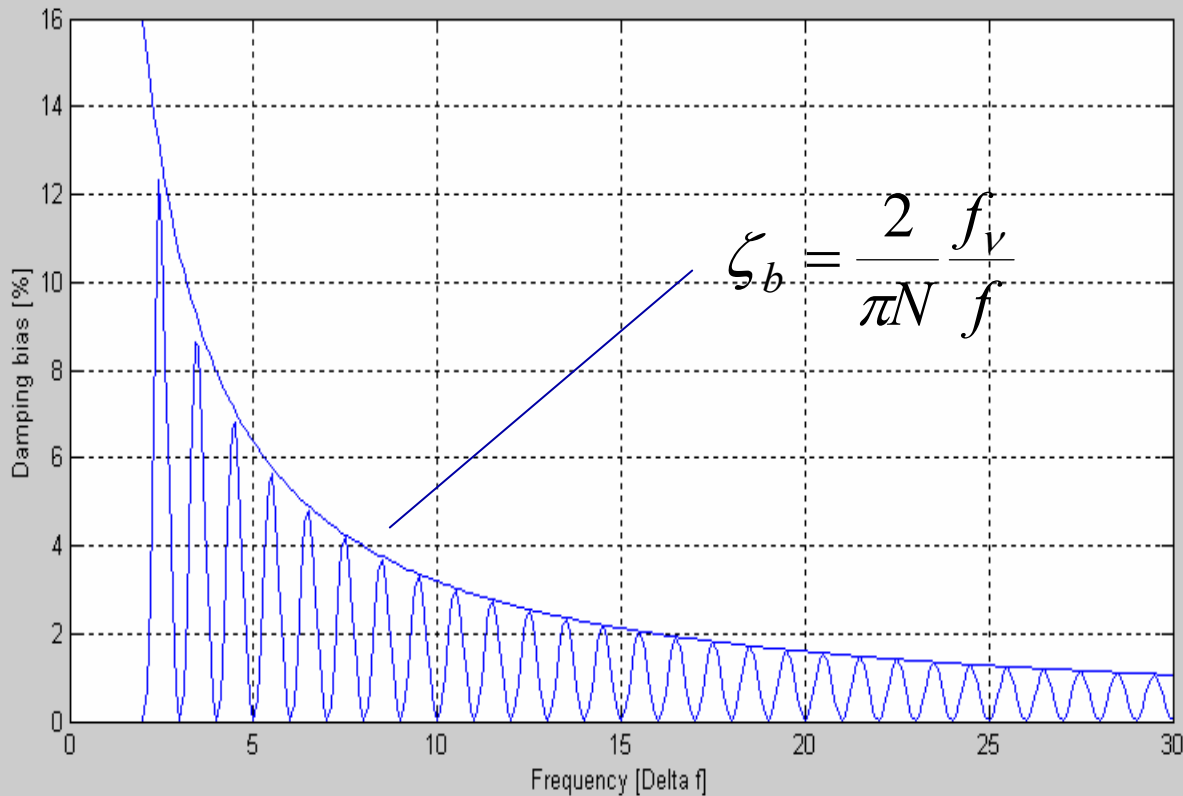
Bias error - illustrated



Bias error - Damping

- Consider the frequency of the harmonic as a multiple of the frequency resolution
- Calculate spectral density of harmonic
- Convert to correlation function by IFFT
- Calculate the bias as the decay = damping

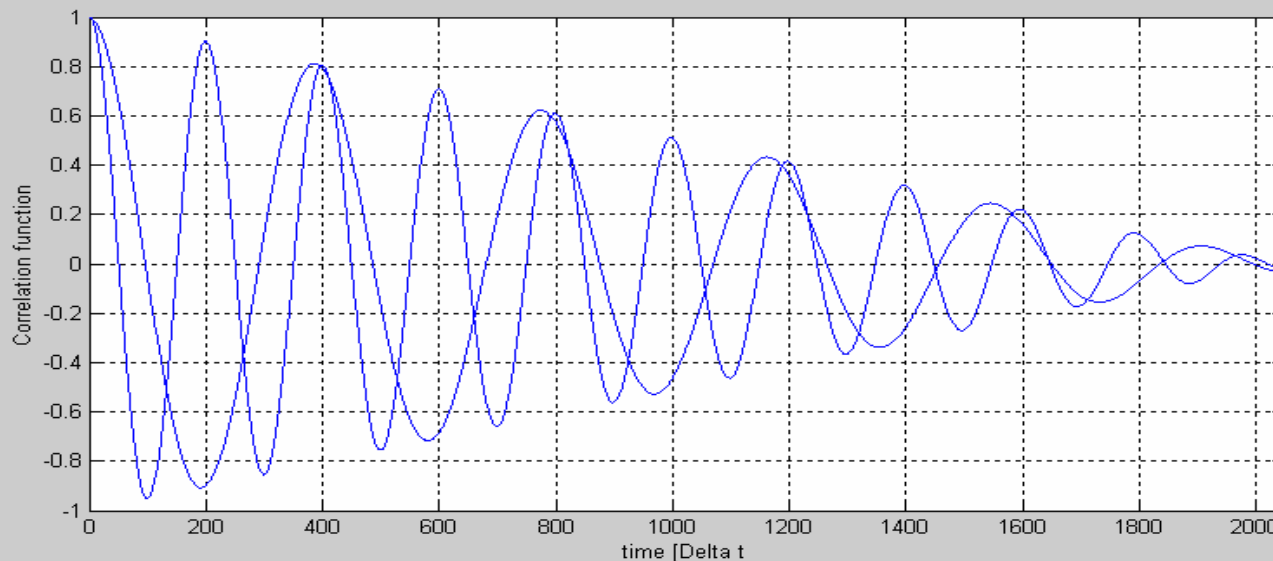
Bias error - Damping



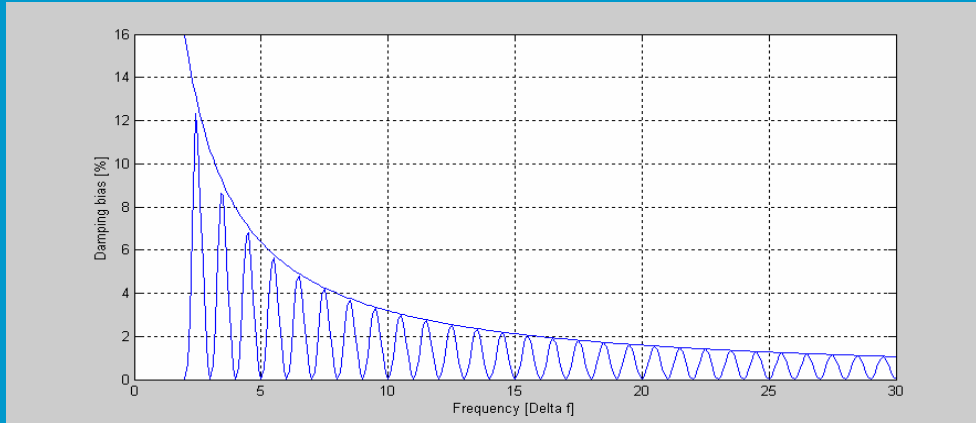
The absolute error on the damping in % (damping bias) as a function of the dimensionless frequency $\alpha = f / \Delta f$

Bias error - Damping

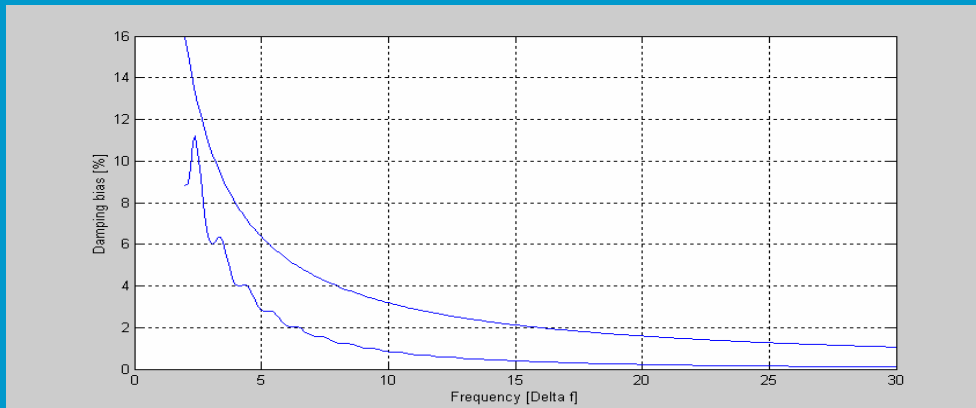
- At maximum bias the correlation function has a linear decay
- Minimum damping is at beginning (smallest relative error)



Bias error - Windows



- No window



- Triangular window

Bias error - Windows

Damping bias in % for different spectral windows as a function of the dimensionless frequency

| $f / \Delta f$ | Boxcar | Blackman | Hanning | Triang. | Hamming |
|----------------|--------|----------|---------|---------|---------|
| 10 | 3.18 | 1.36 | 1.01 | 0.83 | 0.81 |
| 20 | 1.59 | 0.35 | 0.26 | 0.22 | 0.21 |
| 30 | 1.06 | 0.16 | 0.12 | 0.10 | 0.09 |
| 40 | 0.80 | 0.09 | 0.07 | 0.06 | 0.05 |
| 50 | 0.64 | 0.06 | 0.04 | 0.04 | 0.03 |
| 60 | 0.53 | 0.04 | 0.03 | 0.03 | 0.03 |
| 70 | 0.45 | 0.04 | 0.03 | 0.03 | 0.02 |
| 80 | 0.40 | 0.03 | 0.02 | 0.02 | 0.02 |
| 90 | 0.35 | 0.03 | 0.02 | 0.02 | 0.02 |
| 100 | 0.32 | 0.02 | 0.02 | 0.02 | 0.02 |

Conclusions

Random errors

- Overlapping essential
- With proper overlapping the error is "the same" as given by classical analysis

Bias errors

- Windows essential
- Choice of window less important
- Low frequency is dangerous

May Anders and Kjell forgive me...

Thank you