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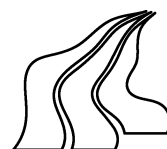
by

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CUBICAL LOCAL PARTIAL ORDERS ON CUBICALLY SUBDIVIDED SPACES - EXISTENCE AND CONSTRUCTION.

LISBETH FAJSTRUP

ABSTRACT. The geometric models of Higher Dimensional Automata and Dijkstra's PV-model are cubically subdivided topological spaces with a local partial order. If a cubicalization of a topological space is free of immersed cubic Möbius bands, then there are consistent choices of direction in all cubes, such that any n-cube in the cubic subdivision is dihomeomorphic to $[0,1]^n$ with the induced partial order from $\mathbb{R}^n$. After subdivision once, any cubicalized space has a cubical local partial order. In particular, all triangularized spaces have a cubical local partial order. This implies in particular that the underlying geometry of an HDA may be quite complicated.

1. INTRODUCTION

In the study of applications of geometry and topology in computer science, the notion of a locally partially ordered cubical complex is introduced [2]. This seems to be a very rigid structure, which would not appear many places in "nature". On the other hand, many spaces can be subdivided into cubes; for instance all triangulizable spaces admit cubical subdivisions in an unordered way, and one may ask whether they admit cubical subdivisions with a cubical local partial order, i.e., such that each cube in the subdivision is dihomeomorphic to $I^n = [0,1]^n$ with partial order induced from the standard partial order on $\mathbb{R}^n$, $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n)$ if $x_i \leq y_i$ for all i.

Given a cubical subdivision of a space, a necessary condition for having compatible partial orders on all the cubes is, that there is a compatible local partial order on the 2-skeleton. For this, all edges, which are opposite edges in some 2-cube have to be oriented consistently. We prove here, that this is in fact also a sufficient condition. Moreover, after subdividing once, all cubicalizations have a compatible local partial order. In particular all triangularizable spaces allow cubical local partial orders. Hence, the underlying geometry of a Higher Dimensional Automaton [4], an HDA, may be very complicated, which is probably a reflection of the computational power of the HDA.

Another consequence is, that we expect calculations of the dihomotopy category for locally partially ordered cubical complexes, to lead to statements about the underlying non-directed fundamental group. This strategy is thus applicable to many spaces. It is expected that calculations of dihomotopy categories will be

possible, whereas calculation of fundamental groups is known to be equivalent to the word problem.

2. Definitions: Local partial orders and cubicalized spaces.

The first four definitions here are from [2]. The other definitions and results are new.

Definition 2.1. *A semi-cubical complex M is a family of sets $\{M_n | n \geq 0\}$ with face maps $\partial_i^k : M_n \rightarrow M_{n-1}$ ($1 \leq i \leq n$, $k = 0, 1$) satisfying the semi-cubical relations:*

$$\partial_i^k \partial_j^l \;\; = \;\; \partial_{j-1}^l \partial_i^k \;\; (i < j)$$

A geometric cubical complex is a semi cubical complex M such that for any pair L_n and K_m of elements of M, there is a (perhaps empty) common face F_r such that any other common face X_k is a face of F_r.

We do not in general have a global partial order on a cubical complex. For instance the circle has a direction by increasing angle, but this is not a transitive relation. Instead, we get local partial orders:

Definition 2.2. *A local partial order on a Hausdorff topological space X is a cover $\mathcal{U} = \{(U_i, \leq_i)\}$ of X by open sets, $U_i \subset X$ each with a partial order $\leq_i$. Such that:*

- *$\leq_i$ is closed: For any pair $x, y \in U_i$, with $x \not\leq y$, there are V_x and V_y, open neighborhoods of x and y, s.t. $z \in V_x$ and $w \in V_y$ implies $z \not\leq w$*
- *For all $x \in X$ there is a non-empty open po-neighborhood, $(W_x, \leq_W)$ s.t. whenever $x, y, z \in U_i \cap W_x$, then $y \leq_W z \Leftrightarrow y \leq_{U_i} z$.*

Two local partial orders $\mathcal{U}$ and $\mathcal{V}$ on X are equivalent if their union is a local partial order.

In [2], we give a local partial order on the geometric realization of a geometric cubical complex. Each k-cube $|m_k| \in |M|$ has a natural homeomorphism to $[0,1]^k$ and these respect the face maps. The local partial order on each k-cube is the induced partial order from $\mathbb{R}^k$, which is $(x_1, \ldots, x_k) \leq (y_1, \ldots, y_k) \Leftrightarrow x_i \leq y_i$, $i = 1, \ldots, k$. In the star of each vertex, one takes the transitive hull of the relation. This provides a local partial order.

Definition 2.3. *Let X and Y be locally partially ordered spaces. A continuous map $f : X \rightarrow Y$ is a dimap, if for all $x \in X$ there are po-neighborhoods U of x and V of $f(x)$ such that for $z, w \in f^{-1}(V) \cap U$, $z \leq_U w$ if and only if $f(z) \leq_V f(w)$*

In other words, a dimap is continuous and locally monotone.

Definition 2.4. *A dihomeomorphism is a bijective dimap $f : X \to Y$ such that the inverse is a dimap.*

To define a cubically subdivided topological space, we follow the definition of a triangulation in [1] p. 112. We let $\square_k$ denote the k-cube $[0, 1]^k$. We do not allow infinite dimensional complexes, since these are not needed for the applications. But is seems that the statements will apply in that case too.

Definition 2.5. *Let X be a Hausdorff topological space and let $\mathcal{C} = \{C_k\}_{k=0}^{k=N}$ be a family of (possibly empty) sets of continuous maps such that the elements $c_k \in C_k$ are maps $c_k : \square_k \to X$ from the k-cube to X. Then $\mathcal{C}$ is a cubicalization of X if*

(1) *X is covered: $X = \bigcup_{c_k \in C_k, k=0,\dots} c_k(\square_k)$*
(2) *All c_k are injective.*
(3) *For all $c_m \in C_m$, $c_n \in C_n$, $c_m^{-1}(c_n(\square_n))$ is a (possibly empty) face of $\square_m$ and $c_n^{-1}(c_m(\square_m))$ is a face of $\square_n$ and moreover*

$$c_n^{-1} \circ c_m : c_m^{-1}(c_n(\square_n)) \to c_n^{-1}(c_m(\square_m))$$

   is a linear isomorphism.
(4) *X has the weak topology with respect to the inclusions: $A \subset X$ is closed if and only if $A \cap c_k(\square_k)$ is closed for all c_k.*

Remark 2.6. Any triangulizable space has a cubicalization. This follows from the fact that there is a cubic barycentric subdivision of a triangulation. Each n-simplex S_n is divided into $n + 1$ n-cubes, each of which have a vertex v at the barycenter and a diagonally opposite vertex at a vertex w_i of the n-simplex. The other vertices are midpoints of those edges in S_n which are in the star of w_i.

Definition 2.7. *Given a space X with a cubicalization $\mathcal{C}$. A local partial order on $X, \mathcal{C}$ consists of the following:*

(1) *For each cube $c_k \in C_k$, there is a dihomeomorphism $\phi_{c_k} : \overset{\to k}{I} \to \square_k$ of the form*

$$\phi_{c_k}(x_1, \dots, x_k) = (\phi_{c_k,1}(x_1, \dots, x_k), \phi_{c_k,2}(x_1, \dots, x_k) \dots, \phi_{c_k,k}(x_1, \dots, x_k)),$$

   where $\phi_{c_k,j}(x_1, \dots, x_k)$ is either x_j or $1 - x_j$.
(2) *For all $c_m \in C_m, c_n \in C_n$, the map*

$$(c_n \circ \phi_{c_n})^{-1} \circ c_m \circ \phi_{c_m} : (c_m \circ \phi_{c_m})^{-1}(c_n(\square_n)) \to \overset{\to n}{I}$$

   is a dimap.

Lemma 2.8. *Let $X = I^n$ be the n-cube with all its subcubes and let $\mathcal{I} = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. Then the partial order $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if $x_i \geq y_i$ for $i \in \mathcal{I}$ and $x_i \leq y_i$ for $i \notin \mathcal{I}$ is a cubical partial order on X*

Proof. The map $\phi_{c_n} : \overrightarrow{I}^n \to X$ has coordinate functions $x_i \to 1 - x_i$ for $i \in \mathcal{I}$ and $x_i \to x_i$ else. The maps to the various subcubes are given by restriction. □

Corollary 2.9. *Suppose in a cubical complex $(X, \mathcal{C})$, we have provided ϕ_{c_2} for all 2-cubes and ϕ_{c_1} for all edges. And suppose that they satisfy the compatibility, 2.7.2. Then there is a unique extension to a local partial order on all of $(X, \mathcal{C})$*

Proof. All cubes have been provided with a direction on the edges such that these agree on edges which are opposite in a 2-face. Hence we get an induced partial order on each cube. By 2.8, the induced local partial order is an fact a cubical local partial order. And it is unique, since we require compatibility. □

Remark 2.10. A cubical complex has several different meanings in the literature. For instance group actions on such complexes - in particular when they are non-positively curved- has been studied intensively following M. Gromov and others. In this case, the metric is important. This is not the main point here. We are interested in the local partial order on $\mathbb{R}^n$ and hence the cubes, and whether this can be transferred consistently to the cubical complex.

3. Construction of a cubical local partial order.

Definition 3.1. *A cubical Möbius strip* [5] *is a 2-complex with k 2-cells $A_1, \ldots, A_k$ glued as pictured in Fig. 1*
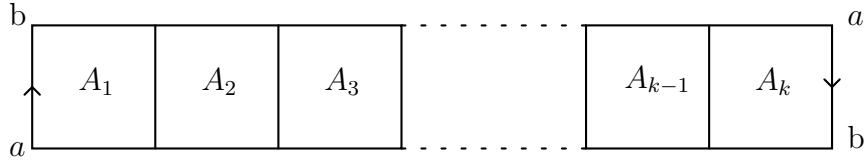

FIGURE 1. Cubical Möbius strip

It is clear that a cubical Möbius strip does not have a consistent cubical partial order, since opposite edges in a 2-cell should be ordered consistently, and there is no consistent choice of an order on the family of edges generated by taking edges opposite the edge ab in the 2-cells.

It is possible however to find a cubical subdivision of the Möbius strip, which has a cubical partial order. There are many ways of doing that - the following example is from [2]:

Example 3.2. *If the identification of edges is made as in Fig.2, then starting with a direction on the edge a, directions are generated on all vertical edges plus the horizontal edges of A_1, and no obstructions arise. It is clear that one can choose directions on the remaining horizontal edges in a consistent way, and this gives a cubical local partial order.*
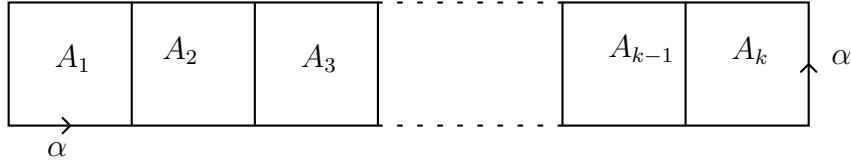
FIGURE 2. A cubicalized Möbius strip which has a cubical local partial order.

Hence, a cubical local partial order does not give an orientation of the manifold. And vice versa: non-orientable manifolds can have cubical local partial orders.

Remark 3.3. A cubical Möbius strip can be subdivided to obtain a cubicalization which can be directed. There are many ways of doing this, but this is the one, which will be useful later: Subdivide all edges once by adding a vertex in the middle, then subdivide 2-cubes by adding a vertex in the center and inserting edges from this vertex to the midpoints of the edges - thus subdividing a 2-cube in 4 cubes. Then in particular the non-compatible edges (vertical in Fig. 3) are subdivided, and the relation generated around the Möbius band implies that either all these edges are oriented away from the zero section of the band or they all point towards it. See Fig. 3. The horizontal edges may certainly be oriented consistently.
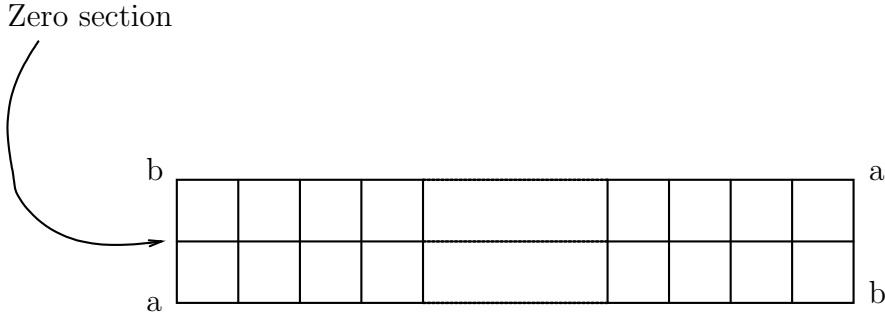

FIGURE 3. A subdivided cubical Möbius strip has a cubical local partial order

Proposition 3.4. *A cubicalized topological space* $(X, \mathcal{C})$ *has a cubical local partial order if and only if it does not contain any immersed cubical Möbius bands*

Proof. Let M be a cubicalized topological space. If there are immersed cubical Möbius bands, there is clearly no consistent orientation of the edges of the 2-cells and hence no consistent local partial order of M.

Suppose M does not contain any cubical Möbius strips. The cubical partial order on a cube is given, once we provide a consistent orientation of the edges and 2-cells, and moreover, all the 2^n consistent orientations of the edges of an n-cube
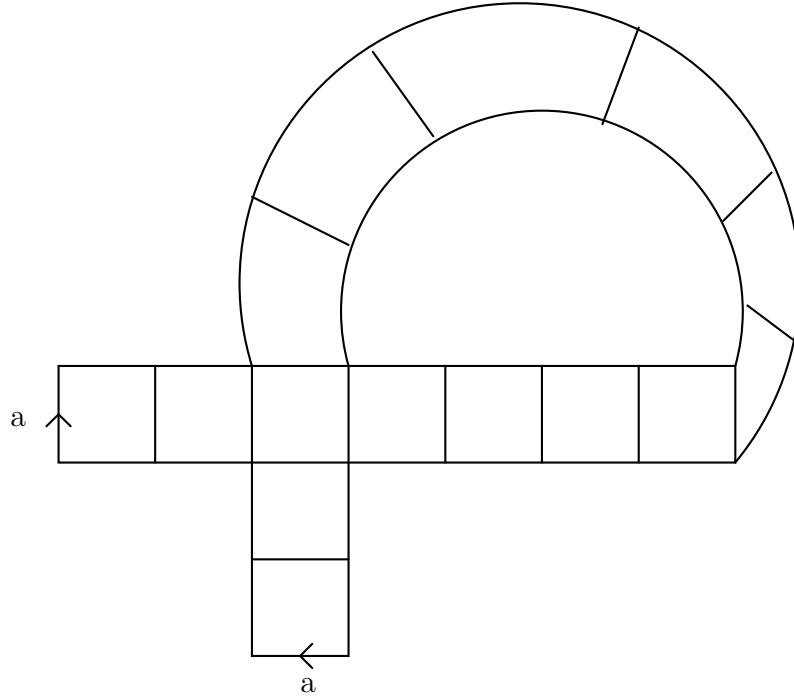
FIGURE 4. A cubical Möbius strip with a selfintersection

give a cubical partial order by Lemma 2.8. Define an equivalence relation on edges
in the cubicalization of M as the transitive hull of the relation of being opposite
edges in a 2-cell. Each such equivalence class has two choices of orientation,
provided there are no cubical Möbius bands: Pick an edge in M and orient it.
Then orient all the edges equivalent to it consistently. Since there are no immersed
cubical Möbius bands, there are no obstructions to this. Now pick another edge
which has not already been oriented, choose an orientation on it and orient edges
equivalent to it consistently. Keep doing this until all edges have an orientation.
This gives the 2-cells a consistent partial order. Give the other cubes the induced
partial order.

$\square$

Remark 3.5. As noticed in [5], the cubical Möbius strip does not have to be
embedded in M. The conflict arising from the relation of being opposite edges
in a 2-cell could reuse some of the 2-cells, as seen in the bowlike shape in Fig.4
This does not affect the above proof.

We remind the reader of some definitions from combinatorics, which may be
found in for instance [3]

Definition 3.6. *A d-polytope is the convex hull of a finite subset of $\mathbb{R}^d$. Such a polytope is* cubical *if all its proper faces, i.e, faces of dimension strictly less than d are combinatorially isomorphic to cubes. A* facet *is a $d-1$ dimensional face.*

With notation from [5], we give

Definition 3.7. *The derivative complex of the boundary complex Q of a cubical d-polytope is a cubical complex $D(Q)$ whose vertices are the midpoints of all edges of Q and whose facets correspond to pairs $F, [e]$, where F is a facet and $[e]$ is an equivalence class of edges in F as defined above. The facets of $D(Q)$ separate opposite facets of Q.*

Each connected component of the derivative complex is a $d-1$ dimensional PL-manifold, the *dual manifold* of Q.

Corollary 3.8. *For every cubical d-polytope, P, ($d \geq 3$) the dual manifolds of P are orientable if and only if P admits a cubical local partial order.*

Proof. This is a direct consequence of our Prop. 3.4 and Proposition 2.1 in [5] □

It is well known that the barycentric subdivision gives rise to a finer cubical (or simplicial) subdivision of a space:

Definition 3.9. *Let $M, \mathcal{C}$ be a cubicalized space. Then the barycentric subdivision of the cubicalization is the cubicalization induced by subdividing $\square_n = I^n$ into the 2^n subcubes $[k_1, l_1] \times \cdots \times [k_n, l_n]$, where $k_i, l_i \in \{0, 1/2, 1\}$ and $k_i < l_i$.*

Theorem 3.10. *Let M be a cubicalized topological space. Then there is a cubical local partial order on M compatible with the once barycentrically subdivided cubical structure on M.*

Proof. We have to see that after subdividing once, there are no cubical Möbius strips left.

Look at the equivalence classes of edges. If an equivalence class of edges contains no cubical Möbius strips, then it gives rise to two equivalence classes in the subdivided complex, neither of which contain cubical Möbius strips. An equivalence class E, which contains a cubical Möbius strip gives rise to one equivalence class in the subdivided complex, since for any edge $[a, b]$ in E, with midpoint c; going around the Möbius strip we see that $[a, c]$ is equivalent to $[b, c]$ in a directed sense. This equivalence class has a compatible orientation: Either orient all subdivided edges in the equivalence class toward the midpoint of the original edges or away from it.

□

Corollary 3.11. *Any triangulizable space has a cubical local partial order.*

Corollary 3.12. *The number of cubical local partial orders on a cubical complex without cubical Möbius bands is 2^e where $e = \#\{$ equivalence classes of edges$\}$*

Proof. See the construction of local partial orders. For each equivalence class, there are two possible choices of direction, and these choices are independent.   □

Corollary 3.13. *For every cubical d-polytope, the polytope obtained by subdividing once has orientable dual manifolds*

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