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ON APPROXIMATION WITH WAVE PACKETS GENERATED FROM A REFINABLE FUNCTION

LASSE BORUP AND MORTEN NIELSEN

ABSTRACT. We consider best m-term approximation in $L_p(\mathbb{R}^d)$ with wave packets generated from a single refinable function. The main examples of such wave packets are orthonormal wavelets or more generally tight wavelet frames based on an MRA (so-called framelets). The approximation classes associated with best m-term approximation in $L_p(\mathbb{R}^d)$ with such wave packets are completely characterized in terms of Besov spaces.

As an application of the main result we show that for m-term approximation in $L_p(\mathbb{R}^d)$ with elements from an oversampled version of a framelet system with compactly supported generators, the associated approximation classes turn out to be (essentially) Besov spaces.

1. INTRODUCTION

The standard method used to generate "atoms" such as wavelets or wavelet frames is to begin with a refinable function $\phi \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ and then construct the generators $\{\psi^i\}_{i=1}^L$ of the wavelet or wavelet frame system by applying a filter to ϕ, i.e.,

$$\hat{\psi}^i(2\xi) = m^i(\xi)\hat{\phi}(\xi). \tag{1.1}$$

Usually m^i is a so-called high-pass filter, but this is not important for the results considered in this paper. We call a function ψ^i of the type given by (1.1) a wave packet generated by the refinable function ϕ.

The main purpose of this paper is to study approximation of smooth functions with m-term approximants formed by dilating and translating one or more wave packets of the form given by (1.1). More precisely, given a finite collection of wave packets $\{\psi^i\}_{i\in F}$ and $K \in \mathbb{Z}$, we consider the dictionary

$$X_K(\{\psi^i\}_{i\in F}) := \{\psi^i(2^j \cdot -k/2^K) \colon j \in \mathbb{Z}, k \in \mathbb{Z}^d, i \in F\} \subset L_2(\mathbb{R}^d),$$

We notice that K is an oversampling ratio, and for regular wavelet systems one usually considers $K = 1$.

The associated (nonlinear) space of all possible m-term expansions by elements from $X_K(\{\psi^i\}_{i\in F})$ is given by

$$\Sigma_m(X_K(\{\psi^i\}_{i\in F}) := \left\{S \colon S = \sum_{j=1}^m a_j g_j, \text{ with } a_j \in \mathbb{C}, g_j \in X_K(\{\psi^i\}_{i\in F})\right\}.$$

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We measure the approximation error in the Triebel-Lizorkin norm $\dot{F}_p^\gamma(L_q(\mathbb{R}^d))$, see [9]. The error of the best m-term approximation of f from $X_K(\psi)$ is then given by

$$\sigma_m(f, X_K(\psi))_{\dot{F}_p^\gamma(L_q)} := \inf_{S\in\Sigma_m(X_K(\psi))} \|f - S\|_{\dot{F}_p^\gamma(L_q)}.$$

We let $\mathcal{A}_t^\gamma(\dot{F}_p^\gamma(L_q(\mathbb{R}^d)), X_K(\psi))$, $0 < t \leq \infty$, denote the approximation class of all functions f such that

$$|f|_{\mathcal{A}_t^\gamma(\dot{F}_p^\gamma(L_q), X_K(\psi))} := \left(\sum_{m=1}^\infty \left(n^\gamma \sigma_m(f, X_K(\psi))_{\dot{F}_p^\gamma(L_q)}\right)^t \frac{1}{m} \right)^{1/t} < \infty, \tag{1.2}$$

with the usual modification when $t = \infty$. Now the fundamental question is whether it is possible to completely characterize $\mathcal{A}_t^\gamma(\dot{F}_p^\gamma(L_q(\mathbb{R}^d)), X_K(\Psi))$ in terms of known smoothness spaces. When $p = 2$ and $1 < q < \infty$ For "nice" bi-orthogonal wavelet systems in $L_2(\mathbb{R}^d)$ and for parameters $\gamma = 0$, $p = 2$, and $1 < q < \infty$ in (1.2) (we recall: $\dot{F}_2^0(L_q(\mathbb{R}^d)) \approx L_q(\mathbb{R}^d)$), the approximation classes are known to be essentially Besov spaces. In the paper [4] it was proved that for tight wavelet frames based on a spline multiresolution for $L_2(\mathbb{R})$, one obtains the same approximation classes if we put $K = 2$.

The main result of this paper is that given one or more compactly supported wave packets of the form (1.1), based on a compactly supported refinable function, it is always possible to find $K_0 \in \mathbb{N}$ such that for $K \geq K_0$, the approximation class $\mathcal{A}_2^0(\dot{F}_p^\gamma(L_q))$ is given by (essentially) a Besov space, just like in the orthonormal wavelet systems. The precise statement (Theorem 3.2) is given in Section 3, and it is perhaps surprising that the existence of such a K_0 does not depend on the particular form of the filters m^i from (1.1).

We use the standard approach to obtain the characterizaion of the approximation classes. First we prove a Jackson inequality for best m-term approximation with elements from the dictionary $X_K(\{\psi^i\}_{i\in F})$ for sufficiently large values of K. The proof in Section 2 is based on an application of the of ϕ-transform machinery by Frazier and Jawerth [3]. To obtain the characterization of the approximation class, we establish a Bernstein inequality for m-term expansions from $X_K(\{\psi^i\}_{i\in F})$. This is done in Section 3. Put together, the Jackson and Bernstein inequality leads to the main result, Theorem 3.2. Our main application of the results is to framalet systems, i.e., to tight wavelet frames based on an MRA.

2. Jackson inequalities for general wave packets

In this section we establish a Jackson inequality for best m-term approximation with elements from the dictionary $X_K(\{\psi^i\}_{i\in F})$ for sufficiently large values of K. The proof is based on the ϕ-transform technique of Frazier and Jawerth. We will state the result from [3] that is needed below, but first we introduce the following notation.

D denotes the set of dyadic cubes $Q = Q_{\nu k} = 2^{-\nu}([0,1]^d + k)$, $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^d$. We will use two index notations in this paper; $\{\phi^Q\}_{Q\in D}$ will denote a sequence of functions indexed by the dyadic cubes while $\phi_Q(x) := 2^{-\nu d/2}\phi(2^\nu x - k)$.

For $\gamma \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leq \infty$, we let $\dot{f}_{p,q}^\gamma(D)$ respectively $\dot{b}_{p,q}^\gamma(D)$ be the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in D}$ such that

$$\|s\|_{\dot{f}_{p,q}^\gamma} := \left\| \Big(\sum_{Q\in D} (|Q|^{-\frac{\gamma}{d}-\frac{1}{2}} |s_Q| \chi_Q)^q \Big)^{\frac{1}{q}} \right\|_{L_p} < \infty,$$

respectively

$$\|s\|_{\dot{b}^{\gamma}_{p,q}} := \Big(\sum_{\nu\in\mathbb{Z}} 2^{\nu q(\gamma+\frac{d}{2}-\frac{d}{p})}\Big(\sum_{Q\in D_\nu} |s_Q|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty.$$

We recall the following result given by Frazier and Jawerth in [3]. For $x \in \mathbb{R}$ we let $\lfloor x \rfloor$ denote the integer satisfying $x-1 < \lfloor x \rfloor \le x$.

Theorem 2.1. *Let $s \ge 0$, $0 < q \le \infty$, $0 < p < \infty$, $J = d/\min(1,p,q)$, and $N = \max(\lfloor J-d-s \rfloor, -1)$. Suppose δ satisfies $s - \lfloor s \rfloor < \delta \le 1$, and suppose $M > J$. Let u be a function satisfying the four conditions:*

$$|\hat{u}(\xi)| \ge c > 0 \qquad \text{if } 2^{-1} \le |\xi| \le 2, \tag{2.1}$$

$$\int x^\gamma u(x)\,dx = 0 \qquad \text{if } |\gamma| \le N, \tag{2.2}$$

$$|\partial^\gamma u(x)| \le c_\gamma (1+|x|)^{-M} \qquad \text{if } |\gamma| \le \lfloor s+1 \rfloor, \tag{2.3}$$

$$|\partial^\gamma u(x) - \partial^\gamma u(y)| \le \sup_{|z|\le|y-x|} \frac{|x-y|^\delta}{(1+|x-z|)^M} \qquad \text{if } |\gamma| = \lfloor s+1 \rfloor. \tag{2.4}$$

Given $\mu \in \mathbb{Z}$, let $g(x) = 2^{\mu d} u(2^\mu x)$. Then there is a $\mu_0 \le 0$ with the property that if $\mu \le \mu_0$, there exists a family of functions $\{\tilde{g}^Q\}_{Q\in D}$, such that for all $f \in \dot{F}^s_q(L_p(\mathbb{R}^d))$, we have

$$f = \sum_{Q\in D} \langle f, \tilde{g}^Q\rangle g_Q,$$

with $\|\{\langle f,\tilde{g}^Q\rangle\}\|_{\dot{f}^s_{p,q}} \le c\|f\|_{\dot{F}^s_q(L_p)}$, and for any sequence $s = \{s_Q\}_{Q\in D}$, we have

$$\Big\|\sum_{Q\in D} s_Q g_Q\Big\|_{\dot{F}^s_q(L_p)} \le c\|s\|_{\dot{f}^s_{p,q}}.$$

Remark 2.2. If (2.3) and (2.4) are substituted by the condition

$$|u(x) - u(y)| \le \sup_{|z|\le|y-x|} \frac{|x-y|^\delta}{(1+|x-z|)^{M-s}}, \tag{2.5}$$

Theorem 2.1 holds true for $s < 0$.

Remark 2.3. It is easy to verify that exactly the same result as given in Theorem 2.1 holds true if the Besov space $\dot{B}^s_q(L_p(\mathbb{R}^d))$ is considered instead of the Triebel-Lizorkin space.

Theorem 2.1 gives sufficient conditions for a function g to generate an atomic decomposition of the Triebel-Lizorkin and Besov spaces. However, many interesting functions do not satisfy the conditions (2.1) and (2.2). For example, whenever $\int \psi \ne 0$ then (2.2) clearly fails, but in the paper [8] it was shown how to build appropriate wave packets from such functions with the required number of vanishing moments. The wave packets then form atomic decompositions of the Triebel-Lizorkin and Besov spaces. Another type of function often encountered in applications is one that is very smooth, but with a comparatively small number of vanishing moments. Often generators of tight wavelet frames will have these characteristics. We can apply Theorem 2.1 directly to such functions, but we will only get atomic decompositions valid for a more restricted range of smoothness parameters than one would expect from the smoothness of the generator. Inspired by [8], we now show how to create wave packets from a function which does not satisfy the conditions (2.1) and (2.2).

Proposition 2.4. *Given $s \geq 0$, let $\phi \in C^s(\mathbb{R}^d)$ be a compactly supported function with $\hat{\phi}(0) \neq 0$ and let ψ be any fixed function in* $\operatorname{span}\{\phi(2\cdot - k)\colon k \in \mathbb{Z}\}$. *Then for $N \in \mathbb{N}_0$ there exists a $J \in \mathbb{N}$ and a finite set of coefficients $\{c_k\}_{k=1}^n$, $n \leq 2N+1$, such that the function $u = \sum_{k=1}^n c_k \psi(2^J(\cdot - k))$ satisfies the conditions* (2.1)–(2.4).

Proof. Since ϕ has compact support we have that both ϕ and ψ satisfy (2.3) and (2.4) for any $M \in \mathbb{N}$. Thus, it suffices to prove that u satisfies (2.1) and (2.2). Since $\psi \in \operatorname{span}\{\phi(\cdot - k)\colon k \in \mathbb{Z}\}$, there exists a trigonometric polynomial τ such that $\hat{\psi}(\xi) = \tau(\xi)\hat{\phi}(\xi)$. In particular, there exists an $r > 0$ such that $\tau(\xi) \neq 0$ for $0 < |\xi| < r$. Likewise there exists an $r' > 0$ such that $\hat{\phi}(\xi) \neq 0$ for $|\xi| < r'$. Now, if we choose $J \in \mathbb{N}$ such that $2^J > 1/\min(r, r')$, then $\tilde{u} = \psi(2^J\cdot)$ satisfies (2.1).

In order to obtain N vanishing moments, we simply apply a high-pass filter with symbol $m(\xi)$ which has a zero of order N at $\xi = 0$ and no zeros on $K := \{\xi \in \mathbb{R}^d\colon 1/2 \leq |\xi| \leq 2\}$. For example, let $u = \left(\sum_{j=1}^d (\Delta_{e_j} + \Delta_{-e_j})\right)^N \tilde{u}$, where Δ_e is the difference operator in the direction $e \in \mathbb{R}^d$, $\Delta_e f(x) = f(x+e) - f(x)$, and e_j is a unit vector in the j-th direction. Then it is easy to see that u satisfies (2.2). Moreover, since $\mathcal{F}(\Delta_{e_j} + \Delta_{-e_j})f(\xi) = 2(\cos(\xi\cdot e_j) - 1)\hat{f}$ and $|\sum_{j=1}^d \cos(\xi\cdot e_j) - 1| \geq 1 - \cos(1/2)$ on K, u satisfies (2.1) too. □

By Proposition 2.4 there exists a $K_0 \in \mathbb{Z}$ such that if $K \geq K_0$ then $\operatorname{span} X_K(\psi)$ is dense in the Triebel-Lizorkin and Besov spaces provided that each generator ψ^i is itself contained in that particular space (which clearly is not always satisfied). In the remaining of this paper we will fix such a K and consider approximation using m-term expansions with elements from the dictionaries $X_K(\psi)$.

As a consequence of Proposition 2.4 we have the following Jackson inequality.

Proposition 2.5. *Given $s \geq 0$, let $\phi \in C^s(\mathbb{R}^d)$ be a compactly supported refinable function and let $\psi \in$* $\operatorname{span}\{\phi(2\cdot - k)\colon k \in \mathbb{Z}\}$. *Suppose $0 < p < \infty$, $0 < t \leq \infty$, $0 \leq \beta < \gamma < s$ and $\psi \in \dot{F}_t^\beta(L_p(\mathbb{R}^d))$, and define $1/\tau := (\gamma - \beta)/d + 1/p$. Then there exists a finite constant C such that*

$$\sigma_m(f, X_K(\psi))_{\dot{F}_t^\beta(L_p)} \leq C m^{-(\gamma-\beta)/d} \|f\|_{\dot{B}_\tau^\gamma(L_\tau)}, \qquad \forall m \in \mathbb{N}, f \in \dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d)).$$

Proof. By Proposition 2.4, we can construct a function $g \in \Sigma_n(X_K(\psi))$ for some finite n, and a sequence $\{\tilde{g}^Q\}_{Q\in D}$, such that

$$f = \sum_{Q\in D} \langle f, \tilde{g}^Q\rangle g_Q,$$

for every $f \in \dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d))$, with $\|\{\langle f, \tilde{g}^Q\rangle\}\|_{\dot{b}_{\tau,\tau}^\gamma} \leq C\|f\|_{\dot{B}_\tau^\gamma(L_\tau)}$. Fix $f \in \dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d))$, and let η be one of the $2^d - 1$ orthonormal Meyer wavelets on $\mathbb{R}^d$. We notice that the function $h := \sum_{Q\in D} \langle f, \tilde{g}^Q\rangle \eta_Q$ belongs to $\dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d))$ with $\|h\|_{\dot{B}_\tau^\gamma(L_\tau)} \leq C\|f\|_{\dot{B}_\tau^\gamma(L_\tau)}$. From [7], we have

$$\sigma_m(h, \{\eta_Q\}_Q)_{\dot{F}_t^\beta(L_p)} \leq C m^{-(\gamma-\beta)/d} \|h\|_{\dot{B}_\tau^\gamma(L_\tau)}, \qquad m \in \mathbb{N}. \tag{2.6}$$

Let $h_m \in \Sigma_m(\{\eta_Q\}_Q)$ be a sequence that realizes (2.6) up to the relaxed constant $2C$. We want to map h_m to an element of $f_m \in \Sigma_m(\{g_Q\}_Q)$. To accomplish this, we consider the operator T with kernel

$$K(x, y) := \sum_{Q\in D} g_Q(x)\eta_Q(y).$$

The matrix of this operator in the Meyer wavelet basis is $M = [\langle g_P, \eta_Q\rangle]_{P,Q\in D}$. It is easy to verify that $M \in \mathbf{ad}_{p,t}^\beta$, where $\mathbf{ad}_{p,t}^\beta$ is the algebra of almost diagonal matrices, see [3, Sec. 9]. We notice that $f_m := Th_m \in \Sigma_m(\{g_Q\}_Q)$, and moreover, $f - f_m = T(h - h_m)$. However, the matrix representation of T shows that T is bounded on $\dot{F}_t^\beta(L_p(\mathbb{R}^d))$ so

$$\begin{aligned}\|f - f_m\|_{\dot{F}_t^\beta(L_p)} &\leq C_1\|h - h_m\|_{\dot{F}_\tau^\beta(L_\tau)}\\ &\leq 2C_1Cm^{-(\gamma-\beta)/d}\|h\|_{\dot{B}_\tau^\gamma(L_\tau)}\\ &\leq C'm^{-(\gamma-\beta)/d}\|f\|_{\dot{B}_\tau^\gamma(L_\tau)}.\end{aligned}$$

Hence, we have the wanted Jackson inequality for the dictionary $\{g_Q\}_{Q\in D}$, and thus for $X_K(\psi)$. □

3. Approximation with framelet systems

In this section we derive a fairly general Bernstein inequality for the system $X_K(\{\phi^i\})$, and then using the Jackson inequality for such systems, valid for large K, we give a complete characterization of $\mathcal{A}_q^{\alpha/d}\big(L_p(\mathbb{R}^d), X_K(\psi)\big)$ for $1 < p < \infty$. The Bernstein inequality for $X_K(\{\phi^i\})$ is given by the following proposition.

Proposition 3.1. *Given $s \geq 0$, let $\phi \in W^s(L_\infty(\mathbb{R}^d))$ be a compactly supported refinable function and let $\psi \in \operatorname{span}\{\phi(\cdot - k)\colon k \in \mathbb{Z}\}$. If $d > 1$ we suppose $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a locally linearly independent set (this condition is void if $d = 1$). Then for each $0 < \gamma < s$ and $K \geq 1$, the Bernstein inequality*

$$|S|_{\dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d))} \leq Cm^{\gamma/d}\|S\|_{L_p(\mathbb{R}^d)}, \qquad \forall S \in \Sigma_m(X_K(\psi)), \tag{3.1}$$

$1/\tau := \gamma/d + 1/p$, $0 < p \leq \infty$, holds true.

Proof. In the case $d = 1$, if the integer shifts of the function ϕ are not already linearly independent, we can always find a perfect generator $\tilde{\phi}$ for the shift invariant space $S_0 := \operatorname{span}\{\phi(\cdot - k)\colon k \in \mathbb{Z}\}$, i.e., $\tilde{\phi}$ is a compactly supported refinable function with linearly independent shifts that generates S_0, see [6]. In the arguments below we may use $\tilde{\phi}$ in place of ϕ.

By the result of Jia [5], for each $0 < \gamma < s$, the Bernstein inequality

$$|S|_{\dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d))} \leq Cm^{\gamma/d}\|S\|_{L_p(\mathbb{R}^d)}, \qquad \forall S \in \Sigma_m(X(\phi)),$$

$1/\tau := \gamma/d + 1/p$, $0 < p \leq \infty$, holds true for the system

$$X(\phi) := \{\phi(2^j x - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}.$$

By assumption there is a finite mask $\{b_k\}_k$ such that

$$\psi(x) = \sum_{\ell\in\mathbb{Z}^d} b_\ell \phi(x - \ell),$$

and since ϕ is compactly supported and refinable there is another finite mask $\{a_k\}$ such that

$$\phi(x) = \sum_{\ell\in\mathbb{Z}^d} a_\ell \phi(2x - \ell).$$

Thus, for $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, we have

$$
\begin{aligned}
&\psi(2^j x - k/2^K) \\
&\quad= \sum_{\ell_0 \in \mathbb{Z}^d} b_{\ell_0} \phi\big(2^j x - k/2^K - \ell_0\big) \\
&\quad= \sum_{\ell_0, \ell_1 \in \mathbb{Z}^d} b_{\ell_0} a_{\ell_1} \phi\big(2^{j+1} x - k/2^{K-1} - 2\ell_0 - \ell_1\big) \\
&\quad\quad \vdots \\
&\quad= \sum_{\ell_0, \ell_1, \ldots, \ell_K \in \mathbb{Z}^d} b_{\ell_0} a_{\ell_1} \cdots a_{\ell_K} \phi\big(2^{j+K} x - k - 2^{K-1}\ell_0 - 2^{K-2}\ell_1 - \cdots - \ell_K\big).
\end{aligned}
$$

That is to say $\psi(2^j x - k/2^K) \in \Sigma_L(X(\phi))$ for some uniform L depending only on K and the length of the finite masks used above. Take any $S \in \Sigma_m(X_K(\Psi))$, then $S \in \Sigma_{Lm}(X(\phi))$. Using the Bernstein inequality for $X(\phi)$ we obtain the wanted inequality,

$$
\begin{aligned}
|S|_{\dot{B}^\gamma_\tau(L_\tau(\mathbb{R}^d))} &\le C(Lm)^{\gamma/d} \|S\|_{L_p(\mathbb{R}^d)} \\
&\le \tilde{C} m^{\gamma/d} \|S\|_{L_p(\mathbb{R}^d)}, \qquad \forall S \in \Sigma_m(X_K(\Psi)).
\end{aligned}
$$

□

We can now apply the Jackson and Bernstein inequalities obtained so far to get the main result of this paper.

Theorem 3.2. *Given $s \ge 0$, let $\phi \in W^s(L_\infty(\mathbb{R}^d))$ be a compactly supported refinable function and let $\psi \in \operatorname{span}\{\phi(\cdot - k) \colon k \in \mathbb{Z}\}$. If $d > 1$ we suppose $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}^d}$ is a locally linearly independent set (condition is void if $d = 1$). Then there exists $K_0 \in \mathbb{N}$ such that for $K \ge K_0$,*

$$\mathcal{A}^{\alpha/d}_q\big(L_p(\mathbb{R}^d), X_K(\psi)\big) = \big(L_p(\mathbb{R}^d), B^\gamma_\tau(L_\tau(\mathbb{R}^d))\big)_{\alpha/\gamma, q},$$

for $1 < p < \infty$, $0 < \alpha < \gamma < s$, and $1/\tau := \gamma/d + 1/p$.

Proof. By Proposition 3.1, we have the Bernstein inequality

$$|S|_{\dot{B}^\gamma_\tau(L_\tau(\mathbb{R}^d))} \le \tilde{C_K} m^{\gamma/d} \|S\|_{L_p(\mathbb{R}^d)}, \qquad \forall S \in \Sigma_m(X_K(\psi)), \quad \gamma < s.$$

Using Proposition 2.5, there is a $K \ge 1$ such that the Jackson inequality

$$\sigma_m(f, X_K(\psi))_{L_p(\mathbb{R}^d)} \le C m^{-\gamma/d} \|f\|_{\dot{B}^\gamma_\tau(L_\tau)}, \qquad \forall m \in \mathbb{N}, f \in \dot{B}^\gamma_\tau(L_\tau),$$

holds for $\gamma < s$. Hence, the result follows by the well known theorem of DeVore and Popov [2]. □

Remark 3.3. For $p \le 1$, we cannot get a Bernstein inequality valid for $H_p(\mathbb{R}^d)$ using the techniques of this paper. This is due to the fact that the refinable function ϕ is not contained in $H_p(\mathbb{R}^d)$. Indeed, whether $\mathcal{A}^{\alpha/d}_q\big(H_p(\mathbb{R}^d), X_K(\psi)\big)$ is even defined depends on the properties of filters m^i from (1.1).

Clearly, Proposition 3.2 also characterize the approximation classes associated with the oversampled scaling system $X_K(\phi)$. However, since ϕ is a refinable function we can actually do much better. We have the following result.

Proposition 3.4. *Consider an oversampled system $X_K(\phi)$ for which the following Jackson inequality holds*

$$\sigma_m(f, X_K(\phi))_{\dot{F}^\beta_t(L_p)} \le C m^{-(\gamma-\beta)/d} \|f\|_{\dot{B}^\gamma_\tau(L_\tau)}, \qquad \forall m \in \mathbb{N}, f \in \dot{B}^\gamma_\tau(L_\tau(\mathbb{R}^d)),$$

where $1/\tau := (\gamma-\beta)/d + 1/p$ and $0 < t \le \infty$. Suppose ϕ is refinable and compactly supported. Then the Jackson inequality (with the same parameters as above)

$$\sigma_m(f, X(\phi))_{\dot{F}_t^\beta(L_p)} \le Cm^{-(\gamma-\beta)/d}\|f\|_{\dot{B}_\tau^\gamma(L_\tau)}, \qquad \forall m \in \mathbb{N}, f \in \dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d)),$$

holds for the non-oversampled system $X(\phi)$.

Proof. We notice that it suffice to prove that $\Sigma_m(X_K(\phi)) \subset \Sigma_{Lm}(X(\phi))$ for some uniform constant L, since then we would obtain the estimate

$$\sigma_{Lm}(f, X(\phi))_{\dot{F}_t^\beta(L_p)} \le \sigma_m(f, X_K(\phi))_{\dot{F}_t^\beta(L_p)}.$$

Using the refinement relation

$$\phi(x) = \sum_{\ell\in\mathbb{Z}^d} a_\ell \phi(2x-\ell)$$

successively, we notice that

$$\begin{aligned}
&\phi(2^j x - k/2^K)\\
&\quad= \sum_{\ell_1\in\mathbb{Z}^d} a_{\ell_1}\phi\big(2^{j+1}x - k/2^{K-1} - \ell_1\big)\\
&\quad= \sum_{\ell_1,\ell_2\in\mathbb{Z}^d} a_{\ell_1}a_{\ell_2}\phi\big(2^{j+2}x - k/2^{K-2} - 2\ell_1 - \ell_2\big)\\
&\qquad\vdots\\
&\quad= \sum_{\ell_1,\ell_2,\dots,\ell_K\in\mathbb{Z}^d} a_{\ell_1}a_{\ell_2}\cdots a_{\ell_K}\phi\big(2^{j+K}x - k - 2^{K-1}\ell_1 - 2^{K-2}\ell_2 - \cdots - \ell_K\big).
\end{aligned}$$

It follows that $\phi(2^j x - k/2^K) \in \Sigma_L(X(\phi))$, where L only depends on the length of the mask $\{a_k\}_k$ and on K. This proves the claim. □

We easily deduce the following result which concludes the paper

Corollary 3.5. *Let $\phi \in W^s(L_\infty(\mathbb{R}^d))$, $s > 0$, be a compactly supported refinable function. Then we have the Jackson inequality*

$$\sigma_m(f, X(\phi))_{\dot{F}_t^\beta(L_p)} \le Cm^{-(\gamma-\beta)/d}\|f\|_{\dot{B}_\tau^\gamma(L_\tau)}, \qquad \forall m \in \mathbb{N}, f \in \dot{B}_\tau^\gamma(L_\tau(\mathbb{R}^d)), \tag{3.2}$$

where $1 < p < \infty$, $0 < \gamma < s$, and $1/\tau := (\gamma-\beta)/d + 1/p$.

Moreover, if for $d > 1$ $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a locally linearly independent set (the condition is void if $d = 1$), then we have for each $K \ge 1$,

$$\mathcal{A}_q^{\alpha/d}\big(L_p(\mathbb{R}^d), X_K(\phi)\big) = \big(L_p(\mathbb{R}^d), B_\tau^\gamma(L_\tau(\mathbb{R}^d))\big)_{\alpha/\gamma,q},$$

for $1 < p < \infty$, $0 < \alpha < \gamma < s$, and $1/\tau := \gamma/d + 1/p$.

Proof. The Jackson inequality for $X_K(\phi)$ (for large K) follows from Proposition 2.5, and then the Jackson inequality for smaller K can be deduced from Proposition 3.4. The Bernstein inequality for $X_K(\phi)$ follows from the same string of arguments as used in the proof of Proposition 3.1. □

Remark 3.6. The Jackson inequality (3.2) given by Corollary 3.5 generalizes [1, Theorem 2.1] to the case where the approximation errors is measured on the full scale of Triebel-Lizorkin spaces and not just in the L_p-spaces.

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