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ABSTRACT

This paper is an attempt to study the possible dynamic behaviors that can emerge as a consequence of increased trade relations.
First, a classical nonlinear macrodynamic model, stemming from the endogenous business cycle tradition of Hicks (1950) and Goodwin (1951), is presented.
Second following a suggestion present in Velupillai (1991), the described macroeconomic structure is coupled, through trade, with another similar economy.
Third, the dynamics that emerge as the consequence of the above coupling is studied.
According to the different stages of the development and intensity of trade, it is shown that the two economies exhibit different dynamic behaviors such as frequency locking, bifurcation cascades and eventually chaotic evolution.
The simplicity of the model and the richness of its dynamic behavior suggests the importance of studying interdependent economies as coupled oscillators, and systems of economies as systems of parallel coupled oscillators.

KEYWORDS: *Nonlinear economics, frequency-locking, period-doubling.*

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INTRODUCTION

Transition periods and higher levels of integration are two of the economic themes of our time. The evolution taking place in the former socialist countries, the new free trade conditions experienced by the EEC members and the recent Nafta agreement show this tendency. Unfortunately in some cases this transition towards high integration levels seems to be associated with higher degrees of a-synchronicity and instability[1].

Nevertheless it should be pointed out that this is not a new phenomena. The evolution occurring in the industrialized as well as in the developing countries seems to have always been characterized by phases of synchronous developments which are from time to time replaced by phases of a-synchronous ones[2].

Here it is suggested that a possible explanation of such evolutions may be attributable to the intrinsic characteristic cyclical behavior of the

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[1]This seems to be the case for the former socialist countries and, with lower intensity, for the EEC members. The fall of the Berlin wall and the fall of the 'trade' walls in the EEC originated great expectations of higher economic stability and growth. Today it seems that these expectations have been largely disappointed. In both economic regions individual member states seem to experience diverse phases of economic evolution and high degrees of instability.

[2]An example may be drawn from the recent evolution of the industrialized countries. In the early sixties the economic evolution was highly synchronized to the point that the implemented policy was known as the 'stop and go' policy. During the end of the sixties through the seventies the development of individual countries has been highly turbulent and, to a certain extend, highly a-synchronous. This phase of a-synchronous developments has been replaced in the second half of the eighties by a phase of seemingly convergent behavior, while today we seem to experience again a-synchronous developments. The above picture is obviously simplified and degrees of a-synchronous evolution for the individual countries have always been observed, but what is important to realize is that convergence and divergence processes seem to be an historical characteristic of many economic regions.

aggregate demand and production of the single countries. It will be shown that the synchronous and a-synchronous evolution of the different economies *may* be ascribed, in some cases and in a non trivial way, to the mode in which these economies happen to be *coupled*, i.e., to the type and intensity of the trade interdependency.

In order to understand whether the cyclical evolution of an economy is to be principally attributed to its intrinsic mode of operation or to the way in which the economy is related with other economies, it is essential that the model describing its dynamic behavior is able to endogenously generate cycles. In other words in order to make the approach as general as possible it is crucial that the functional form describing the behavior of the economies be such as to allow cyclical as well as monotonic developments. If this is done, as we shall see, it will become theoretically possible to verify whether the cyclical evolution of a country is principally attributable to its intrinsic structure or to external factors.

As it will be shown here, even when the model description of each individual economy is very simple, the dynamic behavior of the coupled systems may be very complex. The bifurcation cascade or the strange chaotic dynamic behavior that can emerge from the interdependence is to be attributable to the well known fact that in the case of nonlinear systems the principle of superposition does not hold in general. In nonlinear systems the classical separation of the economic evolution into "cycles" which are described according to their periods is meaningful only in special cases. In non-linear systems a certain period length cycle is contemporaneously influenced and influences all the cycles so that causal separation is not, in principle, possible. This, of course, does not imply that the whole cannot be separated into its parts (which is often the case for the statistical work) but it suggests that much care should be put into the way in which we use our models. In the context of the simple model used here it is shown how higher levels of trade do not always imply higher degrees of stability of the global economic system. On the contrary higher degrees of a-synchronous behavior may in fact be observed.

A Classical Nonlinear Macroeconomic Model

The model chosen as a starting point of our analysis is very simple and traditional. It is based on the flexible-accelerator concept developed by Hicks (1950) and Goodwin (1951). Whenever possible the structure of the model will be kept as close as possible to the original formulation by Goodwin.

The yearly income of country i, $Y_i(t)$, is separated into demand for consumption goods, $C_i(t)$ and demand for investment goods, $I_i(t)$.

$$Y_i(t) = C_i(t) + I_i(t) \tag{1}$$

The consumption demand function, which I will assume to be linear is given by:

$$C_i(t) = C_{0i} + c_i Y_i(t) \tag{2}$$

where c_i is the marginal propensity to consume.

The investment function has a shape which is determined by 'nature' and the decision processes of the agents. When the system is in equilibrium, because there are no discrepancies between the planned level of demand and the planned level of production, the investment will evolve at equilibrium levels or rates. In disequilibrium, i.e., in a situation of change in demand expectations, it is assumed that there is a tendency for the producers to adjust their production capacity in accordance with the newly emerged needs. Therefore the feedback that the decision of investment has on the future decisions of production or consumption is only in part discounted so that it is a consequence of the newly emerged conditions of demand. It can be claimed that the above accelerational relation holds linearly in normal conditions, but does not hold at high or low levels of activity. The reason being that the assumption that a substantial part of the investment decisions is determined by the rate of change of demand is tenable only during normal periods. But when the rate of change in demand is very high this usually means that the system is already producing at a high pace so that most likely limits to the capacity of production are faced and this implies that the system may not be able to produce, above a certain level, at equilibrium rates. On the other hand when demand is low old capital goods are not replaced with new ones but are not destroyed as well so that the (negative) investment cannot be lower than depreciation, and again this implies that the investment level faces another limit.

A reasonable functional form which is consistent with the above description is given by the logistic equation so that:

$$I_i(t) = \dot{K}_i(t) = \varphi_i(\dot{Y}_i(t)) = \frac{k_i^{**} - k_i^{*}}{1 + e^{-D_i(\dot{Y}_i(t) + B_i)}} + k_i^{*} \quad (3)$$

where B_i, D_i, are calibration parameters and k_i^{**} and k_i^{*} represent respectively maximum levels of investment and disinvestment. The slope of the above function at the equilibrium value ($\dot{Y}_i=0$) is usually recognized as the linear accelerator. In the sequel I will follow this tradition and define the accelerator as $v_i=d\varphi_i(0)/d\dot{Y}_i(t)$.

Finally the dynamics of motion of the macroeconomic variables is influenced by the structural lags describing the moment at which expenditure decisions are made and the moment in which they are actually realized (the so called Robertson lag) and the moment at which the investment decision is made and the corresponding outlays (the so called Lundberg lag).

The total demand at time $t+\theta_i+\varepsilon_i$ is given by:

$$Y_i((t+\theta_i)+\varepsilon_i) = C_i((t+\theta_i)+\varepsilon_i) + I_i((t+\theta_i)+\varepsilon_i) \tag{1a}$$

where ε_i is the Robertson lag and $\theta_i+\varepsilon_i$ the Lundberg lag[3].

The lag occurring between the moment at which income is earned and it is spent may be described as follows:

$$C_i((t+\theta)+\varepsilon) = C_{0i} + c_i\, Y_i(t+\theta) \tag{2a}$$

The fact that it takes <u>time to build</u> and consequently there is a lag between the moment at which an investment decision is made and the

[3]This particular choices for the description of the lags are made, as it will be shown in the sequel, to maintain the model as close as possible to the formulation present in Goodwin (1951) and in Velupillai (1991).

capital goods are actually delivered may be described as follows:

$$I(t+\theta_i+\epsilon_i) = \dot{K}(t+\theta_i+\epsilon_i) = \varphi_i(\dot{Y}_i(t)) = \frac{k_i^{**}-k_i^{*}}{1+e^{-D_i(\dot{Y}_i(t)+B_i)}}+k_i^{*} \quad (3a)$$

Substituting equations, (2a) and (3a) into (1a) we have the law of motion of our economy which is described by a mixed nonlinear difference-differential equation[4]. In order to maintain the structure of the model as simple as possible the mixed difference differential equation is approximated by a second order differential equation.

The Taylor series expansion of $Y((t+\theta_i)+\varepsilon_i)$ around the point $(t+\theta_i)$ yields:

$$Y_i(t+\theta_i)+\epsilon_i\dot{Y}_i(t+\theta_i)+\frac{\epsilon_i^2}{2!}\ddot{Y}_i(t+\theta_i)+\ldots = C_{0i} + c_iY(t+\theta_i)+\varphi_i(t) \quad (4)$$

Retaining the first two terms of this series expansion and approximating[5] again around point t and retaining the first two terms we obtain:

[4]A possible approximate solution of the above can be obtained by the implementation of the so called direct method. For an application to Frisch's (1933) model of this method see Zambelli (1992).

[5]The approximating error is obviously a function of theta, epsilon and of the particular truncation of the Taylor series expansion. Therefore the quantitative descriptions of the dynamics of the above system may vary considerably. Fortunately the qualitative behavior is, independently from the different choices, the same. I have tried out different approximation schemes and verified that the qualitative conclusions here reported are unaffected.

$$\epsilon_i \theta_i \ddot{Y}_i(t) = -[\epsilon_i + (1-c_i)\theta_i]\dot{Y}_i(t) - (1-c_i)Y_i(t) + C_{0i} + \varphi_i(\dot{Y}_i(t)) \quad (5)$$

The above equation has the same functional form as in the Goodwin model, but here an explicit definition for $\varphi_i(\dot{Y}_i(t))$ is given[6].

The state space representation of equation (5) is given by:

$$\dot{Y}_i(t) = Z_i(t) \quad (6.1)$$

$$\dot{Z}_i(t) = b_i\,[C_{0i} - (1\text{-}c_i)\,Y_i(t) - a_i Z_i(t) + \varphi_i(Z_i(t))] \quad (6.2)$$

$b_i = 1/(\theta_i \varepsilon_i)$ and $a_i = \varepsilon_i + \theta_i(1 + m_i\text{-}c_i)$

The above model is able to account for cyclical behavior. In figure 1 a set of characteristic curves, depending on the accelerator parameter v (i.e., the middle range slope of $\varphi i(\dot{Y}i(t))$, are reported and in figure 2 an Hopf bifurcation is reproduced.

The functional form of this model is already richer than most standard macroeconomic models because, according to different values of the parameters, it allows for diverse dynamic behaviors. For example as the value of the accelerator increases the equilibrium changes from being an attractor into being a repellor, and a limit-cycle behavior emerges (see fig. 2).

An industrialized economy is likely to operate with a high accelerator value, while a less developed one is likely to operate with a low

[6]To be more precise the functional form of equation 5 in the text is the same as equation 5f of Goodwin's (1951) article.

accelerator value. At the same time it is also likely that the marginal propensity to consume in a developed country is lower than in a less developed country. The parable that can be told with this model is that at depressed stages of structural development, when the accelerator is low the economy evolves non cyclically, but when more mature stages are reached the accelerator increases and cyclical behaviors emerge naturally.

Moreover, if one is willing to agree that the parameters describing the functional forms mirror, at the aggregate macro level, the micro-decisions of agents, the structure of this model may be considered sufficiently general to represent different stages in the evolution of the countries.

Coupled Open Economies.

Given the richness of the model it seems to be appropriate to study the dynamic evolutions of two coupled economies which are characterized by different values of the structural parameters[7].

The flow accounting description of an open economy is given by:

$$M_i(t) + Y_i(t) = I_i(t) + C_i(t) + X_i(t) \tag{7}$$

Following traditional lines I shall assume that the demand for foreign goods, M_i, is a function of the level of income so that:

[7]In the Appendix the possibility that the economy may be studied when driven by an external independent oscillating component is presented to elucidate some of the properties of forced oscillators. In particular the frequency locking property will be shortly discussed.
In fact equation (5c) of Goodwin's (1951) model includes a forcing term.

$$M_i(t) = m_i Y_i(t) \tag{8}$$

Obviously the export of one country is the import of the other so that:

$$X_i(t) = M_j(t) = m_j Y_j(t) \tag{9}$$

In order to keep the model as simple as possible I have kept the traditional assumptions of linearity and that of constant prices. Obviously these are very simplifying assumptions and they could be relaxed. However this would complicate the model and distract attention from the main theme[8], which is to analyze the rich dynamic behavior emerging from a very simple model structure.

Maintaining the model described by system (6), enlarged by (7) (8) and (9), we obtain:

$$\dot{Y}_1(t) = Z_1(t) \tag{10.1}$$

$$\dot{Z}_1(t) = b_1 [C_{01} - (1+m_1-c_1)\, Y_1(t) - a_1 Z_1(t) + m_2(Y_2(t)+Z_2(t)) + \varphi_1(Z_1(t))] \tag{10.2}$$

$$\dot{Y}_2(t) = Z_2(t) \tag{10.3}$$

$$\dot{Z}_2(t) = b_2 [C_{02} - (1+m_2-c_2)\, Y_2(t) - a_2 Z_2(t) + m_1(Y_1(t)+Z_1(t)) + \varphi_2(Z_2(t))] \tag{10.4}$$

where $b_i = 1/(\theta_i \varepsilon_i)$ and $a_i = \varepsilon_i + \theta_i(1+m_i-c_i)$, $i=1,2$.

[8]Obviously if the aim of the present paper was that to construct a more realistic model description of the economy richer functional relationships should be postulated.
As an alternative one could assume, contrary to the approach chosen in this paper, that there are two 'representative' agents 'representing' each economy and that each agent is maximizing an intertemporal utility function whose arguments are ($Y_i(t)$-$X_j(t)$), $M_i(t)$ and eventually $K(t)$, and that c_i, m_i and so on are the solution of the maximization problem, but here it is assumed that these parameters are given, i.e., that they express already taken decisions.

The comparative static analysis of this class of models is well known[9]. In the following a comparative dynamic method is employed[10]. It is assumed that Economy 1 is the most developed while Economy 2 is the least developed, and this is captured by the following:

i. The functions φ_1 and φ_2 are such that $\varphi_1(\dot{Y}(t)) - \varphi_2(\dot{Y}(t)) > 0$ when $\dot{Y}(t) > 0$ and that $\varphi_1(\dot{Y}(t)) - \varphi_2(\dot{Y}(t)) < 0$ when $\dot{Y}(t) < 0$. This implies that the more industrialized economy has greater capacity to produce capital goods. The accelerator $v_i = dK_i(t)/dY_i(t)$, the slope of φ in the middle range, is consequently such that $v_1 > v_2$.

ii. Following a classical Kaldorian assumption, the marginal propensity to consume of the industrialized economy may be assumed to be lower than the marginal propensity to consume of the least developed economy, i.e., $c_1 < c_2$, but the average propensity to consume of the industrialized country is, generally, greater than that of the least developed country, i.e., $C_{01} > C_{02}$.

iii. The industrialized country, while it enjoys a greater capacity to produce capital goods, due to the higher complexity of the production process may take longer to actually construct them so that $\theta_1 > \theta_2$.

The choices of the particular numerical values evidently reflect an a priori view, but the model could obviously be 'calibrated' to fit other a priori views. What it is important to realize is that the two economies have different structures, they could obviously be both industrialized countries, but characterized by different reduced form parameters.

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9 The equilibrium points of the two economies are given by:

$$Y_i^e = \frac{C_{0i}(1-c_j) + (C_{0i} + C_{0j})m_j}{(1-c_i)(1-c_j) + (1-c_i)m_j + (1-c_j)m_i}$$

10 The trajectories are approximated numerically implementing the standard Runge-Kutta approximating procedure (see for example Fröberg, 1985). The programs for the computation of the trajectories as well as all the programs used in the calculations have been written by myself using the programming language MATLAB. They are available on request.

Therefore *the particular choice of the numerical values is not relevant in itself and different values could be chosen*[11].

According to the different numerical values chosen for the parameters different dynamic behaviors for $Y_1(t)$ and $Y_2(t)$ will emerge. For certain sets the two economies will be highly synchronized while for others highly a-synchronized.

Figure 3a through 3c show the different dynamic behaviors that can emerge when the interdependence of the two economies increases, i.e., both m_1 and m_2 increase and the other parameters are kept the same. The x-axis shows the value of the propensity to import of the second country m_2 (and $m_1 = 2\ m_2$). The y-axis reports the invariant set of a Poincaré map[12]. In Fig. 3a and 3b for each value of m_2 the set of points $Y_i(t)$

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[11] When not otherwise stated the values of the parameters for the simulations have been:
for Economy 1 $\theta_1 = 1.0$, $\varepsilon_1 = .25$, $C_{01}=10$, $c_1 = 0.6$, $v_1 = 2.0$, $k^{**}_1 = 9$, $k^{*}_1 = -3$,
for Economy 2 $\theta_2 = 0.5$, $\varepsilon_2 = .25$, $C_{02}= 2$, $c_2 = 0.8$, $v_2 = 1.6$, $k^{**}_2 = 4$, $k^{*}_2 = -2$,
The parameters m_1 m_2 and occasionally v_2 have been considered control parameters. The parameters v_i are the 'slope' of the function $\varphi_i(Z_i)$ at points $(\varphi_i, Z_i) = (0,0)$. Consequently the parameters D_i and B_i, see (3), are computed to fulfill this condition. The step size for the Runge-Kutta approximation has been $h=2*\pi/60$.

[12] A description of the Poincaré map may be found in almost all the books dealing with dynamical systems. Good references may be Thompson and Stewart (1986) or Abraham and Shaw (1985).
The procedure is the following. For a given set of parameters a trajectory is computed. After a transient time the trajectory has converged towards a limiting set, which is either an equilibrium point, a limit cycle or a strange attractor. The Poincaré section (or cut) is represented by the points of the trajectory that go through an (hyper)-plane. The invariant set of a Poincaré map is the set of points passed by the trajectory. For example if the trajectory intersects the plane at constant intervals of time a subarmonic oscillation of order n would appear as a sequense of n dots repeated indefinitely in the same order.
The dimension of the invariant Poincaré section is represented by the number of points in the plane passed by the trajectory and a high dimension of the Poincaré invariant set reveals a remarkably complex structure. Obviously all the graphs reported in the present paper are the result of some numerical approximation. Consequently the limiting invariant set of the Poincaré mapping cannot be precisely

associated with Poincaré cut $Z_i(t)=0$ and $dZ_i(t)/dt>0$ are reported. In Fig 3c for each value of m_2 the set of points $Y_2(t)$ associated with the equilibrium values of the first economy, $Y_1(t)=Y_1^e$ and with $dY_1(t)/dt > 0$, is reported.

In Figures 3a and 3b a single point associated with a value m_2 indicates that the economy behaves cyclically and that the attractor is a simple limit cycle, i.e., the expansion phases and the depression phases endlessly repeat themselves in the same fashion. When more points are associated with m_2, we have quite a different picture. If the number of points are two this means that the economy, before repeating the same dynamical pattern, goes through at least two different expansion and two different contraction phases. By extension we have that when the points are n the economy goes through at least n different expansion and contraction phases.

In figure 3c a single point associated with m_2 indicates that the two economies exhibit synchronous behavior, i.e., they experience contemporaneous phases of expansion and contraction (they are 'locked' with each others). In the case of no interdependency (when $m_2=0$ and consequently also $m_1=0$) the two economies are independent and consequently

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determined. The problems in determining the dimension of the limiting invariant set emerge when this is particularly high. The limiting invariant set of the Poincaré mapping is infinity, (a finite cycle cannot be traced), for example, in two cases: a) when two oscillating variables are incommensurable (the ratios between the periods belong to the irrational numbers, the case $m_1=m_2=0$ would probably be the case); b) when the system is chaotic and the limiting set is a strange attractor. Evidently the infinity and the infinitesimal have to be numerically approximated by the 'large' and the 'small'. The large and the small are not 'measurable' in standard mathematics, but yet they are defined. Different is the situation if one operates inside non-standard mathematics were these magnitudes are not considered. Velupillai (1992) has shown that in the case of the Goodwin's model the dynamics of the nonlinear oscillators may in fact be better understood implementing non-standard analysis.

they are not synchronized (They would be synchronized only by fluke). From an inspection of figures 3a through 3c one could conclude that higher degrees of trade are actually advisable for the stability of the system, but that the road toward higher integration level implies turbulent or at least a-synchronous behavior.

Figure 4 reports the 'degrees' of synchronicity of the two economies as a function of the respective propensities to import (m_1 and m_2). To each pair (m_1, m_2) is associated the number of limiting points relative to the Poincaré map obtained at the point where the first economy is exhibiting a level of production which is compatible with equilibrium (Y_1^e). In other words for each couple (m_1,m_2) the number of point elements of the invariant set of the Poincaré mapping $P:(Y_2(t_n)) \dashrightarrow (Y2(t_{n+1}))$, is reported; $\{t_n\}$ is the sequence of points for which $Y_1(t)=Y_1^e$ and $dY_1(t)/dt > 0$. Obviously if the set of limiting points has only one element the two economies are highly synchronized; if the limiting set is constituted of two points or more the two economies are increasingly a-synchronized[13].

[13]For numerical approximation purposes in this paper two points that are near each other are considered to be equal if they fall inside an approximating interval and are consider to be non equal if they fall outside it. Therefore the actual determination of the dimension of the invariant set is reliable only when its actual dimension is low and the points are distinguishable.
Fortunately, given the problem at hand, it is not essential to determine whether the system can give rise to chaotic behavior or not. In fact an estimated high dimension is anyway an indication of a very long period and of a high degree of a-symmetric behavior.

CONCLUSIONS

The model economy presented in this paper seems to be able to account for different phases in the economic development of two cyclically evolving economies. This is analogous to Day's (1992) attempt to analyze different phases of development of an economy, but here it is shown that the intrinsic oscillating behavior properties that an economy might exhibit, when studied as a closed system, might change quite substantially when studied as an open economy.

In transition periods the structural properties of an economy are obviously subjected to changes and these changes are bound to influence the dynamic evolution of the entire economic region. In some cases the qualitative dynamic behavior is not affected, but there exist many relevant cases in which the qualitative behavior would drastically change and complex dynamic interactions would emerge.

This fact is well encapsulated by both figure 3 and figure 4. As the degrees of interdependency increases the two economies will exhibit quite different cyclical behaviors. While in the closed case a depression is followed by a phase of expansion which mimics the previous one, when interdependency arises the different phases of expansion and contraction may not at all alternate in the same way so that a rather sustained depression may be followed by a rather mild boom and vice versa.

Moreover in the context of the present simple model an increase of exports of the second economy, due to a positive expansion phase of the first may not be associated with any detectable increase in income. For example, if we consider the parameters m_1 and m_2 to be control parameters, somehow influenced by the authorities or determined by a bargaining process, an increase in the propensity to import of the first economy

(m_1) greater than the increase in the propensity to import of the second economy (m_2) may actually lead to a higher degree of instability and of unsynchronized evolutions[14].

The currently observed a-synchronous evolution could be attributed, to some extent, to important recent events, such as those that occurred in 1989, which have determined important structural changes[15]. Therefore the argument that as a consequence of a higher integration the development and stability of an economic region is most likely to improve, should be made with a certain amount of caution[16].

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[14]This phenomena can be seen by an inspection of figure 3c. Each point in that diagram represents, by construction, a situation where the first economy is expanding. An odd number of limiting points implies, therefore, given the continuity of the trajectories, that there are situations for which, as the first economy is going through an expansionary phase, the second is going through a depressive one. Any minor changes in the propensities to import of the countries may not allow the system to reach a higher degree of synchronization. For values of m_2 below 0.042 the system is bound to be a-synchronous.

[15]Many examples are before us. A good one may be that of the German reunification. After the reunification the German economy operated at higher levels of aggregate income, exports and imports and different investment and consumptions behaviors. In terms of the model presented in this paper this implies that all the structural parameters describing the German system have changed. But the reunification should not have determined structural changes in the German trading partners. Therefore the degree of instability and a-synchronicity observed in the European countries could be attributed to the emergence of a new and more complex limit cycle. This being the case it would imply that a-synchronous behavior is a normal permanent recurrent condition and not a 'transitory' one. In the absence of further structural changes we would have that the countries involved would go through, in different moments, to different phases of the business cycle. The recent collapse of the European Monetary System would, in this case, simply mirror the difficulties, imbedded in the economic structure, to generate convergent economic behavior.

[16]After all, the model structure used here, the flexible multiplier-accelerator, is very traditional and still at the foundation of many economic and econometric models currently used.

APPENDIX. *Forced Oscillations and the Devil's Staircase.*

If the cyclical evolution of the balance of payments in current accounts (CA) is exogenously given the study of the system is highly simplified. I will assume that it behaves sinusoidally so that:

$$CA(t) = X_1(t) - M_1(t) = B \sin(t) \tag{1.A}$$

and therefore

$$Y_1(t) = I_1(t) + C_1(t) + B \sin(t) \tag{2.A}$$

$I_1(t)$ and $C_1(t)$ follow the laws of motion described in the text.

The presence of an external oscillating forcing term whose frequency and intensity is independent from the current state of the economy, while not as interesting as in the case of interdependency, may help to elucidate some of the properties relative to the frequency locking condition.

In figure A.1 a bifurcation diagram reporting the Poincaré invariant set relative to the frequency of the forcing term is reported[17]. Therefore the numerical index of the set indicates the period length of the driven oscillator. One point implies the same period as the forcing term, two points double period, three points triple period and so on. Given numerical approximation problems, when the points become too dense a precise estimation of the length of the period is not possible. Here this would be the case for the forcing amplitude $B < .9$.

In figure A.2 the evolution of the rotation number as a function of the coupling strength B is reported. The rotation number is often used to give an indication of the degree of 'twist' that the forcing term exerts on

[17]The trajectories have been computed for Economy 1, the values of the parameters have been the same as reported in footnote 11.

the already oscillating system. First the trajectory of the oscillating system is written in polar coordinates, secondly the Poincaré section relative to the multiples of the period of the forcing term, in this case $2*\pi$, is taken. Given that $\{\theta_n\}$ is the sequence of the angular coefficients relative to the invariant Poincaré set the rotation number for each trajectory is given by:

$$\rho = \lim_{n \to \infty} \frac{\theta_n - \theta_0}{2\pi}$$

In the case in which the invariant set is made of three points the above limit would tend to 1/3. In fact in polar coordinates the invariant set of the Poincaré map is $\{\theta_1, \theta_2, \theta_3\}$. Therefore $(\theta_1 + \theta_2 + \theta_3)/(2*\pi*3)$ would give exactly 1/3. A representation of the frequency locking properties of the system as a function of the rotation number is known as the 'Devil's Staircase'[18]. It is a staircase where the stairs are represented by frequency locking regions associated with rational numbers, i.e. regions for which changes in the control parameter will not determine a change in the frequency of the system, but it is suited for a supernatural being, the Devil, because the jump between two stairs is paved by the infinity of irrational numbers which separate two rational ones.

[18]The devil's staircase is suggestively described in Schroeder (1991).

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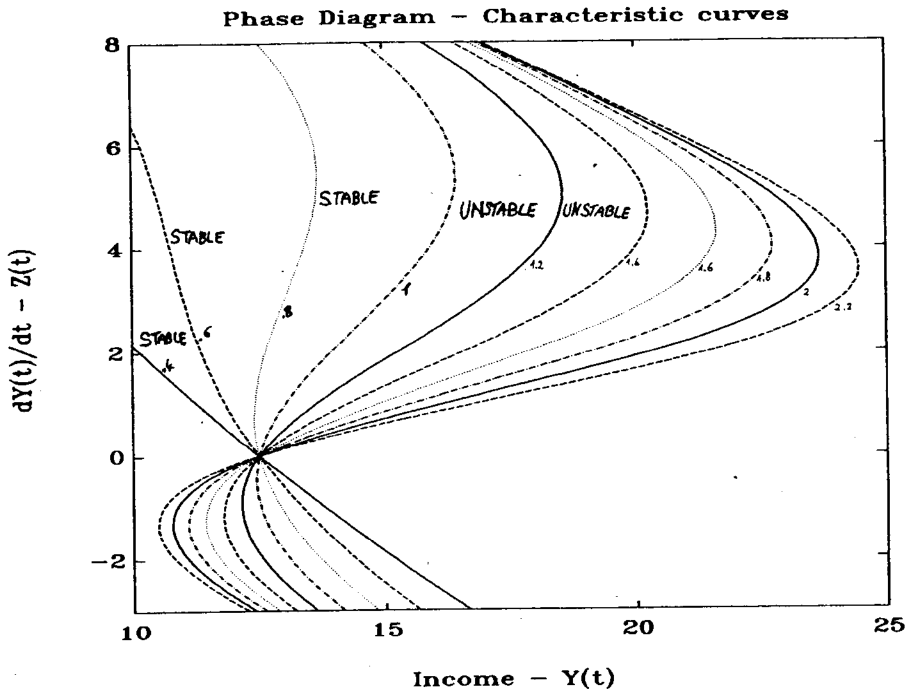


Figure 1.

Set of characteristic curves, functions of the accelerator parameter v = $d\varphi_1(Z_1(t))/dt$, i.e., the 'slope' of the function $\varphi_1(Z_1)$ at points $(\varphi_1, Z_1) = (0,0)$. The parameters D_1 and B_1, see eq. 3, are computed, for given k^{**} and k^*, to fulfill this condition. The numerical values of the remaining parameters are $C_{01} = 10$, $c_1 = 0.6$, $v_1 = 2.0$, $k^{**}_1 = 9$, $k^*_1 = -3$.

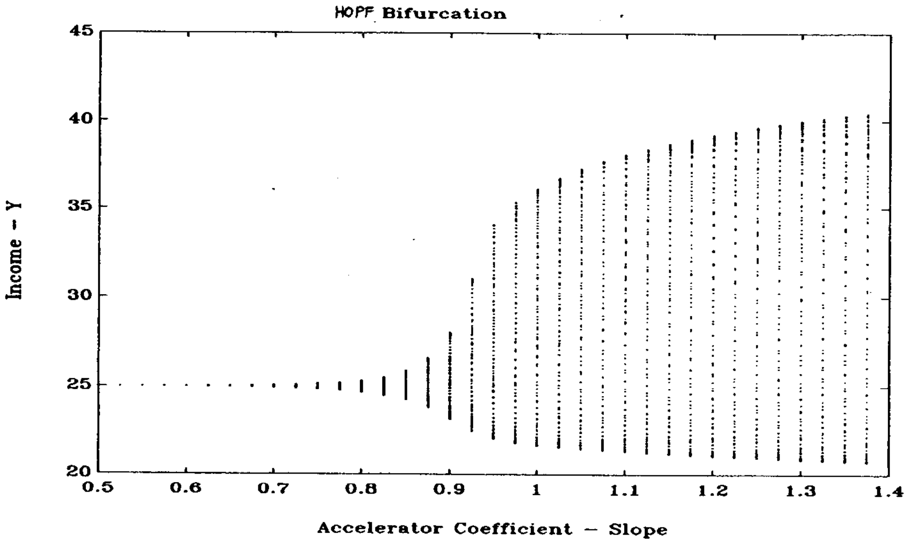
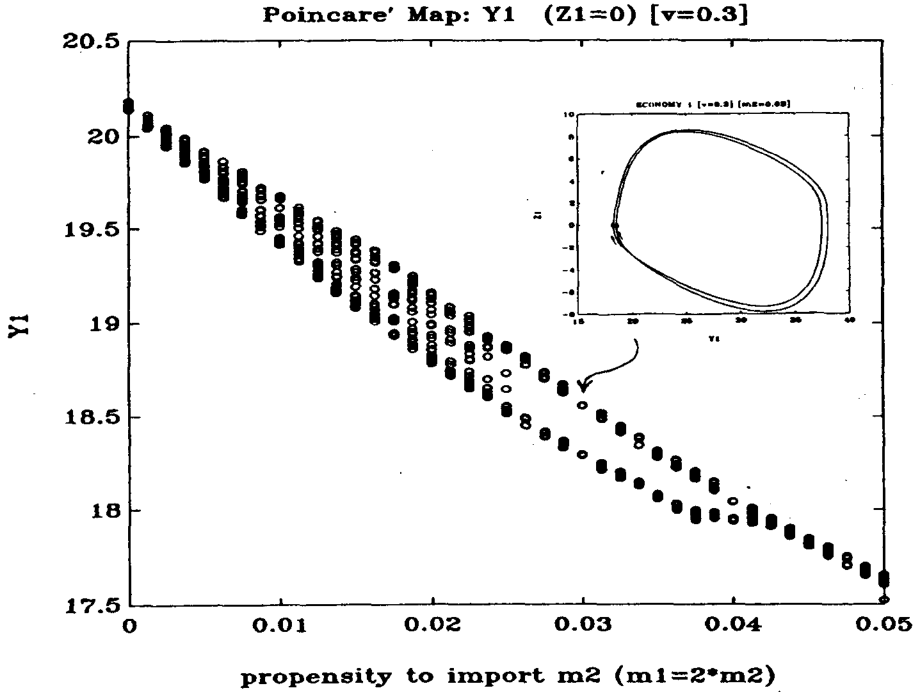
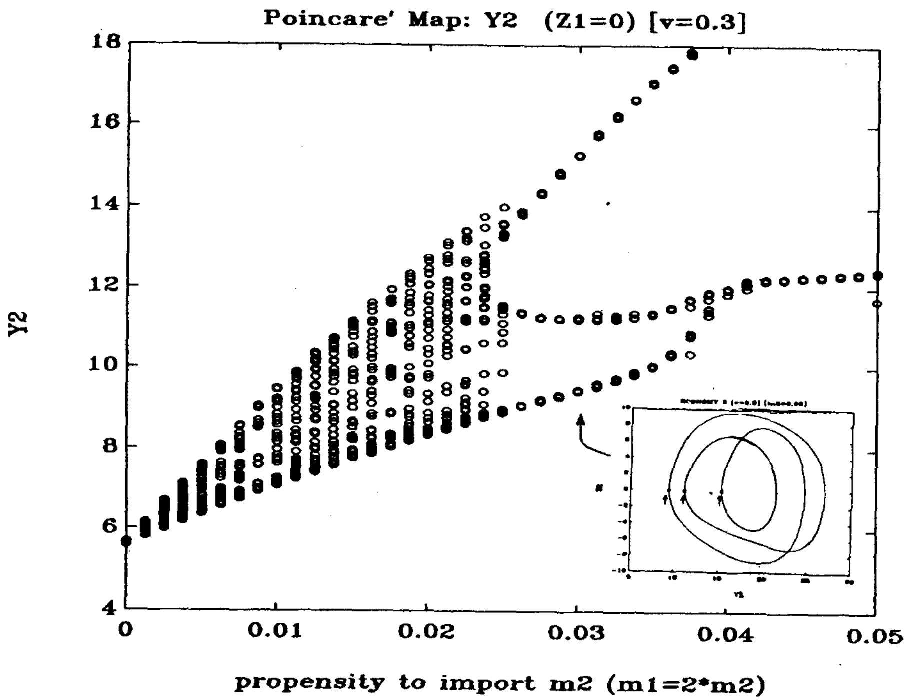


Figure 2
Hopf bifurcation. For values of v_1 less than 0.8 the equilibrium point is an attractor, for values above 0.8 it becomes a repeller and the trajectories are attracted towards a limit cycle. The Y-axis reports the domain of the limiting set.



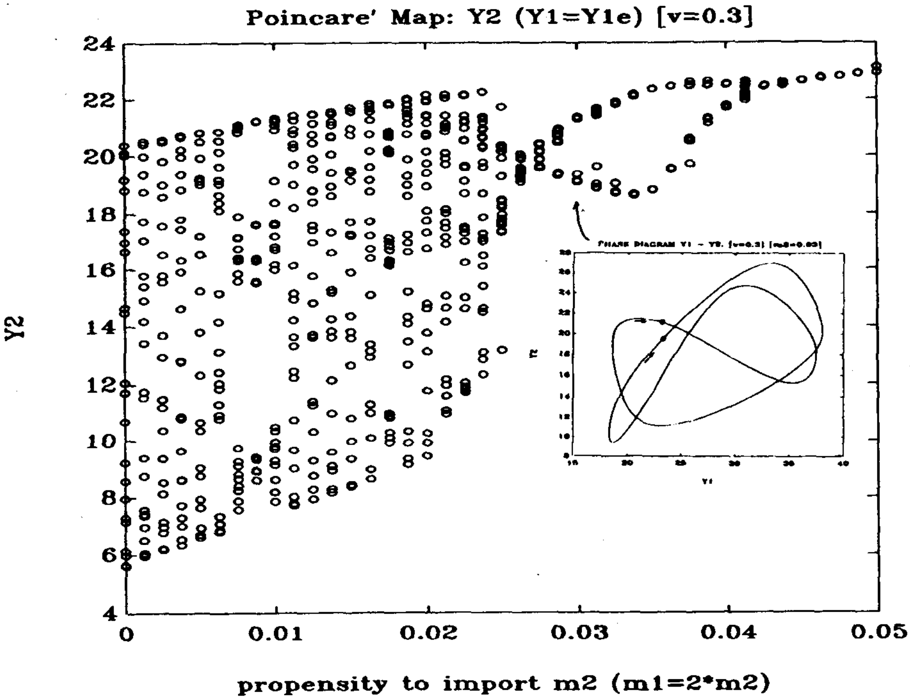
<u>Figure 3a.</u>

Bifurcation Diagram. Estimates of the invariant set of the Poincaré map.
The Poincaré cut is computed at points $Z_1(t)=0$ and $dZ_1(t)/dt>0$
x-axis values of m_2 (with $m_1=2m_1$ and $v_2 = 0.3$).
y-axis values of the invariant set of the Poincaré cut $\{Y_1^P\}$.
The small picture reports the projection in the Y1 - Z1 plane of a typical trajectory (associated with $m_2 = 0.03$).



<u>Figure 3b.</u>

Bifurcation Diagram. Estimates of the invariant set of the Poincaré map.
The Poincaré cut is computed at points $Z_2(t)=0$ and $dZ_2(t)/dt>0$
x-axis values of m_2 (with $m_1=2m_1$ and $v_2 = 0.3$).
y-axis values of the invariant set of the Poincaré cut $\{Y_2^P\}$.
The small picture reports the projection in the Y2 - Z2 plane of a typical trajectory (associated with $m_2 = 0.03$).



<u>Figure 3c.</u>

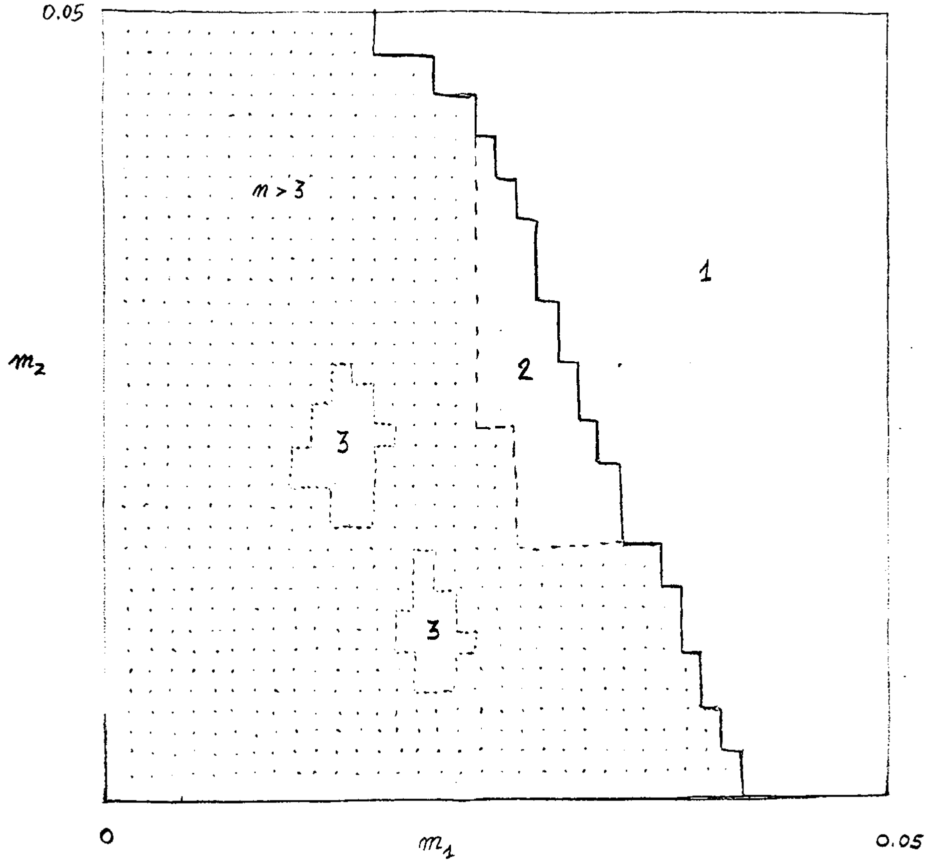
Bifurcation Diagram. Estimates of the invariant set of the Poincaré map. The Poincaré cut is computed at points $Y_1(t)=Y_1^e$ and $dY_1(t)/dt>0$. Where Y_1^e is the equilibrium value of Economy 1.

x-axis: values of m_2 (with $m_1=2m_1$ and $v_2 = 0.3$).

y-axis: values of the invariant set of the Poincaré cut $\{Y_2^P\}$.

<u>Note</u>. Each point Y_2^P represents the state of Economy 2 as Economy 1 is going through an expansion phase.

The small picture reports the projection in the Y1 - Y2 plane of a typical trajectory (associated with $m_2 = 0.03$).



<u>Figure 4</u>

For each couple (m_1,m_2) the dimension of the invariant set of the Poincaré map is reported.

The Poincaré cut is computed at points $Y_1(t) = Y_1^e$ and $dY_1(t)/dt > 0$, Y_1^e is the equilibrium value of Economy 1. The dimention (number index n) of the Poincaré map indicates that Economy 1 has to go through at least n expansion and contraction phase before a cycle repeats.

(NOTE. A more exact computer generated diagram is in preparation. While the new figure will be more detailed, the qualitative content should remain the same).

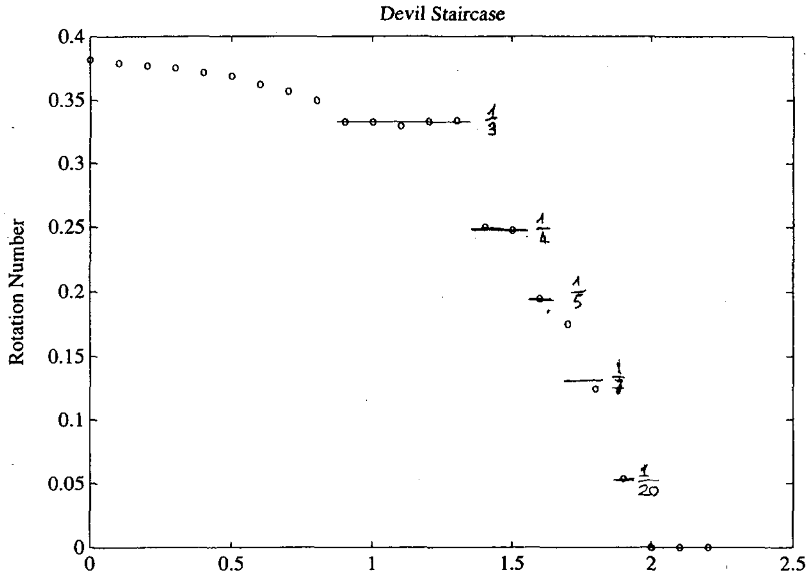

Devil Staircase
Rotation Number
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.5
1
1.5
2
2.5
1/3
1/4
1/5
1/7
1/20

FIG. A 2

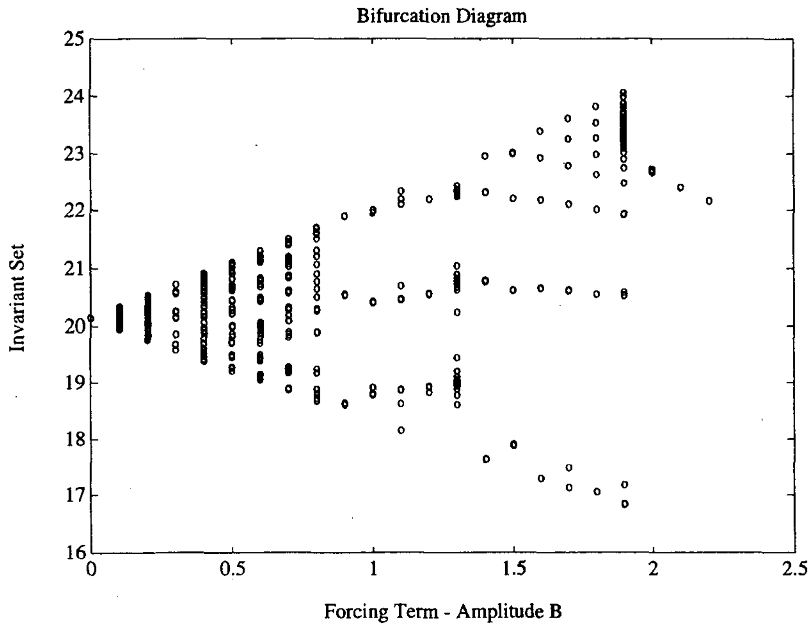

Bifurcation Diagram
Invariant Set
25
24
23
22
21
20
19
18
17
16
0
0.5
1
1.5
2
2.5
Forcing Term - Amplitude B

FIG. A.1

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