# Uncertainty of Modal Parameters Estimated by ARMA Models

Jensen, Jacob Laigaard; Brincker, Rune; Rytter, Anders

*Published in:*
Proceedings of the 9th International Conference on Experimental Mechanics

*Publication date:*
1990

*Document Version*
Publisher's PDF, also known as Version of record

# UNCERTAINTY OF MODAL PARAMETERS ESTIMATED BY ARMA MODELS

Jakob Laigaard Jensen*), Rune Brincker*) and Anders Rytter**)
*) University of Aalborg
Sohngaardsholmsvej 57,DK-9000 Aalborg
Denmark
**) Rambøll & Hannemann
Kjærulfsgade 2, DK-9400 Nørresundby
Denmark

## SUMMARY

In this paper the uncertainties of identified modal parameters such as eigenfrequencies and damping ratios are assessed. From the measured response of dynamic excited structures the modal parameters may be identified and provide important structural knowledge. However the uncertainty of the parameters decides the value of the experimental obtained information. The paper deals with the sources of uncertainty. The different sources of uncertainty have been categorized into physical errors, statistical errors and model errors. The nature of the uncertainty due to those errors has been studied by a simulation study of a lightly damped single degree of freedom system. Identification by ARMA models has been chosen as system identification method. It is concluded that both the sampling interval and number of sampled points may play a significant role with respect to the statistical errors. Furthermore, it is shown that the model errors may also contribute significantly to the uncertainty.

## INTRODUCTION

In civil engineering the dynamic characteristics of structures become increasingly important. Due to limited theoretical knowledge it is useful to identify the dynamic characteristics by analysis of obtained measurements of the structural response.

The subject of this paper is to present and discuss the uncertainty of the dynamic characteristics given by a set of modal parameters $\overline{P}$ which can be estimated from e.g. the observed structural accelerations $\ddot{x}(t)$. The parameters will e.g. be given as:

$$\overline{P}^T = (f_1, f_2, \dots, f_n, \zeta_1, \zeta_2, \dots, \zeta_n) \tag{1}$$

where $f_i$ and $\zeta_i$ are denoted as respectively the $i$th eigenfrequency and damping ratio for a vibrating system of $n$ degrees of freedom. The parameters will be estimated by minimizing the difference between the measured data $\ddot{x}(t)$, transformed into an appropiate form $\mathbf{M}(\ddot{x}(t)|\overline{P}^*)$ and some given model $\mathbf{M}(\ddot{x}(t)|\overline{P})$:

$$\mathbf{ERROR}(\overline{P}, \ddot{x}(t)) = \|\mathbf{M}(\ddot{x}(t)|\overline{P}^*) - \mathbf{M}(\ddot{x}(t)|\overline{P})\| \tag{2}$$

where $\overline{P}^*$ denotes the true parameters of the system. The minimum error will provide the estimate of the parameters $\hat{\overline{P}}$.

Different identification approaches can be chosen. The data can be transformed into the frequency domain to spectral estimates and the parameters can be estimated by curvefitting of a given model. This approach is the conventional which has been widely applied during the last 25 years. However, the approach has shown to be ineffective in estimation of especially the damping of light damped structures, see [1]. In fact, it has been shown that the uncertainty of the damping ratio will typically lie in the range of 20-50 % when the conventional method is applied, see [2]. Thus in this paper the more effective approach called identification by ARMA models has been applied to evaluate the main sources of uncertainty of the modal parameters.

## SOURCES OF UNCERTAINTIES

If human errors are disregarded three different sources of uncertainty of the parameter estimates exist:

- Physical errors.
- Statistical errors.
- Model errors.

The physical errors are due to the fact that the measured data, on which the parameter estimation is based, will essential be realisations of random processes. This means unavoidable that the estimates will be random variables with an associated minimum uncertainty depending upon the given problem and modelling approach. This uncertainty is equivalent to the Rao-Cramér bound of the covariance matrix of the parameters, see e.g. [3].

The statistical errors are caused by the limited available information about the underlying random processes. In practice only realisations of the random processes are observed at discrete time instants $k\Delta$ within a finite time period $T$. This means the parameter estimates also will include statistical uncertainty.

The model errors will depend upon how well the practical available theories can provide models describing the real systems. Typical it is assumed that the structural system can be described by a linear time invariant viscous damped system with a finite number of degrees of freedom.

Thus while the physical errors always will contribute to parameter uncertainty the two other types of errors, the statistical and the model errors will depend solely on the engineer. Consequently, it is those contribution to uncertainty which will be considered in this paper.

## STRUCTURAL MODEL

A proper structural model of $n$ degrees of freedom is assumed to be:

$$\overline{\overline{M}}\ddot{\overline{x}}(t) + \overline{\overline{C}}\dot{\overline{x}}(t) + \overline{\overline{K}}\overline{x}(t) = \overline{a}(t) \tag{3}$$

where $\overline{\overline{M}}$, $\overline{\overline{C}}$ and $\overline{\overline{C}}$ are the $n$ x $n$ mass, viscous damping and stiffness matrix respectively. $\overline{a}(t)$ is the force process vector which is assumed to contain elements of white noise processes. The white noise assumption will often be a proper approximation in civil engineering where structures are randomly excited by wind or waves and the structures themselves can be considered as lightly damped systems, see e.g. [2].



It is furthermore assumed that (3) can be decoupled into $n$ equations which can be solved independently as

$$\ddot{\overline{z}}(t) + (\,2(2\pi f_i)\zeta_i\,)\,\dot{\overline{z}}(t) + (\,(2\pi f_i)^2\,)\,\overline{z}(t) = \frac{\overline{\overline{\Phi}}^T\,\overline{a}(t)}{\overline{\overline{\Phi}}^T\,\overline{\overline{M\Phi}}} \tag{4}$$

where the $n$ eigenfrequencies $f_i$, the $n$ damping ratios $\zeta_i$ and the $nxn$ eigenmode matrix $\overline{\overline{\Phi}}$ are given by the eigenvalue problem ($\overline{a}(t) \equiv \overline{0}$ in (3) ), see e.g. [4].

## IDENTIFICATION BY ARMA MODELS

The modal parameters given by (3) and (4) can be estimated by application of ARMA models. The identification by ARMA model gives a direct relation to the modal parameters while the FFT-analysis gives a nonparametric model which followed by a curvefitting algorithm gives the estimates of the modal parameters.

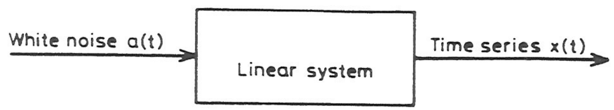


Fig. 1. White noise excited linear system.

An ARMA model can be found from the stationary Gaussian zero mean response of a linear system excited by Gaussian white noise, $a(t)$. The ARMA model of the measured response of a given point, $x(t)$ at discrete time intervals is defined by:

$$x_t = \underbrace{\sum_{i=1}^{n} \Phi_i x_{t-i}}_{AR-part} + a_t - \underbrace{\sum_{i=1}^{m} \Theta_i a_{t-i}}_{MA-part} \tag{5}$$

This is called an ARMA$(n,m)$ model (Auto Regressive Moving Average of order $(n,m)$ ). The parameters in the ARMA model are real numbers.

The appopiate order should be $(2n, 2n-1)$ for a white noise excited system with $n$ degrees of freedom. This choice will be a proper choice since it can be shown that for the assumed white noise excitation the theoretical covariance function of the sampled response will be equivalent to the covariance function derived from an ARMA$(2n, 2n-1)$ model, see [5]. In other words an ARMA model will provide an unbiased estimate of the response spectrum provided that the assumptions hold (linear model, white noise, $n$ degrees of freedom).

The parameters of the ARMA model and the associated covariance matrix are estimated from the time series $x_t \quad t = 1, 2, \ldots N$. This is done by minimizing the error function which in the present paper is identical with the computed variance of $a_t$ :

$$\sigma_a^2(\Phi_i, \Theta_j, x_t) = \frac{1}{N} \sum_{t=1}^{N} a_t^2 \quad (6)$$

The error function will be nonlinear with respect to the parameters which means that e.g. methods of nonlinear least squares have to be applied.

When the ARMA parameters and the residual $a_t$ have been estimated it must be checked whether or not $a_t$ is a realisation of white noise. If not, it indicates that the model order is too low, thus the theoretical expected model order $(2n, 2n-1)$ may be too low. This means that the residual $a_t$ consists of a white noise part plus a model error contribution. However, it can be shown that a proper ARMA model can be obtained, if the model order is increased until the model error has been minimized, see [6]. The two main reasons for increased model order are nonwhite excitation and nonlinearities which both can be considered as model errors.

The modal parameters are found from the $2n$ roots, $\lambda_i$ of the characteristic polynomial of the AR-parameters:

$$\lambda^{2n} - \Phi_1 \lambda^{2n-1} - \cdots - \Phi_{2n-1}\lambda - \Phi_{2n} = 0 \quad (7)$$

In e.g. [7] it is shown that the roots are related to the modal parameters through the $2n$ relations :

$$(\lambda_i) = (\exp(\mu_i \Delta)) \quad (8)$$

where $\Delta$ is the sampling interval and $\mu_i$ is the $i$th eigenvalue of (3) with $\overline{a}(t) \equiv \overline{0}$ related to the modal parameters by the following expression:

$$\mu_{(i)12} = -\omega_i\zeta_i \pm i\omega_i\sqrt{1-\zeta_i^2} \qquad \zeta_i < 1.0 \quad (9)$$

The index (12) refers here to the fact that the $\lambda_i$s are found as complex conjugated pairs if the modes are underdamped. It is seen that this set of equations gives the relation between the estimated AR parameters $\overline{\Phi}^T = (\Phi_1 \quad \Phi_2 \quad \ldots \quad \Phi_{2n})$ and the modal parameters $\overline{P}$.

Evaluation of the Covariance Matrix

The covariance matrix of the modal parameters $f_i$ and $\zeta_i$ can be computed from the covariance matrix of the AR parameters $\overline{\overline{\mathbf{cov}}}_{\overline{\Phi}}$ which will be a subresult of the ARMA model estimation in most software packages. Since the modal parameters are uniquely related to the AR-parameters:

$$\overline{P} = \overline{g}(\overline{\Phi}) \quad (10)$$

a covariance matrix for the modal parameters $\overline{\overline{\mathbf{cov}}}_{\overline{P}}$ can be obtained. $\overline{g}(\overline{\Phi})$ will generally be nonlinear thus to obtain a practical applicable approach, the functional relationship is linearized about the parameter estimates $\hat{\overline{P}}$:



$$\overline{P} = \hat{\overline{P}} + \left(\frac{\partial \overline{g}(\overline{\Phi})}{\partial \overline{\Phi}}\right)\Big|_{\overline{\Phi}=\hat{\overline{\Phi}}} (\overline{\Phi} - \hat{\overline{\Phi}}) \quad (11)$$

which is rewritten as:

$$d\overline{P} = \overline{\overline{S}}\, d\overline{\Phi} \quad (12)$$

$$\overline{\overline{S}} = \begin{pmatrix} \frac{\partial g_1(\overline{\Phi})}{\partial \Phi_1} & \frac{\partial g_1(\overline{\Phi})}{\partial \Phi_2} & \cdots & \frac{\partial g_1(\overline{\Phi})}{\partial \Phi_{2n}} \\ \frac{\partial g_2(\overline{\Phi})}{\partial \Phi_1} & \frac{\partial g_2(\overline{\Phi})}{\partial \Phi_2} & \cdots & \frac{\partial g_2(\overline{\Phi})}{\partial \Phi_{2n}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_{2n}(\overline{\Phi})}{\partial \Phi_1} & \frac{\partial g_{2n}(\overline{\Phi})}{\partial \Phi_2} & \cdots & \frac{\partial g_{2n}(\overline{\Phi})}{\partial \Phi_{2n}} \end{pmatrix} \quad (13)$$

Hereby the covariance matrix of $\overline{P}$ can be obtained:

$$\overline{\overline{\mathbf{cov}}}_{\overline{P}} = \overline{\overline{S}}\,\overline{\overline{\mathbf{cov}}}_{\overline{\Phi}}\,\overline{\overline{S}}^T \quad (14)$$

Naturally this expression will only be accurate when the error due to the linear approximation is small. The gradient matrix $\overline{\overline{S}}$ can easily be found analytical for the single degree of freedom case and otherwise by numerical calculations.

The approach of numerical covariance computation has been compared with a semi-analytical procedure developed in [8]. A good agreement was found for small and moderate uncertainty of the parameters.

## SIMULATION STUDY: THE SDOF CASE

In [8] it has been shown that the uncertainty of the modal parameters is relative insensitive with respect to the number of degrees of freedom of the system, thus in this paper a simulation study of the parameter uncertainty has been performed for an SDOF system:

$$\ddot{x}(t) + (4\pi f_1 \zeta_1)\dot{x}(t) + (2\pi f_1)^2 x(t) = a(t) \quad (15)$$

with $f_1 = 1.114$ Hz and $\zeta_1 = 0.02$ corresponding to a case also studied in [8].

With an SDOF system excited by white noise it follows that the proper model of the measured response $x(t)$ (or $\ddot{x}(t)$, $\dot{x}(t)$) will be an ARMA(2, 1) model.

The relationship between the ARMA(2, 1) model and the modal parameters should be noticed. From (7) to (9) it can be shown, see e.g. [6] that the white noise excited system will give the following relations between the AR parameters $\overline{\Phi}^T = (\Phi_1 \quad \Phi_2)$ and the modal parameters $\overline{P}^T = (f_1 \quad \zeta_1)$:

$$\Phi_1 = -2e^{-\zeta_1(2\pi f_1)\Delta}\cos((2\pi f_1)\Delta\sqrt{1-\zeta_1^2}) \quad (16)$$

$$\Phi_2 = e^{-2\zeta_1(2\pi f_1)\Delta} \quad (17)$$

It is seen that the AR parameters will depend on the two dimensionless quantities $f_1\Delta$ and $\zeta_1$ which will be given by the actual structure and the choice of $\Delta$.

## EFFECT OF STATISTICAL ERRORS

Since the realisation is of finite length and is obtained at discrete time instants there will exist two sources of statistical uncertainty:

- The sampling interval $\Delta$.
- The length of the realisation $T = N\Delta$.

For a given $\Delta$ the length of the realisation will be given by $N$. $N$ will typical be a number which is limited by the capacity of the available computer while $T$ will typically depend upon how long it is possible to measure the response under stationary conditions.

Both sources will influence the parameter uncertainty as well as the estimates themselves.

### The sampling interval $\Delta$

The influence of the sampling interval will depend on the dimensionless quantity $(\Delta f_1)$ which is seen from (16) and (17). This quantity will for a given system affect the AR parameters $\overline{\Phi}$ and their sensitivity to the modal parameters $f_1$ and $\zeta_1$. Thus for a given structure the sampling interval will influence the accuracy with which the AR parameters can be estimated given by the covariance matrix $\overline{\overline{\mathbf{cov}}}_{\overline{\Phi}}$. Furthermore, the sampling interval will also contribute to the sensitivity of the functional relation (7) to (9), mapping the AR-parameters into the modal parameters.

Figure 2 shows the coefficient of variation of the eigenfrequency and the damping ratio with respect to $(\Delta f_1)$ for different applied $N$. Generally, it is seen that the coefficient of variation of $f_1$ obtains its minimum for $(\Delta f_1) \approx 0.25$. With respect to the damping it is seen that the minimum apparently is obtained for $(\Delta f_1) \approx 0.45{-}0.48$. For $(\Delta f_1) \to 0.5$ (the Nyquist frequency

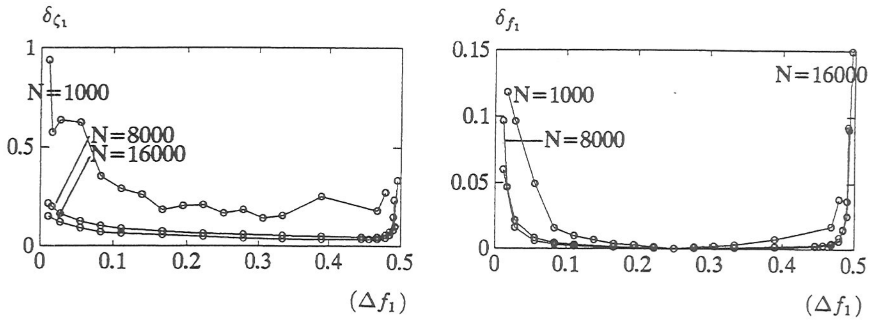


Fig. 2. The relation between the sampling interval $\Delta$ and the coefficient of variation of the eigenfrequency and the damping ratio.



$\frac{1}{2\Delta}$) the uncertainty is seen to increase dramatically corresponding to the fact that unambiguous information about the frequency content in the response is lost. For $(\Delta f_1) \to 0$ the ARMA model degenerates as shown in [6] and the modal parameters become more uncertain since the random character of the response is built into the ARMA model.

Thus with respect to the sampling interval the conclusion is that for a given system, an optimal choice of the sampling interval will exist depending on the eigenfrequency. The optimal choice will depend slightly on whether it is the accuracy of the eigenfrequency or the damping ratio which is optimized. However, it is seen that a good choice of $(\Delta f_1)$ will be in the range $0.2 - 0.3$. Thus for the identification of a system of several eigenmodes the sampling interval should be selected depending on which eigenmode is considered. It should be emphasized that the presented results and conclusions have been found to be valid for the identification of light damping only. For larger damping the optimal choice of the sampling interval is thought to be more sensitive with respect to the magnitude of the damping.

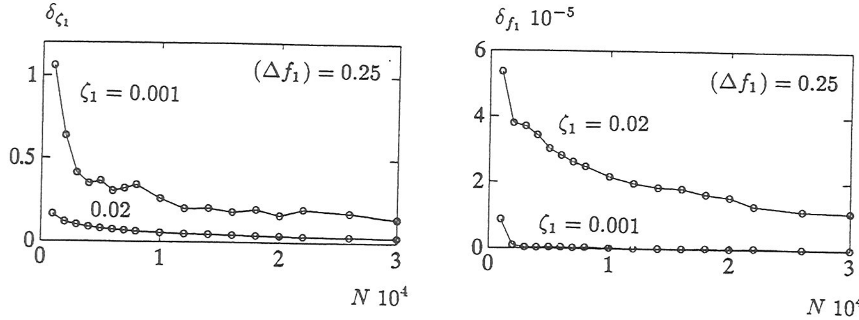


Fig. 3. The relation between the number of applied points $N$ and the coefficient of variation of the eigenfrequency and the damping ratio.

### The number of points $N$

As the length of the realisation increases the information about the underlying random process will increase and the variance will go asymtopically against the Rao-Cramér bound if the chosen estimation method is efficient. The influence of the number of points on the coefficient of variation of the eigenfrequency and the damping ratio was investigated for a sampling interval $(\Delta f_1) = 0.25$ sec. The results are shown in figure 3 for two different damping ratios $\zeta_1 = 0.001$ and $\zeta_1 = 0.02$.

First of all it is seen that the coefficient of variation seems to go asymtopical against a lower Rao-Cramér bound corresponding to the physical uncertainty of the problem. Secondly, a well-known fact is observed: The coefficient of variation of the eigenfrequency is inversely proportional with the damping ratio while the coefficient of variation of the damping ratio is direct proportional with the damping level, see also [8].

It is noticed that for a given $(\Delta f_1)$ the uncertainty can always be reduced by increasing the number of points $N$. This is seen in figure 2 by a wider flat minimum of the coefficient of variation with respect to $(\Delta f_1)$ for increasing $N$. However, a subject which remains to be investigated is the relation between the coefficient of variation and $T = N\Delta$ for different $(\Delta f_1)$.

The conclusion is that considering solely the problem of identification, $N$ should be chosen as large as possible. However, in practice for a given level of damping only limited improvement in accuracy is obtained by increasing $N$ when $N$ has obtained a given magnitude. The optimal choice of $N$ can in principle only be evaluated by a cost-benefit analysis.

## EFFECT OF MODEL ERRORS

Model errors will also be likely to increase the uncertainty of the estimated parameters. The model errors can be due to:

- Nonlinearity of the system.
- Nonwhite excitation.
- Wrong assumption about the number of degrees of freedom of the system.

Due to limited space only the effect of nonlinearities will be discussed. However, the applied approach will also be valid for the other types of model errors.

The system identification based on a linear model assumption can be considered as an equivalent linearization, see e.g. [9]. For the SDOF case a nonlinear structural system can be given by:

$$\ddot{x}(t) + g(x(t), \dot{x}(t)) = f(t) \tag{18}$$

which is approximated by the linear model:

$$\ddot{x}(t) + c_{eq}\dot{x}(t) + k_{eq}x(t) = f(t) + \epsilon(t|c_eq, k_eq) \tag{19}$$

by a minimization of the mean square error:

$$\mathrm{E}[(\epsilon(t|c_{eq}, k_{eq}))^2] == \mathrm{E}[(g(x(t), \dot{x}(t)) - c_{eq}\dot{x}(t) - k_{eq}x(t))^2] \tag{20}$$

with respect to the unknown parameters in the linear model (19). Thus it is seen that the model errors due to nonlinearities can be considered as an additional noise source. Consequently, it leads to an increase in the variance of the estimated parameters. Furthermore, this noise source will definitely in general not be Gaussian distributed, which means that unbiased convergence of the estimates is not guaranteed. Thus, structural nonlinearities will be an important source to parameter uncertainty.

### Example: Coulomb damped system

Since nonlinearities as a concept is quite arbitrary a specific example has been considered: The response of an SDOF system with $f_1 = 1.114$ Hz and a Coulomb damping force $f_d(t) = c\frac{\dot{x}}{|\dot{x}|}$ has been simulated by a Runge Kutta algorithm with a white noise excitation as input. The level of Coulomb damping was chosen so that the equivalent viscous damping ratio for the given excitation was approximately equal to $\zeta_1 = 0.02$.



An equivalent damping ratio was found by a least square reduction of the error given by (20) applying the simulated velocity $\dot{x}(t)$. The estimated equivalent damping ratio was found from an ARMA model of the simulated acceleration $\ddot{x}(t)$. This estimate should be consistent with the least square value obtained from (20) since both are based on a least square approximation of the acceleration of the actual nonlinear system fitted to a linear model. The ARMA model of the measured velocity or displacement would give a completely different approximation due to the nonlinearity.

It was found that the proper ARMA model of the measured response was not an ARMA$(2,1)$ model but an ARMA$(4,3)$ model. This was recognized by considering the autospectrum of the assumed white noise $a(t)$. The ARMA$(2,1)$ model gave a clear nonwhite spectrum typical with a peak occurrence at the frequency corresponding to the eigenfrequency $f_1$. It was interesting to note that even though the ARMA$(2,1)$ was the wrong model it gave anyway modal estimates with an uncertainty much less the modal estimates of the ARMA$(4,3)$. However, the modal estimates of the ARMA$(2,1)$ was significantly biased due to the nonlinearity of the system: In a single case the damping ratio was overestimated by a factor 4 and the eigenfrequency by 5%.

The results are shown in table 1 for three different values $(\Delta f_1)$ with $N = 30000$. It is seen that the eigenfrequency is estimated well, even though a slight overestimation might seem to be the case. The equivalent damping ratio lies in the range of uncertainty of the estimated equivalent damping ratio. The corresponding coefficient of variation of a linear system with $\zeta_1 = 0.02$ is given in the brackets. It is seen that the presence of nonlinearity in the system leads to a significant increase in the coefficient of variation due to the linear approximation.

| $(\Delta f_1)$ | True $f_1$ [Hz] | $\hat{f}_1$ [Hz] | $\delta_{f_1}$ [%] | Equiv. $\zeta_1$ by (20) | $\hat{\zeta}_1$ | $\delta_{\zeta_1}$ [%] |
|---|---|---|---|---|---|---|
| 0.11 | 1.114 | 1.1145 | 0.7 (0.2) | 0.0207 | 0.0226 | 43.0 (6.4) |
| 0.25 | 1.114 | 1.1155 | 0.1 (0.03) | 0.0204 | 0.0190 | 15.2 (3.9) |
| 0.40 | 1.114 | 1.1153 | 4.3 (0.1) | 0.0203 | 0.0204 | 33.0 (3.5) |

Table 1. The modal estimates and uncertainty of Coulomb damped SDOF system with a equivalent damping ratio of $\zeta_1 \approx 0.02$. $N = 30000$.

Due to the limited amount of results no conclusion can be given about the influence of $(\Delta f_1)$ in this nonlinear case. The presented results can not yet be generalized except for one point, namely the fact that the uncertainty of modal estimates will increase dramatically when model errors due to nonlinearities are present.

## CONCLUSION

In this paper the influence of the three principle sources of uncertainty has been pointed out. It has been concluded that:

- by choosing the number of data points $N$ large enough, the uncertainty due to the physical error corresponding to the Rao-Cramér bound can be quantified.

- when estimating modal parameters an optimal choice of the sampling interval $\Delta$ exists. However, the optimal choice will not be the same for the eigenfrequency and the damping ratio but will in both cases be relatively close to the Nyquist frequency.
- the accuracy of the eigenfrequency will increase but the accuracy of the damping ratio will decrease for decreasing level of damping.
- identification by ARMA models may give least square modal estimates of nonlinear systems but will depend upon whether records of accelerations, velocities or displacements have been applied.
- the uncertainty due to model errors of nonlinear system will give rise to a significant increase in the uncertainty of the modal parameters.

The above conclusions are expected to be further documented and generalized in future research.

## ACKNOWLEDGEMENTS

Financial support from the Danish Energy Agency and the Danish Technical Research Council is greatfully acknowledged.

## REFERENCES

1 Bendat, J. S. and Piersol, A. G.: "Random Data: Analysis and Measurement Procedures", Wiley-Interscience, New York, 1971.
2 Jensen, J. L.: "System Identification of Offshore Structures", Ph.D. Thesis, University of Aalborg, Aalborg, 1990.
3 Papoulis, A.: "Probability and Statistics", Prentice Hall, Englewood Cliffs, 1990.
4 Langen, I. and Sigbjornsson, R.: "Dynamisk Analyse av Konstruksjoner", Tapir, Trondheim, 1979.
5 Kozin, F. and Natke, N. G.: "System Identification Techniques", J. Structural Safety, 1986(3), 297-305.
6 Pandit, S. M. and Wu, S.-M.: "Time Series and System Analysis with Applications", John-Wiley, 1983.
7 Pi, Y. L. and Mickleborough, N. C.: "Modal Identification of Vibrating Structures Using ARMA Model", J. Engineering Mechanics, (1989) 115 (10), 2232-2250.
8 Gersch, W.: "On the Achievable Accuracy of Structural System Parameter Estimates", J. Sound and Vibration, (1974) 34(1), 63-79.
9 Lin, Y. K.: "Probabilistic Theory of Structural Dynamics", Mc.Graw-Hill Book Co., New York, 1967.